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Eigenvalue multiplicities of group elements in irreducible representations of simple linear algebraic groups

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Abstract

Let k be an algebraically closed field of characteristic $p \geq 0$, let G be a simple simply connected linear algebraic group of rank $\ell \geq 1$ over k and let V be a rational irreducible tensor-indecomposable finite-dimensional kG-module. For $g \in G$, let $V_g(\mu)$ denote the eigenspace corresponding to the eigenvalue $\mu \in k^*$ of g on V. We set

$$\nu_G(V) = \min\{\dim(V) - \dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in k^*\}.$$

In this thesis we will find all pairs (G, V) with the property that $\nu_G(V) \leq \sqrt{\dim(V)}$. This problem is an extension of the classification result obtained by Guralnick and Saxl for the condition $\nu_G(V) \leq \max\left\{2, \frac{\sqrt{\dim(V)}}{2}\right\}$. Moreover, for all the candidate pairs (G, V) we had to consider in our classification, we will determine the value of $\nu_G(V)$.

Key words: Representation theory, algebraic groups, semisimple elements, unipotent elements.

Résumé

Soit k un corps algébriquement clos de caractéristique $p \geq 0$, soit G un groupe linéaire algébrique simple simplement connexe de rang $\ell \geq 1$ sur k et soit V un kG-module rationnel, irréductible, tenseur-indécomposable de dimension finie. Pour $g \in G$, on désigne par $V_g(\mu)$ l'espace propre correspondant à la valeur propre $\mu \in k^*$ de g sur V. Nous définissons

$$\nu_G(V) = \min\{\dim(V) - \dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in k^*\}.$$

Dans cette thèse, nous trouverons toutes les paires (G, V) ayant la propriété que $\nu_G(V) \leq \sqrt{\dim(V)}$. Ce problème est une extension du résultat de classification obtenu par Guralnick et Saxl pour la condition $\nu_G(V) \leq \max\left\{2, \frac{\sqrt{\dim(V)}}{2}\right\}$. De plus, pour toutes les paires candidats (G, V) que nous avons dû considérer dans notre classification, nous allons déterminer la valeur de $\nu_G(V)$.

Mots clés : Théorie des représentations, groupes algébriques, éléments semisimples, éléments unipotents.

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Contents

Chapter 1

Introduction

The term *algebraic group* is believed to have first appeared in the late 1800s in the work of Émile Picard on the Galois theory of linear differential equations, where the Galois groups he was studying were in fact complex algebraic groups. The Galois theory of linear differential equations was also the starting point for the work of Ellis Kolchin on algebraic groups in [Kol48a] and [Kol48b]. Those results were taken and built upon by Armand Borel in his fundamental paper *Groupes lineaires algebriques* of 1956, [Bor56], where he explores the analogy between linear algebraic groups and Lie groups. This paper played an essential role in Claude Chevalley's classification of semisimple linear algebraic groups, which was announced during the *Séminaire sur la classification des groupes de Lie algébriques*, 1956-1958. This classification is one of the most crucial and important results in the theory of algebraic groups, as it allowed for a lot of progress to be made on topics such as the subgroup structure, conjugacy classes and representation theory of linear algebraic groups. In this thesis we consider such a question in the representation theory of linear algebraic groups.

Let k be an algebraically closed field of characteristic $p \geq 0$, let V be a finite-dimensional k-vector space and let H be a group acting linearly, irreducibly and primitively on V. For $h \in H$ denote by $V_h(\mu)$ the eigenspace corresponding to the eigenvalue $\mu \in k^*$ of h on V. Set $\nu_H(V) = \min\{\dim(V) - \dim(V_h(\mu)) \mid h \in H \setminus \mathrm{Z}(H) \text{ and } \mu \in k^*\}$. In 1991, Gordeev, [Gor91], set out to classify groups H acting linearly, irreducibly and primitively on a vector space V (over a field of characteristic zero) that contain an element h for which $\nu_H(V)$ is small when compared to $\dim(V)$. The following year, Hall, Liebeck and Seitz, [HLS92, Theorems 4 and 5], expanded on Gordeev's result by working over algebraically closed fields of arbitrary characteristic. In [HLS92, Theorem 5] they proved that, in the case of linear algebraic groups, if H is classical, we have $\nu_H(V) \geq \frac{n}{8(2\ell+1)}$, where ℓ is the rank of H and V is a faithful rational irreducible kH-module of dimension n; while, if H is not of classical type, then $\nu_H(V) > \frac{\sqrt{n}}{12}$. Now, with the lower-bounds for $\nu_H(V)$ known, the following natural step was to start the classification of pairs (H, V) with bounded $\nu_H(V)$ from above, in particular the pairs (H, V) with $\nu_H(V) = 1$ or $\nu_H(V) = 2$ have been of great interest, see for example [KW82], [KM97] and [Ver99]. In 2001, Guralnick and Saxl classified irreducible subgroups H of $\mathrm{GL}(V)$, where V is a finite-dimensional k-vector space of dimension $n > 1$, which act primitively and tensor-indecomposably on V and $\nu_H(V) \leq \max\{2, \frac{\sqrt{n}}{2}\}$, see [GS03, Theorem 7.1 and 8.3].

In this thesis we classify simple simply connected linear algebraic groups G and rational irreducible tensor-indecomposable finite-dimensional kG-modules V for which

$$\nu_G(V) < \sqrt{\dim(V)}.$$

We also determine the value of $\nu_G(V)$ for all the aforementioned pairs (G, V).

1.1 Statement of results

We begin this section by setting up the notation needed to state the main results of this thesis, see Theorems 1.1.1 and 1.1.3. The theory will be expanded upon in Chapter 2. With this, let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple, simply connected linear algebraic group of rank $\ell \geq 1$. Let T be a maximal torus in G and let $\mathrm{X}(T)$ be its rational character group. Let V be a nontrivial rational finite-dimensional kG-module and, for $g \in G$, let $V_g(\mu) := \{v \in V \mid g \cdot v = \mu \cdot v\}$ be the eigenspace corresponding to the eigenvalue $\mu \in k^*$ of g on V. We define

$$\nu_G(V) := \min\{\dim(V) - \dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in k^*\}.$$

In Theorem 1.1.1 we classify pairs (G, V), where V is a nontrivial rational irreducible tensor-indecomposable finite-dimensional kG-module, for which $\nu_G(V) \leq \sqrt{\dim(V)}$. The result follows from Theorem 1.1.3, in which the value of $\nu_G(V)$ for all candidate pairs (G, V) is given.

Theorem 1.1.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of rank $\ell \geq 1$. Let T be a maximal torus in G and let $\mathrm{X}(T)$ be its rational character group. Let $\lambda \in \mathrm{X}(T)$ be a nonzero p-restricted dominant weight and let $V = L_G(\lambda)$. Then*

$$\nu_G(V) \leq \sqrt{\dim(V)}$$

if and only if G, ℓ, λ and p are featured in the following list:

(1) $G = A_\ell$, $\ell \geq 1$, $\lambda \in \{\omega_1, \omega_\ell\}$ *and* $p \geq 0$;

(2) $G = B_\ell$, $\ell \geq 2$, $\lambda = \omega_1$ *and* $p \geq 0$;

(3) $G = C_\ell$, $\ell \geq 2$, $\lambda = \omega_1$ *and* $p \geq 0$;

(4) $G = D_\ell$, $\ell \geq 4$, $\lambda = \omega_1$ *and* $p \geq 0$;

(5) $G = A_\ell$, $\ell \leq 3$, $\lambda \in \{2\omega_1, 2\omega_\ell\}$ *and* $p \neq 2$;

(6) $G = A_\ell$, $\ell = 3, 4$, $\lambda \in \{\omega_2, \omega_{\ell-1}\}$ *and* $p \geq 0$;

(7) $G = A_1$, $\lambda \in \{3\omega_1, 4\omega_1\}$ *and* $p \neq 2, 3$;

(8) $G = C_2$, $\lambda = \omega_2$ *and* $p \geq 0$;

(9) $G = D_4$, $\lambda \in \{\omega_3, \omega_4\}$ *and* $p \geq 0$;

(10) $G = B_\ell$, $\ell = 3, 4$, $\lambda = \omega_\ell$ *and* $p \geq 0$;

(11) $G = C_\ell$, $\ell = 3, 4$, $\lambda = \omega_\ell$ *and* $p = 2$;

(12) $G = D_5$, $\lambda \in \{\omega_4, \omega_5\}$ *and* $p \geq 0$;

(13) $G = G_2$, $\lambda = \omega_1$ *and* $p \geq 0$;

(14) $G = G_2$, $\lambda = \omega_2$ *and* $p = 3$.

We compare our result with the result of Guralnick and Saxl, Theorem 8.3 of [GS03], for the case of simple simply connected linear algebraic groups and their respective irreducible tensor-indecomposable modules.

Theorem 1.1.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple, simply connected linear algebraic group of rank $\ell \geq 1$. Let T be a maximal torus in G and let $\mathrm{X}(T)$ be its rational character group. Let $\lambda \in \mathrm{X}(T)$ be a nonzero p-restricted dominant weight and let $V = L_G(\lambda)$. Then*

$$\max\{2, \frac{\sqrt{\dim(V)}}{2}\} < \nu_G(V) \leq \sqrt{\dim(V)}$$

if and only if G, ℓ, λ and p are featured in the following list:

(1) $G = A_3$, $\lambda \in \{2\omega_1, 2\omega_\ell\}$ *and* $p \neq 2$;

(2) $G = A_\ell$, $\ell = 3, 4$, $\lambda \in \{\omega_2, \omega_{\ell-1}\}$ *and* $p \geq 0$;

(3) $G = C_2$, $\lambda = \omega_2$ *and* $p \geq 0$;

(4) $G = B_4$, $\lambda = \omega_\ell$ *and* $p \geq 0$;

(5) $G = C_4$, $\lambda = \omega_\ell$ *and* $p = 2$;

(6) $G = D_5$, $\lambda \in \{\omega_4, \omega_5\}$ *and* $p \geq 0$.

In Section 2.7 we determine the list of nontrivial rational irreducible tensor-indecomposable kG-modules V which are candidates for the classification of Theorem 1.1.1. For classical groups, we will group these modules into two classes: one class will contain the *families of kG-modules*, while the other will consist of the *particular kG-modules*.

Theorem 1.1.3. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G, T and $L_G(\lambda)$ be as in Theorem 1.1.1. Then, the value of $\nu_G(L_G(\lambda))$ is given in the tables below:*

Group	λ	Characteristic	Rank	$\nu_G(L_G(\lambda))$	$\sqrt{\dim(L_G(\lambda))}$
A_ℓ	ω_1, ω_ℓ	$p \geq 0$	$\ell \geq 1$	1	$\sqrt{\ell+1}$
	$2\omega_1$, $2\omega_\ell$	$p \neq 2$	$\ell \geq 1$	ℓ	$\sqrt{\frac{(\ell+1)(\ell+2)}{2}}$
	ω_2, $\omega_{\ell-1}$	$p \geq 0$	$\ell \geq 3$	$\ell - 1$	$\sqrt{\frac{\ell(\ell+1)}{2}}$
	$\omega_1 + \omega_\ell$	$p \nmid \ell + 1$	$\ell \geq 2$	2ℓ	$\sqrt{\ell^2 + 2\ell}$
		$p \mid \ell + 1$	$\ell = 2$	3	$\sqrt{7}$
			$\ell \geq 3$	2ℓ	$\sqrt{\ell^2 + 2\ell - 1}$
C_ℓ	ω_1	$p \geq 0$	$\ell \geq 2$	1	$\sqrt{2\ell}$
	$2\omega_1$	$p \neq 2$	$\ell \geq 2$	2ℓ	$\sqrt{2\ell^2 + \ell}$
	ω_2	$p \geq 0$	$\ell = 2$	1	$\sqrt{5 - \delta_{p,2}}$
			$\ell \geq 3$	$2\ell - 2$	$\sqrt{2\ell^2 - \ell - 1 - \delta_{p,\ell}}$
B_ℓ	ω_1	$p \geq 0$	$\ell \geq 2$	1	$\sqrt{2\ell + 1 - \delta_{p,2}}$
	$2\omega_1$	$p \neq 2$	$\ell \geq 2$	2ℓ	$\sqrt{2\ell^2 + 3\ell - \delta_{p,2\ell+1}}$
	ω_2	$p \geq 0$	$\ell = 2$	1	$\sqrt{5 - \delta_{p,2}}$
		$p \neq 2$	$\ell \geq 3$	2ℓ	$\sqrt{2\ell^2 + \ell}$
		$p = 2$		$2\ell - 2$	$\sqrt{2\ell^2 + \ell - 1 - \delta_{\ell,2}}$
D_ℓ	ω_1	$p \geq 0$	$\ell \geq 4$	2	$\sqrt{2\ell}$
	$2\omega_1$	$p \neq 2$	$\ell \geq 4$	$4\ell - 4$	$\sqrt{2\ell^2 + \ell - 1 - \delta_{p,\ell}}$
	ω_2	$p \geq 0$	$\ell \geq 4$	$4\ell - 6$	$\sqrt{2\ell^2 - \ell - \delta_{p,2}\gcd(2,\ell)}$

Table 1.1.1: The value of $\nu_G(L_G(\lambda))$ for simple classical groups and their respective families of modules.

Group	λ	Characteristic	$\nu_G(L_G(\lambda))$	$\sqrt{\dim(L_G(\lambda))}$
A_1	$m\omega_1$, $3 \leq m \leq 8$	$p = 0$, or $p > m$	$m - \lfloor \frac{m}{2} \rfloor$	$\sqrt{m+1}$
A_3	$\omega_1 + \omega_2$	$p = 3$	6	4
A_5, A_6, A_7	ω_3, $\omega_{\ell-2}$	$p \geq 0$	$\frac{(\ell-2)(\ell-1)}{2}$	$\sqrt{\frac{(\ell-1)\ell(\ell+1)}{6}}$
C_2	$\omega_1 + \omega_2$	$p \geq 0$	$8 - 2\delta_{p,5}$	$\sqrt{16 - 4\delta_{p,5}}$
	$2\omega_2$	$p \neq 2$	4	$\sqrt{14 - \delta_{p,5}}$
	$3\omega_1$	$p \neq 2, 3$	10	$\sqrt{20}$
	$\omega_1 + 2\omega_2$	$p = 7$	12	$\sqrt{24}$
	$3\omega_2$	$p = 7$	9	5
	$2\omega_1 + \omega_2$	$p = 3$	9	5
$C_3, \ldots, C_8$	ω_ℓ	$p = 2$	$2^{\ell-2}$	$\sqrt{2^\ell}$
C_3	ω_3	$p \neq 2$	4	$\sqrt{14}$
	$\omega_1 + \omega_3$	$p = 2$	20	$\sqrt{48}$
C_4	ω_3	$p \geq 0$	$14 - \delta_{p,3}$	$\sqrt{48 - 8\delta_{p,3}}$
	ω_4	$p \neq 2$	$14 - \delta_{p,3}$	$\sqrt{42 - \delta_{p,3}}$
C_5	ω_3	$p = 2$	26	10
B_3	$2\omega_3$	$p \neq 2$	14	$\sqrt{35}$
	$\omega_1 + \omega_3$	$p \geq 0$	$20 - 2\delta_{p,7}$	$\sqrt{48 - 8\delta_{p,7}}$
$B_3, \ldots, B_8$	ω_ℓ	$p \geq 0$	$2^{\ell-2}$	$\sqrt{2^\ell}$
D_4	ω_3, ω_4	$p \geq 0$	2	$\sqrt{8}$
	$\omega_3 + \omega_4$ $\omega_1 + \omega_3$ $\omega_1 + \omega_4$	$p \geq 0$	$22 - 2\delta_{p,2}$	$\sqrt{56 - 8\delta_{p,2}}$
D_5	ω_3	$p = 2$	40	10
$D_5, \ldots, D_9$	$\omega_{\ell-1}$, ω_ℓ	$p \geq 0$	$2^{\ell-3}$	$\sqrt{2^{\ell-1}}$

Table 1.1.2: The value of $\nu_G(L_G(\lambda))$ for simple classical groups and their respective particular modules.

Group	λ	Characteristic	$\nu_G(L_G(\lambda))$	$\sqrt{\dim(L_G(\lambda))}$
G_2	ω_1	$p \geq 0$	2	$\sqrt{7 - \delta_{p,2}}$
	ω_2	$p \geq 0$	$6 - 4\delta_{p,3}$	$\sqrt{14 - 7\delta_{p,3}}$
F_4	ω_4	$p \geq 0$	6	$\sqrt{26 - \delta_{p,3}}$
	ω_1	$p \geq 0$	$16 - 10\delta_{p,2}$	$\sqrt{52 - 26\delta_{p,2}}$
E_6	ω_1 , ω_6	$p \geq 0$	6	$\sqrt{27}$
	ω_2	$p \geq 0$	22	$\sqrt{78 - \delta_{p,3}}$
E_7	ω_7	$p \geq 0$	12	$\sqrt{56}$

Table 1.1.3: The value of $\nu_G(L_G(\lambda))$ for simple exceptional groups and their respective modules.

1.2 The structure of the thesis

This introductory chapter will end with Section 1.3, in which we fix the notation and terminology we will use throughout the thesis. In Chapter 2, we set the theoretical groundwork we require to prove Theorems 1.1.1 and 1.1.3. The main goal of this chapter is to determine the kG-modules V, where G is a simple simply connected linear algebraic group, that are candidates for the classification in Theorem 1.1.1. Chapters 3 through 6 contain the proof of Theorem 1.1.1 in the case of the classical linear algebraic groups. In view of Proposition 2.2.3, the first part of each chapter will investigate eigenspace dimensions corresponding to semisimple elements, while the second will focus on eigenspace dimensions of unipotent elements. The final section of each of these chapters contains the proof of the results presented in Tables 1.1.1 and 1.1.2 for the respective group. Lastly, Chapter 7 completes the proof of Theorem 1.1.1, as it deals with the exceptional linear algebraic groups and their respective modules. The structure of the chapter is similar to the previous ones: in the fist part we study eigenspace dimensions corresponding to semisimple elements, in the second we study eigenspace dimensions of unipotent elements while in the last section we establish Table 1.1.3.

1.3 Notation and terminology

Throughout the thesis, unless otherwise mentioned, k is an algebraically closed field of characteristic $p \geq 0$ with additive group $\boldsymbol{G_a}$ and multiplicative group $\boldsymbol{G_m}$. Note that when we write $p \neq p_0$, for some prime p_0, we allow $p = 0$. We let G be a simple linear algebraic group of rank $\ell \geq 1$. We fix T a <u>maximal torus</u> in G and let $\mathrm{X}(T) := \mathrm{Hom}(T, \boldsymbol{G_m})$ be its group of <u>rational characters</u>. Let $\mathrm{N}_G(T)$ be the normalizer of T in G and set $\mathcal{W} := \mathrm{N}_G(T)/T$. The group $\mathcal{W}$ is called the <u>Weyl group</u> of G corresponding to T. Now, let Φ be the <u>root system</u> of G corresponding to T and let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a <u>set of simple roots</u> in Φ, where we use the standard Bourbaki labeling as given in [Hum72, 11.4, p.58]. All roots $\beta \in \Phi$ can be written as $\beta = \sum_{i=1}^{\ell} a_i\alpha_i$ with coefficients $a_i \in \mathbb{Z}$ all nonnegative, or all nonpositive. The subset $\Phi^+ := \left\{ \sum_{i=1}^{\ell} a_i\alpha_i \mid a_i \geq 0 \text{ for all } 1 \leq i \leq \ell \right\}$ of Φ is the <u>set of positive roots</u> of G. Following [Car89, Section 2.1], we fix a total order $\preceq$ on Φ: for $\alpha, \beta \in \Phi$ we have $\alpha \preceq \beta$ if and only if $\alpha = \beta$, or $\beta - \alpha = \sum_{i=1}^{r} a_i\alpha_i$ with $1 \leq r \leq \ell$, $a_i \in \mathbb{Z}$, $1 \leq i \leq r$, and $a_r > 0$.

Now, for each $\alpha \in \Phi$, there exists a morphism $x_\alpha : \boldsymbol{G_a} \to G$ of linear algebraic groups, which induces an isomorphism $x_\alpha : \boldsymbol{G_a} \to \mathrm{im}(x_\alpha)$ with the property that $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ for all $t \in T$ and all $c \in \boldsymbol{G_a}$. Such a morphism is unique up to multiplication by a scalar in k^*. Set $U_\alpha := \mathrm{im}(x_\alpha) = \{x_\alpha(c) \mid c \in k\}$ and call this one-dimensional subgroup the <u>root subgroup</u> of G, relative to T, associated to the root $\alpha \in \Phi$. An important property of root subgroups is that they generate the group G, i.e. we have that $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$. Lastly, we fix B to be the <u>positive Borel subgroup</u> of G, i.e. $B = \langle T, U_\alpha \mid \alpha \in \Phi^+ \rangle$.

Set $\mathrm{Y}(T) := \mathrm{Hom}(\boldsymbol{G_m}, T)$ to be the group of <u>rational cocharacters</u> of T. For any $\lambda \in$

$\mathrm{X}(T)$ and any $\phi \in \mathrm{Y}(T)$ there exists a unique integer $\langle \lambda, \phi \rangle$ such that the composition map $\lambda \circ \phi : \boldsymbol{G_m} \to \boldsymbol{G_m}$ is given by $c \to c^{\langle \lambda, \phi \rangle}$, where $c \in \boldsymbol{G_m}$. The pairing $\langle -, - \rangle$ on $\mathrm{X}(T) \times \mathrm{Y}(T)$ induces an isomorphism $\mathrm{X}(T) \cong \mathrm{Hom}(\mathrm{Y}(T), \mathbb{Z})$. For any $\alpha \in \Phi$, there exists a homomorphism $\phi_\alpha : \mathrm{SL}_2(k) \to G$ given by:

$$\phi_\alpha \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = x_\alpha(c) \text{ and } \phi_\alpha \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = x_{-\alpha}(c)$$

for all $c \in k^*$. We define the elements $n_\alpha(c) := x_\alpha(c) x_{-\alpha}(-c^{-1}) x_\alpha(c) = \phi_\alpha \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \in \mathrm{N}_G(T)$ and $h_\alpha(c) := n_\alpha(c) n_\alpha(-1) = \phi_\alpha \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \in T$, for all $c \in k^*$. Clearly, we have $h_\alpha \in \mathrm{Y}(T)$ and we call h_α the <u>coroot</u>, or <u>dual root</u>, corresponding to α. The set $\Phi^\vee = \{h_\alpha \mid \alpha \in \Phi\}$ is called the <u>dual root system</u> of Φ.

The elements of $\mathrm{X}(T)$ are called the <u>weights</u> of G. The subset $\mathrm{X}(T)^+ := \{\lambda \in \mathrm{X}(T) | \langle \lambda, h_{\alpha_i} \rangle \geq 0 \text{ for all } 1 \leq i \leq \ell\}$ of $\mathrm{X}(T)$ is called the <u>set of dominant weights</u> of G and an element $\lambda \in \mathrm{X}(T)^+$ is called a <u>dominant weight</u>. Moreover, we call a weight $\lambda \in \mathrm{X}(T)^+$ <u>p-restricted</u> if $0 \leq \langle \lambda, h_{\alpha_i} \rangle \leq p - 1$, for all $1 \leq i \leq \ell$. We adopt the usual convention that when $\mathrm{char}(k) = 0$, all weights are p-restricted. As G is simple, we have that Δ is a basis of $\mathrm{X}(T)^{\mathbb{Q}} := \mathrm{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Consequently, $\{h_{\alpha_i} \mid \alpha_i \in \Delta\}$ is a basis of $\mathrm{Y}(T)^{\mathbb{Q}} := \mathrm{Y}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and so there exist $\omega_{\alpha_i} \in \mathrm{X}(T)^{\mathbb{Q}}$ with $\langle \omega_{\alpha_i}, h_{\alpha_j} \rangle = \delta_{i,j}$, for all $\alpha_i,\ \alpha_j \in \Delta$. We call $\omega_i := \omega_{\alpha_i}$ the <u>fundamental dominant weight</u> of G corresponding to the simple root α_i, $1 \leq i \leq \ell$.

Set $\mathrm{X}(T)^{\mathbb{R}} := \mathrm{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that Φ, the set of roots of G with respect to T, is the set of nonzero weights of T in the action of G on its <u>Lie algebra</u>. These roots form an <u>abstract root system</u>, in the sense of [Hum72, Chapter 3], in an appropriate euclidean space $\mathrm{X}(T)^{\mathbb{R}}$. Let E be a fixed euclidean space. We call a root system Φ in E <u>indecomposable</u> if its base Δ cannot be partitioned into two proper mutually orthogonal subsets. Now, if Φ is an indecomposable root system, then, up to isomorphism, Φ is of one of the following types:

$$A_\ell\ (\ell \geq 1),\ C_\ell\ (\ell \geq 2),\ B_\ell\ (\ell \geq 3),\ D_\ell\ (\ell \geq 4),\ E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

The algebraic groups with root systems of types A_ℓ, C_ℓ, B_ℓ, or D_ℓ are called <u>classical</u>, while the ones with root systems of type E_6, E_7, E_8, F_4, or G_2 are called <u>exceptional</u>. A surjective homomorphism of algebraic groups $\phi : \tilde{G} \to G$ with finite kernel is called an <u>isogeny</u> and the two groups $\tilde{G}$ and G are called <u>isogenous</u>. The various types of simple algebraic groups with the same root system Φ are called the <u>isogeny types</u> of Φ. We remark that simple algebraic groups have indecomposable root systems and each indecomposable root system corresponds to an isogeny class of simple algebraic groups. We say that G is of <u>adjoint type</u> if $\mathrm{X}(T) = \mathbb{Z}\Phi$ and we say that G is of <u>simply connected type</u> if $\mathrm{Y}(T) = \mathbb{Z}\Phi^\vee$. Table 9.2 of [MT11] lists the various isogeny types of simple algebraic groups.

Suppose that G is simply connected. Then, in particular, $\omega_i \in \mathrm{X}(T)$, for all $1 \leq i \leq \ell$, and, in this case, $\{\omega_i \mid 1 \leq i \leq \ell\}$ is a base of $\mathrm{X}(T)$. Consequently, we can write each weight $\lambda \in \mathrm{X}(T)$ as $\lambda = \sum_{i=1}^{\ell} d_i \omega_i$ with $d_i \in \mathbb{Z}$. We have that $\lambda \in \mathrm{X}(T)^+$ if and only if $d_i \geq 0$, for all $1 \leq i \leq \ell$. Moreover, if $\mathrm{char}(k) = p > 0$, then λ is p-restricted if and only if $0 \leq d_i \leq p - 1$, for all $1 \leq i \leq \ell$.

For $\alpha \in \Phi$, let $s_\alpha \in \mathcal{W}$ be the reflection corresponding to α, i.e. $s_\alpha : \mathrm{X}(T)^{\mathbb{R}} \to \mathrm{X}(T)^{\mathbb{R}}$ is given by $s_\alpha(\lambda) = \lambda - \langle \lambda, h_\alpha \rangle \alpha$ for $\lambda \in \mathrm{X}(T)^{\mathbb{R}}$. Now, $\mathcal{W}$ is a finite group and is generated by $\{s_{\alpha_1}, \ldots, s_{\alpha_\ell}\}$. The element $w_0 \in \mathcal{W}$ with the property that $w_0(\Delta) = -\Delta$ is called the longest element of $\mathcal{W}$ with respect to Δ. Note that $w_0 = -\mathrm{id}_{\mathrm{X}(T)}$ when the root system Φ is of type A_1, B_ℓ, C_ℓ, D_ℓ with ℓ even, E_7, E_8, F_4 and G_2.

Let V be a finite-dimensional k-vector space. A morphism $\rho : G \to \mathrm{GL}(V)$ of algebraic groups is called a rational representation of G. Similarly, we call the kG-module V rational if its corresponding representation is rational. From this point onward, all representations and all modules of a linear algebraic group are assumed to be rational. For a nontrivial kG-module V we will use the notation $V = W_1 \mid W_2 \mid \cdots \mid W_m$ to express that V has a composition series $V = V_1 \supset V_2 \supset \cdots \supset V_m \supset V_{m+1} = 0$ with composition factors $W_i \cong V_i/V_{i+1}$, $1 \leq i \leq m$. Another notation we will use is V^m for the direct sum $V \oplus \cdots \oplus V$, in which V occurs $m \geq 2$ times.

We summarize the presentation of the notation used in this thesis in the following list. A more complete list can be found on page 270.

- G is a simple linear algebraic group of rank $\ell \geq 1$ over the algebraically closed field k.
- T is a maximal torus of G and $\mathrm{X}(T)$ is its group of rational characters.
- Φ is the root system of G given by T and, for $\alpha \in \Phi$, U_α is the root subgroup of G corresponding to the root α.
- B is a Borel subgroup of G containing the maximal torus T, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is the set of simple roots in Φ determined by B and Φ^+ is the set of positive roots in Φ with respect to Δ. We usually choose B to be the positive Borel subgroup, i.e. $B = \langle T, U_\alpha \mid \alpha \in \Phi^+ \rangle$.
- $\mathcal{W} = \mathrm{N}_G(T)/T$ is the Weyl group of G with respect to T and $s_\alpha \in \mathcal{W}$ is the reflection corresponding to $\alpha \in \Phi$. Moreover, $w_0 \in \mathcal{W}$ is the longest element.
- $\mathrm{Y}(T)$ is the set of rational cocharacters of T and $\Phi^\vee = \{h_\alpha \mid \alpha \in \Phi\}$ is the dual root system of Φ.
- $\mathrm{X}(T)^+ \subseteq \mathrm{X}(T)$ is the set of dominant weights.
- $\omega_1, \ldots, \omega_\ell \in \mathrm{X}(T)^{\mathbb{Q}}$ are the fundamental dominant weights of G with respect to Δ and we label them with the standard Bourbaki labeling, as given in [Hum72, p.58].
- For $\lambda \in \mathrm{X}(T)^+$, we denote by $L_G(\lambda)$, respectively by $V_G(\lambda)$, the irreducible, respectively the Weyl, kG-module with highest weight λ. Moreover, we let $\operatorname{rad} V_G(\lambda)$ be the unique maximal submodule of $V_G(\lambda)$.

Chapter 2

Theoretical background and preliminary results

The main goal of this introductory chapter is to establish for a simple simply connected linear algebraic group the list of irreducible tensor-indecomposable modules that are candidates for the classification in Theorem 1.1.1. Along the way we establish some preliminary results required in the proofs of Theorems 1.1.1 and 1.1.3, respectively. We begin with a discussion on nondegenerate bilinear forms and we present the way linear algebraic groups of classical type arise as *algebraic groups of automorphisms* of a vector space, [Bor56]. Now, to understand linear algebraic groups, it is only natural to start their study at the level of individual elements. In Section 2.2, we do just this, with emphasis on the classes of semisimple and unipotent elements. Afterwards, in Section 2.3, we discuss the representation theory of linear algebraic groups. Here we recall that, up to isomorphism, irreducible tensor-indecomposable modules of an algebraic group are parametrized by the p-restricted dominant weights of that group. In Section 2.4, we outline two algorithms, one for semisimple elements and one for unipotent elements, which will be used to calculate eigenspace dimensions. This is followed by a brief presentation of Gurlanick and Saxl's generation results for linear algebraic groups, [GS03]. These generation results will be used to establish a dimensional criteria, Section 2.6, which, in turn, will be used to determine the complete list of candidate modules, see Section 2.7. The chapter ends with a section on unipotent elements, in which we present, using various methods, the classification of unipotent conjugacy classes in the classical linear algebraic groups.

2.1 Bilinear forms and isometry groups

Throughout this section k is an algebraically closed field of characteristic $p \geq 0$ and V is an n-dimensional k-vector space, for some $n \geq 1$. Following [Gro02], we will present the simple classical algebraic groups as *algebraic groups of automorphisms of a vector space*, [Bor56], and show how they arise as closed subgroups of $\mathrm{GL}(V)$.

2.1.1 The special linear group

The general linear group of V, denoted $\mathrm{GL}(V)$, is the group of all invertible linear transformations on V. Fixing a basis in V establishes a group isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}_n(k)$, where $\mathrm{GL}_n(k)$ is the group of invertible $n \times n$ matrices with coefficients in k. Now, the determinant $\det : \mathrm{GL}(V) \to k^*$ is a group homomorphism with $\ker(\det) = \mathrm{SL}(V)$, the special linear group. Therefore, by definition, we have:

$$\mathrm{SL}(V) := \{\sigma \in \mathrm{GL}(V) |\ \det(\sigma) = 1\}.$$

We remark that $\mathrm{Z}(\mathrm{SL}(V)) = \{c \cdot \mathrm{id}_V \mid c \in k^*,\ c^n = 1\}$. Let $\mathrm{PGL}(V) = \mathrm{GL}(V)/\,\mathrm{Z}(\mathrm{GL}(V))$, where $\mathrm{Z}(\mathrm{GL}(V))$ is the subgroup of $\mathrm{GL}(V)$ consisting of all scalar transformations of V. Now, $\mathrm{PGL}(V)$ is isomorphic to the simple adjoint linear algebraic group of type A_{n-1}, see [Car89, Theorem 11.3.2], while $\mathrm{SL}(V)$ is isomorphic to the simple simply connected linear algebraic group of type A_{n-1}, see [Car89, p.184].

2.1.2 Bilinear forms

A bilinear form on V is a function $b : V \times V \to k$ which satisfies:

$$b(v_1 + v_2, u) = b(v_1, u) + b(v_2, u) \text{ and } b(cv, u) = cb(v, u)$$

$$b(v, u_1 + u_2) = b(v, u_1) + b(v, u_2) \text{ and } b(v, cu) = cb(v, u)$$

for all $v, v_1, v_2, u, u_1, u_2 \in V$ and all $c \in k$. Let $\{v_1, \ldots, v_n\}$ be a fixed basis in V and let b be a bilinear form on V. The matrix $B = (b_{i,j})_{i,j} \in \mathrm{M}_n(k)$ given by $b_{i,j} = b(v_i, v_j)$, for all $1 \leq i, j \leq n$, is called the representing matrix of b relative to $\{v_1, \ldots, v_n\}$.

We say that the bilinear form b on V is symmetric if $b(v_1, v_2) = b(v_2, v_1)$, for all $v_1, v_2 \in V$. We remark that b is symmetric if and only if its representing matrix B is symmetric, i.e. $B^{\mathrm{tr}} = B$. We say that b is alternating if $b(v, v) = 0$, for all $v \in V$. Hence, if b is an alternating form, then $b(v_1, v_2) = -b(v_2, v_1)$, for all $v_1, v_2 \in V$. We note that when $\mathrm{char}(k) = 2$, an alternating form is, in particular, symmetric.

Let $v_1, v_2 \in V$. We say that v_1 is orthogonal to v_2 and write $v_1 \perp v_2$ if $b(v_1, v_2) = 0$. We call b reflexive if orthogonality is a reflexive relation on V, i.e. $v_1 \perp v_2 \Rightarrow v_2 \perp v_1$. Note that a bilinear form is reflexive if and only if it is either symmetric or alternating, [Gro02, Proposition 2.7]. From this point onward, all the forms we consider will be reflexive.

For a subspace W of V, the set $W^\perp = \{v \in V \mid b(v, u) = 0 \text{ for all } u \in W\}$ is called the orthogonal complement of W in V. We define the radical of W as $\mathrm{Rad}(W) = W \cap W^\perp$. In particular, for $W = V$, the set $\mathrm{Rad}(V) := V^\perp = \{v \in V |\ b(v, u) = 0 \text{ for all } u \in V\}$ is called the radical of V with respect to b. We call the bilinear form b nondegenerate if $\mathrm{Rad}(V) = 0$. Now, if $\mathrm{Rad}(W) = 0$, then we call W a nondegenerate subspace relative to b and we note that $\mathrm{Rad}(W) = 0$ if and only if the restriction of b to W, $b\,|_{W \times W}$, is a nondegenerate form on W.

The dual space of V is defined to be the k-vector space $V^* := \mathrm{Hom}(V, k)$. Now, as V is a finite-dimensional k-vector space, it follows that V^* is also finite-dimensional, where the set $\{v_1^*, \ldots, v_n^*\} \subseteq V^*$ with the property that $v_i^*(v_j) = \delta_{ij}$, for all $1 \leq i, j \leq n$, is a basis of V^*. This basis is called the dual basis. We note that the bilineat form b is nondegenerate if

for all $f \in V^*$, there exist $v_1, v_2 \in V$ such that $f(u) = b(v_1, u) = b(u, v_2)$ for all $u \in V$, see [Gro02, Corollary 2.2].

Let V_1, V_2 be two k-vector spaces equipped with bilinear forms b_1 and b_2, respectively. A k-isomorphism $\phi : V_1 \to V_2$ is an isometry with respect to b_1 and b_2 if

$$b_2(\phi(v_1), \phi(v_2)) = b_1(v_1, v_2), \text{ for all } v_1, v_2 \in V_1.$$

If an isometry exists, then the two forms are called equivalent.

Alternating bilinear forms: We assume that b is an alternating bilinear form on the k-vector space V. If $v, u \in V$ are such that $b(v, u) \neq 0$, then the set $\{v, u\}$ is linearly independent. We call the pair $(v, u) \in V \times V$ a hyperbolic pair if $b(v, u) = 1$ and we call the subspace $\langle v, u\rangle$ a hyperbolic plane. We note that $b\mid_{\langle v,u\rangle}$ has representing matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ relative to the basis $\{v, u\}$.

For $\ell \geq 1$, set K_ℓ to be the $\ell \times \ell$ matrix given by $K_\ell := \begin{pmatrix} 0 & \cdots & 1 \\ & \iddots & \\ 1 & \cdots & 0 \end{pmatrix}$.

Theorem 2.1.1. [Gro02, Theorem 2.10] *Let V be a finite-dimensional k-vector space equipped with an alternating bilinear form b. If $\dim(V) = n$, then V has a basis $\{v_1, \ldots, v_\ell, w_1, \ldots, w_{n-2\ell}, u_\ell, \ldots, u_1\}$ with the property that $\{v_i, u_i\}$ is a hyperbolic pair for all $1 \leq i \leq \ell$, $\{w_1, \ldots, w_{n-2\ell}\}$ is a basis of $\mathrm{Rad}(V)$ and*

$$V = \bigoplus_{i=1}^{\ell} \langle v_i, u_i\rangle \oplus \mathrm{Rad}(V)$$

is an orthogonal direct sum. In particular, with respect to this basis, the representing matrix of b has the form $B = \begin{pmatrix} & & K_\ell \\ & 0_{n-2\ell} & \\ -K_\ell & & \end{pmatrix}$.

Assume that the alternating form b on V is nondegenerate. Then, in particular, we have that $\dim(V) = n = 2\ell$ is even, where $\ell \geq 1$. An ordered basis $\{v_1, \ldots, v_\ell, u_\ell, \ldots, u_1\}$ of V, where $\{v_i, u_i\}$ is a hyperbolic pair for all $1 \leq i \leq \ell$, as in Theorem 2.1.1, is called a symplectic basis for V and V is called a symplectic space. An invertible linear transformation σ of V is called symplectic if $b(\sigma(v_1), \sigma(v_2)) = b(v_1, v_2)$, for all $v_1, v_2 \in V$. We define the symplectic group on V, denoted by $\mathrm{Sp}(V)$, as the subgroup of $\mathrm{GL}(V)$ consisting of all symplectic transformations on V.

Remark 2.1.2. [Gro02, Corollary 2.12] *The symplectic group $\mathrm{Sp}(V)$ does not depend in a significant way on the choice of b, as any nondegenerate alternating bilinear form leads to the same group, up to conjugacy, in $\mathrm{GL}(V)$.*

We finish this paragraph by making a few remarks. Note that $\mathrm{Sp}(V) \leq \mathrm{SL}(V)$, [Gro02, Corollary 3.5]. Moreover, if $\dim(V) = 2$, then $\mathrm{Sp}(V) = \mathrm{SL}(V)$, [Gro02, Proposition 3.1]. We

fix a basis in V and we let $\mathrm{CS_p}(V)$ be the conformal symplectic group, i.e.

$$\mathrm{CS_p}(V) = \{\sigma \in \mathrm{GL}(V) \mid T_\sigma^{\mathrm{tr}} \begin{pmatrix} 0 & K_\ell \\ -K_\ell & 0 \end{pmatrix} T_\sigma = c \begin{pmatrix} 0 & K_\ell \\ -K_\ell & 0 \end{pmatrix}, \text{ for some } c \in k^*\},$$

where T_σ is the matrix representing σ with respect to the basis we fixed in V. Moreover, let $\mathrm{PCS_p}(V) := \mathrm{CS_p}(V)/\,\mathrm{Z}(\mathrm{CS_p}(V))$. Now, by [Car89, Theorem 11.3.2], $\mathrm{PCS_p}(V)$ is isomorphic to the simple adjoint linear algebraic group of type C_ℓ, while $\mathrm{Sp}(V)$ is isomorphic to the simple simply connected linear algebraic group of type C_ℓ, see [Car89, p.184].

Symmetric and quadratic forms in characteristic $\neq 2$: For the moment, we assume that $\mathrm{char}(k) \neq 2$. We let V be a finite-dimensional k-vector space equipped with a symmetric bilinear form b. We define the associated quadratic form to be the map $Q : V \to k$ given by $Q(v) = b(v, v)$, for $v \in V$. We note that b is completely determined by its associated quadratic form Q as:

$$b(v_1, v_2) = \frac{1}{2}\Big[Q(v_1 + v_2) - Q(v_1) - Q(v_2)\Big] \text{ for all } v_1, v_2 \in V.$$

Theorem 2.1.3. [Gro02, Theorem 4.2] *Let V be a finite-dimensional vector space over the algebraically closed field k with* $\mathrm{char}(k) \neq 2$. *Let* $\dim(V) = n$ *and assume that V is equipped with a symmetric bilinear form b. Then V admits an orthogonal basis $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ with the property that the representing matrix of b has the form* $B = \begin{pmatrix} b_1 & 0 & & & & 0 \\ 0 & \ddots & & & & \\ & & b_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix}$, *where $b_i \neq 0$ for all $1 \leq i \leq r$ and $\{v_{r+1}, \ldots, v_n\}$ is a basis of* $\mathrm{Rad}(V)$.

A nonzero vector $v \in V$ is called isotropic if $b(v, v) = 0$ and anisotropic if $b(v, v) \neq 0$. If V contains an isotropic vector, then b (respectively Q) and V are called isotropic. Le W be a subspace of V. If $b(w, w) = 0$ for all $w \in W$, then W is a totally isotropic subspace of V. Similarly, if $b(v, v) = 0$ for all $v \in V$, then we call V totally isotropic. Lastly, we call the pair of isotropic vectors $(v, u) \in V \times V$ a hyperbolic pair if $b(v, u) = 1$. The subspace $\langle v, u\rangle$ is called a hyperbolic plane and the restriction of b to $\langle v, u\rangle$, $b\,|_{\langle v,u\rangle}$, has representing matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the basis $\{v, u\}$.

Corollary 2.1.4. *Let V be a finite-dimensional vector space over the algebraically closed field k with* $\mathrm{char}(k) \neq 2$. *Let* $\dim(V) = n$ *and assume that V is equipped with a nondegenerate symmetric bilinear form b. Let Q be the associated quadratic form. Then one of the following holds:*

(1) *We have $n = 2\ell$ and V admits a basis $\{v_1, \ldots, v_\ell, u_\ell, \ldots, u_1\}$, where (v_i, u_i) is a hyperbolic pair, for all $1 \leq i \leq \ell$, with the property that*

$$V = \bigoplus_{i=1}^{\ell} \langle v_i, u_i\rangle$$

is an orthogonal direct sum of hyperbolic planes. Moreover, the representing matrix of b has the form $B = \begin{pmatrix} & K_\ell \\ K_\ell & \end{pmatrix}$, *where* $K_\ell = \begin{pmatrix} 0 & \cdots & 1 \\ & \iddots & \\ 1 & \cdots & 0 \end{pmatrix}$.

(2) *We have $n = 2\ell + 1$ and V admits a basis $\{v_1, \ldots, v_\ell, w, u_\ell, \ldots, u_1\}$, where w is such that $Q(w) = 1$ and (v_i, u_i) is a hyperbolic pair, for all $1 \leq i \leq \ell$, with the property that*

$$V = \bigoplus_{i=1}^{\ell} \langle v_i, u_i \rangle \oplus \langle w \rangle$$

is an orthogonal direct sum. Moreover, the representing matrix of b has the form $B = \begin{pmatrix} & & K_\ell \\ & 1 & \\ K_\ell & & \end{pmatrix}$.

Proof. Let $\{w_1, \ldots, w_n\}$ be the orthogonal basis of V given in Theorem 2.1.3. Then, the representing matrix of b has the form $B = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix}$, where $b_i \neq 0$ for all $1 \leq i \leq n$. We first consider the case when $n = 2\ell$. For every $1 \leq i \leq \ell$, set $v_i := w_i + cw_{n+1-i} \in V$ and $u_i := \frac{1}{2b_i}(w_i - cw_{n+1-i}) \in V$, where $c \in k$ is such that $c^2 = -\frac{b_i}{b_{n+1-i}}$. In what follows we show that (v_i, u_i) is a hyperbolic pair.

For all $1 \leq i \leq \ell$, we see that $Q(v_i) = Q(w_i + cw_{n+1-i}) = Q(w_i) + c^2 Q(w_{n+1-i}) = b_i + c^2 b_{n+1-i} = 0$ and $Q(u_i) = Q(\frac{1}{2b_i}(w_i - cw_{n+1-i})) = \frac{1}{4b_i^2}[Q(w_i) + c^2 Q(w_{n+1-i})] = 0$, as $b(w_i, w_{n+1-i}) = 0$. Secondly, we have $b((w_i + cw_{n+1-i}), \frac{1}{2b_i}(w_i - cw_{n+1-i})) = \frac{1}{2b_i}[b_i - c^2 b_{n+1-i}] = 1$, as $-c^2 b_{n+1-i} = b_i$, for all $1 \leq i \leq \ell$. Thereby, (v_i, u_i) is a hyperbolic pair for all $1 \leq i \leq \ell$. Therefore, we have $V = \bigoplus_{i=1}^{\ell} \langle v_i, u_i \rangle$. Moreover, as $\{w_1, \ldots, w_n\}$ are orthogonal, it follows from the definition of v_i and u_i that $b(v_i, v_j) = b(u_i, u_j) = 0$ for all $1 \leq i, j \leq \ell$ and that $b(v_i, u_j) = 0$ for all $1 \leq i, j \leq \ell$ with $i \neq j$. Thus, any two hyperbolic planes $\langle v_i, u_i \rangle$ and $\langle v_j, u_j \rangle$, $1 \leq i, j \leq \ell$, are mutually orthogonal. Lastly, we note that the representing matrix of b has the form $B = \begin{pmatrix} & K_\ell \\ K_\ell & \end{pmatrix}$, with respect to the basis $\{v_1, \ldots, v_\ell, u_\ell, \ldots, u_1\}$ of V.

The case of $n = 2\ell + 1$ is similar to that of $n = 2\ell$: we set $w = \frac{1}{\sqrt{b_{\ell+1}}} w_{\ell+1}$ and, for all $1 \leq i \leq \ell$, we set $v_i := w_i + cw_{n+1-i}$ and $u_i := \frac{1}{2b_i}(w_i - cw_{n+1-i})$. Now (v_i, u_i) is a hyperbolic pair and the hyperbolic planes $\langle v_i, u_i \rangle$ are mutually orthogonal. Moreover, we have that $Q(w) = Q(\frac{1}{\sqrt{b_{\ell+1}}} w_{\ell+1}) = \frac{1}{b_{\ell+1}} Q(w_{\ell+1}) = 1$ and $b(w, v_i) = b(w, u_i) = 0$ for all $1 \leq i \leq \ell$, as $b(w_{\ell+1}, w_i) = 0$ for all $1 \leq i \leq n$, $i \neq \ell + 1$. Therefore, $V = \bigoplus_{i=1}^{\ell} \langle v_i, u_i \rangle \oplus \langle w \rangle$ is an orthogonal direct sum of mutually orthogonal subspaces and the representing matrix of b has the form $B = \begin{pmatrix} & & K_\ell \\ & 1 & \\ K_\ell & & \end{pmatrix}$, with respect to the basis $\{v_1, \ldots, v_\ell, w, u_\ell, \ldots, u_1\}$ of V. $\square$

A vector space V equipped with a nondegenerate symmetric bilinear form b is called a quadratic space. An isometry of V is called an orthogonal linear transformation and we define $\mathrm{O}(V)$ to be the group of isometries of V, i.e.

$$\mathrm{O}(V) = \{\sigma \in \mathrm{GL}(V) |\ b(\sigma(v_1), \sigma(v_2)) = b(v_1, v_2) \text{ for all } v_1, v_2 \in V\} \leq \mathrm{GL}(V).$$

We note that $\sigma \in \mathrm{O}(V)$ if and only if $Q(\sigma(v)) = Q(v)$ for all $v \in V$.

We fix a basis in V as in Corollary 2.1.4 and let $\sigma \in \mathrm{GL}(V)$. Then $\sigma \in \mathrm{O}(V)$ if and only if $T_\sigma^{\mathrm{tr}} B T_\sigma = B$, where B is given in Corollary 2.1.4 and T_σ is the matrix representing σ with respect to the given basis. Assume that $\sigma \in \mathrm{O}(V)$. Then, $\det(T_\sigma) = \pm 1$. We define the special orthogonal group $\mathrm{SO}(V)$ to be the subgroup of $\mathrm{O}(V)$ given by

$$\mathrm{SO}(V) := \{\sigma \in \mathrm{O}(V) \mid \det(T_\sigma) = 1\}.$$

Let $0 \neq v \in V$ be an anisotropic vector and define the linear transformation σ_v, called the reflection along v through the hyperplane $v^\perp$, by:

$$\sigma_v(u) = u - 2\frac{b(u,v)}{Q(v)}v, \text{ for all } u \in V.$$

Note that a quadratic space admits anisotropic vectors, [Gro02, Proposition 4.1]. One checks that $\sigma_v \in \mathrm{O}(V)$ and that $\sigma_v \notin \mathrm{SO}(V)$ for all $v \in V$. Now, assume that the vector space V is such that $\dim(V) = 2$. Then, by [Gro02, Proposition 6.1], we have that $\mathrm{SO}(V) \cong k^*$ and $\mathrm{O}(V) = \mathrm{SO}(V) \rtimes < \sigma_v >$ for any reflection $\sigma_v \in \mathrm{O}(V)$. Therefore, we will assume that $\dim(V) = n \geq 3$. Let $\Omega(V) = \mathrm{O}'(V)$ denote the derived group of $\mathrm{O}(V)$. The following results exhibits some of the properties of the group $\Omega(V)$. Proofs for these can be found in [Gro02, Section 6].

Proposition 2.1.5. *Let V be a finite-dimensional k-vector space of dimension $n \geq 3$ equipped with a nondegenerate symmetric bilinear form. Then:*

(a) $\Omega(V) = \mathrm{O}'(V) = \mathrm{SO}'(V)$.

(b) *If* $\dim(V) \geq 5$, *then* $\Omega'(V) = \Omega(V)$.

(c) $\mathrm{Z}(\mathrm{O}(V)) = \{\pm \mathrm{id}_V\}$ *and* $-\mathrm{id}_V \in \mathrm{SO}(V)$ *if and only if n is even.*

(d) *(Dickson-Dieudonné Theorem) Assume that* $\dim(V) \geq 5$. *Then, the projective group* $P\Omega(V) := \Omega(V)/(\Omega(V) \cap \{\pm \mathrm{id}_V\})$ *is simple.*

Lastly, if $n = 2\ell + 1$, then $P\Omega(V)$ is the simple adjoint linear algebraic group of type B_ℓ and if $n = 2\ell$, then $P\Omega(V)$ is the simple adjoint linear algebraic group of type D_ℓ, see [Car89, Theorem 11.3.2].

Symmetric and quadratic forms in characteristic 2: Assume that k is an algebraically closed field of characteristic 2 and let V be a finite-dimensional k-vector space. A quadratic form Q on V is a map $Q : V \to k$ such that

$$Q(cv) = c^2 Q(v), \text{ for all } c \in k^* \text{ and all } v \in V;$$

and the map $b : V \times V \to k$ given by

$$b(v_1, v_2) = Q(v_1 + v_2) + Q(v_1) + Q(v_2), \text{ for all } v_1, v_2 \in V$$

is a bilinear form. Note that the bilinear form b is uniquely determined by Q and that

$$b(v, v) = 0 \text{ for all } v \in V,$$

i.e the bilinear form b is alternating and, therefore, symmetric, since $\mathrm{char}(k) = 2$.

Let $\dim(V) = n$. Since the bilinear form b is alternating, by Theorem 2.1.1, it follows that V admits an ordered basis $\{v_1, \ldots, v_\ell, w_1, \ldots, w_{n-2\ell}, u_\ell, \ldots, u_1\}$, where $\ell \geq 0$, $\{v_i, u_i\}$ is a b-hyperbolic pair for all $1 \leq i \leq \ell$, and $\{w_1, \ldots, w_{n-2\ell}\}$ is a basis of $\mathrm{Rad}(V)$, with the property that

$$V = \bigoplus_{i=1}^{\ell} \langle v_i, u_i \rangle \oplus \mathrm{Rad}(V)$$

is an orthogonal direct sum. Let B be the representing matrix of b with respect to the basis $\{v_1, \ldots, v_\ell, w_1, \ldots, w_{n-2\ell}, u_\ell, \ldots, u_1\}$. Then, by Theorem 2.1.1, $B = \begin{pmatrix} & & K_\ell \\ & 0_{n-2\ell} & \\ K_\ell & & \end{pmatrix}$, thereby $\mathrm{rank}(B) = 2\ell$ and $\dim(\mathrm{Rad}(V)) = n - 2\ell$.

We call V defective if $\mathrm{Rad}(V) \neq 0$ and nondefective if $\mathrm{Rad}(V) = 0$. We note that all vectors $0 \neq v \in V$ are isotropic, since $b(v, v) = 0$, however this does not generally imply that $Q(v) = 0$. We call $0 \neq v \in V$ singular if $Q(v) = 0$ and, similarly, we call a subspace $W \leq V$ singular if W contains a singular vector. Moreover, we call W totally singular if all vectors in W are singular and we note that if W is a totally singular subspace, then W is totally isotropic with respect to b, as:

$$b(w, w') = Q(w + w') + Q(w) + Q(w') = 0 \text{ for all } w, w' \in W.$$

We call the quadratic form Q nondegenerate if for all nonzero $v \in \mathrm{Rad}(V)$ we have $Q(v) \neq 0$. Moreover, if Q is nondegenerate, then $\dim(\mathrm{Rad}(V)) = 0$ or 1, see [Gro02, p.114].

We define a quadratic space to be a k-vector space V equipped with a nondegenerate quadratic form Q. Note that this definition allows the possibility for V to be defective.

Theorem 2.1.6. [Gro02, Theorem 12.9] *Let (V, Q) be a quadratic space over the algebraically closed field k of characteristic 2. Then V has an ordered basis $\{v_1, \ldots, v_n\}$, where $n = \dim(V)$, with respect to which Q has one of the following forms:*

(a) $Q(\sum_{i=1}^{n} c_i v_i) = c_1 c_{2\ell+1} + c_2 c_{2\ell} + \cdots + c_\ell c_{\ell+2} + c_{\ell+1}^2$, *if* $n = 2\ell + 1$.

(b) $Q(\sum_{i=1}^{n} c_i v_i) = c_1 c_{2\ell} + c_2 c_{2\ell-1} + \cdots + c_\ell c_{\ell+1}$, *if* $n = 2\ell$.

We call the ordered pair of singular vectors $(v, u) \in V \times V$ a hyperbolic pair relative to Q if $b(v, u) = 1$ and we call the subspace $\langle v, u \rangle$ a hyperbolic plane. By Theorem 2.1.6, it follows that V is an orthogonal direct sum of one of these types:

$$\textit{Type } 1: \ V = H_1 \oplus \cdots \oplus H_\ell \oplus \langle w \rangle, \text{ where } Q(w) = 1 \text{ and } n = 2\ell + 1$$

and

$$\textit{Type } 2: \ V = H_1 \oplus \cdots \oplus H_\ell, \text{ where } n = 2\ell$$

and each H_i is a hyperbolic plane. Therefore, if $\dim(V)$ is odd, then $\dim(\mathrm{Rad}(V)) = 1$ and so V is a defective space, while, if $\dim(V)$ is even, then V is nondefective space.

Let (V, Q) and (V', Q') be two quadratic spaces and define an isometry from V to V' to be an isomorphism $\phi : V \to V'$ with the property that $Q'(\phi(v)) = Q(v)$ for all $v \in V$. The orthogonal group $\mathrm{O}(V)$ is defined to be the group of all isometries of V. We note that:

$$\mathrm{O}(V) \leq \mathrm{SL}(V),$$

therefore $\det(\sigma) = 1$, for all $\sigma \in \mathrm{O}(V)$.

Theorem 2.1.7. [Gro02, Theorem 14.2] *Let* (V, Q) *be a quadratic space over the algebraically closed field* k *of characteristic* 2. *Let* $\dim(V) = 2\ell + 1$, *for some* $\ell \geq 1$. *Then* $\mathrm{O}(V) \cong \mathrm{Sp}(2\ell, k)$ *as abstract groups.*

We now focus on the case when V is even-dimensional. We define the special orthogonal group, $\mathrm{SO}(V)$, to be $\ker(\delta)$, where $\delta : \mathrm{O}(V) \to \mathbb{F}_2$ is the *Dickson pseudodeterminant*, see [Gro02, Theorem 13.13 and Corollary 14.4]. We note that $[\mathrm{O}(V) : \mathrm{SO}(V)] = 2$. Let $\Omega(V) = \mathrm{O}'(V)$ denote the derived group of $\mathrm{O}(V)$. The following result exhibits some of the properties of the group $\Omega(V)$. Proofs for these can be found in [Gro02, Section 14].

Proposition 2.1.8. *Let* (V, Q) *be a quadratic space over the algebraically closed field* k *of characteristic* 2. *Let* $\dim(V) = 2\ell$, *for some* $\ell \geq 2$. *Then*

(a) $\Omega(V) = \mathrm{O}'(V) = \mathrm{SO}'(V)$.

(b) $\Omega'(V) = \Omega(V)$.

(c) $\Omega(V)$ *is simple, unless* $\dim(V) = 4$ *and* V *has Witt index* 2.

Lastly, [Car89, Theorem 11.3.2] identifies $\Omega(V)$ with the simple adjoint linear algebraic group of type D_ℓ.

2.2 Elements of linear algebraic groups

A first natural step towards understanding linear algebraic groups is to understand properties of individual elements. This section presents some basic structural properties of elements of a linear algebraic group G. In the first subsection we discuss the Jordan decomposition of elements in G with the goal of proving that to determine $\nu_G(V)$, where V is an irreducible kG-module, it suffices to calculate $\max\{\dim(V_s(\mu)) \mid s \in G \setminus \mathrm{Z}(G)$ semisimple, $\mu \in k^*\}$ and $\max\{\dim(V_u(1)) \mid u \in G \setminus \{1\}$ unipotent$\}$, see Proposition 2.2.3. In Subsection 2.2.2 we reintroduce the elements $h_\alpha(c)$, $c \in k^*$, and $x_\alpha(c')$, $c' \in k$, of G, where α is an element in the root system Φ of G. We give a presentation of semisimple, respectively unipotent, elements of G as ordered products $\prod_{\alpha\in\Delta} h_\alpha(c_\alpha)$, where $c_\alpha \in k^*$ and Δ is a set of simple roots in Φ, respectively $\prod_{\alpha\in\Phi^+} x_\alpha(c'_\alpha)$, where $c'_\alpha \in k$ and Φ^+ is the set of positive roots in Φ.

2.2.1 The Jordan decomposition

Let k be an algebraically closed field of characteristic $p \geq 0$ and let V be a finite-dimensional k-vector space. An endomorphism $s \in \mathrm{End}(V)$ is called semisimple if s is diagonalizable over k. An endomorphism $n \in \mathrm{End}(V)$ is called nilpotent if 0 is the sole eigenvalue of n on V. We note that if $x \in \mathrm{End}(V)$ is both semisimple and nilpotent, then $x = 0$. Now, any $x \in \mathrm{End}(V)$ admits a so called additive Jordan decomposition, i.e. there exist unique commuting endomorphisms $x_s, x_n \in \mathrm{End}(V)$ with the property that x_s is semisimple, x_n is nilpotent and $x = x_s + x_n = x_n + x_s$, [Hum75, Lemma A of Section 15.1]. An invertible endomorphism $u \in \mathrm{GL}(V)$ is called unipotent if $u - \mathrm{id}_V$ is nilpotent, or equivalently, if 1 is its sole eigenvalue on V. We also note that if $x \in \mathrm{GL}(V)$ is both semisimple and unipotent, then $x = \mathrm{id}_V$.

We now consider the case when $x \in \mathrm{GL}(V)$. Then its eigenvalues are nonzero, therefore $x_s \in \mathrm{GL}(V)$. We set $x_u := 1 + x_s^{-1}x_n$ and note that $x_u \in \mathrm{GL}(V)$ is unipotent. Moreover, we see that $x = x_s x_u = x_u x_s$ with $x_s \in \mathrm{GL}(V)$ semisimple and $x_u \in \mathrm{GL}(V)$ unipotent. We call this decomposition of $x \in \mathrm{GL}(V)$ the multiplicative Jordan decomposition. The uniqueness of x_s and x_n in the additive Jordan decomposition of $x \in \mathrm{GL}(V)$ gives the uniqueness of x_s and x_u in the multiplicative Jordan decomposition of x, i.e. for any $x \in \mathrm{GL}(V)$, there exists a unique semisimple endomorphism $x_s \in \mathrm{GL}(V)$, called the semisimple part of x, and a unique unipotent endomorphism $x_u \in \mathrm{GL}(V)$, called the unipotent part of x, with the property that $x = x_s x_u = x_u x_s$.

Remark 2.2.1. *Semisimple elements behave almost the same in both characteristic* 0 *and positive characteristic. However, this is not the case for unipotent elements. If* $\mathrm{char}(k) = 0$, *then a unipotent element* $u \neq 1$ *has infinite order. On the other hand, if* $\mathrm{char}(k) = p > 0$, *then* u *is unipotent if and only if its order is a power of* p.

We now state the Jordan decomposition theorem for arbitrary linear algebraic groups, as it is given in [MT11, Theorem 2.5].

Theorem 2.2.2. [Jordan decomposition] *Let* G *be a linear algebraic group.*

(a) *For any embedding ρ of G into some $\mathrm{GL}_n(k)$ and for any $g \in G$, there exist unique $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$, where $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.*

(b) *The decomposition $g = g_s g_u = g_u g_s$ is independent of the chosen embedding.*

(c) *Let $\rho : G_1 \to G_2$ be a morphism of algebraic groups. Then, for $g \in G_1$ with $g = g_s g_u$, as in* (a)*, we have $\rho(g_s) = \rho(g)_s$ and $\rho(g_u) = \rho(g)_u$, i.e.*

$$\rho(g) = \rho(g_s)\rho(g_u) = \rho(g)_s \rho(g)_u$$

is the Jordan decomposition of $\rho(g)$ in G_2.

Let G be a simple linear algebraic group and let $g \in G$. The decomposition $g = g_s g_u = g_u g_s$ of Theorem 2.2.2 is called the multiplicative Jordan decomposition of $g \in G$. We call $g \in G$ semisimple if $g = g_s$ and, similarly, we call $g \in G$ unipotent if $g = g_u$. Now, let G_s be the set of semisimple elements of G and let G_u be the set of unipotent elements of G. Theorem 2.2.2 shows that G_s and G_u are well-defined and that, for any $\phi : G \to G'$ morphism of algebraic groups, we have $\phi(G_s) = \phi(G)_s$ and $\phi(G_u) = \phi(G)_u$.

Let V be an irreducible kG-module. Recall from Section 1.1 that $\nu_G(V) = \min\{\dim(V) - \dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in k^*\}$. We remark that to determine $\nu_G(V)$ it is enough to determine

$$M_V := \max\{\dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in k^*\}.$$

In what follows, we investigate the invariant M_V. Before we begin, we remind the reader that the center of G, $\mathrm{Z}(G)$, consists of semisimple elements, i.e. the only unipotent element in $\mathrm{Z}(G)$ is the identity. Now, let $\rho : G \to \mathrm{GL}(V)$ be the associated representation of G into $\mathrm{GL}(V)$ and let $g \in G \setminus \mathrm{Z}(G)$. We write down the Jordan decomposition of g:

$$g = g_s g_u = g_u g_s,$$

where $g_s \in G_s$ and $g_u \in G_u$. Theorem 2.2.2 gives us the Jordan decomposition of $\rho(g)$ in $\mathrm{GL}(V)$:

$$\rho(g) = \rho(g_s)\rho(g_u) = \rho(g)_s \rho(g)_u.$$

We choose a basis of V with the property that $\rho(g)$ is written in its Jordan normal form. Then, with respect to this basis, $\rho(g)_s$ is the diagonal matrix whose entries are just the diagonal entries of $\rho(g)$, while $\rho(g)_u$ is the unipotent matrix obtained from the Jordan normal form of $\rho(g)$ by dividing all entries of each Jordan block by the diagonal element. We distinguish the following two cases:

Case 1: Assume that $g_s \in \mathrm{Z}(G)$. First, we remark that $g_u \neq 1$, as $g \notin \mathrm{Z}(G)$. Secondly, as $g_s \in \mathrm{Z}(G)$, it follows that $\rho(g)_s = \mathrm{diag}(c, c, \ldots, c)$ for some $c \in k^*$. Thereby, c is the sole eigenvalue of $\rho(g)$ on V and we have:

$$\dim(V_g(c)) = \dim(V_{g_u}(1)) \leq \max_{u \in G_u \setminus \{1\}} \dim(V_u(1)). \tag{2.1}$$

Case 2: Assume that $g_s \notin \mathrm{Z}(G)$. Then, since $\rho(g)_s$ is a diagonal matrix with entries the diagonal entries of $\rho(g)$, we determine that $\rho(g)$ and $\rho(g)_s$ have the same eigenvalues on V and, for any eigenvalue $c \in k^*$ of $\rho(g)$ on V, we have:

$$\dim(V_g(c)) \leq \dim(V_{g_s}(c)) \leq \max_{s \in G_s \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}. \tag{2.2}$$

Therefore, for any $g \in G \setminus \mathrm{Z}(G)$ and any eigenvalue $c \in k^*$ of $\rho(g)$ on V, by (2.1) and (2.2), it follows that

$$\dim(V_g(c)) \leq \max\{\max_{u \in G_u \setminus \{1\}} \dim(V_u(1)), \max_{s \in G_s \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}\}.$$

We fix a maximal torus T in G. Let $s \in G_s$ be semisimple. Now, as any semisimple element of G lies in a maximal torus, [MT11, Corollary 4.5], and, as maximal tori are conjugate in G, [MT11, Theorem 4.4], there exists $s' \in T$ such that s and s' are G-conjugate. It follows that

$$\max_{s \in G_s \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\} = \max_{s' \in T \setminus \mathrm{Z}(G)} \{\dim(V_{s'}(\mu')) \mid \mu' \in k^*\}.$$

We summarize the above discussion in the following result:

Proposition 2.2.3. *Let G be a simple linear algebraic group and let V be an irreducible kG-module. We define*

$$M_s := \max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\} \text{ and } M_u := \max_{u \in G_u \setminus \{1\}} \dim(V_u(1)).$$

Then $M_V = \max\{M_s, M_u\}$ and $\nu_G(V) = \dim(V) - \max\{M_s, M_u\}$.

Remark 2.2.4. *Proposition* 2.2.3 *is one of the essential results of this thesis, as it gives a strategy on how to calculate $\nu_G(V)$. It explains the structure of Chapters* 3 *through* 7*, where the first part of each chapter is dedicated to the calculation of $\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, while the second part is concerned with $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$.*

2.2.2 The presentation of semisimple and unipotent elements of a linear algebraic group

As in the previous subsection, let k be an algebraically closed field of characteristic $p \geq 0$. Let G be a simple simply connected algebraic group of rank $\ell \geq 1$, let T be a maximal torus in G and let $\mathrm{X}(T)$ be its group of rational characters. Moreover, let Φ be the root system of G determined by T. Recall that $\boldsymbol{G_a}$, respectively $\boldsymbol{G_m}$, is the additive, respectively multiplicative, group of k. Now, for each $\alpha \in \Phi$, we have seen that there exists a unique, up to scalar multiplication, morphism $x_\alpha : \boldsymbol{G_a} \to G$ of linear algebraic groups, which induces an isomorphism $x_\alpha : \boldsymbol{G_a} \to \mathrm{im}(x_\alpha)$ with the property that $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ for all $t \in T$ and all $c \in \boldsymbol{G_a}$, see Subsection 1.3. Moreover, for each $\alpha \in \Phi$, $U_\alpha = \mathrm{im}(x_\alpha) = \{x_\alpha(c) \mid c \in k\}$ is the root subgroup of G, relative to T, associated to the root α.

Let B be the positive Borel subgroup of G which contains T and let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be the corresponding base in Φ. Then $B = T \cdot \prod_{\alpha \in \Phi^+} U_\alpha$, where the product respects the total order $\preceq$ on Φ fixed in Section 1.3. Let $u \in G$ be a unipotent element. Now, we can assume, without loss of generality, that $u \in B$, see [MT11, Corollary 6.11]. There exist $c_\alpha \in k$ such that $u = \prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$, where the product respects $\preceq$. To u we associate the

subset $S_u \subseteq \Phi^+$ with the property that $u = \prod_{\alpha \in S_u} x_\alpha(c_\alpha)$, where the product respects $\preceq$ and $c_\alpha \in k^*$ for all $\alpha \in S_u$. Note that S_u is well-defined, as with the order $\preceq$ fixed on Φ, the expression $u = \prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$ is unique.

Having dealt with the unipotent elements of G, we now focus on the semisimple elements. Let $\alpha \in \Phi$ and $c \in k^*$ and recall the elements $n_\alpha(c) := x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c) \in \mathrm{N}_G(T)$, respectively $h_\alpha(c) := n_\alpha(c)n_\alpha(-1) \in T$, of G. As $\{h_{\alpha_i} \mid \alpha_i \in \Delta\}$ is a basis of $\mathrm{Y}(T)$, the group of cocharacters of T, we determine that $T = \langle \mathrm{im}(h_{\alpha_i}) \mid \alpha_i \in \Delta \rangle$. Therefore, for any semisimple element $s \in G$, we will write $s = \prod_{\alpha_i \in \Delta} h_{\alpha_i}(c_{\alpha_i})$, where $c_{\alpha_i} \in k^*$.

2.3 Representation theory of linear algebraic groups

In this section some well-known results of the representation theory of algebraic groups will be presented. As these classical results can be found in most books covering this subject, we will not include their proofs, only references for further reading. We will be following [MT11], but other sources are [Jan07] and [Hum75]. The goal is to understand the irreducible tensor-indecomposable modules of a simple linear algebraic group G. We present the classical results of Chevalley, Theorem 2.3.3, and Steinberg, Theorem 2.3.8, which tell us that, up to isomorphism, irreducible tensor-indecomposable kG-modules are parametrized by the p-restricted dominant weights of G. Lastly, we discuss isogenous groups and the connection between their respective irreducible modules.

2.3.1 Irreducible kG-modules

To begin, recall from Section 1.3 that k is an algebraically closed field of characteristic $p \geq 0$; G is a simple simply-connected linear algebraic group of rank ℓ; B is a Borel subgroup in G which contains T, a fixed maximal torus of G; $\mathrm{X}(T)$ is the rational character group of T; $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is the set of simple roots determined by B in Φ; and Φ is the root system of G associated to T.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. As T consists of commuting semisimple elements, by Theorem 2.2.2, we have that $\rho(T)$ is a subgroup of commuting semisimple elements in $\mathrm{GL}(V)$, hence $\rho(T)$ is diagonalizable. Consequently, the vector space V can be decomposed into a direct sum:

$$V = \bigoplus_{\lambda \in \mathrm{X}(T)} V_\lambda, \text{ where } V_\lambda = \{v \in V \mid \ t \cdot v = \lambda(t)v \text{ for all } t \in T\}.$$

The elements $\lambda \in \mathrm{X}(T)$ for which $V_\lambda \neq \{0\}$ are called the weights of V with respect to the maximal torus T. We will denote by $\Lambda(V)$ the set of weights of a kG-module V. The spaces V_λ, where $\lambda \in \Lambda(V)$, are called T-weight spaces of V, and, for a weight $\lambda \in \Lambda(V)$ we define the multiplicity of λ as $\dim(V_\lambda)$.

We will now describe how certain subgroups of G act on the weight spaces V_λ of the kG-module V. First, recall that $\mathcal{W}$ is the Weyl group of G associated to T. Now, by [MT11,

Section 8.1], the action of $\mathcal{W}$ on $\mathrm{X}(T)$ is given by

$$(w \cdot \lambda)(t) = \lambda(\dot{w}^{-1} t \dot{w}),$$

where $t \in T$, $\lambda \in \mathrm{X}(T)$ and $\dot{w} \in \mathrm{N}_G(T)$ is an arbitrary preimage of $w \in \mathcal{W}$. Therefore, weights in the same $\mathcal{W}$-orbit have the same multiplicity, see [MT11, Lemma 15.3]. For every $\lambda \in \mathrm{X}(T)$ there exists a unique dominant weight $\lambda' \in \mathrm{X}(T)$ such that $\lambda \in \mathcal{W} \cdot \lambda'$. Moreover, if $\lambda \in \mathrm{X}(T)$ is dominant, then for all $w \in \mathcal{W}$ we have $w \cdot \lambda \leq \lambda$, see [MT11, Proposition 15.8 and Lemma B.4]. We now focus our attention on the root subgroups U_α of G, where $\alpha \in \Phi$, and their action on the weight spaces V_λ of V.

Lemma 2.3.1. [MT11, Lemma 15.4] *Let $\lambda \in \Lambda(V)$ and let V_λ be its corresponding weight space. Moreover, let $\alpha \in \Phi$. Then, for all $v \in V_\lambda$, we have:*

$$U_\alpha \cdot v \subseteq v + \sum_{j \in \mathbb{Z}_{>0}} V_{\lambda + j\alpha}.$$

We call a vector $v^+ \in V$ a <u>maximal vector</u> with respect to B if $B \cdot \langle v^+ \rangle = \langle v^+ \rangle$. Note that, by the classical Lie-Kolchin theorem, [MT11, Theorem 4.1], maximal vectors always exist. We call a kG-module V a <u>highest weight module</u> if V is generated by a maximal vector $v^+ \in V$, with respect to B. Let V be a highest weight kG-module with maximal vector $v^+ \in V$. Since, in particular, $\langle v^+ \rangle$ is stabilized by T, there exists a dominant weight $\lambda \in \Lambda(V)$ such that $v^+ \in V_\lambda$, see [MT11, Proposition 15.9]. The dominant weight $\lambda \in \mathrm{X}(T)$ with the property that $v^+ \in V_\lambda$ is called the <u>highest weight</u> of V. Now, if V is irreducible, then, by [MT11, Corollary 15.10], all maximal vectors in V have the same weight λ. Moreover, $\dim(V_\lambda) = 1$ and all weights in $\Lambda(V)$ are of the form $\lambda - \sum_{\alpha_i \in \Delta} c_{\alpha_i} \alpha_i$ with $c_{\alpha_i} \in \mathbb{N}_{\geq 0}$ for all $\alpha_i \in \Delta$.

Proposition 2.3.2. [Jan07, II.2.13] *Let $V_G(\lambda)$ be the <u>Weyl kG-module</u>. Then $V_G(\lambda)$ is a highest weight kG-module and any highest weight kG-module of highest weight $\lambda \in \mathrm{X}(T)$ is a homomorphic image of $V_G(\lambda)$.*

Theorem 2.3.3 (Chevalley)**.** [MT11, Theorem 15.17] *Let G be a simple linear algebraic group.*

(a) *There exists an irreducible kG-module, denoted by $L_G(\lambda)$, of highest weight λ for all dominant weights $\lambda \in \mathrm{X}(T)$.*

(b) *The two irreducible kG-modules $L_G(\lambda_1)$ and $L_G(\lambda_2)$ of respective highest weights λ_1 and λ_2 are isomorphic if and only if $\lambda_1 = \lambda_2$.*

In particular, the set $\{L_G(\lambda) \mid \lambda \in \mathrm{X}(T) \text{ dominant}\}$ is a set of representatives for the isomorphism classes of irreducible kG-modules.

We end this subsection with two results, courtesy of [Jan07, II.2.14] and [Pre88, Theorem 1], respectively, which give the relationship between the irreducible kG-module $L_G(\lambda)$ and the Weyl kG-module $V_G(\lambda)$ and their respective set of weights $\Lambda(L_G(\lambda))$ and $\Lambda(V_G(\lambda))$. They will be used extensively in the proofs of the results in Chapters 3 through 7.

Proposition 2.3.4. [Jan07, II.2.14] *Let $\lambda \in \mathrm{X}(T)$ be a dominant weight. Then:*

$$V_G(\lambda)/\operatorname{Rad}(V_G(\lambda)) \cong L_G(\lambda).$$

Set $e(\Phi) = 1$, if Φ is of type A_ℓ, D_ℓ, E_6, E_7 or E_8; $e(\Phi) = 2$ if Φ is of type B_ℓ, C_ℓ or F_4; and $e(\Phi) = 3$ if Φ is of type G_2. Now, recall that a weight $\lambda \in \mathrm{X}(T)$, $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, is p-restricted if $0 \leq d_i \leq p-1$ for all $1 \leq i \leq \ell$.

Remark 2.3.5. [MT11, Section 16, p.137] *For a fixed simple simply connected linear algebraic group G and a fixed prime p, there exist only finitely many p-restricted weights.*

Theorem 2.3.6. [Pre88, Theorem 1] *Let k be an algebraically closed field of characteristic $p > 0$, let G be a simple linear algebraic group and let T be a fixed maximal torus in G. Let $\lambda \in \mathrm{X}(T)$ be a p-restricted dominant weight. If $p > e(\Phi)$, then*

$$\Lambda(L_G(\lambda)) = \Lambda(V_G(\lambda)).$$

2.3.2 Steinberg's Tensor Product Theorem

Let k be an algebraically closed field of characteristic $p > 0$. Let $F_p : k \to k$ be the Frobenius automorphism, i.e. $F_p(c) = c^p$ for $c \in k$. By [MT11, Proposition 16.5], we know that F_p induces an endomorphism of algebraic groups $F : G \to G$ given by

$$F(x_\alpha(c)) = x_\alpha(c^p), \text{ for all } \alpha \in \Phi \text{ and all } c \in k.$$

Now, let V be a kG-module and let the action of G on V be given by $g \cdot v$, where $g \in G$ and $v \in V$. Now, for all $i \geq 1$, we can define a new action of G on V in the following way:

$$g \cdot^{(p^i)} v := F^i(g) \cdot v, \text{ for all } g \in G \text{ and all } v \in V.$$

We denote this new kG-module by $V^{(p^i)}$.

Proposition 2.3.7. [MT11, Proposition 16.6] *Let k be an algebraically closed field of characteristic $p > 0$ and let G be a simple linear algebraic group. Let T be a maximal torus in G and let $\lambda \in \mathrm{X}(T)$ be a dominant weight. Then, we have the following isomorphism of kG-modules:*

$$L_G(\lambda)^{(p)} \cong L_G(p\lambda).$$

Let $\lambda \in \mathrm{X}(T)$ be a dominant weight. We can express λ uniquely as

$$\lambda = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_r, \tag{2.3}$$

where $r \in \mathbb{Z}_{\geq 0}$ and $\lambda_i \in \mathrm{X}(T)$ is a p-restricted dominant weight for all $0 \leq i \leq r$. The following result, due to Steinberg, shows that there exists a decomposition of the associated irreducible kG-module $L_G(\lambda)$ analogous to (2.3).

Theorem 2.3.8 (Steinberg)**.** *Let G be a simple simply connected linear algebraic group over the algebraically closed field k of characteristic $p > 0$. We fix a maximal torus T in G and we let $\lambda \in \mathrm{X}(T)$ be a dominant weight. We write $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_r$, where $r \in \mathbb{Z}_{\geq 0}$ and $\lambda_i \in \mathrm{X}(T)$ is a p-restricted dominant weight for all $0 \leq i \leq r$. Then we have the following isomorphism of kG-modules:*

$$L_G(\lambda) \cong L_G(\lambda_0) \otimes L_G(\lambda_1)^{(p)} \otimes \cdots \otimes L_G(\lambda_r)^{(p^r)}.$$

Remark 2.3.9. *For p-restricted dominant weights, the associated irreducible kG-modules are called <u>p-restricted</u>. Theorem 2.3.8 allows us to restrict many questions in the study of all rational irreducible kG-modules to the finitely many p-restricted ones.*

2.3.3 Group isogenies and irreducible modules

Let k be an algebraically closed field of characteristic $p \geq 0$ and assume that the simple algebraic group G is not simply connected. Let $\tilde{G}$ be the simple simply connected linear algebraic group of the same type as G. Fix a central isogeny $\phi : \tilde{G} \to G$ with $\ker(\phi) \subseteq \mathrm{Z}(\tilde{G})$ and $d\phi \neq 0$. Let $\tilde{T}$ be a maximal torus in $\tilde{G}$ with the property that $\phi(\tilde{T}) = T$ and, similarly, let $\tilde{B}$ be the Borel subgroup of $\tilde{G}$ given by $\phi(\tilde{B}) = B$. Note that $\tilde{T} \subseteq \tilde{B}$. By [Jan07, II.2.10], we know that each simple kG-module is also a simple $k\tilde{G}$-module. With this in mind, we consider the simple kG-module $L_G(\lambda)$, where $\lambda \in \mathrm{X}(T)$ is a dominant weight. Since $\mathrm{X}(T) \subseteq \mathrm{X}(\tilde{T})$, we will denote by $\tilde{\lambda}$ the weight λ when viewing it as an element of $\mathrm{X}(\tilde{T})$. Moreover, by [Jan07, II.2.10], as $\lambda \in \mathrm{X}(T)$ is dominant, it follows that $\tilde{\lambda} \in \mathrm{X}(\tilde{T})$ is also dominant. Now, since $L_G(\lambda)$ is a simple $k\tilde{G}$-module, it follows that there exists $\tilde{\gamma} \in \mathrm{X}(\tilde{T})$ dominant such that $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\gamma})$ as $k\tilde{G}$-modules. We now use [Jan07, II.2.10] to determine that $\tilde{\gamma} = \tilde{\lambda}$, i.e. $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\lambda})$ as $k\tilde{G}$-modules. In what follows we show that $M_{L_{\tilde{G}}(\tilde{\lambda})} := \max\{\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{g}}(\tilde{\mu})) \mid \tilde{g} \in \tilde{G} \setminus \mathrm{Z}(\tilde{G}), \tilde{\mu} \in k^*\}$ and $M_{L_G(\lambda)} := \max\{\dim((L_G(\lambda)_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G), \mu \in k^*\}$ are equal.

Let $\tilde{g} \in \tilde{G} \setminus \mathrm{Z}(\tilde{G})$ and let $\tilde{\mu} \in k^*$ be an eigenvalue of $\tilde{g}$ on $L_{\tilde{G}}(\tilde{\lambda})$. Let $g = \phi(\tilde{g})$ and note that $g \in G \setminus \mathrm{Z}(G)$. Since $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\lambda})$ as $k\tilde{G}$-modules, we have that:

$$\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{g}}(\tilde{\mu})) = \dim((L_G(\lambda))_{\tilde{g}}(\mu)),$$

where $\mu \in k^*$ is the eigenvalue of $\tilde{g}$ on $L_G(\lambda)$ corresponding to $\tilde{\mu}$ under the $k\tilde{G}$-module isomorphism $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\lambda})$. Moreover, as $\dim((L_G(\lambda))_{\tilde{g}}(\mu)) = \dim((L_G(\lambda))_{\phi(\tilde{g})}(\mu))$, it follows that

$$\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{g}}(\tilde{\mu})) = \dim((L_G(\lambda))_{\phi(\tilde{g})}(\mu)) = \dim((L_G(\lambda))_g(\mu)).$$

Now, since the map $\phi : \tilde{G} \to G$ is surjective, it follows that:

$$\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{g}}(\tilde{\mu})) \leq \max_{g' \in G \setminus \mathrm{Z}(G)} \{\dim((L_G(\lambda))_{g'}(\mu')) \mid \mu' \in k^*\} = M_{L_G(\lambda)}, \tag{2.4}$$

for all $\tilde{g} \in \tilde{G} \setminus \mathrm{Z}(\tilde{G})$. Lastly, let $(g, \mu) \in G \setminus \mathrm{Z}(G) \times k^*$ be such that $\dim((L_G(\lambda))_g(\mu)) = \max_{g' \in G \setminus \mathrm{Z}(G)} \{\dim((L_G(\lambda))_{g'}(\mu')) \mid \mu' \in k^*\}$. Then:

$$\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{g}}(\tilde{\mu})) = \max_{g' \in G \setminus \mathrm{Z}(G)} \{\dim((L_G(\lambda))_{g'}(\mu')) \mid \mu' \in k^*\},$$

where $\tilde{g}$ is an arbitrary preimage of g in $\tilde{G}$ and $\tilde{\mu} \in k^*$ is the eigenvalue of $\tilde{g}$ on $L_{\tilde{G}}(\tilde{\lambda})$ corresponding to μ under the isomorphism $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\lambda})$. This shows that there exist pairs $(\tilde{g}, \tilde{\mu}) \in \tilde{G} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound in (2.4) is attained and thus, we have shown that $M_{L_{\tilde{G}}(\tilde{\lambda})} = M_{L_G(\lambda)}$.

Lastly, we set $\tilde{M}_s := \max\limits_{\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{s}}(\tilde{\mu})) \mid \tilde{\mu} \in k^*\}$. Arguing as above, see Inequality (2.4), we establish that:

$$\tilde{M}_s = \max_{s' \in T \setminus \mathrm{Z}(G)} \{\dim((L_G(\lambda))_{s'}(\mu')) \mid \mu' \in k^*\} = M_s.$$

Similarly, we set $\tilde{M}_u := \max\limits_{\tilde{u} \in \tilde{G}_u \setminus \{1\}} \dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{u}}(1))$, where $\tilde{G}_u$ is the set of unipotent elements in $\tilde{G}$. Then, arguing as for Inequality (2.4), one shows that:

$$\tilde{M}_u = \max_{u' \in G_u \setminus \{1\}} \dim((L_G(\lambda))_{u'}(1)) = M_u.$$

We recall that, by Proposition 2.2.3, we have $\nu_G(V) = \dim(V) - M_V$, where $M_V = \max\{M_s, M_u\}$, for any irreducible kG-module V, respectively $\nu_{\tilde{G}}(\tilde{V}) = \dim(\tilde{V}) - M_{\tilde{V}}$, where $M_{\tilde{V}} = \max\{\tilde{M}_s, \tilde{M}_u\}$, for any irreducible $k\tilde{G}$-module $\tilde{V}$. We summarize the discussion above in the following lemma:

Lemma 2.3.10. *Let k be an algebraically closed field of arbitrary characteristic, let G be a simple algebraic group and let $\tilde{G}$ be the simple simply connected algebraic group of the same type as G. Let V be an irreducible kG-module and let $\tilde{V}$ denote V when viewing it as an irreducible $k\tilde{G}$-module. Set $\tilde{M}_s = \max\limits_{\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) \mid \tilde{\mu} \in k^*\}$, respectively $M_s = \max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, and $\tilde{M}_u = \max\limits_{\tilde{u} \in \tilde{G}_u \setminus \{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$, where $\tilde{G}_u$ is the set of unipotent elements in $\tilde{G}$, respectively $M_u = \max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, where G_u is the set of unipotent elements in G. Then*

$$\tilde{M}_s = M_s \text{ and } \tilde{M}_u = M_u.$$

In particular, we have $M_V = M_{\tilde{V}}$ and $\nu_G(V) = \nu_{\tilde{G}}(\tilde{V})$.

We end this subsection with the following remark, which justifies why we will treat algebraic groups of type B_ℓ, and their respective modules, only over fields of characteristic different than 2.

Remark 2.3.11. *Let k be an algebraically closed field of characteristic 2. For C, a simple simply connected linear algebraic group of type C_ℓ, and B, a simple simply connected linear algebraic group of type B_ℓ, there exists an exceptional isogeny $\phi : C \to B$ between the two groups, see* [Ste16, Theorem 28]. *A consequence of this fact is that we can induce irreducible kC-modules from irreducible kB-modules by twisting with the isogeny ϕ. More specifically, for any 2-restricted dominant weight $\mu = \sum\limits_{i=1}^{\ell} d_i \omega_i^B$, where the ω_i^B's are the fundamental*

dominant weights of B, we have that:

$$L_B(\mu) \cong L_C(2\sum_{i=1}^{\ell-1} d_i\omega_i^C + d_\ell\omega_\ell^C) \cong L_C(\sum_{i=1}^{\ell-1} d_i\omega_i^C)^{(2)} \otimes L_C(d_\ell\omega_\ell^C),$$

in view of Theorem 2.3.8, where the ω_i^C's are the fundamental dominant weights of C.

Thus, for any 2-restricted dominant weight $\mu = \sum_{i=1}^{\ell-1} d_i\omega_i^B$ of B, we have $\nu_B(L_B(\mu)) = \nu_C(L_C(2\lambda)) = \nu_C(L_C(\lambda)^{(2)}) = \nu_C(L_C(\lambda))$, where $\lambda = \sum_{i=1}^{\ell-1} d_i\omega_i^C$. Similarly, for the weight ω_ℓ^B, we have $\nu_B(L_B(\omega_\ell^B)) = \nu_C(L_C(\omega_\ell^C))$. Lastly, in the case of the weights of the form $\mu = \sum_{i=1}^{\ell} d_i\omega_i^B$, where there exists $1 \leq i \leq \ell-1$ such that $d_i = 1$ and $d_\ell = 1$, we will calculate $\nu_C(L_C(\sum_{i=1}^{\ell-1} d_i\omega_i^C)^2 \otimes L_C(\omega_\ell))$ to determine $\nu_B(L_B(\mu))$.

2.4 Parabolic subgroups

In this section we introduce a family of subgroups of the simple linear algebraic group G called parabolic subgroups. They will play an important role in this thesis, as their structure as a semidirect product of a reductive subgroup and a unipotent normal subgroup allows us to use inductive algorithms to prove certain parts of Theorem 1.1.1, see Subsections 2.4.3 and 2.4.4, respectively.

A subgroup P of G which contains a Borel subgroup is called parabolic. In what follows, we will describe these subgroups and present their construction. Fix a proper subset $I \subseteq \{1, \ldots, \ell\}$ and set $\Delta_I := \{\alpha_i \in \Delta \mid i \in I\}$ and $\Phi_I := \Phi \cap \sum_{\alpha_i \in \Delta_I} \mathbb{Z}\alpha_i$. The subgroup $P_I := \langle T, U_\alpha \mid \alpha \in \Phi^+ \cup \Phi_I \rangle$ is a parabolic subgroup in G, called the standard parabolic subgroup of G corresponding to Δ_I. Let $Q_I := R_u(P_I)$ be the unipotent radical of P_I. We have that $Q_I = \langle U_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_I \rangle$, and that $L_I := \langle T, U_\alpha \mid \alpha \in \Phi_I \rangle$ is a complement of Q_I in P_I, see [MT11, Subsection 12.2]. Now, P_I admits a decomposition

$$P_I = Q_I \rtimes L_I,$$

called the Levi decomposition, where the subgroup L_I, called the standard Levi complement of P_I, is reductive and has Φ_I as root system, [MT11, Proposition 12.6]. Therefore, by [MT11, Corollary 8.22], we have that $L_I = \mathrm{Z}(L_I)^\circ[L_I, L_I]$, where $\mathrm{Z}(L_I)^\circ$ is a torus and, since G is simply connected, the derived subgroup $[L_I, L_I]$ is also of simply connected type and is of rank strictly smaller than $\mathrm{rank}(G)$, [MT11, Proposition 12.14].

We now turn our attention to a specific family of parabolic subgroups, called maximal parabolic subgroups. These are the standard parabolic subgroups which correspond to $\Delta_i := \Delta \setminus \{\alpha_i\}$, $1 \leq i \leq \ell$. Therefore, for each $1 \leq i \leq \ell$, set P_i to be the maximal parabolic subgroup of G corresponding to $\Delta_i = \{\alpha_1 \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell\}$. Set $L_i = \langle T, U_{\pm\alpha_1}, \ldots,$

$U_{\pm\alpha_{i-1}}, U_{\pm\alpha_{i+1}}, \ldots, U_{\pm\alpha_\ell}\rangle$ to be a Levi subgroup of P_i. We have seen earlier that L_i has root system $\Phi_i = \Phi \cap (\mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{i-1} + \mathbb{Z}\alpha_{i+1} + \cdots + \mathbb{Z}\alpha_\ell)$, in which Δ_i is a set of simple roots. Now, we have that $L_i = \mathrm{Z}(L_i)^\circ[L_i, L_i]$, where $\mathrm{Z}(L_i)^\circ = \left(\bigcap_{j\neq i} \ker(\alpha_j)\right)^\circ$ is a one-dimensional subtorus of G and $[L_i, L_i]$ is a simply connected linear algebraic group of rank $\ell - 1$. Lastly, we have that $T' = T \cap [L_i, L_i]$ is a maximal torus in $[L_i, L_i]$, contained in the Borel subgroup $B' = B \cap [L_i, L_i]$. Let $\omega_1, \ldots, \omega_\ell$ denote the fundamental dominant weights of G corresponding to Δ. We will abuse notation and denote the fundamental dominant weights of L_i corresponding to Δ_i by $\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_\ell$.

2.4.1 Restriction to Levi subgroups

Let $\mathcal{W}$ be the Weyl group of G corrsponding to T. Let $\lambda \in \mathrm{X}(T)$ be a dominant weight and write $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, where $d_i \geq 0$ for all $1 \leq i \leq \ell$. Let $L_G(\lambda)$ be the associated irreducible kG-module.

Definition 2.4.1. *Fix some $1 \leq i \leq \ell$. We say that a weight μ in $L_G(\lambda)$ has <u>α_i-level</u> $j \geq 0$ if $\mu = \lambda - j\alpha_i - \sum_{r\neq i} c_r\alpha_r$, where $c_r \in \mathbb{Z}_{\geq 0}$. The <u>maximum α_i-level of weights</u> in $L_G(\lambda)$ will be denoted by $e_i(\lambda)$.*

Remark 2.4.2. *By definition, $e_i(\lambda)$ is equal to the α_i-level of the lowest weight in $L_G(\lambda)$, which, by* [Jan07, II, Proposition 2.4(b)]*, is $w_0(\lambda)$, where $w_0 \in \mathcal{W}$ is the longest word.*

Let $V = L_G(\lambda)$. Fix $1 \leq i \leq \ell$ and let L_i be a Levi subgroup of P_i, the maximal parabolic subgroup of G corresponding to $\Delta_i = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell\}$, see Subsection 2.4. For each $0 \leq j \leq e_i(\lambda)$, define the subspace $V^j := \bigoplus_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-j\alpha_i-\gamma}$ of V and note that V^j is invariant under L_i. Then, as a $k[L_i, L_i]$-module, V admits the following decomposition:

$$V\mid_{[L_i,L_i]} = \bigoplus_{j=0}^{e_i(\lambda)} V^j.$$

We note that, by [Jan07, II.2.11], $V^0 = \bigoplus_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-\gamma}$ is the irreducible $k[L_i, L_i]$-module of highest weight $\lambda\mid_{T'}$, where $T' = T \cap [L_i, L_i]$ is a maximal torus in $[L_i, L_i]$.

Lemma 2.4.3. [Duality Lemma] *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group. Let T be a maximal torus in G and let $V = L_G(\lambda)$, where $\lambda \in \mathrm{X}(T)$ is dominant. Assume that V is a self-dual kG-module. Fix $1 \leq i \leq \ell$ and let L_i be a Levi subgroup of the maximal parabolic subgroup P_i of G. Moreover, for all $0 \leq j \leq e_i(\lambda)$, let $V^j = \sum_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-j\alpha_i-\gamma}$. Then, for all $0 \leq j \leq \left\lfloor \frac{e_i(\lambda)}{2} \right\rfloor$, we have the following isomorphism of $k[L_i, L_i]$-modules:*

$$V^{e_i(\lambda)-j} \cong (V^j)^*.$$

Proof. We note that, as V is self-dual, we have that $w_0(\lambda) = -\lambda$ and V is equipped with a nondegenerate bilinear form, which will be denoted by $(-,-)$. Let $\mu, \mu^{'} \in \Lambda(V)$ be such that $\mu^{'} \neq -\mu$. We will show that $V_{\mu^{'}} \subset V_{\mu}^{\perp}$. For this, let $v \in V_{\mu}$ and $v^{'} \in V_{\mu^{'}}$. We have that:

$$(v, v^{'}) = (t \cdot v, t \cdot v^{'}) = (\mu(t)v, \mu^{'}(t)v^{'}) = (\mu + \mu^{'})(t)(v, v^{'}), \text{ for all } t \in T.$$

Therefore $(v, v^{'}) = 0$, as $\mu^{'} \neq -\mu$. Moreover, we note that if $-\mu \notin \Lambda(V)$, then $V = V_{\mu}^{\perp}$, contradicting the fact that $(-,-)$ is nondegenerate. Therefore $-\mu \in \Lambda(V)$ for all $\mu \in \Lambda(V)$.

Secondly, let $\mu \in \Lambda(V)$ be a weight of α_i-level j, where $0 \leq j \leq e_i(\lambda)$. We will show that $-\mu$ has α_i-level $e_i(\lambda) - j$. On one hand, we know that $e_i(\lambda)$ is equal to the α_i-level of $w_0(\lambda)$, see Remark 2.4.2, therefore we have:

$$w_0(\lambda) = -\lambda = \lambda - e_i(\lambda)\alpha_i - \sum_{r \neq i} a_r \alpha_r,$$

where $a_r \in \mathbb{Z}_{\geq 0}$. On the other hand, since $\mu = \lambda - j\alpha_i - \sum_{r \neq i} c_r \alpha_r$, for $c_r \in \mathbb{Z}_{\geq 0}$, we have:

$$-\mu = -\lambda + j\alpha_i + \sum_{r \neq i} c_r \alpha_r = \lambda - (e_i(\lambda) - j)\alpha_i - \sum_{r \neq i} b_r \alpha_r,$$

where $b_r \in \mathbb{Z}_{\geq 0}$ for all $r \neq i$. Therefore, $-\mu$ has α_i-level equal to $e_i(\lambda) - j$. In particular, as $V_{\mu'} \subseteq (V_{\mu})^{\perp}$ for all $\mu' \neq -\mu$, it follows that $(V^j)^{\perp} \supseteq \bigoplus_{r \neq e_i(\lambda) - j} V^r$.

Lastly, as $V\,|_{[L_i, L_i]} = \bigoplus_{j=0}^{e_i(\lambda)} V^j$ is self-dual, it follows that $V\,|_{[L_i, L_i]} \cong \bigoplus_{j=0}^{e_i(\lambda)} (V^j)^*$. Furthermore, as V is equipped with a nondegenerate bilinear form, we have that $(V^j)^* \cong V/(V^j)^{\perp}$, for all $0 \leq j \leq e_i(\lambda)$. As $(V^j)^{\perp} \supseteq \bigoplus_{r \neq e_i(\lambda) - j} V^r$, it follows that $\dim((V^j)^*) \leq \dim(V^{e_i(\lambda) - j})$. By the same argument, this time applied to $V^{e_i(\lambda) - j}$, we determine that $\dim((V^{e_i(\lambda) - j})^*) \leq \dim(V^j)$. Therefore, $\dim((V^j)^*) = \dim(V^{e_i(\lambda) - j})$, thus $(V^j)^{\perp} = \bigoplus_{r \neq e_i(\lambda) - j} V^r$, and we conclude that

$$(V^j)^* \cong V^{e_i(\lambda) - j}.$$

□

2.4.2 Maximum α_i-levels of weights in $L_G(\lambda)$

In this subsection, we exhibit formulas for the maximum α_i-levels of weights in $L_G(\lambda)$, for all $1 \leq i \leq \ell$ and all types of simple classical linear algebraic groups. Now, as $\lambda = \sum_{i=1}^{\ell} d_i \omega_i$, where $d_i \geq 0$ for all $1 \leq i \leq \ell$, we will use [Hum72, Table 1, p.69], which allows us to write the fundamental dominant weights ω_i in terms of the simple roots α_i. Before we begin, we recall that $w_0(\lambda)$ is the lowest weight in $L_G(\lambda)$, where $w_0 \in \mathcal{W}$ denotes the longest word in the Weyl group $\mathcal{W}$ of G.

Lemma 2.4.4. *Let G be of type A_ℓ, $\ell \geq 1$ and let $\lambda \in \mathrm{X}(T)$, $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, be a dominant weight. Then, for all $1 \leq i \leq \left\lfloor \frac{\ell}{2} \right\rfloor$, the maximum α_i-level and the maximum $\alpha_{\ell-i+1}$-level of weights in $L_G(\lambda)$ are equal. Moreover, one of the following holds:*

(1) *for ℓ even, we have $e_i(\lambda) = e_{\ell-i+1}(\lambda) = \sum_{j<i} j(d_j + d_{\ell-j+1}) + i \sum_{j=i}^{\ell-i+1} d_j$, for all $1 \leq i \leq \frac{\ell}{2}$;*

(2) *for ℓ odd, we have* $\begin{cases} e_{\frac{\ell+1}{2}}(\lambda) = \dfrac{\ell+1}{2} d_{\frac{\ell+1}{2}} + \sum_{j=1}^{\frac{\ell-1}{2}} j(d_j + d_{\ell-j+1}); \\ e_i(\lambda) = e_{\ell-i+1}(\lambda) = \sum_{j<i} j(d_j + d_{\ell-j+1}) + i \sum_{j=i}^{\ell-i+1} d_j, \text{ for all } 1 \leq i \leq \dfrac{\ell-1}{2}. \end{cases}$

Proof. In order to determine $e_i(\lambda)$, $1 \leq i \leq \ell$, we have to calculate the α_i-level of $w_0(\lambda)$. Using [Hum72, Table 1, p.69], we write the ω_i's, $1 \leq i \leq \ell$, in terms of the simple roots α_j, $1 \leq j \leq \ell$, and we see that:

$$w_0(\lambda) = \lambda - (\lambda - w_0(\lambda)) = \lambda - \sum_{r=1}^{\ell} d_r(\omega_r - w_0(\omega_r)) = \lambda - \sum_{r=1}^{\ell} d_r(\omega_r + \omega_{\ell-r+1}).$$

Let $1 \leq r \leq \left\lfloor \frac{\ell}{2} \right\rfloor$. Then:

$$\omega_r + \omega_{\ell-r+1} = \sum_{j=1}^{r-1} j\alpha_j + r \sum_{j=r}^{\ell-r+1} \alpha_j + \sum_{j=\ell-r+2}^{\ell} (\ell+1-j)\alpha_j.$$

We now assume that ℓ is even. Then:

$$\begin{aligned}
w_0(\lambda) &= \lambda - \sum_{r=1}^{\frac{\ell}{2}} (d_r + d_{\ell-r+1})(\omega_r + \omega_{\ell-r+1}) \\
&= \lambda - \sum_{r=1}^{\frac{\ell}{2}} (d_r + d_{\ell-r+1}) \left[\sum_{j=1}^{r-1} j\alpha_j + r \sum_{j=r}^{\ell-r+1} \alpha_j + \sum_{j=\ell-r+2}^{\ell} (\ell+1-j)\alpha_j \right] \\
&= \lambda - (d_1 + d_\ell)(\alpha_1 + \cdots + \alpha_\ell) - (d_2 + d_{\ell-1})(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell) - \cdots - \\
&\quad - (d_{\frac{\ell}{2}} + d_{\frac{\ell}{2}+1}) \left[\alpha_1 + \cdots + \left(\frac{\ell}{2} - 1\right)\alpha_{\frac{\ell}{2}-1} + \frac{\ell}{2} \cdot \alpha_{\frac{\ell}{2}} + \frac{\ell}{2} \cdot \alpha_{\frac{\ell}{2}+1} + \left(\frac{\ell}{2} - 1\right)\alpha_{\frac{\ell}{2}+1} + \cdots + \alpha_\ell \right] \\
&= \lambda - \sum_{r=1}^{\frac{\ell}{2}} \left[\sum_{j<r} j(d_j + d_{\ell-j+1}) + r \sum_{j=r}^{\frac{\ell}{2}} (d_j + d_{\ell-j+1}) \right] (\alpha_r + \alpha_{\ell-r+1}) \\
&= \lambda - \sum_{r=1}^{\frac{\ell}{2}} \left[\sum_{j<r} j(d_j + d_{\ell-j+1}) + r \sum_{j=r}^{\ell-r+1} d_j \right] (\alpha_r + \alpha_{\ell-r+1}).
\end{aligned}$$

Therefore, we have proven that, for all $1 \leq i \leq \frac{\ell}{2}$, we have $e_i(\lambda) = e_{\ell-i+1}(\lambda)$, where

$$e_i(\lambda) = \sum_{j<i} j(d_j + d_{\ell-j+1}) + i \sum_{j=i}^{\ell-i+1} d_j.$$

We can now assume that ℓ is odd. Then:

$$\omega_{\frac{\ell+1}{2}} = \frac{1}{2}\left[\alpha_1 + 2\alpha_2 + \cdots + \frac{\ell-1}{2} \cdot \alpha_{\frac{\ell-1}{2}} + \frac{\ell+1}{2} \cdot \alpha_{\frac{\ell+1}{2}} + \frac{\ell-1}{2} \cdot \alpha_{\frac{\ell+1}{2}+1} + \cdots + \alpha_\ell\right].$$

This gives:

$$\begin{aligned}
w_0(\lambda) &= \lambda - \sum_{r=1}^{\frac{\ell-1}{2}} \left[(d_r + d_{\ell-r+1})(\omega_r + \omega_{\ell-r+1})\right] - d_{\frac{\ell+1}{2}}\left(\omega_{\frac{\ell+1}{2}} + \omega_{\frac{\ell+1}{2}}\right) \\
&= \lambda - \sum_{r=1}^{\frac{\ell-1}{2}} \left[(d_r + d_{\ell-r+1})\left(\sum_{j=1}^{r-1} j\alpha_j + r\sum_{j=r}^{\ell-r+1} \alpha_j + \sum_{j=\ell-r+2}^{\ell} (\ell+1-j)\alpha_j\right)\right] - \\
&- d_{\frac{\ell+1}{2}}\left(\sum_{j=1}^{\frac{\ell-1}{2}} j\alpha_j + \frac{\ell+1}{2} \cdot \alpha_{\frac{\ell+1}{2}} + \sum_{j=\frac{\ell+1}{2}+1}^{\ell} (\ell-j+1)\alpha_j\right) \\
&= \lambda - \sum_{r=1}^{\frac{\ell-1}{2}} \left[\sum_{j<r} j(d_j + d_{\ell-j+1}) + r\sum_{j=r}^{\frac{\ell}{2}} (d_j + d_{\ell-j+1}) + rd_{\frac{\ell+1}{2}}\right](\alpha_r + \alpha_{\ell-r+1}) - \\
&- \left(\frac{\ell+1}{2} d_{\frac{\ell+1}{2}} + \sum_{j=1}^{\frac{\ell-1}{2}} j(d_j + d_{\ell-j+1})\right)\alpha_{\frac{\ell+1}{2}} \\
&= \lambda - \sum_{r=1}^{\frac{\ell-1}{2}} \left[\sum_{j<r} j(d_j + d_{\ell-j+1}) + r\sum_{j=r}^{\ell-r+1} d_j\right](\alpha_r + \alpha_{\ell-r+1}) - \\
&- \left(\frac{\ell+1}{2} d_{\frac{\ell+1}{2}} + \sum_{j=1}^{\frac{\ell-1}{2}} j(d_j + d_{\ell-j+1})\right)\alpha_{\frac{\ell+1}{2}}.
\end{aligned}$$

We conclude that $e_{\frac{\ell+1}{2}}(\lambda) = \frac{\ell+1}{2} d_{\frac{\ell+1}{2}} + \sum_{j=1}^{\frac{\ell-1}{2}} j(d_j + d_{\ell-j+1})$ and, for all $1 \leq i \leq \frac{\ell-1}{2}$, we have $e_i(\lambda) = e_{\ell-i+1}(\lambda) = \sum_{j<i} j(d_j + d_{\ell-j+1}) + i \sum_{j=i}^{\ell-i+1} d_j$. □

Lemma 2.4.5. *Let G be of type C_ℓ, $\ell \geq 2$, and let $\lambda \in \mathrm{X}(T)$, $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, be a dominant*

weight. Then the maximum α_i-level of weights in $L_G(\lambda)$ is:

$$e_i(\lambda) = \begin{cases} 2\Big[\sum_{j=1}^{i-1} jd_j + i\sum_{j=i}^{\ell} d_j\Big], \text{ for } 1 \leq i \leq \ell - 1; \\ \sum_{j=1}^{\ell} jd_j, \text{ for } i = \ell. \end{cases}$$

Proof. Note that, as G is a group of type C_ℓ, $\ell \geq 2$, we have $w_0 = -1$, hence $w_0(\lambda) = -\lambda$. We write the ω_i's, $1 \leq i \leq \ell$, in terms of the simple roots α_j, $1 \leq j \leq \ell$, and we see that:

$$\begin{aligned} w_0(\lambda) = -\lambda = \lambda - 2\lambda = &\lambda - 2(d_1 + \cdots + d_\ell)\alpha_1 - 2\Big(d_1 + 2\sum_{j=2}^{\ell} d_j\Big)\alpha_2 - \cdots - \\ &- 2\Big(\sum_{j=1}^{i-1} jd_j + i\sum_{j=i}^{\ell} d_j\Big)\alpha_i - \cdots - \Big(\sum_{j=1}^{\ell} jd_j\Big)\alpha_\ell. \end{aligned}$$

We remark that the coefficient of each α_i, $1 \leq i \leq \ell$, in the above, is a nonnegative integer and the result follows. □

Lemma 2.4.6. *Let G be of type B_ℓ, $\ell \geq 3$, and let $\lambda \in \mathrm{X}(T)$, $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, be a dominant weight. Then the maximum α_i-level of weights in $L_G(\lambda)$ is:*

$$e_i(\lambda) = 2\Big[\sum_{j=1}^{i-1} jd_j + i\Big(\sum_{j=i}^{\ell-1} d_j + \frac{1}{2}d_\ell\Big)\Big], \text{ for all } 1 \leq i \leq \ell.$$

Proof. Note that, as G is a group of type B_ℓ, $\ell \geq 3$, we have that $w_0 = -1$, hence $w_0(\lambda) = -\lambda$. We write the fundamental dominant weights ω_i, $1 \leq i \leq \ell$, in terms of the simple roots α_j, $1 \leq j \leq \ell$, and we determine that:

$$\lambda = \sum_{i=1}^{\ell}\Big[\sum_{j=1}^{i-1} jd_j + i\Big(\sum_{j=i}^{\ell-1} d_j + \frac{1}{2}d_\ell\Big)\Big]\alpha_i.$$

Therefore

$$w_0(\lambda) = -\lambda = \lambda - 2\lambda = \lambda - 2\sum_{i=1}^{\ell}\Big[\sum_{j=1}^{i-1} jd_j + i\Big(\sum_{j=i}^{\ell-1} d_j + \frac{1}{2}d_\ell\Big)\Big]\alpha_i$$

and so $e_i(\lambda) = 2\Big[\sum_{j=1}^{i-1} jd_j + i\Big(\sum_{j=i}^{\ell-1} d_j + \frac{1}{2}d_\ell\Big)\Big]$, for all $1 \leq i \leq \ell$. □

Lemma 2.4.7. *Let G be of type D_ℓ, $\ell \geq 4$, and let $\lambda \in \mathrm{X}(T)$, $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, be a dominant weight. Then, for all $1 \leq i \leq \ell - 2$, the maximum α_i-level of weights in $L_G(\lambda)$ is:*

$$e_i(\lambda) = 2\Bigg[\sum_{j=1}^{i-1} jd_j + i\sum_{j=i}^{\ell-2} d_j + \frac{1}{2}i(d_{\ell-1} + d_\ell)\Bigg].$$

Moreover, if ℓ is even, then:

$$e_{\ell-1}(\lambda) = \sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}\Bigg[\ell d_{\ell-1} + (\ell-2)d_\ell\Bigg] \text{ and } e_\ell(\lambda) = \sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}\Bigg[(\ell-2)d_{\ell-1} + \ell d_\ell\Bigg],$$

while, if ℓ is odd, then:

$$e_{\ell-1}(\lambda) = e_\ell(\lambda) = \sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}(\ell-1)(d_{\ell-1} + d_\ell).$$

Proof. We first assume that ℓ is even. Then $w_0 = -1$ and we have:

$$w_0(\lambda) = -\lambda = \lambda - 2\lambda = \lambda - 2\sum_{j=1}^{\ell-2} d_j\omega_j - 2d_{\ell-1}\omega_{\ell-1} - 2d_\ell\omega_\ell,$$

therefore

$$\sum_{j=1}^{\ell-2} d_j\omega_j = \sum_{r=1}^{\ell-2}\Bigg[\sum_{j=1}^{r-1} jd_j + r\sum_{j=r}^{\ell-2} d_j\Bigg]\alpha_r + \frac{1}{2}\Bigg[\sum_{j=1}^{\ell-2} jd_j\Bigg]\alpha_{\ell-1} + \frac{1}{2}\Bigg[\sum_{j=1}^{\ell-2} jd_j\Bigg]\alpha_\ell$$

and

$$d_{\ell-1}\omega_{\ell-1} + d_\ell\omega_\ell = \frac{1}{2}(d_{\ell-1} + d_\ell)\sum_{j=1}^{\ell-2} j\alpha_j + \frac{1}{4}\Bigg[\ell d_{\ell-1} + (\ell-2)d_\ell\Bigg]\alpha_{\ell-1} + \frac{1}{4}\Bigg[(\ell-2)d_{\ell-1} + \ell d_\ell\Bigg]\alpha_\ell.$$

This gives

$$w_0(\lambda) = \lambda - \sum_{r=1}^{\ell-2} 2\Bigg[\sum_{j=1}^{r-1} jd_j + r\sum_{j=r}^{\ell-2} d_j + \frac{1}{2}r(d_{\ell-1} + d_\ell)\Bigg]\alpha_r - \Bigg[\sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}(\ell d_{\ell-1} + (\ell-2)d_\ell)\Bigg]\alpha_{\ell-1} -$$
$$- \Bigg[\sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}((\ell-2)d_{\ell-1} + \ell d_\ell)\Bigg]\alpha_\ell,$$

thus $e_i(\lambda) = 2\Bigg[\sum_{j=1}^{i-1} jd_j + i\sum_{j=i}^{\ell-2} d_j + \frac{1}{2}i(d_{\ell-1} + d_\ell)\Bigg]$, for all $1 \leq i \leq \ell - 2$, $e_{\ell-1}(\lambda) = \sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}\Bigg[\ell d_{\ell-1} + (\ell-2)d_\ell\Bigg]$ and $e_\ell(\lambda) = \sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}\Bigg[(\ell-2)d_{\ell-1} + \ell d_\ell\Bigg]$.

We now assume that ℓ is odd. We note that $w_0(\omega_j) = -\omega_j$, for all $1 \leq j \leq \ell - 2$, $w_0(\omega_{\ell-1}) = -\omega_\ell$ and $w_0(\omega_\ell) = -\omega_{\ell-1}$. It follows that:

$$\begin{aligned} w_0(\lambda) &= -\sum_{j=1}^{\ell-2} d_j\omega_j - d_{\ell-1}\omega_\ell - d_\ell\omega_{\ell-1} = -\lambda + d_{\ell-1}\omega_{\ell-1} + d_\ell\omega_\ell - d_{\ell-1}\omega_\ell - d_\ell\omega_{\ell-1} \\ &= \lambda - 2\lambda + (d_{\ell-1} - d_\ell)\omega_{\ell-1} + (d_\ell - d_{\ell-1})\omega_\ell \\ &= \lambda - 2\sum_{j=1}^{\ell-2} d_j\omega_j - 2d_{\ell-1}\omega_{\ell-1} - 2d_\ell\omega_\ell + (d_{\ell-1} - d_\ell)\omega_{\ell-1} + (d_\ell - d_{\ell-1})\omega_\ell \\ &= \lambda - 2\sum_{j=1}^{\ell-2} d_j\omega_j - (d_{\ell-1} + d_\ell)(\omega_{\ell-1} + \omega_\ell). \end{aligned}$$

Now, we determine that

$$\sum_{j=1}^{\ell-2} d_j\omega_j = \sum_{r=1}^{\ell-2}\left[\sum_{j=1}^{r-1} jd_j + r\sum_{j=r}^{\ell-2} d_j\right]\alpha_r + \frac{1}{2}\left[\sum_{j=1}^{\ell-2} jd_j\right]\alpha_{\ell-1} + \frac{1}{2}\left[\sum_{j=1}^{\ell-2} jd_j\right]\alpha_\ell$$

and that

$$(d_{\ell-1} + d_\ell)(\omega_{\ell-1} + \omega_\ell) = (d_{\ell-1} + d_\ell)\left[\sum_{j=1}^{\ell-2} j\alpha_j\right] + \frac{\ell-1}{2}(d_{\ell-1} + d_\ell)\alpha_{\ell-1} + \frac{\ell-1}{2}(d_{\ell-1} + d_\ell)\alpha_\ell.$$

Therefore, we have:

$$w_0(\lambda) = \lambda - \sum_{r=1}^{\ell-2} 2\left[\sum_{j=1}^{r-1} jd_j + r\sum_{j=r}^{\ell-2} d_j + \frac{1}{2}r(d_{\ell-1} + d_\ell)\right]\alpha_r - \left[\sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}(\ell-1)(d_{\ell-1} + d_\ell)\right](\alpha_{\ell-1} + \alpha_\ell)$$

and thus $e_i(\lambda) = 2\left[\sum_{j=1}^{i-1} jd_j + i\sum_{j=i}^{\ell-2} d_j + \frac{1}{2}i(d_{\ell-1} + d_\ell)\right]$, for all $1 \leq i \leq \ell - 2$, and $e_{\ell-1}(\lambda) = e_\ell(\lambda) = \sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}(\ell-1)(d_{\ell-1} + d_\ell)$. □

2.4.3 The algorithm for semisimple elements

Recall that k is an algebraically closed field of characteristic $p \geq 0$; G is a simple simply connected linear algebraic group of rank $\ell \geq 1$; T is a fixed maximal torus in G with rational character group $\mathrm{X}(T)$; Φ is the root system of G determined by T; Δ is a set of simple roots in Φ; and B is the positive Borel subgroup of G. Let $\lambda \in \mathrm{X}(T)$ be a nonzero p-restricted dominant weight and let $V = L_G(\lambda)$ be the corresponding irreducible kG-module. Fix $1 \leq i \leq \ell$ and let P_i be the maximal parabolic subgroup of G given by $\Delta_i = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell\}$. Let L_i be a Levi subgroup of P_i. In this subsection, we

outline an inductive algorithm which calculates $\max_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu \in k^*\}$. For this, we consider the restriction:

$$V\mid_{[L_i,L_i]}=\bigoplus_{j=0}^{e_i(\lambda)} V^j,$$

where $e_i(\lambda)$ is the maximum α_i-level of weights in V and, for all $0 \le j \le e_i(\lambda)$, we have $V^j = \bigoplus_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-j\alpha_i-\gamma}$. Now, let $s \in T \setminus \mathrm{Z}(G)$. Then, in particular, $s \in L_i$ and so $s = z \cdot h$, where $z \in \mathrm{Z}(L_i)^\circ$ and $h \in [L_i, L_i]$. As $z \in \mathrm{Z}(L_i)^\circ$ and $\mathrm{Z}(L_i)^\circ$ is a one-dimensional torus, there exists $c \in k^*$ and $k_r \in \mathbb{Z}$, $1 \le r \le \ell$, such that $z = \prod_{r=1}^{\ell} h_{\alpha_r}(c^{k_r})$. Moreover, we have $\alpha_j(z) = 1$ for all $1 \le j \le \ell$, $j \ne i$. On the other hand, as $h \in [L_i, L_i]$, we have $h = \prod_{\substack{1\le r\le \ell\\ r\ne i}} h_{\alpha_r}(a_r)$, where $a_r \in k^*$ for all $1 \le r \le \ell$, $r \ne i$. We write λ as $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, where $0 \le d_i \le p-1$ for all $1 \le i \le \ell$. Now, z acts on each V^j, $0 \le j \le e_i(\lambda)$, as scalar multiplication by

$$\begin{aligned} s_z^j := (\lambda - j\alpha_i - \gamma)(z) &= (\lambda - j\alpha_i - \gamma)(\prod_{r=1}^{\ell} h_{\alpha_r}(c^{k_r})) = (\lambda - j\alpha_i)(\prod_{r=1}^{\ell} h_{\alpha_r}(c^{k_r})) \\ &= \prod_{r=1}^{\ell}\left(c^{k_r\langle\lambda,\alpha_r\rangle}\right)\cdot\prod_{r=1}^{\ell} c^{-jk_r\langle\alpha_i,\alpha_r\rangle} \\ &= \prod_{r=1}^{\ell}\left(c^{k_r d_r}\right)\cdot\prod_{r=1}^{\ell} c^{-jk_r\langle\alpha_i,\alpha_r\rangle}, \end{aligned} \tag{2.5}$$

where we used the fact that $\gamma(z) = 1$, as $\gamma \in \mathbb{N}\Delta_i$ and $\alpha_j(z) = 1$ for all $\alpha_j \in \Delta_i$. Now, let $\mu_1^j, \dots, \mu_{t_j}^j$, $t_j \ge 1$, be the distinct eigenvalues of h on V^j, $0 \le j \le e_i(\lambda)$, and let $n_1^j, \dots, n_{t_j}^j$ be their respective multiplicities. Then, as $s = z \cdot h$ and z acts on V^j as the scalar s_z^j, it follows that the eigenvalues of s on V^j are $s_z^j\mu_1^j, \dots, s_z^j\mu_{t_j}^j$ and they are distinct, as the μ_r^j's are, with respective multiplicities $n_1^j, \dots, n_{t_j}^j$. This discussion yields the following lemma:

Lemma 2.4.8. *Let $s \in T \setminus \mathrm{Z}(G)$ and write $s = z \cdot h$ with $z \in \mathrm{Z}(L_i)^\circ$ and $h \in [L_i, L_i]$. Let $\mu_1^j, \dots, \mu_{t_j}^j$, $t_j \ge 1$, be the distinct eigenvalues of h on V^j, $0 \le j \le e_i(\lambda)$, and let $n_1^j, \dots, n_{t_j}^j$ be their respective multiplicities. Then:*

(1) *z acts on each V^j, $0 \le j \le e_i(\lambda)$, as scalar multiplication by s_z^j, where s_z^j is given in (2.5);*

(2) *the distinct eigenvalues of s on V^j, $0 \le j \le e_i(\lambda)$, are $s_z^j\mu_1^j, \dots, s_z^j\mu_{t_j}^j$, with respective multiplicities $n_1^j, \dots, n_{t_j}^j$;*

(3) *the eigenvalues of s on V are $s_z^j\mu_1^j, \dots, s_z^j\mu_{t_j}^j$, $0 \le j \le e_i(\lambda)$, with respective multiplicities at least $n_1^j, \dots, n_{t_j}^j$.*

2.4.4 The algorithm for unipotent elements

In this subsection, we shift the focus to the unipotent elements of G and outline an inductive algorithm to calculate $\max\limits_{u\in G_u\setminus\{1\}} \dim(V_u(1))$. We will denote by $k[u]$ the group algebra of $\langle u\rangle$ over k. The rest of the notation is the same as in Subsection 2.4.3. We begin with a lemma which will be used extensively in the chapters that follow.

Lemma 2.4.9. *Let $u \in G$ be a unipotent element and let V be a finite-dimensional kG-module. Let $V = M_t \supseteq M_{t-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$, where $t \geq 1$, be a filtration of $k[u]$-submodules of V. Then:*

$$\dim(V_u(1)) \leq \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1)).$$

Moreover, suppose that for each i, we have a u-invariant decomposition $M_i = M_{i-1} \oplus M'_{i-1}$ with $M'_{i-1} \cong M_i/M_{i-1}$ as $k[u]$-modules. Then

$$\dim(V_u(1)) = \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1)).$$

Proof. We start by noting that

$$\begin{aligned} V_u(1) &= \{v \in V \mid u\cdot v = v\} = \{v \in V \mid u\cdot v - v = 0\} \\ &= \{v \in V \mid (u - \mathrm{id}_V)\cdot v = 0\} = \ker(u - \mathrm{id}_V). \end{aligned} \tag{2.6}$$

Now, for each $1 \leq i \leq t$, we fix a basis in M_i with the property that the matrix $(u)_{M_i/M_{i-1}}$ associated to the action of u on M_i/M_{i-1} is upper-triangular. Then, the matrix $(u)_V$ of the action of u on V is the block upper-triangular matrix:

$$(u)_V = \begin{pmatrix} (u)_{M_1} & \star & \star & \cdots & \star \\ 0 & (u)_{M_2/M_1} & \star & \cdots & \star \\ 0 & 0 & (u)_{M_3/M_2} & \cdots & \star \\ & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (u)_{M_t/M_{t-1}} \end{pmatrix}.$$

Using $(u)_V$, we calculate the matrix of the action of $u - \mathrm{id}_V$ on V:

$$(u - \mathrm{id}_V)_V = \begin{pmatrix} (u - \mathrm{id}_{M_1})_{M_1} & \star & \star & \cdots & \star \\ 0 & (u - \mathrm{id}_{M_2/M_1})_{M_2/M_1} & \star & \cdots & \star \\ 0 & 0 & \ddots & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (u - \mathrm{id}_{M_t/M_{t-1}})_{M_t/M_{t-1}} \end{pmatrix},$$

where $(u-\mathrm{id}_{M_i/M_{i-1}})_{M_i/M_{i-1}}$ is the matrix of the action of $u-\mathrm{id}_{M_i/M_{i-1}}$ on M_i/M_{i-1}, $1 \leq i \leq t$, with respect to the basis of M_i we have previously fixed. It follows that:

$$\mathrm{rank}(u - \mathrm{id}_V) \geq \sum_{i=1}^{t} \mathrm{rank}((u - \mathrm{id}_{M_i/M_{i-1}})_{M_i/M_{i-1}})$$

and, consequently, we have

$$\dim(\ker(u - \mathrm{id}_V)) \leq \sum_{i=1}^{t} \dim(\ker((u - \mathrm{id}_V) \mid_{M_i/M_{i-1}})).$$

Using (2.6), we determine that:

$$\dim(V_u(1)) \leq \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1)).$$

Lastly, for all $1 \leq i \leq t$, assume that there exists a $k[u]$-submodule, M'_{i-1}, of M_i such that $M_i = M_{i-1} \oplus M'_{i-1}$. Then $V \mid_{k[u]} = M'_0 \oplus \cdots \oplus M'_{t-1} \cong M_1 \oplus M_2/M_1 \oplus \cdots \oplus M_t/M_{t-1}$, and so there exists a basis of V with the property that:

$$(u - \mathrm{id}_V)_V = \begin{pmatrix} (u - \mathrm{id}_{M_1})_{M_1} & 0 & 0 & \cdots & 0 \\ 0 & (u - \mathrm{id}_{M_2/M_1})_{M_2/M_1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (u - \mathrm{id}_{M_t/M_{t-1}})_{M_t/M_{t-1}} \end{pmatrix},$$

thereby $\mathrm{rank}(u - \mathrm{id}_V) = \sum_{i=1}^{t} \mathrm{rank}((u - \mathrm{id}_{M_i/M_{i-1}})_{M_i/M_{i-1}})$. Arguing as above, we establish that

$$\dim(V_u(1)) = \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1)).$$

□

We have set P_i to be the maximal parabolic subgroup of G associated to $\Delta_i = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell\}$. We write the Levi decomposition of P_i:

$$P_i = L_i \cdot Q_i = \langle T, U_\beta \mid \beta \in \Phi_i \rangle \cdot \langle U_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_i \rangle,$$

where $\Phi_i = \Phi \cap (\mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{i-1} + \mathbb{Z}\alpha_{i+1} + \cdots + \mathbb{Z}\alpha_\ell)$. Let $u \in G$, $u = \prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$, where the product respects the total order $\preceq$ on Φ, see Section 1.3, and $c_\alpha \in k$. Now, as $u \in B$ and $B \subseteq P_i$, it follows that u admits a decomposition:

$$u = \prod_{\alpha \in \Phi_i} x_\alpha(c'_\alpha) \cdot \prod_{\alpha \in \Phi^+ \setminus \Phi_i} x_\alpha(c'_\alpha),$$

where each of the products respects $\preceq$ and $c'_\alpha \in k$, for all $\alpha \in \Phi^+$. We set $u_{L_i} = \prod_{\alpha \in \Phi_i} x_\alpha(c'_\alpha)$ and $u_{Q_i} = \prod_{\alpha \in \Phi^+ \setminus \Phi_i} x_\alpha(c'_\alpha)$, and we note that $u_{L_i} \in L_i$ and $u_{Q_i} \in Q_i$.

Let V be an irreducible kG-module of highest weight $\lambda \in \mathrm{X}(T)$. We consider the restriction:

$$V\mid_{[L_i,L_i]}=\bigoplus_{j=0}^{e_i(\lambda)} V^j,$$

where $e_i(\lambda)$ is the maximum α_i-level of weights in V, and $V^j = \bigoplus_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-j\alpha_i-\gamma}$, for all $0 \leq j \leq e_i(\lambda)$. Let $\mu \in \Lambda(V)$, with corresponding weight space V_μ, and let $\alpha \in \Phi$. As

$$U_\alpha V_\mu \subseteq \bigoplus_{r\in\mathbb{Z}_{\geq 0}} V_{\mu+r\alpha},$$

see Lemma 2.3.1, we determine that

$$u_{L_i}\cdot V^j \subseteq V^j,\ u_{Q_i}\cdot V^j \subseteq \bigoplus_{r=0}^{j} V^r \text{ and } (u_{Q_i}-1)V^j \subseteq \bigoplus_{r=0}^{j-1} V^r, \text{ for all } 0 \leq j \leq e_i(\lambda).$$

Therefore, V admits a filtration $V = M_{e_i(\lambda)} \supseteq M_{e_i(\lambda)-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 \supseteq 0$ of $k[u]$-submodules, where $M_j = \bigoplus_{r=0}^{j} V^r$ for all $0 \leq j \leq e_i(\lambda)$. We see that u acts on each M^j/M^{j-1}, $1 \leq j \leq e_i(\lambda)$, as u_{L_i} and so, by Lemma 2.4.9, we determine that:

$$\dim(V_u(1)) \leq \sum_{j=0}^{e_i(\lambda)} \dim(V^j_{u_{L_i}}(1)) = \dim(V_{u_{L_i}}(1)). \tag{2.7}$$

Lastly, we remark that if $u = u_{L_i}$, i.e. $u_{Q_i} = 1$, then $u \cdot V^j \subseteq V^j$, for all $0 \leq j \leq e_i(\lambda)$, and thus, by Lemma 2.4.9, it follows that:

$$\dim(V_u(1)) = \dim(V_{u_{L_i}}(1)). \tag{2.8}$$

The algorithm: The first step is to identify all unipotent conjugacy classes in G. For simple linear algebraic groups of classical type this is done by Theorem 2.9.2, or Theorem 2.9.11, depending on whether $p \neq 2$ or $p = 2$, while, for exceptional groups, we use [Sim13, Tables 3.1-3.9]. We choose a representative u' for each unipotent conjugacy class in G. Since, in particular, $u' \in P_i$, we write $u' = u'_{L_i} \cdot u'_{Q_i}$, where $u'_{L_i} \in L_i$ and $u'_{Q_i} \in Q_i$. Now, as a $k[L_i, L_i]$-module, V admits the following decomposition:

$$V\mid_{[L_i,L_i]}=\bigoplus_{j=0}^{e_i(\lambda)} V^j,$$

where $V^j = \sum_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-j\alpha_i-\gamma}$ for all $0 \leq j \leq e_i(\lambda)$. The next step of the algorithm is to identify the kL_i-composition factors of each V^j, $0 \leq j \leq e_i(\lambda)$. Afterwards, we use Lemma 2.4.9 and already proven results to determine an upper-bound for each $\dim(V^j_{u'_{L_i}}(1))$, $0 \leq j \leq e_i(\lambda)$.

Now, assuming that $u'_{L_i} \neq 1$, as $\dim(V_{u'_{L_i}}(1)) = \sum_{j=0}^{e_i(\lambda)} \dim(V^j_{u'_{L_i}}(1))$, we established an upper-bound for $\dim(V_{u'_{L_i}}(1))$, hence for $\dim(V_{u'}(1))$, as $\dim(V_{u'}(1)) \leq \dim(V_{u'_{L_i}}(1))$ by Inequality (2.7). Lastly, we take $u \in G$ to be a non-identity unipotent element. Then u is conjugate to exactly one non-identity unipotent class representative $u^{'}$. Therefore, since $\dim(V_u(1)) = \dim(V_{u'}(1))$, we have determined an upper-bound for $\dim(V_u(1))$. To end, we note that if we choose $1 \leq i \leq \ell$ such that L_i has the property that each non-identity unipotent conjugacy class of G admits a representative u' with $u'_{L_i} \neq 1$, then the upper-bound for $\dim(V_u(1))$, where $u \in G$ is a nontrivial unipotent element, given by the algorithm is strictly smaller than $\dim(V)$.

2.5 Generation of linear algebraic groups

In this section we present the generation results for linear algebraic groups established by Guralnick and Saxl in [GS03, Section 8]. For this, let F be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple linear algebraic group. It is said that a subset S of G topologically generates G, in the Zariski topology, if the closure of the subgroup generated by S is the whole of G, i.e. $\overline{\langle S \rangle} = G$. When it is understood from the context, we will delete topological and just say that G is generated.

Remark 2.5.1. *If the field F is algebraic over a finite field, then G will be a locally finite group. Thereby, we assume, for now, that F is an algebraically closed field of characteristic $p \geq 0$ which is not algebraic over a finite field.*

Definition 2.5.2. *For $g \in G \setminus \mathrm{Z}(G)$, let $\alpha(g)$ be the minimal number of G-conjugates of g necessary to (topologically) generate G. We define:*

$$\alpha(G) := \max\{\alpha(g) |\ g \in G \setminus \mathrm{Z}(G)\}.$$

Theorem 2.5.3. [GS03, Theorem 8.1] *Let F be an algebraically closed field of characteristic $p \geq 0$ which is not algebraic over a finite field. Let G be a simple classical linear algebraic group with natural module of dimension n. Then $\alpha(G) = n$, unless one of the following holds:*

(a) *G is of type A_1, in which case $\alpha(G) = 3$;*

(b) *G is of type C_ℓ and $p = 2$, in which case $\alpha(G) = 2\ell + 1$;*

(c) *G is of type C_2, in which case $\alpha(G) = 5$.*

Theorem 2.5.4. [GS03, Theorem 8.2] *Let F be an algebraically closed field of characteristic $p \geq 0$ which is not algebraic over a finite field. Let G be a simple exceptional linear algebraic group of rank ℓ. Then $\alpha(G) = \ell + 3$, unless G is of type F_4, in which case $\alpha(G) = 8$.*

2.6 The dimensional criteria

Let F be an algebraically closed field of characteristic $p \geq 0$ which is not algebraic over a finite field and let G be a simple simply connected linear algebraic group. Following [GS03], we will establish a dimensional criteria for irreducible tensor-indecomposable FG-modules to satisfy in order to be candidates for the classification of Theorem 1.1.1. With that, let V be an irreducible tensor-indecomposable FG-module and recall that $\nu_G(V) = \min\{\dim(V) - \dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in F^*\}$. Let $g \in G \setminus \mathrm{Z}(G)$ have the property that it realizes $\nu_G(V)$, i.e. g affords an eigenvalue $\mu \in F^*$ on V such that $\nu_G(V) = \dim(V) - \dim(V_g(\mu))$. Let $\alpha(g) = n$, where $n \in \mathbb{Z}_{\geq 2}$, and note that $n \leq \alpha(G)$. Let $g_1, \ldots, g_n \in G$ be G-conjugates of g which generate G. Note that each g_i affords μ as an eigenvalue on V, since g does, and we set $V_i := V_{g_i}(\mu)$, for each $1 \leq i \leq n$. Moreover, we have $\dim(V_i) = \dim(V_g(\mu))$, $1 \leq i \leq n$.

Now, $V_1 + \bigcap_{i=2}^{n} V_i$ is a subspace of V, hence $\dim(V_1 + \bigcap_{i=2}^{n} V_i) \leq \dim(V)$. This gives

$$\dim(V_1 + \bigcap_{i=2}^{n} V_i) = \dim(V_1) + \dim(\bigcap_{i=2}^{n} V_i) - \dim(\bigcap_{i=1}^{n} V_i) \leq \dim(V). \tag{2.9}$$

Assume $\bigcap_{i=1}^{n} V_i \neq \{0\}$ and fix $0 \neq v \in \bigcap_{i=1}^{n} V_i$. Set $\Gamma = \{x \in G \mid x \cdot v \notin \langle v \rangle\}$ and note that Γ is an open subset of G. Since $\overline{\langle g_1, \ldots, g_n \rangle} = G$, every nonempty open subset of G intersects $\langle g_1, \ldots, g_n \rangle$ nontrivially. We deduce that $\Gamma = \emptyset$, therefore $\langle v \rangle$ is a G-stable subspace of V, contradicting the fact that V is irreducible. Thus, we have $\bigcap_{i=1}^{n} V_i = \{0\}$ and so $\dim(\bigcap_{i=1}^{n} V_i) = 0$. We come back to Inequality (2.9) and see that

$$\dim(V_1) + \dim(\bigcap_{i=2}^{n} V_i) \leq \dim(V). \tag{2.10}$$

Now, $\dim(\bigcap_{i=2}^{n} V_i) = \dim(V_2) + \dim(\bigcap_{i=3}^{n} V_i) - \dim(V_2 + \bigcap_{i=3}^{n} V_i)$, $\dim(V_2) = \dim(V_1)$ and $\dim(V_2 + \bigcap_{i=3}^{n} V_i) \leq \dim(V)$, therefore, by Inequality (2.10), we have:

$$2\dim(V_1) + \dim(\bigcap_{i=3}^{n} V_i) \leq 2\dim(V).$$

Since, for all $3 \leq j \leq n$, we have $\dim(V_i) = \dim(V_1)$, while, for all $3 \leq j \leq n-1$, we have $\dim(\bigcap_{i=j}^{n} V_i) = \dim(V_j) + \dim(\bigcap_{i=j+1}^{n} V_i) - \dim(V_j + \bigcap_{i=j+1}^{n} V_i)$ and $\dim(V_j + \bigcap_{i=j+1}^{n} V_i) \leq \dim(V)$, recursively, we deduce that:

$$n \cdot \dim(V_1) \leq (n-1) \cdot \dim(V).$$

As $\dim(V_1) = \dim(V_g(\mu))$ and $\dim(V_g(\mu)) = \dim(V) - \nu_G(V)$, the above gives:

$$\dim(V) \leq n \cdot \nu_G(V) \leq \alpha(G) \cdot \nu_G(V). \tag{2.11}$$

Lastly, as we are interested in identifying the irreducible tensor-indecomposable FG-modules V for which $\nu_G(V) \leq \sqrt{\dim(V)}$, Inequality (2.11) establishes the following *dimensional criteria*:

$$\dim(V) \leq \alpha(G)^2. \tag{2.12}$$

To end this section, we will show that the representation theoretic results for algebraic groups over $k' = \overline{\mathbb{F}_p}$, the algebraic closure of $\mathbb{F}_p$, follow from the results over F, where F is our fixed algebraically closed field which is not algebraic over $\mathbb{F}_p$. We may assume that $k' \subseteq F$ and we let $\iota : k' \to F$ denote this injective homomorphism. Let $V_{k'}$ be a finite-dimensional k'-vector space with $\dim(V_{k'}) = r$ and let $V_F := V_{k'} \otimes_{k'} F$ be the vector space over F obtained from $V_{k'}$ by extension of the ground field via ι. Let G be a group and let $\rho : G \to \mathrm{GL}(V_{k'})$ be a representation of G. Then $\rho^\iota : G \to \mathrm{GL}(V_F)$ given by $g \to \rho(g) \otimes 1$, where $g \in G$, is a representation of G into $\mathrm{GL}(V_F)$. If we identify $V_{k'}$ with $(k^{'})^r$, then V_F is canonically identified with F^r, i.e. V_F is a finite-dimensional F-vector space with $\dim_F(V_F) = \dim_{k'}(V_{k'})$. We also remark that for $g \in G$, the coefficients of the matrix $\rho^\iota(g)$ are obtained by applying ι to the coefficients of $\rho(g)$. Hence, if $\mu \in (k')^*$ is an eigenvalue of $g \in G$ on $V_{k'}$, then $\iota(\mu)$ is an eigenvalue of g on V_F and, moreover, we have $\dim_{k'}((V_{k'})_g(\mu)) = \dim_F((V_F)_g(\iota(\mu)))$.

We write $G_{k'}$ for G, when G is a simply connected linear algebraic group over k', and we let G_F denote the simply connected linear algebraic group G over F. We want to establish a dimensional criteria similar to (2.12) for irreducible tensor-indecomposable $k'G_{k'}$-modules. For this, we will require the following result:

Theorem 2.6.1. [Jan07, II, Corollary 2.9] *Let $V_{k'}$ be an irreducible tensor-indecomposable $k'G_{k'}$-module. Then, the FG_F-module $V_{k'} \otimes_{k'} F$ is irreducible and tensor-indecomposable.*

In view of the above theorem, let $V_{k'}$ be an irreducible tensor-indecomposable $k'G_{k'}$-module and assume that $\nu_{G_{k'}}(V_{k'}) \leq \sqrt{\dim(V_{k'})}$. Then, $V_F = V_{k'} \otimes_{k'} F$ is an irreducible tensor-indecomposable FG_F-module. Since $\dim_F(V_F) = \dim_{k'}(V_{k'})$ and $\dim_F((V_F)_g(\iota(\mu))) = \dim_{k'}((V_{k'})_g(\mu))$ for all $g \in G_{k'}$, it follows that $\nu_{G_F}(V_F) \leq \nu_{G_{k'}}(V_{k'})$. In particular, we have $\nu_{G_F}(V_F) \leq \sqrt{\dim_F(V_F)}$ and, consequently, $\dim(V_F) \leq \alpha(G_F)^2$, hence $\dim(V_{k'}) \leq \alpha(G_F)^2$.

In conclusion, we have shown that if G is a simply connected linear algebraic group over $k' = \overline{\mathbb{F}_p}$ and V is an irreducible tensor-indecomposable $k'G$-module with $\nu_G(V) \leq \sqrt{\dim(V)}$, then V has to satisfy the following dimensional criteria:

$$\dim(V) \leq \alpha(G_F)^2,$$

where G_F is the simply connected linear algebraic group G over F, an algebraically closed field which is not algebraic over $\mathbb{F}_p$.

From this point onward, unless stated explicitly, the ground field k will be an arbitrary algebraically closed field of arbitrary characteristic.

2.7 The list of modules

In this section we will identify the nontrivial irreducible tensor-indecomposable kG-modules V which satisfy the dimensional criteria (2.12). Now, by Chevalley's classical result, Theorem 2.3.3, Proposition 2.3.7 and by Steinberg's tensor product theorem, Theorem 2.3.8, we may assume that $V = L_G(\lambda)$, where $\lambda \in \mathrm{X}(T)$ is a nonzero p-restricted dominant weight.

2.7.1 G of type A_ℓ, $\ell \geq 1$

First, let $\ell = 1$ and let $V = L_G(m\omega_1)$, where $m \in \mathbb{Z}_{\geq 1}$. As V is p-restricted, we have $p = 0$, or $p > m$. Now, by Theorem 2.5.3, we have $\alpha(G) = 3$ and, by substituting in the dimensional criteria (2.12), we deduce that

$$m + 1 = \dim(V) \leq 9. \tag{2.13}$$

Therefore, $V = L_G(m\omega_1)$ with $1 \leq m \leq 8$.

We can now assume that $\ell \geq 2$. Let $V = L_G(\lambda)$ for some nonzero p-restricted dominant weight $\lambda \in \mathrm{X}(T)$. By Theorem 2.5.3 we have $\alpha(G) = \ell + 1$ and, substituting in the dimensional criteria (2.12), gives

$$\dim(V) \leq (\ell + 1)^2. \tag{2.14}$$

We define F^{A_ℓ} to be the set of all nonzero p-restricted dominant weights $\lambda \in \mathrm{X}(T)$, up to duality of the associated irreducible module, with the property that $L_G(\lambda)$ satisfies the dimensional criteria (2.13) for $\ell = 1$ and (2.14) for all $\ell \geq 2$. Using [Lü01a, Theorems 5.1 and 4.4], we determine that $F^{A_\ell} := \{\omega_1, \omega_2, 2\omega_1, \omega_1 + \omega_\ell\}$. Moreover, for $\ell \geq 8$ we see that the only kG-modules V that satisfy (2.14) are the ones corresponding to highest weights $\lambda \in F^{A_\ell}$. Lastly, for $2 \leq \ell \leq 7$ there exist additional kG-modules which satisfy (2.14) and we list their corresponding highest weights in Table 2.7.1.

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell = 3$	$\omega_1 + \omega_2$	$p = 3$	16
$\ell = 5$	ω_3	all	20
$\ell = 6$	ω_3	all	35
$\ell = 7$	ω_3	all	56

Table 2.7.1: The particular highest weight modules for groups of type A_ℓ that satisfy (2.14).

2.7.2 G of type C_ℓ, $\ell \geq 2$

First, let $\ell = 2$ and $V = L_G(\lambda)$, where $\lambda \in \mathrm{X}(T)$ is nonzero p-restricted and dominant. By Theorem 2.5.3, we have $\alpha(G) = 5$ and, substituting in the dimensional criteria (2.12), gives:

$$\dim(V) \leq 25. \tag{2.15}$$

We now use [Lü01a, Theorem 4.4] to determine the kG-modules $L_G(\lambda)$ which satisfy (2.15) and we list their corresponding highest weights in Table 2.7.2.

λ	p	$\dim(L_G(\lambda))$
ω_1	all	4
ω_2	all	$5-\delta_{p,2}$
$2\omega_1$	$p \neq 2$	10
$\omega_1+\omega_2$	all	$16-4\delta_{p,5}$
$2\omega_2$	$p \neq 2$	$14-\delta_{p,5}$
$3\omega_1$	$p \neq 2,3$	20
$\omega_1+2\omega_2$	$p=7$	24
$3\omega_2$	$p=7$	25
$2\omega_1+\omega_2$	$p=3$	25

Table 2.7.2: The particular highest weight modules for groups of type C_2 that satisfy (2.15).

We can now assume that $\ell \geq 3$. By Theorem 2.5.3, we have $\alpha(G) = 2\ell$, if $p \neq 2$, respectively $\alpha(G) = 2\ell+1$, if $p = 2$. Substituting in the dimensional criteria (2.12), we deduce that:

$$\dim(V) \leq \begin{cases} 4\ell^2, \text{ if } p \neq 2; \\ (2\ell+1)^2, \text{ if } p = 2. \end{cases} \tag{2.16}$$

We define F^{C_ℓ} to be the set of all nonzero p-restricted dominant weights $\lambda \in \mathrm{X}(T)$ with the property that the associated irreducible module $L_G(\lambda)$ satisfies the dimensional criteria (2.15), for $\ell = 2$, and the dimensional criteria (2.16), for all $\ell \geq 3$. Once more, using [Lü01a, Theorems 5.1 and 4.4], we determine that $F^{C_\ell} := \{\omega_1, \omega_2, 2\omega_1\}$. We note that for $\ell \geq 9$, the only kG-modules $L_G(\lambda)$ that satisfy (2.16) are the ones corresponding to highest weights $\lambda \in F^{C_\ell}$. Lastly, for $3 \leq \ell \leq 8$ the additional kG-modules which satisfy (2.16) have corresponding highest weights listed in Table 2.7.3.

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell=3$	ω_3	$p=2$	8
	ω_3	$p \neq 2$	14
	$\omega_1+\omega_3$	$p=2$	48
	$2\omega_1+\omega_3$	$p=2$	48
$\ell=4$	ω_4	$p=2$	16
	ω_4	$p=3$	41
	ω_4	$p \neq 2,3$	42
	ω_3	$p=3$	40
	ω_3	$p \neq 3$	48
$\ell=5$	ω_5	$p=2$	32
	ω_3	$p=2$	100
$\ell=6,7,8$	ω_ℓ	$p=2$	2^ℓ

Table 2.7.3: The particular highest weight modules for groups of type C_ℓ that satisfy (2.16).

Remark 2.7.1. *We see that in Table 2.7.3, for the group C_3, we listed the weight $2\omega_1+\omega_3$. We added this module to the list, as we made the choice to not treat groups of type B_ℓ in*

characteristic 2, and, in view of Remark 2.3.11, this is the only module for groups of type B_ℓ which satisfies the dimensional criteria (2.17) and which, when viewed as a module for the group of type C_ℓ, is not isomorphic to (a twist of) a module already listed.

2.7.3 G of type B_ℓ, $\ell \geq 3$

Recall that for groups of type B_ℓ we are assuming that the characteristic of k is different than 2. Let $V = L_G(\lambda)$, where $\lambda \in \mathrm{X}(T)$ is nonzero p-restricted and dominant. By Theorem 2.5.3, we have $\alpha(G) = 2\ell + 1$ and substituting in the dimensional criteria (2.12) gives

$$\dim(V) \leq (2\ell + 1)^2. \tag{2.17}$$

We define F^{B_ℓ} to be the set of all nonzero p-restricted dominant weights $\lambda \in \mathrm{X}(T)$ with the property that the associated irreducible module $L_G(\lambda)$ satisfies the dimensional criteria (2.17), for all $\ell \geq 3$. Using [Lü01a, Theorems 5.1 and 4.4], we determine that $F^{B_\ell} := \{\omega_1, \omega_2, 2\omega_1\}$. Furthermore, for $\ell \geq 9$ we see that the only kG-modules V that satisfy (2.17) are the ones corresponding to highest weights $\lambda \in F^{B_\ell}$. Lastly, for $3 \leq \ell \leq 8$, the additional kG-modules which satisfy (2.17) correspond to highest weights given in Table 2.7.4.

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell = 3$	ω_3	$p \neq 2$	8
	$2\omega_3$	$p \neq 2$	35
	$\omega_1 + \omega_3$	$p = 7$	40
	$\omega_1 + \omega_3$	$p \neq 2, 7$	48
$\ell = 4, 5, 6, 7, 8$	ω_ℓ	$p \neq 2$	2^ℓ

Table 2.7.4: The particular highest weight modules for groups of type B_ℓ that satisfy (2.17).

2.7.4 G of type D_ℓ, $\ell \geq 4$

Let $V = L_G(\lambda)$, where $\lambda \in \mathrm{X}(T)$ is nonzero p-restricted and dominant. By Theorem 2.5.3, we have $\alpha(G) = 2\ell$ and substituting in the dimensional criteria (2.12) gives

$$\dim(V) \leq 4\ell^2. \tag{2.18}$$

We define F^{D_ℓ} to be the set of all nonzero p-restricted dominant weights $\lambda \in \mathrm{X}(T)$ with the property that the associated irreducible module $L_G(\lambda)$ satisfies the dimensional criteria (2.18) for all $\ell \geq 4$. Using [Lü01a, Theorems 5.1 and 4.4], we determine that $F^{D_\ell} := \{\omega_1, \omega_2, 2\omega_1\}$. Moreover, for $\ell \geq 10$ we see that the only kG-modules V that satisfy (2.18) are the ones corresponding to highest weights $\lambda \in F^{D_\ell}$. Lastly, for $4 \leq \ell \leq 9$, the additional kG-modules which satisfy (2.18) have highest weights given in Table 2.7.5.

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell = 4$	ω_3, ω_4	all	8
	$2\omega_3$, $2\omega_4$	$p \neq 2$	35
	$\omega_3 + \omega_4$, $\omega_1 + \omega_3$, $\omega_1 + \omega_4$	$p = 2$	48
	$\omega_3 + \omega_4$, $\omega_1 + \omega_3$, $\omega_1 + \omega_4$	$p \neq 2$	56
$\ell = 5$	ω_4, ω_5	all	16
	ω_3	$p = 2$	100
$\ell = 6, 7, 8, 9$	$\omega_{\ell-1}$, ω_ℓ	all	$2^{\ell-1}$

Table 2.7.5: The particular highest weight modules for groups of type D_ℓ that satisfy (2.18).

Remark 2.7.2. (1) *Assume that $\ell = 4$. Let $\sigma : \Delta \to \Delta$ be the triality automorphism of Δ given by $\alpha_1 \to \alpha_3$, $\alpha_2 \to \alpha_2$, $\alpha_3 \to \alpha_4$ and $\alpha_4 \to \alpha_1$. Now, using* [Hum72, Table 1, p.69]*, we see that $\sigma(r\omega_1) = r\omega_3$ and $\sigma(r\omega_3) = r\omega_4$, where $r = 1, 2$. Therefore, we have $L_G(r\omega_3) = L_G(\sigma(r\omega_1))$ and $L_G(r\omega_4) = L_G(\sigma^2(r\omega_1))$, where $r = 1, 2$, thus the result for $L_G(r\omega_3)$ and $L_G(r\omega_4)$ will follows from that for $L_G(r\omega_1)$, where $r = 1, 2$.*

Similarly, we see that $L_G(\omega_1 + \omega_3) = L_G(\sigma^2(\omega_3 + \omega_4))$ and $L_G(\omega_1 + \omega_4) = L_G(\sigma(\omega_3 + \omega_4))$, therefore, the result for $L_G(\omega_1 + \omega_3)$ and $L_G(\omega_1 + \omega_4)$ will follow from that for $L_G(\omega_3 + \omega_4)$.

(2) *Consider the case when $\ell \geq 5$. Let $\sigma' : \Delta \to \Delta$ be the automorphism of Δ given by $\alpha_i \to \alpha_i$, for all $1 \leq i \leq \ell - 2$, $\alpha_{\ell-1} \to \alpha_\ell$ and $\alpha_\ell \to \alpha_{\ell-1}$. Once more, using* [Hum72, Table 1, p.69]*, we see that $L_G(\omega_\ell) = L_G(\sigma'(\omega_{\ell-1}))$. Therefore, the result for $L_G(\omega_\ell)$ will follow from the result for $L_G(\omega_{\ell-1})$.*

2.8 Identifying irreducible modules as composition factors of certain tensor products

Recall that if the simple simply connected linear algebraic group G is of type B_ℓ, we assume that $p \neq 2$, where p is the characteristic of the algebraically closed field k. The following lemmas will enable us to identify the irreducible kG-modules $L_G(\lambda)$ corresponding to p-restricted dominant weights $\lambda \in F^G$ as composition factors of either $W \otimes W$ or $W \otimes W^*$, where W is the natural module of G, i.e. $W \cong L_G(\omega_1)$. Before, we state these results, we remind the reader that we use the notation $V = W_1 \mid W_2 \mid \cdots \mid W_m$, $m \geq 2$, to express that V has a composition series $V = V_1 \supset V_2 \supset \cdots \supset V_m \supset V_{m+1} = \{0\}$ with composition factors $W_i \cong V_i/V_{i+1}$, $1 \leq i \leq m$.

Lemma 2.8.1. [McN98, Propositions 4.2.2 and 4.6.10] *Let k be an algebraically closed field of characteristic $p \geq 0$ and let W be an $\ell + 1$-dimensional k-vector space, where $\ell \geq 1$. Set $G = \mathrm{SL}(W)$. Then as kG-modules, we have $\mathrm{S}^2(W) \cong L_G(2\omega_1)$ (if $p \neq 2$), $\wedge^2(W) \cong L_G(\omega_2)$ (if $\ell > 1$) and*

$$W \otimes W^* \cong \begin{cases} L_G(\omega_1 + \omega_\ell) \oplus L_G(0), & \text{if } p \nmid \ell + 1; \\ L_G(0) \mid L_G(\omega_1 + \omega_\ell) \mid L_G(0), & \text{if } p \mid \ell + 1. \end{cases}$$

Lemma 2.8.2. [McN98, Proposition 4.2.2 and Lemma 4.8.2] *Let k be an algebraically closed field of characteristic $p \geq 0$ and let W be a 2ℓ-dimensional k-vector space, where $\ell \geq 2$, equipped with a nondegenerate alternating bilinear form. Set $G = \mathrm{Sp}(W)$. Then, as kG-modules, we have*

$$\wedge^2(W) \cong \begin{cases} L_G(\omega_2) \oplus L_G(0), & \text{if } p \nmid \ell, \\ L_G(0) \mid L_G(\omega_2) \mid L_G(0), & \text{if } p \mid \ell. \end{cases}$$

Moreover, if $p \neq 2$, then $\mathrm{S}^2(W) \cong L_G(2\omega_1)$.

Remark 2.8.3. *In the setting of Lemma 2.8.2, first, consider the case when $p \nmid \ell$. Then, since $\wedge^2(W) \cong L_G(\omega_2) \oplus L_G(0)$ and since both $L_G(\omega_2)$ and $L_G(0)$ are self-dual kG-modules, see* [MT11, Proposition 16.1]*, we deduce that $\wedge^2(W)$ is a self-dual kG-module. Now, in the case of $p \mid \ell$, we use* [Kor17, Lemma 4.2 and Table 1]*, to determine that $\wedge^2(W)$ is a self-dual kG-module.*

Lemma 2.8.4. [McN98, Propositions 4.2.2 and 4.7.3] *Let k be an algebraically closed field of characteristic $p \neq 2$ and let W be a finite-dimensional k-vector space equipped with a nondegenerate symmetric bilinear form. Set $G = \mathrm{SO}(W)$. Then as kG-modules, we have $\wedge^2(W) \cong L_G(\omega_2)$ and*

$$\mathrm{S}^2(W) \cong \begin{cases} L_G(2\omega_1) \oplus L_G(0), & \text{if } p \nmid \dim(W); \\ L_G(0) \mid L_G(2\omega_1) \mid L_G(0), & \text{if } p \mid \dim(W). \end{cases}$$

Lemma 2.8.5. *Let k be an algebraically closed field of characteristic $p = 2$ and let W be a 2ℓ-dimensional k-vector space, where $\ell \geq 4$, equipped with a nondegenerate quadratic form Q. Set $G = \mathrm{SO}(W, Q)$. Then, one of the following holds:*

(a) *ℓ is odd and $\wedge^2(W) \cong L_G(\omega_2) \oplus L_G(0)$ as kG-modules.*

(b) *ℓ is even and, as a kG-module, $\wedge^2(W)$ has three composition factors: one isomorphic to $L_G(\omega_2)$ and two isomorphic to $L_G(0)$.*

Proof. By [Sei87, 1.15], we know that, as a kG-module, $\wedge^2(W)$ admits a unique nontrivial composition factor with corresponding highest weight ω_2. Since $\dim(L_G(\omega_2)) = 2\ell^2 - \ell - \gcd(2, \ell)$, see [Lü01a, Table 2], we determine that, if ℓ is odd, then $\wedge^2(W)$ has two composition factors: one isomorphic to $L_G(\omega_2)$ and one isomorphic to $L_G(0)$, while, if ℓ is even, then $\wedge^2(W)$ has three composition factors: one isomorphic to $L_G(\omega_2)$ and two isomorphic to $L_G(0)$. Lastly, we focus on the case of ℓ odd. Let a be the nondegenerate alternating bilinear form on W given by $a(w_1, w_2) = Q(w_1) + Q(w_2) + Q(w_1 + w_2)$, for all $w_1, w_2 \in W$. Set $H = \mathrm{Sp}(W, a)$ and note that $G < H$. Now, as $\wedge^2(W)$ is a self-dual kH-module, see Remark 2.8.4, by [Sch19, Lemma 1.4.1], we have that, in particular, $\wedge^2(W)$ is a self-dual kG-module and so, we apply [Sch19, Lemma 1.4.3], to conclude that $\wedge^2(W) \cong L_G(\omega_2) \oplus L_G(0)$. □

2.9 Unipotent elements

This section is devoted to the study of unipotent elements of simple classical linear algebraic groups. In the first two subsections, we will present basic facts concerning unipotent conjugacy

classes. In Subsection 2.9.1 we discuss the *Jordan normal form* of a unipotent element and show that over fields of good characteristic, i.e. $p \neq 2$ for groups of type B_ℓ, C_ℓ and D_ℓ, with a few exceptions in groups of type D_ℓ, this form completely determines unipotent conjugacy classes. However, when $p = 2$, this is no longer the case and we will use a different tool to realize this classification. In Subsections 2.9.3 and 2.9.4, respectively, we introduce the *Hesselink normal form*, respectively the *distinguished normal form*, of a unipotent element. We will show that each of these forms gives a complete characterization of unipotent conjugacy classes over fields of characteristic 2 and moreover, we will give a method to translate between the two.

2.9.1 The Jordan normal form

We begin this subsection with the following basic lemma, whose proof quickly follows from [Car93, Proposition 5.1.1].

Lemma 2.9.1. [Kor18, Lemma 2.1.2] *Let $\phi : G_1 \to G_2$ be an isogeny between two simple algebraic groups G_1 and G_2. Then the map ϕ restricts to a bijection between the unipotent varieties of G_1 and G_2 and ϕ induces a bijection between the unipotent conjugacy classes of G_1 and G_2.*

In light of Lemma 2.9.1, it is enough to describe unipotent conjugacy classes in simple algebraic groups for some fixed isogeny type. In what follows, we will give this description for the classical groups $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$ and $\mathrm{SO}(V)$.

We let k be an algebraically closed field of characteristic $p \geq 0$ and G be a simple classical linear algebraic group. Recall that, when G is of type B_ℓ, we assume that $p \neq 2$. Let T, Φ, B, Δ and $\omega_1, \ldots, \omega_\ell$ be as usual. Let u be a unipotent element of G and let $k[u]$ be the group algebra of $\langle u \rangle$ over k. For each $i \geq 0$, we will denote by V_i the indecomposable $k[u]$-module with $\dim(V_i) = i$ and on which u acts as the full Jordan block J_i of size i. We note that $\{V_i \mid i \geq 0\}$ is a set of representatives of the isomorphisms classes of indecomposable $k[u]$-modules.

Let W be the natural module for G, i.e. $W \cong L_G(\omega_1)$, as $p \neq 2$ when G is of type B_ℓ. Moreover, let $\dim(W) = n$. Then, in particular, W is a $k[u]$-module and so admits a decomposition $W \mid_{k[u]} \cong V_{n_1}^{r_1} \oplus \cdots \oplus V_{n_m}^{r_m}$, where $m \geq 1$, $n_1 > \cdots > n_m \geq 1$, $r_i \geq 1$ for all $1 \leq i \leq m$ and $\sum_{i=1}^{m} n_i r_i = n$. As u acts as J_i on each V_i, we determine that the action of u on W is given by $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$. We call $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ the <u>Jordan normal form</u> of u on W. We will see in Theorem 2.9.2 that the Jordan normal form of a unipotent element plays an essential role in determining unipotent conjugacy classes in G.

Theorem 2.9.2. [LS12, Theorem 3.1, Corollary 3.6, Lemma 3.11] *Let k be an algebraically closed field of characteristic $p \geq 0$ and let $G = \mathrm{SL}_n(k)$, $\mathrm{Sp}_n(k)$, or $\mathrm{O}_n(k)$. Assume that $p \neq 2$ when G is symplectic or orthogonal. Moreover, let W be the natural module for G and let $\dim(W) = n$.*

(1) *Two unipotent elements of G are G-conjugate if and only if they are $\mathrm{GL}_n(k)$-conjugate, hence if and only if they have the same Jordan form on W.*

(2) *Let $u \in \mathrm{GL}_n(k)$ be a unipotent element with Jordan form $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$. Then*

(2.1) *$u \in \mathrm{Sp}_n(k)$ if and only if r_i is even for each odd n_i;*

(2.2) *$u \in \mathrm{O}_n(k)$ if and only if r_i is even for each even n_i.*

(3) *The class $u^{\mathrm{O}_n(k)}$ splits into two $\mathrm{SO}_n(k)$-classes if and only if n_i is even for all $1 \leq i \leq m$.*

In view of Theorem 2.9.2, we make a few remarks.

Remark 2.9.3. (1) *When G is a simple group of type A_ℓ, the Jordan normal form of a unipotent element on W completely determines its conjugacy class in G.*

(2) *When $p \neq 2$ and G is a simple group of type C_ℓ, the Jordan normal form of a unipotent element on W completely determines its conjugacy class in G. However, when $p = 2$, this is no longer the case. For example, in $G = \mathrm{Sp}_4(k)$ there are two unipotent conjugacy classes whose Jordan form on W is J_2^2, however elements in one class act on $L_G(\omega_2)$ as J_2^2, while elements of the other class act on $L_G(\omega_2)$ as $J_2 \oplus J_1^2$, see the proof of Proposition 4.3.10. Therefore, we will require a different tool to distinguish between unipotent conjugacy classes in G, see Subsection 2.9.3.*

(3) *As we only consider algebraic groups of type B_ℓ over fields k with $\mathrm{char}(k) \neq 2$, Theorem 2.9.2 gives a complete characterization of unipotent conjugacy classes in simple groups of type B_ℓ.*

(4) *When G is a simple group of type D_ℓ and $p \neq 2$, we see that there exist two unipotent conjugacy classes whose Jordan form on W is $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$, where n_i is even for all $1 \leq i \leq m$. We will refer to these classes as split. Furthermore, as in the case of groups of type C_ℓ when $p = 2$, the Jordan normal form no longer suffices to characterize conjugacy classes.*

We end this subsection with two results which we will use extensively in the chapters to come. We have seen in Section 2.8 that the families of kG-modules of a simple simply connected classical linear algebraic group G can be identified with certain composition factors of various tensor products. Therefore, it will prove extremely useful to know how the unipotent elements of G act on tensor products and on their composition factors. Luckily, the following result due to Liebeck and Seitz, see [LS12, Lemma 3.4], gives us an almost complete answer:

Lemma 2.9.4. [LS12, Lemma 3.4] *Let V_i, V_j be vector spaces of dimensions i, j over k and let u_i, u_j denote unipotent elements acting as a single Jordan block in $\mathrm{GL}(V_i)$, $\mathrm{GL}(V_j)$, respectively.*

(a) *Then $\dim((V_i \otimes V_j)_{u_i \otimes u_j}(1)) = \min\{i, j\}$.*

(b) *Suppose* $p \neq 2$. *Then:*

(b.1) $\dim((\wedge^2(V_i))_{u_i}(1)) = \lfloor \frac{i}{2} \rfloor$;

(b.2) $\dim((S^2(V_i))_{u_i}(1)) = i - \lfloor \frac{i}{2} \rfloor$.

The only case we need, which is not covered by Lemma 2.9.4, is the case of $\wedge^2(V_i)$ in characteristic $p = 2$. We treat it in the result below:

Lemma 2.9.5. *Let k be a field of characteristic $p = 2$ and let V be a vector space of dimension $i \geq 1$ over k. Let u be a unipotent element acting as a single Jordan block in* $\mathrm{GL}(V)$. *Then*

$$\dim((\wedge^2(V))_u(1)) = \left\lfloor \frac{i}{2} \right\rfloor.$$

Proof. We will prove the result by induction on $i \geq 1$. First, we note that both cases $i = 1$ and $i = 2$ follow directly from the structure of $\wedge^2(V)$. Hence, we assume that $i \geq 3$ and that the result holds for all $1 \leq r < i$.

Let m be the unique nonnegative integer for which $2^{m-1} < i \leq 2^m$ and set $q = 2^m$. Now, up to isomorphism, there exist exactly q indecomposable $k[u]$-modules: $V_1, V_2, \ldots, V_q$, where $\dim(V_j) = j$ and u acts on V_j as the full Jordan block of size j. Therefore, as $k[u]$-modules, we have $V \cong V_i$. We now use [GL06, Theorem 2], by which, the following isomorphism of $k[u]$-modules holds:

$$\wedge^2(V_i) = \wedge^2(V_{q-i}) \oplus (i - \frac{q}{2} - 1)V_q \oplus V_{3\frac{q}{2}-i}.$$

This gives

$$\dim((\wedge^2(V_i))_u(1)) = \dim((\wedge^2(V_{q-i}))_u(1)) + (i - \frac{q}{2} - 1)\dim((V_q)_u(1)) + \dim((V_{3\frac{q}{2}-i})_u(1)). \tag{2.19}$$

As $3\frac{q}{2} - i < q$ and as u acts as a single Jordan block on V_q and $V_{3\frac{q}{2}-i}$, respectively, it follows that $\dim((V_q)_u(1)) = 1$ and $\dim((V_{3\frac{q}{2}-i})_u(1)) = 1$. Furthermore, we note that, as $\frac{q}{2} < i$, we have $q - i < i$ and, by applying induction, it follows that $\dim((\wedge^2(V_{q-i}))_u(1)) = \left\lfloor \frac{q-i}{2} \right\rfloor$. Substituting in (2.19) we obtain:

$$\dim((\wedge^2(V_i))_u(1)) = \left\lfloor \frac{q-i}{2} \right\rfloor + i - \frac{q}{2} - 1 + 1 = \left\lfloor \frac{i}{2} \right\rfloor.$$

This concludes the proof of the lemma. □

2.9.2 Paired modules

We have noted in Remark 2.9.3, that when G is of type C_ℓ or D_ℓ and the field k has characteristic $p = 2$, the Jordan normal form no longer suffices to distinguish between unipotent conjugacy classes in G. Therefore, when $p = 2$, we require new methods of realizing the classification of unipotent classes. To begin, we give a brief overview, following [Kor20, Section 5], on bilinear kG-modules. We require this theoretic part to introduce

the Hesselink, respectively, the distinguished, normal form of a unipotent element in the subsections that follow.

For the remainder of this subsection, we assume that the algebraically closed field k has characteristic $p = 2$. Let V be a finite-dimensional k-vector space and let $b : V \times V \to k$ be a nondegenerate bilinear form on V. We recall, from Subsection 2.1.2, that we call a nondegenerate bilinear form symmetric if $b(v_1, v_2) = b(v_2, v_1)$ for all $v_1, v_2 \in V$, respectively alternating if $b(v, v) = 0$ for all $v \in V$. We also recall that over fields of characteristic 2 any alternating bilinear form is, in particular, symmetric.

Let H be a group and assume that V is a finite-dimensional kH-module equipped with a bilinear form b. We say that b is H-invariant if $b(h \cdot v_1, h \cdot v_2) = b(v_1, v_2)$ for all $v_1, v_2 \in V$ and all $h \in H$. A bilinear kH-module (V, b) is a kH-module V equipped with a H-invariant bilinear form b. Two bilinear kH-modules (V, b) and $(V^{'}, b^{'})$ are isomorphic if there exists an isomorphism of kH-modules $\phi : V \to V^{'}$ with the property that $b^{'}(\phi(v_1), \phi(v_2)) = b(v_1, v_2)$ for all $v_1, v_2 \in V$. We call the bilinear kH-module (V, b) nondegenerate if b is nondegenerate and, assuming that b is nondegenerate, we call the bilinear kH-module (V, b) symmetric, respectively, alternating, if b is symmetric, respectively alternating.

Let (V, b) and $(V^{'}, b^{'})$ be two bilinear kH-modules. We define the orthogonal direct sum of (V, b) and $(V^{'}, b^{'})$ to be the bilinear kH-module $(V \oplus V^{'}, b \perp b^{'})$, where $(b \perp b^{'})(v_1 + v_1^{'}, v_2 + v_2^{'}) = b(v_1, v_2) + b^{'}(v_1^{'}, v_2^{'})$ for all $v_1, v_2 \in V$ and all $v_1^{'}, v_2^{'} \in V^{'}$. We denote $(V \oplus V^{'}, b \perp b^{'})$ by $(V, b) \perp (V^{'}, b^{'})$. A nonzero bilinear kH-module (V, b) is orthogonally indecomposable if whenever $V = V_1 \perp V_2$, where V_1 and V_2 are kH-submodules of V, we have $V_1 = 0$, or $V_2 = 0$. Lastly, any bilinear kH-module decomposes into an orthogonal direct sum of orthogonally indecomposable bilinear kH-modules. However, in this setting, we do not have an analog of the Krull-Schmidt Theorem, see [Mur16, Example 2.1] .

Definition 2.9.6. *Let V be a kH-module. The paired module associated to V is the bilinear kH-module $(V \oplus V^*, a)$, where*

$$a(v_1 + f_1, v_2 + f_2) = f_1(v_2) + f_2(v_1), \text{ for all } v_1, v_2 \in V \text{ and all } f_1, f_2 \in V^*.$$

In [Mur16, Section 3.2], Murray showed that for any kH-module V, the associated paired module $(V \oplus V^*, a)$ is always an alternating bilinear kH-module. The following result tells us when the converse is true. Before we state it, we recall from Subsection 2.1.2 that over fields of characteristic 2, we call a subspace W' of (W, b), where b is a nondegenerate alternating bilinear form, totally isotropic if $b(w, w') = 0$ for all $w, w' \in W$. Now, we say that W admits a totally isotropic decomposition $W = W' \oplus W''$ whenever W' and W'' are two proper totally isotropic subspaces of W.

Lemma 2.9.7. [Kor20, Lemma 5.12] *Let (V, a) be a nondegenerate alternating bilinear kH-module. Then (V, a) is a paired module if and only if there exists a totally isotropic decomposition $V = W \oplus W^{'}$, where W and $W^{'}$ are kH-submodules of V. In this case (V, a) is the paired module associated to W.*

We finish this summary on bilinear kH-modules with the following two lemmas, which will be required in the sequel.

Lemma 2.9.8. [PM18, Section 2.3] *Let V be an indecomposable kH-module. Then the paired module $(V \oplus V^*, a)$ associated to V is indecomposable.*

Lemma 2.9.9. [Kor20, Lemma 5.15] *Let (V, b) be a bilinear kH-module and let $(V \oplus V^*, a)$ be the paired module associated to V. Then, as bilinear kH-modules, we have:*

$$(V, b) \perp (V, b) \perp (V, b) \cong (V, b) \perp (V \oplus V^*, a).$$

2.9.3 The Hesselink normal form

Let k be an algebraically closed field of characteristic $p = 2$ and let V be a finite-dimensional k-vector space equipped with a quadratic form Q. Let b be the nondegenerate alternating bilinear form on V given by $b(v_1, v_2) = Q(v_1 + v_2) + Q(v_1) + Q(v_2)$ for all $v_1, v_2 \in V$. Set $G = \mathrm{Sp}(V, b) = \{g \in \mathrm{GL}(V) \mid b(g \cdot v_1, g \cdot v_2) = b(v_1, v_2), \text{ for all } v_1, v_2 \in V\}$. Now, over fields of characteristic 2, we always have $\mathrm{O}(V, Q) < G$. The following theorem shows that if we classify the unipotent conjugacy classes in G, we also classify the ones in $\mathrm{O}(V, Q)$:

Theorem 2.9.10. [Dye79, Theorems 4 and 5]

(a) *Each conjugacy class of G contains one conjugacy class of $\mathrm{O}(V, Q)$.*

(b) *Two elements $g, g' \in \mathrm{O}(V, Q)$ are conjugate in $\mathrm{O}(V, Q)$ if and only if they are conjugate in G.*

Let u be a unipotent element in G. As $\mathrm{char}(k) = 2$, the order of u is $q = 2^t$, for some $t \geq 0$. Recall that we have denoted by $V_1, \ldots, V_q$ the q nonisomorphic indecomposable $k[u]$-modules, where $\dim(V_i) = i$ and on which u acts as the full Jordan block of size i. In [Hes79, Section 2.1], Hesselink proves that there exists a one-to-one correspondence between unipotent conjugacy classes in G and decompositions of $V|_{k[u]}$ into orthogonal direct sums of orthogonally indecomposable $k[u]$-modules. Furthermore, he identifies the two unique families of orthogonally indecomposable summands that can occur and he denotes them by $V(d)$, where $d \geq 2$ is even, and $W(d)$, where $d \geq 1$, see [Hes79, Proposition 3.5]. In what follows we give the definitions of $V(d)$ and $W(d)$, as they appear in [Kor20, Definitions 6.1 and 6.2], and afterwards we describe the Hesselink normal form of a unipotent element of G.

Let $d \geq 2$ be an even integer, $d = 2f$ for some $f \geq 1$, and consider the indecomposable $k[u]$-module V_d. Fix a basis $B_e = \{e_1, \ldots, e_d\}$ of V_d with the property that

$$\begin{cases} u \cdot e_1 = e_1; \\ u \cdot e_i = e_i + e_{i-1} + \cdots + e_1, \text{ for all } 2 \leq i \leq f + 1; \\ u \cdot e_i = e_i + e_{i-1}, \text{ for all } f + 2 \leq i \leq d. \end{cases} \tag{2.20}$$

We define $V(d)$ to be the bilinear $k[u]$-module (V_d, b_d), where $b_d(e_i, e_j) = 1$, if $i + j = d + 1$, and 0, otherwise. We remark that $\dim(V(d)) = d$ and that u acts on $V(d)$ as a single Jordan block of size d. Furthermore, $V(d)$ is orthogonally indecomposable, as V_d is indecomposable as a $k[u]$-module.

Now, let $d \geq 1$ and define $W(d)$ to be the paired module $(V_d \oplus V_d^*, a)$ associated to V_d. We remark that $\dim(W(d)) = 2d$ and that u acts on $W(d)$ as J_d^2. Moreover, since V_d is an indecomposable $k[u]$-module, it follows that $W(d)$ is orthogonally indecomposable, see Lemma 2.9.8.

Theorem 2.9.11. [Kor20, Theorem 6.4] *Let k be an algebraically closed field of characteristic 2, let V be a finite-dimensional k-vector space equipped with a nondegenerate alternating bilinear form b and let $G = \mathrm{Sp}(V, b)$. Let $u \in G$ be a unipotent element and let $V|_{k[u]} \cong V_{n_1}^{r_1} \oplus \cdots \oplus V_{n_m}^{r_m}$, where $m \geq 1$, $n_1 > \cdots > n_m \geq 1$ and $r_i \geq 1$.*

There exists a unique sequence $W_1, \ldots, W_m$ of nondegenerate alternating bilinear $k[u]$-modules such that $V|_{k[u]} \cong W_1 \perp \cdots \perp W_m$ and the following hold for all $1 \leq i \leq m$:

(a) *if n_i is odd, then r_i is even and $W_i \cong W(n_i)^{\frac{r_i}{2}}$;*

(b) *if n_i is even, then either r_i is even and $W_i \cong W(n_i)^{\frac{r_i}{2}}$, or $W_i \cong V(n_i)^{r_i}$.*

The decomposition $V|_{k[u]} \cong W_1 \perp \cdots \perp W_m$ of Theorem 2.9.11 is called the <u>Hesselink normal form</u> of u on V. In [Hes79, Section 3.7], Hesselink proved that this form completely determines the unipotent conjugacy class of u in G.

Remark 2.9.12. *The number of distinct Jordan block sizes occurring in the Jordan decomposition of u on V is equal to the number of nondegenerate alternating bilinear $k[u]$-modules W_i that occur in the Hesselink normal form of u on V. Thus, for distinct odd block sizes d, with respective multiplicities r_d, occurring in the Jordan form of u on V, there exists a unique $1 \leq i \leq m$ such that $W_i \cong W(d)^{\frac{r_d}{2}}$. Similarly, for distinct even block sizes d, with respective multiplicities r_d, occurring in the Jordan form of u on V, there exists a unique $1 \leq i \leq m$ such that either $W_i \cong W(d)^{\frac{r_d}{2}}$, or $W_i \cong V(d)^{r_d}$, i.e. there do not exist $i \neq j$ such that $W_i \cong W(d)^{\frac{r'_d}{2}}$ and $W_j \cong V(d)^{r''_d}$ with $r'_d, r''_d > 0$.*

We define $\varepsilon_{V,b} : \mathbb{Z}_{\geq 1} \to \{0, 1\}$ by:

$$\varepsilon_{V,b}(d) = \begin{cases} 0, & \text{if } b((u-1)^{d-1} \cdot v, v) = 0 \text{ for all } v \in V \text{ such that } (u-1)^d \cdot v = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 2.9.13. [Kor20, Lemmas 6.9 and 6.10] *Let $u \in \mathrm{Sp}(V, b)$ be a unipotent element and let $V|_{k[u]} \cong V_{n_1}^{r_1} \oplus \cdots \oplus V_{n_m}^{r_m}$, where $m \geq 1$, $n_1 > \cdots > n_m \geq 1$ and $r_i \geq 1$.*

(a) *If n_i is odd, then $\varepsilon_{V,b}(n_i) = 0$.*

(b) *The following statements are equivalent:*

  (b.1) *$\varepsilon_{V,b}(n_i) = 1$.*

  (b.2) *n_i is even and $V(n_i)$ occurs as an orthogonal direct summand of V.*

  (b.3) *n_i is even and $V(n_i)$ is isomorphic to an orthogonal direct summand of any decomposition of V into a direct sum of orthogonally indecomposable $k[u]$-submodules.*

In what follows we will see that the Hesselink normal form of u on V, hence its conjugacy class in $\mathrm{Sp}(V, b)$, is completely determined by the Jordan form of u on V and the values of $\varepsilon_{V,b}$ on the Jordan block sizes of u.

Remark 2.9.14. *Let $u \in \mathrm{Sp}(V,b)$ be a unipotent element and let $V|_{k[u]} \cong V_{n_1}^{r_1} \oplus \cdots \oplus V_{n_m}^{r_m}$, where $m \geq 1$, $n_1 > \cdots > n_m \geq 1$ and $r_i \geq 1$. Furthermore, let $V|_{k[u]} \cong W_1 \perp \cdots \perp W_m$ be the Hesselink normal form of u on V, as given in Theorem 2.9.11. First, assume that there exists $1 \leq i \leq m$ such that $W_i \cong W(n_i)^{\frac{r_i}{2}}$. If n_i is odd, then by Lemma 2.9.13 (a), we have that $\varepsilon_{V,b}(n_i) = 0$. Similarly, if n_i is even, then by Lemma 2.9.13 (b), it follows that $\varepsilon_{V,b}(n_i) = 0$, since, otherwise, there would exist $1 \leq j \leq m$, $j \neq i$, such that $W_j \cong V(n_i)^{r_i}$, contradicting Remark 2.9.12. Conversely, if $\varepsilon_{V,b}(n_i) = 0$ for some $1 \leq i \leq m$, then, by Lemma 2.9.13 and Theorem 2.9.11, it follows that $W_i \cong W(n_i)^{\frac{r_i}{2}}$. We have just proven that $\varepsilon_{V,b}(n_i) = 0$ if and only if $W_i \cong W(n_i)^{\frac{r_i}{2}}$. Similarly, one shows that $\varepsilon_{V,b}(n_i) = 1$ if and only if $W_i \cong V(n_i)^{r_i}$.*

Theorem 2.9.15. [Remark 2.9.14 and [Kor20, Theorem 6.7]] *Let $u \in \mathrm{Sp}(V,b)$ be a unipotent element and set $\varepsilon = \varepsilon_{V,b}$. Let $V|_{k[u]} \cong V_{n_1}^{r_1} \oplus \cdots \oplus V_{n_m}^{r_m}$, where $m \geq 1$, $n_1 > \cdots > n_m \geq 1$ and $r_i \geq 1$, i.e. u has Jordan normal form on V given by $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$. Moreover, let $V|_{k[u]} \cong W_1 \perp \cdots \perp W_m$ be the Hesselink normal form of u on V, as given in Theorem 2.9.11. Then for all $1 \leq i \leq m$, we have $W_i \cong W(n_i)^{\frac{r_i}{2}}$ if and only if $\varepsilon(n_i) = 0$ and $W_i \cong V(n_i)^{r_i}$ if and only if $\varepsilon(n_i) = 1$.*

In particular, the Hesselink normal form of u on V is uniquely determined by the tuple $(n_{1_{\varepsilon(n_1)}}^{r_1}, \ldots, n_{m_{\varepsilon(n_m)}}^{r_m})$.

Let $u \in G$ be a unipotent element. From this point onward, we will abuse notation and call the tuple

$$(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots, n_{m_1}^{r_m})$$

the Hesselink normal form of u on V, where $m \geq 1$, $t \geq 0$ $n_1 > \cdots > n_t \geq 1$, $n_{t+1} > \cdots > n_m \geq 1$ and $r_i \geq 1$ for all $1 \leq i \leq m$.

We end this subsection by returning to the example in item (2) of Remark 2.9.3, where we noted that, over fields of characteristic 2, there exist two unipotent conjugacy classes in $\mathrm{Sp}_4(k)$ whose Jordan form on W is J_2^2. Now, using Theorem 2.9.15, we can actually identify these two classes by their Hesselink normal form. Let u and u' be representatives of each of these classes. Then the Hesselink normal form of u is (2_0^2), i.e. $W|_{k[u]} \cong W(2)$, and the Hesselink normal form of u' is (2_1^2), i.e. $W|_{k[u']} \cong V(2) \oplus V(2)$.

2.9.4 The distinguished normal form

Let k be an algebraically closed field of characteristic 2, let V be a finite-dimensional k-vector space equipped with a nondegenerate quadratic form Q. Let b be the nondegenerate alternating bilinear form on V given by $b(v_1, v_2) = Q(v_1 + v_2) + Q(v_1) + Q(v_2)$, for all $v_1, v_2 \in V$. Set $G = \mathrm{Sp}(V,b)$. In the previous subsection, we saw that the Hesselink normal form completely determines unipotent conjugacy classes in G. Moreover, by Theorem 2.9.10, we know that each conjugacy class of G contains one conjugacy class of $\mathrm{O}(V,Q)$ and that two elements of $\mathrm{O}(V,Q)$ are conjugate in $\mathrm{O}(V,Q)$ if and only if they are conjugate in G. We now consider the subgroup $H = \mathrm{SO}(V,Q)$ of G. In this subsection, we will give a criteria, in terms of the Hesselink normal form, to determine when unipotent conjugacy classes of G intersect H. Moreover, we will also determine when unipotent conjugacy classes in G

split into distinct unipotent classes in H. In order to achieve this, we will exhibit another normal form which can be used to characterize unipotent conjugacy classes in G, called the *distinguished normal form*. Lastly, we will present methods of translation between the two, see Lemma 2.9.18 and Remark 2.9.19, respectively, and, using [LS12, Proposition 6.22], we state the classification of unipotent conjugacy classes in $\mathrm{SO}(V, Q)$, see Proposition 2.9.20.

Let u be a unipotent element in G. According to [LS12, Table 4.1], viewed as a $k[u]$-module, V decomposes into an orthogonal direct sum of orthogonally indecomposable bilinear $k[u]$-modules of the form $V_D(m)$, where $m \geq 2$ is even, and $W_D(m)$, where $m \geq 1$, that we define below.

Let $m \geq 2$ be an even integer, $m = 2n$ for some $n \geq 1$. In [LS12, Section 6.1], the orthogonally indecomposable $k[u]$-module $V_D(m)$ is defined to be the m-dimensional k-vector space equipped with a bilinear form b in which we fix a basis $\{x_1, \ldots, x_m\}$ with the property that $b(x_i, x_j) = 1$, if $i + j = m + 1$, and 0, otherwise; $Q(x_i) = 1$, if $i = n$, and 0, otherwise; and on which u acts as:

$$\begin{cases} u \cdot x_1 = x_1; \\ u \cdot x_i = x_i + x_{i-1}, \text{ for } 2 \leq i \leq n; \\ u \cdot x_{n+i} = x_{n+i} + x_{n+i-1} + \cdots + x_n, \text{ for } 1 \leq i \leq n. \end{cases} \tag{2.21}$$

We note that the bilinear form b is nondegenerate and alternating and that u fixes both b and Q. Therefore $V_D(m)$ is a nondegenerate alternating bilinear $k[u]$-module on which u acts as a single Jordan block of size m, see [LS12, Table 4.1].

Now let $m \geq 1$. In [LS12, Section 6.1], the orthogonally indecomposable $k[u]$-module $W_D(m)$ is defined to be the $2m$-dimensional vector space equipped with a bilinear form b in which we fix a basis $\{x_{m-1}, x_{m-3}, \ldots, x_{-(m-1)}, y_{m-1}, y_{m-3}, \ldots, y_{-(m-1)}\}$ with the property that $b(x_i, y_{-i}) = 1$, for all $-(m-1) \leq i \leq m-1$, and all other inner products between basis vectors are 0; $Q(x_i) = Q(y_i) = 0$, for all $-(m-1) \leq i \leq m-1$; and on which u acts as:

$$\begin{cases} u \cdot x_{m-1} = x_{m-1}; \\ u \cdot x_i = x_i + x_{i+2}, \text{ for } -(m-1) \leq i \leq m-3; \\ u \cdot y_{m-1} = y_{m-1}; \\ u \cdot y_i = y_i + y_{i+2} + \cdots + y_{m-1}, \text{ for } -(m-1) \leq i \leq m-3. \end{cases}$$

We note that the bilinear form b is nondegenerate and alternating and that u fixes both b and Q. Therefore, $W_D(m)$ is a nondegenerate alternating bilinear $k[u]$-module on which u acts as J_m^2, see [LS12, Table 4.1].

The following proposition shows that the decomposition of V into an orthogonal direct sum of orthogonally indecomposable bilinear $k[u]$-modules of the form $V_D(m)$ and $W_D(m)$ completely determines the unipotent conjugacy class of u in G:

Proposition 2.9.16. [LS12, Lemma 6.2 and Proposition 6.22]. *Let $u \in G$ be a unipotent element. Then, there exists an orthogonal decomposition*

$$V \mid_{k[u]} \cong \coprod_{i=1}^{s} W_D(n_i)^{r_i} \perp \coprod_{j=s+1}^{m} V_D(n_j)^{r_j}, \tag{2.22}$$

where $m \geq 1$ and $0 \leq s \leq m$ are such that the n_i's are distinct and $r_i \geq 1$, for all $1 \leq i \leq s$, the n_j's are even and distinct and $r_j \leq 2$, for all $s+1 \leq j \leq m$. Moreover, the following hold:

(a) *The summands in (2.22) are unique and they completely determine the unipotent conjugacy class of u in G.*

(b) *We have that $u \in \mathrm{SO}(V,Q)$ if and only if $\sum_{j=s+1}^{m} r_j$ is even.*

(c) *If $u \in \mathrm{SO}(V,Q)$, then the conjugacy class of u in G splits into two $\mathrm{SO}(V,Q)$ classes if and only if $s = m$ and n_i is even for all $1 \leq i \leq m$.*

Remark 2.9.17. *We will refer to the two unipotent conjugacy classes of $\mathrm{SO}(V,Q)$ from item (c) of Proposition 2.9.16 as split.*

The decomposition in (2.22) is called the distinguished normal form of u on V. As this form is different from the Hesselink normal form, for our purpose, it is useful to have a method of translating between the two. For this, we first have to prove the following isomorphisms of bilinear $k[u]$-modules:

$$(W_D(d), b) \cong (W(d), a), \text{ for all } d \geq 1$$

and

$$(V_D(d), b) \cong (V(d), b_d), \text{ for all } d \geq 2 \text{ even.}$$

Recall that for $1 \leq i \leq \mathrm{ord}(u)$ we have denoted by V_i the indecomposable i-dimensional $k[u]$-module on which u acts as the full Jordan block of size i. Let $d \geq 1$ and let N, respectively N', be the $k[u]$-submodule of $W_D(d)$ generated by $\{x_{d-1}, x_{d-3}, \ldots, x_{-(d-1)}\}$, respectively by $\{y_{d-1}, y_{d-3}, \ldots, y_{-(d-1)}\}$. As $b(x_i, x_j) = 0$ and $b(y_i, y_j) = 0$, for all $-(d-1) \leq i, j \leq d-1$, it follows that $W_D(d) = N \oplus N'$ is a totally isotropic decomposition of $W_D(d)$. Hence, by Lemma 2.9.7, we have that $W_D(d)$ is the paired module associated to N. Moreover, as $\dim(N) = d$ and as u acts as a single Jordan block of size d on N, it follows that N is indecomposable as a $k[u]$-module and therefore N and V_d are isomorphic $k[u]$-modules. We conclude that $W_D(d)$ is the paired module associated to V_d.

Now, recall that in Subsection 2.9.3, $W(d)$ has been defined as the paired module $(V_d \oplus V_d^*, a)$ associated to V_d. We apply Lemma 2.9.7 and deduce that $W(d) = V_d \oplus V_d^*$ is a totally isotropic decomposition with respect to a. Let $\{v_{d-1}, v_{d-3}, \ldots, v_{-(d-1)}\}$ be a basis in V_d and let $\{v_{d-1}^*, v_{d-3}^*, \ldots, v_{-(d-1)}^*\}$ be the dual basis of V_d^*, i.e. , for all $-(d-1) \leq i \leq d-1$, we have $v_i^*(v_j) = 1$, if $i = -j$, and 0, otherwise. Then $\{v_{d-1}, v_{d-3}, \ldots, v_{-(d-1)}, v_{d-1}^*, v_{d-3}^*, \ldots, v_{-(d-1)}^*\}$ is a basis in $W(d)$ with the property that, for all $-(d-1) \leq i \leq d-1$, we have $a(v_i, v_j^*) = 1$, if $j = -i$, and 0, otherwise. Now, the action of u on $\{v_{d-1}, v_{d-3}, \ldots, v_{-(d-1)}\}$ is given by:

$$\begin{cases} u \cdot v_{d-1} = v_{d-1}; \\ u \cdot v_i = v_i + v_{i+2}, \text{ for all } -(d-1) \leq i \leq d-3. \end{cases}$$

Recursively, one shows that the action of u^{-1} on $\{v_{d-1}, \dots, v_{-(d-1)}\}$ is $u^{-1} \cdot v_i = v_i + v_{i+2} + \cdots + v_{d-1}$, for all $-(d-1) \leq i \leq d-1$, and, using this, determines that the action of u on $\{v^*_{d-1}, \dots, v^*_{-(d-1)}\}$ is given by:

$$\begin{cases} u. \cdot v^*_{d-1} = v^*_{d-1}; \\ u \cdot v^*_i = v^*_i + v^*_{i+2} + \cdots + v^*_{d-1}, \text{ for all } -(d-1) \leq i \leq d-3. \end{cases}$$

One checks that the map $\phi : W(d) \to W_D(d)$ defined by $\phi(v_i) = x_i$ and $\phi(v^*_i) = y_i$, for all $-(d-1) \leq i \leq d-1$, is a $k[u]$-module isomorphism and, furthermore, as $a(w, w') = b(\phi(w), \phi(w'))$, for all $w, w' \in \{v_{d-1}, v_{d-3}, \dots, v_{-(d-1)}, v^*_{d-1}, v^*_{d-3}, \dots, v^*_{-(d-1)}\}$, we conclude that $(W(d), a)$ and $(W_D(d), b)$ are isomorphic bilinear $k[u]$-modules.

Let $d \geq 2$ be an even integer with $d = 2f$, for some $f \geq 1$. Recall that, in Subsection 2.9.3, we have fixed a basis $B_e = \{e_1, e_2, \dots, e_d\}$ on V_d with the property that $b_d(e_i, e_j) = 1$, if $i + j = d + 1$, and 0, otherwise, and we have defined $V(d)$ to be the bilinear $k[u]$-module (V_d, b_d). Moreover, recall that we have described the action of u on B_e in (2.20). Set A to be the block matrix $\left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right) \in \mathrm{M}_d(k)$, where $B = (b_{i,j})_{i,j} \in \mathrm{M}_f(k)$ is such that $b_{i,j} = \binom{f-i}{f-j}$, for all $i \leq j \leq f$, and $b_{i,j} = 0$, for all $i > j$; and $C = (c_{i,j})_{i,j} \in \mathrm{M}_f(k)$ is such that $c_{i,j} = b_{f+1-j,f+1-i}$, for all $i \leq j \leq f$, and $c_{i,j} = 0$, for all $i > j$. We see that A has the following form:

$$A = \begin{pmatrix} 1 & b_{1,2} & b_{1,3} & \cdots & b_{1,f-1} & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & b_{2,3} & \cdots & b_{2,f-1} & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{f-2,f-1} & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & b_{f-2,f-1} & b_{f-3,f-1} & \cdots & b_{2,f-1} & b_{1,f-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & b_{f-3,f-2} & \cdots & b_{2,f-2} & b_{1,f-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & b_{1,2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $B_x = \{x_1, x_2, \dots x_d\}$ be the basis of $V(d)$ with the property that A is the change of basis matrix from B_e to B_x, i.e. we have

$$\begin{cases} x_j = \sum_{i=1}^{j} b_{i,j} e_i, \text{ for } 1 \leq j \leq f; \\ x_{f+j} = \sum_{i=1}^{f} c_{i,j} e_{f+i} = \sum_{i=1}^{j} b_{f+1-j,f+1-i} e_{f+i}, \text{ for } 1 \leq j \leq f. \end{cases}$$

We will first show that the matrix A is symplectic, i.e. it preserves the form b_d. As $A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right)$, we have that $A^{\mathrm{tr}} S_d A = \left(\begin{array}{c|c} 0 & B^{\mathrm{tr}} K_f C \\ \hline (B^{\mathrm{tr}} K_f C)^{\mathrm{tr}} & 0 \end{array}\right)$, where $S_d = \left(\begin{array}{c|c} 0 & K_f \\ \hline K_f & 0 \end{array}\right)$

and $(K_f)_{i,j} = \delta_{j,f+1-i}$. Therefore, to show that A is symplectic, we only need to show that $B^{\mathrm{tr}} K_f C = K_f$. Let $1 \leq i, j \leq f$. Then:

$$\begin{aligned}
(B^{\mathrm{tr}} K_f C)_{i,j} &= \sum_{r=1}^{f} (B^{\mathrm{tr}} K_f)_{i,r} \cdot c_{r,j} = \sum_{r=1}^{f} \left[\sum_{q=1}^{f} (B^{\mathrm{tr}})_{i,q} (K_f)_{q,r} \right] \cdot b_{f+1-j,f+1-r} \\
&= \sum_{r=1}^{f} \left[\sum_{q=1}^{f} b_{q,i} \cdot \delta_{q,f+1-r} \right] \cdot b_{f+1-j,f+1-r} = \sum_{r=1}^{f} b_{f+1-r,i} \cdot b_{f+1-j,f+1-r} \\
&= \sum_{q=1}^{f} b_{q,i} \cdot b_{f+1-j,q} = \sum_{q=1}^{i} b_{q,i} \cdot b_{f+1-j,q},
\end{aligned}$$

as $b_{q,i} = 0$, for all $q > i$. Now, if $f + 1 - j > i$, then $(B^{\mathrm{tr}} K_f C)_{i,j} = 0$. If $f + 1 - j = i$, then

$$(B^{\mathrm{tr}} K_f C)_{i,f+1-i} = \sum_{q=1}^{i} b_{q,i} \cdot b_{i,q} = b_{i,i} \cdot b_{i,i} = \binom{f-i}{f-i} \cdot \binom{f-i}{f-i} = 1,$$

as $b_{i,j} = 0$, for $i > j$, and $b_{i,j} = \binom{f-i}{f-j}$, for $i \leq j$. We thus assume that $f + 1 - j < i$. Then

$$\begin{aligned}
(B^{\mathrm{tr}} K_f C)_{i,j} &= \left[\sum_{q=f+1-j}^{i} \binom{f-q}{f-i} \binom{j-1}{f-q} \right] \\
&= \left[\frac{(j-1)!}{(f-i)!} \left(\sum_{q=f+1-j}^{i} \frac{1}{(i-q)!(q+j-f-1)!} \right) \right] \\
&= \left[\frac{(j-1)!}{(f-i)!(i+j-f-1)!} \left(\sum_{q=f+1-j}^{i} \binom{i+j-f-1}{i-q} \right) \right] \\
&= \left[\frac{(j-1)!}{(f-i)!(i+j-f-1)!} \cdot 2^{i+j-f-1} \right] \\
&= 0.
\end{aligned}$$

We conclude that $B^{\mathrm{tr}} K_f C = K_f$ and, consequently, $A^{\mathrm{tr}} S_d A = S_d$. Therefore the matrix A is symplectic.

We will now show that the action of u on $\{x_1, \ldots, x_d\}$ is as in (2.21). Let $1 \leq j \leq f$ and, using relations (2.20), we have that

$$\begin{aligned}
u \cdot x_j = u \cdot (\sum_{i=1}^{j} b_{i,j} e_i) &= \sum_{i=1}^{j} b_{i,j} \left(\sum_{r=1}^{i} e_r \right) = x_j + \sum_{i=2}^{j} \sum_{r=1}^{i-1} b_{i,j} e_r = x_j + \sum_{r=1}^{j-1} \Big(\sum_{i=r+1}^{j} b_{i,j} \Big) e_r \\
&= x_j + \sum_{r=1}^{j-1} \sum_{i=r+1}^{j} \binom{f-i}{f-j} e_r.
\end{aligned}$$

Now, for all $0 < m < n$, we have that $\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$. Thus, for all $1 \leq r \leq j-1$, one can show that

$$\begin{aligned}
\binom{f-r}{f-j+1} &= \binom{f-r-1}{f-j} + \binom{f-r-1}{f-j+1} = \binom{f-r-1}{f-j} + \left[\binom{f-r-2}{f-j} + \binom{f-r-2}{f-j+1}\right] \\
&= \sum_{i=r+1}^{r+2} \left[\binom{f-i}{f-j}\right] + \left[\binom{f-r-3}{f-j} + \binom{f-r-3}{f-j+1}\right] \\
&\vdots \\
&= \sum_{i=r+1}^{j} \binom{f-i}{f-j}.
\end{aligned} \tag{2.23}$$

It follows that

$$u \cdot x_j = x_j + \sum_{r=1}^{j-1} \binom{f-r}{f-j+1} e_r = x_j + \sum_{r=1}^{j-1} b_{r,j-1} e_r$$

and so

$$u \cdot x_j = x_j + x_{j-1}, \text{ for all } 1 \leq j \leq f. \tag{2.24}$$

Similarly, for $1 \leq j \leq f$, we compute

$$\begin{aligned}
u \cdot x_{f+j} &= u \cdot (\sum_{i=1}^{j} b_{f+1-j,f+1-i} e_{f+i}) = b_{f+1-j,f} \cdot (e_1 + \cdots + e_{f+1}) + \sum_{i=2}^{j} b_{f+1-j,f+1-i}(e_{f+i} + e_{f+i-1}) \\
&= x_{f+j} + b_{f+1-j,f} \cdot (e_1 + \cdots + e_f) + \sum_{i=2}^{j} b_{f+1-j,f+1-i} e_{f+i-1}.
\end{aligned}$$

We remark that $b_{f+1-j,f} = 1$, for all $1 \leq j \leq f$, and, as $b_{i,f} = 1$, for all $1 \leq i \leq f$, we deduce that $x_f = b_{f+1-j,f} \cdot (e_1 + \cdots + e_f)$. Therefore, we have:

$$u \cdot x_{f+j} = x_{f+j} + x_f + \sum_{i=2}^{j} b_{f+1-j,f+1-i} e_{f+i-1}.$$

On the other hand:

$$\begin{aligned}
x_{f+1} + \cdots + x_{f+j-1} &= \sum_{i=1}^{j-1} \left(\sum_{r=1}^{i} b_{f+1-i,f+1-r} e_{f+r} \right) (\text{ we interchange the order of the sums}) \\
&= \sum_{r=1}^{j-1} \left(\sum_{i=r}^{j-1} b_{f+1-i,f+1-r} e_{f+r} \right) = \sum_{r=1}^{j-1} \left(\sum_{i=r}^{j-1} b_{f+1-i,f+1-r} \right) e_{f+r} \\
&= \sum_{r=1}^{j-1} \left[\sum_{i=r}^{j-1} \binom{i-1}{r-1} \right] e_{f+r}.
\end{aligned}$$

For $1 \leq r \leq j-1$, we remark that $\binom{j-1}{r} = \binom{f-(f-j+1)}{f-(f-r+1)+1}$ and so, by (2.23), it follows that $\binom{j-1}{r} = \sum_{q=f+2-j}^{f+1-r} \binom{f-q}{r-1}$. We make the variable change $i = f+1-q$, i.e. $q = f+1-i$. Hence we have $f-q = i-1$ and, for $q = f+2-j$, we get $i = j-1$, while, for $q = f+1-r$, we get $i = r$. Thus, $\binom{j-1}{r} = \sum_{i=r}^{j-1} \binom{i-1}{r-1}$ and so

$$\begin{aligned} x_{f+1} + \cdots + x_{f+j-1} &= \sum_{r=1}^{j-1} \binom{j-1}{r} e_{f+r} = \sum_{r=1}^{j-1} b_{f+1-j,f-r} e_{f+r} \\ &= \sum_{i=2}^{j} b_{f+1-j,f+1-i} e_{f+i-1}. \end{aligned}$$

We have shown that

$$u \cdot x_{f+j} = x_{f+j} + x_{f+j-1} + \cdots + x_f, \text{ for all } 1 \leq j \leq f. \tag{2.25}$$

Equations (2.24) and (2.25) show that the action of u on B_x is as in (2.21).

Using the fact that A is a symplectic matrix with respect to b_d, one is able to show that the map $\phi : V(d) \to V_D(d)$ given by $\phi(e_i) = x_i$ for all $1 \leq i \leq d$ is an isomorphism of bilinear $k[u]$-modules.

Lemma 2.9.18. *Let k be an algebraically closed field of characteristic 2 and let $G = \mathrm{Sp}(V, b)$. Let u be a unipotent element of G whose Hesselink normal form on V is*

$$(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots, n_{m_1}^{r_m}),$$

where $m \geq 1$ and $t \geq 0$. For all $t+1 \leq i \leq m$, write $r_i = x_i + 2y_i$, where $1 \leq x_i \leq 2$ and $y_i \geq 0$. Then the distinguished normal form of u on V is:

$$V\mid_{k[u]} \cong \coprod_{i=1}^{t} W(n_i)^{\frac{r_i}{2}} \perp \coprod_{i=t+1}^{m} W(n_i)^{y_i} \perp \coprod_{i=t+1}^{m} V(n_i)^{x_i},$$

where $\coprod_{i=1}^{t} W(n_i)^{\frac{r_i}{2}}$ is empty if $t = 0$.

Proof. To begin, following Theorem 2.9.15, we write down the decomposition of V as an orthogonal direct sum of indecomposable $k[u]$-modules:

$$V\mid_{k[u]} \cong \coprod_{i=1}^{t} W(n_i)^{\frac{r_i}{2}} \perp \coprod_{i=t+1}^{m} V(n_i)^{r_i}.$$

If $r_i \leq 2$ for all $t+1 \leq i \leq m$, we set $x_i = r_i$ and $y_i = 0$. Then:

$$V\mid_{k[u]} \cong \coprod_{i=1}^{t} W(n_i)^{\frac{r_i}{2}} \perp \coprod_{i=t+1}^{m} V(n_i)^{x_i}$$

is the distinguished normal form of u on V.

We can thus assume that there exists some $t+1 \leq j \leq m$ such that $r_j \geq 3$. Then $r_j = 3q_1^j + p_1^j$, where $q_1^j \geq 1$ and $0 \leq p_1^j \leq 2$. By Lemma 2.9.9, it follows that

$$V(n_j)^{r_j} \cong W(n_j)^{q_1^j} \perp V(n_j)^{q_1^j+p_1^j}.$$

If $q_1^j + p_1^j \leq 2$, set $x_j = q_1^j + p_1^j$ and $y_j = q_1^j$. We note that $1 \leq x_j \leq 2$, as $q_1^j \geq 1$, and that $r_j = x_j + 2y_j$. On the other hand, if $q_1^j + p_1^j \geq 3$, we write $q_1^j + p_1^j = 3q_2^j + p_2^j$, where $q_2^j \geq 1$ and $0 \leq p_2^j \leq 2$. Once more, by Lemma 2.9.9, it follows that:

$$V(n_j)^{r_j} \cong W(n_j)^{q_1^j} \perp V(n_j)^{q_1^j+p_1^j} \cong W(n_j)^{q_1^j+q_2^j} \perp V(n_j)^{q_2^j+p_2^j}.$$

If $q_2^j + p_2^j \leq 2$, set $x_j = q_2^j + p_2^j$ and $y_j = q_1^j + q_2^j$. We note that $1 \leq x_j \leq 2$, as $q_2^j \geq 1$, and that $r_j = 3q_1^j + p_1^j = 2q_1^j + 3q_2^j + p_2^j = x_j + 2y_j$. On the other hand, if $q_2^j + p_2^j \geq 3$, we repeat the above procedure. Now, as r_j is finite, it follows that there exists $s \geq 1$ with the property that $q_s^j + p_s^j \leq 2$, where $q_s^j \geq 1$ and $0 \leq p_s^j \leq 2$ are given by $q_{s-1}^j + p_{s-1}^j = 3q_s^j + p_s^j$. Then, by Lemma 2.9.9, we have that

$$V(n_j)^{r_j} \cong W(n_j)^{q_1^j+\cdots+q_{s-1}^j} \perp V(n_j)^{q_{s-1}^j+p_{s-1}^j} \cong W(n_j)^{q_1^j+\cdots+q_s^j} \perp V(n_j)^{q_s^j+p_s^j}.$$

Set $x_j = q_s^j + p_s^j$ and $y_j = q_1^j + \cdots + q_s^j$. We note that $1 \leq x_j \leq 2$, as $q_s^j \geq 1$. Lastly, as $r_j = 3q_1^j + p_1^j$, we have that $p_1^j + q_1^j = r_j - 2q_2^j$ and, as $p_1^j + q_1^j = 3q_2^j + p_2^j$, we deduce that $p_2^j + q_2^j = r_j - 2(q_1^j + q_2^j)$. Recursively, we show that $p_i^j + q_i^j = r_j - 2(q_1^j + \cdots + q_i^j)$, for all $1 \leq i \leq s$ and thus, for $i = s$, we obtain $x_j = r_j - 2y_j$.

In conclusion, the distinguished normal form of u on V is:

$$V\mid_{k[u]} \cong \bigperp_{i=1}^{t} W(n_i)^{\frac{r_i}{2}} \perp \bigperp_{i=t+1}^{m} W(n_i)^{y_i} \perp \bigperp_{i=t+1}^{m} V(n_i)^{x_i},$$

where, for all $t+1 \leq i \leq m$, the integers $1 \leq x_i \leq 2$ and $y_i \geq 0$ are such that $r_i = x_i + 2y_i$. $\square$

Remark 2.9.19. *Let k be an algebraically closed field of characteristic 2 and let $G = \mathrm{Sp}(V, b)$. Let u be a unipotent element of G.*

(a) *If the distinguished normal form of u on V is*

$$V\mid_{k[u]} \cong \bigperp_{j=1}^{m} V(m_j)^{b_j},$$

where $m \geq 1$, the m_j's are even and distinct and $b_j \leq 2$, for all $1 \leq j \leq m$, then the Hesselink normal form of u on V is $(m_{1_1}^{b_1}, \ldots, m_{m_1}^{b_m})$.

(b) *If the distinguished normal form of u on V is*

$$V\mid_{k[u]} \cong \bigperp_{i=1}^{m} W(n_i)^{a_i},$$

where $m \geq 1$, the n_i's are distinct and $a_i \geq 1$, for all $1 \leq i \leq m$, then the Hesselink normal form of u on V is $(n_{1_0}^{2a_1}, \ldots, n_{m_0}^{2a_m})$.

(c) *Let the distinguished normal form of u on V be*

$$V\mid_{k[u]}\cong \bigperp_{i=1}^{s} W(n_i)^{a_i} \perp \bigperp_{j=s+1}^{m} V(m_j)^{b_j},$$

where $m \geq 2$, $1 \leq s \leq m-1$, the n_i's are distinct and $a_i \geq 1$, for all $1 \leq i \leq s$, and the m_j's are even and distinct and $b_j \leq 2$, for all $s+1 \leq j \leq m$.

Let $0 \leq d \leq s$ be the number of n_i's, $1 \leq i \leq s$, with the property that $n_i \neq m_j$ for all $s+1 \leq j \leq m$. If $d = 0$, then we relabel the m_j's, $s+1 \leq j \leq m$, such that $n_i = m_{s+i}$ for all $1 \leq i \leq s$. We rewrite the distinguished normal form of u on V as:

$$V|_{k[u]} \cong \bigperp_{i=1}^{s} \left[W(n_i)^{a_i} \perp V(m_{s+i})^{b_{s+i}}\right] \perp \bigperp_{j=2s+1}^{m} V(m_j)^{b_j}$$

and, keeping in mind that $n_i = m_{s+i}$ for all $1 \leq i \leq s$, we apply Lemma 2.9.9 to determine that:

$$V|_{k[u]} \cong \bigperp_{i=1}^{s} V(n_i)^{2a_i+b_{s+i}} \perp \bigperp_{j=2s+1}^{m} V(m_j)^{b_j}.$$

Hence, by Theorem 2.9.15, the Hesselink normal form of u on V is

$$(n_{1_1}^{2a_1+b_{s+1}}, \ldots, n_{s_1}^{2a_s+b_{2s}}, m_{2s+1_1}^{b_{2s+1}}, \ldots, m_{m_1}^{b_m}).$$

If $d = s$, then the Hesselink normal form of u on V is

$$(n_{1_0}^{2a_1}, \ldots, n_{s_0}^{2a_s}, m_{s+1_1}^{b_{s+1}}, \ldots, m_{m_1}^{b_m}).$$

Lastly, if $1 \leq d \leq s-1$, we relabel the n_i's, $1 \leq i \leq s$, and the m_j's, $s+1 \leq j \leq m$, in the following way:

$$\begin{cases} n_1, \ldots, n_d \text{ are such that } n_i \neq m_j \text{ for all } s+1 \leq j \leq m; \\ n_{d+1}, \ldots, n_s \text{ and } m_{s+1}, \ldots, m_{2s-d} \text{ are such that } n_{d+r} = m_{s+r} \text{ for all } 1 \leq r \leq s-d; \\ m_{2s-d+1}, \ldots, m_m \text{ are such that } m_j \neq n_i \text{ for all } 1 \leq i \leq s. \end{cases}$$

Moreover, we rewrite the distinguished normal form of u on V in the following way:

$$V\mid_{k[u]} \cong \bigperp_{i=1}^{d} W(n_i)^{a_i} \perp \bigperp_{r=1}^{s-d} \left[W(n_{d+r})^{a_{d+r}} \perp V(m_{s+r})^{b_{s+r}}\right] \perp \bigperp_{j=2s-d+1}^{m} V(m_j)^{b_j}.$$

Now, by Lemma 2.9.9 and keeping in mind that $n_{d+r} = m_{s+r}$ for all $1 \leq r \leq s-d$, we have:

$$W(n_{d+r})^{a_{d+r}} \perp V(m_{s+r})^{b_{s+r}} \cong V(n_{d+r})^{2a_{d+r}+b_{s+r}}$$

It follows that the Hesselink normal form of u on V is

$$(n_{1_0}^{2a_1}, \ldots, n_{d_0}^{2a_d}, n_{d+1_1}^{2a_{d+1}+b_{s+1}}, \ldots, n_{s_1}^{2a_s+b_{2s-d}}, m_{2s-d+1_1}^{b_{2s-d+1}}, \ldots, m_{m_1}^{b_m}).$$

We recall that we have fixed k to be an algebraically closed field of characteristic 2, V to be a finite-dimensional k-vector space equipped with a nondegenerate quadratic form Q, b to be the nondegenerate alternating bilinear form on V given by $b(v_1, v_2) = Q(v_1 + v_2) + Q(v_1) + Q(v_2)$ for all $v_1, v_2 \in V$, $G = \mathrm{Sp}(V, b)$ and $H = \mathrm{SO}(V, Q)$. Let $u \in G$ be a unipotent element and let $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots, n_{m_1}^{r_m})$ be its Hesselink normal form on V. Then, by Lemma 2.9.18, it follows that the distinguished normal form of u on V is:

$$V|_{k[u]} \cong \bigperp_{i=1}^{t} W(n_i)^{\frac{r_i}{2}} \perp \bigperp_{i=t+1}^{m} W(n_i)^{y_i} \perp \bigperp_{i=t+1}^{m} V(n_i)^{x_i},$$

where, for all $t+1 \leq i \leq m$, the integers $1 \leq x_i \leq 2$ and $y_i \geq 0$ are such that $r_i = x_i + 2y_i$.

Now, by Proposition 2.9.16 (b), it follows that $u \in H$ if and only if the sum $\sum_{i=t+1}^{m} x_i$ is even. Hence, we deduce that $u \in H$ if and only if $\sum_{i=t+1}^{m} r_i$ is even, as x_i and r_i have the same parity, for all $t+1 \leq i \leq m$.

Lastly, let $u \in H$ be a unipotent element. Then, in particular, $u \in G$. By Proposition 2.9.16 (c), it follows that the conjugacy class of u in G splits into two H-classes if and only if the distinguished normal form of u on V is

$$V|_{k[u]} \cong \bigperp_{i=1}^{m} W(n_i)^{r_i},$$

where $m \geq 1$ and the n_i's are even and distinct for all $1 \leq i \leq m$. We now use Remark 2.9.19 (b) to deduce that the Hesselink normal form of u on V is $(n_{1_0}^{2r_1}, \ldots, n_{m_0}^{2r_m})$. Therefore, the conjugacy class of u in G splits into two H-classes if and only if n_i is even and $\varepsilon_{V,b}(n_i) = 0$, for all $1 \leq i \leq m$. We have proven the following result:

Proposition 2.9.20. *Let k be an algebraically closed field of characteristic 2, let V be a finite-dimensional k-vector space equipped with a nondegenerate quadratic form Q and let b be the nondegenerate alternating bilinear form on V given by $b(v_1, v_2) = Q(v_1 + v_2) + Q(v_1) + Q(v_2)$, for all $v_1, v_2 \in V$. Set $G = \mathrm{Sp}(V, b)$ and $H = \mathrm{SO}(V, Q)$. Let $u \in G$ be a unipotent element and let $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots, n_{m_1}^{r_m})$ be its Hesselink normal form on V. The following statements hold:*

(a) *$u \in H$ if and only if $\sum_{i=t+1}^{m} r_i$ is even;*

(b) *for $u \in H$, the conjugacy class of u in G splits into two H-classes if and only if, for all $1 \leq i \leq m$, we have that n_i is even and $t = m$, i.e. if and only if, for all $1 \leq i \leq m$, n_i is even and $\varepsilon_{V,b}(n_i) = 0$.*

Chapter 3

Groups of type A_ℓ

In this chapter we prove Theorems 1.1.1 and 1.1.3 for the simple simply connected linear algebraic groups of type A_ℓ, $\ell \geq 1$. The structure of the chapter is as follows: in the first section we construct such a group and exhibit some properties of its semisimple and unipotent elements. In Section 3.2 we determine $\max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, where V runs through the list of kG-modules we identified in Subsection 2.7.1. Similarly, in Section 3.3, we determine $\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, where G_u denotes the set of unipotent elements of G, for the same kG-modules V. Lastly, Section 3.4 records all the results of this chapter.

We now give some notation that will be used throughout the chapter. We fix k to be an algebraically closed field of characteristic $p \geq 0$ and G to be a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$. We will use the notation T, Φ, B, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\omega_1, \ldots, \omega_\ell$ to denote a fixed maximal torus of G, the root system of G determined by T, the positive Borel subgroup of G, the set of simple roots in Φ given by B, and the fundamental dominant weights of G corresponding to α_i, $1 \leq i \leq \ell$.

3.1 Construction of linear algebraic groups of type A_ℓ

Let W be an $\ell+1$-dimensional k-vector space, for some $\ell \geq 1$, and fix an ordered basis B_W in W. Set $G = \mathrm{SL}(W)$ and note that G is a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$, see Subsection 2.1.1. Let T be the set of diagonal matrices in G, and note that T is a maximal torus in G. Further, let B be the set of upper-triangular matrices in G, and note that B is a Borel subgroup of G with the property that $T \subseteq B$.

Let $s \in T$, $s = \mathrm{diag}(a_1, a_2, \ldots, a_{\ell+1})$ with $a_i \in k^*$ and $\prod\limits_{i=1}^{\ell+1} a_i = 1$. Let $m \geq 1$ and let $\mu_1, \mu_2, \ldots \mu_m$ denote the distinct a_i's. For all $1 \leq i \leq m$, let n_i denote the multiplicity of each μ_i in s. We have that $\sum\limits_{i=1}^{m} n_i = \ell+1$ and we can assume, without loss of generality, that $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$. Furthermore, by conjugating s by an element of $\mathrm{N}_G(T)$, we can assume that $s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \mu_2 \cdot \mathrm{I}_{n_2}, \ldots, \mu_m \cdot \mathrm{I}_{n_m})$. Lastly, we remark that, since $\prod\limits_{i=1}^{\ell+1} a_i = 1$,

we have $\prod_{i=1}^{m} \mu_i^{n_i} = 1$. Moreover, if $s \notin \mathrm{Z}(G)$, then $m \geq 2$.

We now turn our attention to the unipotent elements of G. By Theorem 2.9.2, we know that two unipotent elements of G are G-conjugate if and only if they are $\mathrm{GL}(W)$-conjugate, i.e. if and only if they have the same Jordan form on W. We write $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ for the Jordan form of a unipotent element of G on W, where $n_i \geq 1$ and $r_i \geq 1$, for all $1 \leq i \leq m$, see Section 2.9.1. We can assume, without loss of generality, that $\ell + 1 \geq n_1 > n_2 > \cdots > n_m \geq 1$. Moreover, if $u \neq 1$, then $n_1 \geq 2$.

3.2 Eigenspace dimensions for semisimple elements

Before we state the main results of this section, we recall that $F^{A_\ell} = \{\omega_1, \omega_2, 2\omega_1, \omega_1 + \omega_\ell\}$, see Subsection 2.7.1.

Theorem 3.2.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$. Let T be a fixed maximal torus in G. If $\ell = 1$, let $V = L_G(m\omega_1)$, where $1 \leq m \leq 8$, and assume that $p = 0$ or $p > m$. If $\ell \geq 2$, let $V = L_G(\lambda)$, where $\lambda \in F^{A_\ell}$ or λ appears in Table 2.7.1. Then there exist $s \in T \setminus \mathrm{Z}(G)$ and $\mu \in k^*$, an eigenvalue of s on V, such that*

$$\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)},$$

if and only if ℓ, λ and p appear in the following list:

(1) *$\ell \geq 1$, $\lambda = \omega_1$ and $p \geq 0$;*

(2) *$\ell \leq 3$, $\lambda = 2\omega_1$ and $p \neq 2$;*

(3) *$\ell \leq 3$, $\lambda = \omega_2$ and $p \geq 0$;*

(4) *$\ell = 1$, $\lambda \in \{3\omega_1, 4\omega_1\}$ and $p \neq 2, 3$.*

Theorem 3.2.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$. Let T and V be as in Theorem 3.2.1. Then the value of $\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ is given in the table below:*

V	Char.	Rank	$\max\limits_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$
$L_G(\omega_1)$	$p\geq 0$	$\ell\geq 1$	ℓ
${}^\star L_G(2\omega_1)$	$p\neq 2$	$\ell\geq 1$	$\frac{\ell^2+\ell+2}{2}$
${}^\star L_G(\omega_2)$	$p\geq 0$	$\ell=3$	4
		$\ell\geq 4$	$\frac{\ell(\ell-1)}{2}$
${}^\dagger L_G(\omega_1+\omega_\ell)$	$p\nmid \ell+1$	$\ell\geq 2$	ℓ^2
${}^\dagger L_G(\omega_1+\omega_\ell)$	$p\mid \ell+1$	$\ell=2$	4
		$\ell\geq 3$	ℓ^2-1
${}^\ddagger L_G(m\omega_1)$, $3\leq m\leq 8$	$p=0$, or $p>m$	$\ell=1$	$1+\left\lfloor\frac{m}{2}\right\rfloor$
${}^\dagger L_G(\omega_1+\omega_2)$	$p=3$	$\ell=3$	10
${}^\dagger L_G(\omega_3)$	$p\geq 0$	$\ell=5$	12
${}^\dagger L_G(\omega_3)$	$p\geq 0$	$\ell=6$	20
${}^\dagger L_G(\omega_3)$	$p\geq 0$	$\ell=7$	35

Table 3.2.1: The value of $\max\limits_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$.

In particular, for each V of Table 3.2.1 labeled as follows: ${}^\dagger V$; ${}^\star V$ with $\ell\geq 4$; and ${}^\ddagger V$ with $m\geq 5$; we have $\dim(V_s(\mu))<\dim(V)-\sqrt{\dim(V)}$ for all $s\in T\setminus \mathrm{Z}(G)$ and all eigenvalues $\mu\in k^*$ of s on V.

We will give the proof of Theorems 3.2.1 and 3.2.2 in a series of results, each treating one of the candidate-modules. In Subsection 3.2.1, we focus on the irreducible kG-modules $L_G(\lambda)$ corresponding to p-restricted dominant weights $\lambda\in F^{A_\ell}$. As these modules need to be considered for all $\ell\geq 1$, we will refer to them as *families of modules*. In Subsection 3.2.2, we will treat the irreducible kG-modules $L_G(m\omega_1)$, where $3\leq m\leq 8$, of the simple simply connected linear algebraic group G of type A_1, as well as the irreducible kG-modules $L_G(\lambda)$, where G is a simple simply connected linear algebraic group of type A_ℓ of rank $\ell\geq 2$ and the p-restricted dominant weight λ is featured in Table 2.7.1.

3.2.1 The families of modules

For the rest of this chapter, we fix the following hypothesis on semisimple elements in G:

$({}^\dagger H_s)$: any $s\in T\setminus \mathrm{Z}(G)$ is such that $s=\mathrm{diag}(\mu_1\cdot \mathrm{I}_{n_1},\mu_2\cdot \mathrm{I}_{n_2},\ldots,\mu_m\cdot \mathrm{I}_{n_m})$, where $m\geq 2$, $\mu_i\neq\mu_j$ for all $1\leq i<j\leq m$, $\ell\geq n_1\geq n_2\geq\cdots\geq n_m\geq 1$ and $\prod\limits_{i=1}^{m}\mu_i^{n_i}=1$.

Lemma 3.2.3. *Let $V=L_G(\omega_1)$. Then for all $s\in T\setminus \mathrm{Z}(G)$ and all eigenvalues $\mu\in k^*$ of s on V we have*

$$\dim(V_s(\mu))\leq \ell,$$

where equality holds if and only if, up to conjugation, $s=\mathrm{diag}(\mu_1,\mu_1,\ldots,\mu_1,\mu_1^{-\ell})$ with $\mu_1^{\ell+1}\neq 1$, and $\mu=\mu_1$.

In particular, there exist $s\in T\setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu\in k^*$ on V for which $\dim(V_s(\mu))\geq \dim(V)-\sqrt{\dim(V)}$.

Proof. We first note that $V \cong W$ as kG-modules, thus $\dim(V) = \ell + 1$. As $s \notin \mathrm{Z}(G)$, it follows that $\dim(V_s(\mu)) \leq \dim(V) - 1 = \ell$. Now equality holds if and only if, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_2)$, where $\mu_1 \neq \mu_2$. Lastly, as $\mu_1^\ell \mu_2 = 1$ and $\mu_2 \neq \mu_1$, we have $\mu_2 = \mu_1^{-\ell}$ and $\mu_1^{\ell+1} \neq 1$.

To conclude, we have shown that $\dim(V_s(\mu)) \leq \ell$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. Now, as the inequality $1 \leq \sqrt{\ell + 1}$ holds for all $\ell \geq 1$, it follows that $\ell + 1 - \sqrt{\ell + 1} \leq \ell$ holds for all $\ell \geq 1$ and thus we have shown that there exist $s \in T \setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. □

Proposition 3.2.4. *Let $V = L_G(2\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq \frac{\ell^2 + \ell + 2}{2},$$

where equality holds if and only if, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$, with $\mu_1^{\ell+1} = -1$, and $\mu = \mu_1^2$.

In particular, for $\ell \leq 3$, there exist $s \in T \setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 4$, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^\dagger H_s)$. By Lemma 2.8.1, since $p \neq 2$, we have that $V \cong \mathrm{S}^2(W)$ and so $\dim(V) = \frac{(\ell+1)(\ell+2)}{2}$. Moreover, we deduce that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} \mu_i^2,\ 1 \leq i \leq m, \text{ with multiplicity at least } \frac{n_i(n_i+1)}{2}; \\ \mu_i\mu_j,\ 1 \leq i < j \leq m, \text{ with multiplicity at least } n_i n_j. \end{cases} \tag{3.1}$$

Fix some $1 \leq i \leq m$ and consider the eigenvalue μ_i^2 of s on V. Since the μ_r's are distinct, it follows that $\mu_i^2 \neq \mu_i\mu_j$, for all $i \neq j$. Hence, by (3.1), we find at least $n_i(\ell + 1 - n_i)$ eigenvalues of s on V not equal to μ_i^2. It follows that:

$$\dim(V_s(\mu_i^2)) \leq \frac{(\ell+1)(\ell+2)}{2} - n_i(\ell + 1 - n_i). \tag{3.2}$$

If $\dim(V_s(\mu_i^2)) \geq \dfrac{\ell^2 + \ell + 2}{2}$, then

$$\ell - n_i(\ell + 1 - n_i) \geq 0,$$

and so

$$(\ell - n_i)(1 - n_i) \geq 0. \tag{3.3}$$

Since $\ell \geq n_i \geq 1$, it follows that $(\ell - n_i)(1 - n_i) \leq 0$ and so (3.3) holds if and only if $n_i \in \{1, \ell\}$. In both cases, substituting in (3.2) yields $\dim(V_s(\mu_i^2)) \leq \dfrac{\ell^2 + \ell + 2}{2}$. Now, by (3.1), equality holds if and only if $\mu_i^2 = \mu_j^2$ for all $j \neq i$. If $\mu_i^2 = \mu_j^2$ for all $j \neq i$, it follows that $m = 2$, as the μ_r's are distinct. Therefore, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_2)$, where

$\mu_1 \neq \mu_2$, $\mu_1^\ell \mu_2 = 1$ and $\mu_1^2 = \mu_2^2$. Then $\mu_2 = \mu_1^{-\ell}$, hence $\mu_1^2 = \mu_1^{-2\ell}$ and so $\mu_1 \neq \mu_1^{-\ell}$. It follows that, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} = -1$, as in the statement of the result. Conversely, let $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell}) \in T$ with $\mu_1^{\ell+1} = -1$. Then $s \notin \mathrm{Z}(G)$ and $\dim(V_s(\mu_1^2)) = \frac{\ell^2+\ell+2}{2}$, by (3.1).

Fix some $1 \leq i < j \leq m$ and consider the eigenvalue $\mu_i\mu_j$ of s on V. Since the μ_r's are distinct, we remark that:

$$\begin{cases} \mu_i\mu_j \neq \mu_i^2 \text{ and } \mu_i\mu_j \neq \mu_j^2; \\ \mu_i\mu_j \neq \mu_i\mu_r, \text{ for } i < r \leq m \text{ and } r \neq j, \text{ and } \mu_i\mu_j \neq \mu_r\mu_i, \text{ for } 1 \leq r < i; \\ \mu_i\mu_j \neq \mu_r\mu_j, \text{ for } 1 \leq r < j \text{ and } r \neq i, \text{ and } \mu_i\mu_j \neq \mu_j\mu_r, \text{ for } j < r \leq m. \end{cases}$$

By (3.1), all of the above account for at least $\dfrac{n_i(n_i+1)}{2} + \dfrac{n_j(n_j+1)}{2} + (n_i+n_j)(\ell+1-n_i-n_j)$ eigenvalues of s on V which are different than $\mu_i\mu_j$. Hence, we have:

$$\dim(V_s(\mu_i\mu_j)) \leq \frac{(\ell+1)(\ell+2)}{2} - \frac{n_i(n_i+1)}{2} - \frac{n_j(n_j+1)}{2} - (n_i+n_j)(\ell+1-n_i-n_j).$$

Assume $\dim(V_s(\mu_i\mu_j)) \geq \dfrac{\ell^2+\ell+2}{2}$. It follows that:

$$\ell - (n_i+n_j)(\ell+1-n_i-n_j) - \frac{n_i(n_i+1)+n_j(n_j+1)}{2} \geq 0.$$

We rewrite the above as:

$$(\ell - n_i - n_j)(1 - n_i - n_j) - \frac{n_i(n_i+1)+n_j(n_j+1)}{2} \geq 0 \tag{3.4}$$

and note that $\dfrac{n_i(n_i+1)+n_j(n_j+1)}{2} \geq 2$, as $n_i \geq n_j \geq 1$. Therefore, by (3.4), we have:

$$(\ell - n_i - n_j)(1 - n_i - n_j) - 2 \geq 0. \tag{3.5}$$

If $n_i + n_j \leq \ell$, then $(\ell - n_i - n_j)(1 - n_i - n_j) \leq 0$ and Inequality (3.5) does not hold. If $n_i + n_j = \ell + 1$, then $m = 2$, $n_2 = \ell + 1 - n_1$ and substituting in Inequality (3.4) gives:

$$\ell - \frac{n_1(n_1+1) + (\ell+1-n_1)(\ell+2-n_1)}{2} \geq 0$$

and thus

$$-2n_1^2 + 2n_1\ell - \ell^2 + 2n_1 - \ell - 2 \geq 0. \tag{3.6}$$

But, $-2n_1^2 + 2n_1\ell - \ell^2 + 2n_1 - \ell - 2 = -[(\ell - n_1)^2 + (n_1 - 1)^2 + \ell + 1] < 0$ and so Inequality (3.6) does not hold. Therefore, $\dim(V_s(\mu_i\mu_j)) < \dfrac{\ell^2+\ell+2}{2}$ for all $1 \leq i < j \leq m$.

In conclusion, for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, we have shown that $\dim(V_s(\mu)) \leq \frac{\ell^2+\ell+2}{2}$, where equality holds if and only if, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} = -1$, and $\mu = \mu_1^2$. In particular, as the inequality

$\ell^2 - 3\ell - 2 \leq 0$ holds for all $\ell \leq 3$, it follows that $\frac{\ell^2+3\ell+2}{2} - \sqrt{\frac{\ell^2+3\ell+2}{2}} \leq \frac{\ell^2+\ell+2}{2}$ for all $\ell \leq 3$. This shows that there exist $s \in T \setminus \mathrm{Z}(G)$ which afford an eigenvalue $\mu \in k^*$ on V, for example $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} = -1$ and $\mu = \mu_1^2$, such that $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 4$ we have $\frac{\ell^2+\ell+2}{2} < \frac{\ell^2+3\ell+2}{2} - \sqrt{\frac{\ell^2+3\ell+2}{2}}$ and therefore $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 3.2.5. *Let $\ell \geq 3$ and let $V = L_G(\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V one of the following holds:*

(1) *$\ell = 3$ and $\dim(V_s(\mu)) \leq 4$, where equality holds if and only if, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \pm\mu_1^{-1}, \pm\mu_1^{-1})$, with $\mu_1^2 \neq \pm 1$, and $\mu = \pm 1$.*

(2) *$\ell \geq 4$ and $\dim(V_s(\mu)) \leq \frac{\ell(\ell-1)}{2}$, where we have equality if and only if one of the following holds:*

  (2.1) *$\ell = 4$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_2, \mu_2)$, with $\mu_1 \neq \mu_2$ and $\mu_1^3 = \mu_2^{-2}$, and $\mu = \mu_1\mu_2$.*

  (2.2) *$\ell \geq 4$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$, with $\mu_1^{\ell+1} \neq 1$, and $\mu = \mu_1^2$.*

In particular, for $\ell = 3$ there exist $s \in T \setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 4$, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^\dagger H_s)$. By Lemma 2.8.1, since $\ell \geq 3$, we have that $V \cong \wedge^2(W)$ and so we deduce that $\dim(V) = \frac{\ell(\ell+1)}{2}$ and that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} \mu_i^2,\ 1 \leq i \leq m, \text{ with multiplicity at least } \frac{n_i(n_i-1)}{2}; \\ \mu_i\mu_j,\ 1 \leq i < j \leq m, \text{ with multiplicity at least } n_in_j. \end{cases} \tag{3.7}$$

We note that if $n_i = 1$, then μ_i^2 does not occur as an eigenvalue of s on V. We thus suppose that there exists some $1 \leq i \leq m$ such that $n_i \geq 2$ and consider the eigenvalue μ_i^2 of s on V. Now, since the μ_r's are distinct, it follows that $\mu_i^2 \neq \mu_i\mu_j$ for all $i \neq j$, hence:

$$\dim(V_s(\mu_i^2)) \leq \frac{\ell(\ell+1)}{2} - (\ell+1-n_i)n_i.$$

Let $\ell = 3$, and assume $\dim(V_s(\mu_i^2)) \geq 4$. Then:

$$2 - (4-n_i)n_i = (n_i-2)^2 - 2 \geq 0$$

and so $n_i \geq 4$, contradicting $n_i \leq \ell$.

We now let $\ell \geq 4$ and assume $\dim(V_s(\mu_i^2)) \geq \frac{\ell(\ell-1)}{2}$. It follows that:

$$\ell - (\ell + 1 - n_i)n_i \geq 0.$$

By the arguments following Inequality (3.3) and keeping in mind that $n_i \geq 2$, it follows that the above holds if and only if $n_i = \ell$. Hence, $m = 2$, $n_1 = \ell$, $n_2 = 1$ and, as $\mu_1 \neq \mu_2$ and $\mu_1^\ell \mu_2 = 1$, we have $\mu_2 = \mu_1^{-\ell}$ and $\mu_1^{\ell+1} \neq 1$. Moreover, we note that in this case we have $\dim(V_s(\mu_1^2)) = \frac{\ell(\ell-1)}{2}$. Thus, for $\ell \geq 4$ we showed that $\dim(V_s(\mu_i^2)) \leq \frac{\ell(\ell-1)}{2}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all $1 \leq i \leq m$ and that equality holds if and only if $i = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$, with $\mu_1^{\ell+1} \neq 1$, as in (2.2).

Fix some $1 \leq i < j \leq m$ and consider the eigenvalue $\mu_i\mu_j$ of s on V. Since the μ_r's are distinct, we remark that:

$$\begin{cases} \mu_i\mu_j \neq \mu_i^2 \text{ and } \mu_i\mu_j \neq \mu_j^2; \\ \mu_i\mu_j \neq \mu_i\mu_r, \text{ for } i < r \leq m \text{ and } r \neq j, \text{ and } \mu_i\mu_j \neq \mu_r\mu_i, \text{ for } 1 \leq r < i; \\ \mu_i\mu_j \neq \mu_r\mu_j, \text{ for } 1 \leq r < j \text{ and } r \neq i, \text{ and } \mu_i\mu_j \neq \mu_j\mu_r, \text{ for } j < r \leq m. \end{cases}$$

By (3.7), all of the above account for at least $\dfrac{n_i(n_i-1)}{2} + \dfrac{n_j(n_j-1)}{2} + (n_i+n_j)(\ell+1-n_i-n_j)$ eigenvalues of s on V different than $\mu_i\mu_j$. Hence, we have:

$$\dim(V_s(\mu_i\mu_j)) \leq \frac{\ell(\ell+1)}{2} - \frac{n_i(n_i-1)}{2} - \frac{n_j(n_j-1)}{2} - (n_i+n_j)(\ell+1-n_i-n_j).$$

Let $\ell = 3$. Then, since $3 \geq n_i \geq n_j \geq 1$ and $n_i + n_j \leq 4$, we deduce that $(n_i, n_j) \in \{(3,1), (2,2), (2,1), (1,1)\}$. Assume that $\dim(V_s(\mu_i\mu_j)) \geq 4$. It follows that:

$$2 - \frac{n_i(n_i-1)}{2} - \frac{n_j(n_j-1)}{2} - (n_i+n_j)(4-n_i-n_j) \geq 0. \tag{3.8}$$

Substituting all possible values for (n_i, n_j) in (3.8), we see that the inequality holds if and only if $n_i = n_j = 2$. Assume that $n_i = n_j = 2$. Then $m = 2$, $n_1 = n_2 = 2$ and, as $\mu_1^2\mu_2^2 = 1$, we have $\mu_2 = \pm\mu_1^{-1}$. Now, if $\mu_2 = \mu_1^{-1}$, then, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1^2 \neq 1$, as $\mu_1 \neq \mu_2$. Conversely, for $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1^2 \neq 1$, we have $s \in T \setminus \mathrm{Z}(G)$ and $\dim(V_s(1)) = 4$, by (3.7). Similarly, if $\mu_2 = -\mu_1^{-1}$, then, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, -\mu_1^{-1}, -\mu_1^{-1})$ with $\mu_1^2 \neq -1$. Conversely, for $s = \mathrm{diag}(\mu_1, \mu_1, -\mu_1^{-1}, -\mu_1^{-1})$ with $\mu_1^2 \neq -1$, we have $s \in T \setminus \mathrm{Z}(G)$ and $\dim(V_s(-1)) = 4$, by (3.7).

In conclusion, for $\ell = 3$, we have shown that $\dim(V_s(\mu)) \leq 4$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist elements $s \in T \setminus \mathrm{Z}(G)$ which afford an eigenvalue $\mu \in k^*$ on V for which the bound is achieved, for example $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1})$, with $\mu_1^2 \neq 1$, and $\mu = 1$. Lastly, we note that, since $4 > \dim(V) - \sqrt{\dim(V)}$, there exist $s \in T \setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$.

We now let $\ell \geq 4$ and assume $\dim(V_s(\mu_i\mu_j)) \geq \frac{\ell(\ell-1)}{2}$. Then

$$\ell - \frac{n_i(n_i-1)}{2} - \frac{n_j(n_j-1)}{2} - (n_i+n_j)(\ell+1-n_i-n_j) \geq 0.$$

We rewrite the above as:

$$(\ell - n_i - n_j)(1 - n_i - n_j) - \frac{n_i(n_i-1)}{2} - \frac{n_j(n_j-1)}{2} \geq 0. \tag{3.9}$$

Since $n_i \geq n_j \geq 1$, we have $\frac{n_i(n_i-1)}{2} + \frac{n_j(n_j-1)}{2} \geq 0$ and $1 - n_i - n_j < 0$, therefore, by Inequality (3.9), we have

$$\ell - n_i - n_j \leq 0.$$

Assume that $n_i + n_j = \ell$. Then, for Inequality (3.9) to hold, we need $\frac{n_i(n_i-1)}{2} + \frac{n_j(n_j-1)}{2} = 0$, hence $n_i = n_j = 1$, contradicting $\ell \geq 4$. We can thus assume that $n_i + n_j = \ell + 1$, hence $m = 2$. By (3.9) we have:

$$n_1 + n_2 - 1 - \frac{n_1(n_1-1) + n_2(n_2-1)}{2} \geq 0,$$

which we rewrite as:

$$n_1(3 - n_1) - (n_2 - 1)(n_2 - 2) \geq 0. \tag{3.10}$$

Now, Inequality (3.10) holds if and only if $n_1 \leq 3$ and $n_2 \leq 2$, hence if and only if $n_1 = 3$ and $n_2 = 2$, as $\ell \geq 4$. In this case, $\ell = 4$ and so $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_2, \mu_2)$ with $\mu_1 \neq \mu_2$ and $\mu_1^3 = \mu_2^{-2}$. Therefore, $\dim(V_s(\mu_i\mu_j)) \leq \frac{\ell(\ell-1)}{2}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all $1 \leq i < j \leq m$, where equality holds if and only if $\ell = 4$, $m = 2$, $\mu_i\mu_j = \mu_1\mu_2$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_2, \mu_2)$ with $\mu_1 \neq \mu_2$ and $\mu_1^3 = \mu_2^{-2}$.

In conclusion, for $\ell \geq 4$, we have shown that $\dim(V_s(\mu)) \leq \frac{\ell(\ell-1)}{2}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is achieved. In particular, as the inequality $0 < \ell^2 - \ell$ holds for all $\ell \geq 4$, it follows that $\frac{\ell(\ell-1)}{2} < \frac{\ell(\ell+1)}{2} - \sqrt{\frac{\ell(\ell+1)}{2}}$ for all $\ell \geq 4$, and so $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 3.2.6. *Let $\ell \geq 2$ and let $V' = W \otimes W^*$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V' we have*

$$\dim(V_s'(\mu)) \leq \ell^2 + 1,$$

where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} \neq 1$.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. Since $V' = W \otimes W^*$, we deduce that $\dim(V') = (\ell+1)^2$ and that the eigenvalues of s on V', not necessarily distinct, are:

$$\begin{cases} 1 \text{ with multiplicity at least } \sum_{i=1}^{m} n_i^2; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j, \text{ where } 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j. \end{cases} \tag{3.11}$$

We first consider the eigenvalue 1 of s on V'. Since the μ_i's are distinct, it follows that $1 \neq \mu_i\mu_j^{-1}$ and $1 \neq \mu_i^{-1}\mu_j$ for all $1 \leq i < j \leq m$. Therefore:

$$\dim(V_s'(1)) = \sum_{i=1}^{m} n_i^2 = (\sum_{i=1}^{m} n_i)^2 - 2\sum_{i<j} n_i n_j = (\ell+1)^2 - 2\sum_{i<j} n_i n_j.$$

Assume $\dim(V_s'(1)) \geq \ell^2 + 1$. Then:

$$\ell - \sum_{i<j} n_i n_j \geq 0$$

and, since $\ell = \sum_{i=1}^{m} n_i - 1$, we have that:

$$(1-n_2)(n_1-1) + \sum_{i=3}^{m} n_i(1 - \sum_{j=1}^{i-1} n_j) \geq 0. \tag{3.12}$$

But $\sum_{i=3}^{m} n_i(1 - \sum_{j=1}^{i-1} n_j) \leq 0$ and $(1-n_2)(n_1-1) \leq 0$, since $n_i \geq 1$ for all $1 \leq i \leq m$, and so (3.12) holds if and only if $m = 2$, $n_2 = 1$ and $n_1 = \ell$. Then $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_2)$, with $\mu_1 \neq \mu_2$ and $\mu_1^\ell \mu_2 = 1$, and $\dim(V_s'(1)) = \ell^2 + 1$. We deduce that $\dim(V_s'(1)) \leq \ell^2 + 1$ for all $s \in T \backslash \mathrm{Z}(G)$ and that equality holds if and only if, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} \neq 1$, as in the statement of the proposition.

We now fix some $1 \leq i < j \leq m$ and consider the eigenvalue $\mu_i\mu_j^{-1}$ of s on V'. If $\mu_i\mu_j^{-1} \neq \mu_i^{-1}\mu_j$, then:

$$\dim(V_s'(\mu_i\mu_j^{-1})) \leq (\ell+1)^2 - \sum_{r=1}^{m} n_r^2 - \dim(V_s'(\mu_i^{-1}\mu_j)). \tag{3.13}$$

Since $n_r \geq 1$ for all $1 \leq r \leq m$, we have that $\sum_{r=1}^{m} n_r^2 \geq \sum_{r=1}^{m} n_r = \ell + 1$. Furthermore, since V' is a self-dual kG-module, we have $\dim(V_s'(\mu_i\mu_j^{-1})) = \dim(V_s'(\mu_i^{-1}\mu_j))$ and so Inequality (3.13) becomes:

$$\dim(V_s'(\mu_i\mu_j^{-1})) \leq \frac{(\ell+1)^2 - (\ell+1)}{2} = \frac{\ell(\ell+1)}{2}. \tag{3.14}$$

Since $0 < \ell^2 - \ell + 2$ for all $\ell \geq 2$, we have $\dim(V_s'(\mu_i\mu_j^{-1})) < \ell^2 + 1$ for all eigenvalues $\mu_i\mu_j^{-1} \neq \pm 1$. We can thus assume that $\mu_i\mu_j^{-1} = \mu_i^{-1}\mu_j$ and so $p \neq 2$ and $\mu_i\mu_j^{-1} = -1$. Since the μ_r's are distinct, we remark that:

(1) $-1 \neq \mu_i\mu_r^{-1}$, for $i < r \leq m$, $r \neq j$, and $-1 \neq \mu_r^{-1}\mu_i$, for $1 \leq r < i$, hence $-1 \neq \mu_i^{-1}\mu_r$, for $i < r \leq m$, $r \neq j$, and $-1 \neq \mu_r\mu_i^{-1}$, for $1 \leq r < i$;

(2) $-1 \neq \mu_r\mu_j^{-1}$, for $1 \leq r < j$, $r \neq i$, and $-1 \neq \mu_j^{-1}\mu_r$, for $j < r \leq m$, hence $-1 \neq \mu_r^{-1}\mu_j$, for $1 \leq r < j$, $r \neq i$, and $-1 \neq \mu_j\mu_r^{-1}$, for $j < r \leq m$.

It follows that:

$$\dim(V_s^{'}(-1)) \leq (\ell+1)^2 - \sum_{r=1}^{m} n_r^2 - 2(n_i+n_j)(\ell+1-n_i-n_j). \tag{3.15}$$

Assume that $\dim(V_s^{'}(-1)) \geq \ell^2+1$. Then:

$$2\ell - \sum_{r=1}^{m} n_r^2 - 2(n_i+n_j)(\ell+1-n_i-n_j) \geq 0,$$

which we rewrite as

$$2(\ell-n_i-n_j)(1-n_i-n_j) - \sum_{r=1}^{m} n_r^2 \geq 0. \tag{3.16}$$

Since $\sum_{r=1}^{m} n_r^2 > 0$, for Inequality (3.16) to hold, we must have $(\ell-n_i-n_j)(1-n_i-n_j) > 0$. But then, as $n_i \geq n_j \geq 1$, it follows that $\ell - n_i - n_j < 0$ and so $m=2$ and $n_1+n_2 = \ell+1$. Substituting in Inequality (3.16) gives:

$$-2+2n_1+2n_2-n_1^2-n_2^2 \geq 0,$$

which we rewrite as

$$-(n_1-1)^2-(n_2-1)^2 \geq 0$$

and deduce that $n_1 = n_2 = 1$, contradicting $\ell \geq 2$. We conclude that $\dim(V_s'(-1)) < \ell^2+1$ for all $s \in T \setminus \mathrm{Z}(G)$. This completes the proof of the proposition. □

Corollary 3.2.7. *Let $\ell \geq 2$, $p \nmid \ell+1$ and let $V = L_G(\omega_1+\omega_\ell)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq \ell^2.$$

Moreover, we have equality if and only if one of the following holds:

(1) *$p \neq 2$, $\ell = 2$, $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, -\mu_1)$ with $\mu_1^3 = -1$.*

(2) *$\ell \geq 2$, $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} \neq 1$.*

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. Let $V^{'} = W \otimes W^*$. Then, by Lemma 2.8.1, since $p \nmid \ell+1$, we have $V^{'} = V \oplus L_G(0)$. It follows that $\dim(V) = \ell^2+2\ell$, $\dim(V_s(1)) = \dim(V_s^{'}(1)) - 1$ and $\dim(V_s(\mu)) = \dim(V_s^{'}(\mu))$ for all eigenvalues $\mu \neq 1$ of s on V.

For the eigenvalue 1 and any eigenvalue μ, $\mu \neq \mu^{-1}$, of s on V, Proposition 3.2.6 and Inequality (3.14) give the result. Now assume $p \neq 2$ and let $\mu = -1$. By (3.15), we have:

$$\dim(V_s(-1)) \leq (\ell+1)^2 - \sum_{r=1}^{m} n_r^2 - 2(n_i+n_j)(\ell+1-n_i-n_j).$$

Assume $\dim(V_s(-1)) \geq \ell^2$. It follows that:

$$2\ell + 1 - \sum_{r=1}^{m} n_r^2 - 2(n_i + n_j)(\ell + 1 - n_i - n_j) \geq 0.$$

We proceed as for $V_s^{'}(-1)$, see (3.16), and arrive at

$$2(\ell - n_i - n_j)(1 - n_i - n_j) + 1 - \sum_{r=1}^{m} n_r^2 \geq 0. \tag{3.17}$$

We have that $1 - \sum_{r=1}^{m} n_r^2 < 0$, as $m \geq 2$ and $n_r \geq 1$ for all $1 \leq r \leq m$. Thus, for Inequality (3.17) to hold, we must have $(\ell - n_i - n_j)(1 - n_i - n_j) > 0$ and so $m = 2$ and $n_1 + n_2 = \ell + 1$. Substituting in (3.17) gives:

$$-1 + 2n_1 + 2n_2 - n_1^2 - n_2^2 \geq 0,$$

which we rewrite as:

$$-(n_2 - 1)^2 + n_1(2 - n_1) \geq 0. \tag{3.18}$$

If $n_2 \geq 2$, then $n_1 \geq 2$, as $n_1 \geq n_2$, and we have $-(n_2 - 1)^2 + n_1(2 - n_1) < 0$. It follows that $n_2 = 1$, $n_1 = \ell$, where $\ell \geq 2$, and, by (3.18), we deduce that $n_1 = 2$ and $\ell = 2$. Then, we also have that $\mu_1^2\mu_2 = 1$ and $\mu_1\mu_2^{-1} = -1$, hence, $\mu_2 = \mu_1^{-2}$ and $\mu_1^3 = -1$. Therefore, we have shown that $\dim(V_s(-1)) \leq \ell^2$ for all $s \in T \setminus \mathrm{Z}(G)$ and that equality holds if and only if $\ell = 2$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-2})$ with $\mu_1^3 = -1$, as in (1).

We conclude that $\dim(V_s(\mu)) \leq \ell^2$, for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. In particular, since the inequality $0 < 3\ell^2 - 2\ell$ holds for all $\ell \geq 2$, we have $\ell^2 < \ell^2 + 2\ell - \sqrt{\ell^2 + 2\ell}$ for all $\ell \geq 2$, and therefore $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Corollary 3.2.8. *Let $\ell \geq 2$, $p \mid \ell + 1$ and let $V = L_G(\omega_1 + \omega_\ell)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V one of the following holds*

(1) *$\ell = 2$ and $\dim(V_s(\mu)) \leq 4$ with equality if and only if $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(-1, -1, 1)$.*

(2) *$\ell \geq 3$ and $\dim(V_s(\mu)) \leq \ell^2 - 1$ with equality if and only if $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} \neq 1$.*

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. Let $V^{'} = W \otimes W^*$. Then, by Lemma 2.8.1, since $p \mid \ell + 1$, we have $V^{'} = L_G(0) \mid V \mid L_G(0)$. Therefore, $\dim(V) = \ell^2 + 2\ell - 1$, $\dim(V_s(\mu)) = \dim(V_s^{'}(\mu))$, for all eigenvalues $\mu \neq 1$ of s on V, and $\dim(V_s(1)) = \dim(V_s^{'}(1)) - 2$.

For the eigenvalue 1 and any eigenvalue μ with $\mu \neq \mu^{-1}$ of s on V, Proposition 3.2.6 and Inequality (3.14) give the result. So, we now assume that $p \neq 2$ and consider the eigenvalue $\mu = -1$ of s on V.

If $\ell = 2$, the proof of Corollary 3.2.8 gives the result, see (3.17) and (3.18). We can thus assume $\ell \geq 4$, as $p \mid \ell + 1$ and $p \neq 2$, and, by (3.15), we have:

$$\dim(V_s(-1)) \leq (\ell+1)^2 - \sum_{r=1}^{m} n_r^2 - 2(n_i + n_j)(\ell + 1 - n_i - n_j).$$

Assume $\dim(V_s(-1)) \geq \ell^2 - 1$. Then

$$2\ell + 2 - \sum_{r=1}^{m} n_r^2 - 2(n_i + n_j)(\ell + 1 - n_i - n_j) \geq 0.$$

We proceed as for $V_s^{'}(-1)$, see (3.16), and arrive at

$$2(\ell - n_i - n_j)(1 - n_i - n_j) + 2 - \sum_{r=1}^{m} n_r^2 \geq 0. \tag{3.19}$$

Now, $2 - \sum_{r=1}^{m} n_r^2 \leq 0$, as $m \geq 2$ and $n_r \geq 1$ for all $1 \leq r \leq m$. Thus, for Inequality (3.19) to hold we must have:

$$(\ell - n_i - n_j)(1 - n_i - n_j) \geq 0. \tag{3.20}$$

But $1 - n_i - n_j < 0$ and so, by (3.20), we have $\ell - n_i - n_j \leq 0$. If $n_i + n_j = \ell$, then $m = 3$, $n_3 = 1$ and $2(\ell - n_i - n_j)(1 - n_i - n_j) + 2 - \sum_{r=1}^{m} n_r^2 < 0$. Similarly, if $n_i + n_j = \ell + 1$, then $m = 2$ and, by Inequality (3.19), it follows that:

$$n_1(2 - n_1) + n_2(2 - n_2) \geq 0. \tag{3.21}$$

Since $\ell \geq 4$ and $n_1 \geq n_2$, we have $n_1 \geq 3$, therefore $n_1(2 - n_1) < 0$. Hence, by (3.21), $n_2(2 - n_2) > 0$ and so $n_2 = 1$. Substituting in (3.21) gives:

$$-(n_1 - 1)^2 + 2 \geq 0,$$

contradicting $n_1 \geq 3$. We deduce that $\dim(V_s(-1)) < \ell^2 - 1$ for all $s \in T \setminus \mathrm{Z}(G)$.

In conclusion, for $\ell = 2$, we have shown that $\dim(V_s(\mu)) \leq 4 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Similarly, for $\ell \geq 3$, we have shown that $\dim(V_s(\mu)) \leq \ell^2 - 1$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Since the inequality $0 < 3\ell^2 - 2\ell + 1$ holds for all $\ell \geq 3$, it follows that $\ell^2 - 1 < \ell^2 + 2\ell - 1 - \sqrt{\ell^2 + 2\ell - 1}$ for all $\ell \geq 3$, and therefore $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. This completes the proof of the corollary. □

To conclude this subsection, we remark that Lemma 3.2.3, Propositions 3.2.4 and 3.2.5, and Corollaries 3.2.7 and 3.2.8 give the proof of Theorems 3.2.1 and 3.2.2 for the families of kG-modules corresponding to p-restricted dominant weights $\lambda \in F^{A_\ell}$.

3.2.2 The particular modules

First, for $\ell = 1$, we will inspect the irreducible highest weight kG-module $V = L_G(m\omega_1)$, where $3 \leq m \leq 8$, and determine whether there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. Afterwards, we will assume that $\ell \geq 2$ and we will focus on the irreducible kG-modules $L_G(\lambda)$ corresponding to p-restricted dominant weights λ appearing in Table 2.7.1 and answer the same question. Lastly, although we do not mention the result explicitly, we make great use of the data in [Lü01b], when discussing weights and weight multiplicities in this subsection.

Proposition 3.2.9. *Let k be an algebraically closed field of characteristic $p = 0$ or $p > m$. Assume $\ell = 1$ and let $V = L_G(m\omega_1)$, where $3 \leq m \leq 8$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 1 + \left\lfloor \frac{m}{2} \right\rfloor,$$

where there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

Moreover, for $m = 3$ and $m = 4$ there exist $s \in T \setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $m \geq 5$ we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. First, we note that, as $V = L_G(m\omega_1)$, we have $\dim(V) = m+1$. Now, the eigenvalues of $s = \mathrm{diag}(\mu_1, \mu_1^{-1})$ with $\mu_1 \neq \pm 1$, on V, not necessarily distinct, are $\mu_1^m, \mu_1^{m-2}, \ldots, \mu_1^{-m+2}$, μ_1^{-m}. Since $\mu_1^2 \neq 1$, we have that $\mu_1^i \neq \mu_1^{i-2}$ for all $-m+2 \leq i \leq m$.

We first assume that m is even. We remark that, in this case, 1 occurs as an eigenvalue of s on V, with multiplicity at least 1. Now, let μ be an eigenvalue of s on V. If $\mu \neq \mu^{-1}$, then $\dim(V_s(\mu)) = \dim(V_s(\mu^{-1}))$ and we have

$$\dim(V_s(\mu)) \leq \frac{\dim(V) - \dim(V_s(1))}{2} \leq \frac{m}{2}.$$

We can now assume that the eigenvalue μ is such that $\mu = \mu^{-1}$. First, let $\mu = 1$. If $\mu_1^i = 1$ for some $2 \leq i \leq m$, then we also have $\mu_1^{-i} = 1$ and therefore we will focus on the $\frac{m}{2}$ eigenvalues $\mu_1^m, \mu_1^{m-2}, \ldots, \mu_1^2$. As $\mu_1^2 \neq 1$, at most $\frac{m}{2} - 1$ of the eigenvalues $\mu_1^m, \mu_1^{m-2}, \ldots, \mu_1^4$ can equal 1. As $\mu_1^i \neq \mu_1^{i-2}$, for all $4 \leq i \leq m$, it follows that at most $\left\lfloor \frac{m-2}{4} \right\rfloor + \varepsilon$ of these eigenvalues can equal 1, where $\varepsilon = 1$ if $4 \nmid m-2$ and $\varepsilon = 0$ if $4 \mid m-2$. Therefore:

$$\dim(V_s(1)) \leq \begin{cases} 1 + 2(\left\lfloor \frac{m-2}{4} \right\rfloor + 1), \text{ if } 4 \nmid m-2 \\ 1 + \frac{m-2}{2}, \text{ if } 4 \mid m-2 \end{cases} = \begin{cases} 1 + \frac{m}{2}, \text{ if } 4 \nmid m-2 \\ \frac{m}{2}, \text{ if } 4 \mid m-2 \end{cases} \leq 1 + \left\lfloor \frac{m}{2} \right\rfloor.$$

Now, let $\mu = -1$ and, again, we focus on the eigenvalues of the form μ_1^i, where $2 \leq i \leq m$. As $\mu_1^i \neq \mu_1^{i-2}$ for all $2 \leq i \leq m$, it follows that at most $\left\lfloor \frac{m}{4} \right\rfloor + \xi$ of these eigenvalues can equal -1, where $\xi = 1$ if $4 \nmid m$ and $\xi = 0$ if $4 \mid m$. Therefore:

$$\dim(V_s(-1)) \leq \begin{cases} 2(\left\lfloor \frac{m}{4} \right\rfloor + 1), \text{ if } 4 \nmid m \\ \frac{m}{2}, \text{ if } 4 \mid m \end{cases} = \begin{cases} 1 + \frac{m}{2}, \text{ if } 4 \nmid m \\ \frac{m}{2}, \text{ if } 4 \mid m \end{cases} \leq 1 + \left\lfloor \frac{m}{2} \right\rfloor.$$

We now assume that m is odd. Let $\mu \in k^*$ be an eigenvalue of s on V. If $\mu \neq \mu^{-1}$, then, arguing as in the case of m even, we deduce:

$$\dim(V_s(\mu)) \leq \frac{\dim(V)}{2} = 1 + \frac{m-1}{2}.$$

We can now assume that the eigenvalue μ is such that $\mu = \mu^{-1}$, i.e. $\mu = \pm 1$. If $\mu_1^i = \pm 1$ for some $1 \leq i \leq m$, then we also have $\mu_1^{-i} = \pm 1$ and therefore we will focus on the $\frac{m+1}{2}$ eigenvalues $\mu_1^m, \mu_1^{m-2}, \ldots, \mu_1$. Since $\mu_1 \neq \pm 1$, at most $\frac{m-1}{2}$ of these eigenvalues can equal ± 1 and, moreover, since $\mu_1^i \neq \mu_1^{i-2}$, for all $3 \leq i \leq m$, we deduce that at most $\left\lfloor \frac{m-1}{4} \right\rfloor + \zeta$ of these eigenvalues can equal ± 1, where $\zeta = 1$ if $4 \nmid m-1$ and $\zeta = 0$ if $4 \mid m-1$. Therefore:

$$\dim(V_s(\pm 1)) \leq \begin{cases} 2(\left\lfloor \frac{m-1}{4} \right\rfloor + 1), \text{ if } 4 \nmid m-1 \\ \frac{m-1}{2}, \text{ if } 4 \mid m-1 \end{cases} = \begin{cases} 1 + \frac{m-1}{2}, \text{ if } 4 \nmid m-1 \\ \frac{m-1}{2}, \text{ if } 4 \mid m-1 \end{cases} \leq 1 + \left\lfloor \frac{m}{2} \right\rfloor.$$

We have shown that $\dim(V_s(\mu)) \leq 1 + \left\lfloor \frac{m}{2} \right\rfloor$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. We will now show that there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. For this, let $s = \mathrm{diag}(\mu_1, \mu_1^{-1}) \in T$ be such that $\mu_1^2 = -1$. First, we note that $s \notin \mathrm{Z}(G)$. Now, the eigenvalues of s on V are $\mu_1^m, \mu_1^{m-2}, \ldots, \mu_1^{-m+2}, \mu_1^{-m}$. As $\mu_1^2 = -1$, it follows that $\mu_1^m = \mu_1^{m-4i}$, for all $0 \leq i \leq \frac{m-1}{2}$ if m is odd, respectively for all $0 \leq i \leq \frac{m}{2}$ if m even. It follows that $\dim(V_s(\mu_1^m)) = 1 + \left\lfloor \frac{m}{2} \right\rfloor$.

Lastly, in the cases of $m = 3$ and $m = 4$, one sees that there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$, for example $s = \mathrm{diag}(\mu_1, \mu_1^{-1})$ with $\mu_1^2 = -1$ and $\mu = \mu_1^m$. On the other hand, for $5 \leq m \leq 8$, we have $0 < m^2 - 4m - 4$, therefore $\sqrt{m+1} < m - \frac{m}{2}$. Moreover, as $\left\lfloor \frac{m}{2} \right\rfloor \leq \frac{m}{2}$ for all $m \geq 1$, we have $1 + \left\lfloor \frac{m}{2} \right\rfloor < m + 1 - \sqrt{m+1}$, for all $5 \leq m \leq 8$. It follows that $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

We now turn our attention to the irreducible kG-modules V with highest weights listed in Table 2.7.1. To treat these modules, we will use the inductive algorithm for calculating $\max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ presented in Subsection 2.4.3. To begin, we refer the reader to the construction of the Levi subgroup L_ℓ of the maximal parabolic subgroup P_ℓ of G, as given in Section 2.4. We recall that $L_\ell = \mathrm{Z}(L_\ell)^\circ [L_\ell, L_\ell]$, where $\mathrm{Z}(L_\ell)^\circ$ is a one-dimensional torus and $[L_\ell, L_\ell]$ is a simple simply connected group of type $A_{\ell-1}$. We also recall that we have denoted by T' the maximal torus $T \cap [L_\ell, L_\ell]$ of $[L_\ell, L_\ell]$.

Let $s \in T$. Then $s = z \cdot h$, where $z \in \mathrm{Z}(L_\ell)^\circ$ and $h \in [L_\ell, L_\ell]$. As $z \in \mathrm{Z}(L_\ell)^\circ$, we have $z = \prod\limits_{j=1}^{\ell} h_{\alpha_j}(c^{k_j})$, where $c \in k^*$ and $k_j \in \mathbb{Z}$ for all $1 \leq j \leq \ell$. Moreover, as $\alpha_j(z) = 1$ for all $1 \leq j \leq \ell - 1$, it follows that $z = \prod\limits_{j=1}^{\ell} h_{\alpha_j}(c^j)$, where $c \in k^*$. As $h \in [L_\ell, L_\ell]$, we have $h = \prod\limits_{j=1}^{\ell-1} h_{\alpha_j}(a_j)$, where $a_j \in k^*$ for all $1 \leq j \leq \ell - 1$, and therefore $s = \left(\prod\limits_{j=1}^{\ell-1} h_{\alpha_j}(c^j a_j) \right) h_{\alpha_\ell}(c^\ell)$ with $c \in k^*$ and $a_j \in k^*$ for all $1 \leq j \leq \ell - 1$.

Let V be an irreducible kG-module of p-restricted dominant highest weight $\lambda \in \mathrm{X}(T)$, where $\lambda = \sum_{r=1}^{\ell} d_r\omega_r$ with $0 \leq d_r \leq p-1$ for all $1 \leq r \leq \ell$. We consider the decomposition:

$$V\mid_{[L_\ell,L_\ell]} = \bigoplus_{i=0}^{e_\ell(\lambda)} V^i,$$

where $e_\ell(\lambda)$ is the maximum α_ℓ-level of weights in V, see Definition 2.4.1, and $V^i = \bigoplus_{\gamma\in\mathbb{N}\Delta_\ell} V_{\lambda-i\alpha_\ell-\gamma}$ for all $0 \leq i \leq e_\ell(\lambda)$. Let $s \in T$ and write $s = z \cdot h$, as above. Then, by (2.5), we have:

$$s_z^i = (\lambda - i\alpha_\ell - \gamma)(z) = (\lambda - i\alpha_\ell)(\prod_{j=1}^{\ell} h_{\alpha_j}(c^j)) = \prod_{j=1}^{\ell} c^{jd_j} \cdot c^{-(-i(\ell-1)+2i\ell)} = \prod_{j=1}^{\ell} c^{jd_j} \cdot c^{-(\ell+1)i}.$$

Therefore, z acts on V^i, $0 \leq i \leq e_\ell(\lambda)$, as the scalar $s_z^i = \prod_{j=1}^{\ell} c^{jd_j} \cdot c^{-(\ell+1)i}$. Now, let $\mu_1^i, \ldots, \mu_{t_i}^i$, $t_i \geq 1$, be the distinct eigenvalues of h on V^i, $0 \leq i \leq e_\ell(\lambda)$, and let $n_1^i, \ldots, n_{t_i}^i$ be their respective multiplicities. Then, by Lemma 2.4.8, we determine that the distinct eigenvalues of s on V^i are $s_z^i\mu_1^i, \ldots, s_z^i\mu_{t_i}^i$ with respective multiplicities $n_1^i, \ldots, n_{t_i}^i$.

Proposition 3.2.10. *Let k be an algebraically closed field of characteristic $p = 3$. Assume $\ell = 3$ and let $V = L_G(\omega_1 + \omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 10,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $L = L_3$ and let $\lambda = \omega_1 + \omega_2$. Then $\dim(V) = 16$ and, by Lemma 2.4.4, we have $e_3(\lambda) = 2$, therefore:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma\in\mathbb{N}\Delta_3} V_{\lambda-i\alpha_3-\gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_1+\omega_2)$. In V^1, the weight $(\lambda-\alpha_2-\alpha_3)\mid_{T'} = 2\omega_1$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_L(2\omega_1)$ and $\dim(V^1) \geq \dim(L_L(2\omega_1)) = 6$, as $p = 3$. Similarly, the weight $(\lambda - \alpha_1 - 2\alpha_2 - 2\alpha_3)\mid_{T'} = \omega_1$ admits a maximal vector in V^2, therefore V^2 has a composition factor isomorphic to $L_L(\omega_1)$ and $\dim(V^2) \geq \dim(L_L(\omega_1)) = 3$. By dimensional considerations, we deduce that $V^1 \cong L_L(2\omega_1)$, $V^2 \cong L_L(\omega_1)$ and:

$$V\mid_{[L,L]} \cong L_L(\omega_1 + \omega_2) \oplus L_L(2\omega_1) \oplus L_L(\omega_1). \tag{3.22}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$ with $c^4 \neq 1$. In this case, as s acts on each V^i, $i = 0, 1, 2$, as scalar multiplication by $s_z^i = \prod_{j=1}^{3} c^{jd_j} \cdot c^{-4i} = c^{3-4i}$, we determine that the eigenvalues of s on V, not necessarily distinct are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 7; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^1) = 6; \\ c^{-5} \text{ with } \dim(V_s(c^{-5})) \geq \dim(V^2) = 3. \end{cases}$$

As $c^4 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 10$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s \in \mathrm{Z}(L)^\circ$ with $c^4 = -1$, we have $s \notin \mathrm{Z}(G)$ and $\dim(V_s(c^3)) = 10$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $i = 0, 1, 2$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $i = 0, 1, 2$, where μ_h is any eigenvalue of h on V^i. Now, as $p = 3$, by Corollary 3.2.8, we have $\dim(V_h^0(\mu_h)) \leq 4$ for all eigenvalues μ_h of h on V^0. Similarly, by Proposition 3.2.4, respectively by Lemma 3.2.3, it follows that $\dim(V_h^1(\mu_h)) \leq 4$, respectively $\dim(V_h^2(\mu_h)) \leq 2$, for all eigenvalues μ_h of h on V^1, respectively on V^2. This gives $\dim(V_h(\mu_h)) \leq 10$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 10$ for all eigenvalues μ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 10$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 3.2.11. *Let $\ell = 5$ and let $V = L_G(\omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 12.$$

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. We first note that the kG-module V is self-dual, as $V^* \cong L_G(-w_0(\omega_3))$, see [MT11, Proposition 16.1], and $w_0(\omega_3) = -\omega_3$. Now, let $L = L_5$ and let $\lambda = \omega_3$. Then $\dim(V) = 20$ and, by Lemma 2.4.4, we have $e_5(\lambda) = 1$, therefore

$$V\mid_{[L,L]} = V^0 \oplus V^1,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_5} V_{\lambda - i\alpha_5 - \gamma}$ for $i = 0$ and $i = 1$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_3)$ and thus, by Lemma 2.4.3, we also have $V^1 \cong (L_L(\omega_3))^* \cong L_L(\omega_2)$. It follows that

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_2). \tag{3.23}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 1$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ and so $s = z$ with $c^6 \neq 1$. In this case, as s acts on V^i as scalar

multiplication by $s_z^i = \prod_{j=1}^{5} c^{jd_j} \cdot c^{-6i} = c^{3-6i}$ and, as $c^6 \neq 1$, we determine that the distinct eigenvalues of s on V are

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) = \dim(V^0) = 10; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) = \dim(V^1) = 10. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 1$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $i = 0, 1$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for both $i = 0$ and $i = 1$, where μ_h is any eigenvalue of h on V^i. Now, by Proposition 3.2.5, we have $\dim(V_h^1(\mu_h)) \leq 6$, hence $\dim(V_h^0(\mu_h)) \leq 6$, as $V^1 \cong (V^0)^*$, for all eigenvalues μ_h of h on V^1 and V^0, respectively. It follows that $\dim(V_h(\mu_h)) \leq 12$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 12$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 12 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

We will require the following corollary in the proof of Proposition 3.2.13.

Corollary 3.2.12. *Let $\ell = 5$ and let $V = L_G(\omega_3)$. If*

$$\dim(V_s(\mu)) = 12$$

for some $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$, then one of the following holds:

(1) *$\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(d, d, d, d, d^{-2}, d^{-2})$ with $d^3 \neq 1$.*

(2) *$\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(d, d, d, d, -d^{-2}, -d^{-2})$ with $d^3 \neq -1$.*

Moreover, if $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ are such that $\dim(V_s(\mu)) \neq 12$, then $\dim(V_s(\mu)) \leq 10$.

Proof. Let $L = L_5$ and recall from Proposition 3.2.11 that V is a self-dual kG-module with $V|_{[L,L]} = V^0 \oplus V^1$, where $V^0 \cong L_L(\omega_3)$ and $V^1 \cong L_L(\omega_2)$. Moreover, we have $\dim(V_s(\mu)) \leq 12$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Let $s \in T \setminus \mathrm{Z}(G)$ be such that $\dim(V_s(\mu)) = 12$ for some eigenvalue $\mu \in k^*$ on V. Then, as $\dim(V) = 20$ and V is self-dual, it follows that $\mu = \pm 1$. Now, as $V^1 \cong (V^0)^*$, by the proof of Proposition 3.2.11, we have $\dim(V_s^0(\pm 1)) = 6$ and $\dim(V_s^1(\pm 1)) = 6$. Secondly, by Proposition 3.2.5, as $\dim(V_s^1(\mu)) = 6$, one of the following holds:

(a) $\mu = c^{-3}\mu_1\mu_2$ and, up to conjugation, $s = \mathrm{diag}(c\mu_1, c\mu_1, c\mu_1, c\mu_2, c\mu_2, c^{-5})$ with $\mu_1 \neq \mu_2$ and $\mu_1^3 = \mu_2^{-2}$.

(b) $\mu = c^{-3}\mu_1^2$ and, up to conjugation, $s = \mathrm{diag}(c\mu_1, c\mu_1, c\mu_1, c\mu_1, c\mu_1^{-4}, c^{-5})$ with $\mu_1^5 \neq 1$.

If s and μ are as in (a), then $c^{-3}\mu_1\mu_2 = \pm 1$ and so $\mu_2 = \pm c^3\mu_1^{-1}$. Moreover, since $\mu_2^2 = \mu_1^{-3}$ and $\mu_1 \neq \mu_2$, we deduce that $\mu_1 = c^{-6}$ and $\mu_2 = \pm c^9$. Now, if $\mu_2 = c^9$, then $c^{15} \neq 1$, as $\mu_1 \neq \mu_2$. Similarly, if $\mu_2 = -c^9$, then $c^{15} \neq -1$. Let $d = c^{-5}$. Then, up to conjugation, we have

that $s = \mathrm{diag}(d, d, d, d, d^{-2}, d^{-2})$ with $d^3 \neq 1$, respectively $s = \mathrm{diag}(d, d, d, d, -d^{-2}, -d^{-2})$ with $d^3 \neq -1$.

If s and μ are as in (b), we have $c^{-3}\mu_1^2 = \pm 1$. Set $e = c^{-1}\mu_1$. Now, if $c^{-3}\mu_1^2 = 1$, then $c = e^2$ and therefore $\mu_1 = e^3$, as $\mu_1 = ec$. Since $\mu_1^5 \neq 1$, it follows that $e^{15} \neq 1$. Set $d = e^5$. Then, up to conjugation, we have $s = \mathrm{diag}(d, d, d, d, d^{-2}, d^{-2})$ with $d^3 \neq 1$. Analogously, one shows that if $c^{-3}\mu_1^2 = -1$, then, up to conjugation, we have $s = \mathrm{diag}(d, d, d, d, -d^{-2}, -d^{-2})$ with $d^3 \neq -1$.

Therefore, we have shown that if $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ are such that $\dim(V_s(\mu)) = 12$, then either $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(d, d, d, d, d^{-2}, d^{-2})$ with $d^3 \neq 1$, or $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(d, d, d, d, -d^{-2}, -d^{-2})$ with $d^3 \neq -1$.

To prove the last statement of the result, we assume by contradiction that there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ such that $\dim(V_s(\mu)) = 11$. First, we argue as in the previous case to determine that $\mu = \pm 1$. Secondly, for $\dim(V_s(\pm 1)) = 11$ to hold, by the proof of Proposition 3.2.11, we either have $\dim(V_s^0(\pm 1)) = 6$ and $\dim(V_s^1(\pm 1)) = 5$, or $\dim(V_s^0(\pm 1)) = 5$ and $\dim(V_s^1(\pm 1)) = 6$. However, since $V^1 \cong (V^0)^*$, neither of the two cases holds. We conclude that for $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ with $\dim(V_s(\mu)) \neq 12$, we have $\dim(V_s(\mu)) \leq 10$. □

Proposition 3.2.13. *Let $\ell = 6$ and let $V = L_G(\omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 20,$$

where there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which equality holds.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $L = L_6$ and let $\lambda = \omega_3$. Then $\dim(V) = 35$ and, by Lemma 2.4.4, we have $e_6(\lambda) = 1$, therefore:

$$V\mid_{[L,L]} = V^0 \oplus V^1,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_6} V_{\lambda - i\alpha_6 - \gamma}$ for $i = 0$ and $i = 1$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_3)$. Since the weight $(\lambda - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6)\mid_{T'} = \omega_2$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_2)$. By dimensional considerations, we deduce $V^1 \cong L_L(\omega_2)$ and

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_2). \tag{3.24}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 1$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$ with $c^7 \neq 1$. In this case, as s acts on V^i as scalar multiplication by $s_z^i = \prod_{j=1}^{6} c^{jd_j} \cdot c^{-7i} = c^{3-7i}$ and, as $c^7 \neq 1$, we determine that the distinct eigenvalues of s on V are

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) = \dim(V^0) = 20; \\ c^{-4} \text{ with } \dim(V_s(c^{-4})) = \dim(V^1) = 15. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 1$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $i = 0, 1$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for both $i = 0$ and $i = 1$, where μ_h is any eigenvalue of h on V^i. Now, by Proposition 3.2.5, respectively by Proposition 3.2.11, we have $\dim(V_h^1(\mu_h)) \leq 10$, respectively $\dim(V_h^0(\mu_h)) \leq 12$, for all eigenvalues μ_h of h on V^1, respectively on V^0. It follows that $\dim(V_h(\mu_h)) \leq 22$ for all eigenvalues μ_h of h on V, hence $\dim(V_s(\mu)) \leq 22$ for all eigenvalues $\mu \in k^*$ of s on V.

Assume there exists $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which $\dim(V_s(\mu)) = 22$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Now, as $\dim(V_s(\mu)) = 22$, we have $\dim(V_s^0(\mu)) = 12$ and $\dim(V_s^1(\mu)) = 10$. Moreover, as $\dim(V_s^0(\mu)) = 12$, it follows that h admits an eigenvalue μ_h on V^0 with the property that $\mu = c^3\mu_h$, as z acts on V^0 as multiplication by c^3, and $\dim(V_h^0(\mu_h)) = 12$. Thus, by Corollary 3.2.12, we either have $\mu_h = 1$ and, up to conjugation, $h = \mathrm{diag}(d, d, d, d,\ d^{-2}, d^{-2}, 1)$ with $d^3 \neq 1$, or $\mu_h = -1$ and, up to conjugation, $h = \mathrm{diag}(d, d, d, d,\ -d^{-2}, -d^{-2}, 1)$ with $d^3 \neq -1$. We first consider the case when $\mu_h = 1$ and $h = \mathrm{diag}(d, d, d, d,\ d^{-2}, d^{-2}, 1)$ with $d^3 \neq 1$. We want to determine the eigenvalues of h on V^1. To achieve this, we use (3.7) and we see that the eigenvalues of h on V^1, not necessarily distinct, are d^2, with multiplicity at least 6, d^{-4} with multiplicity at least 1, and d^{-1} with multiplicity at least 8. As $d^3 \neq 1$, it follows that $\dim(V_h^1(\nu_h)) \leq 8$ for all eigenvalues ν_h of h on V^1, thus $\dim(V_s^{'}(\nu)) \leq 8$ for all eigenvalues ν of s on V^1, contradicting our assumption. Analogously, one shows that when $\mu_h = -1$ and $h = \mathrm{diag}(d, d, d, d,\ -d^{-2}, -d^{-2}, 1)$ with $d^3 \neq -1$, we also get $\dim(V_s(\mu)) \leq 20$. Therefore, $\dim(V_s(\mu)) \leq 21$ for all eigenvalues $\mu \in k^*$ of s on V.

Assume there exists $s \in T \setminus \mathrm{Z}(G)$ that admits an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) = 21$. Then, either $\dim(V_s^0(\mu)) = 12$ and $\dim(V_s^1(\mu)) = 9$, or $\dim(V_s^0(\mu)) = 11$ and $\dim(V_s^1(\mu)) = 10$. If $\dim(V_s^0(\mu)) = 12$, we have seen earlier that $\dim(V_s(\mu)) \leq 20$, while, by Corollary 3.2.12, we know that the second case does not occur. It follows that $\dim(V_s(\mu)) \leq 20$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 20$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 3.2.14. *Let $\ell = 7$ and let $V = L_G(\omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 35,$$

where there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which equality holds.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $L = L_7$ and let $\lambda = \omega_3$. Then $\dim(V) = 56$ and, by Lemma 2.4.4, we have $e_7(\lambda) = 1$, therefore:

$$V\mid_{[L,L]} = V^0 \oplus V^1,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_7} V_{\lambda - i\alpha_7 - \gamma}$ for $i = 0$ and $i = 1$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_3)$. Since the weight $(\lambda - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7)\mid_{T'} = \omega_2$ admits a maximal vector in

V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_2)$ and therefore $\dim(V^1) \geq \dim(L_L(\omega_2)) = 21$. By dimensional considerations, we deduce $V^1 \cong L_L(\omega_2)$ and

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_2). \tag{3.25}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 1$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$ with $c^8 \neq 1$. In this case, as s acts on V^i as scalar multiplication by $s_z^i = \prod_{j=1}^{7} c^{jd_j} \cdot c^{-8i} = c^{3-8i}$ and, as $c^8 \neq 1$, we determine that the distinct eigenvalues of s on V are

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) = \dim(V^0) = 35; \\ c^{-5} \text{ with } \dim(V_s(c^{-5})) = \dim(V^1) = 21. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 1$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $i = 0, 1$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for both $i = 0$ and $i = 1$, where μ_h is any eigenvalue of h on V^i. Now, by Proposition 3.2.13, respectively by Proposition 3.2.5, we have $\dim(V_h^0(\mu_h)) \leq 20$, respectively $\dim(V_h^1(\mu_h)) \leq 15$, for all eigenvalues μ_h of h on V^0, respectively on V^1. It follows that $\dim(V_h(\mu_h)) \leq 35$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 35$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 35$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Although the following result is not required for the proof of Theorems 3.2.1 and 3.2.2, it is a nice a generalization for all $\ell \geq 6$ of Propositions 3.2.13 and 3.2.14.

Proposition 3.2.15. *Let $\ell \geq 6$ and let $V = L_G(\omega_3)$. Then, for all $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ we have*

$$\dim(V_s(\mu)) \leq \frac{(\ell-2)(\ell-1)\ell}{6},$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which equality is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. We will use induction to prove this result. The base step for $\ell = 6$ is given by Proposition 3.2.13. We thus assume that $\ell \geq 7$ and that the statement holds for all $r \leq \ell - 1$, and we proceed to prove it for ℓ.

Set $\lambda = \omega_3$ and note that $\dim(V) = \frac{(\ell-1)\ell(\ell+1)}{6}$. By Lemma 2.4.4, we have $e_\ell(\lambda) = 1$, therefore:

$$V\mid_{[L_\ell, L_\ell]} = V^0 \oplus V^1,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_\ell} V_{\lambda - i\alpha_\ell - \gamma}$ for $i = 0$ and $i = 1$. By [Smi82, Proposition], we have $V^0 \cong L_{L_\ell}(\omega_3)$. Since the weight $(\lambda - \alpha_3 - \cdots - \alpha_\ell)\mid_{T'} = \omega_2$ admits a maximal vector in V^1, it

follows that V^1 has a composition factor isomorphic to $L_{L_\ell}(\omega_2)$ and therefore $\dim(V^1) \geq \dim(L_{L_\ell}(\omega_2)) = \frac{(\ell-1)\ell}{2}$. By dimensional considerations, we deduce $V^1 \cong L_{L_\ell}(\omega_2)$ and

$$V\mid_{[L_\ell,L_\ell]} \cong L_{L_\ell}(\omega_3) \oplus L_{L_\ell}(\omega_2). \tag{3.26}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 1$, then $s \in \mathrm{Z}(L_\ell)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$ with $c^{\ell+1} \neq 1$. In this case, as s acts on V^i as scalar multiplication by $s_z^i = \prod_{j=1}^{\ell} c^{jd_j} \cdot c^{-(\ell+1)i} = c^{3-(\ell+1)i}$ and, as $c^{\ell+1} \neq 1$, we determine that the distinct eigenvalues of s on V are

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) = \dim(V^0) = \frac{(\ell-2)(\ell-1)\ell}{6}; \\ c^{2-\ell} \text{ with } \dim(V_s(c^{2-\ell})) = \dim(V^1) = \frac{(\ell-1)\ell}{2}. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 1$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L_\ell)^\circ$ and $h \in [L_\ell, L_\ell]$. Since z acts by scalar multiplication on V^i, $i = 0, 1$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for both $i = 0$ and $i = 1$, where μ_h is any eigenvalue of h on V^i. Now, by induction, we have $\dim(V_h^0(\mu_h)) \leq \frac{(\ell-3)(\ell-2)(\ell-1)}{6}$, while, by Proposition 3.2.5, we have $\dim(V_h^1(\mu_h)) \leq \frac{(\ell-2)(\ell-1)}{2}$, for all eigenvalues μ_h of h on V^0, respectively on V^1. It follows that $\dim(V_h(\mu_h)) \leq \frac{(\ell-2)(\ell-1)\ell}{6}$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq \frac{(\ell-2)(\ell-1)\ell}{6}$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq \frac{(\ell-2)(\ell-1)\ell}{6}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. Moreover, as the inequality $0 < 3\ell^2 - 5\ell - 2$ holds for all $\ell \geq 6$, it follows that $\frac{(\ell-2)(\ell-1)\ell}{6} < \frac{(\ell-1)\ell(\ell+1)}{6} - \sqrt{\frac{(\ell-1)\ell(\ell+1)}{6}}$ for all $\ell \geq 6$, therefore $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

We conclude this subsection by noting that Proposition 3.2.9 completes the proof of Theorems 3.2.1 and 3.2.2 for simple simply connected linear algebraic groups of type A_1. Furthermore, for G of type A_ℓ, $\ell \geq 2$, Propositions 3.2.10 through 3.2.14 cover all the irreducible kG-modules corres-ponding to p-restricted dominant weights featured in Table 2.7.1. This completes the proof of Theorems 3.2.1 and 3.2.2, respectively.

3.3 Eigenspace dimensions for unipotent elements

In this section we prove the following two theorems, analogs of Theorems 3.2.1 and 3.2.2 in the case of unipotent elements. As in the semisimple case, the proofs will be given in a series of results, each treating one of the candidate-modules. In Subsection 3.3.1, we determine $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ for the families of irreducible kG-modules $V = L_G(\lambda)$ with $\lambda \in F^{A_\ell}$, where $F^{A_\ell} = \{\omega_1, 2\omega_1, \omega_2, \omega_1 + \omega_\ell\}$, see Subsection 2.7.1. For the irreducible kG-modules $V = L_G(\lambda)$, where either $\ell = 1$ and $\lambda = m\omega_1$ with $3 \leq m \leq 8$, or $\ell \geq 2$ and λ is featured in Table 2.7.1, we determine $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ in Subsection 3.3.2.

Theorem 3.3.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$. Let T be a fixed maximal torus in G. If $\ell = 1$, let $V = L_G(m\omega_1)$, where $1 \leq m \leq 8$, and assume that $p = 0$ or $p > m$. If $\ell \geq 2$, let $V = L_G(\lambda)$, where $\lambda \in F^{A_\ell}$ or λ appears in Table 2.7.1. Then, there exist non-identity unipotent elements $u \in G$ for which*

$$\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)},$$

if and only if ℓ, λ and p appear in the following list:

(1) *$\ell \geq 1$, $\lambda = \omega_1$ and $p \geq 0$;*

(2) *$\ell \leq 4$, $\lambda = \omega_2$ and $p \geq 0$.*

Theorem 3.3.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$. Let T and V be as in Theorem 3.3.1. Then the value of $\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ is given in the table below:*

V	Char.	Rank	$\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$
$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 1$	ℓ
${}^\dagger L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 1$	$\frac{\ell(\ell+1)}{2}$
${}^\star L_G(\omega_2)$	$p \geq 0$	$\ell \geq 3$	$\frac{\ell^2-\ell+2}{2}$
${}^\dagger L_G(\omega_1 + \omega_\ell)$	$p \nmid \ell + 1$	$\ell \geq 2$	ℓ^2
${}^\dagger L_G(\omega_1 + \omega_\ell)$	$p \mid \ell + 1$	$\ell \geq 2$	$\ell^2 - 1$
${}^\dagger L_G(m\omega_1)$, $3 \leq m \leq 8$	$p = 0$, or $p > m$	$\ell = 1$	1
${}^\dagger L_G(\omega_1 + \omega_2)$	$p = 3$	$\ell = 3$	≤ 8
${}^\dagger L_G(\omega_3)$	$p \geq 0$	$\ell = 5, 6, 7$	$\frac{(\ell-1)\cdot(\ell^2-2\ell+6)}{6}$

Table 3.3.1: The value of $\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$.

In particular, for each V in Table 3.3.1 labeled as ${}^\dagger V$, respectively as ${}^\star V$ with $\ell \geq 5$, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

3.3.1 The families of modules

For the rest of this chapter, we fix the following hypothesis on unipotent elements in G:

$({}^\dagger H_u)$: every $u \in G_u \setminus \{1\}$ has Jordan form on W given by $\bigoplus\limits_{i=1}^{m} J_{n_i}^{r_i}$, where $\sum\limits_{i=1}^{m} n_i r_i = \ell + 1$,

$r_i \geq 1$ for all $1 \leq i \leq m$ and $n_1 \geq 2$.

Lemma 3.3.3. *Let $V = L_G(\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq \ell,$$

where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.

In particular, there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. To begin, we note that $V \cong W$ as kG-modules. Now, let the unipotent element $u \in G$ be as in $({}^{\dagger}H_u)$. Let u_W denote the action of u on W. Then:

$$\dim(V_u(1)) = \dim(W_{u_W}(1)) = \sum_{i=1}^{m} r_i. \tag{3.27}$$

As $u \neq 1$, it follows that $\dim(V_u(1)) \leq \ell$, therefore, we have shown that $\dim(V_u(1)) \leq \ell$ for all non-identity unipotent elements $u \in G$. To complete the proof, we will identify the unipotent classes in G for which equality holds. For this, we assume that $\dim(V_u(1)) = \ell$. Then, by (3.27) and keeping in mind that $\sum_{i=1}^{m} n_i r_i = \ell + 1$, we have:

$$1 = \sum_{i=1}^{m} n_i r_i - \sum_{i=1}^{m} r_i = \sum_{i=1}^{m} (n_i - 1) r_i \tag{3.28}$$

and, in particular, $1 \geq (n_1 - 1)r_1 \geq n_1 - 1$, hence $n_1 = 2$, as $n_1 \geq 2$. Furthermore, by (3.28) and keeping in mind that $r_1 \geq 1$, we deduce that $r_1 = 1$ and $\sum_{i=2}^{m} (n_i - 1) r_i = 0$. It follows that $m \leq 2$. If $m = 1$, then, as $n_1 = 2$ and $r_1 = 1$, we have $\ell = 1$, in which case the Jordan form of u on W is J_2. If $m = 2$, then $n_2 = 1$ and $r_2 = \ell - 1$, in which case the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$. Conversely, let u be a unipotent element of G whose Jordan form on W is $J_2 \oplus J_1^{\ell-1}$. Then, by (3.27), we have $\dim(V_u(1)) = \ell$.

In conclusion, we have shown that $\dim(V_u(1)) \leq \ell$ for all non-identity unipotent elements $u \in G$ and that equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$. Now, let u be such an element of G. Then, as $\sqrt{\ell+1} \geq 1$ for all $\ell \geq 1$, it follows that $\ell \geq \ell + 1 - \sqrt{\ell+1}$ for all $\ell \geq 1$ and, consequently, $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. □

Proposition 3.3.4. *Let $\ell \geq 3$ and let $V = L_G(\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq \frac{\ell^2 - \ell + 2}{2}.$$

Moreover, we have equality if and only if one of the following holds:

(1) *$\ell = 3$ and the Jordan form of u on W is one of J_2^2 and $J_2 \oplus J_1^2$.*

(2) *$\ell \geq 4$ and the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.*

In particular, in both cases $\ell = 3$ and $\ell = 4$ there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 5$ we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we note that, by Lemma 2.8.1, we have the following kG-module isomorphism: $V \cong \wedge^2(W)$. Now, let the unipotent element $u \in G$ be as in $({}^{\dagger}H_u)$.

We first assume that the Jordan form of u on W is $J_{\ell+1}$. Then, by applying either Lemma 2.9.4 if $p \neq 2$, or Lemma 2.9.5 if $p = 2$, we have $\dim(V_u(1)) = \left\lfloor \frac{\ell+1}{2} \right\rfloor = \frac{\ell + 1 + \xi}{2}$, where

$\xi = 0$ if ℓ is odd, or $\xi = -1$ if ℓ is even. Since $0 < (\ell-1)^2 - \xi$ for all $\ell \geq 3$, it follows that $\dim(V_u(1)) < \dfrac{\ell^2 - \ell + 2}{2}$. We can now assume that the Jordan form of u on W consists of at least two blocks.

Secondly, we consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. Then $2 \leq n_1 \leq \ell$ and we write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = \ell+1-n_1$ and u acts trivially on W_2. As $V \cong \wedge^2(W)$, we have the following $k[u]$-modules isomorphism:

$$V \cong \wedge^2(W_1) \oplus (W_1 \otimes W_2) \oplus \wedge^2(W_2),$$

which gives

$$\dim(V_u(1)) = \dim((\wedge^2(W_1))_u(1)) + \dim((W_1 \otimes W_2)_u(1)) + \dim((\wedge^2(W_2))_u(1)). \tag{3.29}$$

As u acts as a single Jordan block on W_1, by Lemma 2.9.4 if $p \neq 2$, or by Lemma 2.9.5 if $p = 2$, we have $\dim((\wedge^2(W_1))_u(1)) = \left\lfloor \dfrac{n_1}{2} \right\rfloor = \dfrac{n_1 + \epsilon}{2}$, where $\epsilon = 0$ if n_1 is even, or $\epsilon = -1$ if n_1 is odd. As u acts trivially on W_2, it also acts trivially on $\wedge^2(W_2)$ and so $\dim((\wedge^2(W_2))_u(1)) = \dfrac{(\ell - n_1)(\ell - n_1 + 1)}{2}$. Lastly, as u acts as $J_{n_1} \otimes J_1^{\ell+1-n_1}$ on $W_1 \otimes W_2$, by Lemma 2.9.4, we have $\dim((W_1 \otimes W_2)_u(1)) = \ell + 1 - n_1$. Substituting in (3.29) gives

$$\begin{aligned}
\dim(V_u(1)) &= \frac{n_1 + \epsilon}{2} + \ell + 1 - n_1 + \frac{(\ell - n_1)(\ell - n_1 + 1)}{2} \\
&= \frac{\ell^2 - 2\ell n_1 + 3\ell - 2n_1 + n_1^2 + 2 + \epsilon}{2} \\
&= \frac{\ell^2 - \ell + 2}{2} + \frac{n_1^2 - 2\ell n_1 - 2n_1 + 4\ell + \epsilon}{2}.
\end{aligned}$$

One checks that the inequality

$$n_1^2 - 2\ell n_1 - 2n_1 + 4\ell + \epsilon \leq 0 \tag{3.30}$$

holds for all $n_1 \in [\ell + 1 - \sqrt{(\ell-1)^2 - \epsilon},\ \ell + 1 + \sqrt{(\ell-1)^2 - \epsilon}]$ and all $\ell \geq 1$. Since $\ell + 1 + \sqrt{(\ell-1)^2 - \epsilon} > \ell$ and $\ell + 1 - \sqrt{(\ell-1)^2 - \epsilon} \leq \ell + 1 - \sqrt{(\ell-1)^2} = 2$, as $\epsilon \leq 0$, it follows that, in particular, Inequality (3.30) holds for all $2 \leq n_1 \leq \ell$ and all $\ell \geq 3$. Therefore $\dim(V_u(1)) \leq \dfrac{\ell^2 - \ell + 2}{2}$ for all unipotent elements u of G whose Jordan form on W is $J_{n_1} \oplus J_1^{\ell+1-n_1}$, where $2 \leq n_1 \leq \ell$. Moreover, equality holds if and only if $n_1 = 2$.

Lastly, we assume that the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq \ell - 1$ and we write $W = W_1' \oplus W_2'$, where $\dim(W_1') = n_1$ and u acts as J_{n_1} on W_1', and $\dim(W_2') = \ell + 1 - n_1$ and u acts as $J_{n_1}^{r_1 - 1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on W_2'. Now, by (3.29), to determine $\dim(V_u(1))$ comes down to determining $\dim((\wedge^2(W_1'))_u(1))$, $\dim((W_1' \otimes W_2')_u(1))$ and $\dim((\wedge^2(W_2'))_u(1))$. Again, either by Lemma 2.9.4 if $p \neq 2$, or by Lemma 2.9.5 if $p = 2$,

we have $\dim((\wedge^2(W_1^{'}))_u(1)) = \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1+\varsigma}{2}$, where $\varsigma = 0$ if n_1 is even, or $\varsigma = -1$ if n_1 is odd. As u acts as $(J_{n_1} \otimes J_{n_1})^{r_1-1} \oplus \bigoplus_{i=2}^{m} (J_{n_1} \otimes J_{n_i})^{r_i}$ on $W_1^{'} \otimes W_2^{'}$, by Lemma 2.9.4, we have

$$\dim((W_1^{'} \otimes W_2^{'})_u(1)) = (r_1-1)n_1 + \sum_{i=2}^{m} n_i r_i = -n_1 + \sum_{i=1}^{m} n_i r_i = \ell + 1 - n_1. \tag{3.31}$$

Substituting in (3.29) gives:

$$\dim(V_u(1)) = \frac{n_1+\varsigma}{2} + \ell + 1 - n_1 + \dim((\wedge^2(W_2^{'}))_u(1)). \tag{3.32}$$

Using induction, we will show that $\dim((\wedge^2(W))_u(1)) \leq \frac{\ell^2-\ell+2}{2}$, where $\dim(W) = \ell+1$, for all $\ell \geq 3$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks . First, let $\ell = 3$. Then u has Jordan form J_2^2 and thus acts as a single Jordan block on $W_2^{'}$. By Lemma 2.9.4 if $p \neq 2$, or by Lemma 2.9.5 if $p = 2$, we have $\dim((\wedge^2(W_2^{'}))_u(1)) = 1$. Substituting in (3.32) gives $\dim((\wedge^2(W))_u(1)) = 4$.

We now assume that $\ell \geq 4$. If u acts on $W_2^{'}$ as a single Jordan block, we have shown earlier that $\dim((\wedge^2(W_2^{'}))_u(1)) < \frac{(\ell-n_1)^2-(\ell-n_1)+2}{2}$. If the Jordan form of the action of u on $W_2^{'}$ consists of at least two blocks and if exactly one of these blocks is nontrivial, then we have shown earlier that $\dim((\wedge^2(W_2^{'}))_u(1)) \leq \frac{(\ell-n_1)^2-(\ell-n_1)+2}{2}$. Lastly, if the Jordan form of the action of u on $W_2^{'}$ admits at least two nontrivial blocks, then, by induction, it follows that $\dim((\wedge^2(W_2^{'}))_u(1)) \leq \frac{(\ell-n_1)^2-(\ell-n_1)+2}{2}$. In all cases, substituting in (3.32) gives:

$$\begin{aligned}
\dim(V_u(1)) &\leq \frac{n_1+\varsigma}{2} + \ell + 1 - n_1 + \frac{(\ell-n_1)^2-(\ell-n_1)+2}{2} \\
&= \frac{\ell^2 - 2\ell n_1 + n_1^2 + \ell + 4 + \varsigma}{2} \\
&= \frac{\ell^2-\ell+2}{2} + \frac{n_1^2 - 2\ell n_1 + 2\ell + 2 + \varsigma}{2}.
\end{aligned}$$

One checks that the inequality

$$n_1^2 - 2\ell n_1 + 2\ell + 2 + \varsigma < 0$$

holds for all $n_1 \in (\ell - \sqrt{\ell^2 - 2\ell - 2 - \varsigma},\ \ell + \sqrt{\ell^2 - 2\ell - 2 - \varsigma})$ and all $\ell \geq 3$. Since $\ell + \sqrt{\ell^2 - 2\ell - 2 - \varsigma} > \ell - 1$ and since $\ell - \sqrt{\ell^2 - 2\ell - 2 - \varsigma} < 2$, as $6 + \varsigma < 2\ell$ for all $\ell \geq 4$, it follows that, in particular, the inequality holds for all $2 \leq n_1 \leq \ell - 1$ and all $\ell \geq 4$. We deduce that $\dim(V_u(1)) < \frac{\ell^2-\ell+2}{2}$ for all $\ell \geq 4$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks. This completes the induction.

Having treated all possible cases, we conclude that $\dim(V_u(1)) \leq \dfrac{\ell^2-\ell+2}{2}$, for all non-identity unipotent elements $u \in G$. Moreover, we have shown that for $\ell = 3$ equality holds if and only if the Jordan form of u on W is one of J_2^2 and $J_2 \oplus J_1^2$; while, for $\ell \geq 4$, equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$. In particular, since the inequality $0 < \ell^2 - 5\ell + 2$ holds for all $\ell \geq 5$, it follows that in both cases $\ell = 3$ and $\ell = 4$ we have $\dfrac{\ell^2-\ell+2}{2} \geq \dfrac{\ell(\ell+1)}{2} - \sqrt{\dfrac{\ell(\ell+1)}{2}}$ and thus there exist non-identity unipotent elements $u \in G$, for example those whose Jordan form on W is $J_2 \oplus J_1^{\ell-1}$, for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 5$ we have $\dfrac{\ell^2-\ell+2}{2} < \dfrac{\ell(\ell+1)}{2} - \sqrt{\dfrac{\ell(\ell+1)}{2}}$ and we conclude that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 3.3.5. *Let k be an algebraically closed field of characteristic $p \neq 2$. Let $V = L_G(2\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq \frac{\ell(\ell+1)}{2},$$

where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. We first note that, as $p \neq 2$, by Lemma 2.8.1, we have $V \cong \mathrm{S}^2(W)$, as kG-modules. Let the unipotent element $u \in G$ be as in $({}^{\dagger}H_u)$.

We first assume that the Jordan form of u on W is $J_{\ell+1}$. Then, by applying Lemma 2.9.4, as $p \neq 2$, we get $\dim(V_u(1)) = \ell + 1 - \left\lfloor \dfrac{\ell+1}{2} \right\rfloor = \dfrac{\ell+1-\xi}{2}$, where $\xi = -1$ if ℓ is even, or $\xi = 0$ if ℓ is odd. Since $0 \leq \ell^2 - 1 + \xi$ holds for all $\ell \geq 1$, it follows that $\dfrac{\ell+1-\xi}{2} \leq \dfrac{\ell(\ell+1)}{2}$ and so $\dim(V_u(1)) \leq \dfrac{\ell(\ell+1)}{2}$ for all $\ell \geq 1$ and all unipotent elements u of G whose Jordan form on W is $J_{\ell+1}$. Moreover, we have equality if and only if $\ell^2 - 1 + \xi = 0$, hence, if and only if $\ell = 1$ and u has Jordan form J_2 on W. We can now assume that the Jordan form of u on W admits at least two blocks and we note that, as $u \neq 1$, we then have $\ell \geq 2$.

We consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. Then u has Jordan form $J_{n_1} \oplus J_1^{\ell+1-n_1}$ on W, where $2 \leq n_1 \leq \ell$, and we write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = \ell+1-n_1$ and u acts trivially on W_2. Then, as $k[u]$-modules, we have

$$V \cong \mathrm{S}^2(W_1) \oplus (W_1 \otimes W_2) \oplus \mathrm{S}^2(W_2)$$

and this gives:

$$\dim(V_u(1)) = \dim((\mathrm{S}^2(W_1))_u(1)) + \dim((W_1 \otimes W_2)_u(1)) + \dim((\mathrm{S}^2(W_2))_u(1)). \tag{3.33}$$

As u acts as a single Jordan block of size n_1 on W_1 and since $p \neq 2$, by Lemma 2.9.4, we have $\dim((\mathrm{S}^2(W_1))_u(1)) = n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 - \epsilon}{2}$, where $\epsilon = -1$ if n_1 is odd, or $\epsilon = 0$ if n_1 is even. Furthermore, as u acts trivially on W_2, it also acts trivially on $\mathrm{S}^2(W_2)$, and we have $\dim((\mathrm{S}^2(W_2))_u(1)) = \frac{(\ell+1-n_1)(\ell+2-n_1)}{2} = \frac{\ell^2 - 2\ell n_1 + n_1^2 + 3\ell - 3n_1 + 2}{2}$. Lastly, u acts as $J_{n_1} \otimes J_1^{\ell+1-n_1}$ on $W_1 \otimes W_2$ and so, by (3.31), $\dim((W_1 \otimes W_2)_u(1)) = \ell + 1 - n_1$. Substituting in (3.33) gives:

$$\begin{aligned}\dim(V_u(1)) &= \frac{n_1 - \epsilon}{2} + \ell + 1 - n_1 + \frac{\ell^2 - 2\ell n_1 + n_1^2 + 3\ell - 3n_1 + 2}{2} \\ &= \frac{\ell(\ell+1)}{2} + \frac{n_1^2 - 2\ell n_1 + 4\ell - 4n_1 + 4 - \epsilon}{2}.\end{aligned}$$

We write

$$\frac{n_1^2 - 2\ell n_1 + 4\ell - 4n_1 + 4 - \epsilon}{2} = \frac{n_1^2 - 2\ell n_1 - 2n_1 + 4\ell + \epsilon}{2} + \frac{4 - 2n_1 - 2\epsilon}{2}$$

and, by (3.30), which holds for all $2 \leq n_1 \leq \ell$ and all $\ell \geq 2$, we have

$$\frac{n_1^2 - 2\ell n_1 + 4\ell - 4n_1 + 4 + \epsilon}{2} \leq \frac{4 - 2n_1 - 2\epsilon}{2}.$$

Now, since $2 \leq n_1$ and since $\epsilon = -1$ or $\epsilon = 0$ according to whether n_1 is odd or even, we have $\frac{4 - 2n_1 - 2\epsilon}{2} \leq 0$. Therefore $\frac{n_1^2 - 2\ell n_1 + 4\ell - 4n_1 + 4 - \epsilon}{2} \leq 0$ and, consequently, $\dim(V_u(1)) \leq \frac{\ell(\ell+1)}{2}$ for all $\ell \geq 2$ and all unipotent elements u of G whose Jordan form on W is $J_{n_1} \oplus J_1^{\ell+1-n_1}$, where $2 \leq n_1 \leq \ell$. We note that equality holds if and only if we have equality in (3.30) and $\frac{4 - 2n_1 - 2\epsilon}{2} = 0$, hence, if and only if $n_1 = 2$, as in the statement of the result.

Lastly, we consider the case when the Jordan form of u on W admits at least two nontrivial blocks. We note that in this case we have $\ell \geq 3$ and that $2 \leq n_1 \leq \ell - 1$. We write $W = W_1' \oplus W_2'$, where $\dim(W_1') = n_1$ and u acts as J_{n_1} on W_1', and $\dim(W_2') = \ell + 1 - n_1$ and u acts as $J_{n_1}^{r_1 - 1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on W_2'. By (3.33) it follows that in order to determine $\dim(V_u(1))$ we only need to know $\dim((\mathrm{S}^2(W_1'))_u(1))$, $\dim((W_1' \otimes W_2')_u(1))$ and $\dim((\mathrm{S}^2(W_2'))_u(1))$. We can apply Lemma 2.9.4, as $p \neq 2$, to deduce that $\dim((\mathrm{S}^2(W_1'))_u(1)) = n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 - \varsigma}{2}$, where $\varsigma = -1$ if n_1 is odd, or $\varsigma = 0$ if n_1 is even. Furthermore, by (3.31) we have $\dim((W_1' \otimes W_2')_u(1)) = \ell + 1 - n_1$. It follows that:

$$\dim(V_u(1)) = \frac{n_1 - \varsigma}{2} + \ell + 1 - n_1 + \dim((\mathrm{S}^2(W_2'))_u(1)). \tag{3.34}$$

Inductively, we will show that $\dim((\mathrm{S}^2(W))_u(1)) < \frac{\ell(\ell+1)}{2}$, where $\dim(W) = \ell + 1$, for all $\ell \geq 3$ and all unipotent elements u of G whose Jordan form admits at least two nontrivial

blocks on W. First, let $\ell = 3$. Then u has Jordan form J_2^2 and thus acts as a single Jordan block on $W_2^{'}$. By applying Lemma 2.9.4, as $p \neq 2$, we have $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) = 1$ and substituting in (3.34) gives $\dim((\mathrm{S}^2(W))_u(1)) = 4$.

We now assume that $\ell \geq 4$. If u acts on $W_2^{'}$ as a single Jordan block, we have shown earlier that $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) < \dfrac{(\ell - n_1)(\ell + 1 - n_1)}{2}$. If the Jordan form of the action of u on $W_2^{'}$ consists of at least two blocks and if exactly one of these blocks is nontrivial, then we have shown earlier that $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) \leq \dfrac{(\ell - n_1)(\ell - n_1 + 1)}{2}$. Lastly, if the Jordan form of the action of u on $W_2^{'}$ admits at least two nontrivial blocks, then, by induction, we have $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) < \dfrac{(\ell - n_1)(\ell - n_1 + 1)}{2}$. In all cases, substituting in (3.34) gives:

$$\begin{aligned}
\dim(V_u(1)) &\leq \frac{n_1 - \varsigma}{2} + \ell + 1 - n_1 + \frac{(\ell - n_1)(\ell + 1 - n_1)}{2} \\
&= \frac{\ell^2 + 3\ell - 2\ell n_1 - 2n_1 + n_1^2 + 2 - \varsigma}{2} \\
&= \frac{\ell(\ell+1)}{2} + \frac{n_1^2 - 2\ell n_1 - 2n_1 + 2\ell + 2 - \varsigma}{2}.
\end{aligned}$$

We write $\dfrac{n_1^2 - 2\ell n_1 - 2n_1 + 2\ell + 2 - \varsigma}{2} = \dfrac{n_1^2 - 2\ell n_1 - 2n_1 + 4\ell + \varsigma}{2} + \dfrac{2 - 2\varsigma - 2\ell}{2}$. By (3.30), which holds for all $n_1 \in [\ell + 1 - \sqrt{(\ell-1)^2 - \varsigma},\ \ell + 1 + \sqrt{(\ell-1)^2 - \varsigma}]$ and all $\ell \geq 1$, hence, in particular, holds for all $2 \leq n_1 \leq \ell - 1$ and all $\ell \geq 4$, we have $\dfrac{n_1^2 - 2\ell n_1 - 2n_1 + 2\ell + 2 - \varsigma}{2} \leq \dfrac{2 - 2\varsigma - 2\ell}{2}$. Furthermore, as $\ell \geq 4$ and as $\varsigma = 0$ or $\varsigma = -1$ according to whether n_1 is even or odd, we have $\dfrac{2 - 2\varsigma - 2\ell}{2} < 0$. It follows that $\dfrac{n_1^2 - 2\ell n_1 - 2n_1 + 2\ell + 2 - \varsigma}{2} < 0$ and, consequently, $\dim(V_u(1)) < \dfrac{\ell(\ell+1)}{2}$ for all $\ell \geq 4$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks. This completes the induction.

Having treated all possible cases, we conclude that $\dim(V_u(1)) \leq \dfrac{\ell(\ell+1)}{2}$ for all non-identity unipotent elements $u \in G$. Moreover, we have shown that equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$. Since $0 < \ell^2 + \ell$ for all $\ell \geq 1$, it follows that $\dfrac{\ell(\ell+1)}{2} < \dfrac{(\ell+1)(\ell+2)}{2} - \sqrt{\dfrac{(\ell+1)(\ell+2)}{2}}$ for all $\ell \geq 1$ and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$, for all non-identity unipotent elements $u \in G$. This completes the proof of the proposition. $\square$

Before we proceed with the proofs of Theorems 3.3.1 and 3.3.2, we recall that the irreducible kG-module $L_G(\omega_1 + \omega_\ell)$ is a composition factor of the kG-module $W \otimes W^*$, see Lemma 2.8.1. This is a relevant fact, since by Lemma 2.9.4, we can calculate the dimension of the fixed point space of a unipotent element $u \in G$ on $W \otimes W^*$. Furthermore, Theorem 6.1 of [Kor19], shows how to deduce $\dim((L_G(\omega_1 + \omega_\ell))_u(1))$ from $\dim((W \otimes W^*)_u(1))$. Before we give this result, we remind the reader that $r_t(u)$, where $u \in G$ is a unipotent element and

$t \geq 1$, is the number of Jordan blocks of size t appearing in the Jordan decomposition of u, and that ν_p is the p-adic valuation on the integers.

Theorem 3.3.6. [Kor19, Theorem 6.1] *Let $u \in G$ be a unipotent element and let $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ be the Jordan form of u on W, where $m \geq 1$, $n_i \geq 1$ and $r_i \geq 1$ for all $1 \leq i \leq m$. Set $\alpha = \nu_p(\gcd(n_1, \ldots, n_m))$. Let $u^{'}$ be the action of u on $W \otimes W^*$ and let u_V be the action of u on $V := L_G(\omega_1 + \omega_\ell)$. Then the Jordan block sizes of u_V are determined from those of $u^{'}$ in the following way:*

(a) *if $p \nmid \ell + 1$, then $r_1(u_V) = r_1(u^{'}) - 1$ and $r_t(u_V) = r_t(u^{'})$ for all $t \neq 1$.*

(b) *if $p \mid \ell + 1$ and $\alpha = 0$, then $r_1(u_V) = r_1(u^{'}) - 2$ and $r_t(u_V) = r_t(u^{'})$ for all $t \neq 1$.*

(c) *if $p \mid \ell + 1$ and $\alpha > 0$:*

  (c.1) *if $p \mid \dfrac{\ell+1}{p^\alpha}$, then $r_{p^\alpha}(u_V) = r_{p^\alpha}(u^{'}) - 2$, $r_{p^\alpha - 1}(u_V) = 2$ and $r_t(u_V) = r_t(u^{'})$ for all $t \neq p^\alpha, p^\alpha - 1$.*

  (c.2) *if $p \nmid \dfrac{\ell+1}{p^\alpha}$ and $p^\alpha > 2$, then $r_{p^\alpha}(u_V) = r_{p^\alpha}(u^{'}) - 1$, $r_{p^\alpha - 2}(u_V) = 1$ and $r_t(u_V) = r_t(u^{'})$ for all $t \neq p^\alpha, p^\alpha - 2$.*

  (c.3) *if $p \nmid \dfrac{\ell+1}{p^\alpha}$ and $p^\alpha = 2$, then $r_2(u_V) = r_2(u^{'}) - 1$ and $r_t(u_V) = r_t(u^{'})$ for all $t \neq 2$.*

Remark 3.3.7. *By Theorem 3.3.6, we determine that:*

(1) *if $p \nmid \ell + 1$, then $\dim(V_u(1)) = \dim((W \otimes W^*)_u(1)) - 1$;*

(2) *if $p \mid \ell + 1$ and $\alpha = 0$, then $\dim(V_u(1)) = \dim((W \otimes W^*)_u(1)) - 2$;*

(3) *if $p \mid \ell + 1$, $\alpha > 0$ and*

  (3.1) *$p \mid \dfrac{\ell+1}{p^\alpha}$, then $\dim(V_u(1)) = \dim((W \otimes W^*)_u(1))$;*

  (3.2) *$p \nmid \dfrac{\ell+1}{p^\alpha}$ and $p^\alpha > 2$, then $\dim(V_u(1)) = \dim((W \otimes W^*)_u(1))$;*

  (3.3) *$p \nmid \dfrac{\ell+1}{p^\alpha}$ and $p^\alpha = 2$, then $\dim(V_u(1)) = \dim((W \otimes W^*)_u(1)) - 1$.*

Proposition 3.3.8. *Let $\ell \geq 2$ and let $V^{'} = W \otimes W^*$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u^{'}(1)) \leq \ell^2 + 1,$$

where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.

Proof. Let the unipotent element $u \in G$ be as in hypothesis $({}^{\dagger}H_u)$. We first consider the case when the Jordan form of u on W is $J_{\ell+1}$. Then, u acts on W^* as $J_{\ell+1}$ and, by applying Lemma 2.9.4 and keeping in mind that $\ell \geq 2$, we deduce that

$$\dim(V_u^{'}(1)) = \ell + 1 < \ell^2 + 1. \tag{3.35}$$

We can now assume that the Jordan form of u on W admits at least two blocks.

Secondly, we consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. Then u has Jordan form $J_{n_1} \oplus J_1^{\ell+1-n_1}$ on W, where $2 \leq n_1 \leq \ell$, and we write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = \ell + 1 - n_1$ and u acts trivially on W_2. Then, as $k[u]$-modules, we have:

$$V^{'} \cong (W_1 \otimes W_1^*) \oplus (W_1 \otimes W_2^*) \oplus (W_2 \otimes W_1^*) \oplus (W_2 \otimes W_2^*). \tag{3.36}$$

Since $W_1 \otimes W_2^* \cong (W_1^* \otimes W_2)^*$ and since the action of u on $(W_1^* \otimes W_2)^*$ has the same Jordan form as the action of u on $W_1 \otimes W_2^*$, it follows that

$$\dim(V_u^{'}(1)) = \dim((W_1 \otimes W_1^*)_u(1)) + 2\dim((W_1 \otimes W_2^*)_u(1)) + \dim((W_2 \otimes W_2^*)_u(1)). \tag{3.37}$$

Since u acts as J_{n_1} on W_1, it also acts as J_{n_1} on W_1^* and, by Lemma 2.9.4, we get $\dim((W_1 \otimes W_1^*)_u(1)) = n_1$. Moreover, as u acts as $J_1^{\ell+1-n_1}$ on W_2, it also acts as $J_1^{\ell+1-n_1}$ on W_2^*, and so $\dim((W_2 \otimes W_2^*)_u(1)) = (\ell + 1 - n_1)^2$. Lastly, as u acts as $J_{n_1} \otimes J_1^{\ell+1-n_1}$ on $W_1 \otimes W_2^*$, we have $\dim((W_1 \otimes W_2^*)_u(1)) = \ell + 1 - n_1$. Substituting in (3.37) gives:

$$\begin{aligned} \dim(V_u^{'}(1)) &= n_1 + 2(\ell + 1 - n_1) + (\ell + 1 - n_1)^2 \\ &= \ell^2 + 1 + n_1^2 - 2\ell n_1 - 3n_1 + 4\ell + 2 \\ &= \ell^2 + 1 + (n_1 - 2)(n_1 - 2\ell - 1). \end{aligned} \tag{3.38}$$

As $2 \leq n_1 \leq \ell$, we have $(n_1 - 2)(n_1 - 2\ell - 1) \leq 0$ and therefore $\dim(V_u^{'}(1)) \leq \ell^2 + 1$ for all $\ell \geq 2$ and all unipotent elements u of G with Jordan form $J_{n_1} \oplus J_1^{\ell+1-n_1}$ on W, where $2 \leq n_1 \leq \ell$. Moreover, equality holds if and only if $(n_1 - 2)(n_1 - 2\ell - 1) = 0$, hence, if and only if $n_1 = 2$.

We can now assume that the Jordan form of u on W admits at least two nontrivial blocks. We note that, in this case, we have $\ell \geq 3$ and $2 \leq n_1 \leq \ell - 1$. We write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = n_1$ and u acts as J_{n_1} on $W_1^{'}$, and $\dim(W_2^{'}) = \ell + 1 - n_1$ and u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$. By (3.37), to determine $\dim(V_u^{'}(1))$ we only need to know $\dim((W_1^{'} \otimes (W_1^{'})^*)_u(1))$, $\dim((W_1^{'} \otimes (W_2^{'})^*)_u(1))$ and $\dim((W_2^{'} \otimes (W_2^{'})^*)_u(1))$. As u acts as J_{n_1} on $W_1^{'}$, by Lemma 2.9.4, we have $\dim((W_1^{'} \otimes (W_1^{'})^*)_u(1)) = n_1$. We note that since u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$, it also acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $(W_2^{'})^*$. Therefore, u acts as $(J_{n_1} \otimes J_{n_1})^{r_1-1} \oplus \bigoplus_{i=2}^{m} (J_{n_1} \otimes J_{n_i})^{r_i}$ on $W_1^{'} \otimes (W_2^{'})^*$ and, by (3.31), we have $\dim((W_1^{'} \otimes (W_2^{'})^*)_u(1)) = \ell + 1 - n_1$. Substituting in (3.37) gives:

$$\dim(V_u^{'}(1)) = n_1 + 2(\ell + 1 - n_1) + \dim((W_2^{'} \otimes (W_2^{'})^*)_u(1)). \tag{3.39}$$

Using induction, we will show that $\dim((W \otimes W^*)_u(1)) < \ell^2 + 1$, where $\dim(W) = \ell + 1$, for all $\ell \geq 3$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial Jordan blocks.

First, assume that $\ell = 3$. Then u has Jordan form J_2^2 on W and thus acts as J_2 on $W_2^{'}$ and $(W_2^{'})^*$, respectively. Thus, by Lemma 2.9.4, we have $\dim((W_2^{'} \otimes (W_2^{'})^*)_u(1)) = 2$ and substituting in (3.39) gives $\dim(V_u^{'}(1)) = 8$.

We can now assume that $\ell \geq 4$. If u acts on $W_2^{'}$ as a single Jordan block, we have shown earlier that $\dim((W_2^{'} \otimes (W_2^{'})^*)_u(1)) < (\ell - n_1)^2 + 1$. If the action of u on $W_2^{'}$ consists of at least two blocks and if exactly one of these blocks is nontrivial, then we have seen earlier that $\dim((W_2^{'} \otimes (W_2^{'})^*)_u(1)) \leq (\ell - n_1)^2 + 1$. Lastly, if the Jordan form of the action of u on $W_2^{'}$ admits at least two nontrivial blocks, then, by induction, it follows that $\dim((W_2^{'} \otimes (W_2^{'})^*)_u(1)) < (\ell - n_1)^2 + 1$. In all cases, substituting in (3.39) gives:

$$\begin{aligned}\dim(V_u^{'}(1)) &\leq n_1 + 2(\ell + 1 - n_1) + (\ell - n_1)^2 + 1 \\ &= \ell^2 + 1 + (n_1^2 - 2\ell n_1 + 2\ell + 2 - n_1) \\ &= \ell^2 + 1 + [n_1(n_1 - \ell) + (\ell + 1)(2 - n_1)].\end{aligned}$$

Since $2 \leq n_1 \leq \ell - 1$, it follows that $n_1(n_1 - \ell) < 0$ and $(\ell + 1)(2 - n_1) \leq 0$. Therefore $n_1(n_1 - \ell) + (\ell + 1)(2 - n_1) < 0$ for all $\ell \geq 4$ and all $2 \leq n_1 \leq \ell - 1$. We deduce that $\dim(V_u^{'}(1)) < \ell^2 + 1$ for all $\ell \geq 4$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks. This completes the induction.

In conclusion, we have shown that $\dim(V_u^{'}(1)) \leq \ell^2 + 1$ for all $\ell \geq 2$ and all non-identity unipotent elements $u \in G$, where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$. □

Corollary 3.3.9. *Let $\ell \geq 2$, $p \nmid \ell + 1$ and let $V = L_G(\omega_1 + \omega_\ell)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq \ell^2,$$

where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Set $V^{'} = W \otimes W^*$ and note that, by Lemma 2.8.1, since $p \nmid \ell + 1$, we have $V^{'} = V \oplus L_G(0)$ and, consequently, $\dim(V_u(1)) = \dim(V_u^{'}(1)) - 1$. We now use Proposition 3.3.8 to deduce that $\dim(V_u(1)) \leq \ell^2$ for all non-identity unipotent elements $u \in G$. Moreover, we achieve equality if and only if $\dim(V_u^{'}(1)) = \ell^2 + 1$, hence, again by Proposition 3.3.8, if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.

In conclusion, we have shown that $\dim(V_u(1)) \leq \ell^2$ for all non-identity unipotent elements $u \in G$ and that equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$. In particular, since $0 < 3\ell^2 - 2\ell$ for all $\ell \geq 2$, it follows that the inequality $\ell^2 < \ell^2 + 2\ell - \sqrt{\ell^2 + 2\ell}$ holds for all $\ell \geq 2$ and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

We will require the following result in the proof of Corollary 3.3.11.

Proposition 3.3.10. *Let $\ell \geq 2$ and let $V' = W \otimes W^*$. Let u be a nontrivial unipotent element of G whose Jordan form on W is different than $J_2 \oplus J_1^{\ell-1}$. Then*

$$\dim(V_u'(1)) \leq \ell^2 - 1,$$

where we have equality if and only if one of the following holds

(1) *$\ell = 2$ and the Jordan form of u on W is J_3.*

(2) *$\ell = 3$ and the Jordan form of u on W is J_2^2.*

Proof. Let the unipotent element $u \in G$ be as in hypothesis $({}^{\dagger}H_u)$ and assume that its Jordan form on W is different than $J_2 \oplus J_1^{\ell-1}$. Thus, if $n_1 = 2$, then, $r_1 \geq 2$. We first consider the case when the Jordan form of u on W is $J_{\ell+1}$. We proceed as in the proof of Proposition 3.3.8, see (3.35), to deduce that $\dim(V_u'(1)) = \ell + 1$, hence $\dim(V_u'(1)) \leq \ell^2 - 1$, as $\ell \geq 2$. Moreover, equality holds if and only if $\ell = 2$, in which case u has Jordan form J_3 on W. We can now assume that the Jordan form of u on W consists of at least two blocks and, since it is different than $J_2 \oplus J_1^{\ell-1}$ and $u \neq 1$, we then have $\ell \geq 3$.

We consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. Then u has Jordan form $J_{n_1} \oplus J_1^{\ell+1-n_1}$, where $3 \leq n_1 \leq \ell$. We proceed as in the proof of Proposition 3.3.8, see (3.36), (3.37) and (3.38), to deduce that

$$\dim(V_u'(1)) = \ell^2 - 1 + n_1^2 - 2\ell n_1 - 3n_1 + 4\ell + 4.$$

One checks that the inequality

$$n_1^2 - 2\ell n_1 - 3n_1 + 4\ell + 4 < 0 \tag{3.40}$$

holds for all $n_1 \in \big(\frac{2\ell + 3 - \sqrt{4\ell^2 - 4\ell - 7}}{2}, \frac{2\ell + 3 + \sqrt{4\ell^2 - 4\ell - 7}}{2}\big)$ and all $\ell \geq 2$. Since $\frac{2\ell + 3 + \sqrt{4\ell^2 - 4\ell - 7}}{2} > \ell$ and since $\frac{2\ell + 3 - \sqrt{4\ell^2 - 4\ell - 7}}{2} < 3$, as $2 < 8\ell$, it follows that, in particular, Inequality (3.40) holds for all $3 \leq n_1 \leq \ell$ and all $\ell \geq 3$. We deduce that $\dim(V_u'(1)) < \ell^2 - 1$ for all unipotent elements u of G with Jordan form $J_{n_1} \oplus J_1^{\ell+1-n_1}$, where $3 \leq n_1 \leq \ell$.

Lastly, we consider the case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq \ell - 1$ and we write $W = W_1' \oplus W_2'$, where $\dim(W_1') = n_1$ and u acts as J_{n_1} on W_1', and $\dim(W_2') = \ell + 1 - n_1$ and u acts as $J_{n_1}^{r_1 - 1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on W_2'. Now, by (3.39) of Proposition 3.3.8, we have $\dim(V_u'(1)) = n_1 + 2(\ell + 1 - n_1) + \dim((W_2' \otimes (W_2')^*)_u(1))$. Furthermore, by the induction argument of same result, we have $\dim((W_2' \otimes (W_2')^*)_u(1)) \leq (\ell - n_1)^2 + 1$, where equality holds if and only if u acts as $J_2 \oplus J_1^{\ell - n_1 - 1}$ on $W_2' \otimes (W_2')^*$. Therefore

$$\begin{aligned}\dim(V_u'(1)) &\leq n_1 + 2(\ell + 1 - n_1) + (\ell - n_1)^2 + 1 \\ &= \ell^2 - 1 + n_1^2 - 2\ell n_1 - n_1 + 2\ell + 4.\end{aligned}$$

One checks that the inequality

$$n_1^2 - 2\ell n_1 - n_1 + 2\ell + 4 \leq 0 \tag{3.41}$$

holds for all $n_1 \in \left[\frac{2\ell+1-\sqrt{4\ell^2-4\ell-15}}{2}, \frac{2\ell+1+\sqrt{4\ell^2-4\ell-15}}{2}\right]$ and all $\ell \geq 3$. Since $\frac{2\ell+1+\sqrt{4\ell^2-4\ell-15}}{2} > \ell - 1$ and since $\frac{2\ell+1-\sqrt{4\ell^2-4\ell-15}}{2} \leq 2$, as $3 \leq \ell$, it follows that, in particular, Inequality (3.41) holds for all $2 \leq n_1 \leq \ell - 1$ and all $\ell \geq 3$. Moreover, we achieve equality in (3.41) if and only if $n_1 = 2$, in which case $\ell = 3$, as $\frac{2\ell+1-\sqrt{4\ell^2-4\ell-15}}{2} = 2$. We conclude that $\dim(V_u^{'}(1)) \leq \ell^2 - 1$ for all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks. Furthermore, we have equality if and only if $\ell = 3$ and the Jordan form of u on W is J_2^2. This completes the proof of the proposition. □

Corollary 3.3.11. *Let $\ell \geq 2$, $p \mid \ell + 1$ and let $V = L_G(\omega_1 + \omega_\ell)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq \ell^2 - 1.$$

Moreover, we have equality if and only if one of the following holds:

(1) *$\ell = 2$ and the Jordan form of u on W is one of J_3 and $J_2 \oplus J_1$.*

(2) *$\ell = 3$ and the Jordan form of u on W is one of J_2^2 and $J_2 \oplus J_1^2$.*

(3) *$\ell \geq 4$ and the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.*

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, set $V^{'} = W \otimes W^*$ and let the unipotent element $u \in G$ be as in hypothesis $({}^{\dagger}H_u)$. If we denote by $u^{'}$, respectively by u_V, the action of u on $V^{'}$, respectively on V, then by Theorem 3.3.6 we know that we can determine the Jordan form of u_V from that of $u^{'}$.

Set $\alpha = \nu_p(\gcd(n_1, \ldots, n_m))$. If $\alpha = 0$, then, by item (2) of Remark 3.3.7, we have $\dim(V_u(1)) = \dim(V_u^{'}(1)) - 2$. By Proposition 3.3.8, it then follows that $\dim(V_u(1)) \leq \ell^2 - 1$, where equality holds if and only if $\dim(V_u^{'}(1)) = \ell^2 + 1$, hence, if and only if the Jordan form of u on W is $J_2 \oplus J_1^{\ell-1}$.

We can assume that $\alpha > 0$. Then, by item (3) of Remark 3.3.7, we have $\dim(V_u(1)) \leq \dim(V_u^{'}(1))$. Moreover, since $\alpha > 0$, the Jordan form of u on W is different than $J_2 \oplus J_1^{\ell-1}$ and therefore, by Proposition 3.3.10, we have $\dim(V_u^{'}(1)) \leq \ell^2 - 1$, hence $\dim(V_u(1)) \leq \ell^2 - 1$. Now, in order for $\dim(V_u(1)) = \ell^2 - 1$, we must have $\dim(V_u^{'}(1)) = \ell^2 - 1$.

Assume that $\dim(V_u^{'}(1)) = \ell^2 - 1$. Then, by Proposition 3.3.10, either $\ell = 2$ and the Jordan form of u on W is J_3, or $\ell = 3$ and the Jordan form of u on W is J_2^2. In the first case, since $p = 3$, $\alpha = 1$ and $p \nmid \frac{\ell+1}{p^\alpha}$, by item (3.2) of Remark 3.3.7, we determine that $\dim(V_u(1)) = \dim(V_u^{'}(1)) = 3$. Similarly, in the second case, since $p = 2$ and $\alpha = 1$, we have $p \mid \frac{\ell+1}{p^\alpha}$ and so $\dim(V_u(1)) = \dim(V_u^{'}(1)) = 8$, by item (3.1) of Remark 3.3.7.

In conclusion, we have shown that $\dim(V_u(1)) \leq \ell^2 - 1$ for all non-identity unipotent elements $u \in G$. In particular, since the inequality $0 < 3\ell^2 - 2\ell + 1$ holds for all $\ell \geq 2$, it follows that $\ell^2 - 1 < \ell^2 + 2\ell - 1 - \sqrt{\ell^2 + 2\ell - 1}$ for all $\ell \geq 2$, and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

To conclude this subsection, we remark that Lemma 3.3.3, Propositions 3.3.4 and 3.3.5, and Corollaries 3.3.9 and 3.3.11 give the proof of Theorems 3.3.1 and 3.3.2 for the families of kG-modules corresponding to p-restricted dominant weights $\lambda \in F^{A_\ell}$.

3.3.2 The particular modules

As previously mentioned, in this subsection, we will prove Theorems 3.3.1 and 3.3.2 in the case of the particular kG-modules. In the first part, we determine $\dim(V_u(1))$, where $u \in G$ is a non-identity unipotent element, for the irreducible kG-module $V = L_G(m\omega_1)$, where $3 \leq m \leq 8$ and $p = 0$ or $p > m$, of the simple simply connected linear algebraic group G of type A_1, see Proposition 3.3.12. Afterwards, we assume that $\ell \geq 2$ and we establish an upper-bound for $\dim(V_u(1))$, where $u \in G$ is a non-identity unipotent element and $V = L_G(\lambda)$, where the p-restricted dominant weight λ appears in Table 2.7.1, see Propositions 3.3.16, 3.3.17, 3.3.18 and 3.3.19, respectively.

Proposition 3.3.12. *Let k be an algebraically closed field of characteristic $p = 0$ or $p > m$. Let $\ell = 1$ and let $V = L_G(m\omega_1)$, where $3 \leq m \leq 8$. Lastly, let $u \in G$ be a non-identity unipotent element. Then*

$$\dim(V_u(1)) = 1.$$

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all $3 \leq m \leq 8$ and for all non-identity unipotent elements $u \in G$.

Proof. The result is proven in [Sup95, Theorem 1.9]. □

We now assume that $\ell \geq 3$ and we focus on the irreducible kG-modules V with highest weights featured in Table 2.7.1. In order to determine $\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, where G_u is the set of unipotent elements of G, we will use the inductive algorithm presented in Subsection 2.4.4. Following this algorithm, we first determine the unipotent conjugacy classes of G and for each non-identity class we identify a representative u' with the property that $u'_L \neq 1$, where $L := L_\ell$ is the Levi subgroup of the maximal parabolic subgroup P_ℓ of G constructed in Section 2.4. By Theorem 2.9.2 we know that two unipotent elements of G are conjugate if and only if they have the same Jordan form on W. Therefore, we can label unipotent conjugacy classes in G by symbols $\bigoplus\limits_{i=1}^{m} J_{d_i}$, where $1 \leq m$, $1 \leq d_1 \leq \cdots \leq d_m$ and $\sum\limits_{i=1}^{m} d_i = \ell + 1$. Now, in order to identify a representative for each class, we use [Kor18, Lemma 2.8.8], which shows how to associate a unipotent element u to a given symbol $\bigoplus\limits_{i=1}^{m} J_{d_i}$. Before we state this lemma, we recall that, to each unipotent element $u \in G$, we have associated the subset $S_u \subseteq \Phi^+$ with the property that $u = \prod\limits_{\alpha \in S_u} x_\alpha(c_\alpha)$, where the product is taken with respect to the total order $\preceq$ on Φ, see Section 1.3, and $c_\alpha \in k^*$ for all $\alpha \in S_u$.

Lemma 3.3.13. [Kor18, Lemma 2.8.8] *Let $m \geq 1$ and let $d_1 \leq \cdots \leq d_m$ be such that $\sum_{i=1}^{m} d_i = \ell + 1$. Set $k_1 = 1$ and $k_i = 1 + \sum_{j=1}^{i-1} d_j$ for all $2 \leq i \leq m$. Moreover, for all $1 \leq i \leq m$, define:*

$$u_i = \begin{cases} 1, \text{ if } d_i = 1; \\ \prod_{j=k_i}^{k_i+d_i-2} x_{\alpha_j}(1), \text{ if } d_i > 1. \end{cases}$$

Then $u = u_1 \cdots u_m$ lies in the unipotent G-conjugacy class labeled by $\bigoplus_{i=1}^{m} J_{d_i}$.

In the following two results, Remark 3.3.14 and Proposition 3.3.15, we will show that each non-identity unipotent conjugacy class admits a representative u' with the property that $S_{u'} \cap \{\alpha_1, \ldots, \alpha_{\ell-1}\} \neq \emptyset$.

Remark 3.3.14. *Let $\bigoplus_{i=1}^{m} J_{d_i}$ be the label of a nontrivial unipotent G-conjugacy class. Then, there exists $1 \leq j \leq m$ such that $d_{j-1} = 1$ and $d_j > 1$, where we set $d_0 := 1$. Now, by Lemma 3.3.13, this class admits a representative $u = u_1 \cdots u_m$, where $u_i = 1$, for all $1 \leq i \leq j-1$, and $u_i = \prod_{j=k_i}^{k_i+d_i-2} x_{\alpha_j}(1)$, for all $j \leq i \leq m$. Since $S_{u_i} = \{\alpha_{k_i}, \ldots, \alpha_{k_i+d_i-2}\}$ for all $j \leq i \leq m$, it follows that for any $\alpha_r \in S_{u_i}$ and any $\alpha_q \in S_{u_j}$, where $1 \leq i < j \leq m$, we have $r < q$ and so $S_{u_i} \cap S_{u_j} = \emptyset$ for all $1 \leq i < j \leq m$. Therefore, $S_u = \{\alpha_{k_j}, \ldots, \alpha_{k_j+d_j-2}, \alpha_{k_j+d_j}, \ldots, \alpha_{k_j+d_j+d_{j+1}-2}, \ldots, \alpha_{k_j+d_j+\cdots+d_{m-1}}, \ldots, \alpha_\ell\}$, where $1 \leq j \leq m$ is such that $d_{j-1} = 1$ and $d_j > 1$.*

Proposition 3.3.15. *Let $\ell \geq 2$. Then, each nontrivial unipotent conjugacy class in G admits a representative u' with the property that $S_{u'} \cap \{\alpha_1, \ldots, \alpha_{\ell-1}\} \neq \emptyset$.*

Proof. Let $\bigoplus_{i=1}^{m} J_{d_i}$, where $1 \leq m$ and $1 \leq d_1 \leq \cdots \leq d_m$ are such that $\sum_{i=1}^{m} d_i = \ell + 1$, be the label of a nontrivial unipotent conjugacy class in G. Let $1 \leq j \leq m$ be such that $d_{j-1} = 1$ and $d_j > 1$, where we set $d_0 := 1$. Let u' be the representative of this class given by Lemma 3.3.13. We have seen in Remark 3.3.14 that $S_{u'} = \{\alpha_{k_j}, \ldots, \alpha_{k_j+d_j-2}, \alpha_{k_j+d_j}, \ldots, \alpha_{k_j+d_j+d_{j+1}-2}, \ldots, \alpha_{k_j+d_j+\cdots+d_{m-1}}, \ldots, \alpha_\ell\}$. If $j < m$, then $\alpha_{k_j} \in S_{u'}$, where $k_j < k_m \leq k_m + d_m - 2 = \ell$, as $d_m \geq d_j > 1$, and so $S_{u'} \cap \{\alpha_1, \ldots, \alpha_{\ell-1}\} \neq \emptyset$. We can thus assume that $j = m$. If $d_m > 2$, then $k_m < k_m + d_m - 2 = \ell$, and, as $\alpha_{k_m} \in S_{u'}$, we determine that $S_{u'} \cap \{\alpha_1, \ldots, \alpha_{\ell-1}\} \neq \emptyset$. Lastly, we consider the case when $d_m = 2$. As $d_i = 1$ for all $1 \leq i \leq m-1$ and as $\sum_{i=1}^{m} d_i = \ell + 1$, it follows that $m = \ell$, thereby the label of the unipotent conjugacy class of $u' = u_\ell = x_{\alpha_\ell}(1)$ is $\underbrace{J_1 \oplus \cdots \oplus J_1}_{m-1} \oplus J_2$. In this case, since u' and $x_{\alpha_1}(1)$ are G-conjugate (they have the same Jordan form on W), we choose $x_{\alpha_1}(1)$ as representative of this class and note that $S_{x_{\alpha_1}(1)} \cap \{\alpha_1, \ldots, \alpha_{\ell-1}\} \neq \emptyset$. This completes the proof of the proposition. $\square$

Having finalized the first step of the algorithm, we will now begin the process of determining an upper-bound for $\dim(V_u(1))$, where $u \in G$ is a non-identity unipotent element and $V = L_G(\lambda)$ with λ a p-restricted dominant weight listed in Table 2.7.1. Recall that we have denoted by L the Levi subgroup L_ℓ of the maximal parabolic subgroup $P := P_\ell$ of G constructed in Section 2.4. Set $Q := R_u(P_\ell)$. Note that, by Proposition 3.3.15, we know that each non-identity unipotent G-conjugacy class admits a representative u' with the property that $u'_L \neq 1$.

Proposition 3.3.16. *Let k be an algebraically closed field of characteristic $p = 3$, let $\ell = 3$ and let $V = L_G(\omega_1 + \omega_2)$. Then for all non-identity unipotent elements $u \in G$, we have*

$$\dim(V_u(1)) \leq 8.$$

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (3.22) of Proposition 3.2.10 which states:

$$V\mid_{[L,L]} \cong L_L(\omega_1 + \omega_2) \oplus L_L(2\omega_1) \oplus L_L(\omega_1).$$

Let $u \in G$ be a non-identity unipotent element and let u' be a representative of the unipotent G-conjugacy class of u with the property that $u'_L \neq 1$, see Proposition 3.3.15. Then, as $\dim(V_u(1)) = \dim(V_{u'}(1))$, by Inequality (2.7) and Decomposition (3.22), we determine that

$$\dim(V_u(1)) \leq \dim(V_{u'_L}(1)) = \dim((L_L(\omega_1 + \omega_2))_{u'_L}(1)) + \dim((L_L(2\omega_1))_{u'_L}(1)) + \dim((L_L(\omega_1))_{u'_L}(1)).$$

Now, as $p = 3$, by Corollary 3.3.11, we have $\dim((L_L(\omega_1 + \omega_2))_{u'_L}(1)) \leq 3$. Similarly, by Proposition 3.3.5, we have $\dim((L_L(2\omega_1))_{u'_L}(1)) \leq 3$, while by Lemma 3.3.3, it follows that $\dim((L_L(\omega_1))_{u'_L}(1)) \leq 2$. We determine that $\dim(V_u(1)) \leq 8$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 8 < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 3.3.17. *Let $\ell = 5$ and let $V = L_G(\omega_3)$. Then for all non-identity unipotent elements $u \in G$, we have*

$$\dim(V_u(1)) \leq 14,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (3.23) of Proposition 3.2.11, which states:

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_2).$$

Let $u \in G$ be a non-identity unipotent element and let u' be a representative of the unipotent G-conjugacy class of u with the property that $u'_L \neq 1$, see Proposition 3.3.15. Then, as $\dim(V_u(1)) = \dim(V_{u'}(1))$, by Inequality (2.7) and Decomposition (3.23), we get

$$\dim(V_u(1)) \leq \dim(V_{u'_L}(1)) = \dim((L_L(\omega_3))_{u'_L}(1)) + \dim((L_L(\omega_2))_{u'_L}(1)).$$

Now, using Proposition 3.3.4 and keeping in mind that $L_L(\omega_3) \cong L_L(\omega_2)^*$, we determine that $\dim((L_L(\omega_2))_{u'_L}(1)) \leq 7$ and $\dim((L_L(\omega_3))_{u'_L}(1)) \leq 7$, therefore $\dim(V_u(1)) \leq 14$.

Lastly, we consider the unipotent element $x_{\alpha_1}(1) \in G$. We first note that $(x_{\alpha_1}(1))_L = x_{\alpha_1}(1)$ and $(x_{\alpha_1}(1))_Q = 1$. Therefore, by Equality (2.8) and Decomposition (3.23), we have $\dim(V_{x_{\alpha_1}(1)}(1)) = \dim((L_L(\omega_2))_{x_{\alpha_1}(1)}(1)) + \dim((L_L(\omega_3))_{x_{\alpha_1}(1)}(1))$, thus $\dim(V_{x_{\alpha_1}(1)}(1)) = 14$, by [LS12, Subsection 3.3.2] and Proposition 3.3.4. This shows that there exist unipotent elements $u \in G$ for which $\dim(V_u(1)) = 14$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 14$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound in attained, for example $x_{\alpha_1}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 3.3.18. *Let $\ell = 6$ and let $V = L_G(\omega_3)$. Then for all non-identity unipotent elements $u \in G$, we have*

$$\dim(V_u(1)) \leq 25,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (3.24) of Proposition 3.2.13 which states:

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_2).$$

Let $u \in G$ be a non-identity unipotent element and let u' be a representative of the unipotent G-conjugacy class of u with the property that $u'_L \neq 1$, see Proposition 3.3.15. Then, as $\dim(V_u(1)) = \dim(V_{u'}(1))$, by Inequality (2.7) and identity (3.24), we get

$$\dim(V_u(1)) \leq \dim(V_{u'_L}(1)) = \dim((L_L(\omega_3))_{u'_L}(1)) + \dim((L_L(\omega_2))_{u'_L}(1)).$$

Now, by Proposition 3.3.4, we have $\dim((L_L(\omega_2))_{u'_L}(1)) \leq 11$ and, similarly, by Proposition 3.3.17, we have $\dim((L_L(\omega_3))_{u'_L}(1)) \leq 14$. It follows that $\dim(V_u(1)) \leq 25$.

Lastly, we consider the unipotent element $x_{\alpha_1}(1) \in G$. We first note that $(x_{\alpha_1}(1))_L = x_{\alpha_1}(1)$ and $(x_{\alpha_1}(1))_Q = 1$. Therefore, by Equality (2.8) and Decomposition (3.24), we have $\dim(V_{x_{\alpha_1}(1)}(1)) = \dim((L_L(\omega_2))_{x_{\alpha_1}(1)}(1)) + \dim((L_L(\omega_3))_{x_{\alpha_1}(1)}(1))$, thus $\dim(V_{x_{\alpha_1}(1)}(1)) = 25$, by [LS12, Subsection 3.3.2] and Propositions 3.3.4 and 3.3.17. This shows that there exist unipotent elements $u \in G$ for which $\dim(V_u(1)) = 25$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 25$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound in attained, for example $x_{\alpha_1}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 3.3.19. *Let $\ell = 7$ and let $V = L_G(\omega_3)$. Then for all non-identity unipotent elements $u \in G$, we have*

$$\dim(V_u(1)) \leq 41,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (3.25) of Proposition 3.2.14 which states

$$V\mid_{[L,L]}\cong L_L(\omega_3)\oplus L_L(\omega_2).$$

Let $u\in G$ be a non-identity unipotent element and let $u^{'}$ be a representative of the unipotent G-conjugacy class of u with the property that $u'_L\neq 1$, see Proposition 3.3.15. Then, as $\dim(V_u(1))=\dim(V_{u'}(1))$, by Inequality (2.7) and Identity (3.25), we get

$$\dim(V_u(1))\leq\dim(V_{u'_L}(1))=\dim((L_L(\omega_3))_{u'_L}(1))+\dim((L_L(\omega_2))_{u'_L}(1)).$$

Now, by Proposition 3.3.4, we have $\dim((L_L(\omega_2))_{u'_L}(1))\leq 16$, while, by Proposition 3.3.18, we have $\dim((L_L(\omega_3))_{u'_L}(1))\leq 25$. It follows that $\dim(V_u(1))\leq 41$.

Lastly, we consider the unipotent element $x_{\alpha_1}(1)\in G$. We first note that $(x_{\alpha_1}(1))_L=x_{\alpha_1}(1)$ and $(x_{\alpha_1}(1))_Q=1$. Therefore, by Equality (2.8) and Decomposition (3.25), we have $\dim(V_{x_{\alpha_1}(1)}(1))=\dim((L_L(\omega_2))_{x_{\alpha_1}(1)}(1))+\dim((L_L(\omega_3))_{x_{\alpha_1}(1)}(1))$, thus $\dim(V_{x_{\alpha_1}(1)}(1))=41$, by [LS12, Subsection 3.3.2] and Propositions 3.3.4 and 3.3.18. This shows that there exist unipotent elements $u\in G$ for which $\dim(V_u(1))=41$.

In conclusion, we have shown that $\dim(V_u(1))\leq 41$ for all non-identity unipotent elements $u\in G$ and that there exist $u\in G$ for which the bound in attained, for example $x_{\alpha_1}(1)$. Lastly, we have $\dim(V_u(1))<\dim(V)-\sqrt{\dim(V)}$ for all non-identity unipotent elements $u\in G$. □

Although the following result is not required for the proof of Theorems 3.3.1 and 3.3.2, it is a nice a generalization for all $\ell\geq 5$ of Propositions 3.3.17, 3.3.18 and 3.3.19.

Proposition 3.3.20. *Let $\ell\geq 5$ and let $V=L_G(\omega_3)$. Then, for all non-identity unipotent elements $u\in G$, we have*

$$\dim(V_u(1))\leq\frac{(\ell-1)(\ell^2-2\ell+6)}{6},$$

where there exist $u\in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1))<\dim(V)-\sqrt{\dim(V)}$ for all non-identity unipotent elements $u\in G$.

Proof. The base case for $\ell=5$ is given by Proposition 3.3.17. Thus, we assume that $\ell\geq 6$ and that the result holds for all $r<\ell$. We proceed to prove it for ℓ. For this, we recall the Decomposition (3.26) of Proposition 3.2.15 which states

$$V\mid_{[L_\ell,L_\ell]}\cong L_{L_\ell}(\omega_3)\oplus L_{L_\ell}(\omega_2).$$

Let $u\in G$ be a non-identity unipotent element and let $u^{'}$ be a representative of the unipotent G-conjugacy class of u with the property that $u'_L\neq 1$, see Proposition 3.3.15. Then, as $\dim(V_u(1))=\dim(V_{u'}(1))$, by Inequality (2.7) and identity (3.26), we get

$$\dim(V_u(1))\leq\dim(V_{u'_{L_\ell}}(1))=\dim((L_{L_\ell}(\omega_3))_{u'_{L_\ell}}(1))+\dim((L_{L_\ell}(\omega_2))_{u'_{L_\ell}}(1)).$$

Now, by Proposition 3.3.4, we have $\dim((L_{L_\ell}(\omega_2))_{u'_{L_\ell}}(1))\leq\frac{(\ell-1)^2-(\ell-1)+2}{2}$, while, by induction, we have $\dim((L_{L_\ell}(\omega_3))_{u'_{L_\ell}}(1))\leq\frac{(\ell-2)[(\ell-1)^2-2(\ell-1)+6]}{6}$. It follows that $\dim(V_u(1))\leq\frac{(\ell-1)(\ell^2-2\ell+6)}{6}$.

Lastly, we consider the unipotent element $x_{\alpha_1}(1) \in G$. We first note that $(x_{\alpha_1}(1))_{L_\ell} = x_{\alpha_1}(1)$ and $(x_{\alpha_1}(1))_{Q_\ell} = 1$. Therefore, by Equality (2.8) and Decomposition (3.26), we have $\dim(V_{x_{\alpha_1}(1)}(1)) = \dim((L_{L_\ell}(\omega_2))_{x_{\alpha_1}(1)}(1)) + \dim((L_{L_\ell}(\omega_3))_{x_{\alpha_1}(1)}(1))$, thus $\dim(V_{x_{\alpha_1}(1)}(1)) = \frac{(\ell-1)(\ell^2-2\ell+6)}{6}$, by [LS12, Subsection 3.3.2], Proposition 3.3.4 and induction. This shows that there exist unipotent elements $u \in G$ for which $\dim(V_u(1)) = \frac{(\ell-1)(\ell^2-2\ell+6)}{6}$.

In conclusion, we have shown that $\dim(V_u(1)) \leq \frac{(\ell-1)(\ell^2-2\ell+6)}{6}$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound in attained, for example $x_{\alpha_1}(1)$. Moreover, as the inequality $0 < 3\ell^3 - 17\ell^2 + 22\ell - 12$ holds for all $\ell \geq 5$, we have that $\frac{(\ell-1)(\ell^2-2\ell+6)}{6} < \frac{(\ell-1)\ell(\ell+1)}{6} - \sqrt{\frac{(\ell-1)\ell(\ell+1)}{6}}$ for all $\ell \geq 5$, and so $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

We conclude this subsection by noting that Proposition 3.3.12 completes the proof of Theorems 3.3.1 and 3.3.2 for simple simply connected linear algebraic groups of type A_1. Furthermore, Propositions 3.3.16, 3.3.17, 3.3.18 and 3.3.19 treat all the irreducible kG-modules, where G is of type A_ℓ with $\ell \geq 2$, corresponding to p-restricted dominant weights featured in Table 2.7.1. This completes the proofs of Theorems 3.3.1 and 3.3.2.

3.4 Results

In this section we collect the results proven in this chapter. In Proposition 3.4.1 we give the values of $\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ and $\nu_G(V)$ for all kG-modules V belonging to one of the families we had to consider.

Proposition 3.4.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$. Let T be a fixed maximal torus in G and let $V = L_G(\lambda)$, where $\lambda \in F^{A_\ell}$. Then the value of $\nu_G(V)$ is as given in the table below:*

V	Char.	Rank	$\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$	$\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$	$\nu_G(V)$
$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 1$	ℓ	ℓ	1
$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 1$	$\frac{\ell^2+\ell+2}{2}$	$\frac{\ell(\ell+1)}{2}$	ℓ
$L_G(\omega_2)$	$p \geq 0$	$\ell = 3$	4	4	2
		$\ell \geq 4$	$\frac{\ell(\ell-1)}{2}$	$\frac{\ell^2-\ell+2}{2}$	$\ell - 1$
$L_G(\omega_1 + \omega_\ell)$	$p \nmid \ell + 1$	$\ell \geq 2$	ℓ^2	ℓ^2	2ℓ
$L_G(\omega_1 + \omega_\ell)$	$p \mid \ell + 1$	$\ell = 2$	4	3	3
		$\ell \geq 3$	$\ell^2 - 1$	$\ell^2 - 1$	2ℓ

Table 3.4.1: The value of $\nu_G(V)$ for the families of modules of groups of type A_ℓ.

Proof. The result follows by Proposition 2.2.3 from Lemmas 3.2.3 and 3.3.3, for $V = L_G(\omega_1)$; Propositions 3.2.4 and 3.3.5, for $V = L_G(2\omega_1)$; Propositions 3.2.5 and 3.3.4, for $V = L_G(\omega_2)$; and Corollaries 3.2.7, 3.2.8, 3.3.9 and 3.3.11, for $V = L_G(\omega_1 + \omega_\ell)$. □

Similarly, Proposition 3.4.2 records the values of $\max_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu \in k^*\}$, $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ and $\nu_G(V)$ for all the particular kG-modules treated in this chapter.

Proposition 3.4.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ, $\ell \geq 1$. Let T be a fixed maximal torus in G. If $\ell = 1$, let $V = L_G(m\omega_1)$ with $3 \leq m \leq 8$ and $p = 0$ or $p > m$. If $\ell \geq 2$, let $V = L_G(\lambda)$, where λ is featured in Table 2.7.1. Then the value of $\nu_G(V)$ is given in the table below:*

Group	$L_G(\lambda)$	Char.	$\max_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu \in k^*\}$	$\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$	$\nu_G(V)$
A_1	$L_G(m\omega_1)$ $3 \leq m \leq 8$	$p = 0$, or $p > m$	$1 + \lfloor \frac{m}{2} \rfloor$	1	$m - \lfloor \frac{m}{2} \rfloor$
A_3	$L_G(\omega_1 + \omega_2)$	$p = 3$	10	≤ 8	6
A_5	$L_G(\omega_3)$	$p \geq 0$	12	14	6
A_6	$L_G(\omega_3)$	$p \geq 0$	20	25	10
A_7	$L_G(\omega_3)$	$p \geq 0$	35	41	15

Table 3.4.2: The value of $\nu_G(V)$ for the particular modules of groups of type A_ℓ.

Proof. The result follows by Proposition 2.2.3, using the detailed results of Subsections 3.2.2 and 3.3.2. □

Lastly, we state the following additional result, whose proof follows by Propositions 2.2.3, 3.2.15 and 3.3.20.

Proposition 3.4.3. *Let $\ell \geq 6$ and $V = L_G(\omega_3)$. Then $\max_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu \in k^*\} = \dfrac{(\ell-2)(\ell-1)\ell}{6}$, $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1)) = \dfrac{(\ell-1)(\ell^2-2\ell+6)}{6}$ and $\nu_G(V) = \frac{(\ell-2)(\ell-1)}{2}$.*

Chapter 4

Groups of type C_ℓ

In this chapter we prove Theorems 1.1.1 and 1.1.3 for the simple simply connected linear algebraic groups of type C_ℓ, $\ell \geq 2$. The structure is as follows: in the first section we construct such a group and exhibit some properties of its semisimple and unipotent elements. In Section 4.2 we determine $\max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, where V runs through the list of kG-modules we identified in Subsection 2.7.2. Similarly, in Section 4.3, we determine $\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, where G_u is the set of unipotent elements of G, for the same kG-modules V. Lastly, Section 4.4 records all the results of this chapter.

We now fix some notation which will be used throughout this chapter. The field k is an algebraically closed field of characteristic $p \geq 0$, unless otherwise specified, and the group G is a simple simply connected linear algebraic group of type C_ℓ, $\ell \geq 2$. By T, Φ, B, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\omega_1, \omega_2, \ldots, \omega_\ell$ we denote a fixed maximal torus in G, the root system of G determined by T, the positive Borel subgroup of G, the set of simple roots in Φ given by B, and the fundamental dominant weights of G corresponding to Δ.

4.1 Construction of linear algebraic groups of type C_ℓ

Let W be a 2ℓ-dimensional, $\ell \geq 2$, k-vector space equipped with a nondegenerate alternating bilinear form b. We fix $B_W = \{u_1, u_2, \ldots, u_\ell, v_\ell, \ldots, v_2, v_1\}$ to be an ordered basis in W with the property that $W = \bigoplus\limits_{i=1}^{\ell} \langle u_i, v_i \rangle$ is an orthogonal direct sum, where $\{u_i, v_i\}$, $1 \leq i \leq \ell$, is a hyperbolic pair, see Theorem 2.1.1. Let D, respectively U, denote the set of diagonal, respectively upper-triangular, matrices in $\mathrm{GL}(W)$. Set $G = \mathrm{Sp}(W)$ and note that G is a simple simply connected linear algebraic group of type C_ℓ, see [Car89, p.184]. Moreover, $B := U \cap G$ is a Borel subgroup of G which contains the maximal torus $T =: D \cap G$ of G.

4.1.1 Semisimple elements

Let $s \in T$, $s = \mathrm{diag}(a_1, \ldots, a_\ell, a_\ell^{-1}, \ldots, a_1^{-1})$ with $a_i \in k^*$ for all $1 \leq i \leq \ell$. Let $\{\mu_1, \mu_2, \ldots, \mu_m\}$, where $1 \leq m \leq \ell$, be the set of distinct a_i's, and let n_i, $1 \leq i \leq m$, be the multiplicity

of each μ_i in s. It follows that $\sum_{i=1}^{m} n_i = \ell$ and we can assume, without loss of generality, that $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$. By conjugating s by an element of $\mathrm{N}_G(T)$, we can also assume that $s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \mu_2 \cdot \mathrm{I}_{n_2}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \mu_2^{-1} \cdot \mathrm{I}_{n_2}, \mu_1^{-1} \cdot \mathrm{I}_{n_1})$.

Lemma 4.1.1. *Assume there exists $1 \leq i < j \leq m$ such that $\mu_j = \mu_i^{-1}$. Then there exists $g \in G$ such that*

$$gsg^{-1} = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix},$$

where

$$A = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \ldots, \mu_{i-1} \cdot \mathrm{I}_{n_{i-1}}, \mu_i \cdot \mathrm{I}_{n_i+n_j}, \mu_{i+1} \cdot \mathrm{I}_{n_{i+1}}, \ldots, \mu_{j-1} \cdot \mathrm{I}_{n_{j-1}}, \mu_{j+1} \cdot \mathrm{I}_{n_{j+1}}, \ldots, \mu_m \cdot \mathrm{I}_{n_m})$$

and $A^* = (A^*_{i,j})_{i,j}$ is a diagonal matrix with $A^*_{r,r} = A^{-1}_{n_1+\cdots+n_m+1-r, n_1+\cdots+n_m+1-r}$, for all $1 \leq r \leq n_1 + \cdots + n_m$.

Proof. For any $r \geq 1$, set K_r to be the $r \times r$ matrix $K_r := \begin{pmatrix} 0 & \cdots & 1 \\ & \iddots & \\ 1 & \cdots & 0 \end{pmatrix}$. We now consider the element $g_1 \in \mathrm{SL}(W)$ given by $g_1 = \begin{pmatrix} \mathrm{I}_a & 0 & \cdots & \cdots & 0 \\ 0 & 0_{n_j} & \cdots & K_{n_j} & \vdots \\ \vdots & 0 & \mathrm{I}_{2b} & 0 & \vdots \\ \vdots & -K_{n_j} & \cdots & 0_{n_j} & 0 \\ 0 & \cdots & \cdots & 0 & \mathrm{I}_a \end{pmatrix}$, where $a = \sum_{i=1}^{j-1} n_i$ and $b = \sum_{i=j+1}^{m} n_j$. We calculate and determine that

$$g_1^{-1} = g_1^{\mathrm{tr}} = \begin{pmatrix} \mathrm{I}_a & 0 & \cdots & \cdots & 0 \\ 0 & 0_{n_j} & \cdots & -K_{n_j} & \vdots \\ \vdots & \vdots & \mathrm{I}_{2b} & \vdots & \vdots \\ \vdots & K_{n_j} & \cdots & 0_{n_j} & 0 \\ 0 & \cdots & \cdots & 0 & \mathrm{I}_a \end{pmatrix}.$$

We denote by $[b]$ the representing matrix of b with respect to the basis B_W. By Theorem 2.1.1, we have that $[b] = \begin{pmatrix} 0 & K_\ell \\ -K_\ell & 0 \end{pmatrix}$. One calculates and determines that $g_1^{\mathrm{tr}}[b]g_1 = [b]$, hence $g_1 \in G$, and

$$\begin{aligned} g_1 s g_1^{-1} = \mathrm{diag}(&\mu_1 \cdot \mathrm{I}_{n_1}, \ldots, \mu_{j-1} \cdot \mathrm{I}_{n_{j-1}}, \mu_i \cdot \mathrm{I}_{n_j}, \mu_{j+1} \cdot \mathrm{I}_{n_{j+1}}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \\ &\mu_{j+1}^{-1} \cdot \mathrm{I}_{n_{j+1}}, \mu_i^{-1} \cdot \mathrm{I}_{n_j}, \mu_{j-1}^{-1} \cdot \mathrm{I}_{n_{j-1}}, \ldots, \mu_1^{-1} \cdot \mathrm{I}_{n_1}). \end{aligned}$$

Finally, reordering as before, we deduce that there exists $g \in G$ such that gsg^{-1} has the desired matrix form. $\square$

Now, let $s \in T$, $s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \mu_1^{-1} \cdot \mathrm{I}_{n_1})$ with $\mu_i \neq \mu_j$ for all $1 \leq i < j \leq m$. By Lemma 4.1.1, we may further assume that $\mu_i \neq \mu_j^{-1}$ for all $1 \leq i < j \leq m$. For the rest of this chapter, we fix the following hypothesis on semisimple elements of G:

$(^{\dagger}H_s)$: any $s \in T \setminus \mathrm{Z}(G)$ is such that $s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \mu_1^{-1} \cdot \mathrm{I}_{n_1})$,

where $\mu_i \neq \mu_j^{\pm 1}$ for all $1 \leq i < j \leq m$, $\displaystyle\sum_{i=1}^{m} n_i = \ell$ and $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$.

Moreover, if $m = 1$, then $\mu_1 \neq \pm 1$.

4.1.2 Unipotent elements

Let u be a unipotent element of G and let $\displaystyle\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ be its Jordan form on W, where $\displaystyle\sum_{i=1}^{m} n_i r_i = 2\ell$ and $r_i \geq 1$ is even for all odd n_i, see Theorem 2.9.2. If $p \neq 2$, we know that the conjugacy class of u in G is completely determined by its Jordan form on W. However, when $p = 2$, the Jordan form is no longer enough to characterize conjugacy classes in G and so we will use the Hesselink normal form to distinguish between unipotent conjugacy classes. Now, by Theorem 2.9.15, the Hesselink normal form of u is $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots, n_{m_1}^{r_m})$, where $m \geq 1$, $t \geq 0$ and $r_i \geq 1$ is even for all odd n_i, see Lemma 2.9.13.

In conclusion, regardless of the characteristic of k, if u is a unipotent element of G with Jordan form $\displaystyle\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ on W, then $\displaystyle\sum_{i=1}^{m} r_i n_i = 2\ell$, r_i is even for all odd n_i and, moreover, we can assume, without loss of generality, that $2\ell \geq n_1 > \cdots > n_m \geq 1$.

4.2 Eigenspace dimensions for semisimple elements

Before we state the main results of this section, we recall that $F^{C_\ell} = \{\omega_1, 2\omega_1, \omega_2\}$, see Subsection 2.7.2.

Theorem 4.2.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ, $\ell \geq 2$. Let T be a fixed maximal torus in G and let $V = L_G(\lambda)$, where $\lambda \in F^{C_\ell}$ or λ is given in Tables 2.7.2 and 2.7.3. Then, there exist $s \in T \setminus \mathrm{Z}(G)$ and $\mu \in k^*$, an eigenvalue of s on V, such that*

$$\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$$

if and only if ℓ, λ and p appear in the following list:

(1) *$\ell \geq 2$, $\lambda = \omega_1$ and $p \geq 0$;*

(2) *$\ell = 2$, $\lambda = \omega_2$ and $p \geq 0$.*

Theorem 4.2.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ, $\ell \geq 2$. Let T and V be as in Theorem 4.2.1. Then the value of $\displaystyle\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ is given in the table below:*

V	Char.	Rank	$\max\limits_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$
$L_G(\omega_1)$	$p\geq 0$	$\ell\geq 2$	$2\ell-2$
${}^{\dagger}L_G(2\omega_1)$	$p\neq 2$	$\ell\geq 2$	$2\ell^2-3\ell+4$
${}^{\star}L_G(\omega_2)$	$p\nmid \ell$	$\ell=2,3,4$	$2\ell^2-6\ell+8$
		$\ell\geq 5$	$2\ell^2-5\ell+3$
	$p\mid \ell$	$\ell=2,3$	$2\ell^2-4\ell+2$
		$\ell\geq 4$	$2\ell^2-5\ell+2$
${}^{\dagger}L_G(\omega_1+\omega_2)$	$p\geq 0$	$\ell=2$	$8-2\delta_{p,5}$
${}^{\dagger}L_G(2\omega_2)$	$p\neq 2$	$\ell=2$	$10-\delta_{p,5}$
${}^{\dagger}L_G(3\omega_1)$	$p\neq 2,3$	$\ell=2$	10
${}^{\dagger}L_G(\omega_1+2\omega_2)$	$p=7$	$\ell=2$	12
${}^{\dagger}L_G(3\omega_2)$	$p=7$	$\ell=2$	16
${}^{\dagger}L_G(2\omega_1+\omega_2)$	$p=3$	$\ell=2$	16
${}^{\dagger}L_G(\omega_\ell)$	$p=2$	$3\leq\ell\leq 8$	$2^{\ell-1}$
${}^{\dagger}L_G(\omega_3)$	$p\neq 2$	$\ell=3$	10
${}^{\dagger}L_G(\omega_1+\omega_3)$	$p=2$	$\ell=3$	≤ 24
${}^{\dagger}L_G(2\omega_1+\omega_3)$	$p=2$	$\ell=3$	20
${}^{\dagger}L_G(\omega_3)$	$p\geq 0$	$\ell=4$	$\leq 30-4\delta_{p,3}$
${}^{\dagger}L_G(\omega_4)$	$p\neq 2$	$\ell=4$	28
${}^{\dagger}L_G(\omega_3)$	$p=2$	$\ell=5$	≤ 58

Table 4.2.1: The value of $\max\limits_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$.

In particular, for each V in Table 4.2.1 labeled as ${}^{\dagger}V$, respectively as each ${}^{\star}V$ with $\ell\geq 3$, we have $\dim(V_s(\mu))<\dim(V)-\sqrt{\dim(V)}$ for all $s\in T\setminus \mathrm{Z}(G)$ and all eigenvalues $\mu\in k^*$ of s on V.

We will give the proof of Theorems 4.2.1 and 4.2.2 in a series of results, each treating one of the candidate-modules. In Subsection 4.2.1, we determine $\max\limits_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$, where V belongs to one of the families of modules, i.e. V is an irreducible kG-module $L_G(\lambda)$ with p-restricted dominant weight $\lambda\in F^{C_\ell}$. In Subsection 4.2.2, we determine $\max\limits_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$ for the irreducible kG-modules $V=L_G(\lambda)$ with p-restricted dominant weight λ featured in one of the Tables 2.7.2 and 2.7.3.

4.2.1 The families of modules

Lemma 4.2.3. *Let $V=L_G(\omega_1)$. Then for all $s\in T\setminus \mathrm{Z}(G)$ and all eigenvalues $\mu\in k^*$ of s on V we have*

$$\dim(V_s(\mu))\leq 2\ell-2,$$

where equality holds if and only if $\mu=\pm 1$ and, up to conjugation, $s=\mathrm{diag}(\pm 1,\ldots,\pm 1,d,d^{-1},\pm 1,\ldots,\pm 1)$ with $d\neq\pm 1$.

In particular, there exist $s\in T\setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu\in k^*$ on V for which $\dim(V_s(\mu))\geq\dim(V)-\sqrt{\dim(V)}$.

Proof. We first note that $V \cong W$ as kG-modules, hence $\dim(V) = 2\ell$. Now, let $s \in T \setminus \mathrm{Z}(G)$ and let $\mu \in k^*$ be an eigenvalue of s on V with $\mu \neq \mu^{-1}$. Then, $\dim(V_s(\mu)) \leq \frac{\dim(V)}{2} = \ell$, as $\dim(V_s(\mu)) = \dim(V_s(\mu^{-1}))$. On the other hand, for $\mu = \pm 1$, as $s \notin \mathrm{Z}(G)$, we have $\dim(V_s(\mu)) \leq 2\ell - 2$. Now equality holds if and only if, up to conjugation, $s = \mathrm{diag}(\pm 1, \ldots, \pm 1, d, d^{-1}, \pm 1, \ldots, \pm 1)$ with $d \neq \pm 1$, as in the statement of the result.

In conclusion, we proved that $\dim(V_s(1)) \leq 2\ell - 2$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. Now, as the inequality $2 \leq \sqrt{2\ell}$ holds for all $\ell \geq 2$, it follows that $2\ell - 2 \geq 2\ell - \sqrt{2\ell}$ for all $\ell \geq 2$ and thus, we have shown that there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ with the property that $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. □

Proposition 4.2.4. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V = L_G(2\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell^2 - 3\ell + 4.$$

Furthermore, we have equality if and only if one of the following holds:

(1) *$\ell = 2$, $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1^2 = -1$.*

(2) *$\ell \geq 2$, $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, -1, -1, 1, \ldots, 1)$.*

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. By Lemma 2.8.2, since $p \neq 2$, we have that $V \cong \mathrm{S}^2(W)$ and so we deduce that $\dim(V) = 2\ell^2 + \ell$ and that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2},\ 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i+1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1},\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j,\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ 1 \text{ with multiplicity at least } \displaystyle\sum_{i=1}^{m} n_i^2. \end{cases} \tag{4.1}$$

Let $\mu \in k^*$ be an eigenvalue of s on V such that $\mu \neq \mu^{-1}$. Then:

$$\dim(V_s(\mu)) \leq \dim(V) - \dim(V_s(1)) - \dim(V_s(\mu^{-1})).$$

Since $\dim(V_s(1)) \geq \sum_{i=1}^{m} n_i^2$ and $n_i \geq 1$ for all $1 \leq i \leq m$, we have that $\dim(V_s(1)) \geq \sum_{i=1}^{m} n_i = \ell$. Furthermore, V is a self-dual module, hence $\dim(V_s(\mu)) = \dim(V_s(\mu^{-1}))$. Now, as $\ell \geq 2$, it follows that:

$$\dim(V_s(\mu)) \leq \frac{2\ell^2 + \ell - \ell}{2} = \ell^2 < 2\ell^2 - 3\ell + 4. \tag{4.2}$$

Therefore we can assume that μ is such that $\mu = \mu^{-1}$.

Let $m = 1$. Then, $n_1 = \ell$ and, since $s \notin \mathrm{Z}(G)$, we have $\mu_1 \neq \pm 1$, hence $\mu_1^2 \neq 1$. Now, by (4.1), the eigenvalues of s on V, not necessarily distinct, are μ_1^2 and μ_1^{-2}, each with multiplicity at least $\frac{\ell(\ell+1)}{2}$, and 1 with multiplicity ℓ^2. It follows that $\dim(V_s(-1)) \leq \ell^2 + \ell$. If $\ell \geq 3$, then $\dim(V_s(-1)) < 2\ell^2 - 3\ell + 4$. On the other hand, if $\ell = 2$, then $\dim(V_s(-1)) \leq 2\ell^2 - 3\ell + 4$, where equality holds if and only if $-1 = \mu_1^2$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1})$, as in (1).

We now assume that $m \geq 2$. First, we consider the eigenvalue 1 of s on V. Since $\mu_i \neq \mu_j^{\pm 1}$ for all $1 \leq i < j \leq m$, it follows that $\mu_i^{\pm 1}\mu_j^{\pm 1} \neq 1$ for all $1 \leq i < j \leq m$. By (4.1), these account for at least $4\sum_{i<j} n_i n_j$ eigenvalues of s on V which are different than 1, therefore

$$\dim(V_s(1)) \leq 2\ell^2 + \ell - 4\sum_{i<j} n_i n_j.$$

Assume $\dim(V_s(1)) \geq 2\ell^2 - 3\ell + 4$. Then:

$$\ell - 1 - \sum_{i<j} n_i n_j \geq 0. \tag{4.3}$$

Since $\ell = \sum_{i=1}^{m} n_i$, by Inequality (4.3), we have:

$$\sum_{i=1}^{m-2} n_i(1 - \sum_{i<j} n_j) + (n_{m-1} - 1)(1 - n_m) \geq 0. \tag{4.4}$$

But $\sum_{i=1}^{m-2} n_i(1 - \sum_{i<j} n_j) \leq 0$ and $(n_{m-1} - 1)(1 - n_m) \leq 0$, as $n_i \geq 1$ for all $1 \leq i \leq m$, and so Inequality (4.4) holds if and only if $m = 2$, $n_2 = 1$ and $n_1 = \ell - 1$. In this case, $\dim(V_s(1)) = 2\ell^2 - 3\ell + 4$ if and only if all eigenvalues of s on V different than $\mu_1^{\pm 1}\mu_2^{\pm 1}$ are equal to 1. Hence, by (4.1), it follows that $\mu_1^2 = \mu_2^2 = 1$, where $\mu_1 \neq \mu_2$, and so, we deduce that, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, -1, -1, 1, \ldots, 1)$, as in (2).

Finally, we consider the eigenvalue -1 of s on V. We first remark that

$$\dim(V_s(-1)) \leq 2\ell^2 + \ell - \sum_{i=1}^{m} n_i^2.$$

If $\mu_i\mu_j \neq -1$ for all $1 \leq i < j \leq m$, then also $\mu_i^{-1}\mu_j^{-1} \neq -1$ for all $1 \leq i < j \leq m$. By (4.1), these account for at least $2\sum_{i<j} n_i n_j$ additional eigenvalues of s on V different than -1 and so:

$$\begin{aligned} \dim(V_s(-1)) &\leq 2\ell^2 + \ell - \sum_{i=1}^{m} n_i^2 - 2\sum_{i<j} n_i n_j \\ &= 2\ell^2 + \ell - (\sum_{i=1}^{m} n_i)^2 \\ &= \ell^2 + \ell. \end{aligned} \tag{4.5}$$

If $\ell \geq 3$, then $\dim(V_s(-1)) < 2\ell^2 - 3\ell + 4$, while, for $\ell = 2$, we have that $\dim(V_s(-1)) \leq 2\ell^2 - 3\ell + 4$ where equality holds if and only if all eigenvalues of s on V different than 1, $\mu_1\mu_2$ and $\mu_1^{-1}\mu_2^{-1}$ are equal to -1. But then, by (4.1), it follows that $\mu_1^2 = \mu_1\mu_2^{-1}$ and so $\mu_1 = \mu_2^{-1}$, a contradiction.

We can thus assume that there exists some $1 \leq i < j \leq m$ such that $\mu_i\mu_j = -1$. In this case, we also have $\mu_i^{-1}\mu_j^{-1} = -1$. Furthermore, since the μ_i's are distinct, it follows that:

$$\begin{cases} \mu_i^2 \neq -1,\ \mu_i^{-2} \neq -1 \text{ and } \mu_j^2 \neq -1,\ \mu_j^{-2} \neq -1; \\ \mu_i\mu_r \neq -1,\ \text{for } i < r \leq m,\ r \neq j;\ \text{and } \mu_r\mu_i \neq -1,\ \text{for } 1 \leq r < i; \\ \mu_i^{-1}\mu_r^{-1} \neq -1,\ \text{for } i < r \leq m,\ r \neq j;\ \text{and } \mu_r^{-1}\mu_i^{-1} \neq -1,\ \text{for } 1 \leq r < i; \\ \mu_j\mu_r \neq -1,\ \text{for } j < r \leq m,\ \text{and } \mu_r\mu_j \neq -1,\ \text{for } 1 \leq r < j,\ r \neq i; \\ \mu_j^{-1}\mu_r^{-1} \neq -1,\ \text{for } j < r \leq m,\ \text{and } \mu_r^{-1}\mu_j^{-1} \neq -1,\ \text{for } 1 \leq r < j,\ r \neq i. \end{cases}$$

By (4.1), these account for at least $n_i(n_i+1)+n_j(n_j+1)+2(n_i+n_j)(\ell-n_i-n_j)$ additional eigenvalues of s on V which are different than -1. Thus, we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + \ell - \sum_{r=1}^{m} n_r^2 - n_i(n_i+1) - n_j(n_j+1) - 2(n_i+n_j)(\ell - n_i - n_j). \quad (4.6)$$

Assume $\dim(V_s(-1)) \geq 2\ell^2 - 3\ell + 4$. Then:

$$4\ell - 4 - \sum_{r=1}^{m} n_r^2 - n_i(n_i+1) - n_j(n_j+1) - 2(n_i+n_j)(\ell - n_i - n_j) \geq 0 \quad (4.7)$$

and therefore

$$4\ell - 4 - \sum_{r\neq i,j} n_r^2 - 2n_i^2 - 2n_j^2 - (n_i+n_j)(\ell - n_i - n_j) - (n_i+n_j) - (n_i+n_j)(\ell - n_i - n_j) \geq 0,$$

which gives:

$$\ell(4 - n_i - n_j) - \sum_{r\neq i,j} n_r^2 - (n_i - n_j)^2 - (n_i+n_j)(\ell + 1 - n_i - n_j) - 4 \geq 0. \quad (4.8)$$

We remark that $\ell + 1 > n_i + n_j$, as $\sum_{r=1}^{m} n_r = \ell$, and so $(-\sum_{r\neq i,j} n_r^2 - (n_i - n_j)^2 - (n_i+n_j)(\ell + 1 - n_i - n_j) - 4) < 0$. Therefore, by (4.8), we have:

$$\ell(4 - n_i - n_j) > 0$$

and, since $n_i \geq n_j \geq 1$, it follows that $(n_i, n_j) \in \{(2,1),(1,1)\}$. If $(n_i, n_j) = (2,1)$, then the left-hand side of Inequality (4.7) becomes:

$$4\ell - 4 - \sum_{r\neq i,j} n_r^2 - 4 - 1 - 6 - 2 - 6(\ell - 3) = 1 - 2\ell - \sum_{r\neq i,j} n_r^2 < 0,$$

while if $(n_i, n_j) = (1, 1)$ we get:

$$4\ell - 4 - \sum_{r \neq i,j} n_r^2 - 2 - 2 - 2 - 4(\ell - 2) = -2 - \sum_{r \neq i,j} n_r^2 < 0.$$

Having covered all cases, we deduce that $\dim(V_s(-1)) < 2\ell^2 - 3\ell + 4$ for all $s \in T \setminus \mathrm{Z}(G)$.

In conclusion, we showed that $\dim(V_s(\mu)) \leq 2\ell^2 - 3\ell + 4$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. In particular, since the inequality $0 < 14\ell^2 - 33\ell + 16$ holds for all $\ell \geq 2$, it follows that $2\ell^2 - 3\ell + 4 < 2\ell^2 + \ell - \sqrt{2\ell^2 + \ell}$ for all $\ell \geq 2$, thereby $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$. □

Proposition 4.2.5. *Let $V' = \wedge^2(W)$. Moreover, let $s \in T \setminus \mathrm{Z}(G)$ and let $\mu \in k^*$ be an eigenvalue of s on V'. Then one of the following holds:*

(1) *$\ell = 2$ and $\dim(V_s'(\mu)) \leq 4$, where we have equality if and only if one of the following holds:*

  (1.1) *$\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq \pm 1$.*

  (1.2) *$p \neq 2$, $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(1, -1, -1, 1)$.*

(2) *$\ell = 3$ and $\dim(V_s'(\mu)) \leq 9$, where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq \pm 1$.*

(3) *$\ell \geq 4$ and $\dim(V_s'(\mu)) \leq 2\ell^2 - 5\ell + 4$, where we have equality if and only if one of the following holds:*

  (3.1) *$\ell = 4$, $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq \pm 1$.*

  (3.2) *$\ell = 4$, $p \neq 2$, $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(1, 1, -1, -1, -1, -1, 1, 1)$.*

  (3.3) *$\ell \geq 4$, $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, \mu_2, \mu_2^{-1}, 1, \ldots, 1)$ with $\mu_2 \neq 1$.*

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. We first remark that the kG-module $V' = \wedge^2(W)$ is self-dual, see Remark 2.8.3. Secondly, we note that $\dim(V') = 2\ell^2 - \ell$ and we determine that the eigenvalues of s on V', not necessarily distinct, are:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2},\ 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i-1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1},\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j,\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ 1 \text{ with multiplicity at least } \displaystyle\sum_{i=1}^{m} n_i^2. \end{cases} \tag{4.9}$$

Let $\mu \in k^*$ be an eigenvalue of s on V' such that $\mu \neq \mu^{-1}$. Then:

$$\dim(V_s'(\mu)) \leq \dim(V') - \dim(V_s'(1)) - \dim(V_s'(\mu^{-1})).$$

Since $n_i \geq 1$ for all $1 \leq i \leq m$, we have $\dim(V_s^{'}(1)) \geq \sum_{i=1}^{m} n_i^2 \geq \sum_{i=1}^{m} n_i = \ell$. Moreover, we have that $\dim(V_s^{'}(\mu)) = \dim(V_s^{'}(\mu^{-1}))$, as $V^{'}$ is a self-dual kG-module, and so, substituting in the above yields:

$$\dim(V_s^{'}(\mu)) \leq \frac{2\ell^2 - \ell - \ell}{2} = \ell^2 - \ell. \tag{4.10}$$

If $\ell \geq 4$, then $\dim(V_s^{'}(\mu)) < 2\ell^2 - 5\ell + 4$, while, if $\ell \leq 3$, then $\dim(V_s^{'}(\mu)) < \ell^2$. Thus, we can assume from this point onward that the eigenvalue μ is such that $\mu = \mu^{-1}$.

We now consider the case of $m = 1$, hence $n_1 = \ell$. Since $s \notin \mathrm{Z}(G)$, we have that $\mu_1^2 \neq 1$. By (4.9), we deduce that $\dim(V_s^{'}(1)) = \ell^2$ and $\dim(V_s^{'}(-1)) \leq \ell^2 - \ell$. Therefore, if $\ell \geq 5$, it follows that $\dim(V_s^{'}(\pm 1)) \leq \ell^2 < 2\ell^2 - 5\ell + 4$. On the other hand, if $\ell \leq 4$, then $\dim(V_s^{'}(\pm 1)) \leq \ell^2$, where equality holds if and only if the eigenvalue is 1 and, up to conjugation, $s = \mathrm{diag}(\underbrace{\mu_1, \ldots, \mu_1}_{\ell}, \underbrace{\mu_1^{-1}, \ldots, \mu_1^{-1}}_{\ell})$ with $\mu_1 \neq \pm 1$, as in (1.1), (2) and (3.1).

Now we can assume that $m \geq 2$.

Let $\ell = 2$. Then $m = 2$, hence $n_1 = n_2 = 1$, and, by (4.9), the eigenvalues of s on $V^{'}$ are 1 with multiplicity at least 2, $\mu_1\mu_2$, $\mu_1^{-1}\mu_2^{-1}$, $\mu_1^{-1}\mu_2$ and $\mu_1\mu_2^{-1}$. Therefore, $\dim(V_s^{'}(1)) = 2$, as $\mu_1 \neq \mu_2^{\pm 1}$ and $\dim(V_s^{'}(-1)) \leq 4$ where equality holds if and only if $\mu_1\mu_2^{\pm 1} = -1$, hence if and only if $\mu_1 = -\mu_2$ and $\mu_2^2 = 1$. We conclude that $\dim(V_s^{'}(\mu)) \leq 4$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on $V^{'}$ and that equality holds if and only if $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(1, -1, -1, 1)$, as in (1.2).

Let $\ell = 3$.

<u>Case 1.1</u>: Assume that $m = 2$. Then $n_1 = 2$ and $n_2 = 1$, as $n_1 \geq n_2$. For $\mu = 1$, since $\mu_1 \neq \mu_2^{\pm 1}$, it follows that $\mu_1^{\pm 1}\mu_2^{\pm 1} \neq 1$, hence $\dim(V_s^{'}(1)) \leq 7$. For $\mu = -1$ we have $\dim(V_s^{'}(-1)) \leq 10$, as the eigenvalue 1 occurs with multiplicity at least 5, see (4.9). Since $\mu_1 \neq \mu_2$, it follows that $\mu_1^2 \neq \mu_1\mu_2$, hence $\mu_1^{-2} \neq \mu_1^{-1}\mu_2^{-1}$, and so $\dim(V_s^{'}(-1)) \leq 8$.

<u>Case 1.2</u>: Assume that $m = 3$. For $\mu = 1$, since $\mu_i \neq \mu_j^{\pm 1}$ for all $1 \leq i < j \leq 3$, we have that $\mu_i^{\pm 1}\mu_j^{\pm 1} \neq 1$ for all $1 \leq i < j \leq 3$, hence $\dim(V_s^{'}(1)) = 3$. For $\mu = -1$ we have $\dim(V_s^{'}(-1)) \leq 12$, see (4.9). Since the μ_i's are distinct, it follows that -1 can equal at most one eigenvalue of the form $\mu_i\mu_j$ and at most one of the form $\mu_i\mu_j^{-1}$, thus $\dim(V_s^{'}(-1)) \leq 4$.

Having dealt with the cases of $\ell = 2$ and $\ell = 3$, we can now assume that $\ell \geq 4$. Recall that we are still in the case of $m \geq 2$ and $\mu = \mu^{-1}$. For $\mu = 1$, since $\mu_i \neq \mu_j^{\pm 1}$ for all $1 \leq i < j \leq m$, it follows that $\mu_i^{\pm 1}\mu_j^{\pm 1} \neq 1$ for all $1 \leq i < j \leq m$. Therefore:

$$\dim(V_s^{'}(1)) \leq 2\ell^2 - \ell - 4\sum_{i<j} n_i n_j. \tag{4.11}$$

Assume $\dim(V_s^{'}(1)) \geq 2\ell^2 - 5\ell + 4$. Then:

$$\ell - 1 - \sum_{i<j} n_i n_j \geq 0$$

and this is just Inequality (4.3), which we have shown to hold if and only if $m = 2$, $n_2 = 1$ and $n_1 = \ell - 1$. In this case, by (4.11), we have $\dim(V_s^{'}(1)) \leq 2\ell^2 - 5\ell + 4$ where equality holds if and only if all eigenvalues of s on V different than $\mu_1^{\pm 1}\mu_2^{\pm 1}$ are equal to 1. Hence $\mu_1^2 = 1$

and, since $\mu_1\mu_2 \neq 1$, it follows that, up to conjugation, $s = \pm\,\mathrm{diag}(1,\ldots,1,\mu_2,\mu_2^{-1},1,\ldots,1)$ with $\mu_2 \neq 1$, as in (3.3).

Lastly, let $\mu = -1$. We remark that $\dim(V_s^{'}(-1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2$, see (4.9). If $\mu_i\mu_j \neq -1$ for all $1 \leq i < j \leq m$, then $\mu_i^{-1}\mu_j^{-1} \neq -1$ for all $1 \leq i < j \leq m$, and we have:

$$\dim(V_s^{'}(-1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2 - 2\sum_{i<j} n_i n_j = 2\ell^2 - \ell - (\sum_{r=1}^{m} n_r)^2 = \ell^2 - \ell. \tag{4.12}$$

Therefore $\dim(V_s^{'}(-1)) < 2\ell^2 - 5\ell + 4$, as $\ell \geq 4$. We can thus assume that there exist $1 \leq i < j \leq m$ such that $\mu_i\mu_j = -1$. Then $\mu_i^{-1}\mu_j^{-1} = -1$ and, since the μ_i's are distinct, we have that:

$$\begin{cases} \mu_i^2 \neq -1 \text{ and } \mu_j^2 \neq -1, \text{ hence } \mu_i^{-2} \neq -1 \text{ and } \mu_j^{-2} \neq -1; \\ \mu_i\mu_r \neq -1,\ i < r \leq m,\ r \neq j, \text{ and } \mu_r\mu_i \neq -1,\ 1 \leq r < i; \\ \mu_i^{-1}\mu_r^{-1} \neq -1,\ i < r \leq m,\ r \neq j, \text{ and } \mu_r^{-1}\mu_i^{-1} \neq -1,\ 1 \leq r < i; \\ \mu_r\mu_j \neq -1,\ 1 \leq r < j,\ r \neq i, \text{ and } \mu_j\mu_r \neq -1,\ j < r \leq m; \\ \mu_r^{-1}\mu_j^{-1} \neq -1,\ 1 \leq r < j,\ r \neq i, \text{ and } \mu_j^{-1}\mu_r^{-1} \neq -1,\ j < r \leq m. \end{cases} \tag{4.13}$$

By (4.9), all of the above account for at least $n_i(n_i-1) + n_j(n_j-1) + 2(n_i+n_j)(\ell - n_i - n_j)$ additional eigenvalues of s on $V^{'}$ different than -1. Therefore, we have:

$$\dim(V_s^{'}(-1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2 - n_i(n_i-1) - n_j(n_j-1) - 2(n_i+n_j)(\ell - n_i - n_j). \tag{4.14}$$

Assume $\dim(V_s^{'}(-1)) \geq 2\ell^2 - 5\ell + 4$. Then:

$$4\ell - 4 - \sum_{r=1}^{m} n_r^2 - n_i(n_i-1) - n_j(n_j-1) - 2(n_i+n_j)(\ell - n_i - n_j) \geq 0. \tag{4.15}$$

After simplifications, this becomes:

$$4\ell - 4 - \sum_{r\neq i,j} n_r^2 - 2n_i^2 - 2n_j^2 + (n_i+n_j) - 2(n_i+n_j)(\ell - n_i - n_j) \geq 0.$$

Since $\sum_{r\neq i,j} n_r^2 \geq 0$, we must have:

$$4\ell - 4 - 2n_i^2 - 2n_j^2 - (n_i+n_j)(\ell - n_i - n_j) - (n_i+n_j)(\ell - n_i - n_j - 1) \geq 0.$$

Again, after simplifications, this becomes:

$$4\ell - 4 - (n_i+n_j)\ell - n_i^2 + 2n_in_j - n_j^2 - (n_i+n_j)(\ell - n_i - n_j - 1) \geq 0,$$

therefore

$$\ell(4 - n_i - n_j) - 4 - (n_i - n_j)^2 - (n_i+n_j)(\ell - n_i - n_j - 1) \geq 0. \tag{4.16}$$

If $\ell - n_i - n_j - 1 < 0$, then, as $\sum_{r=1}^{m} n_r = \ell$, we have $m = 2$ and so $\ell = n_1 + n_2$. By Inequality (4.16), it follows that :

$$\ell(5-\ell) - 4 - (2n_1 - \ell)^2 \geq 0.$$

As $\ell \geq 4$ and $-(2n_1 - \ell)^2 \leq 0$, the above inequality holds if and only if $\ell = 4$ and $2n_1 = \ell$, hence if and only if $\ell = 4$ and $n_1 = n_2 = 2$. Substituting in (4.14) gives $\dim(V_s^{'}(-1)) \leq 16$ and we note that equality holds if and only if all eigenvalues of s on $V^{'}$ different than 1 and the ones listed in (4.13) are equal to -1. Hence $\mu_1\mu_2^{-1} = -1$ and, as $\mu_1\mu_2 = -1$, it follows that $\mu_1^2 = \mu_2^2 = 1$ and so, up to conjugation, $s = \mathrm{diag}(1, 1, -1, -1, -1, -1, 1, 1)$, as in (3.2).

On the other hand, if $\ell - n_i - n_j - 1 \geq 0$, then, by (4.16), it follows that $\ell(4 - n_i - n_j) > 0$ and so $n_i + n_j \leq 3$. Since $n_i \geq n_j \geq 1$, we deduce that $(n_i, n_j) \in \{(2, 1), (1, 1)\}$. If $(n_i, n_j) = (2, 1)$, then, by (4.16), we have $7 - 2\ell \geq 0$, contradicting $\ell \geq 4$. If $(n_i, n_j) = (1, 1)$, then, by (4.15), we have

$$2 - \sum_{r \neq i,j} n_r^2 \geq 0.$$

As $\sum_{r \neq i,j} n_r^2 \geq \ell - 2$ and $\ell \geq 4$, it follows that $\sum_{r \neq i,j} n_r^2 \geq 2$. We deduce that $\sum_{r \neq i,j} n_r^2 = 2$, hence $m = 4$, $n_i = 1$ for all $1 \leq i \leq 4$ and $\ell = 4$. Substituting in (4.14) gives $\dim(V_s^{'}(-1)) \leq 16$ and we note that equality holds if and only if all eigenvalues of s on $V^{'}$ different than 1 and the ones listed in (4.13) are equal to -1. Therefore $\mu_i\mu_j^{-1} = -1$ for all $1 \leq i \leq 4$ and all $i < j \leq 4$, contradicting the fact that the μ_i's are distinct. This completes the proof of the proposition. □

Corollary 4.2.6. *Assume $p \nmid \ell$ and let $V = L_G(\omega_2)$. Moreover, let $s \in T \setminus \mathrm{Z}(G)$ and let $\mu \in k^*$ be an eigenvalue of s on V. Then one of the following holds:*

(1) *$\ell = 2$ and $\dim(V_s(\mu)) \leq 4$, where equality holds if and only if $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(1, -1, -1, 1)$.*

(2) *$\ell = 3$ and $\dim(V_s(\mu)) \leq 8$, where we have equality if and only if one of the following holds:*

  (2.1) *$\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq \pm 1$.*

  (2.2) *$p \neq 2$, $\mu = -1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, 1, -1, -1, 1, 1)$.*

(3) *$\ell = 4$ and $\dim(V_s(\mu)) \leq 16$, where equality holds if and only if $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(1, 1, -1, -1, -1, -1, 1, 1)$.*

(4) *$\ell \geq 5$ and $\dim(V_s(\mu)) \leq 2\ell^2 - 5\ell + 3$, where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, \mu_2, \mu_2^{-1}, 1, \ldots, 1)$ with $\mu_2 \neq 1$.*

In particular, for $\ell = 2$ there exist $s \in T \setminus \mathrm{Z}(G)$ which afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 3$ we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. Set $V' = \wedge^2(W)$. By Lemma 2.8.2, since $p \nmid \ell$, it follows that $V' = V \oplus L_G(0)$, thus $\dim(V) = 2\ell^2 - \ell - 1$, $\dim(V_s(1)) = \dim(V_s'(1)) - 1$ and $\dim(V_s(\mu)) = \dim(V_s'(\mu))$ for all eigenvalues $\mu \neq 1$ of s on V.

We remark that for any eigenvalue $\mu \in k^*$ of s on V such that $\mu \neq \mu^{-1}$, by Inequality (4.10), we have $\dim(V_s(\mu)) \leq \ell^2 - \ell$ and so $\dim(V_s(\mu)) < \ell^2$ for $\ell \leq 4$, respectively $\dim(V_s(\mu)) < 2\ell^2 - 5\ell + 3$ for $\ell \geq 5$. Furthermore, for the eigenvalue 1 of s on V, items (2) and (3) of Proposition 4.2.5 establish statements (2.1) and (4) of this corollary.

In order to complete the proof, we assume $p \neq 2$ and consider the eigenvalue -1 of s on V. For $\ell = 2$ and $\ell = 4$, items (1) and (3) of Proposition 4.2.5 establish statements (1) and (3) of this corollary. For the case of $\ell = 3$, by <u>Case 1.1</u> and <u>Case 1.2</u> of the proof of Proposition 4.2.5, we have that $\dim(V_s'(-1)) \leq 8$, where equality holds if and only if $m = 2$ and $\mu_1\mu_2^{\pm 1} = -1$. We determine that $\dim(V_s(-1)) \leq 8$ and equality holds if and only if, up to conjugation, $s = \pm\,\mathrm{diag}(1, 1, -1, -1, 1, 1)$, as in (2.2).

We can now assume that $\ell \geq 5$. First, if $\mu_i\mu_j \neq -1$ for all $1 \leq i < j \leq m$, then we proceed as for Inequality (4.12) to determine that

$$\dim(V_s(-1)) \leq \ell^2 - \ell.$$

Since $0 < \ell^2 - 4\ell + 3$ for all $\ell \geq 5$, it follows that $\dim(V_s(-1)) < 2\ell^2 - 5\ell + 3$. We can thus assume that there exist $1 \leq i < j \leq m$ such that $\mu_i\mu_j = -1$. Then, we argue as for Inequality (4.14) to determine that

$$\dim(V_s(-1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2 - n_i(n_i - 1) - n_j(n_j - 1) - 2(n_i + n_j)(\ell - n_i - n_j). \quad (4.17)$$

Suppose that $\dim(V_s(-1)) \geq 2\ell^2 - 5\ell + 3$. Then

$$4\ell - 3 - \sum_{r=1}^{m} n_r^2 - n_i(n_i - 1) - n_j(n_j - 1) - 2(n_i + n_j)(\ell - n_i - n_j) \geq 0. \quad (4.18)$$

We proceed as in the proof for $V_s'(-1)$, see (4.16), and we arrive at

$$\ell(4 - n_i - n_j) - 3 - (n_i - n_j)^2 - (n_i + n_j)(\ell - n_i - n_j - 1) \geq 0. \quad (4.19)$$

Assume that $\ell - n_i - n_j - 1 < 0$. Then $m = 2$, $\ell = n_1 + n_2$ and, by Inequality (4.19), we get

$$\frac{8}{13}(n_1 - \frac{5}{4})^2 + \frac{8}{13}(n_2 - \frac{5}{4})^2 \leq 1.$$

It follows that $(n_1, n_2) \in \{(2, 2), (2, 1), (1, 1)\}$, contradicting $\ell \geq 5$. Therefore $\ell - n_i - n_j - 1 \geq 0$ and so, by (4.19), we have

$$\ell(4 - n_i - n_j) > 0.$$

Hence $n_i + n_j \leq 3$ and, since $n_i \geq n_j$, we have $(n_i, n_j) \in \{(2, 1), (1, 1)\}$. If $(n_i, n_j) = (2, 1)$, then Inequality (4.19) gives $-2\ell + 8 \geq 0$, contradicting $\ell \geq 5$. If $(n_i, n_j) = (1, 1)$, then Inequality (4.18) gives $\sum_{r \neq i,j} n_r^2 \leq 3$, therefore $\sum_{r \neq i,j} n_r^2 = 3$, as $\sum_{r=1}^{m} n_r^2 \geq \ell$ and $\ell \geq 5$. It follows

that $m = 5$, $n_i = 1$, for all $1 \leq i \leq 5$, and $\ell = 5$. Substituting in Inequality (4.17) gives $\dim(V_s(-1)) \leq 28$. Equality holds if and only if all eigenvalues of s on V different than 1 and the ones listed in (4.13) are equal to -1. By (4.9), it follows that $\mu_i\mu_j^{-1} = \mu_i\mu_r^{-1}$ for all $1 \leq i \leq 5$ and $i < j < r \leq 5$, contradicting the fact that the μ_i's are distinct.

In conclusion, for $\ell = 2$ we have shown that $\dim(V_s(\mu)) \leq 4$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which equality holds, for example $s = \mathrm{diag}(1, -1, -1, 1)$ and $\mu = -1$. Therefore, there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ such that $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell = 3$ and $\ell = 4$ we have proven that $\dim(V_s(\mu)) \leq 2\ell^2 - 6\ell + 8$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, thus $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$. Lastly, for $\ell \geq 5$ we have proven that $\dim(V_s(\mu)) \leq 2\ell^2 - 5\ell + 3$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Now, as the inequality $0 < 14\ell^2 - 31\ell + 17$ holds for all $\ell \geq 5$, we have $2\ell^2 - 5\ell + 3 < 2\ell^2 - \ell - 1 - \sqrt{2\ell^2 - \ell - 1}$ for all $\ell \geq 5$, hence $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Corollary 4.2.7. *Assume $p \mid \ell$ and let $V = L_G(\omega_2)$. Moreover, let $s \in T \setminus \mathrm{Z}(G)$ and let $\mu \in k^*$ be an eigenvalue of s on V. Then one of the following holds:*

(1) *$\ell = 2$ and $\dim(V_s(\mu)) \leq 2$, where we have equality if and only if one of the following holds:*

  (1.1) *$\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq 1$.*

  (1.2) *$\mu = \mu_1^{\pm 1}$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, 1, 1, \mu_1^{-1})$ with $\mu_1 \neq 1$.*

(2) *$\ell = 3$ and $\dim(V_s(\mu)) \leq 8$, where equality holds if and only if $\mu = -1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, 1, -1, -1, 1, 1)$.*

(3) *$\ell = 4$ and $\dim(V_s(\mu)) \leq 14$, where we have equality if and only if one of the following holds:*

  (3.1) *$\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq 1$.*

  (3.2) *$\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(1, 1, 1, \mu_2, \mu_2^{-1}, 1, 1, 1)$ with $\mu_2 \neq 1$.*

(4) *$\ell \geq 5$ and $\dim(V_s(\mu)) \leq 2\ell^2 - 5\ell + 2$, where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, \mu_2, \mu_2^{-1}, 1, \ldots, 1)$ with $\mu_2 \neq 1$.*

In particular, for $\ell = 2$ there exist $s \in T \setminus \mathrm{Z}(G)$ which afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 3$ we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. Let $V' = \wedge^2(W)$. By Lemma 2.8.2, since $p \mid \ell$, we have that $V' = L_G(0) \mid V \mid L_G(0)$, hence $\dim(V) = 2\ell^2 - \ell - 2$, $\dim(V_s(1)) = \dim(V_s'(1)) - 2$ and $\dim(V_s(\mu)) = \dim(V_s'(\mu))$ for all eigenvalues $\mu \neq 1$ of s on V.

Let $\ell = 2$ and note that, in this case, $p = 2$. For the eigenvalue 1 item (1.1) of Proposition 4.2.5 gives the result, while for any eigenvalue $\mu \neq 1$ of s on V, by Inequality (4.10), we have $\dim(V_s(\mu)) \leq 2$. Now, if $m = 1$, then $\dim(V_s(\mu)) = 2$ if and only if $\mu_1^2 = \mu_1^{-2}$, contradicting

$s \notin \mathrm{Z}(G)$. On the other hand, if $m = 2$, then the eigenvalues of s on V, not necessarily distinct, are $\mu_1\mu_2$, $\mu_1\mu_2^{-1}$, $\mu_1^{-1}\mu_2$ and $\mu_1^{-1}\mu_2^{-1}$, see (4.9). One checks that $\dim(V_s(\mu)) = 2$ if and only if $\mu = \mu_1^{\pm 1}$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, 1, 1, \mu_1^{-1})$ with $\mu_1 \neq 1$, as in (1.2).

We can now assume that $\ell \geq 3$. Then for any eigenvalue $\mu \in k^*$ of s on V such that $\mu \neq \mu^{-1}$, by Inequality (4.10), we have $\dim(V_s(\mu)) \leq \ell^2 - \ell$, therefore $\dim(V_s(\mu)) < \ell^2$ for $\ell = 3, 4$, respectively $\dim(V_s(\mu)) < 2\ell^2 - 5\ell + 2$ for $\ell \geq 5$. Moreover, Proposition 4.2.5 solves the case of the eigenvalue $\mu = 1$ and, in particular, establishes statements (3) and (4) of the corollary. Hence, to complete the proof, we only need to investigate the dimension of the eigenspace corresponding to the eigenvalue -1 of s on V.

We first consider the case of $\ell = 3$. By Case 1.1 and Case 1.2 of the proof of Proposition 4.2.5, we determine that $\dim(V_s(-1)) \leq 8$ for all $s \in T \setminus \mathrm{Z}(G)$ and that equality holds if and only if, up to conjugation, $s = \pm\,\mathrm{diag}(1, 1, -1, -1, 1, 1)$, as in (2).

As $p \mid \ell$ and as we want to determine $\dim(V_s(-1))$, we can assume that $\ell \geq 5$. If $\mu_i\mu_j \neq -1$ for all $1 \leq i < j \leq m$. We argue as for Inequality (4.12) to determine that

$$\dim(V_s(-1)) \leq \ell^2 - \ell.$$

As $0 < \ell^2 - 4\ell + 2$ for all $\ell \geq 4$, it follows that $\dim(V_s(-1)) < 2\ell^2 - 5\ell + 2$. Hence, we can assume that there exist $1 \leq i < j \leq m$ such that $\mu_i\mu_j = -1$. Then, we argue as for Inequality (4.14) to determine that:

$$\dim(V_s(-1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2 - n_i(n_i - 1) - n_j(n_j - 1) - 2(n_i + n_j)(\ell - n_i - n_j). \quad (4.20)$$

Assume $\dim(V_s(-1)) \geq 2\ell^2 - 5\ell + 2$. Then:

$$4\ell - 2 - \sum_{r=1}^{m} n_r^2 - n_i(n_i - 1) - n_j(n_j - 1) - 2(n_i + n_j)(\ell - n_i - n_j) \geq 0. \quad (4.21)$$

Once again, we proceed as for $V_s'(-1)$, see (4.16), to arrive at

$$\ell(4 - n_i - n_j) - 2 - (n_i - n_j)^2 - (n_i + n_j)(\ell - n_i - n_j - 1) \geq 0. \quad (4.22)$$

Assume that $\ell - n_i - n_j - 1 < 0$. Then $m = 2$, $\ell = n_1 + n_2$ and, by Inequality (4.22), we get

$$\frac{8}{17}(n_1 - \frac{5}{4})^2 + \frac{8}{17}(n_2 - \frac{5}{4})^2 \leq 1.$$

Therefore $(n_1, n_2) \in \{(2, 2), (2, 1), (1, 1)\}$, contradicting $\ell \geq 5$. Thus $\ell - n_i - n_j - 1 \geq 0$ and so by (4.22), we get

$$\ell(4 - n_i - n_j) > 0.$$

Hence $n_i + n_j \leq 3$ and, since $n_i \geq n_j$, we have $(n_i, n_j) \in \{(2, 1), (1, 1)\}$. If $(n_i, n_j) = (2, 1)$, then, by (4.22), we get $-2\ell + 9 \geq 0$, contradicting $\ell \geq 5$. On the other hand, if $(n_i, n_j) = (1, 1)$, then, by Inequality (4.21), we get $\sum_{r \neq i,j} n_r^2 \leq 4$. But, as $\sum_{r=1}^{m} n_r^2 \geq \ell$ and $\ell \geq 5$, it follows

that $\sum_{r\neq i,j} n_r^2 \in \{3,4\}$. In both cases, since $\ell \geq 5$, it follows that $m = \ell$, $n_i = 1$, for all $1 \leq i \leq \ell$, and $\ell \leq 6$.

Assume that $\ell = 5$. Then, up to conjugation, $s = \mathrm{diag}(\mu_1, \ldots, \mu_5, \mu_5^{-1}, \ldots, \mu_1^{-1})$ with $\mu_i \neq \mu_j^{\pm 1}$ for all $1 \leq i < j \leq 5$ and there exist $1 \leq r < q \leq 5$ such that $\mu_r\mu_q = -1$. We can assume without loss of generality that $\mu_1\mu_2 = -1$. Now, by (4.9), the eigenvalues of s on V, not necessarily distinct, are: $\mu_i\mu_j$, $\mu_i^{-1}\mu_j^{-1}$, $\mu_i\mu_j^{-1}$, $\mu_i^{-1}\mu_j$, $1 \leq i < j \leq 5$ and 1 with multiplicity 3. Since $\mu_1\mu_2 = -1$, it follows that $\mu_1\mu_i \neq -1$, $\mu_1^{-1}\mu_i^{-1} \neq -1$, $\mu_2\mu_i \neq -1$ and $\mu_2^{-1}\mu_i^{-1} \neq -1$ for all $3 \leq i \leq 5$, as the μ_j's are distinct. This totals 12 eigenvalues of s on V that are different than -1. Similarly, since $\mu_i \neq \mu_j^{-1}$ for all $1 \leq i < j \leq 5$, we determine that $\mu_1^{-1}\mu_i \neq -1$, $\mu_1\mu_i^{-1} \neq -1$, $\mu_2^{-1}\mu_i \neq -1$ and $\mu_2\mu_i^{-1} \neq -1$ for all $3 \leq i \leq 5$. This amounts to another 12 eigenvalues of s on V that are different than -1. It follows that $\dim(V_s(-1)) \leq 16$, contradicting our assumption that $\dim(V_s(-1)) \geq 27$.

If $\ell = 6$, then, substituting in Inequality (4.20) gives $\dim(V_s(-1)) \leq 44$. Equality holds if and only if all eigenvalues of s on V different than 1 and those listed in (4.13) are equal to -1. Therefore, $\mu_i\mu_j^{-1} = \mu_i\mu_r^{-1}$ for all $1 \leq i \leq 6$ and $i < j < r \leq 6$, contradicting the fact that the μ_i's are distinct.

In conclusion, for $\ell = 2$ we have shown that $\dim(V_s(\mu)) \leq 2$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V and that there exist $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which equality holds, for example $s = \mathrm{diag}(\mu_1, 1, 1, \mu_1^{-1})$, with $\mu_1 \neq 1$, and $\mu = 1$. Therefore, there exist $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ such that $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell = 3$ we have proven that $\dim(V_s(\mu)) \leq 8$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, thus $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$. Lastly, for $\ell \geq 4$ we have proven that $\dim(V_s(\mu)) \leq 2\ell^2 - 5\ell + 2$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Now, as the inequality $0 < 14\ell^2 - 31\ell + 18$ holds for all $\ell \geq 4$, it follows that $2\ell^2 - 5\ell + 2 < 2\ell^2 - \ell - 2 - \sqrt{2\ell^2 - \ell - 2}$ for all $\ell \geq 4$, hence $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

To conclude this subsection, we remark that Lemma 4.2.3, Proposition 4.2.4 and Corollaries 4.2.6 and 4.2.7 give the proof of Theorems 4.2.1 and 4.2.2 for the families of kG-modules given by p-restricted dominant weights $\lambda \in F^{C_\ell}$.

4.2.2 The particular modules

As previously mentioned, in this subsection we will give an upper-bound for $\dim(V_s(\mu))$, where $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ and V is an irreducible kG-module with p-restricted dominant highest weight featured in one of the Tables 2.7.2 and 2.7.3. In order to determine $\max_{s\in T\setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ we will use the inductive algorithm of Subsection 2.4.3. For this, we will use the properties of the Levi subgroup L_1 of the maximal parabolic subgroup P_1 of G given in Section 2.4. We recall that $L_1 = \mathrm{Z}(L_1)^\circ[L_1, L_1]$, where $\mathrm{Z}(L_1)^\circ$ is a one-dimensional torus and $[L_1, L_1]$ is a simply connected group of type $C_{\ell-1}$; and that we have denote by T' the maximal torus $T \cap [L_1, L_1]$ of $[L_1, L_1]$. Moreover, for $\ell = 2$, we also recall that L_2, a Levi subgroup of the maximal parabolic subgroup P_2 of G, is such that $L_2 = \mathrm{Z}(L_2)^\circ[L_2, L_2]$, where $\mathrm{Z}(L_2)^\circ$ is a one-dimensional torus and $[L_2, L_2]$ is a simply connected group of type A_1. We abuse notation and denote by T' the maximal torus $T \cap [L_2, L_2]$

of $[L_2, L_2]$, as it will be clear from the context which derived subgroup we are refering to. Lastly, although we do not mention the result explicitly, we make great use of the data in [Lü01b], when discussing weights and weight multiplicities in this subsection.

Let $s \in T$. Then $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. As $z \in \mathrm{Z}(L_1)^\circ$, we have $z = \prod_{j=1}^{\ell} h_{\alpha_j}(c^{k_j})$, where $c \in k^*$ and $k_j \in \mathbb{Z}$ for all $1 \leq j \leq \ell$. Moreover, we have $\alpha_j(z) = 1$ for all $2 \leq j \leq \ell$ and so $z = \prod_{i=1}^{\ell} h_{\alpha_i}(c)$ for some $c \in k^*$. As $h \in [L_1, L_1]$, we have $h = \prod_{j=2}^{\ell} h_{\alpha_j}(a_j)$ with $a_j \in k^*$ for all $2 \leq j \leq \ell$. Therefore, $s = h_{\alpha_1}(c) \cdot (\prod_{j=2}^{\ell} h_{\alpha_j}(ca_j))$ with $c \in k^*$ and $a_j \in k^*$ for all $2 \leq j \leq \ell$.

Let V be an irreducible kG-module of p-restricted dominant highest weight $\lambda \in \mathrm{X}(T)$, where $\lambda = d_1\omega_1 + \cdots + d_\ell\omega_\ell$ with $0 \leq d_1, \ldots, d_\ell \leq p-1$. We consider the decomposition:

$$V\mid_{[L_1,L_1]} = \bigoplus_{i=0}^{e_1(\lambda)} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq e_1(\lambda)$. Let $s \in T$ and write $s = z \cdot h$ as above. By (2.5), we have that:

$$s_z^i = (\lambda - i\alpha_1 - \gamma)(z) = (\lambda - i\alpha_1)(\prod_{j=1}^{\ell} h_{\alpha_j}(c)) = \prod_{j=1}^{\ell} c^{d_j} \cdot c^{-i}.$$

Therefore, z acts on V^i, $0 \leq i \leq e_1(\lambda)$, as the scalar s_z^i. Now, let $\mu_1^i, \ldots, \mu_{t_i}^i$, $t_i \geq 1$, be the distinct eigenvalues of h on V^i, $0 \leq i \leq e_1(\lambda)$, and let $n_1^i, \ldots, n_{t_i}^i$ be their respective multiplicities. Then, by Lemma 2.4.8, it follows that the distinct eigenvalues of s on V^i are $s_z^i\mu_1^i, \ldots, s_z^i\mu_{t_i}^i$ with respective multiplicities $n_1^i, \ldots, n_{t_i}^i$.

Now, we consider the case of $\ell = 2$ and let $s \in T$. Then $s = z' \cdot h'$, where $z' \in \mathrm{Z}(L_2)^\circ$ and $h' \in [L_2, L_2]$. As $z' \in \mathrm{Z}(L_2)^\circ$, we have $\alpha_1(z') = 1$ and so $z' = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$ with $c \in k^*$. As $h' \in [L_2, L_2]$, we have $h' = h_{\alpha_1}(a_1)$, where $a_1 \in k^*$. Therefore, $s = h_{\alpha_1}(ca_1)h_{\alpha_2}(c^2)$ with $c, a_2 \in k^*$. As before, let V be an irreducible kG-module of p-restricted dominant highest weight $\lambda \in \mathrm{X}(T)$, $\lambda = d_1\omega_1 + \cdots + d_\ell\omega_\ell$ with $0 \leq d_1, \ldots, d_\ell \leq p-1$. We have the following decomposition:

$$V\mid_{[L_2,L_2]} = \bigoplus_{i=0}^{e_2(\lambda)} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for all $0 \leq i \leq e_2(\lambda)$. Let $s \in T$ and write $s = z' \cdot h'$ as above. By (2.5), we have that:

$$s_{z'}^i := (\lambda - i\alpha_2 - \gamma)(z') = (\lambda - i\alpha_2)(h_{\alpha_1}(c)h_{\alpha_2}(c^2)) = c^{d_1+2d_2} \cdot c^{-2i}.$$

Therefore, z' acts on V^i, $0 \leq i \leq e_2(\lambda)$, as the scalar $s^i_{z'}$. Now, let $\mu^i_1, \ldots, \mu^i_{t_i}$, $t_i \geq 1$, be the distinct eigenvalues of h' on V^i, $0 \leq i \leq e_2(\lambda)$, and let $n^i_1, \ldots, n^i_{t_i}$ be their respective multiplicities. Then, as in the previous case, by Lemma 2.4.8, the distinct eigenvalues of s on V^i are $s^i_{z'}\mu^i_1, \ldots, s^i_{z'}\mu^i_{t_i}$, with respective multiplicities $n^i_1, \ldots, n^i_{t_i}$.

Proposition 4.2.8. *Let k be an algebraically closed field of characteristic $p = 5$. Assume $\ell = 2$ and let $V = L_G(\omega_1 + \omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 6,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_1 + \omega_2$ and let $L = L_2$. Then $\dim(V) = 12$, as $p = 5$, and, by Lemma 2.4.5, we have $e_2(\lambda) = 3$, therefore

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for all $0 \leq i \leq 3$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_1)$ and, by Lemma 2.4.3, we also have $V^3 \cong (L_L(\omega_1))^* \cong L_L(\omega_1)$. Now, the weight $(\lambda - \alpha_2)\mid_{T'} = 3\omega_1$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_L(3\omega_1)$ and thus $\dim(V^1) \geq \dim(L_L(3\omega_1)) = 4$, since $p = 5$. Since $V^2 \cong (V^1)^*$, see Lemma 2.4.3, it follows that $\dim(V^1) = 4$ and so $V^1 \cong L_L(3\omega_1)$, hence $V^2 \cong (L_L(3\omega_1))^* \cong L_L(3\omega_1)$. Therefore, we have:

$$V\mid_{[L,L]} \cong L_L(\omega_1) \oplus L_L(3\omega_1) \oplus L_L(3\omega_1) \oplus L_L(\omega_1). \tag{4.23}$$

If $\dim(V^i_s(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1, 2$ or $i = 3$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$ with $c^2 \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 3$, as scalar multiplication by c^{3-2i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 2; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 4; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 4; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^3) = 2. \end{cases}$$

As $c^2 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 6$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(c)h_{\alpha_2}(-1)$ with $c^2 = -1$, we have $\dim(V_s(\pm c)) = 6$.

We can now assume that $\dim(V^i_s(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 3$. We write $s = z' \cdot h'$, where $z' \in \mathrm{Z}(L)^\circ$ and $h' \in [L, L]$. Since z' acts by scalar multiplication on V^i, $0 \leq i \leq 3$, it follows that $\dim(V^i_{h'}(\mu_{h'})) < \dim(V^i)$ for all $0 \leq i \leq 3$, where $\mu_{h'}$ is any eigenvalue of h' on V^i. Now, by Lemma 3.2.3, we have $\dim(V^0_{h'}(\mu_{h'})) \leq 1$, respectively $\dim(V^3_{h'}(\mu_{h'})) \leq 1$, for all eigenvalues $\mu_{h'}$ of h' on V^0, respectively on V^3. Similarly, by Proposition 3.2.9, we have $\dim(V^1_{h'}(\mu_{h'})) \leq 2$, respectively $\dim(V^2_{h'}(\mu_{h'})) \leq 2$,

for all eigenvalues $\mu_{h'}$ of h' on V^1, respectively on V^2. This gives $\dim(V_{h'}(\mu_{h'})) \leq 6$ for all eigenvalues $\mu_{h'}$ of h' on V, therefore $\dim(V_s(\mu)) \leq 6$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 6$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.9. *Let k be an algebraically closed field of characteristic $p \neq 5$. Assume $\ell = 2$ and let $V = L_G(\omega_1 + \omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 8,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $L = L_1$ and $\lambda = \omega_1 + \omega_2$. Then $\dim(V) = 16$, as $p \neq 5$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 4$, therefore

$$V \mid_{[L,L]} = V^0 \oplus \cdots \oplus V^4,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 4$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_2)$ and, by Lemma 2.4.3, we also have $V^4 \cong (L_L(\omega_2))^* \cong L_L(\omega_2)$. Now, the weight $(\lambda - \alpha_1) \mid_{T'} = 2\omega_2$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_L(2\omega_2)$. Moreover, we also note that the weight $(\lambda - \alpha_1 - \alpha_2) \mid_{T'} = 0$ occurs with multiplicity 2 in V^1, where it has multiplicity $1 - \delta_{p,2}$ in the composition factor of V^1 isomorphic to $L_L(2\omega_2)$. It follows that $\dim(V^1) \geq 4$ and, by Lemma 2.4.3, we also have $\dim(V^3) \geq 4$, hence $\dim(V^2) \leq 4$. Similarly, the weight $(\lambda - 2\alpha_1 - \alpha_2) \mid_{T'} = \omega_2$ is the highest weight in V^2, in which it occurs with multiplicity 2, and admits a maximal vector. It follows that V^2 has two composition factors, both isomorphic to $L_L(\omega_2)$. As $\dim(L_L(\omega_2)) = 2$, we deduce that V^2 has exactly two composition factors, both isomorphic to $L_L(\omega_2)$, and, by [Jan07, II.2.14], we have $V^2 \cong L_L(\omega_2) \oplus L_L(\omega_2)$. Moreover, it also follows that $\dim(V^1) = 4$ and $\dim(V^3) = 4$. If $p \neq 2$, then V^1, hence V^3, by Lemma 2.4.3, consists of exactly two composition factors: one isomorphic to $L_L(2\omega_2)$ and one isomorphic to $L_L(0)$. Then, by [Jan07, II.2.14], we have $V^1 \cong L_L(2\omega_2) \oplus L_L(0)$ and $V^3 \cong L_L(2\omega_2) \oplus L_L(0)$. Therefore, in the case of $p \neq 2$, we have:

$$V \mid_{[L,L]} \cong L_L(\omega_2) \oplus L_L(2\omega_2) \oplus L_L(0) \oplus L_L(\omega_2) \oplus L_L(\omega_2) \oplus L_L(2\omega_2) \oplus L_L(0) \oplus L_L(\omega_2).$$

On the other hand, if $p = 2$, then V^1, hence V^3, by Lemma 2.4.3, has three composition factors: one isomorphic to $L_L(\omega_2)^{(2)}$ and two isomorphic to $L_L(0)$.

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1, 2, 3$ or $i = 4$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 4$, as scalar multiplication by c^{2-i}, we determine that the eigenvalues of s

on V, not necessarily distinct, are:

$$\begin{cases} c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^0) = 2; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 4; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^2) = 4; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^3) = 4; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^4) = 2. \end{cases}$$

Since $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 8$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(\pm 1)) = 8$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 4$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 4$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 4$, where μ_h is any eigenvalue of h on V^i. Using (3.1) of the proof of Proposition 3.2.4, we determine that the eigenvalues of h on V^1 are of the form a_1^2, 1 with multiplicity 2, and a_1^{-2}. Note that, if $a_1^2 = 1$, or $a_1^{-2} = 1$, then $\dim(V_h^1(1)) = \dim(V^1)$, hence $\dim(V_s^1(c)) = \dim(V^1)$, contradicting our assumption. Therefore, $\dim(V_h^1(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^1. Moreover, as $V^3 \cong (V^1)^*$, we also have $\dim(V_h^3(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^3. Lastly, by Lemma 3.2.3, we have $\dim(V_h^0(\mu_h)) \leq 1$, $\dim(V_h^2(\mu_h)) \leq 2$ and $\dim(V_h^4(\mu_h)) \leq 1$ for all eigenvalues μ_h of h on V^0, V^2 and V^4, respectively. It follows that $\dim(V_h(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V and so $\dim(V_s(\mu)) \leq 8$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 8$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.10. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 2$ and let $V = L_G(2\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 10 - \delta_{p,5},$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = 2\omega_2$ and let $L = L_1$. Then $\dim(V) = 14 - \delta_{p,5}$ and, by Lemma 2.4.5, we have $e_1(\lambda) = 4$, therefore

$$V \mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3 \oplus V^4,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 4$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(2\omega_2)$ and, by Lemma 2.4.3, we also have $V^4 \cong (L_L(2\omega_2))^* \cong L_L(2\omega_2)$. Now, the weight $(\lambda - \alpha_1 - \alpha_2) \mid_{T'} = \omega_2$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_L(\omega_2)$ and so $\dim(V^1) \geq \dim(L_L(\omega_2)) = 2$. Since $V^3 \cong (V^1)^*$, see Lemma 2.4.3, we have $\dim(V^3) \geq 2$, and so $\dim(V^2) \leq 4 - \delta_{p,5}$. Lastly, in V^2 the weight $(\lambda - 2\alpha_1 - \alpha_2) \mid_{T'} = 2\omega_2$ admits a maximal vector, thus V^2 has a composition factor isomorphic

to $L_L(2\omega_2)$ and $\dim(V^2) \geq \dim L_L(2\omega_2) = 3$, as $p \neq 2$. If $p = 5$, then $V^1 \cong L_L(\omega_2)$, $V^3 \cong (L_L(\omega_2))^* \cong L_L(\omega_2)$ and $V^2 \cong L_L(2\omega_2)$. Therefore

$$V\mid_{[L,L]} \cong L_L(2\omega_2) \oplus L_L(\omega_2) \oplus L_L(2\omega_2) \oplus L_L(\omega_2) \oplus L_L(2\omega_2). \tag{4.24}$$

Similarly, if $p \neq 5$, then, as $3 \leq \dim(V^2) \leq 4$, it follows that $V^1 \cong L_L(\omega_2)$, $V^3 \cong (L_L(\omega_2))^* \cong L_L(\omega_2)$ and $\dim(V^2) = 4$. Therefore, V^2 consists of exactly two composition factors: one isomorphic to $L_L(2\omega_2)$ and one to $L_L(0)$. We use [Jan07, II.2.14] to show that $V^2 \cong L_L(2\omega_2) \oplus L_L(0)$, and so

$$V\mid_{[L,L]} \cong L_L(2\omega_2) \oplus L_L(\omega_2) \oplus L_L(2\omega_2) \oplus L_L(0) \oplus L_L(\omega_2) \oplus L_L(2\omega_2). \tag{4.25}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1, 2, 3$ or $i = 4$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 4$, as scalar multiplication by c^{2-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^0) = 3; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 2; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^2) = 4 - \delta_{p,5}; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^3) = 2; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^4) = 3. \end{cases}$$

As $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 10 - \delta_{p,5}$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(1)) = 10 - \delta_{p,5}$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 4$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 4$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 4$, where μ_h is any eigenvalue of h on V^i. By Proposition 3.2.4, we have $\dim(V_h^0(\mu_h)) \leq 2$, $\dim(V_h^4(\mu_h)) \leq 2$ and $\dim(V_h^2(\mu_h)) \leq 3 - \delta_{p,5}$ for all eigenvalues μ_h of h on V^0, V^4 and V^2, respectively. Similarly, by Lemma 3.2.3, we have $\dim(V_h^1(\mu_h)) \leq 1$, respectively $\dim(V_h^3(\mu_h)) \leq 1$, for all eigenvalues μ_h of h on V^1, respectively on V^3. It follows that $\dim(V_h(\mu_h)) \leq 9 - \delta_{p,5}$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 9 - \delta_{p,5}$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we proved that $\dim(V_s(\mu)) \leq 10 - \delta_{p,5}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. Therefore, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.11. *Let k be an algebraically closed field of characteristic $p \neq 2, 3$. Assume $\ell = 2$ and let $V = L_G(3\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 10,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = 3\omega_1$ and let $L = L_2$. Then $\dim(V) = 20$, as $p \neq 2, 3$, and, by Lemma 2.4.5, we have $e_2(\lambda) = 3$, therefore:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for all $0 \leq i \leq 3$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(3\omega_1)$ and so $V^3 \cong (L_L(3\omega_1))^* \cong L_L(3\omega_1)$, see Lemma 2.4.3. Therefore $\dim(V^1) + \dim(V^2) = 12$ and, as $V^2 \cong (V^1)^*$, see Lemma 2.4.3, we deduce that $\dim(V^1) = \dim(V^2) = 6$. Now, the weight $(\lambda - \alpha_1 - \alpha_2)\mid_{T'} = 3\omega_1$ admits a maximal vector in V^1, thus V^1 has a composition factor isomorphic to $L_L(3\omega_1)$. Moreover, the dominant weight $(\lambda - 2\alpha_1 - \alpha_2)\mid_{T'} = \omega_1$, occurring with multiplicity 2 in V^1, has multiplicity 1 in the composition factor of V^1 isomorphic to $L_L(3\omega_1)$. As $\dim(V^1) = 6$, we determine that V^1 consists of exactly two composition factors: one isomorphic to $L_L(3\omega_1)$ and one isomorphic to $L_L(\omega_1)$. As $p \neq 2, 3$, we use [Jan07, II.2.14] to show that $V^1 \cong L_L(3\omega_1) \oplus L_L(\omega_1)$. Lastly, as $V^2 \cong (V^1)^*$, it follows that $V^2 \cong L_L(3\omega_1) \oplus L_L(\omega_1)$, and so:

$$V\mid_{[L,L]} \cong L_L(3\omega_1) \oplus L_L(3\omega_1) \oplus L_L(\omega_1) \oplus L_L(3\omega_1) \oplus L_L(\omega_1) \oplus L_L(3\omega_1). \tag{4.26}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1, 2$ or $i = 3$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$ with $c^2 \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 3$, as scalar multiplication by c^{3-2i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 4; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 6; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 6; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^3) = 4. \end{cases}$$

As $c^2 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 10$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(c)h_{\alpha_2}(-1)$ with $c^2 = -1$ we have $\dim(V_s(\pm c)) = 10$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 3$. We write $s = z' \cdot h'$, where $z' \in \mathrm{Z}(L)^\circ$ and $h' \in [L, L]$. Since z' acts by scalar multiplication on V^i, $0 \leq i \leq 3$, it follows that $\dim(V_{h'}^i(\mu_{h'})) < \dim(V^i)$ for all $0 \leq i \leq 3$, where $\mu_{h'}$ is any eigenvalue of h' on V^i. Now, by Proposition 3.2.9, we have $\dim(V_{h'}^0(\mu_{h'})) \leq 2$, respectively $\dim(V_{h'}^3(\mu_{h'})) \leq 2$, for all eigenvalues $\mu_{h'}$ of h' on V^0, respectively on V^3. Moreover, by the same result together with Lemma 3.2.3, it follows that $\dim(V_{h'}^1(\mu_{h'})) \leq 3$ and $\dim(V_{h'}^2(\mu_{h'})) \leq 3$ for all eigenvalues $\mu_{h'}$ of h' on V^1 and V^2, respectively. This gives $\dim(V_{h'}(\mu_{h'})) \leq 10$ for all eigenvalues $\mu_{h'}$ of h' on V, therefore $\dim(V_s(\mu)) \leq 10$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 10$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

Proposition 4.2.12. *Let k be an algebraically closed field of characteristic $p = 7$. Assume $\ell = 2$ and let $V = L_G(\omega_1 + 2\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 12,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_1 + 2\omega_2$ and let $L = L_1$. Then $\dim(V) = 24$, as $p = 7$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 6$, therefore:

$$V \mid_{[L,L]} = V^0 \oplus V^1 \oplus \cdots \oplus V^6,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 6$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(2\omega_2)$ and, moreover, by Lemma 2.4.3, we also have $V^6 \cong (L_L(2\omega_2))^* \cong L_L(2\omega_2)$. Now, in V^1, the weight $(\lambda - \alpha_1) \mid_{T'} = 3\omega_2$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_L(3\omega_2)$ and $\dim(V^1) \geq \dim(L_L(3\omega_2)) = 4$, since $p = 7$. Moreover, by Lemma 2.4.3, we also have $\dim(V^5) \geq 4$. Similarly, the weight $(\lambda - 2\alpha_1 - \alpha_2) \mid_{T'} = 2\omega_2$ admits a maximal vector in V^2, thus V^2 has a composition factor isomorphic to $L_L(2\omega_2)$ and $\dim(V^2) \geq \dim(L_L(2\omega_2)) = 3$, as $p = 7$. Once more, by Lemma 2.4.3, we also have $\dim(V^4) \geq 3$. Lastly, the weight $(\lambda - 3\alpha_1 - \alpha_2) \mid_{T'} = 3\omega_2$ admits a maximal vector in V^3 and so V^3 has a composition factor isomorphic to $L_L(3\omega_2)$, hence $\dim(V^3) \geq 4$, since $p = 7$. We deduce that:

$$V \mid_{[L,L]} \cong L_L(2\omega_2) \oplus L_L(3\omega_2) \oplus L_L(2\omega_2) \oplus L_L(3\omega_2) \oplus L_L(2\omega_2) \oplus L_L(3\omega_2) \oplus L_L(2\omega_2). \quad (4.27)$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $0 \leq i \leq 5$ or $i = 6$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 6$, as scalar multiplication by c^{3-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 3; \\ c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^1) = 4; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^2) = 3; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^3) = 4; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^4) = 3; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^5) = 4; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^6) = 3. \end{cases}$$

As $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 12$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(-1)) = \dim(V_s(1)) = 12$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 6$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 6$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all

$0 \leq i \leq 6$, where μ_h is any eigenvalue of h on V^i. Now, by Propositions 3.2.4 and 3.2.9, we have $\dim(V_h^i(\mu_h)) \leq 2$ for all $0 \leq i \leq 6$ and all eigenvalues μ_h of h on V^i. It follows that $\dim(V_h(\mu_h)) \leq 14$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 14$ for all eigenvalues $\mu \in k^*$ of s on V. However, we will show that, in fact, we have $\dim(V_s(\mu)) \leq 12$ for all pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$.

Assume there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ with the property that $\dim(V_s(\mu)) > 12$. Then, as V is a self-dual kG-module and $\dim(V) = 24$, it follows that $\mu = \pm 1$. Moreover, since $\dim(V_s(\pm 1)) > 12$, by the arguments of the previous paragraph, it follows that there exist at least 6 V^i's such that $\dim(V_s^{'}(\pm 1)) = 2$. Furthermore, as $V^{6-i} \cong (V^i)^*$ for all $0 \leq i \leq 6$, we determine that $\dim(V_s^i(\pm 1)) = 2$ for $i = 0, 1, 2, 4, 5$ and $i = 6$. We write $s = z \cdot h$, where $z = h_{\alpha_1}(c)h_{\alpha_2}(c)$ and $h \in [L, L]$. Let μ_h^i, $0 \leq i \leq 6$, be the eigenvalue of h on V^i with the property that $\mu = c^{3-i}\mu_h^i$. We have that $\dim(V_h^i(\mu_h^i)) = 2$ for $i = 0, 1, 2, 4, 5$ and $i = 6$. We now use Proposition 3.2.4 to determine that $\mu_h^2 = -1$ and so, we get $c = -\mu$. Similarly, we use the proof of Proposition 3.2.9 to determine that $\mu_h^1 = d^{\pm 1}$, where $d^2 = -1$. Therefore, we have $\mu = c^2 d^{\pm 1} = (-\mu)^2 d^{\pm 1} = d^{\pm 1}$, contradicting the fact that $d^2 = -1$. This shows that there do not exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ such that $\dim(V_s(\mu)) > 12$.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 12$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

Proposition 4.2.13. *Let k be an algebraically closed field of characteristic $p = 7$. Assume $\ell = 2$ and let $V = L_G(3\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, we have*

$$\dim(V_s(\mu)) \leq 16,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = 3\omega_2$ and let $L = L_1$. Then $\dim(V) = 25$, as $p = 7$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 6$, therefore

$$V \mid_{[L,L]} = V^0 \oplus \cdots \oplus V^6,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 6$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(3\omega_2)$ and thus $V^6 \cong (L_L(3\omega_2))^* \cong L_L(3\omega_2)$, see Lemma 2.4.3. Now, the weight $(\lambda - \alpha_1 - \alpha_2) \mid_{T'} = 2\omega_2$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_L(2\omega_2)$ and $\dim(V^1) \geq \dim(L_L(2\omega_2)) = 3$, since $p = 7$. Moreover, by Lemma 2.4.3, we also have $\dim(V^5) \geq 3$. The weight $(\lambda - 2\alpha_1 - \alpha_2) \mid_{T'} = 3\omega_2$ admits a maximal vector in V^2, therefore V^2 has a composition factor isomorphic to $L_L(3\omega_2)$ and $\dim(V^2) \geq \dim(L_L(3\omega_2)) = 4$, since $p = 7$. Once more, by Lemma 2.4.3, we have $\dim(V^4) \geq 4$. Lastly, the weight $(\lambda - 3\alpha_1 - 2\alpha_2) \mid_{T'} = 2\omega_2$ admits a maximal vector in V^3, therefore V^3 has a composition factor isomorphic to $L_L(2\omega_2)$ and $\dim(V^3) \geq 3$. We deduce that:

$$V \mid_{[L,L]} \cong L_L(3\omega_2) \oplus L_L(2\omega_2) \oplus L_L(3\omega_2) \oplus L_L(2\omega_2) \oplus L_L(3\omega_2) \oplus L_L(2\omega_2) \oplus L_L(3\omega_2). \quad (4.28)$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $0 \leq i \leq 5$ or $i = 6$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 6$, as scalar multiplication by c^{3-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 4; \\ c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^1) = 3; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^2) = 4; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^3) = 3; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^4) = 4; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^5) = 3; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^6) = 4. \end{cases}$$

As $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 16$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(-1)) = 16$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 6$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 6$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 6$, where μ_h is any eigenvalue of h on V^i. Now, by Propositions 3.2.9 and 3.2.4, we have $\dim(V_h^i(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^i, $0 \leq i \leq 6$. It follows that $\dim(V_h(\mu_h)) \leq 14$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 14$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 16$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.14. *Let k be an algebraically closed field of characteristic $p = 3$. Assume $\ell = 2$ and let $V = L_G(2\omega_1 + \omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 16,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = 2\omega_1 + \omega_2$ and let $L = L_2$. Then $\dim(V) = 25$, as $p = 3$, and, by Lemma 2.4.5, we have $e_2(\lambda) = 4$, therefore

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus \cdots \oplus V^4,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for all $0 \leq i \leq 4$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(2\omega_1)$ and therefore $V^4 \cong (L_L(2\omega_1))^* \cong L_L(2\omega_1)$, see Lemma 2.4.3. Now, the weight $(\lambda - \alpha_2)\mid_{T'} = 4\omega_1$ admits a maximal vector in V^1, thus V^1 has a composition factor isomorphic to $L_L(4\omega_1)$. We remark that, as $p = 3$, we have $L_L(4\omega_1) \cong L_L(\omega_1) \otimes L_L(\omega_1)^{(3)}$, see Theorem

2.3.8, thus $\dim(V^1) \geq 4$. Moreover, we also note that the dominant weight $(\lambda - 2\alpha_1 - \alpha_2)\mid_{T'} = 0$, occurring with multiplicity 2 in V^1, is not a sub-dominant weight in the composition factor of V^1 isomorphic to $L_L(\omega_1) \otimes L_L(\omega_1)^{(3)}$. It follows that $\dim(V^1) \geq 6$, hence $\dim(V^3) \geq 6$, by Lemma 2.4.3, thereby $\dim(V^2) \leq 7$. Similarly, the weight $(\lambda - \alpha_1 - 2\alpha_2)\mid_{T'} = 4\omega_1$ admits a maximal vector in V^2, thus V^2 has a composition factor isomorphic to $L_L(\omega_1) \otimes L_L(\omega_1)^{(3)}$. Moreover, the dominant weight $(\lambda - 2\alpha_1 - 2\alpha_2)\mid_{T'} = 2\omega_1$, occurring with multiplicity 2 in V^2, is a sub-dominant weight in the composition factor of V^2 isomorphic to $L_L(\omega_1) \otimes L_L(\omega_1)^{(3)}$, in which it has multiplicity 1. As $p = 3$, we have $\dim(L_L(2\omega_1)) = 3$ and so V^2 consists of exactly two composition factors: one isomorphic to $L_L(\omega_1) \otimes L_L(\omega_1)^{(3)}$ and one isomorphic to $L_L(2\omega_1)$. It follows that $\dim(V^1) = 6$, therefore V^1 is composed of exactly three composition factors: two isomorphic to $L_L(0)$ and one isomorphic to $L_L(\omega_1) \otimes L_L(\omega_1)^{(3)}$. Lastly, we apply Lemma 2.4.3 once more to determine that V^3 also consists of exactly three composition factors: two isomorphic to $L_L(0)$ and one isomorphic to $L_L(\omega_1) \otimes L_L(\omega_1)^{(3)}$.

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $0 \leq i \leq 3$ or $i = 4$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$ with $c^2 \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 4$, as scalar multiplication by c^{4-2i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^4 \text{ with } \dim(V_s(c^4)) \geq \dim(V^0) = 3; \\ c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^1) = 6; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^2) = 7; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^3) = 6; \\ c^{-4} \text{ with } \dim(V_s(c^{-4})) \geq \dim(V^4) = 3. \end{cases}$$

As $c^2 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 13$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 4$. We write $s = z' \cdot h'$, where $z' \in \mathrm{Z}(L)^\circ$, $z' = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$ with $c \in k^*$, and $h' \in [L, L]$. Since z' acts by scalar multiplication on V^i, $0 \leq i \leq 4$, it follows that $\dim(V_{h'}^i(\mu_{h'})) < \dim(V^i)$ for all $0 \leq i \leq 4$, where $\mu_{h'}$ is any eigenvalue of h' on V^i. First, we will show that $\dim(V_{h'}^1(\mu_{h'})) \leq 4$. For this, we recall that V^1 has three composition factors: two isomorphic to $L_L(0)$ and one isomorphic to $L_L(4\omega_1)$. Now, by the proof of Proposition 3.2.9, we determine that the eigenvalues of h' on V^1, not necessarily distinct, are 1 with multiplicity 2, a_1^4, a_1^2, a_1^{-2} and a_1^{-4}, where $a_1 \in k^*$. If $a_1^2 = 1$, then $\dim(V_{h'}^1(1)) = 6$ and so $\dim(V_s^1(c^2)) = 6$, contradicting our assumption. Therefore, $a_1^2 \neq 1$ and we have $\dim(V_{h'}^1(\mu_{h'})) \leq 4$ for all eigenvalues $\mu_{h'}$ of h' on V^1. Moreover, as $V^3 \cong (V^1)^*$, we also have $\dim(V_{h'}^3(\mu_{h'})) \leq 4$ for all eigenvalues $\mu_{h'}$ of h' on V^3. We now focus on V^2 and we will show that $\dim(V_{h'}^2(\mu_{h'})) \leq 4$ for all eigenvalues $\mu_{h'}$ of h' on V^2. For this, we recall that V^2 consists of two composition factors: one isomorphic to $L_L(2\omega_1)$ and one isomorphic to $L_L(4\omega_1)$. Thus, the eigenvalues of h' on V^2, not necessarily distinct, are a_1^2, 1, a_1^{-2}, by (3.1), and a_1^4, a_1^2, a_1^{-2}, a_1^{-4}, by the proof of Proposition 3.2.9. As in the case of V^1, we argue that $a_1^2 \neq 1$. Therefore, $\dim(V_{h'}^2(\mu_{h'})) \leq 4$ for all eigenvalues $\mu_{h'}$ of h' on V^2. Lastly, by Proposition 3.2.4, we have $\dim(V_{h'}^0(\mu_{h'})) \leq 2$ and $\dim(V_{h'}^4(\mu_{h'})) \leq 2$ for all eigenvalues $\mu_{h'}$ of h' on V^0 and V^4, respectively. We conclude that $\dim(V_{h'}(\mu_{h'})) \leq 16$ for all eigenvalues $\mu_{h'}$ of h' on V, therefore $\dim(V_s(\mu)) \leq 16$ for all eigenvalues $\mu \in k^*$ of s on V.

We will now show that there exist $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ such that $\dim(V_s(\mu)) = 16$. Let $s = z' \cdot h'$, where $z' = h_{\alpha_1}(c)h_{\alpha_2}(-1)$ with $c^2 = -1$ and $h' = h_{\alpha_2}(\mu_1)$ with $\mu_1^2 = -1$. Then $s \in T \setminus \mathrm{Z}(G)$. Using Propositions 3.2.4 and 3.2.9, we determine that the distinct eigenvalues of s on V are 1 and -1 with $\dim(V_s^0(-1)) = \dim(V_s^4(-1)) = 2$ and $\dim(V_s^0(1)) = \dim(V_s^4(1)) = 1$; $\dim(V_s^1(-1)) = \dim(V_s^3(-1)) = 4$ and $\dim(V_s^1(1)) = \dim(V_s^3(1)) = 2$; and $\dim(V_s^2(-1)) = 4$ and $\dim(V_s^2(1)) = 3$. Therefore, $\dim(V_s(-1)) = 16$ and $\dim(V_s(1)) = 9$.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 16$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.15. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell \geq 3$ and let $V = L_G(\omega_\ell)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2^{\ell-1},$$

where there exist pairs $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_\ell$ and let $L = L_1$. Then, $\dim(V) = 2^\ell$, as $p = 2$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 2$, therefore

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. Now, as $p = 2$, we have $V^1 = \{0\}$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_\ell)$ and thus $V^2 \cong (L_L(\omega_\ell))^* \cong L_L(\omega_\ell)$, see Lemma 2.4.3. Therefore, we have:

$$V\mid_{[L,L]} \cong L_L(\omega_\ell) \oplus L_L(\omega_\ell). \tag{4.29}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c) \cdots h_{\alpha_\ell}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $i = 0$ and $i = 2$, as scalar multiplication by c^{1-i}, we determine that the distinct eigenvalues of s on V are c and c^{-1} with $\dim(V_s(c^{\pm 1})) = 2^{\ell-1}$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L,L]$. Since z acts by scalar multiplication on V^0 and V^2, respectively, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for both $i = 0$ and $i = 2$, where μ_h is any eigenvalue of h on V^i. First, suppose that $\ell = 3$. Then, by Corollary 4.2.7, as $p = 2$, we have $\dim(V_h^0(\mu_h)) \leq 2$ and $\dim(V_h^2(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. It follows that $\dim(V_h(\mu_h)) \leq 2^2$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 2^2$ for all eigenvalues $\mu \in k^*$ of s on V. Suppose now that $\ell \geq 4$. Then, by recurrence, we have $\dim(V_h^0(\mu_h)) \leq 2^{\ell-2}$ and $\dim(V_h^2(\mu_h)) \leq 2^{\ell-2}$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. This gives $\dim(V_h(\mu_h)) \leq 2^{\ell-1}$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 2^{\ell-1}$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 2^{\ell-1}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which equality

holds. Since the inequality $0 < 2^\ell(2^{\ell-2}-1)$ holds for all $\ell \geq 3$, we have $2^{\ell-1} < 2^\ell - \sqrt{2^\ell}$ for all $\ell \geq 3$, and thus $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.16. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 3$ and let $V = L_G(\omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 10,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_3$ and let $L = L_1$. Then $\dim(V) = 14$, as $p \neq 2$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 2$, therefore:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_3)$, therefore $V^2 \cong (L_L(\omega_3))^* \cong L_L(\omega_3)$, by Lemma 2.4.3. Since the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3)\mid_{T'} = \omega_2$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_2)$ and $\dim(V^1) \geq \dim(L_L(\omega_2)) = 4$. By dimensional considerations, we deduce that $V^1 \cong L_L(\omega_2)$ and

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_2) \oplus L_L(\omega_3). \tag{4.30}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c)h_{\alpha_3}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $i = 0, 1, 2$, as scalar multiplication by c^{1-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are

$$\begin{cases} c \text{ with } \dim(V_s(c)) \geq \dim(V^0) = 5; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^1) = 4; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 5. \end{cases}$$

As $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 10$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(-1)) = 10$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 2$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where μ_h is any eigenvalue of h on V^i. As $p \neq 2$, by Corollary 4.2.6, we have $\dim(V_h^0(\mu_h)) \leq 4$ and $\dim(V_h^2(\mu_h)) \leq 4$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. Similarly, by Lemma 4.2.3, we have $\dim(V_h^1(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^1. This implies that $\dim(V_h(\mu_h)) \leq 10$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 10$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 10$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.17. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 3$ and let $V = L_G(\omega_1 + \omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 24.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_1 + \omega_3$ and let $L = L_1$. Then $\dim(V) = 48$, as $p = 2$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 4$, therefore

$$V \mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3 \oplus V^4,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 4$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_3)$ and so, by Lemma 2.4.3, we also have $V^4 \cong (L_L(\omega_3))^* \cong L_L(\omega_3)$. Since the weight $(\lambda - \alpha_1) \mid_{T'} = \omega_2 + \omega_3$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_2 + \omega_3)$ and $\dim(V^1) \geq \dim(L_L(\omega_2 + \omega_3)) = 16$, since $p = 2$. Hence, by Lemma 2.4.3, we also have $\dim(V^3) \geq 16$. Lastly, as the weight $(\lambda - 2\alpha_1 - 2\alpha_2 - \alpha_3) \mid_{T'} = \omega_3$ is the highest weight in V^2, in which it has multiplicity 2, and admits a maximal vector, it follows that V^2 has two composition factors, both isomorphic to $L_L(\omega_3)$. By dimensional considerations, it follows that $V^1 \cong L_L(\omega_2 + \omega_3)$, hence $V^3 \cong (L_L(\omega_2 + \omega_3))^* \cong L_L(\omega_2 + \omega_3)$, $V^2 \cong L_L(\omega_3) \oplus L_L(\omega_3)$, by [Jan07, II.2.14], and so:

$$V \mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_2 + \omega_3) \oplus L_L(\omega_3) \oplus L_L(\omega_3) \oplus L_L(\omega_2 + \omega_3) \oplus L_L(\omega_3). \tag{4.31}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $0 \leq i \leq 3$ or $i = 4$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c)h_{\alpha_3}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 4$, as scalar multiplication by c^{2-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are

$$\begin{cases} c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^0) = 4; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 16; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^2) = 8; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^3) = 16; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^4) = 4. \end{cases}$$

As $c \neq 1$ and $p = 2$, it follows that $\dim(V_s(\mu)) \leq 20$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 4$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 4$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 4$, where μ_h is any eigenvalue of h on V^i. Now, as $p = 2$, by Corollary 4.2.7, we have $\dim(V_h^0(\mu_h)) \leq 2$, $\dim(V_h^2(\mu_h)) \leq 4$ and $\dim(V_h^4(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^0, V^2 and V^4, respectively. Furthermore, by Proposition 4.2.9, we have $\dim(V_h^1(\mu_h)) \leq 8$ and $\dim(V_h^3(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^1 and V^3, respectively. This implies that $\dim(V_h(\mu_h)) \leq 24$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 24$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 24 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

Proposition 4.2.18. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 3$ and let $V = L_G(2\omega_1 + \omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 20,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = 2\omega_1 + \omega_3$ and let $L = L_1$. Then $\dim(V) = 48$, as $p = 2$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 6$, therefore

$$V\mid_{[L,L]} = V^0 \oplus \cdots \oplus V^6,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 6$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_3)$ and so, by Lemma 2.4.3, we also have $V^6 \cong (L_L(\omega_3))^* \cong L_L(\omega_3)$. As $p = 2$, we have $V^1 = \{0\}$, $V^3 = \{0\}$ and $V^5 = \{0\}$. This gives $\dim(V^2) = 20$, as $V^4 \cong (V^2)^*$, by Lemma 2.4.3. Since the weight $(\lambda - 2\alpha_1)\mid_{T'} = 2\omega_2 + \omega_3$ admits a maximal vector in V^2, it follows that V^2 has a composition factor isomorphic to $L_L(2\omega_2 + \omega_3)$. Remark that $\dim(L_L(2\omega_2 + \omega_3)) = 16$, as $L_L(2\omega_2 + \omega_3) \cong L_L(\omega_2)^{(2)} \otimes L_L(\omega_3)$, by Theorem 2.3.8. Moreover, we note that the dominant weight $(\lambda - 2\alpha_1 - 2\alpha_2 - \alpha_3)\mid_{T'} = \omega_3$, which occurs with multiplicity 3 in V^2, is a sub-dominant weight in the composition factor of V^2 isomorphic to $L_L(2\omega_2 + \omega_3)$, in which it has multiplicity 2. Therefore, as $\dim(V^2) = 20$, we determine that V^2 has two composition factors: one isomorphic to $L_L(2\omega_2 + \omega_3)$ and one isomorphic to $L_L(\omega_3)$. Lastly, by Lemma 2.4.3, it follows that V^4 also consists of exactly two composition factors: one isomorphic to $L_L(2\omega_2 + \omega_3)$ and one isomorphic to $L_L(\omega_3)$.

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 2, 4$ or $i = 6$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c)h_{\alpha_3}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $i = 0, 2, 4, 6$, as scalar multiplication by c^{3-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 4; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^2) = 20; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^4) = 20; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^6) = 4. \end{cases}$$

As $c \neq 1$ and $p = 2$, it follows that $\dim(V_s(\mu)) \leq 20$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $i = 0, 2, 4$ and $i = 6$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $i = 0, 2, 4, 6$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $i = 0, 2, 4$ and $i = 6$, where μ_h is any eigenvalue of h on V^i. In what follows, we will show that $\dim(V_h^2(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^2. First, we determine the eigenvalues

of h on the composition factor of V^2 isomorphic to $L_L(2\omega_2+\omega_3)$. For this, we recall that, by Theorem 2.3.8, we have

$$L_L(2\omega_2+\omega_3) \cong L_L(\omega_2)^{(2)} \otimes L_L(\omega_3).$$

Now, the eigenvalues of h on $L_L(\omega_2)^{(2)}$ have the form a_2^2, a_3^2, a_2^{-2} and a_3^{-2}, while the eigenvalues of h on $L_L(\omega_3)$, by (4.9), have the form a_2a_3, $a_2a_3^{-1}$, $a_2^{-1}a_3$ and $a_2^{-1}a_3^{-1}$, where $a_2, a_3 \in k^*$ are not both simultaneously equal to 1. Therefore, keeping in mind that the other composition factor of V^2 is isomorphic to $L_L(\omega_3)$, we determine that the eigenvalues of h on V^2, not necessarily distinct, are

$$\begin{cases} a_2a_3, a_2a_3^{-1}, a_2^{-1}a_3 \text{ and } a_2^{-1}a_3^{-1}, \text{ each with multiplicity at least 3;} \\ a_2^3a_3, a_2a_3^3, a_2^3a_3^{-1}, a_2^{-1}a_3^3, a_2a_3^{-3}, a_2^{-3}a_3, a_2^{-1}a_3^{-3} \text{ and } a_2^{-3}a_3^{-1}, \text{ each with multiplicity at least 1.} \end{cases}$$

<u>Case 1</u>: Consider the eigenvalue $\mu_h = 1$ of h on V^2.

(1) If $a_2a_3 = 1$, then $a_3 = a_2^{-1}$ and the eigenvalues of h on V^2, not necessarily distinct, are: a_2^2 and a_2^{-2}, each with multiplicity at least 5, a_2^4 and a_2^{-4}, each with multiplicity at least 2, and 1 with multiplicity 6. It follows that $\dim(V_h^2(1)) = 6$ and $\dim(V_h^2(\nu_h)) \leq 7$ for all eigenvalues ν_h with $\nu_h \neq \nu_h^{-1}$ of h on V^2.

(2) If $a_2^{-1}a_3^{-1} = 1$, then, as above, we obtain $\dim(V_h^2(1)) = 6$ and $\dim(V_h^2(\nu_h)) \leq 7$ for all eigenvalues ν_h with $\nu_h \neq \nu_h^{-1}$ of h on V^2.

(3) The cases of $a_2a_3^{-1} = 1$ and $a_2^{-1}a_3 = 1$ are analogs of (1) and (2).

Lastly, if each a_2a_3, $a_2^{-1}a_3^{-1}$, $a_2a_3^{-1}$ and $a_2^{-1}a_3$ is different that 1, then we get $\dim(V_h^2(1)) \leq 8$.

<u>Case 2</u>: Consider the eigenvalue μ_h of h on V^2 with $\mu_h \neq \mu_h^{-1}$. We first note that $\dim(V_h^2(\mu_h)) \leq 10$, as V^2 is self-dual.

(1) Suppose that $\mu_h = a_2a_3$.

(1.1) If $\mu_h = a_2a_3^{-1}$, then $a_3 = 1$, hence $a_2 \neq 1$, $\mu_h = a_2$ and the distinct eigenvalues of h on V^2 are a_2 and a_2^{-1}, each with multiplicity 8; and a_2^3 and a_2^{-3}, each with multiplicity 2.

(1.2) If $\mu_h = a_2^{-1}a_3$, then $a_2 = 1$, hence $a_3 \neq 1$, $\mu_h = a_3$, and, as in the previous case, we have $\dim(V_h^2(a_3)) = \dim(V_h^2(a_3^{-1})) = 8$ and $\dim(V_h^2(a_3^3)) = \dim(V_h^2(a_3^{-3})) = 2$.

(1.3) If $\mu_h \neq a_2a_3^{-1}$ and $\mu_h \neq a_2^{-1}a_3$, then $\dim(V_h^2(\mu_h)) \leq 7$. This completes the case of $\mu_h = a_2a_3$.

(2) If $\mu_h = a_2^{-1}a_3^{-1}$, then, we argue as in (1) to show that $\dim(V_h^2(\mu_h)) \leq 8$.

(3) If $\mu_h \neq a_2a_3$ and $\mu_h \neq a_2^{-1}a_3^{-1}$, then $\dim(V_h^2(\mu_h)) \leq 7$.

To summarize all of the above, we have shown that $\dim(V_h^2(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^2. Furthermore, we note that, as $V^4 \cong (V^2)^*$, we also have $\dim(V_h^4(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^4. Lastly, as $p = 2$, by Corollary 4.2.7, we have $\dim(V_h^0(\mu_h)) \leq 2$ and $\dim(V_h^6(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^0 and V^6, respectively. Therefore,

$\dim(V_h(\mu_h)) \leq 20$ for all eigenvalues μ_h of h on V, and, consequently, $\dim(V_s(\mu)) \leq 20$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 20$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.19. *Let k be an algebraically closed field of characteristic $p = 3$. Assume $\ell = 4$ and let $V = L_G(\omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 26,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_3$ and let $L = L_1$. Then $\dim(V) = 40$, as $p = 3$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 2$, therefore:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_3)$ and so, by Lemma 2.4.3, we also have $V^2 \cong (L_L(\omega_3))^* \cong L_L(\omega_3)$. Therefore, $\dim(V^1) = 14$, as $\dim(V^0) = \dim(V^2) = 13$. Since the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3)\mid_{T'} = \omega_4$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_4)$ and $\dim(V^1) \geq \dim(L_L(\omega_4)) = 14$, since $p \neq 2$. Therefore, $V^1 \cong L_L(\omega_4)$ and

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_4) \oplus L_L(\omega_3). \tag{4.32}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c) \cdots h_{\alpha_4}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{1-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are

$$\begin{cases} c \text{ with } \dim(V_s(c)) \geq \dim(V^0) = 13; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^1) = 14; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 13. \end{cases}$$

As $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 26$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1) \cdots h_{\alpha_4}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(-1)) = 26$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 2$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where μ_h is any eigenvalue of h on V^i. As $p = 3$, by Corollary 4.2.7, it follows that $\dim(V_h^0(\mu_h)) \leq 8$ and $\dim(V_h^2(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. Similarly, as $p \neq 2$, by Proposition 4.2.16, we have $\dim(V_h^1(\mu_h)) \leq 10$ for all

eigenvalues μ_h of h on V^1. This implies that $\dim(V_h(\mu_h)) \leq 26$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 26$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 26$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is reched. Therefore, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.20. *Let k be an algebraically closed field of characteristic $p \neq 3$. Assume $\ell = 4$ and let $V = L_G(\omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 30.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_3$ and let $L = L_1$. Then $\dim(V) = 48$, as $p \neq 3$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 2$, therefore

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_3)$ and so, by Lemma 2.4.3, we also have $V^2 \cong (L_L(\omega_3))^* \cong L_L(\omega_3)$. This gives $\dim(V^1) = 20$. Now, in V^1 the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3)\mid_{T'} = \omega_4$ admits a maximal vector, hence V^1 has a composition factor isomorphic to $L_L(\omega_4)$. Furthermore, the dominant weight $(\lambda - \alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4)\mid_{T'} = \omega_2$ occurs with multiplicity 2 in V^1 and it has multiplicity $1 - \delta_{p,2}$ in the composition factor of V^1 isomorphic to $L_L(\omega_4)$. By dimensional considerations, we deduce that, if $p \neq 2$, then V^1 consists of exactly two composition factors: one isomorphic to $L_L(\omega_4)$ and one isomorphic to $L_L(\omega_2)$, while, if $p = 2$, then V^1 consists of three composition factors: one isomorphic to $L_L(\omega_4)$ and two isomorphic to $L_L(\omega_2)$.

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c) \cdots h_{\alpha_4}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{1-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are

$$\begin{cases} c \text{ with } \dim(V_s(c)) \geq \dim(V^0) = 14; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1) = 20; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 14. \end{cases}$$

As $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 28$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 2$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where μ_h is any eigenvalue of h on V^i. As $p \neq 3$, by Corollary 4.2.6, we have $\dim(V_h^0(\mu_h)) \leq 8$ and $\dim(V_h^2(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. Similarly, by Lemma 4.2.3 and Proposition 4.2.16 if $p \neq 2$, respectively by Proposition 4.2.15 if $p = 2$, we have $\dim(V_h^1(\mu_h)) \leq 14$ if $p \neq 2$, respectively $\dim(V_h^1(\mu_h)) \leq 12$ if $p = 2$, for all eigenvalues

μ_h of h on V^1. In both cases it follows that $\dim(V_h(\mu_h)) \leq 30$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 30$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 30 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.21. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 4$ and let $V = L_G(\omega_4)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 28,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_4$ and let $L = L_1$. Then, as $p \neq 2$, we have $\dim(V) = 42 - \delta_{p,3}$ and, by Lemma 2.4.5, we have $e_1(\lambda) = 2$, therefore

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_4)$ and so, by Lemma 2.4.3, we also have $V^2 \cong (L_L(\omega_4))^* \cong L_L(\omega_4)$. Moreover, as $\dim(V^0) = \dim(V^2) = 14$, since $p \neq 2$, we have $\dim(V^1) = 14 - \delta_{p,3}$. Now, since the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)\mid_{T'} = \omega_3$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_3)$ and $\dim(V^1) \geq \dim(L_L(\omega_3))$. We deduce that $V^1 \cong L_L(\omega_3)$ and

$$V\mid_{[L,L]} \cong L_L(\omega_4) \oplus L_L(\omega_3) \oplus L_L(\omega_4). \tag{4.33}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c) \cdots h_{\alpha_4}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{1-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are

$$\begin{cases} c \text{ with } \dim(V_s(c)) \geq \dim(V^0) = 14; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^1) = 14 - \delta_{p,3}; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 14. \end{cases}$$

As $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 28$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1) \cdots h_{\alpha_4}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(-1)) = 28$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 2$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where μ_h is any eigenvalue of h on V^i. Now, either by Corollary 4.2.6 if $p \neq 3$, or by Corollary 4.2.7 if $p = 3$, it follows that $\dim(V_h^1(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^1. Similarly, as $p \neq 2$, by Proposition 4.2.16, we have $\dim(V_h^0(\mu_h)) \leq 10$ and $\dim(V_h^2(\mu_h)) \leq 10$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. This implies that $\dim(V_h(\mu_h)) \leq 28$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 28$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 28$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. Therefore, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 4.2.22. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 5$ and let $V = L_G(\omega_3)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 58.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\lambda = \omega_3$ and let $L = L_1$. Then $\dim(V) = 100$, as $p = 2$, and, by Lemma 2.4.5, we have $e_1(\lambda) = 2$, therefore:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], it follows that $V^0 \cong L_L(\omega_3)$ and so, by Lemma 2.4.3, we also have $V^2 \cong (L_L(\omega_3))^* \cong L_L(\omega_3)$. This gives $\dim(V^1) = 48$, as $\dim(V^0) = \dim(V^2) = 26$, since $p = 2$. Now, in V^1 the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3)\mid_{T'} = \omega_4$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_L(\omega_4)$ and $\dim(V^1) \geq \dim(L_L(\omega_4)) = 48$, since $p \neq 3$. We deduce that $V^1 \cong L_L(\omega_4)$ and

$$V\mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_4) \oplus L_L(\omega_3). \tag{4.34}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c) \cdots h_{\alpha_5}(c)$ with $c \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{1-i}, we determine that the distinct eigenvalues of s on V are

$$\begin{cases} c \text{ with } \dim(V_s(c)) = \dim(V^0) = 26; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1) = 48; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) = \dim(V^2) = 26. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 2$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where μ_h is any eigenvalue of h on V^i. As $p = 2$, by Corollary 4.2.7, it follows that $\dim(V_h^0(\mu_h)) \leq 14$ and $\dim(V_h^2(\mu_h)) \leq 14$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. Similarly, as $p = 2$, by Proposition 4.2.20, we have $\dim(V_h^1(\mu_h)) \leq 30$ for all eigenvalues μ_h of h on V^1. This implies that $\dim(V_h(\mu_h)) \leq 58$ for all eigenvalues μ_h of h on V, therefore $\dim(V_s(\mu)) \leq 58$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 58 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

We conclude this subsection by noting that Propositions 4.2.8 through 4.2.22 complete the proofs of Theorems 4.2.1 and 4.2.2, as they cover all the irreducible kG-modules $L_G(\lambda)$ corresponding to p-restricted dominant weights λ featured in Tables 2.7.2 and 2.7.3.

4.3 Eigenspace dimensions for unipotent elements

This section is dedicated to the proof of the following two theorems, analogs of Theorems 4.2.1 and 4.2.2 in the case of unipotent elements. Similar to the semisimple case, the proofs will be given in a series of results, each treating one of the candidate-modules. In Subsection 4.3.1, we determine $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, where G_u is the set of unipotent elements in G and V belongs to one of the families of kG-modules that satisfies the dimensional criteria (2.16), i.e. V is an irreducible kG-modules $L_G(\lambda)$ for which $\lambda \in F^{C_\ell}$, where $F^{C_\ell} = \{\omega_1, 2\omega_1, \omega_2\}$. We complete the proofs of the two theorems in Subsection 4.3.2, where we establish $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ for the irreducible kG-modules $V = L_G(\lambda)$ corresponding to p-restricted dominant weights λ featured in one of the Tables 2.7.2 and 2.7.3.

Theorem 4.3.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ, $\ell \geq 2$. Let T be a fixed maximal torus in G and let $V = L_G(\lambda)$, where either $\lambda \in F^{C_\ell}$, or λ is featured in one of the Tables* 2.7.2 *and* 2.7.3. *Then there exist non-identity unipotent elements $u \in G$ for which:*

$$\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$$

if and only if ℓ, λ and p appear in the following list:

(1) $\ell \geq 2$, $\lambda = \omega_1$ *and* $p \geq 0$;

(2) $\ell = 2$, $\lambda = \omega_2$ *and* $p \geq 0$;

(3) $\ell = 3, 4$, $\lambda = \omega_\ell$ *and* $p = 2$.

Theorem 4.3.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ, $\ell \geq 2$. Let T and V be as in Theorem* 4.3.1. *Then the value of $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ is given in the table below:*

V	Char.	Rank	$\max\limits_{u\in G_u\setminus\{1\}} \dim(V_u(1))$
$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 2$	$2\ell - 1$
${}^{\dagger}L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 2$	$2\ell^2 - \ell$
${}^{\star}L_G(\omega_2)$	$p \nmid \ell$	$\ell \geq 2$	$2\ell^2 - 3\ell + 1$
	$p \mid \ell$	$\ell = 2$	3
		$\ell \geq 3$	$2\ell^2 - 3\ell$
${}^{\dagger}L_G(\omega_1 + \omega_2)$	$p \geq 0$	$\ell = 2$	$8 - 3\delta_{p,5}$
${}^{\dagger}L_G(2\omega_2)$	$p \neq 2$	$\ell = 2$	≤ 8
${}^{\dagger}L_G(3\omega_1)$	$p \neq 2, 3$	$\ell = 2$	10
${}^{\dagger}L_G(\omega_1 + 2\omega_2)$	$p = 7$	$\ell = 2$	7
${}^{\dagger}L_G(3\omega_2)$	$p = 7$	$\ell = 2$	7
${}^{\dagger}L_G(2\omega_1 + \omega_2)$	$p = 3$	$\ell = 2$	≤ 13
${}^{\ddagger}L_G(\omega_\ell)$	$p = 2$	$3 \leq \ell \leq 8$	$3 \cdot 2^{\ell-2}$
${}^{\dagger}L_G(\omega_3)$	$p \neq 2$	$\ell = 3$	9
${}^{\dagger}L_G(\omega_1 + \omega_3)$	$p = 2$	$\ell = 3$	28
${}^{\dagger}L_G(2\omega_1 + \omega_3)$	$p = 2$	$\ell = 3$	28
${}^{\dagger}L_G(\omega_3)$	$p \geq 0$	$\ell = 4$	$34 - 7\delta_{p,3}$
${}^{\dagger}L_G(\omega_4)$	$p \neq 2$	$\ell = 4$	$\leq 28 - \delta_{p,3}$
${}^{\dagger}L_G(\omega_3)$	$p = 2$	$\ell = 5$	74

Table 4.3.1: The value of $\max\limits_{u\in G_u\setminus\{1\}} \dim(V_u(1))$.

In particular, for each V in Table 4.3.1 labeled as ${}^{\dagger}V$; as ${}^{\star}V$ with $\ell \geq 3$; and as ${}^{\ddagger}V$ with $\ell \geq 5$; we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

4.3.1 The families of modules

For the rest of this chapter, we fix the following hypothesis on unipotent elements in G:

$$({}^{\dagger}H_u) : \text{every } u \in G_u \setminus \{1\}, \text{ has Jordan form on } W \text{ given by } \bigoplus_{i=1}^{m} J_{n_i}^{r_i}, \text{ where } \sum_{i=1}^{m} n_i r_i = 2\ell,$$

$$r_i \geq 1 \text{ is even for all odd } n_i,\ 2\ell \geq n_1 > \cdots > n_m \geq 1 \text{ and } n_1 \geq 2.$$

Lemma 4.3.3. *Let $V = L_G(\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell - 1,$$

where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$.

In particular, there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. To begin, we note that $V \cong W$ as kG-modules. Now, let the unipotent element $u \in G$ be as in $({}^{\dagger}H_u)$. Let u_W denote the action of u on W. Then:

$$\dim(V_u(1)) = \dim(W_{u_W}(1)) = \sum_{i=1}^{m} r_i. \tag{4.35}$$

As $u \neq 1$, it follows that $\dim(V_u(1)) \leq 2\ell - 1$. Moreover, by Lemma 3.3.3, we have $\dim(V_u(1)) = 2\ell - 1$ if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 2\ell - 1$ for all non-identity unipotent elements $u \in G$ and that equality holds if and only if u has Jordan form $J_2 \oplus J_1^{2\ell-2}$ on W. Now, let u be such an element of G. Since the inequality $\sqrt{2\ell} \geq 1$ holds for all $\ell \geq 2$, it follows that $2\ell - 1 \geq 2\ell - \sqrt{2\ell}$ for all $\ell \geq 2$ and thus $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. □

The following corollary, although not relevant for this subsection, will be the fundamental result used in the proof of Lemma 6.3.4.

Corollary 4.3.4. *Assume $\ell \geq 4$ and let $V = L_G(\omega_1)$. Let u be a non-identity unipotent element of G and assume that its Jordan form on W is different than $J_2 \oplus J_1^{2\ell-2}$. Then:*

$$\dim(V_u(1)) \leq 2\ell - 2,$$

where equality holds if and only if u has Jordan form $J_2^2 \oplus J_1^{2\ell-4}$ on W.

Proof. Let the unipotent element $u \in G$ be as in $({}^{\dagger}H_u)$. We note that if $n_1 = 2$, then, by hypothesis, we have $r_1 \geq 2$. Moreover, by Lemma 4.3.3, as the Jordan form of u on W is different than $J_2 \oplus J_1^{2\ell-2}$, we have $\dim(V_u(1)) \leq 2\ell - 2$. Assume that $\dim(V_u(1)) = 2\ell - 2$. Then, by (4.35) and keeping in mind that $\sum_{i=1}^{m} n_i r_i = 2\ell$, it follows that $\sum_{i=1}^{m}(n_i - 1)r_i = 2$ and, in particular, that $2 \geq (n_1 - 1)r_1 \geq n_1 - 1$, hence $3 \geq n_1$.

Assume that $n_1 = 3$. Then $r_1 \geq 2$ and thus $\sum_{i=1}^{m}(n_i - 1)r_i \geq 4$, a contradiction. Therefore $n_1 = 2$, hence $m \leq 2$ and $r_1 \geq 2$. Moreover, as $\sum_{i=1}^{m}(n_i - 1)r_i \geq 2$, it follows that $r_1 = 2$. Lastly, as $\ell \geq 4$, we deduce that $m = 2$, $n_2 = 1$, $r_2 = 2\ell - 4$ and, consequently, the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$. Conversely, let $u \in G$ be a unipotent element whose Jordan form on W is $J_2^2 \oplus J_1^{2\ell-4}$. Then, by (4.35), we have that $\dim(V_u(1)) = 2\ell - 2$. This completes the proof of the corollary. □

Before, we continue the proofs of Theorems 4.3.1 and 4.3.2, we recall that the irreducible kG-module $L_G(\omega_2)$ is a composition factor of the kG-module $\wedge^2(W)$, see Lemma 2.8.2. This is a relevant fact, since, using Proposition 3.3.4, we can calculate the dimension of the fixed point space on $\wedge^2(W)$ of any unipotent element $u \in G$. With $\dim((\wedge^2(W))_u(1))$ known, we can deduce $\dim((L_G(\omega_2))_u(1))$ using either [Kor19, Corollary 6.2], or [Kor20, Theorem B], depending on whether $p \neq 2$ or $p = 2$. Before we state these two results, we recall that $r_t(u)$ is the number of Jordan blocks of size $t \geq 1$ appearing in the Jordan form of the unipotent element u; and that ν_p denotes the p-adic valuation on the integers.

Theorem 4.3.5. [Kor19, Corollary 6.2] *Let k be an algebraically closed field of characteristic $p \neq 2$. Let $u \in G$ be a unipotent element and let $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ be its Jordan form on W, where $m \geq 1$, $n_i \geq 1$ and $r_i \geq 1$ for all $1 \leq i \leq m$. Set $\alpha = \nu_p(\gcd(n_1, \ldots, n_m))$. Let u' be the action of u on $\wedge^2(W)$ and let u_V be the action of u on $V := L_G(\omega_2)$. Then the Jordan block sizes of u_V are determined from those of u' in the following way:*

(a) *If* $p \nmid \ell$, *then* $r_1(u_V) = r_1(u^{'}) - 1$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq 1$.

(b) *If* $p \mid \ell$ *and* $\alpha = 0$, *then* $r_1(u_V) = r_1(u^{'}) - 2$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq 1$.

(c) *If* $p \mid \ell$ *and* $\alpha > 0$, *then:*

(c.1) *If* $p \mid \dfrac{2\ell}{p^\alpha}$, *then* $r_{p^\alpha}(u_V) = r_{p^\alpha}(u^{'}) - 2$, $r_{p^\alpha - 1}(u_V) = 2$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq p^\alpha, p^\alpha - 1$.

(c.2) *If* $p \nmid \dfrac{2\ell}{p^\alpha}$, *then* $r_{p^\alpha}(u_V) = r_{p^\alpha}(u^{'}) - 1$, $r_{p^\alpha - 2}(u_V) = 1$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq p^\alpha, p^\alpha - 2$.

Theorem 4.3.6. [Kor20, Theorem B] *Let* k *be an algebraically closed field of characteristic* $p = 2$. *Let* $u \in G$ *be a unipotent element and let* $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, 2n_{t+1_1}^{r_{t+1}}, \ldots, 2n_{m_1}^{r_m})$ *be its Hesselink normal form, where* $m \geq 1$, $t \geq 0$ *and* $r_i \geq 1$ *for all* $1 \leq i \leq m$. *Set* $\alpha = \nu_2(\gcd(n_1, \ldots, n_t, n_{t+1}, \ldots, n_m))$. *Let* $u^{'}$ *denote the action of* u *on* $\wedge^2(W)$ *and let* u_V *denote the action of* u *on* $V := L_G(\omega_2)$. *Then the Jordan block sizes of* u_V *are determined from those of* $u^{'}$ *in the following way:*

(a) *If* $2 \nmid \ell$, *then* $r_1(u_V) = r_1(u^{'}) - 1$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq 1$.

(b) *If* $2 \mid \ell$ *and* $\alpha = 0$, *then* $r_1(u_V) = r_1(u^{'}) - 2$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq 1$.

(c) *If* $2 \mid \ell$ *and* $\alpha > 0$, *then*

(c.1) *If* $2 \mid \dfrac{\ell}{2^\alpha}$, *then* $r_{2^\alpha}(u_V) = r_{2^\alpha}(u^{'}) - 2$, $r_{2^\alpha - 1}(u_V) = 2$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq 2^\alpha, 2^\alpha - 1$.

(c.2) *If* $\alpha > 1$ *and* $2 \nmid \dfrac{\ell}{2^\alpha}$, *then* $r_{2^\alpha}(u_V) = r_{2^\alpha}(u^{'}) - 1$, $r_{2^\alpha - 2}(u_V) = 1$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq 2^\alpha, 2^\alpha - 2$.

(c.3) *If* $\alpha = 1$ *and* $2 \nmid \dfrac{\ell}{2^\alpha}$, *then* $r_2(u_V) = r_2(u^{'}) - 1$ *and* $r_t(u_V) = r_t(u^{'})$ *for all* $t \neq 2$.

Proposition 4.3.7. *Let* $V^{'} = \wedge^2(W)$. *Then for all non-identity unipotent elements* $u \in G$ *we have*

$$\dim(V_u^{'}(1)) \leq 2\ell^2 - 3\ell + 2.$$

Moreover, we have equality if and only if one of the following holds:

(1) $\ell = 2$ *and the Jordan form of* u *on* W *is one of* J_2^2 *and* $J_2 \oplus J_1^2$.

(2) $\ell \geq 3$ *and the Jordan form of* u *on* W *is* $J_2 \oplus J_1^{2\ell-2}$.

Proof. Let u be a non-identity unipotent element of G. We apply Proposition 3.3.4, keeping in mind that $\dim(W) = 2\ell$, to deduce that $\dim(V_u^{'}(1)) \leq \dfrac{(2\ell-1)^2 - (2\ell-1) + 2}{2} = 2\ell^2 - 3\ell + 2$ for all non-identity unipotent elements $u \in G$. Moreover, by the same result, for $\ell = 2$ equality holds if and only if the Jordan form of u on W is one of J_2^2 and $J_2 \oplus J_1^2$, while, for $\ell \geq 3$ equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$. □

Corollary 4.3.8. *Assume $p \nmid \ell$ and let $V = L_G(\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - 3\ell + 1.$$

Moreover, we have equality if and only if one of the following holds:

(1) *$\ell = 2$ and the Jordan form of u on W is one of J_2^2 and $J_2 \oplus J_1^2$.*

(2) *$\ell \geq 3$ and the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$.*

In particular, for $\ell = 2$ there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 3$, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we set $V' = \wedge^2(W)$. By Lemma 2.8.2, since $p \nmid \ell$, we have that $V' \cong V \oplus L_G(0)$ and therefore $\dim(V_u(1)) = \dim(V_u'(1)) - 1$. We now apply Proposition 4.3.7 to deduce that $\dim(V_u(1)) \leq 2\ell^2 - 3\ell + 1$ for all non-identity unipotent elements $u \in G$. Moreover, equality holds if and only if $\dim(V_u'(1)) = 2\ell^2 - 3\ell + 2$ and we use Proposition 4.3.7 once more to obtain the result.

To conclude, we first assume that $\ell = 2$, hence $\dim(V) = 5$, and we let u be a unipotent element of G whose Jordan form on W is J_2^2. Then $\dim(V_u(1)) = 3$ and so, we have shown that there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 3$, since the inequality $0 < (2\ell - 5)(\ell - 1)$ holds for all $\ell \geq 3$, it follows that $2\ell^2 - 3\ell + 1 < 2\ell^2 - \ell - 1 - \sqrt{2\ell^2 - \ell - 1}$ for all $\ell \geq 3$ and, consequently, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

As with Corollary 4.3.4, the following result, although not relevant for this subsection, will be one of the fundamental tools used in the proof of Proposition 6.3.6.

Lemma 4.3.9. *Let u be a non-identity unipotent element of G whose Jordan form on W is different than $J_2 \oplus J_1^{2\ell-2}$. If $V' = \wedge^2(W)$, then one of the following holds:*

(1) *$\ell = 2$ and $\dim(V_u'(1)) \leq 4$, where equality holds if and only if the Jordan form of u on W is J_2^2.*

(2) *$\ell = 3$ and $\dim(V_u'(1)) \leq 9$, where equality holds if and only if the Jordan form of u on W is one of J_2^3 and $J_2^2 \oplus J_1^2$.*

(3) *$\ell \geq 4$ and $\dim(V_u'(1)) \leq 2\ell^2 - 5\ell + 6$, where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$.*

Proof. We note that for $\ell = 2$ the result follows from Proposition 4.3.7 and thus, we can assume that $\ell \geq 3$.

Let the unipotent element $u \in G$ be as in $({}^{\dagger}H_u)$. First, we assume that the Jordan form of u on W is $J_{2\ell}$. Then, either by Lemma 2.9.4 if $p \neq 2$, or by Lemma 2.9.5 if $p = 2$, it follows that $\dim(V_u'(1)) = \ell$. Hence, for $\ell = 3$ we have $\dim(V_u'(1)) < 9$, while for $\ell \geq 4$ we have $\dim(V_u'(1)) < 2\ell^2 - 5\ell + 6$.

We can now assume that the Jordan form of u on W consists of at least two blocks and we first consider the case when exactly one of these blocks, J_{n_1}, is nontrivial. Then the Jordan form of u on W is $J_{n_1} \oplus J_1^{2\ell-n_1}$, where $4 \leq n_1 \leq 2\ell - 2$ is even, since $r_1 = 1$. We write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = 2\ell - n_1$ and u acts trivially on W_2. Then, we have the following isomorphism of $k[u]$-modules:

$$V^{'} \cong \wedge^2(W_1) \oplus (W_1 \otimes W_2) \oplus \wedge^2(W_2),$$

which gives:

$$\dim(V_u^{'}(1)) = \dim((\wedge^2(W_1))_u(1)) + \dim((W_1 \otimes W_2)_u(1)) + \dim((\wedge^2(W_2))_u(1)). \tag{4.36}$$

We now apply either Lemma 2.9.4 if $p \neq 2$, or Lemma 2.9.5 if $p = 2$, to deduce that $\dim((\wedge^2(W_1))_u(1)) = \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1}{2}$ and that $\dim((W_1 \otimes W_2)_u(1)) = 2\ell - n_1$. Lastly, as u acts trivially on W_2, it acts trivially on $\wedge^2(W_2)$, and so $\dim((\wedge^2(W_2))_u(1)) = \frac{(2\ell - n_1 - 1)(2\ell - n_1)}{2}$. Substituting in (4.36) gives:

$$\begin{aligned} \dim(V_u^{'}(1)) &= \frac{n_1}{2} + 2\ell - n_1 + \frac{(2\ell - n_1 - 1)(2\ell - n_1)}{2} \\ &= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 2\ell}{2}. \end{aligned} \tag{4.37}$$

Assume $\ell = 3$. Then u has Jordan form $J_4 \oplus J_1^2$ on W and by (4.37) it follows that $\dim(V_u^{'}(1)) = 5$. We can now assume that $\ell \geq 4$ and, by (4.37), we have

$$\dim(V_u^{'}(1)) = 2\ell^2 - 5\ell + 6 + \frac{n_1^2 - 4\ell n_1 + 12\ell - 12}{2}.$$

One checks that the inequality

$$n_1^2 - 4\ell n_1 + 12\ell - 12 < 0$$

holds for all $n_1 \in (2\ell - 2\sqrt{\ell^2 - 3\ell + 3},\ 2\ell + 2\sqrt{\ell^2 - 3\ell + 3})$ and all $\ell \geq 1$. Since $2\ell + 2\sqrt{\ell^2 - 3\ell + 3} > 2\ell - 2$ and since $2\ell - 2\sqrt{\ell^2 - 3\ell + 3} < 4$, it follows that, in particular, the inequality holds for all $4 \leq n_1 \leq 2\ell - 2$ and all $\ell \geq 4$. Therefore $\dim(V_u^{'}(1)) < 2\ell^2 - 5\ell + 6$ for all $\ell \geq 4$ and all unipotent elements u of G whose Jordan form on W is $J_{n_1} \oplus J_1^{2\ell-n_1}$, where $4 \leq n_1 \leq 2\ell - 2$.

Lastly, we consider the case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell - 2$ and we write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = n_1$ and u acts as J_{n_1} on $W_1^{'}$, and $\dim(W_2^{'}) = 2\ell - n_1$ and u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$. Now, either by Lemma 2.9.4 if $p \neq 2$, or by Lemma 2.9.5 if $p = 2$, we have $\dim((\wedge^2(W_1^{'}))_u(1)) = \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 + \epsilon}{2}$, where $\epsilon = 0$ if n_1 is even, or $\epsilon = -1$ if n_1 is odd, and, furthermore, since u acts as $(J_{n_1} \otimes J_{n_1})^{r_1-1} \oplus \bigoplus_{i=2}^{m} (J_{n_1} \otimes J_{n_i})^{r_i}$ on $W_1^{'} \otimes W_2^{'}$, we also deduce that

$$\dim((W_1^{'} \otimes W_2^{'})_u(1)) = (r_1 - 1)n_1 + \sum_{i=2}^{m} n_i r_i = 2\ell - n_1. \tag{4.38}$$

Substituting in (4.36) gives:

$$\dim(V_u^{'}(1)) = \frac{n_1+\epsilon}{2} + 2\ell - n_1 + \dim((\wedge^2(W_2^{'}))_u(1)). \tag{4.39}$$

Assume $\ell = 3$. Then the Jordan form of u on W is one of $J_4 \oplus J_2$, J_3^2, J_2^3 and $J_2^2 \oplus J_1^2$. In the first two cases, as u acts as a single Jordan block on $W_2^{'}$, we apply either Lemma 2.9.4 if $p \neq 2$, or Lemma 2.9.5 if $p = 2$, to determine that, in both cases, $\dim((\wedge^2(W_2^{'}))_u(1)) = 1$. Therefore, by (4.39), in both cases, we have $\dim(V_u^{'}(1)) = 5$. Similarly, for the last two cases, as u acts on $W_2^{'}$ as J_2^2 and as $J_2 \oplus J_1^2$, respectively, we apply Proposition 3.3.4 to determine that, in both cases, $\dim((\wedge^2(W_2^{'}))_u(1)) = 4$. Now, by (4.39), in both cases, we get $\dim(V_u^{'}(1)) = 9$. Therefore $\dim(V_u^{'}(1)) \leq 9$ for all unipotent elements $u \in G$ whose Jordan form on W admits at least two nontrivial blocks. Moreover, equality holds if and only if the Jordan form of u on W is one of J_2^3 and $J_2^2 \oplus J_1^2$.

We can now assume that $\ell \geq 4$. We use Proposition 3.3.4 to deduce that $\dim((\wedge^2(W_2^{'}))_u(1)) \leq \frac{(2\ell - n_1 - 1)^2 - (2\ell - n_1 - 1) + 2}{2}$, where equality holds if and only if u acts on $W_2^{'}$ as $J_2 \oplus J_1^{2\ell - n_1 - 2}$. Substituting in (4.39) gives:

$$\begin{aligned}\dim(V_u^{'}(1)) &\leq \frac{n_1+\epsilon}{2} + 2\ell - n_1 + \frac{(2\ell - n_1 - 1)^2 - (2\ell - n_1 - 1) + 2}{2} \\ &= \frac{4\ell^2 - 4\ell n_1 + n_1^2 - 2\ell + 2n_1 + 4 + \epsilon}{2} \\ &= 2\ell^2 - 5\ell + 6 + \frac{n_1^2 - 4\ell n_1 + 2n_1 + 8\ell - 8 + \epsilon}{2}.\end{aligned} \tag{4.40}$$

If $n_1 = 2$, then $\epsilon = 0$ and, by (4.40), we have $\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 6$, where, as previously noted, equality holds if and only if u acts on $W_2^{'}$ as $J_2 \oplus J_1^{2\ell-4}$. We can thus assume that $n_1 \geq 3$. One checks that the inequality

$$n_1^2 - 4\ell n_1 + 2n_1 + 8\ell - 8 + \epsilon < 0$$

holds for all $n_1 \in (2\ell - 1 - \sqrt{4\ell^2 - 12\ell + 9 - \epsilon},\ 2\ell - 1 + \sqrt{4\ell^2 - 12\ell + 9 - \epsilon})$ and all $\ell \geq 1$. Since $2\ell - 1 - \sqrt{4\ell^2 - 12\ell + 9 - \epsilon} < 3$, as $7 + \epsilon < 4\ell$ for all $\ell \geq 4$, and since $2\ell - 1 + \sqrt{4\ell^2 - 12\ell + 9 - \epsilon} > 2\ell - 2$, it follows that, in particular, the inequality holds for all $3 \leq n_1 \leq 2\ell - 2$ and all $\ell \geq 4$. Therefore, $\dim(V_u^{'}(1)) < 2\ell^2 - 5\ell + 6$ for all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks and $n_1 \geq 3$. This completes the proof of the lemma. □

Proposition 4.3.10. *Assume $p \mid \ell$ and let $V = L_G(\omega_2)$. Then for all non-identity unipotent elements $u \in G$ one of the following holds:*

(1) *$\ell = 2$ and $\dim(V_u(1)) \leq 3$, where equality holds if and only if the Hesselink normal form of u is (2_0^2).*

(2) *$\ell \geq 3$ and $\dim(V_u(1)) \leq 2\ell^2 - 3\ell$, where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$.*

In particular, for $\ell = 2$ there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. *On the other hand, for $\ell \geq 3$, we have* $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ *for all non-identity unipotent elements $u \in G$.*

Proof. To begin, set $V' = \wedge^2(W)$ and let u be a non-identity unipotent element of G. If we denote by u', respectively by u_V, the action of u on V', respectively on V, then either by Theorem 4.3.5 if $p \neq 2$, or by Theorem 4.3.6 if $p = 2$, we know that we can determine the Jordan form of u_V from that of u'.

First, assume $\ell = 2$. Then $p = 2$ and the Hesselink normal form of u is one of (4_1), (2_1^2), (2_0^2) and $(1_0^2, 2_1)$. In each case, one determines that the Jordan form of u' is $J_4 \oplus J_2$, $J_2^2 \oplus J_1^2$, $J_2^2 \oplus J_1^2$ and $J_2^2 \oplus J_1^2$, respectively. We apply Theorem 4.3.6, cases (c.3), (b), (c.3) and (b), respectively, to deduce that $\dim(V_u(1)) = 1, 2, 3$ and 2, respectively. We conclude that for $\ell = 2$, we have $\dim(V_u(1)) \leq 3$ and equality holds if and only if the Hesselink normal form of u is (2_0^2). We can now assume that $\ell \geq 3$.

First, we consider the case when $p \neq 2$. Let $n_1, \ldots, n_m$ be the distinct Jordan block sizes of u on W, where $m \geq 1$, and set $\alpha = \nu_p(\gcd(n_1, \ldots, n_m))$. If $\alpha = 0$, then, since $p \mid \ell$, by Theorem 4.3.5 (b), it follows that $\dim(V_u(1)) = \dim(V_u'(1)) - 2$. We now use Proposition 4.3.7 to deduce that $\dim(V_u(1)) \leq 2\ell^2 - 3\ell$ for all unipotent elements u, $u \neq 1$, of G with $\alpha = 0$. Moreover, by the same result, equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$. On the other hand, if $\alpha > 0$, then by Theorem 4.3.5 (c), it follows that $\dim(V_u(1)) = \dim(V_u'(1))$. Let $\ell = 3$. As $\alpha > 0$, the Jordan form of u on W is either J_6 or J_3^2. In both cases, by Proposition 4.3.9, it follows that $\dim(V_u'(1)) < 9$ and, consequently, $\dim(V_u(1)) < 9$. We now assume that $\ell \geq 4$. Again, as $\alpha > 0$, the Jordan form of u on W is different than $J_2 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-4}$, therefore, by Proposition 4.3.9, we have $\dim(V_u'(1)) < 2\ell^2 - 5\ell + 6$. As the inequality $2\ell^2 - 5\ell + 6 < 2\ell^2 - 3\ell$ holds for all $\ell \geq 4$, we deduce that $\dim(V_u'(1)) < 2\ell^2 - 3\ell$ and, consequently, $\dim(V_u(1)) < 2\ell^2 - 3\ell$ for all unipotent elements u of G with $\alpha > 0$.

We now consider the case when $p = 2$. For $t \geq 0$, let $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, 2n_{t+1_1}^{r_{t+1}}, \ldots, 2n_{m_1}^{r_m})$ be the Hesselink normal form of u. Set $\alpha = \nu_2(\gcd(n_1, \ldots, n_t, n_{t+1}, \ldots, n_m))$. If $\alpha = 0$, we use Theorem 4.3.6 (b), Proposition 4.3.7 and proceed as in the analogous case of $p \neq 2$ to deduce that $\dim(V_u(1)) \leq 2\ell^2 - 3\ell$ for all unipotent elements u, $u \neq 1$, of G. Moreover, equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$. On the other hand, if $\alpha > 0$, we use Theorem 4.3.6 (c), Proposition 4.3.9 and proceed as in the analogous case of $p \neq 2$ to deduce that $\dim(V_u(1)) < 2\ell^2 - 3\ell$ for all unipotent elements u of G with $\alpha > 0$.

In conclusion, we have shown that for $\ell = 2$ there exist non-identity unipotent elements $u \in G$, for example those with Hesselink normal form (2_0^2), for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 3$, since the inequality $0 < (\ell - 2)(2\ell - 3)$ holds for all $\ell \geq 3$, we have $2\ell^2 - 3\ell < 2\ell^2 - \ell - 2 - \sqrt{2\ell^2 - \ell - 2}$ and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

As with Corollary 4.3.4 and Lemma 4.3.9, the following result, although not relevant for this subsection, will be one of the fundamental tools used in the proof of Proposition 6.3.6.

Proposition 4.3.11. *Let u be a non-identity unipotent element of G whose Jordan form on W is different than J_2^2, J_2^3, $J_2 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-4}$. If $V' = \wedge^2(W)$, then:*

$$\dim(V_u'(1)) \leq 2\ell^2 - 5\ell + 4.$$

Furthermore, we have equality if and only if one of the following holds:

(1) $\ell = 2$ *and the Jordan form of* u *on* W *is* J_4.

(2) $\ell = 4$ *and the Jordan form of* u *on* W *is one of* J_2^4 *and* $J_2^3 \oplus J_1^2$.

Proof. Let the unipotent element $u \in G$ be as in $({}^{\dagger}H_u)$. We note that, if $n_1 = 2$, then, by hypothesis, $\ell \geq 4$ and $r_1 \geq 3$. First, assume that the Jordan form of u on W is $J_{2\ell}$. Then, either by Lemma 2.9.4 if $p \neq 2$, or by Lemma 2.9.5 if $p = 2$, we have $\dim(V_u^{'}(1)) = \ell$ and, as the inequality $0 \leq \ell^2 - 3\ell + 2$ holds for all $\ell \geq 2$, it follows that $\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 4$. Moreover, equality holds if and only if $\ell = 2$, in which case u has Jordan form J_4 on W. We can now assume that the Jordan form of u on W admits at least two blocks. Furthermore, we can also assume that $\ell \geq 3$.

Secondly, we consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. Then $4 \leq n_1 \leq 2\ell - 2$ is even and we write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = 2\ell - n_1$ and u acts trivially on W_2. We proceed as in the proof of Lemma 4.3.9, see arguments leading to (4.37), to deduce that:

$$\dim(V_u^{'}(1)) = \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 2\ell}{2} = 2\ell^2 - 5\ell + 4 + \frac{n_1^2 - 4\ell n_1 + 12\ell - 8}{2}.$$

One checks that the inequality

$$n_1^2 - 4\ell n_1 + 12\ell - 8 < 0$$

holds for all $n_1 \in (2\ell - 2\sqrt{\ell^2 - 3\ell + 2},\ 2\ell + 2\sqrt{\ell^2 - 3\ell + 2})$ and all $\ell \geq 2$. Since $2\ell - 2\sqrt{\ell^2 - 3\ell + 2} < 4$ and since $2\ell + 2\sqrt{\ell^2 - 3\ell + 2} > 2\ell - 2$, it follows that, in particular, the inequality holds for all $4 \leq n_1 \leq 2\ell - 2$ and all $\ell \geq 3$. Hence, $\dim(V_u^{'}(1)) < 2\ell^2 - 5\ell + 4$ for all unipotent elements $u \in G$ whose Jordan form on W is $J_{n_1} \oplus J_1^{2\ell - n_1}$, where $4 \leq n_1 \leq 2\ell - 2$.

Lastly, we consider the case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell - 2$. Assume that $n_1 = 2$. Then $\ell \geq 4$, $r_1 \geq 3$ and we write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = 4$ and u acts as J_2^2 on $W_1^{'}$, and $\dim(W_2^{'}) = 2\ell - 4$ and u acts as $J_2^{r_1 - 2} \oplus J_1^{2\ell - 2r_1}$ on $W_2^{'}$. By (4.36), to determine $\dim(V_u^{'}(1))$ comes down to determining $\dim((\wedge^2(W_1^{'}))_u(1))$, $\dim((W_1^{'} \otimes W_2^{'})_u(1))$ and $\dim((\wedge^2(W_2^{'}))_u(1))$. By Proposition 4.3.7 we have $\dim((\wedge^2(W_1^{'}))_u(1)) = 4$ and, furthermore, as u acts nontrivially on $W_2^{'}$, since $r_1 \geq 3$, we also have $\dim((\wedge^2(W_2^{'}))_u(1)) \leq 2(\ell - 2)^2 - 3(\ell - 2) + 2 = 2\ell^2 - 11\ell + 16$. Moreover, by the same result, equality holds if and only if u acts on $W_2^{'}$ as one of J_2^2 and $J_2 \oplus J_1^{2\ell - 6}$. Finally, as u acts as $(J_2 \otimes J_2)^{2r_1 - 4} \oplus (J_2 \otimes J_1)^{4\ell - 4r_1}$ on $W_1^{'} \otimes W_2^{'}$, we use Lemma 2.9.4 to deduce that $\dim((W_1^{'} \otimes W_2^{'})_u(1)) = 2(2r_1 - 4) + 4\ell - 4r_1 = 4\ell - 8$. By (4.36), it follows that:

$$\dim(V_u^{'}(1)) \leq 4 + 4\ell - 8 + 2\ell^2 - 11\ell + 16 = 2\ell^2 - 7\ell + 12.$$

As the inequality $2\ell^2 - 7\ell + 12 \leq 2\ell^2 - 5\ell + 4$ holds for all $\ell \geq 4$, we have $\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 4$ for all unipotent elements u of G whose Jordan form on W is $J_2^{r_1} \oplus J_1^{2\ell - 2r_1}$, where $r_1 \geq 3$. Furthermore, equality holds if and only if $2\ell^2 - 7\ell + 12 = 2\ell^2 - 5\ell + 4$, hence $\ell = 4$, and $\dim((\wedge^2(W_2^{'}))_u(1)) = 2\ell^2 - 11\ell + 16$, hence u acts on $W_2^{'}$ as one of J_2^2 and $J_2 \oplus J_1^2$, see Proposition 4.3.7. Therefore, $\dim(V_u^{'}(1)) = 2\ell^2 - 5\ell + 4$ if and only if $\ell = 4$ and the Jordan form of u on W is one of J_2^4 and $J_2^3 \oplus J_1^2$.

We now assume that $n_1 \geq 3$. We write $W = W_1^{''} \oplus W_2^{''}$, where $\dim(W_1^{''}) = n_1$ and u acts as J_{n_1} on $W_1^{''}$, and $\dim(W_2^{''}) = 2\ell - n_1$ and u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{''}$. We then proceed as in the proof of Lemma 4.3.9, see (4.38), (4.39) and (4.40), to deduce that:

$$\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 4 + \frac{n_1^2 - 4\ell n_1 + 2n_1 + 8\ell - 4 + \epsilon}{2}.$$

One checks that the inequality

$$n_1^2 - 4\ell n_1 + 2n_1 + 8\ell - 4 + \epsilon < 0$$

holds for all $n_1 \in (2\ell - 1 - \sqrt{4\ell^2 - 12\ell + 5 - \epsilon},\ 2\ell - 1 + \sqrt{4\ell^2 - 12\ell + 5 - \epsilon})$ and all $\ell \geq 3$. Since $2\ell - 1 + \sqrt{4\ell^2 - 12\ell + 5 - \epsilon} > 2\ell - 2$ and since $2\ell - 1 - \sqrt{4\ell^2 - 12\ell + 5 - \epsilon} < 3$, as $11 + \epsilon < 4\ell$ holds for all $\ell \geq 3$, it follows that, in particular, the inequality holds for all $3 \leq n_1 \leq 2\ell - 2$ and all $\ell \geq 3$. We deduce that $\dim(V_u^{'}(1)) < 2\ell^2 - 5\ell + 4$ for all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks and $n_1 \geq 3$. This completes the proof of the proposition. □

Proposition 4.3.12. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V = L_G(2\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - \ell,$$

where equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. We first note that, as $p \neq 2$, by Lemma 2.8.2, we have $V \cong \mathrm{S}^2(W)$. Keeping in mind that $\dim(W) = 2\ell$, we apply Proposition 3.3.5 to determine that $\dim(V_u(1)) \leq \dfrac{2\ell(2\ell-1)}{2} = 2\ell^2 - \ell$ for all non-identity unipotent elements $u \in G$. Furthermore, equality holds if and only if the Jordan form of u on W is $J_2 \oplus J_1^{2\ell-2}$.

In conclusion, as $\sqrt{2\ell^2 + \ell} < 2\ell$ for all $\ell \geq 2$, it follows that the inequality $2\ell^2 + \ell - \sqrt{2\ell^2 + \ell} > 2\ell^2 - \ell$ holds for all $\ell \geq 2$, and so $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

To conclude this subsection, we remark that Lemma 4.3.3, Propositions 4.3.10 and 4.3.12 and Corollary 4.3.8 give the proof of Theorems 4.3.1 and 4.3.2 for the families of kG-modules corresponding to p-restricted dominant weights $\lambda \in F^{C_\ell}$.

4.3.2 The particular modules

As previously mentioned, this subsection is dedicated to the proof of Theorems 4.3.1 and 4.3.2 for the particular kG-modules, i.e. the kG-modules $V = L_G(\lambda)$ for which the corresponding p-restricted dominant weight λ is listed in one of the Tables 2.7.2 and 2.7.3. In order to determine an upper-bound for $\dim(V_u(1))$, where $u \in G$ is a non-identity unipotent element,

we will use the inductive algorithm described in Subsection 2.4.4. In the first part of this subsection, we will consider the case of $\ell = 2$ and for each $V = L_G(\lambda)$ with λ featured in Table 2.7.2 we will establish $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, see Propositions 4.3.13 through 4.3.19. In the second part, we assume that $\ell \geq 3$ and we focus our attention on the irreducible kG-modules $V = L_G(\lambda)$, where λ is listed in Table 2.7.3, for which we will establish an upper-bound for $\dim(V_u(1))$, see Propositions 4.3.13 through 4.3.31.

For the moment, let us assume that $\ell = 2$. Let L_1 and L_2, respectively, be Levi subgroups of the maximal parabolic subgroups P_1 and P_2, respectively, of G, see Section 2.4. Now, if $p \neq 2$, we have seen in Theorem 2.9.2 that unipotent conjugacy classes in G are completely determined by the Jordan normal form of a class representative on W. In Table 4.3.2 we list all unipotent conjugacy classes of G and we give a representative. Note that for each non-identity class, the representative $u^{'}$ has been chosen such that either $u^{'}_{L_1} \neq 1$, or $u^{'}_{L_2} \neq 1$. On the other hand, when $p = 2$, we have seen in Theorem 2.9.11 that unipotent conjugacy classes in G are completely determined by the Hesselink normal form of a class representative. In Table 4.3.3 we list all unipotent conjugacy classes of G and for each class we give a representative. Once more, note that for each non-identity class, the representative $u^{'}$ has been chosen such that either $u^{'}_{L_1} \neq 1$, or $u^{'}_{L_2} \neq 1$.

[LS12, Subsection 3.3.2][MKT21, Table 7]

Class representative	Jordan normal form
1	J_1^4
$x_{\alpha_1}(1)$	J_2^2
$x_{\alpha_2}(1)$	$J_2 \oplus J_1^2$
$x_{\alpha_1}(1)x_{\alpha_2}(1)$	J_4

Table 4.3.2: Unipotent class representatives in C_2 when $p \neq 2$.

[Remark 2.9.19][MKT21, Table 7]

Class representative	Hesselink normal form
1	(1_0^4)
$x_{\alpha_1}(1)$	(2_0^2)
$x_{\alpha_2}(1)$	$(1_0^2, 2_1)$
$x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)$	(2_1^2)
$x_{\alpha_1}(1)x_{\alpha_2}(1)$	(4_1)

Table 4.3.3: Unipotent class representatives in C_2 when $p = 2$.

Let $u \in G$ be a non-identity unipotent element and let $u^{'}$ be the class representative given in either Table 4.3.2 or Table 4.3.3, depending on whether $p \neq 2$, or $p = 2$, of the unipotent conjugacy class of u. Because of the choice of representatives, we either have $u^{'}_{L_1} \neq 1$, or $u^{'}_{L_2} \neq 1$. First, suppose that $u^{'}_{L_1} \neq 1$, thus $u^{'}_{L_1} = x_{\alpha_2}(1)$. Now, as $\dim(V_u(1)) = \dim(V_{u'}(1))$, $\dim(V_{u'}(1)) \leq \dim(V_{x_{\alpha_2}(1)}(1))$, see Inequality (2.7), it follows that $\dim(V_u(1)) \leq \dim(V_{x_{\alpha_2}(1)}(1))$. Now, suppose that $u^{'}_{L_1} = 1$, hence $u^{'}_{L_2} \neq 1$ and so $u^{'}_{L_2} = x_{\alpha_1}(1)$. We argue exactly as in the case of $u^{'}_{L_1} \neq 1$ to show that $\dim(V_u(1)) \leq \dim(V_{x_{\alpha_1}(1)}(1))$. Lastly, as either

$u'_{L_1} \neq 1$, or $u'_{L_2} \neq 1$, for any pair (u, u') of a non-identity unipotent element $u \in G$ and a class representative u' of the G-conjugacy class of u, we have:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}. \tag{4.41}$$

Moreover, by Identity (2.8), we deduce that there exist unipotent elements $u \in G$ for which the bound in (4.41) is attained, for example $x_{\alpha_1}(1)$ or $x_{\alpha_2}(1)$, depending on whether $\dim(V_{x_{\alpha_1}(1)}(1))$, or $\dim(V_{x_{\alpha_2}(1)}(1))$ realizes the maximum in (4.41). Hence, in what follows, we concentrate on determining $\max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}$.

Proposition 4.3.13. *Let k be an algebraically closed field of characteristic $p = 5$. Assume $\ell = 2$ and let $V = L_G(\omega_1 + \omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 5,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Let $u \in G$ be a non-identity unipotent element. We recall Inequality (4.41), which states that:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}.$$

Therefore, in order to establish a bound for $\dim(V_u(1))$, we need to determine $\dim(V_{x_{\alpha_1}(1)}(1))$ and $\dim(V_{x_{\alpha_2}(1)}(1))$. We begin with the former. For this, we recall the Decomposition (4.23) of Proposition 4.2.8, which states:

$$V\mid_{[L_2,L_2]} \cong L_{L_2}(\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1).$$

By applying Lemma 3.3.3 and Proposition 3.3.12 it follows that $\dim(V_{x_{\alpha_1}(1)}(1)) = 4$.

We will now determine $\dim(V_{x_{\alpha_2}(1)}(1))$. For this, we consider the Levi subgroup L_1 of the maximal parabolic subgroup P_1 of G. Let T' denote the maximal torus $T \cap [L_1, L_1]$ of $[L_1, L_1]$. Set $\lambda = \omega_1 + \omega_2$ and note that $\dim(V) = 12$, as $p = 5$. By Lemma 2.4.5, we have $e_1(\lambda) = 4$, therefore

$$V\mid_{[L_1,L_1]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3 \oplus V^4,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for $0 \leq i \leq 4$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\omega_2)$ and, by Lemma 2.4.3, we also have $V^4 \cong (L_{L_1}(\omega_2))^* \cong L_{L_1}(\omega_2)$. Now, the weight $(\lambda - \alpha_1)\mid_{T'} = 2\omega_2$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_{L_1}(2\omega_2)$, thus $\dim(V^1) \geq 3$. Moreover, as $V^3 \cong (V^1)^*$, see Lemma 2.4.3, we have $\dim(V^2) \leq 2$. Lastly, since the weight $\lambda - 2\alpha_1 - \alpha_2 \mid_{T'} = \omega_2$ admits a maximal vector in V^2, it follows that $V^2 \cong L_{L_1}(\omega_2)$. Therefore, $V^1 \cong L_{L_1}(2\omega_2)$, $V^3 \cong L_{L_1}(2\omega_2)$ and

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(\omega_2).$$

We now apply Lemma 3.3.3 and Proposition 3.3.5 to determine that $\dim(V_{x_{\alpha_2}(1)}(1)) = 5$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 5$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Moreover, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.14. *Let k be an algebraically closed field of characteristic $p \neq 5$. Assume $\ell = 2$ and let $V = L_G(\omega_1 + \omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 8,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Let $u \in G$ be a non-identity unipotent element. We recall Inequality (4.41), which states that:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}.$$

Therefore, in order to establish a bound for $\dim(V_u(1))$, we need to determine $\dim(V_{x_{\alpha_1}(1)}(1))$ and $\dim(V_{x_{\alpha_2}(1)}(1))$.

First, we assume that $p = 2$. In this case, by [Sei87, (1.6)], we have the following isomorphism of kG-modules:

$$V = L_G(\omega_1) \otimes L_G(\omega_2).$$

We start with determining $\dim(V_{x_{\alpha_1}(1)}(1))$. Now, $x_{\alpha_1}(1)$ has Hesselink normal form (2_0^2), see Table 4.3.3. Thereby, the Jordan form of $x_{\alpha_1}(1)$ on W, hence on $L_G(\omega_1)$, is J_2^2. This gives rise to the following $k[x_{\alpha_1}(1)]$-module isomorphism: $W\mid_{k[x_{\alpha_1}(1)]} \cong V_2 \oplus V_2$, where V_i, $i = 1, 2$, is the unique, up to isomorphism, indecomposable $k[x_{\alpha_1}(1)]$-module with $\dim(V_i) = i$ and on which $x_{\alpha_1}(1)$ acts as the full Jordan block of size i. Then

$$\begin{aligned}\wedge^2(W) \cong \wedge^2(V_2 \oplus V_2) &= \wedge^2(V_2) \oplus V_2 \otimes V_2 \oplus \wedge^2(V_2)\\ &= V_2^2 \oplus V_1^2,\end{aligned}$$

as $k[x_{\alpha_1}(1)]$-modules. Thus, the Jordan form of the action of $x_{\alpha_1}(1)$ on $\wedge^2(W)$ is $J_2^2 \oplus J_1^2$. Since the Hesselink normal form of $x_{\alpha_1}(1)$ is (2_0^2), it follows that $\alpha = \nu_2(2) = 1$, and since $2 \nmid \frac{\ell}{2^\alpha}$, we use case (c.3) of Theorem 4.3.6 to determine that the Jordan form of the action of $x_{\alpha_1}(1)$ on $L_G(\omega_2)$ is $J_2 \oplus J_1^2$. Therefore, the Jordan form of the action of $x_{\alpha_1}(1)$ on V is $J_2^2 \otimes (J_2 \oplus J_1^2) = J_2^8$, since $p = 2$, and so $\dim(V_{x_{\alpha_1}(1)}(1)) = 8$.

We will now determine $\dim(V_{x_{\alpha_2}(1)}(1))$. As $x_{\alpha_2}(1)$ has Hesselink normal form $(1_0^2, 2_1)$, see Table 4.3.3, it follows that the Jordan form of $x_{\alpha_2}(1)$ on W, hence on $L_G(\omega_1)$, is $J_2 \oplus J_1^2$. This gives rise to the following $k[x_{\alpha_2}(1)]$-module isomorphism: $W\mid_{k[x_{\alpha_2}(1)]} \cong V_2 \oplus V_1^2$, where V_i, $i = 1, 2$, is the unique, up to isomorphism, indecomposable $k[x_{\alpha_2}(1)]$-module with $\dim(V_i) = i$ and on which $x_{\alpha_2}(1)$ acts as the full Jordan block of size i. Then:

$$\begin{aligned}\wedge^2(W) \cong \wedge^2(V_2 \oplus V_1^2) &= \wedge^2(V_2) \oplus V_2 \otimes V_1^2 \oplus \wedge^2(V_1^2)\\ &= V_2^2 \oplus V_1^2,\end{aligned}$$

as $k[x_{\alpha_2}(1)]$-modules. Thus, the Jordan form of the action of $x_{\alpha_2}(1)$ on $\wedge^2(W)$ is $J_2^2 \oplus J_1^2$. Since the Hesselink normal form of $x_{\alpha_2}(1)$ is $(1_0^2, 2_1)$, it follows that $\alpha = \nu_2(1) = 0$, and, since $2 \mid \ell$, we use case (b) of Theorem 4.3.6 to determine that the Jordan form of the action of $x_{\alpha_2}(1)$ on $L_G(\omega_2)$ is J_2^2. Therefore, the Jordan form of the action of $x_{\alpha_2}(1)$ on V is $(J_2 \oplus J_1^2) \otimes J_2^2 = J_2^8$, since $p = 2$, and so $\dim(V_{x_{\alpha_2}(1)}(1)) = 8$.

We can now assume that $p \neq 2$. We will first determine $\dim(V_{x_{\alpha_2}(1)}(1))$. For this, we recall the decomposition of $V\mid_{[L_1,L_1]}$ from Proposition 4.2.9, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(0) \oplus L_{L_1}(\omega_2) \oplus L_{L_1}(\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(0) \oplus L_{L_1}(\omega_2).$$

By applying Lemma 3.3.3 and Proposition 3.3.5, we deduce that $\dim(V_{x_{\alpha_2}(1)}(1)) = 8$.

We will now determine $\dim(V_{x_{\alpha_1}(1)}(1))$. For this, we consider the Levi subgroup L_2 of the maximal parabolic subgroup P_2 of G. Let T' denote the maximal torus $T \cap [L_2, L_2]$ of $[L_2, L_2]$. Set $\lambda = \omega_1 + \omega_2$ and note that $\dim(V) = 16$. By Lemma 2.4.5, we have $e_2(\lambda) = 3$, therefore

$$V\mid_{[L_2,L_2]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for $i = 0, 1, 2$ and $i = 3$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_2}(\omega_1)$, hence, by Lemma 2.4.3, we also have $V^3 \cong (L_{L_2}(\omega_1))^* = L_{L_2}(\omega_1)$. Then $\dim(V^1) + \dim(V^2) = 12$ and, as $V^2 \cong (V^1)^*$, see Lemma 2.4.3, it follows that $\dim(V^1) = \dim(V^2) = 6$. Now, the weight $(\lambda - \alpha_2)\mid_{T'} = 3\omega_1$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_{L_2}(3\omega_1)$. Moreover, we note that the dominant weight $(\lambda - \alpha_1 - \alpha_2)\mid_{T'} = \omega_1$ occurs in V^1 with multiplicity 2. We now distinguish the following two cases:

<u>Case 1</u>: Assume that $p \neq 3$. Then ω_1 is a sub-dominant weight with multiplicity 1 in the composition factor of V^1 isomorphic to $L_{L_2}(3\omega_1)$. Now, by dimensional considerations, we deduce that V^1 admits exactly two composition factors, one isomorphic to $L_{L_2}(3\omega_1)$ and one isomorphic to $L_{L_2}(\omega_1)$. Using [Jan07, II.2.14], we determine that $V^1 \cong L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1)$, therefore $V^2 \cong L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1)$, by Lemma 2.4.3, and so:

$$V\mid_{[L_2,L_2]} \cong L_{L_2}(\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1) \oplus L_{L_2}(\omega_1).$$

We apply Lemma 3.3.3 and Proposition 3.3.12 to determine that $\dim(V_{x_{\alpha_1}(1)}(1)) = 6$.

<u>Case 2</u>: Assume that $p = 3$. Then V^1, respectively V^2, admits exactly three composition factors: two isomorphic to $L_{L_2}(\omega_1)$ and one isomorphic to $L_{L_2}(\omega_1)^{(3)}$. In this case, using Lemmas 2.4.9 and 3.3.3, we determine that $\dim(V_{x_{\alpha_1}(1)}(1)) \leq 8$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 8$ for all non-identity unipotent elements $u \in G$ and that there exist unipotent elements $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Therefore, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.15. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 2$ and let $V = L_G(2\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 8.$$

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Let $u \in G$ be a non-identity unipotent element. We recall Inequality (4.41), which states that:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}.$$

Therefore, in order to establish a bound for $\dim(V_u(1))$, we need to determine $\dim(V_{x_{\alpha_1}(1)}(1))$ and $\dim(V_{x_{\alpha_2}(1)}(1))$. We begin with the latter. For this, we recall the decomposition of $V\mid_{[L_1,L_1]}$ from Proposition 4.2.10, which, in the case of $p=5$, states that:

$$V\mid_{[L_1,L_1]}\cong L_{L_1}(2\omega_2)\oplus L_{L_1}(\omega_2)\oplus L_{L_1}(2\omega_2)\oplus L_{L_1}(\omega_2)\oplus L_{L_1}(2\omega_2),$$

see (4.24). Then, by Lemma 3.3.3 and Proposition 3.3.5, we determine that $\dim(V_{x_{\alpha_2}(1)}(1))=5$. Similarly, by (4.25), in the case of $p\neq 5$, we have

$$V\mid_{[L_1,L_1]}\cong L_{L_1}(2\omega_2)\oplus L_{L_1}(\omega_2)\oplus L_{L_1}(2\omega_2)\oplus L_{L_1}(0)\oplus L_{L_1}(\omega_2)\oplus L_{L_1}(2\omega_2).$$

Then, by Lemma 3.3.3 and Proposition 3.3.5, it follows that $\dim(V_{x_{\alpha_2}(1)}(1))=6$.

We will now determine $\dim(V_{x_{\alpha_1}(1)}(1))$. For this, we consider the Levi subgroup L_2 of the maximal parabolic subgroup P_2 of G. Let T' denote the maximal torus $T\cap[L_2,L_2]$ of $[L_2,L_2]$. Now, as $p\neq 2$, we have $\dim(V)=14-\delta_{p,5}$. Set $\lambda=2\omega_2$. By Lemma 2.4.5 we have $e_2(\lambda)=4$ and so:

$$V\mid_{[L_2,L_2]}=V^0\oplus\cdots\oplus V^4,$$

where $V^i=\bigoplus_{\gamma\in\mathbb{N}\Delta_2}V_{\lambda-i\alpha_2-\gamma}$ for $i=0,1,2,3$ and $i=4$. By [Smi82, Proposition], it follows that $V^0\cong L_{L_2}(0)$ and so $V^4\cong L_{L_2}(0)$, by Lemma 2.4.3. Now, the weight $(\lambda-\alpha_2)\mid_{T'}=2\omega_1$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_{L_2}(2\omega_1)$ and $\dim(V^1)\geq 3$, as $p\neq 2$. By Lemma 2.4.3, we also have $\dim(V^3)\geq 3$, thus $\dim(V^2)\leq 6+\delta_{p,5}$. In V^2, the dominant weight $(\lambda-2\alpha_2)\mid_{T'}=4\omega_1$ admits a maximal vector, therefore V^2 has a composition factor isomorphic to $L_{L_2}(4\omega_1)$. We also note that the dominant weight $(\lambda-2\alpha_1-2\alpha_2)\mid_{T'}=0$ occurs with multiplicity $2-\delta_{p,5}$ in V^2. We distinguish the following three case:

<u>Case 1</u>: Assume that $p\geq 7$. Then $\dim(V^2)\leq 6$ and the multiplicity of the weight 0 in V^2 is 2. As the weight 0 is a sub-dominant weight with multiplicity 1 in the composition factor of V^2 isomorphic to $L_{L_2}(4\omega_1)$ and as $\dim(V^2)\leq 6$, it follows that V^2 consists of exactly two composition factors: one isomorphic to $L_{L_2}(4\omega_1)$ and one isomorphic to $L_{L_2}(0)$. Furthermore, by [Jan07, II.2.14], we deduce that $V^2\cong L_{L_2}(4\omega_1)\oplus L_{L_2}(0)$. Moreover, we have $V^1\cong L_{L_2}(2\omega_1)$, hence $V^3\cong L_{L_2}(2\omega_1)$, and so:

$$V\mid_{[L_2,L_2]}\cong L_{L_2}(0)\oplus L_{L_2}(2\omega_1)\oplus L_{L_2}(4\omega_1)\oplus L_{L_2}(0)\oplus L_{L_2}(2\omega_1)\oplus L_{L_2}(0).$$

We now use Proposition 3.3.5 and Proposition 3.3.12 to determine that $\dim(V_{x_{\alpha_1}(1)}(1))=6$.

<u>Case 2</u>: Assume that $p=5$. Then $\dim(V^2)\leq 5$ and, consequently, $V^2\cong L_{L_2}(4\omega_1)$. Furthermore, we have $V^1\cong L_{L_2}(2\omega_1)$, hence $V^3\cong L_{L_2}(2\omega_1)$, and so:

$$V\mid_{[L_2,L_2]}\cong L_{L_2}(0)\oplus L_{L_2}(2\omega_1)\oplus L_{L_2}(4\omega_1)\oplus L_{L_2}(2\omega_1)\oplus L_{L_2}(0).$$

Once more, by Propositions 3.3.5 and 3.3.12, we determine that $\dim(V_{x_{\alpha_1}(1)}(1))=5$.

<u>Case 3</u>: Assume that $p=3$. Then, as in <u>Case 1</u>, we have $\dim(V^2)\leq 6$ and the multiplicity of the weight 0 in V^2 is 2. However, as $p=3$, the weight 0 is not a sub-dominant weight in the composition factor of V^2 isomorphic to $L_{L_2}(4\omega_1)$. Therefore, V^2 consists of 3 composition factors: one isomorphic to $L_{L_2}(4\omega_1)\cong L_{L_2}(\omega_1)\otimes L_{L_2}(\omega_1)^{(3)}$, see Theorem 2.3.8, and two

isomorphic to $L_{L_2}(0)$. Moreover, as in the previous two cases, we have $V^1 \cong L_{L_2}(2\omega_1)$, hence $V^3 \cong L_{L_2}(2\omega_1)$. Lastly, by Proposition 3.3.5, we establish that $\dim(V_{x_{\alpha_1}(1)}(1)) = 4 + \dim(V^2_{x_{\alpha_1}(1)}(1))$. Now, as $x_{\alpha_1}(1)$ acts as J_2 on $L_{L_2}(\omega_1)$ and on $L_{L_2}(\omega_1)^{(3)}$, respectively, by Lemmas 2.9.4 and 2.4.9, we determine that $\dim(V^2_{x_{\alpha_1}(1)}(1)) \leq 4$, hence $\dim(V_{x_{\alpha_1}(1)}(1)) \leq 8$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 8 < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.16. *Let k be an algebraically closed field of characteristic $p \neq 2, 3$. Assume $\ell = 2$ and let $V = L_G(3\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 10,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Let $u \in G$ be a non-identity unipotent element. We recall Inequality (4.41), which states that:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}.$$

Therefore, in order to establish a bound for $\dim(V_u(1))$, we need to determine $\dim(V_{x_{\alpha_1}(1)}(1))$ and $\dim(V_{x_{\alpha_2}(1)}(1))$. We begin with the former. For this, we recall the Decomposition (4.26) of $V\mid_{[L_2,L_2]}$ from Proposition 4.2.11, which states:

$$V\mid_{[L_2,L_2]} \cong L_{L_2}(3\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1) \oplus L_{L_2}(3\omega_1).$$

Applying Lemma 3.3.3 and Proposition 3.3.12, it follows that $\dim(V_{x_{\alpha_1}(1)}(1)) = 6$.

We will now determine $\dim(V_{x_{\alpha_2}(1)}(1))$. For this we consider the Levi factor L_1 of the maximal parabolic subgroup P_1 of G. Let T' denote the maximal torus $T \cap [L_1, L_1]$ of $[L_1, L_1]$. Set $\lambda = 3\omega_1$ and note that $\dim(V) = 20$. By Lemma 2.4.5, we have $e_1(\lambda) = 6$, therefore:

$$V\mid_{[L_1,L_1]} = V^0 \oplus V^1 \oplus \cdots \oplus V^6,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for $0 \leq i \leq 6$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(0)$ and so $V^6 \cong L_{L_1}(0)$, by Lemma 2.4.3. Now, in V^1, the weight $(\lambda - \alpha_1)\mid_{T'} = \omega_2$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_{L_1}(\omega_2)$ and $\dim(V^1) \geq \dim(L_{L_1}(\omega_2)) = 2$. By Lemma 2.4.3, we also have $\dim(V^5) \geq 2$. Similarly, the weight $(\lambda - 2\alpha_1)\mid_{T'} = 2\omega_2$ admits a maximal vector in V^2, thus V^2 has a composition factor isomorphic to $L_{L_1}(2\omega_2)$. Moreover, we also note that the dominant weight $(\lambda - 2\alpha_1 - \alpha_2)\mid_{T'} = 0$, which occurs with multiplicity 2 in V^2, as $p \neq 2$, is a sub-dominant weight in the composition factor of V^2 isomorphic to $L_{L_1}(2\omega_2)$, in which it has multiplicity 1. Therefore $\dim(V^2) \geq 4$, hence $\dim(V^4) \geq 4$, by Lemma 2.4.3. It follows that $\dim(V^3) \leq 6$. Lastly, the weight $(\lambda - 3\alpha_1)\mid_{T'} = 3\omega_2$ admits a maximal vector in V^3, thus V^3 has a composition factor isomorphic to $L_{L_1}(3\omega_2)$. Moreover, the dominant weight $(\lambda - 3\alpha_1 - \alpha_2)\mid_{T'} = \omega_2$, which occurs with multiplicity 2 in V^3, as $p \neq 2, 3$, is a sub-dominant weight in the composition factor of V^3 isomorphic to $L_{L_1}(3\omega_2)$, in which it has multiplicity 1. Therefore, as $\dim(V^3) \leq 6$, we

determine that V^3 consists of exactly two composition factors: one isomorphic to $L_{L_1}(3\omega_2)$ and one isomorphic to $L_{L_1}(\omega_2)$. Moreover, by [Jan07, II.2.14], we have $V^3 \cong L_{L_1}(3\omega_2) \oplus L_{L_1}(\omega_2)$. Now, by dimensional considerations, it follows that $V^1 \cong L_{L_1}(\omega_2)$, $V^5 \cong L_{L_1}(\omega_2)$, by Lemma 2.4.3, $V^2 \cong L_{L_1}(2\omega_2) \oplus L_{L_1}(0)$, by [Jan07, II.2.14], and $V^4 \cong L_{L_1}(2\omega_2) \oplus L_{L_1}(0)$, by Lemma 2.4.3. We have shown that:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(0) \oplus L_{L_1}(\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(0) \oplus L_{L_1}(3\omega_2) \oplus L_{L_1}(\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(0) \oplus L_{L_1}(\omega_2) \oplus L_{L_1}(0).$$

We now use Lemma 3.3.3, Proposition 3.3.5 and Proposition 3.3.12 to determine that $\dim(V_{x_{\alpha_2}(1)}(1)) = 10$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 10$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Lastly we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.17. *Let k be an algebraically closed field of characteristic $p = 7$. Assume $\ell = 2$ and let $V = L_G(\omega_1 + 2\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 7,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Let $u \in G$ be a non-identity unipotent element. We recall Inequality (4.41), which states that:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}.$$

Therefore, in order to establish a bound for $\dim(V_u(1))$, we need to determine $\dim(V_{x_{\alpha_1}(1)}(1))$ and $\dim(V_{x_{\alpha_2}(1)}(1))$. We begin with the latter. For this, we recall the Decomposition (4.27) of Proposition 4.2.12, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(2\omega_2) \oplus L_{L_1}(3\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(3\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(3\omega_2) \oplus L_{L_1}(2\omega_2).$$

Using Proposition 3.3.5 and Proposition 3.3.12, it follows that $\dim(V_{x_{\alpha_2}(1)}(1)) = 7$.

We will now determine $\dim(V_{x_{\alpha_1}(1)}(1))$. For this, we consider the Levi subgroup L_2 of the maximal parabolic subgroup P_2 of G. Let T' denote the maximal torus $T \cap [L_2, L_2]$ of $[L_2, L_2]$. Set $\lambda = \omega_1 + 2\omega_2$ and note that $\dim(V) = 24$, as $p = 7$. By Lemma 2.4.5, we have $e_2(\lambda) = 5$, therefore

$$V\mid_{[L_2,L_2]} = V^0 \oplus V^1 \oplus \cdots \oplus V^5,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for $0 \leq i \leq 5$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_2}(\omega_1)$ and so $V^5 \cong (L_{L_2}(\omega_1))^* \cong L_{L_2}(\omega_1)$, by Lemma 2.4.3. Now, in V^1, the weight $(\lambda - \alpha_2)\mid_{T'} = 3\omega_1$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_{L_2}(3\omega_1)$ and $\dim(V^1) \geq \dim(L_{L_2}(3\omega_1)) = 4$, since $p = 7$. Moreover, by Lemma 2.4.3, we also have $\dim(V^4) \geq 4$, therefore $\dim(V^2) = \dim(V^3) \leq 6$, as $V^3 \cong (V^2)^*$. Lastly, the weight $(\lambda - 2\alpha_2)\mid_{T'} = 5\omega_1$ admits a maximal vector in V^2, thus V^2 has a composition

factor isomorphic to $L_{L_2}(5\omega_1)$ and, as $\dim(L_{L_2}(5\omega_1)) = 6$, since $p = 7$, it follows that $V^2 \cong L_{L_2}(5\omega_1)$. Therefore, we also have $V^3 \cong L_{L_2}(5\omega_1)$. Now, since $\dim(V^1) = 4$, it follows that $V^1 \cong L_{L_2}(3\omega_1)$ and so $V^4 \cong L_{L_2}(3\omega_1)$, by Lemma 2.4.3. We have shown that:

$$V\mid_{[L_2,L_2]} \cong L_{L_2}(\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(5\omega_1) \oplus L_{L_2}(5\omega_1) \oplus L_{L_2}(3\omega_1) \oplus L_{L_2}(\omega_1).$$

Using Lemma 3.3.3 and Proposition 3.3.12 we determine that $\dim(V_{x_{\alpha_1}(1)}(1)) = 6$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 7$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.18. *Let k be an algebraically closed field of characteristic $p = 7$. Assume $\ell = 2$ and let $V = L_G(3\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 7,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Let $u \in G$ be a non-identity unipotent element. We recall Inequality (4.41), which states that:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}.$$

Therefore, in order to establish a bound for $\dim(V_u(1))$, we need to determine $\dim(V_{x_{\alpha_1}(1)}(1))$ and $\dim(V_{x_{\alpha_2}(1)}(1))$. We begin with the latter. For this, we recall the Decomposition (4.28) of Proposition 4.2.13, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(3\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(3\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(3\omega_2) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(3\omega_2).$$

Using Proposition 3.3.5 and Proposition 3.3.12, it follows that $\dim(V_{x_{\alpha_2}(1)}(1)) = 7$.

We will now determine $\dim(V_{x_{\alpha_1}(1)}(1))$. For this we consider the Levi subgroup L_2 of the maximal parabolic subgroup P_2 of G. Let T' denote the maximal torus $T \cap [L_2, L_2]$ of $[L_2, L_2]$. Set $\lambda = 3\omega_2$ and note that $\dim(V) = 25$, as $p = 7$. By Lemma 2.4.5, we have $e_2(\lambda) = 6$, therefore

$$V\mid_{[L_2,L_2]} = V^0 \oplus \cdots \oplus V^6,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for $0 \leq i \leq 6$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_2}(0)$ and thus $V^6 \cong L_{L_2}(0)$, by Lemma 2.4.3. Now, the weight $(\lambda - \alpha_2)\mid_{T'} = 2\omega_1$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_{L_2}(2\omega_1)$ and $\dim(V^1) \geq \dim(L_{L_2}(2\omega_1)) = 3$, since $p = 7$. Moreover, by Lemma 2.4.3, we also have $\dim(V^5) \geq 3$. The weight $(\lambda - 2\alpha_2)\mid_{T'} = 4\omega_1$ admits a maximal vector in V^2, therefore V^2 has a composition factor isomorphic to $L_{L_2}(4\omega_1)$ and $\dim(V^2) \geq \dim(L_{L_2}(4\omega_1)) = 5$, since $p = 7$. Once more, by Lemma 2.4.3, we also have $\dim(V^4) \geq 5$. Lastly, the weight $(\lambda - 3\alpha_2)\mid_{T'} = 6\omega_1$ admits a maximal vector in V^3, therefore V^3 has a composition factor isomorphic to $L_{L_2}(6\omega_1)$ and $\dim(V^3) \geq 7$, as $p = 7$. We deduce that:

$$V\mid_{[L_2,L_2]} \cong L_{L_2}(0) \oplus L_{L_2}(2\omega_1) \oplus L_{L_2}(4\omega_1) \oplus L_{L_2}(6\omega_1) \oplus L_{L_2}(4\omega_1) \oplus L_{L_2}(2\omega_1) \oplus L_{L_2}(0).$$

Using Proposition 3.3.5 and Proposition 3.3.12 we determine that $\dim(V_{x_{\alpha_1}(1)}(1)) = 7$.

In conclusion, we showed that $\dim(V_u(1)) \leq 7$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_1}(1)$ and $x_{\alpha_2}(1)$. Moreover, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.19. *Let k be an algebraically closed field of characteristic $p = 3$. Assume $\ell = 2$ and let $V = L_G(2\omega_1 + \omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 13.$$

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Let $u \in G$ be a non-identity unipotent element. We recall Inequality (4.41), which states that:

$$\dim(V_u(1)) \leq \max\{\dim(V_{x_{\alpha_1}(1)}(1)), \dim(V_{x_{\alpha_2}(1)}(1))\}.$$

Therefore, in order to establish a bound for $\dim(V_u(1))$, we need to determine $\dim(V_{x_{\alpha_1}(1)}(1))$ and $\dim(V_{x_{\alpha_2}(1)}(1))$. We begin with the former. For this, we recall the decomposition of $V\mid_{[L_2,L_2]}$ from Proposition 4.2.14, which states:

$$V\mid_{[L_2,L_2]} \cong L_{L_2}(2\omega_1) \oplus V^1 \oplus V^2 \oplus V^3 \oplus L_{L_2}(2\omega_1),$$

where V^1 and V^3 each have three composition factors: one isomorphic to $L_{L_2}(\omega_1) \otimes L_{L_2}(\omega_1)^{(3)}$ and two isomorphic to $L_{L_2}(0)$; and V^2 has two composition factors: one isomorphic to $L_{L_2}(\omega_1) \otimes L_{L_2}(\omega_1)^{(3)}$ and one isomorphic to $L_{L_2}(2\omega_1)$. Now, as $x_{\alpha_1}(1)$ acts as J_2 on both $L_{L_2}(\omega_1)$ and $L_{L_2}(\omega_1)^{(3)}$, by Lemma 2.9.4, it follows that $\dim((L_{L_2}(\omega_1) \otimes L_{L_2}(\omega_1)^{(3)})_{x_{\alpha_1}(1)}(1)) = 2$. Thus, by Lemma 2.4.9, we have $\dim(V^1_{x_{\alpha_1}(1)}(1)) \leq 4$, $\dim(V^3_{x_{\alpha_1}(1)}(1)) \leq 4$ and, by Proposition 3.3.5, we also have $\dim(V^2_{x_{\alpha_1}(1)}(1)) \leq 3$. Lastly, applying Proposition 3.3.5 one more time, we deduce that $\dim(V_{x_{\alpha_1}(1)}(1)) \leq 13$.

We will now determine $\dim(V_{x_{\alpha_2}(1)}(1))$. For this we consider the Levi factor L_1 of the maximal parabolic subgroup P_1 of G. Let T' denote the maximal torus $T \cap [L_1, L_1]$ of $[L_1, L_1]$. Set $\lambda = 2\omega_1 + \omega_2$ and note that $\dim(V) = 25$, as $p = 3$. By Lemma 2.4.5, we have $e_1(\lambda) = 6$, therefore:

$$V\mid_{[L_1,L_1]} = V^0 \oplus V^1 \oplus \cdots \oplus V^6,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for $0 \leq i \leq 6$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\omega_2)$ and so $V^6 \cong (L_{L_1}(\omega_2))^* \cong L_{L_1}(\omega_2)$, by Lemma 2.4.3. Now, the weight $(\lambda - \alpha_1)\mid_{T'} = 2\omega_2$ admits a maximal vector in V^1, thus V^1 has a composition factor isomorphic to $L_{L_1}(2\omega_2)$ and $\dim(V^1) \geq 3$, as $p = 3$. Then, $\dim(V^5) \geq 3$, by Lemma 2.4.3. Similarly, in V^3, the dominant weight $(\lambda - 3\alpha_1 - \alpha_2)\mid_{T'} = 2\omega_2$ admits a maximal vector, therefore V^3 has a composition factor isomorphic to $L_{L_1}(2\omega_2)$ and $\dim(V^3) \geq 3$, as $p = 3$. Then, by Lemma 2.4.3, it follows that $\dim(V^2) = \dim(V^4) \leq 6$. Now, the weight $(\lambda - 2\alpha_1)\mid_{T'} = 3\omega_2$ admits a maximal vector in V^2, thus V^2 has a composition factor isomorphic to $L_{L_1}(3\omega_2) \cong L_{L_1}(\omega_2)^{(3)}$, as $p = 3$. Moreover, since $p = 3$, the dominant weight $(\lambda - 2\alpha_1 - \alpha_2)\mid_{T'} = \omega_2$, which occurs

with multiplicity 2 in V^2, does not appear as a sub-dominant weight in the composition factor of V^2 isomorphic to $L_{L_1}(\omega_2)^{(3)}$. Since $\dim(V^2) \leq 6$, it follows that V^2 consists of exactly three composition factors: one isomorphic to $L_{L_1}(\omega_2)^{(3)}$ and another two isomorphic to $L_{L_1}(\omega_2)$. Thus, by Lemma 2.4.3, V^4 also consists of exactly three composition factors: one isomorphic to $L_{L_1}(\omega_2)^{(3)}$ and two isomorphic to $L_{L_1}(\omega_2)$. Lastly, as $\dim(V^3) = 3$ and $\dim(V^1) = 3$, we determine that $V^3 \cong L_{L_1}(2\omega_2)$, $V^1 \cong L_{L_1}(2\omega_2)$ and so $V^5 \cong L_{L_1}(2\omega_2)$, by Lemma 2.4.3. Having determined the decomposition $V\mid_{[L_1,L_1]}$, we use Lemma 3.3.3, Proposition 3.3.5 and Lemma 2.4.9 to deduce that $\dim(V_{x_{\alpha_2}(1)}(1)) \leq 11$.

In conclusion, we showed that $\dim(V_u(1)) \leq 13 < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. $\square$

At this point, we have determined $\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ for all the irreducible modules V of G of type C_2 featured in Table 2.7.2. We will now assume that $\ell \geq 3$ and focus on the kG-modules of Table 2.7.3. Let $u \in G$ be a unipotent element and write $u = \prod\limits_{\alpha \in S_u} x_\alpha(c_\alpha)$, where the product is taken with respect to the total order $\preceq$ on Φ, see Section 1.3, $S_u \subseteq \Phi^+$ and $c_\alpha \in k^*$ for all $\alpha \in S_u$. In what follows, we will prove that each non-identity unipotent conjugacy class in G admits a representative $u^{'}$ with the property that $S_{u'} \cap \{\alpha_2, \ldots, \alpha_\ell\}$, see Proposition 4.3.22.

First, assume that $p \neq 2$. Theorem 2.9.2 tells us that unipotent conjugacy classes in G are completely determined by the Jordan form on W of a representative. Moreover, we know that odd sized Jordan blocks occur with even multiplicity. With that in mind, let $u \in G$ be a non-identity unipotent element and let V_i, $1 \leq i \leq \mathrm{ord}(u)$, be the unique, up to isomorphism, indecomposable $k[u]$-module with $\dim(V_i) = i$ and on which u acts as the full Jordan block of size i. Following [Kor18, Subsection 2.8.2], we associate to u the (possibly empty) sequences $(o_i)_{1 \leq i \leq t}$ and $(e_j)_{1 \leq j \leq s}$ such that:

$$W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{o_i}^2 \oplus \bigoplus_{j=1}^{s} V_{2e_j},$$

where $1 \leq o_1 \leq \cdots \leq o_t$ are odd and $1 \leq e_1 \leq \cdots \leq e_s$ are such that $\sum\limits_{i=1}^{t} o_i + \sum\limits_{j=1}^{s} e_j = \ell$. Note that the above decomposition completely determines the conjugacy class of u in G.

We now assume that $p = 2$. Theorems 2.9.11 and 2.9.15 tell us that unipotent conjugacy classes in G are completely determined by the Hesselink normal form of a representative. With this in mind, let $u \in G$ be a non-identity unipotent element and let $(o_{1_0}, \ldots, o_{t_0}, 2e_{1_1}, \ldots, 2e_{s_1})$ be its Hesselink normal form, where $t \geq 0$, $s \geq 0$, $1 \leq o_1 \leq \cdots \leq o_t$ and $1 \leq e_1 \leq \cdots \leq e_s$ are such that $\sum\limits_{i=1}^{t} o_i + \sum\limits_{j=1}^{s} e_j = \ell$. Then W admits the following decomposition as on orthogonal direct sum of indecomposable $k[u]$-modules:

$$W\mid_{k[u]} = \bigoplus_{i=1}^{t} W(o_i) \oplus \bigoplus_{j=1}^{s} V(2e_j).$$

Note that the above decomposition completely determines the conjugacy class of u in G.

Lemma 4.3.20. [Kor18, Lemmas 2.8.11 and 2.8.12] *Let k be an algebraically closed field of characteristic $p \geq 0$. Let $1 \leq o_1 \leq \cdots \leq o_t$, where the o_i's are odd if $p \neq 2$, and let $1 \leq e_1 \leq \cdots \leq e_s$ be such that $\sum_{i=1}^{t} o_i + \sum_{j=1}^{s} e_j = \ell$. Set $o_{t+j} = e_j$, for all $1 \leq j \leq s$. Further, set $k_1 = 1$ and $k_i = 1 + o_1 + \cdots + o_{i-1}$, $2 \leq i \leq t+s$. Lastly, for all $1 \leq i \leq t+s-1$, define:*

$$u_i = \begin{cases} 1, \text{ if } 1 \leq i \leq t \text{ and } o_i = 1; \\ \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1), \text{ if } 1 \leq i \leq t \text{ and } o_i > 1; \\ x_{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1), \text{ if } t+1 \leq i \leq t+s-1 \text{ and } o_i = 1; \\ \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1) \cdot x_{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1), \text{ if } t+1 \leq i \leq t+s-1 \text{ and } o_i > 1; \end{cases}$$

and $u_{t+s} = \begin{cases} x_{\alpha_\ell}(1), \text{ if } o_{t+s} = 1; \\ \prod_{j=k_{t+s}}^{\ell} x_{\alpha_j}(1), \text{ if } o_{t+s} > 1. \end{cases}$

Then, one of the following holds:

(1) *If $p \neq 2$, then $u = u_1 \cdots u_{t+s}$ lies in the unipotent conjugacy class of G determined by the decomposition $W|_{k[u]} = \bigoplus_{i=1}^{t} V_{o_i}^2 \oplus \bigoplus_{j=1}^{s} V_{2e_j}$.*

(2) *If $p = 2$, then $u = u_1 \cdots u_{t+s}$ lies in the unipotent conjugacy class of G determined by the decomposition $W|_{k[u]} = \bigoplus_{i=1}^{t} W(o_i) \oplus \bigoplus_{j=1}^{s} V(2e_j)$.*

Remark 4.3.21. *In this remark we will determine $S_{u'}$ for a non-identity unipotent element $u' \in G$. If $p \neq 2$, let $W|_{k[u']} = \bigoplus_{i=1}^{t} V_{o_i}^2 \oplus \bigoplus_{j=1}^{s} V_{2e_j}$ be the corresponding decomposition of W as a $k[u']$-module. Similarly, if $p = 2$, let $W|_{k[u']} = \bigoplus_{i=1}^{t} W(o_i) \oplus \bigoplus_{j=1}^{s} V(2e_j)$ be the corresponding decomposition of W as a $k[u']$-module. In both situations, by Lemma 4.3.20, there exists a representative u of the unipotent G-conjugacy class of u' with the property that $u = u_1 \cdots u_{t+s}$.*

Case 1: Assume that $t = 0$. If $o_s = 1$, then $o_i = 1$ for all $1 \leq i \leq s-1$, and we have $S_{u_i} = \{2\alpha_{k_{i+1}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell\}$, for all $1 \leq i \leq s-1$, and $S_{u_s} = \{\alpha_\ell\}$. Clearly, $S_{u_i} \cap S_{u_j} = \emptyset$, for all $1 \leq i < j \leq s$. We now use the commutator relations, see [MT11,

Theorem 11.8], *to determine that the u_i's commute, i.e. we have:*

$$u = u_1 \cdot u_2 \cdots u_s = \prod_{i=1}^{s-1} x_{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \cdot x_{\alpha_\ell}(1)$$
$$= x_{\alpha_\ell}(1) \cdot \left(x_{2\alpha_{k_s-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \cdots x_{2\alpha_{k_2-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \right) = u_s \cdot u_{s-1} \cdots u_1.$$

This gives $S_u = \{\alpha_\ell, 2\alpha_{k_s-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \ldots, 2\alpha_{k_2-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell\}$.

On the other hand, if $o_s \geq 2$, let $1 \leq j \leq s$ be such that $o_{j-1} = 1$ and $o_j > 1$. Now, for all $1 \leq i \leq j-1$, we have $S_{u_i} = \{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell\}$, while, for all $j \leq i \leq s-1$, we have $S_{u_j} = \{\alpha_{k_j}, \ldots, \alpha_{k_{j+1}-2}, 2\alpha_{k_{j+1}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell\}$ and $S_{u_s} = \{\alpha_{k_s}, \ldots, \alpha_\ell\}$. To determine S_u, we once again apply the commutator relations, [MT11, Theorem 11.8], *by which all the terms in the product $u_1 \cdots u_s$ commute, and it follows that:*

$$\begin{aligned} u &= u_1 \cdots u_{j-1} \cdot u_j \cdots u_s \\ &= \prod_{i=1}^{j-1} x_{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \cdot \prod_{i=j}^{s-1} \left[\left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot x_{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \right] \cdot \prod_{r=k_s}^{\ell} x_{\alpha_r}(1) \\ &= \prod_{i=j}^{s-1} \left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{r=k_s}^{\ell} x_{\alpha_r}(1) \cdot \left(x_{2\alpha_{k_s-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \cdots x_{2\alpha_{k_{j+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \right) \cdot \\ &\cdot \left(x_{2\alpha_{k_j-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \cdots x_{2\alpha_{k_2-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \right). \end{aligned}$$

This shows that

$$\begin{aligned} S_u = \{&\alpha_{k_j}, \ldots, \alpha_{k_{j+1}-2}, \alpha_{k_{j+1}}, \ldots, \alpha_{k_{j+2}-2}, \ldots, \alpha_{k_{s-1}}, \ldots, \alpha_{k_s-2}, \alpha_{k_s}, \ldots, \alpha_\ell, 2\alpha_{k_s-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \\ &\ldots, 2\alpha_{k_{j+1}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, 2\alpha_{k_j-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \ldots, 2\alpha_{k_2-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell\}. \end{aligned}$$

<u>*Case 2*</u>*: Assume $t \geq 1$. If $o_t = 1$, then $o_i = 1$, for all $1 \leq i \leq t$, and so $u_i = 1$, for all $1 \leq i \leq t$. In this case, S_u is as in one of the two situations in* <u>*Case 1*</u>*. On the other hand, if $o_t \geq 2$, let $1 \leq j_1 \leq t$ be such that $o_{j_1-1} = 1$ and $o_{j_1} \geq 2$. Then $u_i = 1$, for all $1 \leq i \leq j_1 - 1$, and $u_i = \prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1)$, for all $j_1 \leq i \leq t$. In this case, we have*

$$\begin{aligned} u = u_1 \cdots u_{j_1-1} \cdot u_{j_1} \cdots u_t \cdot u_{t+1} \cdots u_{t+j_2} \cdot u_{t+j_2+1} \cdots u_{t+s} = \prod_{i=j_1}^{t} \left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \\ \cdot \prod_{i=t+1}^{t+j_2} x_{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \cdot \prod_{i=t+j_2+1}^{t+s-1} \left[\left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot x_{2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-1}+\alpha_\ell}(1) \right] \cdot u_{t+s}, \end{aligned}$$

where $1 \leq j_2 \leq s$ is such that $o_{t+j_2} = 1$ and $o_{t+j_2+1} > 1$. We determine that

$$S_{u_1 \cdots u_t} = \{\alpha_{k_{j_1}}, \ldots, \alpha_{k_{j_1+1}-2}, \alpha_{k_{j_1+1}}, \ldots, \alpha_{k_{j_1+2}-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}\}.$$

Moreover, we argue as we did in <u>*Case 1*</u> *to show that, if* $j_2 = s$*, then:*

$$S_{u_{t+1}\cdots u_{t+s}} = \{\alpha_\ell, 2\alpha_{k_{t+s}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \ldots, 2\alpha_{k_{t+2}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell\},$$

while, if $j_2 < s$*, then:*

$$\begin{aligned} S_{u_{t+1}\cdots u_{t+s}} = \{&\alpha_{k_{t+j_2+1}}, \ldots, \alpha_{k_{t+j_2+2}-2}, \alpha_{k_{t+j_2+2}}, \ldots, \alpha_{k_{t+j_2+3}-2}, \ldots, \alpha_{k_{t+s-1}}, \ldots, \alpha_{k_{t+s}-2}, \alpha_{k_{t+s}}, \\ &\ldots, \alpha_\ell, 2\alpha_{k_{t+s}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \ldots, 2\alpha_{k_{t+j_2+2}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \\ &2\alpha_{k_{t+j_2+1}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \ldots, 2\alpha_{k_{t+2}-1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell\}. \end{aligned}$$

Therefore, if we write $S_{u_1\cdots u_t} = \{\beta_1, \ldots, \beta_{m_1}\}$ *and* $S_{u_{t+1}\cdots u_{t+s}} = \{\gamma_1, \ldots, \gamma_{m_2}\}$*, where* $m_1, m_2 \geq 1$*, we have that* $\beta_i \preceq \gamma_j$ *and* $\beta_i \neq \gamma_j$*, for all* $1 \leq i \leq m_1$ *and all* $1 \leq j \leq m_2$*, thus* $S_u = \{\beta_1, \ldots, \beta_{m_1}, \gamma_1, \ldots, \gamma_{m_2}\}$*.*

Proposition 4.3.22. *Let* k *be an algebraically closed field of characteristic* $p \geq 0$*. Assume that* $\ell \geq 3$*. Then all non-identity unipotent conjugacy classes in* G *admit a representative* u *with the property that* $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$*.*

Proof. Let $u' \in G$ be a non-identity unipotent element. If $p \neq 2$, let $W\mid_{k[u']} = \bigoplus_{i=1}^{t} V_{o_i}^2 \oplus \bigoplus_{j=1}^{s} V_{2e_j}$ be the corresponding decomposition of W as a $k[u']$-module. Similarly, if $p = 2$, let $W\mid_{k[u]} = \bigoplus_{i=1}^{t} W(o_i) \oplus \bigoplus_{j=1}^{s} V(2e_j)$ be the corresponding decomposition of W as a $k[u']$-module. Then, there exists a representative u in the unipotent G-conjugacy class of u' such that $u = u_1 \cdots u_{t+s}$ with u_i's as in Lemma 4.3.20.

We first consider the case when there are no odd sized blocks in the Jordan form of u on W, i.e. $t = 0$. Then S_u is as in <u>Case 1</u> of Remark 4.3.21. Therefore, we see that $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$. We can now assume that there exist odd blocks in the Jordan form of u on W, i.e. $t \geq 1$. Then S_u is as given in <u>Case 2</u> of Remark 4.3.21. We distinguish the following two cases:

<u>Case 1</u>: Assume that $s = 0$. Then, since u is nontrivial, it follows that $o_t > 1$. Let $1 \leq j_1 \leq t$ be such that $o_{j_1-1} = 1$ and $o_{j_1} > 1$. Therefore,

$$S_u = S_{u_1\cdots u_t} = \{\alpha_{k_{j_1}}, \ldots, \alpha_{k_{j_1+1}-2}, \alpha_{k_{j_1+1}}, \ldots, \alpha_{k_{j_1+2}-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}\},$$

by <u>Case 2</u> of Remark 4.3.21. Now, since $\sum_{i=1}^{t} o_t = \ell$ and $k_{t+1} = 1 + o_1 + \cdots + o_t$, we have $k_{t+1} - 2 = \ell - 1$ and so $\alpha_{\ell-1} \in S_u$, thereby $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$.

<u>Case 2</u>: Assume that $s > 1$. Then, by <u>Case 2</u> of Remark 4.3.21, it follows that $\alpha_\ell \in S_u$, hence $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$. This completes the proof of the proposition. □

We now continue with the proofs of Theorems 4.3.1 and 4.3.2 in the case of the irreducible kG-modules $L_G(\lambda)$, where the p-restricted dominant weight λ is listed in Table 2.7.3.

Proposition 4.3.23. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 3$ and let $V = L_G(\omega_3)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 6,$$

where there exist $u \in G$ for which equality holds.

In particular, there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. To begin, we recall the Decomposition (4.29) in the case of $\ell = 3$ of Proposition 4.2.15, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_3). \tag{4.42}$$

Let $u \in G$ be a non-identity unipotent element and let $u^{'}$ be a representative of the unipotent G-conjugacy class of u with the property that $u^{'}_{L_1} \neq 1$. Note that, by Proposition 4.3.22, such a representative always exists. Then, by Inequality (2.7) and Decomposition (4.42), it follows that:

$$\dim(V_u(1)) \leq \dim(V_{u^{'}_{L_1}}(1)) = 2\dim((L_{L_1}(\omega_3))_{u^{'}_{L_1}}(1)).$$

Since $p = 2$, we apply Proposition 4.3.10 to determine that $\dim((L_{L_1}(\omega_3))_{u^{'}_{L_1}}(1)) \leq 3$, thereby $\dim(V_u(1)) \leq 6$. Now, consider the unipotent element $x_{\alpha_2}(1) \in G$. We first note that $(x_{\alpha_2}(1))_{L_1} = x_{\alpha_2}(1)$ and $(x_{\alpha_2}(1))_{Q_1} = 1$. Therefore, by Equality (2.8) and Decomposition (4.42), we have $\dim(V_{x_{\alpha_2}(1)}(1)) = 2\dim((L_{L_1}(\omega_3))_{x_{\alpha_2}(1)}(1))$, thus $\dim(V_{x_{\alpha_2}(1)}(1)) = 6$, by Proposition 4.3.10 and Table 4.3.3.

In conclusion, we showed that $\dim(V_u(1)) \leq 6$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Therefore, there exist non-identity unipotent elements $u \in G$ such that $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. □

Proposition 4.3.24. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 3$ and let $V = L_G(\omega_3)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 9,$$

where there exist unipotent elements $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ *for all non-identity unipotent elements $u \in G$.*

Proof. To begin, we recall the Decomposition (4.30) of Proposition 4.2.16, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_2) \oplus L_{L_1}(\omega_3).$$

Let $u \in G$ be a non-identity unipotent element and let $u^{'}$ be a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1} \neq 1$. Note that, by Proposition 4.3.22, such a representative always exists. Then, by Inequality (2.7) and Decomposition (4.30), we have:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1)) = 2\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) + \dim((L_{L_1}(\omega_2))_{u'_{L_1}}(1)).$$

Since $p \neq 2$, we apply Corollary 4.3.8 to determine that $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) \leq 3$, while, by Lemma 4.3.3, it follows that $\dim((L_{L_1}(\omega_2))_{u'_{L_1}}(1)) \leq 3$. Therefore, $\dim(V_u(1)) \leq 9$. Now, consider the unipotent element $x_{\alpha_3}(1) \in G$. We first note that $(x_{\alpha_3}(1))_{L_1} = x_{\alpha_3}(1)$ and $(x_{\alpha_3}(1))_{Q_1} = 1$. Thus, by Equality (2.8) and Decomposition (4.30), it follows that $\dim(V_{x_{\alpha_3}(1)}(1)) = 2\dim((L_{L_1}(\omega_3))_{x_{\alpha_3}(1)}(1)) + \dim((L_{L_1}(\omega_2))_{x_{\alpha_3}(1)}(1))$ and so $\dim(V_{x_{\alpha_3}(1)}(1)) = 9$, by Corollary 4.3.8, Lemma 4.3.3 and Table 4.3.2. This shows that there exist $u \in G$ for which $\dim(V_u(1)) = 9$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 9$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_3}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.25. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 3$ and let $V = L_G(\omega_1 + \omega_3)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 28,$$

where there exist $u \in G$, $u \neq 1$, for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (4.31) from Proposition 4.2.17, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_2+\omega_3) \oplus L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_2+\omega_3) \oplus L_{L_1}(\omega_3).$$

Let $u \in G$ be a non-identity unipotent element and let u' be a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1} \neq 1$, see Proposition 4.3.22. Then, by Inequality (2.7) and Decomposition (4.31), we have:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1)) \leq 4\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) + 2\dim((L_{L_1}(\omega_2+\omega_3))_{u'_{L_1}}(1)).$$

Using Proposition 4.3.10, we determine that $\dim((L_{L_1}(\omega_3))_{u_{L_1}}(1)) \leq 3$, while, by Proposition 4.3.14, we have $\dim((L_{L_1}(\omega_2+\omega_3))_{u_{L_1}}(1)) \leq 8$. It follows that $\dim(V_u(1)) \leq 28$. Now, consider the unipotent element $x_{\alpha_2}(1) \in G$. We first note that $(x_{\alpha_2}(1))_{L_1} = x_{\alpha_2}(1)$ and $(x_{\alpha_2}(1))_{Q_1} = 1$. Therefore, by Equality (2.8) and Decomposition (4.31), it follows that $\dim(V_{x_{\alpha_2}(1)}(1)) = 4\dim((L_{L_1}(\omega_3))_{x_{\alpha_2}(1)}(1)) + 2\dim((L_{L_1}(\omega_2+\omega_3))_{x_{\alpha_2}(1)}(1))$ and this gives $\dim(V_{x_{\alpha_2}(1)}(1)) = 28$, by Proposition 4.3.10, the second paragraph of the proof of Proposition 4.3.14 and Table 4.3.3. This shows that there exist $u \in G$ for which $\dim(V_u(1)) = 28$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 28$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.26. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 3$ and let $V = L_G(2\omega_1 + \omega_3)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 28,$$

where there exist $u \in G$, $u \neq 1$, for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we note that, by Theorem 2.3.8, as $p = 2$, we have the following isomorphism of kG-modules:

$$V \cong L_G(\omega_1)^{(2)} \otimes L_G(\omega_3). \tag{4.43}$$

We first focus on the kG-module $L_G(\omega_1)^{(2)}$. We remark that $\dim(L_G(\omega_1)^{(2)}) = 6$ and, by Lemma 2.4.5, we have $e_1(2\omega_1) = 4$, therefore

$$L_G(\omega_1)^{(2)} \mid_{[L_1,L_1]} = V^0 \oplus \cdots \oplus V^4,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{2\omega_1 - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 4$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(0)$ and so, by Lemma 2.4.3, we also have $V^4 \cong L_{L_1}(0)$. As $p = 2$, we have $V^1 = \{0\}$ and $V^3 = \{0\}$. Lastly, as the weight $(\lambda - 2\alpha_1) \mid_{T'} = 2\omega_2$ admits a maximal vector in V^2, it follows that V^2 has a composition factor isomorphic to $L_{L_1}(2\omega_2)$ and, as $\dim(L_{L_1}(2\omega_2)) = 4$, since $p = 2$, we determine that $V^2 \cong L_{L_1}(2\omega_2)$. Therefore, we have:

$$L_G(\omega_1)^{(2)} \mid_{[L_1,L_1]} \cong L_{L_1}(0) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(0). \tag{4.44}$$

In the case of the kG-module $L_G(\omega_3)$, by Decomposition (4.29), we have:

$$L_G(\omega_3) \mid_{[L_1,L_1]} \cong L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_3). \tag{4.45}$$

Coming back to (4.43), by (4.44) and (4.45), using Theorem 2.3.8, we determine that:

$$\begin{aligned} L_G(2\omega_1 + \omega_3) \mid_{[L_1,L_1]} &\cong (L_{L_1}(0) \oplus L_{L_1}(2\omega_2) \oplus L_{L_1}(0)) \otimes (L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_3)) \\ &\cong L_{L_1}(\omega_3)^4 \oplus (L_{L_1}(2\omega_2) \otimes L_{L_1}(\omega_3))^2 \\ &\cong L_{L_1}(\omega_3)^4 \oplus L_{L_1}(2\omega_2 + \omega_3)^2. \end{aligned} \tag{4.46}$$

Let $u \in G$ be a non-identity unipotent element and let u' be, as usual, a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1} \neq 1$. Then, by Inequality (2.7) and Decomposition (4.46), we have:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1)) \leq 4\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) + 2\dim((L_{L_1}(2\omega_2 + \omega_3))_{u'_{L_1}}(1)).$$

We will first determine $\dim((L_{L_1}(2\omega_2 + \omega_3))_{u'_{L_1}}(1))$. By Theorem 2.3.8, we note that:

$$L_{L_1}(2\omega_2 + \omega_3) \cong L_{L_1}(\omega_2)^{(2)} \otimes L_{L_1}(\omega_3).$$

Now, by Table 4.3.3, we have that the Hesselink normal form of u'_{L_1} on $L_{L_1}(\omega_2)$ is one of $(1_0^2, 2_1)$, (2_0^2), (2_1^2) and (4_1). Thus, the Jordan form of the action of u'_{L_1} on $L_{L_1}(\omega_2)^{(2)}$ is one of $J_2 \oplus J_1^2$, J_2^2, J_2^2 and J_4, respectively. Using the second paragraph of the proof of Proposition 4.3.10 and Theorem 5.3.5 cases (b), (c.3), (b) and (c.3), respectively, we determine that the Jordan form of the action of u'_{L_1} on $L_{L_1}(\omega_3)$ is J_2^2, $J_2 \oplus J_1^2$, J_2^2 and J_4, respectively. We now calculate the Jordan form of the action of u'_{L_1} on $L_{L_1}(\omega_2)^{(2)} \otimes L_{L_1}(\omega_3)$, either by hand or using a computer, keeping in mind that $p = 2$, and we get that it is J_2^8, J_2^8, J_2^8 and J_4^4, respectively. Thus, $\dim((L_{L_1}(2\omega_2 + \omega_3))_{u'_{L_1}}(1)) \leq 8$. Furthermore, by Proposition 4.3.10, we determine that $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) \leq 3$, and so, it follows that $\dim(V_u(1)) \leq 28$.

Now, consider the unipotent element $x_{\alpha_2}(1) \in G$. We first note that $(x_{\alpha_2}(1))_{L_1} = x_{\alpha_2}(1)$ and $(x_{\alpha_2}(1))_{Q_1} = 1$. Therefore, by Equality (2.8) and Decomposition (4.46), it follows that $\dim(V_{x_{\alpha_2}(1)}(1)) = 4\dim((L_{L_1}(\omega_3))_{x_{\alpha_2}(1)}(1)) + 2\dim((L_{L_1}(2\omega_2+\omega_3))_{x_{\alpha_2}(1)}(1))$ and this gives $\dim(V_{x_{\alpha_2}(1)}(1)) = 28$, by the above, Proposition 4.3.10 and Table 4.3.3. This shows that there exist $u \in G$ for which $\dim(V_u(1)) = 28$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 28$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.27. *Let k be an algebraically closed field of characteristic $p = 3$. Assume $\ell = 4$ and let $V = L_G(\omega_3)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 27,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (4.32) of Proposition 4.2.19, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_4) \oplus L_{L_1}(\omega_3).$$

Let $u \in G$ be a non-identity unipotent element and let u' be, as usual, a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1} \neq 1$. Then, by Inequality (2.7) and Decomposition (4.32), we have:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1)) \leq 2\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) + \dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1)).$$

Since $p = 3$, we apply Propositions 4.3.10 and 4.3.24 to determine that $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) \leq 9$ and $\dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1)) \leq 9$, respectively. It follows that $\dim(V_u(1)) \leq 27$. Now consider the unipotent element $x_{\alpha_4}(1) \in G$. First, we note that $(x_{\alpha_4}(1))_{L_1} = x_{\alpha_4}(1)$ and $(x_{\alpha_4}(1))_{Q_1} = 1$. Therefore, by Equality (2.8) and Decomposition (4.32), we have $\dim(V_{x_{\alpha_4}(1)}(1)) = 2\dim((L_{L_1}(\omega_3))_{x_{\alpha_4}(1)}(1)) + \dim((L_{L_1}(\omega_4))_{x_{\alpha_4}(1)}(1))$ and so $\dim(V_{x_{\alpha_4}(1)}(1)) = 27$, by Proposition 4.3.10, the proof of Proposition 4.3.24 and [LS12, Subsection 3.3.2]. This shows that there exist $u \in G$ with $\dim(V_u(1)) = 27$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 27$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_4}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.28. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 4$ and let $V = L_G(\omega_4)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 28 - \delta_{p,3}.$$

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (4.33) of Proposition 4.2.21, which states:

$$V\mid_{[L_1,L_1]}\cong L_{L_1}(\omega_4)\oplus L_{L_1}(\omega_3)\oplus L_{L_1}(\omega_4).$$

Let $u\in G$ be a non-identity unipotent element and let $u^{'}$ be, as usual, a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1}\neq 1$. Then, by Inequality (2.7) and Decomposition (4.33), we have:

$$\dim(V_u(1))\leq \dim(V_{u'_{L_1}}(1))=\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1))+2\dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1).$$

Using Proposition 4.3.24, we have $\dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1))\leq 9$. If $p=3$, then by Proposition 4.3.10, it follows that $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1))\leq 9$, while, if $p\neq 3$, then by Corollary 4.3.8, we have $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1))\leq 10$. We obtain $\dim(V_u(1))\leq 28-\delta_{p,3}$.

In conclusion, we showed that $\dim(V_u(1))\leq 28-\delta_{p,3}<\dim(V)-\sqrt{\dim(V)}$ for all non-identity unipotent elements $u\in G$. □

Proposition 4.3.29. *Let k be an algebraically closed field of characteristic $p\neq 3$. Assume $\ell=4$ and let $V=L_G(\omega_3)$. Then for all non-identity unipotent elements $u\in G$ we have*

$$\dim(V_u(1))\leq 34,$$

where there exist $u\in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1))<\dim(V)-\sqrt{\dim(V)}$ for all non-identity unipotent elements $u\in G$.

Proof. To begin, we recall the decomposition of $V\mid_{[L_1,L_1]}$ from Proposition 4.2.20, which states:

$$V\mid_{[L_1,L_1]}\cong L_{L_1}(\omega_3)\oplus V^1\oplus L_{L_1}(\omega_3),$$

where, if $p\neq 2$, V^1 has two composition factors: one isomorphic to $L_{L_1}(\omega_4)$ and one isomorphic to $L_{L_1}(\omega_2)$; while, if $p=2$, then V^1 has three composition factors: one isomorphic to $L_{L_1}(\omega_4)$ and two isomorphic to $L_{L_1}(\omega_2)$.

We first consider the case when $p\neq 2$. Then, by [Jan07, II.2.14], it follows that $V^1\cong L_{L_1}(\omega_4)\oplus L_{L_1}(\omega_2)$ and so

$$V\mid_{[L_1,L_1]}\cong L_{L_1}(\omega_3)\oplus L_{L_1}(\omega_4)\oplus L_{L_1}(\omega_2)\oplus L_{L_1}(\omega_3). \tag{4.47}$$

Let $u\in G$ be a non-identity unipotent element and let $u^{'}$ be a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1}\neq 1$, see Proposition 4.3.22. Then, by Inequality (2.7) and Decomposition (4.47), we have:

$$\begin{aligned}\dim(V_u(1))\leq \dim(V_{u'_{L_1}}(1))=2\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1))+\dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1))+\\ +\dim((L_{L_1}(\omega_2))_{u'_{L_1}}(1)).\end{aligned}$$

Since $p\neq 2,3$, we use Corollary 4.3.8, Lemma 4.3.3 and Proposition 4.3.24 to determine that $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1))\leq 10$, $\dim((L_{L_1}(\omega_2))_{u'_{L_1}}(1))\leq 5$ and $\dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1))\leq 9$,

respectively. It follows that $\dim(V_u(1)) \leq 34$. Now, consider the unipotent element $x_{\alpha_4}(1) \in G$. First, we note that $(x_{\alpha_4}(1))_{L_1} = x_{\alpha_4}(1)$ and $(x_{\alpha_4}(1))_{Q_1} = 1$. Therefore, by Identity (2.8) and Decomposition (4.47), it follows that $\dim(V_{x_{\alpha_4}(1)}(1)) = 2\dim((L_{L_1}(\omega_3))_{x_{\alpha_4}(1)}(1)) + \dim((L_{L_1}(\omega_4))_{x_{\alpha_4}(1)}(1)) + \dim((L_{L_1}(\omega_2))_{x_{\alpha_4}(1)}(1))$ and so $\dim(V_{x_{\alpha_4}(1)}(1)) = 34$, by Corollary 4.3.8, Lemma 4.3.3, proof of Proposition 4.3.24 and [LS12, Subsection 3.3.2]. This shows that there exist $u \in G$ for which $\dim(V_u(1)) = 34$.

We can now assume that $p = 2$. Again, let $u \in G$ be a non-identity unipotent element and let $u^{'}$ be a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1} \neq 1$, see Proposition 4.3.22. We apply Proposition 4.3.23 and Lemma 4.3.3 to determine that $\dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1)) \leq 6$ and that $\dim((L_{L_1}(\omega_2))_{u'_{L_1}}(1)) \leq 5$. Therefore, by Lemma 2.4.9, we obtain $\dim(V^1_{u'_{L_1}}(1)) \leq 16$. Moreover, by Corollary 4.3.8, as $p \neq 3$, we have $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) \leq 10$, therefore $\dim(V_u(1)) \leq 36$. However, we will show that $\dim(V_u(1)) \leq 34$ for all unipotent elements $u \in G$, $u \neq 1$.

Assume there exists $u \in G$, $u \neq 1$, such that $\dim(V_u(1)) > 34$. Then, by the above discussion, keeping in mind that V^1 has two composition factors isomorphic to $L_{L_1}(\omega_2)$, it follows that $\dim((L_{L_1}(\omega_2))_{u'_{L_1}}(1)) = 5$, where u' is a representative of the unipotent G-conjugacy class of u such that $u'_{L_1} \neq 1$. Furthermore, by Lemma 4.3.3, we have that $\dim((L_{L_1}(\omega_2))_{u'_{L_1}}(1)) = 5$ if and only if the Jordan form of the action of u'_{L_1} on $L_{L_1}(\omega_2)$ is $J_2 \oplus J_1^4$. Thus, by [Kor18, Lemma 3.2.3], the possibilities for the Jordan form of u' on W are $J_2 \oplus J_1^6$, $J_2^2 \oplus J_1^4$ and $J_2^3 \oplus J_1^2$. Now, as $p = 2$, by [McN98, Lemma 4.8.2], we have the following kG-module isomorphism:

$$\wedge^3(W) \cong L_G(\omega_3) \oplus L_G(\omega_1).$$

Therefore, in order to determine $\dim(V_{u'}(1))$, we only need to know $\dim((\wedge^3(W))_{u'}(1))$ and $\dim((L_G(\omega_1))_{u'}(1))$.

<u>Case 1</u>: The Jordan form of u' on W is $J_2 \oplus J_1^6$. Then $W\mid_{k[u']} \cong V_2 \oplus V_1^6$, where V_i is the unique, up to isomorphism, indecomposable $k[u']$-module with $\dim(V_i) = i$ and on which u' acts as J_i. This gives rise to the following $k[u']$-module isomorphism:

$$\begin{aligned}\wedge^3(W) \cong \wedge^3(V_2 \oplus V_1^6) &\cong \wedge^3(V_2) \oplus \wedge^2(V_2) \otimes V_1^6 \oplus V_2 \otimes \wedge^2(V_1^6) \oplus \wedge^3(V_1^6)\\ &\cong V_1 \otimes V_1^6 \oplus V_2 \otimes V_1^{15} \oplus V_1^{20}\\ &\cong V_2^{15} \oplus V_1^{26}.\end{aligned}$$

Therefore, u' acts on $\wedge^3(W)$ as $J_2^{15} \oplus J_1^{26}$, hence $\dim((\wedge^3(W))_{u'}(1)) = 41$. It follows that $\dim(V_{u'}(1)) = 41 - 7 = 34$, as $\dim((L_G(\omega_1))_{u'}(1)) = 7$.

<u>Case 2</u>: The Jordan form of u' on W is $J_2^2 \oplus J_1^4$. Then $W\mid_{k[u']} \cong V_2^2 \oplus V_1^4$, where V_i is the unique, up to isomorphism, indecomposable $k[u']$-module with $\dim(V_i) = i$ and on which u' acts as J_i. This gives rise to the following $k[u']$-module isomorphism:

$$\begin{aligned}\wedge^3(W) &\cong \wedge^3(V_2^2) \oplus \wedge^2(V_2^2) \otimes V_1^4 \oplus V_2^2 \otimes \wedge^2(V_1^4) \oplus \wedge^3(V_1^4)\\ &\cong [\wedge^3(V_2) \oplus \wedge^2(V_2) \otimes V_2]^2 \oplus [\wedge^2(V_2)^2 \oplus V_2 \otimes V_2] \otimes V_1^4 \oplus V_2^2 \otimes V_1^6 \oplus V_1^4\\ &\cong [V_1 \otimes V_2]^2 \oplus [V_1^2 \oplus V_2^2] \otimes V_1^4 \oplus V_2^{12} \oplus V_1^4\\ &\cong V_2^{22} \oplus V_1^{12}.\end{aligned}$$

Therefore, u' acts on $\wedge^3(W)$ as $J_2^{22} \oplus J_1^{12}$, hence $\dim((\wedge^3(W))_{u'}(1)) = 34$. It follows that $\dim(V_{u'}(1)) = 34 - 6 = 28$, as $\dim((L_G(\omega_1))_{u'}(1)) = 6$.

Case 3: The Jordan form of u' on W is $J_2^3 \oplus J_1^2$. Then $W\mid_{k[u']} \cong V_2^3 \oplus V_1^2$, where V_i is the unique, up to isomorphism, indecomposable $k[u']$-module with $\dim(V_i) = i$ and on which u' acts as J_i. This gives rise to the following $k[u']$-module isomorphism:

$$\begin{aligned}
\wedge^3(W) &\cong \wedge^3(V_2^3) \oplus \wedge^2(V_2^3) \otimes V_1^2 \oplus V_2^3 \otimes \wedge^2(V_1^2) \oplus \wedge^3(V_1^2) \\
&\cong [\wedge^3(V_2^2) \oplus \wedge^2(V_2^2) \otimes V_2 \oplus V_2^2 \otimes \wedge^2(V_2) \oplus \wedge^3(V_2)] \oplus \\
&\oplus [\wedge^2(V_2^2) \oplus V_2^2 \otimes V_2 \oplus \wedge^2(V_2)] \otimes V_1^2 \oplus V_2^3 \otimes V_1 \\
&\cong [V_2^2 \oplus (V_2^2 \oplus V_1^2) \otimes V_2 \oplus V_2^2 \otimes V_1] \oplus [V_2^2 \oplus V_1^2 \oplus V_2^4 \oplus V_1] \otimes V_1^2 \oplus V_2^3 \\
&\cong V_2^{25} \oplus V_1^6.
\end{aligned}$$

Therefore, u' acts on $\wedge^3(W)$ as $J_2^{25} \oplus J_1^6$, hence $\dim((\wedge^3(W))_{u'}(1)) = 31$. It follows that $\dim(V_{u'}(1)) = 31 - 5 = 26$, as $\dim((L_G(\omega_1))_{u'}(1)) = 5$.

In conclusion, by Cases 1, 2 and 3, it follows that $\dim(V_u(1)) \leq 34$ for all non-identity unipotent elements $u \in G$. Moreover, we have showed that there exist $u \in G$ for which the bound is attained, for example those with Jordan form on W given by $J_2 \oplus J_1^6$. Lastly, we note that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.30. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 5$ and let $V = L_G(\omega_3)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 74,$$

where there exist $u \in G$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (4.34) of Proposition 4.2.22, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_3) \oplus L_{L_1}(\omega_4) \oplus L_{L_1}(\omega_3).$$

Let $u \in G$ be a non-identity unipotent element and let u' be a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1} \neq 1$, see Proposition 4.3.22. Then, by Inequality (2.7) and Decomposition (4.34), we have:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1)) = 2\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) + \dim(L_{L_1}(\omega_4))_{u'_{L_1}}(1)).$$

By Proposition 4.3.10, as $p = 2$, we have $\dim((L_{L_1}(\omega_3))_{u'_{L_1}}(1)) \leq 20$, while, by Proposition 4.3.29, we have $\dim((L_{L_1}(\omega_4))_{u'_{L_1}}(1)) \leq 34$. It follows that $\dim(V_u(1)) \leq 74$. Now, consider the unipotent element $x_{\alpha_5}(1) \in G$. First, we note that $(x_{\alpha_5}(1))_{L_1} = x_{\alpha_5}(1)$ and $(x_{\alpha_5}(1))_{Q_1} = 1$. Therefore, by Identity (2.8) and Decomposition (4.34), it follows that $\dim(V_{x_{\alpha_5}(1)}(1)) = 2\dim((L_{L_1}(\omega_3))_{x_{\alpha_5}(1)}(1)) + \dim((L_{L_1}(\omega_4))_{x_{\alpha_5}(1)}(1))$ and this gives $\dim(V_{x_{\alpha_5}(1)}(1)) = 74$, by [LS12, Section 6.1], using Proposition 4.3.10 and the proof of Proposition 4.3.29. This shows that there exist $u \in G$ for which $\dim(V_u(1)) = 74$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 74$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_5}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 4.3.31. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell \geq 4$ and let $V = L_G(\omega_\ell)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 3 \cdot 2^{\ell-2},$$

where there exist $u \in G$ for which the bound is attained.

In particular, in the case of $\ell = 4$ there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$, while, for $\ell \geq 5$ we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we recall the Decomposition (4.29) of Proposition 4.2.15, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\omega_\ell) \oplus L_{L_1}(\omega_\ell).$$

Let $u \in G$ be a non-identity unipotent element and let u' be a representative of the unipotent G-conjugacy class of u with the property that $u'_{L_1} \neq 1$, see Proposition 4.3.22. Then, by Inequality (2.7) and Decomposition (4.29), we have:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1)) = 2\dim((L_{L_1}(\omega_\ell))_{u'_{L_1}}(1)).$$

Recursively and using Proposition 4.3.23 for the base case of $\ell = 4$, we get $\dim((L_{L_1}(\omega_\ell))_{u'_{L_1}}(1)) \leq 3 \cdot 2^{\ell-3}$, therefore $\dim(V_u(1)) \leq 3 \cdot 2^{\ell-2}$. Now, consider the unipotent element $x_{\alpha_2}(1) \in G$. First, we note that $(x_{\alpha_2}(1))_{L_1} = x_{\alpha_2}(1)$ and $(x_{\alpha_2}(1))_{Q_1} = 1$. Therefore, by Identity (2.8) and Decomposition (4.29), it follows that $\dim(V_{x_{\alpha_2}(1)}(1)) = 2\dim((L_{L_1}(\omega_\ell))_{x_{\alpha_2}(1)}(1))$ and so $\dim(V_{x_{\alpha_2}(1)}(1)) = 3 \cdot 2^{\ell-2}$, by [LS12, Section 6.1], the proof of Proposition 4.3.23 in the case of $\ell = 4$ and recursively for $\ell \geq 5$.

In conclusion, we showed that $\dim(V_u(1)) \leq 3 \cdot 2^{\ell-2}$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained, for example $x_{\alpha_2}(1)$. Therefore, in the case of $\ell \geq 4$, we see that there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 5$, we have $0 < 2^\ell(2^{\ell-4} - 1)$, therefore the inequality $3 \cdot 2^{\ell-2} < 2^\ell - \sqrt{2^\ell}$ holds for all $\ell \geq 5$ and thus $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

We conclude this subsection by noting that Proposition 4.3.13 through 4.3.31 complete the proof of Theorems 4.3.1 and 4.3.2, as they cover all the irreducible kG-modules corresponding to p-restricted dominant weights featured in one of the Tables 2.7.2 and 2.7.3.

4.4 Results

In this section we collect the results proven in this chapter. In Proposition 4.4.1 we give the values of $\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ and $\nu_G(V)$ for all kG-modules V belonging to one of the families we had to consider. Similarly, Proposition 4.4.2 records the same data for the particular kG-modules treated in this chapter.

Proposition 4.4.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ, $\ell \geq 2$. Let T be a fixed maximal torus in G and let $V = L_G(\lambda)$, where $\lambda \in F^{C_\ell}$. Then the value of $\nu_G(V)$ is given in the table below:*

V	Char.	Rank	$\max_{s\in T\setminus Z(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$	$\max_{u\in G_u\setminus\{1\}}\dim(V_u(1))$	$\nu_G(V)$
$L_G(\omega_1)$	$p\geq 0$	$\ell\geq 2$	$2\ell-2$	$2\ell-1$	1
$L_G(2\omega_1)$	$p\neq 2$	$\ell\geq 2$	$2\ell^2-3\ell+4$	$2\ell^2-\ell$	2ℓ
$L_G(\omega_2)$	$p\nmid\ell$	$\ell=2$	4	3	1
		$\ell=3$	8	10	4
		$\ell=4$	16	21	6
		$\ell\geq 5$	$2\ell^2-5\ell+3$	$2\ell^2-3\ell+1$	$2\ell-2$
	$p\mid\ell$	$\ell=2$	2	3	1
		$\ell=3$	8	9	4
		$\ell\geq 4$	$2\ell^2-5\ell+2$	$2\ell^2-3\ell$	$2\ell-2$

Table 4.4.1: The value of $\nu_G(V)$ for the families of modules of groups of type C_ℓ.

Proof. The result follows by Proposition 2.2.3 from Lemmas 4.2.3 and 4.3.3 for $V=L_G(\omega_1)$; from Propositions 4.2.4 and 4.3.12 for $V=L_G(2\omega_1)$; and from Corollaries 4.2.6 and 4.3.8 in the case of $p\nmid\ell$, respectively from Corollary 4.2.7 and Proposition 4.3.10 in the case of $p\mid\ell$, for $V=L_G(\omega_2)$. □

Proposition 4.4.2. *Let k be an algebraically closed field of characteristic $p\geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ, $\ell\geq 2$. Let T be a fixed maximal torus in G and let $V=L_G(\lambda)$, where λ is featured in one of the Tables* 2.7.2 *and* 2.7.3. *The value of $\nu_G(V)$ is given in the table below:*

Rank	λ	Char.	$\max_{s\in T\setminus Z(G)}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$	$\max_{u\in G_u\setminus\{1\}}\dim(V_u(1))$	$\nu_G(V)$
$\ell=2$	$L_G(\omega_1+\omega_2)$	$p\geq 0$	$8-2\delta_{p,5}$	$8-3\delta_{p,5}$	$8-2\delta_{p,5}$
$\ell=2$	$L_G(2\omega_2)$	$p\neq 2$	$10-\delta_{p,5}$	≤ 8	4
$\ell=2$	$L_G(3\omega_1)$	$p\neq 2,3$	10	10	10
$\ell=2$	$L_G(\omega_1+2\omega_2)$	$p=7$	12	7	12
$\ell=2$	$L_G(3\omega_2)$	$p=7$	16	7	9
$\ell=2$	$L_G(2\omega_1+\omega_2)$	$p=3$	16	≤ 13	9
$3\leq\ell\leq 8$	$L_G(\omega_\ell)$	$p=2$	$2^{\ell-1}$	$3\cdot 2^{\ell-2}$	$2^{\ell-2}$
$\ell=3$	$L_G(\omega_3)$	$p\neq 2$	10	9	4
$\ell=3$	$L_G(\omega_1+\omega_3)$	$p=2$	≤ 24	28	20
$\ell=3$	$L_G(2\omega_1+\omega_3)$	$p=2$	20	28	20
$\ell=4$	$L_G(\omega_3)$	$p\geq 0$	$\leq 30-4\delta_{p,3}$	$34-7\delta_{p,3}$	$14-\delta_{p,3}$
$\ell=4$	$L_G(\omega_4)$	$p\neq 2$	28	$\leq 28-\delta_{p,3}$	$14-\delta_{p,3}$
$\ell=5$	$L_G(\omega_3)$	$p=2$	≤ 58	74	26

Table 4.4.2: The value of $\nu_G(V)$ for the particular modules of groups of type C_ℓ.

Proof. The result follows by Proposition 2.2.3, using the detailed results of Subsections 4.2.2 and 4.3.2. □

Chapter 5

Groups of type B_ℓ

In this chapter we prove Theorems 1.1.1 and 1.1.3 for simple simply connected linear algebraic groups of type B_ℓ, $\ell \geq 3$. The structure is as follows: in the first section we construct G the simple adjoint group of type B_ℓ and exhibit some properties of its semisimple and unipotent elements. In Section 5.2 we determine $\max\limits_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, where $\tilde{G}$ is a simple simply connected linear algebraic group of type B_ℓ, $\ell \geq 3$, $\tilde{T}$ is a maximal torus in $\tilde{G}$ and V runs through the list of $k\tilde{G}$-modules we identified in Subsection 2.7.3. Similarly, in Section 5.3, we determine $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ for the same $k\tilde{G}$-modules V. Lastly, Section 5.4 records all the results of this chapter.

We will now give some notation which will be used throughout the chapter. We fix k to be an algebraically closed field of characteristic $p \neq 2$, G to be a simple adjoint linear algebraic group of type B_ℓ, $\ell \geq 3$, and $\tilde{G}$ to be the simple simply connected linear algebraic group of the same type as G. We also fix $\phi : \tilde{G} \to G$ a central isogeny with $d\phi \neq 0$ and $\ker(\phi) = \mathrm{Z}(\tilde{G})$. In G, we let T, $\mathrm{X}(T)$, Φ, B, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\omega_1, \ldots, \omega_\ell$ be as usual. Moreover, we let $\tilde{T}$, respectively $\tilde{B}$, be a preimage of T, respectively of B, in $\tilde{G}$, and we note that $\tilde{T}$ is a maximal torus of $\tilde{G}$ contained in the Borel subgroup $\tilde{B}$ of $\tilde{G}$. As for G, we let $\mathrm{X}(\tilde{T})$, $\tilde{\Phi}$, $\tilde{\Delta} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_\ell\}$ and $\tilde{\omega}_1, \ldots, \tilde{\omega}_\ell$ be the rational character group of $\tilde{T}$, the root system of $\tilde{G}$ determined by $\tilde{T}$, the set of simple roots in $\tilde{\Phi}$ given by $\tilde{B}$, and the fundamental dominant weights of $\tilde{G}$ corresponding to $\tilde{\Delta}$.

5.1 Construction of linear algebraic groups of type B_ℓ

Let W be a $2\ell+1$-dimensional k-vector space, for some $\ell \geq 3$, equipped with a nondegenerate symmetric bilinear form with associated quadratic form Q. We fix $B_W = \{u_1, u_2, \ldots, u_\ell, w, v_\ell, \ldots, v_2, v_1\}$ to be an ordered basis in W with the property that $W = \bigoplus\limits_{i=1}^{\ell} \langle u_i, v_i \rangle \oplus \langle w \rangle$ is an orthogonal direct sum, where $\{u_i, v_i\}$ is a hyperbolic pair for all $1 \leq i \leq \ell$, and w is such that $Q(w) = 1$, see Corollary 2.1.4. Let D, respectively U, be the set of diagonal matrices, respectively the set of upper-triangular matrices, in $\mathrm{SL}(W)$. Let $G = \mathrm{SO}(W)$ and note that G is a simple adjoint group of type B_ℓ, see [Car89, Theorem 11.3.2]. Now, we have that $T = G \cap D$ is a maximal torus in G contained in the Borel subgroup $B = G \cap U$ of G.

Remark 5.1.1. *We recall from Subsection 2.7.3 that F^{B_ℓ}, the set of p-restricted dominant weights $\tilde{\lambda} \in \mathrm{X}(\tilde{T})$ with the property that the associated irreducible $k\tilde{G}$-module $L_{\tilde{G}}(\tilde{\lambda})$ satisfies the dimensional criteria (2.17) for all $\ell \geq 3$, is given by $F^{B_\ell} = \{\tilde{\omega}_1, 2\tilde{\omega}_1, \tilde{\omega}_2\}$. We remark that for all $\tilde{\lambda} \in \{\tilde{\omega}_1, 2\tilde{\omega}_1, \tilde{\omega}_2\}$, there exists $\lambda \in \mathrm{X}(T)$ such that $\tilde{\lambda}$ is the image of λ when viewed as an element of $\mathrm{X}(\tilde{T})$, see Subsection 2.3.3. More precisely, the weight $\omega_1 \in \mathrm{X}(T)$ is denoted by $\tilde{\omega}_1$ when viewed as an element of $\mathrm{X}(\tilde{T})$, the weight $2\omega_1 \in \mathrm{X}(T)$ is $2\tilde{\omega}_1 \in \mathrm{X}(\tilde{T})$ and the weight $\omega_2 \in \mathrm{X}(T)$ is $\tilde{\omega}_2 \in \mathrm{X}(\tilde{T})$. In all of these cases, by Lemma 2.3.10, we have:*

(1) $\tilde{M}_s = \max\limits_{\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(L_{\tilde{G}}(\tilde{\lambda})_{\tilde{s}}(\tilde{\mu})) \mid \tilde{\mu} \in k^*\} = \max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(L_G(\lambda)_s(\mu)) \mid \mu \in k^*\} = M_s$.

(2) $\tilde{M}_u = \max\limits_{\tilde{u} \in \tilde{G}_u \setminus \{1\}} \dim(L_{\tilde{G}}(\tilde{\lambda})_{\tilde{u}}(1)) = \max\limits_{u \in G_u \setminus \{1\}} \dim(L_G(\lambda)_u(1)) = M_u$, *where G_u is the set of unipotent elements in G.*

(3) $\nu_{\tilde{G}}(L_{\tilde{G}}(\tilde{\lambda})) = \nu_G(L_G(\lambda))$.

5.1.1 Semisimple elements

Let $s \in T$. Then $s = \mathrm{diag}(a_1, a_2, \ldots, a_\ell, 1, a_\ell^{-1}, \ldots, a_2^{-1}, a_1^{-1})$ with $a_i \in k^*$. Let $\mu_1, \mu_2, \ldots, \mu_m$, where $m \geq 1$, denote the distinct a_j's, where $\mu_j \neq 1$, and, for all $1 \leq i \leq m$, let n_i denote the multiplicity of μ_i in s. Furthermore, let n be the multiplicity of 1 in s. Then n is odd and, if $s \notin \mathrm{Z}(G)$, then $1 \leq n \leq 2\ell - 1$. Moreover, we have $n + 2\sum\limits_{i=1}^{m} n_i = 2\ell + 1$. Further, we can assume without loss of generality that $\ell \geq n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$ and, by conjugating s by an element of $\mathrm{N}_G(T)$, we can also assume that

$$s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \mu_2 \cdot \mathrm{I}_{n_2}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, 1 \cdot \mathrm{I}_n, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \mu_2^{-1} \cdot \mathrm{I}_{n_2}, \mu_1^{-1} \cdot \mathrm{I}_{n_1}).$$

We recall that $\mathcal{W} = \mathrm{N}_G(T)/T$ is the Weyl group of G associated to T and that $s_\alpha \in \mathcal{W}$ is the reflection corresponding to $\alpha \in \Phi$. Let $n_\alpha \in \mathrm{N}_G(T)$ be an arbitrary fixed preimage of s_α.

Lemma 5.1.2. *With the notation introduced above, assume there exist $1 \leq i < j \leq m$ with $\mu_j = \mu_i^{-1}$. Then there exists $w \in \mathcal{W}$ with arbitrary fixed preimage $n \in \mathrm{N}_G(T)$ such that*

$$n \cdot s \cdot n^{-1} = \begin{pmatrix} A & & \\ & \mathrm{I}_n & \\ & & A^\star \end{pmatrix},$$

where

$$A = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \ldots, \mu_{i-1} \cdot \mathrm{I}_{n_{i-1}}, \mu_i \cdot \mathrm{I}_{n_i + n_j}, \mu_{i+1} \cdot \mathrm{I}_{n_{i+1}}, \ldots, \mu_{j-1} \cdot \mathrm{I}_{n_{j-1}}, \mu_{j+1} \cdot \mathrm{I}_{n_{j+1}}, \ldots, \mu_m \cdot \mathrm{I}_{n_m})$$

and $A^* = (A^*_{i,j})_{i,j}$ is the diagonal matrix with $A^*_{r,r} = A^{-1}_{n_1 + \cdots + n_m + 1 - r, n_1 + \cdots + n_m + 1 - r}$, for all $1 \leq r \leq n_1 + \cdots + n_m$.

Proof. Let $s_{\alpha_r} \in \mathcal{W}$ be the reflection corresponding to the simple root $\alpha_r \in \Delta$. We remark that when we conjugate s by n_{α_r}, where $1 \leq r \leq \ell - 1$, we interchange the entry in position (r, r) with the one in position $(r+1, r+1)$, the entry in position $(2\ell + 1 - r, 2\ell + 1 - r)$ with

the one in $(2\ell+2-r, 2\ell+2-r)$, and all the other entries are fixed. When we conjugate s by n_{α_ℓ} we interchange the entry in position (ℓ,ℓ) with the one in position $(\ell+2,\ell+2)$ and all the other entries are fixed. Hence, in order to interchange the entry in position (r,r) with the one in position $(2\ell+2-r, 2\ell+2-r)$ we conjugate s by

$$n_r := n_{\alpha_r} n_{\alpha_{r+1}} \cdots n_{\alpha_\ell} n_{\alpha_{\ell-1}} \cdots n_{\alpha_r}.$$

Therefore, conjugating s by $\prod_{r=n_1+\cdots+n_{j-1}+1}^{n_1+\cdots+n_j} n_r$ gives a matrix of the form

$$\begin{aligned}\mathrm{diag}(&\mu_1\cdot \mathrm{I}_{n_1},\ldots,\mu_{j-1}\cdot \mathrm{I}_{n_{j-1}},\mu_i\cdot \mathrm{I}_{n_j},\mu_{j+1}\cdot \mathrm{I}_{n_{j+1}},\ldots,\mu_m\cdot \mathrm{I}_{n_m},1\cdot \mathrm{I}_n,\mu_m^{-1}\cdot \mathrm{I}_{n_m},\ldots,\mu_{j+1}^{-1}\cdot \mathrm{I}_{n_{j+1}},\\ &\mu_i^{-1}\cdot \mathrm{I}_{n_j},\mu_{j-1}^{-1}\cdot \mathrm{I}_{n_{j-1}},\ldots,\mu_1^{-1}\cdot \mathrm{I}_{n_1}).\end{aligned}$$

Finally, reordering as before, we arrive at the desired matrix form. □

Now, let $s\in T$, $s=\mathrm{diag}(\mu_1\cdot \mathrm{I}_{n_1},\mu_2\cdot \mathrm{I}_{n_2},\ldots,\mu_m\cdot \mathrm{I}_{n_m},1\cdot \mathrm{I}_n,\mu_m^{-1}\cdot \mathrm{I}_{n_m},\ldots,\mu_2^{-1}\cdot \mathrm{I}_{n_2},\mu_1^{-1}\cdot \mathrm{I}_{n_1})$ with $\mu_i\neq\mu_j$ for all $1\le i<j\le m$. Lemma 5.1.2 allows us to assume as well that $\mu_i\neq\mu_j^{-1}$ for all $1\le i<j\le m$. Therefore, for the remainder of the chapter, we fix the following hypothesis on semisimple elements in G:

$$\begin{aligned}({}^{\dagger}H_s):\ &\text{any } s\in T\setminus \mathrm{Z}(G) \text{ is such that } s=\mathrm{diag}(\mu_1\cdot \mathrm{I}_{n_1},\ldots,\mu_m\cdot \mathrm{I}_{n_m},1\cdot \mathrm{I}_n,\mu_m^{-1}\cdot \mathrm{I}_{n_m},\ldots,\\ &\mu_1^{-1}\cdot \mathrm{I}_{n_1}) \text{ with } \mu_i\neq\mu_j^{\pm1},\ \text{for all } 1\le i<j\le m,\ \mu_i\neq 1,\ \text{for all } 1\le i\le m,\ \text{and}\\ &n+2\sum_{i=1}^m n_i=2\ell+1,\ \text{where } 1\le n\le 2\ell-1 \text{ and } \ell\ge n_1\ge\cdots\ge n_m\ge 1.\end{aligned}$$

5.1.2 Unipotent elements

Since the algebraically closed field k has characteristic $p\neq 2$, by Theorem 2.9.2, we know that unipotent elements in G are G-conjugate if and only if they are $\mathrm{GL}(W)$-conjugate, i.e. if and only if they have the same Jordan form on W. Let u be a unipotent element of G and let $\bigoplus_{i=1}^m J_{n_i}^{r_i}$ be its Jordan form on W. Then, $\sum_{i=1}^m n_i r_i = 2\ell+1$ and $r_i\ge 1$ is even for all even n_i, see Theorem 2.9.2. We can assume without loss of generality that $2\ell+1\ge n_1>n_2>\cdots>n_m\ge 1$ and we also note that if $u\neq 1$ and $m=1$, then $n_1\ge 3$.

5.2 Eigenspace dimensions for semisimple elements

Before we state the main results of this section, we recall that $F^{B_\ell}=\{\tilde\omega_1,\tilde\omega_2,2\tilde\omega_1\}$, see Subsection 2.7.3.

Theorem 5.2.1. *Let k be an algebraically closed field of characteristic $p\neq 2$ and let $\tilde G$ be a simple simply connected linear algebraic group of type B_ℓ, $\ell\ge 3$. Let $\tilde T$ be a fixed maximal torus in $\tilde G$ and let $V=L_{\tilde G}(\tilde\lambda)$, where $\tilde\lambda\in F^{B_\ell}$ or $\tilde\lambda$ is given in Table 2.7.4. Then there exist $s\in\tilde T\setminus \mathrm{Z}(\tilde G)$ and $\mu\in k^*$, an eigenvalue of s on V, such that*

$$\dim(V_s(\mu))\ge \dim(V)-\sqrt{\dim(V)}$$

if and only if $\tilde{\lambda} = \tilde{\omega}_1$.

Theorem 5.2.2. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type B_ℓ, $\ell \geq 3$. Let $\tilde{T}$ and V be as in Theorem 5.2.1. Then the value of* $\max_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ *is given in the table below:*

V	Char.	Rank	$\max_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$
$L_{\tilde{G}}(\tilde{\omega}_1)$	$p \neq 2$	$\ell \geq 3$	2ℓ
${}^\dagger L_{\tilde{G}}(\tilde{\omega}_2)$	$p \neq 2$	$\ell \geq 3$	$2\ell^2 - \ell$
${}^\dagger L_{\tilde{G}}(2\tilde{\omega}_1)$	$p \neq 2$ and $p \nmid 2\ell + 1$	$\ell \geq 3$	$2\ell^2 + \ell$
	$p \neq 2$ and $p \mid 2\ell + 1$	$\ell \geq 3$	$2\ell^2 + \ell - 1$
${}^\dagger L_G(2\tilde{\omega}_3)$	$p \neq 2$	$\ell = 3$	20
${}^\dagger L_G(\tilde{\omega}_1 + \tilde{\omega}_3)$	$p \neq 2$	$\ell = 3$	$24 - 4\delta_{p,7}$
${}^\dagger L_G(\tilde{\omega}_\ell)$	$p \neq 2$	$3 \leq \ell \leq 8$	$2^{\ell-1}$

Table 5.2.1: The value of $\max_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$.

In particular, for each V in Table 5.2.1 labeled ${}^\dagger V$ we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ *for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.*

We will give the proofs of Theorems 5.2.1 and 5.2.2 in a series of results, each treating one of the candidate-modules. In Subsection 5.2.1, we determine $\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, see Remark 5.1.1, where V belongs to one of the families of kG-modules we have to consider, i.e. V is an irreducible kG-module $L_G(\lambda)$ with p-restricted dominant weight $\lambda \in \{\omega_1, 2\omega_1, \omega_2\}$. In Subsection 5.2.2, we establish $\max_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ for the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with p-restricted dominant weight $\tilde{\lambda}$ featured in Table 2.7.4.

5.2.1 The families of modules

Lemma 5.2.3. *Let $V = L_G(\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell,$$

where equality holds if and only if $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$.

In particular, there exist $s \in T \setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. To begin, note that $V \cong W$ as kG-modules, therefore V is self-dual and $\dim(V) = 2\ell + 1$. Let $s \in T \setminus \mathrm{Z}(G)$. Then $\dim(V_s(\mu)) \leq 2\ell$ for all eigenvalues $\mu \in k^*$ of s on V. Now, as $\dim(V_s(1)) = n$, where n is odd, and $\dim(V_s(\mu)) \leq \frac{\dim(V)}{2}$ for all eigenvalues $\mu \neq \mu^{-1}$ of s on V, it follows that $\dim(V_s(\mu)) = 2\ell$ if and only if $\mu = -1$ and $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 2\ell$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained, for example $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$ and $\mu = -1$. Now, as the inequality $2\ell \geq 2\ell + 1 - \sqrt{2\ell + 1}$ holds for all $\ell \geq 2$, we have shown that there exist $s \in T \setminus \mathrm{Z}(G)$ such that $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$ for some eigenvalue $\mu \in k^*$ on V. This completes the proof of the lemma. □

Proposition 5.2.4. *Let $V = L_G(\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell^2 - \ell,$$

where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$.

In particular, $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothsis $({}^{\dagger}H_s)$. By Lemma 2.8.4, since $p \neq 2$, it follows that $V \cong \wedge^2(W)$, therefore $\dim(V) = 2\ell^2 + \ell$, and we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2}, \text{ where } 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i-1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1}, \text{ where } 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j, \text{ where } 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i \text{ and } \mu_i^{-1}, \text{ where } 1 \leq i \leq m, \text{ each with multiplicity at least } nn_i; \\ 1 \text{ with multiplicity at least } \frac{n(n-1)}{2} + \sum_{r=1}^{m} n_r^2. \end{cases} \tag{5.1}$$

Let $\mu \in k^*$ be an eigenvalue of s on V. If μ is such that $\mu \neq \mu^{-1}$, then:

$$\dim(V_s(\mu)) \leq \dim(V) - \dim(V_s(\mu^{-1})),$$

where, since V is self-dual, we have that $\dim(V_s(\mu)) = \dim(V_s(\mu^{-1}))$. Keeping in mind that $\ell \geq 3$, we deduce that:

$$\dim(V_s(\mu)) \leq \frac{2\ell^2 + \ell}{2} < 2\ell^2 - \ell.$$

Therefore, we can assume that $\mu = \pm 1$.

First, consider the case of $m = 1$. As $s \notin \mathrm{Z}(G)$, it follows that $\mu_1 \neq 1$. Moreover, by (5.1), we determine that the eigenvalues of s on V, not necessarily distinct, are μ_1^2 and μ_1^{-2}, each with multiplicity at least $\frac{n_1(n_1-1)}{2}$, μ_1 and μ_1^{-1}, each with multiplicity at least nn_1, and 1 with multiplicity at least $\frac{n(n-1)}{2} + n_1^2$.

Let $\mu = 1$. Then, as $s \notin \mathrm{Z}(G)$, we have $\mu_1^{\pm 1} \neq 1$ and it follows that

$$\dim(V_s(1)) \leq 2\ell^2 + \ell - 2nn_1. \tag{5.2}$$

Suppose that $\dim(V_s(1)) \geq 2\ell^2 - \ell$. Then, keeping in mind that $2n_1 = 2\ell + 1 - n$, we have:

$$(2\ell - n)(1 - n) \geq 0. \tag{5.3}$$

As $1 \leq n \leq 2\ell - 1$, Inequality (5.3) holds if and only if $n = 1$. Then $n_1 = \ell$ and, by (5.2), we have $\dim(V_s(1)) \leq 2\ell^2 - \ell$. Now, equality holds if and only if all eigenvalues of s on V different than $\mu_1^{\pm 1}$ are equal to 1, hence if and only if $\mu_1^2 = 1$. Therefore $\dim(V_s(1)) = 2\ell^2 - \ell$ if and only if $\mu_1 = -1$ and, up to conjugation, $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$.

Let $\mu = -1$. Then, as $\mu_1 \neq 1$, we distinguish the two cases:

Case 1: $\mu_1 = -1$. Then $\dim(V_s(-1)) = 2nn_1$. Keeping in mind that $2n_1 = 2\ell + 1 - n$, $1 \leq n \leq 2\ell - 1$, and that $\ell \geq 3$, we have

$$\begin{aligned} 2\ell^2 - \ell - 2nn_1 &= 2\ell^2 - \ell - (2\ell + 1 - n)n = (\ell - n)^2 + \ell^2 - \ell - n \\ &\geq (\ell - n)^2 + \ell^2 - 3\ell + 1 > 0, \end{aligned}$$

therefore $2\ell^2 - \ell > \dim(V_s(-1))$.

Case 2: $\mu_1^2 = -1$. Then $\dim(V_s(-1)) = n_1(n_1 - 1)$. Since $1 \leq n_1 \leq \ell$, we have $2\ell^2 - \ell - n_1^2 + n_1 = (\ell - n_1)(\ell + n_1 - 1) + \ell^2 > 0$, therefore $2\ell^2 - \ell > \dim(V_s(-1))$.

We have shown that for all $s \in T \setminus \mathrm{Z}(G)$ with $m = 1$ we have $\dim(V_s(1)) \leq 2\ell^2 - \ell$, where equality holds if and only if $s = \mathrm{diag}(-1, \cdots, -1, 1, -1, \cdots, -1)$; and that $\dim(V_s(-1)) < 2\ell^2 - \ell$.

We can now assume that $m \geq 2$ and start by considering the eigenvalue 1 of s on V. Since $\mu_i \neq \mu_j^{\pm 1}$ for all $1 \leq i < j \leq m$, we have that $\mu_i^{\pm 1}\mu_j^{\pm 1} \neq 1$ for all $1 \leq i < j \leq m$. Furthermore, we also have $\mu_i^{\pm 1} \neq 1$. By (5.1), all of the above account for at least $4\sum_{i<j} n_i n_j + 2n\sum_{i=1}^{m} n_i$ additional eigenvalues of s on V different than 1. Therefore, we have:

$$\dim(V_s(1)) \leq 2\ell^2 + \ell - 4\sum_{i<j} n_i n_j - 2n\sum_{i=1}^{m} n_i.$$

If $\dim(V_s(1)) \geq 2\ell^2 - \ell$, then:

$$2\ell - 4\sum_{i<j} n_i n_j - 2n\sum_{i=1}^{m} n_i \geq 0 \tag{5.4}$$

and, since $2\sum_{i=1}^{m} n_i = 2\ell + 1 - n$, it follows that:

$$(2\ell - n)(1 - n) - 4\sum_{i<j} n_i n_j \geq 0. \tag{5.5}$$

Since $1 \leq n \leq 2\ell - 1$, we have $(2\ell - n)(1 - n) \leq 0$, while, since $m \geq 2$, we have $-4\sum_{i<j} n_i n_j < 0$, therefore Inequality (5.5) does not hold, hence $\dim(V_s(1)) < 2\ell^2 - \ell$.

Finally, let $\mu = -1$. If $\mu_i \neq -1$ for all $1 \leq i \leq m$, then:

$$\dim(V_s(-1)) \leq 2\ell^2 + \ell - \frac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n\sum_{r=1}^{m} n_r.$$

Suppose that $\dim(V_s(-1)) \geq 2\ell^2 - \ell$. Then

$$2\ell - \frac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n \sum_{r=1}^{m} n_r \geq 0.$$

Since $2\sum_{r=1}^{m} n_r = 2\ell + 1 - n$, we have:

$$(2\ell - n)(1-n) - \frac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 \geq 0. \tag{5.6}$$

But $(2\ell - n)(1-n) \leq 0$, $-\frac{n(n-1)}{2} \leq 0$ and $-\sum_{r=1}^{m} n_r^2 < 0$, as $n \geq 1$ and $n_r \geq 1$ for all $1 \leq r \leq m$. Therefore, Inequality (5.6) does not hold. We can now assume that there exist $1 \leq i \leq m$ such that $\mu_i = -1$. Then, since the μ_i's are distinct, we have $\mu_j^{\pm 1} \neq -1$ for all $1 \leq j \leq m$, $j \neq i$. Moreover, since $\mu_j \neq 1$ for all $1 \leq j \leq m$, we also have $\mu_i^{\pm 1}\mu_j^{\pm 1} = -\mu_j^{\pm 1} \neq -1$ for all $1 \leq j \leq m$, $j \neq i$. By (5.1), the latter account for at least $4n_i \sum_{r \neq i} n_r$ additional eigenvalues of s on V different than -1. Further, we have $\mu_i^2 = \mu_i^{-2} = 1$ and so:

$$\dim(V_s(-1)) \leq 2\ell^2 + \ell - \frac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n \sum_{r \neq i} n_r - 4n_i \sum_{r \neq i} n_r - n_i(n_i - 1).$$

Suppose that $\dim(V_s(-1)) \geq 2\ell^2 - \ell$. It follows that:

$$2\ell - \frac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n \sum_{r \neq i} n_r - 4n_i \sum_{r \neq i} n_r - n_i(n_i - 1) \geq 0. \tag{5.7}$$

We have that $\sum_{r=1}^{m} n_r^2 \geq \sum_{r=1}^{m} n_r$, as $n_r \geq 1$ for all $1 \leq r \leq m$, and that $2\ell = 2\sum_{r=1}^{m} n_r + n - 1$. By (5.7), it follows:

$$2\sum_{r=1}^{m} n_r + n - 1 - \frac{n(n-1)}{2} - \sum_{r=1}^{m} n_r - 2n \sum_{r \neq i} n_r - 4n_i \sum_{r \neq i} n_r - n_i(n_i - 1) \geq 0,$$

which we rewrite as:

$$\sum_{r \neq i} n_r(1 - 2n) + n_i(2 - n_i - 4\sum_{r \neq i} n_r) - \frac{(n-1)(n-2)}{2} \geq 0. \tag{5.8}$$

As $n \geq 1$, it follows that $-\frac{(n-1)(n-2)}{2} \leq 0$ and $\sum_{r \neq i} n_r(1 - 2n) < 0$. Moreover, since $m \geq 2$ and $n_r \geq 1$ for all $1 \leq r \leq m$, we have $n_i(2 - n_i - 4\sum_{r \neq i} n_r) < 0$, thus Inequality (5.8) does not hold. This proves that $\dim(V_s(-1)) < 2\ell^2 - \ell$ for all $s \in T \setminus \mathrm{Z}(G)$ with $m \geq 2$.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 2\ell^2 - \ell$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. In particular, as the inequality $0 < 2\ell^2 - \ell$ holds for all $\ell \geq 3$, it follows that $2\ell^2 - \ell < 2\ell^2 + \ell - \sqrt{2\ell^2 + \ell}$ for all $\ell \geq 3$ and therefore $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 5.2.5. *Let $V' = \mathrm{S}^2(W)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V' we have*

$$\dim(V_s'(\mu)) \leq 2\ell^2 + \ell + 1,$$

where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. We first remark that, as $p \neq 2$, $V' = \mathrm{S}^2(W)$ is a self-dual module, see [McN98, Lemma 4.7.1(b)]. Secondly, we note that $\dim(V) = 2\ell^2 + 3\ell + 1$ and we determine that the eigenvalues of s on V', not necessarily distinct, are:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2},\ 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i+1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1},\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j,\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i \text{ and } \mu_i^{-1},\ 1 \leq i \leq m, \text{ each with multiplicity at least } nn_i; \\ 1 \text{ with multiplicity at least } \displaystyle\sum_{r=1}^{m} n_r^2 + \frac{n(n+1)}{2}. \end{cases} \tag{5.9}$$

Let μ be an eigenvalue of s on V'. If $\mu \neq \mu^{-1}$, then:

$$\dim(V_s'(\mu)) \leq 2\ell^2 + 3\ell + 1 - \dim(V_s'(\mu^{-1}))$$

and, since $\dim(V_s'(\mu)) = \dim(V_s'(\mu^{-1}))$, as V' is self-dual, and $\ell \geq 3$, we have that

$$\dim(V_s'(\mu)) \leq \frac{2\ell^2 + 3\ell + 1}{2} < 2\ell^2 + \ell + 1. \tag{5.10}$$

Therefore we can assume that $\mu = \pm 1$.

Suppose that $m = 1$. We note that, in this case, as $s \notin \mathrm{Z}(G)$, we have $\mu_1 \neq 1$. Then, by (5.9), the eigenvalues of s on V', not necessarily distinct, are μ_1^2 and μ_1^{-2}, each with multiplicity at least $\frac{n_1(n_1+1)}{2}$, μ_1 and μ_1^{-1}, each with multiplicity at least nn_1, and 1 with multiplicity at least $\frac{n(n+1)}{2} + n_1^2$.

For $\mu = 1$, since $s \notin \mathrm{Z}(G)$, we have $\mu_1^{\pm 1} \neq 1$ and so

$$\dim(V_s'(1)) \leq 2\ell^2 + 3\ell + 1 - 2nn_1. \tag{5.11}$$

If $\dim(V_s'(1)) \geq 2\ell^2 + \ell + 1$, then, keeping in mind that $2n_1 = 2\ell + 1 - n$, we have:

$$(2\ell - n)(1 - n) \geq 0.$$

But, this is just Inequality (5.3) which holds if and only if $n = 1$. Then $n_1 = \ell$ and, by (5.11), we have $\dim(V_s'(1)) \leq 2\ell^2 + \ell + 1$. Equality holds if and only if all eigenvalues

of s on V' different than $\mu_1^{\pm 1}$ are equal to 1, hence if and only if $\mu_1^2 = 1$. We deduce that $\dim(V_s'(1)) = 2\ell^2+\ell+1$ if and only if, up to conjugation, $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$, as in the statement of the proposition.

For $\mu = -1$ we have that

$$\dim(V_s'(-1)) \leq 2\ell^2 + 3\ell + 1 - n_1^2 - \frac{n(n+1)}{2}. \tag{5.12}$$

If $\dim(V_s'(-1)) \geq 2\ell^2 + \ell + 1$, then:

$$2\ell - n_1^2 - \frac{n(n+1)}{2} \geq 0$$

and, since $2\ell + 1 = 2n_1 + n$, we have:

$$-(n_1 - 1)^2 + n\frac{1-n}{2} \geq 0. \tag{5.13}$$

Inequality (5.13) holds if and only if $n = 1$ and $n_1 = 1$, contradicting $\ell \geq 3$. Therefore $\dim(V_s'(-1)) < 2\ell^2 + \ell + 1$ for all $s \in T \setminus \mathrm{Z}(G)$ with $m = 1$.

We can now assume that $m \geq 2$. For $\mu = 1$, we first recall that $\mu_i^{\pm 1} \neq 1$ for all $1 \leq i \leq m$. Secondly, as $\mu_i \neq \mu_j^{\pm 1}$ for all $1 \leq i < j \leq m$, we have that $\mu_i^{\pm 1}\mu_j^{\pm 1} \neq 1$ for all $1 \leq i < j \leq m$. Hence, by (5.9), it follows that:

$$\dim(V_s'(1)) \leq 2\ell^2 + 3\ell + 1 - 2n\sum_{i=1}^{m} n_i - 4\sum_{i<j} n_i n_j.$$

If $\dim(V_s'(1)) \geq 2\ell^2 + \ell + 1$, then:

$$2\ell - 2n\sum_{i=1}^{m} n_i - 4\sum_{i<j} n_i n_j \geq 0.$$

We see that this is just Inequality (5.4), which does not hold. Therefore, $\dim(V_s'(1)) < 2\ell^2 + \ell + 1$ for all $s \in T \setminus \mathrm{Z}(G)$ with $m \geq 2$.

Lastly, let $\mu = -1$. Suppose that $\mu_i \neq -1$ for all $1 \leq i \leq m$. Then $\mu_i^{-1} \neq -1$ for all $1 \leq i \leq m$ and, by (5.9), it follows that:

$$\dim(V_s'(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r=1}^{m} n_r. \tag{5.14}$$

If $\dim(V_s'(-1)) \geq 2\ell^2 + \ell + 1$, then:

$$2\ell - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r=1}^{m} n_r \geq 0$$

and, since $2\ell + 1 = n + 2\sum_{r=1}^{m} n_r$, we have:

$$n\frac{1-n}{2} + \sum_{r=1}^{m} n_r(2 - n_r - 2n) \geq 1, \tag{5.15}$$

contradicting $n \geq 1$ and $n_r \geq 1$ for all $1 \leq r \leq m$. Therefore we can assume that there exists $1 \leq i \leq m$ such that $\mu_i = -1$. Then $\mu_r^{\pm 1} \neq -1$ for all $r \neq i$. By (5.9), these account for at least $2n \sum_{r \neq i} n_r$ additional eigenvalues which are different than -1. Furthermore, we have $\mu_i^2 = \mu_i^{-2} = 1$ and so:

$$\dim(V_s^{'}(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n \sum_{r \neq i} n_r - n_i(n_i + 1). \tag{5.16}$$

Assume $\dim(V_s^{'}(-1)) \geq 2\ell^2 + \ell + 1$. Then

$$2\ell - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n \sum_{r \neq i} n_r - n_i(n_i + 1) \geq 0$$

and, since $2\ell + 1 = n + 2 \sum_{r=1}^{m} n_r$, we have:

$$n\frac{1-n}{2} + \sum_{r \neq i} n_r(2 - n_r - 2n) - (2n_i^2 - n_i + 1) \geq 0. \tag{5.17}$$

Since $n \geq 1$ and $n_r \geq 1$ for all $1 \leq r \leq m$, one sees that

$$n\frac{1-n}{2} + \sum_{r \neq i} n_r(2 - n_r - 2n) - (2n_i^2 - n_i + 1) < 0,$$

therefore, Inequality (5.17) does not hold. We conclude that $\dim(V_s^{'}(-1)) < 2\ell^2 + \ell + 1$ for all $s \in T \setminus \mathrm{Z}(G)$ with $m \geq 2$, completing the proof of the proposition. □

Corollary 5.2.6. *Assume that $p \nmid 2\ell + 1$ and let $V = L_G(2\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell^2 + \ell,$$

where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \setminus \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. Let $V^{'} = \mathrm{S}^2(W)$. Then, since $p \nmid 2\ell + 1$, by Lemma 2.8.4, it follows that $V^{'} = V \oplus L_G(0)$ and so $\dim(V) = 2\ell^2 + 3\ell$, $\dim(V_s(\mu)) = \dim(V_s^{'}(\mu))$ for all eigenvalues $\mu \neq 1$ of s on V, and $\dim(V_s(1)) = \dim(V_s^{'}(1)) - 1$.

For the eigenvalue 1, by Proposition 5.2.5, we have $\dim(V_s(1)) \leq 2\ell^2 + \ell$, where equality holds if and only if s is as in the statement of the result. Furthermore, as $\ell \geq 3$, by Inequality (5.10), we have $\dim(V_s(\mu)) \leq \frac{2\ell^2+3\ell+1}{2} < 2\ell^2 + \ell$ for all eigenvalues $\mu \neq \mu^{-1}$ of s on V. What is left is to determine $\dim(V_s(-1))$.

Suppose that $m = 1$. Then, by (5.12), we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - n_1^2 - \frac{n(n+1)}{2}.$$

If $\dim(V_s(-1)) \geq 2\ell^2 + \ell$, we proceed as in the proof for $V_s^{'}(-1)$, see (5.13), and arrive at

$$n\frac{1-n}{2} + n_1(2 - n_1) \geq 0. \tag{5.18}$$

Since $n \geq 1$, it follows that $n_1 \leq 2$. If $n_1 = 2$, then, by (5.18), we have $n = 1$, contradicting $\ell \geq 3$. On the other hand, if $n_1 = 1$, then, by (5.18), we have $n - n^2 + 2 \geq 0$ and, since n is odd, it follows that $n = 1$, again contradicting $\ell \geq 3$.

We can now assume that $m \geq 2$. If $\mu_i \neq -1$ for all $1 \leq i \leq m$, then, by (5.14), we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r=1}^{m} n_r.$$

If $\dim(V_s(\mu)) \geq 2\ell^2 + \ell$, we proceed as in the proof for $V_s^{'}(-1)$, see (5.15), and arrive at

$$n\frac{1-n}{2} + \sum_{r=1}^{m} n_r(2 - n_r - 2n) \geq 0,$$

contradicting $n \geq 1$ and $n_r \geq 1$ for all $1 \leq r \leq m$. On the other hand, if there exists $1 \leq i \leq m$ such that $\mu_i = -1$, then, by (5.16), we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r\neq i} n_r - n_i(n_i + 1).$$

If $\dim(V_s(-1)) \geq 2\ell^2 + \ell$, we proceed as in the proof for $V_s^{'}(-1)$, see (5.17), and arrive at

$$n\frac{1-n}{2} + \sum_{r\neq i} n_r(2 - n_r - 2n) - n_i(2n_i - 1) \geq 0.$$

Since $n \geq 1$ and $n_r \geq 1$ for all $1 \leq r \leq m$, the above inequality does not hold. It follows that $\dim(V_s(-1)) < 2\ell^2 + \ell$ for all $s \in T \setminus \mathrm{Z}(G)$.

We conclude that $\dim(V_s(\mu)) \leq 2\ell^2 + \ell$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Moreover, as the inequality $0 < 2\ell^2 - 3\ell$ holds for all $\ell \geq 3$, it follows that $2\ell^2 + \ell < 2\ell^2 + 3\ell - \sqrt{2\ell^2 + 3\ell}$ for all $\ell \geq 3$, and so $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Corollary 5.2.7. *Assume that $p \mid 2\ell + 1$ and let $V = L_G(2\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell^2 + \ell - 1,$$

where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $s \in T \backslash \mathrm{Z}(G)$ be as in hypothesis $({}^{\dagger}H_s)$. Let $V^{'} = \mathrm{S}^2(W)$. Then, since $p \mid 2\ell+1$, by Lemma 2.8.4, we have $V^{'} = L_G(0) \mid V \mid L_G(0)$ and so $\dim(V) = 2\ell^2 + 3\ell - 1$, $\dim(V_s(\mu)) = \dim(V_s^{'}(\mu))$ for all eigenvalues $\mu \neq 1$ of s on V, and $\dim(V_s(1)) = \dim(V_s^{'}(1)) - 2$.

For the eigenvalue 1, by Proposition 5.2.5, we have $\dim(V_s(1)) \leq 2\ell^2 + \ell - 1$, where equality holds if and only if s is as in the statement of the result. Furthermore, as $\ell \geq 3$, by Inequality (5.10), we have $\dim(V_s(\mu)) \leq \frac{2\ell^2+3\ell+1}{2} < 2\ell^2 + \ell - 1$ for all eigenvalues $\mu \neq \mu^{-1}$ of s on V. What is left is to determine $\dim(V_s(-1))$.

Suppose that $m = 1$. Then, by (5.12), we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - n_1^2 - \frac{n(n+1)}{2}.$$

If $\dim(V_s(-1)) \geq 2\ell^2 + \ell - 1$, we proceed as for $V_s^{'}(-1)$, see (5.13), and arrive at

$$n\frac{1-n}{2} + (2n_1 - n_1^2 + 1) \geq 0. \tag{5.19}$$

Since $n \geq 1$, then $2n_1 - n_1^2 + 1 \geq 0$ and so, we have $n_1 \in \{1, 2\}$. If $n_1 = 2$, then, by (5.19), it follows that $n - n^2 + 2 \geq 0$ and, since n is odd, we have $n = 1$, contradicting $\ell \geq 3$. On the other hand, if $n_1 = 1$, then, by (5.19), we have $n - n^2 + 4 \geq 0$ and so $n = 1$, which again contradicts $\ell \geq 3$.

We can now assume that $m \geq 2$. If $\mu_i \neq -1$ for all $1 \leq i \leq m$, then, by (5.14), we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r=1}^{m} n_r.$$

If $\dim(V_s(\mu)) \geq 2\ell^2 + \ell - 1$, we proceed as in the proof for $V_s^{'}(-1)$, see (5.15), and arrive at

$$n\frac{1-n}{2} + 1 + \sum_{r=1}^{m} n_r(2 - n_r - 2n) \geq 0. \tag{5.20}$$

We have that $\sum_{r=1}^{m} n_r(2 - n_r - 2n) < 0$, as $n \geq 1$ and $n_r \geq 1$ for all $1 \leq r \leq m$, therefore $n\frac{1-n}{2} + 1 \geq 0$. Since n is odd, it follows that $n = 1$ and substituting in (5.20) gives:

$$1 - \sum_{r=1}^{m} n_r^2 \geq 0,$$

contradicting $m \geq 2$. On the other hand, if there exists $1 \leq i \leq m$ such that $\mu_i = -1$, then, by (5.16), we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r \neq i} n_r - n_i(n_i + 1).$$

If $\dim(V_s(-1)) \geq 2\ell^2 + \ell - 1$, we proceed as in the proof for $V_s^{'}(-1)$, see (5.17), and arrive at

$$n\frac{1-n}{2} + \sum_{r \neq i} n_r(2 - n_r - 2n) + (-2n_i^2 + n_i + 1) \geq 0.$$

Since $n \geq 1$ and $n_r \geq 1$ for all $1 \leq r \leq m$, the above inequality does not hold. It follows that $\dim(V_s(-1)) < 2\ell^2 + \ell - 1$.

We conclude that $\dim(V_s(\mu)) \leq 2\ell^2+\ell-1$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues μ of s on V. Moreover, as the inequality $0 < 2\ell^2-3\ell+1$ holds for all $\ell \geq 3$, it follows that $2\ell^2+\ell-1 < 2\ell^2+3\ell-1-\sqrt{2\ell^2+3\ell-1}$ for all $\ell \geq 3$, therefore $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

To conclude this subsection, we remark that Lemma 5.2.3, Proposition 5.2.4 and Corollaries 5.2.6 and 5.2.7 give the proof of Theorems 5.2.1 and 5.2.2 for the families of kG-modules corresponding to p-restricted dominant weights $\lambda \in \{\omega_1, \omega_2, 2\omega_1\}$. Therefore, in view of Remark 5.1.1, they prove Theorems 5.2.1 and 5.2.2 for the families of $k\tilde{G}$-modules with p-restricted dominant weights $\tilde{\lambda} \in F^{B_\ell}$.

5.2.2 The particular modules

As previously mentioned, in this subsection we will give an upper-bound for $\dim(V_s(\mu))$, where $(s,\mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ and V is an irreducible $k\tilde{G}$-module with associated highest weight featured in Table 2.7.4. In order to achieve our goal, we will use the same inductive algorithm as for groups of type A_ℓ and C_ℓ, see Subsection 2.4.3 for a description of this method. To begin, let L_1 be a Levi subgroup of the maximal parabolic subgroup P_1 of $\tilde{G}$, as given in Section 2.4. We recall that $L_1 = \mathrm{Z}(L_1)^\circ[L_1, L_1]$, where $\mathrm{Z}(L_1)^\circ$ is a one-dimensional torus and $[L_1, L_1]$ is a simple simply connected linear algebraic group of type $B_{\ell-1}$ with maximal torus $T' = \tilde{T} \cap [L_1, L_1]$. We note that, although we do not mention the result explicitly, we make great use of the data in [Lü01b] when discussing weights and weight multiplicities in this subsection.

Let $s \in \tilde{T}$. Then $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. As $z \in \mathrm{Z}(L_1)^\circ$, it follows that $z = \prod_{i=1}^{\ell} h_{\tilde{\alpha}_i}(c^{k_i})$, where $c \in k^*$ and $k_i \in \mathbb{Z}$ for all $1 \leq i \leq \ell$. Moreover, we have that $\tilde{\alpha}_j(z) = 1$ for all $2 \leq j \leq \ell$. It follows that $z = \prod_{i=1}^{\ell-1} h_{\tilde{\alpha}_i}(c^2) \cdot h_{\tilde{\alpha}_\ell}(c)$ with $c \in k^*$. As $h \in [L_1, L_1]$, we have $h = \prod_{i=2}^{\ell} h_{\tilde{\alpha}_i}(a_i)$, where $a_i \in k^*$ for all $2 \leq i \leq \ell$. Hence, $s = h_{\tilde{\alpha}_1}(c^2) \cdot \prod_{i=2}^{\ell-1} h_{\tilde{\alpha}_i}(c^2 a_i) \cdot h_{\tilde{\alpha}_\ell}(c a_\ell)$ with $c \in k^*$ and $a_i \in k^*$ for all $2 \leq i \leq \ell$.

Let V be an irreducible $k\tilde{G}$-module of highest weight $\tilde{\lambda} \in X(\tilde{T})$, $\tilde{\lambda} = d_1\tilde{\omega}_1 + \cdots + d_\ell\tilde{\omega}_\ell$ with $0 \leq d_1, \ldots, d_\ell \leq p-1$. We consider the decomposition:

$$V\mid_{[L_1,L_1]} = \bigoplus_{i=0}^{e_1(\tilde{\lambda})} V^i,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda}-i\tilde{\alpha}_1-\tilde{\gamma}}$ for all $0 \leq i \leq e_1(\tilde{\lambda})$. Let $s \in \tilde{T}$ and write $s = z \cdot h$, as above. By

(2.5), we have that:

$$s_z^i := (\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma})(z) = (\tilde{\lambda} - i\tilde{\alpha}_1)\left[\prod_{i=1}^{\ell-1} h_{\tilde{\alpha}_i}(c^2) \cdot h_{\tilde{\alpha}_\ell}(c)\right] = \prod_{j=1}^{\ell-1} c^{2d_j} \cdot c^{d_\ell} \cdot c^{-2i}.$$

Therefore, z acts on V^i, $0 \leq i \leq e_1(\tilde{\lambda})$, as the scalar $s_z^i = c^{2d_1+\cdots+2d_{\ell-1}+d_\ell-2i}$. Now, let $\mu_1^i, \ldots, \mu_{t_i}^i$, $t_i \geq 1$, be the distinct eigenvalues of h on V^i, $0 \leq i \leq e_1(\tilde{\lambda})$, and let $n_1^i, \ldots, n_{t_i}^i$ be their respective multiplicities. Then, by Lemma 2.4.8, it follows that the distinct eigenvalues of s on V^i are $s_z^i\mu_1^i, \ldots, s_z^i\mu_{t_i}^i$, with respective multiplicities $n_1^i, \ldots, n_{t_i}^i$.

Proposition 5.2.8. *Assume that $\ell = 3$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3)$. Then for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 4,$$

where there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\tilde{\lambda} = \tilde{\omega}_3$. Then $\dim(V) = 8$ and, by Lemma 2.4.6, we have $e_1(\tilde{\lambda}) = 1$, therefore

$$V\mid_{[L_1,L_1]} = V^0 \oplus V^1,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma}}$ for $i = 0$ and $i = 1$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\tilde{\omega}_3)$ and so, by Lemma 2.4.3, we also have $V^1 \cong (L_{L_1}(\tilde{\omega}_3))^* \cong L_{L_1}(\tilde{\omega}_3)$. Therefore

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_3). \tag{5.21}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 1$, then $s \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$, and so $s = z$ with $c^2 \neq 1$. In this case, as s acts on each V^i, $i = 0, 1$, as scalar multiplication by c^{1-2i}, we determine that the distinct eigenvalues of s on V are:

$$\begin{cases} c \text{ with } \dim(V_s(c)) = \dim(V^0) = 4; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) = \dim(V^1) = 4. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 1$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. Since z acts by scalar multiplication on V^i, $i = 0, 1$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for $i = 0, 1$, where μ_h is any eigenvalue of h on V^i. Now, by Lemma 4.2.3, we have $\dim(V_h^0(\mu_h)) \leq 2$ and $\dim(V_h^1(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^0 and V^1, respectively. We deduce that $\dim(V_s(\mu)) \leq 4$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 4$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained. Moreover, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

Proposition 5.2.9. *Assume $\ell \geq 4$ and let $V = L_{\tilde{G}}(\tilde{\omega}_\ell)$. Then for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2^{\ell-1},$$

where there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\tilde{\lambda} = \tilde{\omega}_\ell$. Then $\dim(V) = 2^\ell$ and, by Lemma 2.4.6, we have $e_1(\tilde{\lambda}) = 1$, therefore

$$V\mid_{[L_1,L_1]} = V^0 \oplus V^1,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma}}$ for $i = 0$ and $i = 1$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\tilde{\omega}_\ell)$ and so, by Lemma 2.4.3, we also have $V^1 \cong (L_{L_1}(\tilde{\omega}_\ell))^* \cong L_{L_1}(\tilde{\omega}_\ell)$. Therefore

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\tilde{\omega}_\ell) \oplus L_{L_1}(\tilde{\omega}_\ell). \tag{5.22}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 1$, then $s \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$, and so $s = z$, with $c^2 \neq 1$. In this case, as s acts on each V^i, $i = 0, 1$, as scalar multiplication by c^{1-2i}, we determine that the distinct eigenvalues of s on V are:

$$\begin{cases} c \text{ with } \dim(V_s(c)) = \dim(V^0) = 2^{\ell-1}; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) = \dim(V^1) = 2^{\ell-1}. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 1$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. Since z acts by scalar multiplication on V^i, $i = 0, 1$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for $i = 0, 1$, where μ_h is any eigenvalue of h on V^i. In the case of $\ell = 4$, by Proposition 5.2.8, we have $\dim(V_h^0(\mu_h)) \leq 2^2$ and $\dim(V_h^1(\mu_h)) \leq 2^2$ for all eigenvalues μ_h of h on V^0 and V^1, respectively. This gives $\dim(V_s(\mu)) \leq 2^3$ for all eigenvalues $\mu \in k^*$ of s on V. Now, for $\ell \geq 5$, by recurrence and using the result for $\ell = 4$ as base case, one shows that $\dim(V_h^0(\mu_h)) \leq 2^{\ell-2}$ and $\dim(V_h^1(\mu_h)) \leq 2^{\ell-2}$ for all eigenvalues μ_h of h on V^0 and V^1, respectively. It follows that $\dim(V_s(\mu)) \leq 2^{\ell-1}$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 2^{\ell-1}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained. Moreover, as the inequality $0 < 2^{\ell-2} - 1$ holds for all $\ell \geq 4$, we have that $2^{\ell-1} < 2^\ell - \sqrt{2^\ell}$ for all $\ell \geq 4$, hence $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 5.2.10. *Assume $\ell = 3$ and let $V = L_{\tilde{G}}(2\tilde{\omega}_3)$. Then for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 20,$$

where there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\tilde{\lambda} = 2\tilde{\omega}_3$. Then $\dim(V) = 35$ and, by Lemma 2.4.6, we have $e_1(\tilde{\lambda}) = 2$, therefore

$$V\mid_{[L_1,L_1]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\tilde{\gamma}\in\mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda}-i\tilde{\alpha}_1-\tilde{\gamma}}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(2\tilde{\omega}_3)$ and so, by Lemma 2.4.3, we also have $V^2 \cong (L_{L_1}(2\tilde{\omega}_3))^* \cong L_{L_1}(2\tilde{\omega}_3)$. This gives $\dim(V^1) = 15$. Now, in V^1, the weight $(\tilde{\lambda}-\tilde{\alpha}_1-\tilde{\alpha}_2-\tilde{\alpha}_3)\mid_{T'} = 2\tilde{\omega}_3$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_{L_1}(2\tilde{\omega}_3)$. Moreover, the dominant weight $(\tilde{\lambda}-\tilde{\alpha}_1-\tilde{\alpha}_2-2\tilde{\alpha}_3)\mid_{T'} = \tilde{\omega}_2$, which occurs with multiplicity 2 in V^1, is a sub-dominant weight in the composition factor of V^1 isomorphic to $L_{L_1}(2\tilde{\omega}_3)$, in which it has multiplicity 1. Comparing dimensions, we determine that V^1 has two composition factors: one isomorphic to $L_{L_1}(2\tilde{\omega}_3)$ and one isomorphic to $L_{L_1}(\tilde{\omega}_2)$; therefore $V^1 \cong L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_2)$, by [Jan07, II.2.14]. This gives

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_2) \oplus L_{L_1}(2\tilde{\omega}_3). \tag{5.23}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$, and so $s = z$, with $c^2 \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{2-2i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^0) = 10; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1) = 15; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^2) = 10. \end{cases}$$

As $c^2 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 20$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(c) \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$ with $c^2 = -1$ we have $\dim(V_s(-1)) = 20$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 2$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where μ_h is any eigenvalue of h on V^i. We will first show that $\dim(V_h^1(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^1. For this, we use (4.1) of Proposition 4.2.4 and (4.9) of Proposition 4.2.5, keeping in mind that $p \neq 2$, to determine that the eigenvalues of h on V^1, not necessarily distinct, are:

$$\begin{cases} d^2, d^{-2}, e^2 \text{ and } e^{-2}, \text{ each with multiplicity at least 1;} \\ de, d^{-1}e^{-1}, d^{-1}e \text{ and } de^{-1} \text{ each with multiplicity at least 2;} \\ 1 \text{ with multiplicity at least 3;} \end{cases}$$

where $d, e \in k^*$ not both simultaneously equal to 1. Thus, one can show that $\dim(V_h^1(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^1. Now, by Proposition 4.2.4, we have $\dim(V_h^0(\mu_h)) \leq 6$ and $\dim(V_h^2(\mu_h)) \leq 6$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. It follows that $\dim(V_s(\mu)) \leq 20$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 20$ for all $s \in \tilde{T}\setminus\mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained. Moreover, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

Proposition 5.2.11. *Let k be an algebraically closed field of characteristic $p = 7$. Assume $\ell = 3$ and let $V = L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_3)$. Then for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 20,$$

where there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\tilde{\lambda} = \tilde{\omega}_1 + \tilde{\omega}_3$. Then $\dim(V) = 40$, as $p = 7$, and, by Lemma 2.4.6, we have $e_1(\tilde{\lambda}) = 3$, therefore

$$V\mid_{[L_1,L_1]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma}}$ for all $0 \leq i \leq 3$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\tilde{\omega}_3)$ and so, by Lemma 2.4.3, we also have $V^3 \cong (L_{L_1}(\tilde{\omega}_3))^* \cong L_{L_1}(\tilde{\omega}_3)$. This gives $\dim(V^1) = 16$, since, by Lemma 2.4.3, we have $V^2 \cong (V^1)^*$. Now, in V^1 the weight $(\tilde{\lambda} - \tilde{\alpha}_1)\mid_{T'} = \tilde{\omega}_2 + \tilde{\omega}_3$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$ and, as $\dim(L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)) = 16$, since $p = 7$, we deduce that $V^1 \cong L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$. Lastly, we use Lemma 2.4.3 once more to determine that $V^2 \cong (L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3))^* \cong L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$, and so

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_3). \tag{5.24}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1, 2$ or $i = 3$, then $s \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$, and so $s = z$ with $c^2 \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 3$, as scalar multiplication by c^{3-2i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 4; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 16; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 16; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^3) = 4. \end{cases}$$

As $c^2 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 20$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(c) \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$ with $c^2 = -1$ we have $\dim(V_s(\pm c)) = 20$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 3$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 3$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 3$, where μ_h is any eigenvalue of h on V^i. Now, by Lemma 4.2.3, it follows that $\dim(V_h^0(\mu_h)) \leq 2$ and $\dim(V_h^3(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^0 and V^3, respectively. By Proposition 4.2.9, we have $\dim(V_h^1(\mu_h)) \leq 8$ and $\dim(V_h^2(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^1 and V^2, respectively. It follows that $\dim(V_s(\mu)) \leq 20$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 20$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained. Moreover, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

Proposition 5.2.12. *Let k be an algebraically closed field of characteristic $p \neq 2, 7$. Assume $\ell = 3$ and let $V = L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_3)$. Then for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 24,$$

where there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $\tilde{\lambda} = \tilde{\omega}_1 + \tilde{\omega}_3$. Then $\dim(V) = 48$, as $p \neq 7$, and, by Lemma 2.4.6, we have $e_1(\tilde{\lambda}) = 3$, therefore

$$V \mid_{[L_1, L_1]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma}}$ for all $0 \leq i \leq 3$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\tilde{\omega}_3)$ and so, by Lemma 2.4.3, we also have $V^3 \cong (L_{L_1}(\tilde{\omega}_3))^* \cong L_{L_1}(\tilde{\omega}_3)$. This gives $\dim(V^1) = 20$, since $V^2 \cong (V^1)^*$, by Lemma 2.4.3. Now, in V^1 the weight $(\tilde{\lambda} - \tilde{\alpha}_1) \mid_{T'} = \tilde{\omega}_2 + \tilde{\omega}_3$ admits a maximal vector, thus V^1 has a composition factor isomorphic to $L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$. We also note that the dominant weight $(\tilde{\lambda} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3) \mid_{T'} = \tilde{\omega}_3$ occurs with multiplicity 3 in V^1. Now, if $p \neq 5$, then $\tilde{\omega}_3$ is a sub-dominant weight in the composition factor of V^1 isomorphic to $L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$, in which it has multiplicity 2. Thereby, when $p \neq 5$, V^1, hence V^2 by Lemma 2.4.3, has exactly two compositions factors: one isomorphic to $L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$ and one isomorphic to $L_{L_1}(\tilde{\omega}_3)$. On the other hand, if $p = 5$, then $\tilde{\omega}_3$ has multiplicity 1 in the composition factor of V^1 isomorphic to $L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$. In this case, we determine that V^1, hence V^2 by Lemma 2.4.3, has exactly three compositions factors: one isomorphic to $L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3)$ and two isomorphic to $L_{L_1}(\tilde{\omega}_3)$.

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1, 2$ or $i = 3$, then $s \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$, and so $s = z$, with $c^2 \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 3$, as scalar multiplication by c^{3-2i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 4; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 20; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 20; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^3) = 4. \end{cases}$$

As $c^2 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 24$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(c) \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$ with $c^2 = -1$ we have $\dim(V_s(\pm c)) = 24$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 3$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. Since z acts by scalar multiplication on V^i, $0 \leq i \leq 3$, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 3$, where μ_h is any eigenvalue of h on V^i. Now, by Lemma 4.2.3, we have $\dim(V_h^0(\mu_h)) \leq 2$ and $\dim(V_h^3(\mu_h)) \leq 2$ for all eigenvalues μ_h of h on V^0 and V^3, respectively. By Lemma 4.2.3 and, if $p = 5$ by Proposition 4.2.8, or, if $p \neq 5$ by Proposition 4.2.9, we have $\dim(V_h^1(\mu_h)) \leq 10$ and $\dim(V_h^2(\mu_h)) \leq 10$ for all eigenvalues μ_h of h on V^1 and V^2, respectively. We deduce that $\dim(V_s(\mu)) \leq 24$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 24$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V, and that there exist pairs $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which the bound is attained. Moreover, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

We conclude this subsection by noting that Propositions 5.2.8 through 5.2.12 complete the proofs of Theorems 5.2.1 and 5.2.2, as they cover all the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with p-restricted dominant weight $\tilde{\lambda}$ featured in Table 2.7.4.

5.3 Eigenspace dimensions for unipotent elements

This section is dedicated to the proofs of the following two theorems, analogs of Theorems 5.2.1 and 5.2.2, in the case of the unipotent elements. Similar to the semisimple case, the proofs will be given in a series of results, each treating one of the candidate.modules. In Subsection 5.3.1, we determine $\max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, see Remark 5.1.1, where V belongs to one of the families of kG-modules we have to consider, i.e. V is an irreducible kG-module $L_G(\lambda)$ with $\lambda \in \{\omega_1, 2\omega_1, \omega_2\}$. We complete the proofs of these two results in Subsection 5.3.2, where we establish $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ for the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with p-restricted dominant weight $\tilde{\lambda}$ listed in Table 2.7.4.

Theorem 5.3.1. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type B_ℓ, $\ell \geq 3$. Let $\tilde{T}$ be a fixed maximal torus in $\tilde{G}$ and let $V = L_{\tilde{G}}(\tilde{\lambda})$, where either $\tilde{\lambda} \in F^{B_\ell}$, or $\tilde{\lambda}$ is featured in Table 2.7.4. Then there exist unipotent elements $\tilde{u} \in \tilde{G}$, $\tilde{u} \neq 1$, for which*

$$\dim(V_{\tilde{u}}(1)) \geq \dim(V) - \sqrt{\dim(V)}$$

if and only if ℓ and $\tilde{\lambda}$ appear in the following list:

(1) *$\ell \geq 3$ and $\tilde{\lambda} = \tilde{\omega}_1$;*

(2) *$\ell = 3, 4$ and $\tilde{\lambda} = \tilde{\omega}_\ell$.*

Theorem 5.3.2. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type B_ℓ, $\ell \geq 3$. Let $\tilde{T}$ and V be as in Theorem 5.3.1. Then the value of $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ is given in the table below:*

V	Char.	Rank	$\max\limits_{u\in\tilde{G}_u\setminus\{1\}} \dim(V_u(1))$
$L_{\tilde{G}}(\tilde{\omega}_1)$	$p\neq 2$	$\ell\geq 3$	$2\ell-1$
${}^{\dagger}L_{\tilde{G}}(\tilde{\omega}_2)$	$p\neq 2$	$\ell\geq 3$	$2\ell^2-3\ell+4$
${}^{\dagger}L_{\tilde{G}}(2\tilde{\omega}_1)$	$p\neq 2$ and $p\nmid 2\ell+1$	$\ell\geq 3$	$2\ell^2-\ell$
	$p\neq 2$ and $p\mid 2\ell+1$	$\ell\geq 3$	$2\ell^2-\ell-1$
${}^{\dagger}L_{\tilde{G}}(2\tilde{\omega}_3)$	$p\neq 2$	$\ell=3$	21
${}^{\dagger}L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_3)$	$p\neq 2$	$\ell=3$	$28-6\delta_{p,7}$
${}^{\ddagger}L_{\tilde{G}}(\tilde{\omega}_\ell)$	$p\neq 2$	$3\leq\ell\leq 8$	$3\cdot 2^{\ell-2}$

Table 5.3.1: The value of $\max\limits_{u\in\tilde{G}_u\setminus\{1\}} \dim(V_u(1))$.

In particular, for each V in Table 5.3.1 labeled as ${}^{\dagger}V$, respectively as ${}^{\ddagger}V$ with $\ell\geq 5$, we have $\dim(V_u(1))<\dim(V)-\sqrt{\dim(V)}$ for all unipotent elements u, $u\neq 1$, of $\tilde{G}$

5.3.1 The families of modules

For the rest of this chapter, we fix the following hypothesis on unipotent elements in G:

$({}^{\dagger}H_u)$: every $u\in G_u\setminus\{1\}$ has Jordan normal form on W given by $\bigoplus\limits_{i=1}^{m} J_{n_i}^{r_i}$, where

$$\sum_{i=1}^{m} n_i r_i = 2\ell+1,\ r_i\geq 1 \text{ is even for all even } n_i \text{ and } 2\ell+1\geq n_1>n_2>\cdots>n_m\geq 1.$$

Moreover, if $m=1$, then $n_1\geq 3$.

Lemma 5.3.3. *Let $V=L_G(\omega_1)$. Then for all non-identity unipotent elements $u\in G$ we have*

$$\dim(V_u(1))\leq 2\ell-1,$$

where we have equality if and only if the Jordan form of u on W is one of $J_3\oplus J_1^{2\ell-2}$ and $J_2^2\oplus J_1^{2\ell-3}$.

In particular, there exist non-identity unipotent elements $u\in G$ for which $\dim(V_u(1))\geq \dim(V)-\sqrt{\dim(V)}$.

Proof. To begin, we note that $V\cong W$ as kG-modules. Now, let u be a unipotent element of G as in hypothesis $({}^{\dagger}H_u)$. Let u_W denote the action of u on W. Then, keeping in mind that $\sum\limits_{i=1}^{m} n_i r_i = 2\ell+1$, we have:

$$\dim(V_u(1))=\dim(W_{u_W}(1))=\sum_{i=1}^{m} r_i = 2\ell+1-\sum_{i=1}^{m}(n_i-1)r_i. \tag{5.25}$$

Assume that $\dim(V_u(1))\geq 2\ell-1$. Then, by (5.25), it follows that

$$2\geq \sum_{i=1}^{m}(n_i-1)r_i \tag{5.26}$$

and, in particular, that $3 \geq n_1$, as $2 \geq (n_1 - 1)r_1 \geq n_1 - 1$. Now, if $n_1 = 3$, then, by (5.26), it follows that $r_1 = 1$ and $\sum_{i=2}^{m}(n_i - 1)r_i = 0$, hence $m = 2$, $n_2 = 1$ and $r_2 = 2\ell - 2$, as $\ell \geq 3$. Thus, u has Jordan form $J_3 \oplus J_1^{2\ell-2}$ on W, and, by (5.25), $\dim(V_u(1)) = 2\ell - 1$. Similarly, if $n_1 = 2$, then r_1 is even and, by (5.26), it follows that $r_1 = 2$ and $\sum_{i=2}^{m}(n_i - 1)r_i = 0$. We argue as before to deduce that the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-3}$ and, by (5.25), that $\dim(V_u(1)) = 2\ell - 1$.

Having treated all possible cases, we conclude that $\dim(V_u(1)) \leq 2\ell - 1$ for all non-identity unipotent elements $u \in G$. Moreover, we have shown that equality holds if and only if the Jordan form of u on W is one of $J_2^2 \oplus J_1^{2\ell-3}$ and $J_3 \oplus J_1^{2\ell-2}$. Then, as the inequality $\sqrt{2\ell+1} \geq 2$ holds for all $\ell \geq 3$, we have shown that there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. □

Proposition 5.3.4. *Let $V = L_G(\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - 3\ell + 4,$$

where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-3}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. First, we note that, by Lemma 2.8.4, we have $V \cong \wedge^2(W)$, as kG-modules. Now, let u be a unipotent element of G as in hypothesis $({}^{\dagger}H_u)$. We first consider the case when u has Jordan form $J_{2\ell+1}$ on W. Then u acts as a single Jordan block on W and so, by Lemma 2.9.4, we have:

$$\dim(V_u(1)) = \left\lfloor \frac{2\ell+1}{2} \right\rfloor = \ell < 2\ell^2 - 3\ell + 4,$$

since $0 < 2\ell^2 - 4\ell + 4$ for all $\ell \geq 3$. We now assume that the Jordan form of u on W consists of at least two blocks.

Secondly, we consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. Then the Jordan form of u is $J_{n_1} \oplus J_1^{2\ell+1-n_1}$, where, since $r_1 = 1$, n_1 is odd, thus $3 \leq n_1 \leq 2\ell - 1$. We write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = 2\ell + 1 - n_1$ and u acts trivially on W_2. Then, as $k[u]$-modules, we have

$$V \cong \wedge^2(W_1) \oplus (W_1 \otimes W_2) \oplus \wedge^2(W_2)$$

and so

$$\dim(V_u(1)) = \dim((\wedge^2(W_1))_u(1)) + \dim((W_1 \otimes W_2)_u(1)) + \dim((\wedge^2(W_2))_u(1)). \tag{5.27}$$

Now, by Lemma 2.9.4, we have $\dim((\wedge^2(W_1))_u(1)) = \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 - 1}{2}$ and, moreover, as u acts as $J_{n_1} \otimes J_1^{2\ell+1-n_1}$ on $W_1 \otimes W_2$, we also have $\dim((W_1 \otimes W_2)_u(1)) = 2\ell + 1 - n_1$. Lastly,

as u acts trivially on W_2, it also acts trivially on $\wedge^2(W_2)$, and so $\dim((\wedge^2(W_2))_u(1)) = \dfrac{(2\ell - n_1)(2\ell - n_1 + 1)}{2}$. It follows that:

$$\begin{aligned}\dim(V_u(1)) &= \frac{n_1 - 1}{2} + 2\ell + 1 - n_1 + \frac{(2\ell - n_1)(2\ell - n_1 + 1)}{2} \\ &= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 6\ell - 2n_1 + 1}{2} \\ &= 2\ell^2 - 3\ell + 4 + \frac{n_1^2 - 4\ell n_1 - 2n_1 + 12\ell - 7}{2}.\end{aligned}$$

One checks that the inequality

$$n_1^2 - 4\ell n_1 - 2n_1 + 12\ell - 7 < 0 \tag{5.28}$$

holds for all $n_1 \in (2\ell + 1 - 2\sqrt{(\ell-1)^2 + 1},\ 2\ell + 1 + 2\sqrt{(\ell-1)^2 + 1})$ and all $\ell \geq 1$. Since $2\ell+1-2\sqrt{(\ell-1)^2+1} < 2\ell+1-2\sqrt{(\ell-1)^2} = 3$ and since $2\ell+1+2\sqrt{(\ell-1)^2+1} > 2\ell-1$, it follows that, in particular, Inequality (5.28) holds for all $3 \leq n_1 \leq 2\ell - 1$ and all $\ell \geq 3$. We conclude that $\dim(V_u(1)) < 2\ell^2 - 3\ell + 4$ for all $\ell \geq 3$ and all unipotent elements u of G with Jordan form $J_{n_1} \oplus J_1^{2\ell+1-n_1}$ on W, where $3 \leq n_1 \leq 2\ell - 1$.

Lastly, we consider the case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell - 1$. We write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = n_1$ and u acts as J_{n_1} on $W_1^{'}$, and $\dim(W_2^{'}) = 2\ell+1-n_1$ and u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$. Now, by (5.27), in order to determine $\dim(V_u(1))$, we only need to know $\dim((\wedge^2(W_1^{'}))_u(1))$, $\dim((W_1^{'} \otimes W_2^{'})_u(1))$ and $\dim((\wedge^2(W_2^{'}))_u(1))$. As u acts as a single Jordan block on $W_1^{'}$, by Lemma 2.9.4, we have $\dim((\wedge^2(W_1^{'}))_u(1)) = \left\lfloor \dfrac{n_1}{2} \right\rfloor = \dfrac{n_1 + \epsilon}{2}$, where $\epsilon = 0$ if n_1 is even, or $\epsilon = -1$ if n_1 is odd. Now, since u acts as $(J_{n_1} \otimes J_{n_1})^{r_1-1} \oplus \bigoplus_{i=2}^{m} (J_{n_1} \otimes J_{n_i})^{r_i}$ on $W_1^{'} \otimes W_2^{'}$, we again use Lemma 2.9.4 to deduce:

$$\dim((W_1^{'} \otimes W_2^{'})_u(1)) = (r_1 - 1)n_1 + \sum_{i=2}^{m} n_i r_i = 2\ell + 1 - n_1. \tag{5.29}$$

Furthermore, since the Jordan form of u admits at least two nontrivial blocks, it follows that u acts nontrivially on $W_2^{'}$ and by, Proposition 3.3.4, we have $\dim((\wedge^2(W_2^{'}))_u(1)) \leq \dfrac{(2\ell - n_1)^2 - (2\ell - n_1) + 2}{2}$. Moreover, we note that equality holds if and only if u acts on $W_2^{'}$ as $J_2 \oplus J_1^{2\ell-n_1-1}$. Now, by (5.27) and keeping in mind that $\epsilon \leq 0$, we have:

$$\begin{aligned}\dim(V_u(1)) &\leq \frac{n_1+\epsilon}{2} + 2\ell + 1 - n_1 + \frac{(2\ell-n_1)^2-(2\ell-n_1)+2}{2}\\ &= \frac{4\ell^2-4\ell n_1+n_1^2+2\ell+4+\epsilon}{2}\\ &\leq \frac{4\ell^2-4\ell n_1+n_1^2+2\ell+4}{2}\\ &= 2\ell^2-3\ell+4+\frac{n_1^2-4\ell n_1+8\ell-4}{2}\\ &= 2\ell^2-3\ell+4+\frac{(n_1-2)(n_1+2-4\ell)}{2}.\end{aligned}$$

Since $2 \leq n_1 \leq 2\ell - 1$, it follows that $(n_1-2)(n_1+2-4\ell) \leq 0$ for all $\ell \geq 3$, thus $\dim(V_u(1)) \leq 2\ell^2 - 3\ell + 4$ for all $\ell \geq 3$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks. Moreover, equality holds if and only if $\dim((\wedge^2(W_2^{'}))_u(1)) = \dfrac{(2\ell-n_1)^2-(2\ell-n_1)+2}{2}$, n_1 is even and $(n_1-2)(n_1+2-4\ell) = 0$, hence, if and only if u acts as $J_2 \oplus J_1^{2\ell-n_1-1}$ on $W_2^{'}$ and as J_2 on $W_1^{'}$. We deduce that u has Jordan form $J_2^2 \oplus J_1^{2\ell-3}$ on W.

Having considered all possible cases, we conclude that $\dim(V_u(1)) \leq 2\ell^2 - 3\ell + 4$ for all non-identity unipotent elements $u \in G$, where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-3}$. Furthermore, since the inequality $14\ell^2 - 33\ell + 16 > 0$ holds for all $\ell \geq 3$, it follows that $2\ell^2 - 3\ell + 4 < 2\ell^2 + \ell - \sqrt{2\ell^2+\ell}$ for all $\ell \geq 3$, thus $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

We will now consider the irreducible kG-module $L_G(2\omega_1)$. We have seen in Lemma 2.8.4 that $L_G(2\omega_1)$ is a composition factor of the kG-module $\mathrm{S}^2(W)$. This is a relevant fact, as we can use Lemma 2.9.4 to calculate the dimension of the fixed point space on $\mathrm{S}^2(W)$ of any unipotent element $u \in G$, see Proposition 5.3.7. Having determined $\dim((\mathrm{S}^2(W))_u(1))$, where $u \in G$ is unipotent, we apply [Kor19, Corollary 6.3] to deduce $\dim(L_G(2\omega_1)_u(1))$. Before we give the statement of this result, we recall that $r_t(u)$ is the number of Jordan blocks of size $t \geq 1$ appearing in the Jordan form of u, and that ν_p is the p-adic valuation on the integers.

Theorem 5.3.5. [Kor19, Corollary 6.3] *Assume $p > 2$ and let $H = \mathrm{SO}(V)$, where $\dim(V) = n$ for some $n \geq 5$. Let $u \in H$ be a unipotent element and let $V\,|_{k[u]} = V_{n_1} \oplus \cdots \oplus V_{n_m}$, where $m \geq 1$ and $n_i \geq 1$ for all $1 \leq i \leq m$. Set $\alpha = \nu_p(\gcd(n_1, \ldots, n_m))$. Let $u^{'}$ be the action of u on $\mathrm{S}^2(V)$ and let u_0 be the action of u on $L_H(2\omega_1)$. Then the Jordan block sizes of u_0 are determined from those of $u^{'}$ in the following way:*

(a) *If $p \nmid n$, then $r_1(u_0) = r_1(u^{'}) - 1$ and $r_t(u_0) = r_t(u^{'})$ for all $t \neq 1$.*

(b) *If $p \mid n$ and $\alpha = 0$, then $r_1(u_0) = r_1(u^{'}) - 2$ and $r_t(u_0) = r_t(u^{'})$ for all $t \neq 1$.*

(c) *If $p \mid n$ and $\alpha > 0$, then:*

(c.1) *If* $p \mid \dfrac{n}{p^\alpha}$, *then* $r_{p^\alpha}(u_0) = r_{p^\alpha}(u') - 2$, $r_{p^\alpha - 1}(u_0) = 2$ *and* $r_t(u_0) = r_t(u')$ *for all* $t \neq p^\alpha, p^\alpha - 1$.

(c.2) *If* $p \nmid \dfrac{n}{p^\alpha}$, *then* $r_{p^\alpha}(u_0) = r_{p^\alpha}(u') - 1$, $r_{p^\alpha - 2}(u_0) = 1$ *and* $r_t(u_0) = r_t(u')$ *for all* $t \neq p^\alpha, p^\alpha - 2$.

Remark 5.3.6. *Let* $V = L_G(2\omega_1)$. *By Theorem* 5.3.5, *for all unipotent elements* $u \in G$, *we have that:*

(1) *If* $p \nmid 2\ell + 1$, *then* $\dim(V_u(1)) = \dim((\mathrm{S}^2(W))_u(1)) - 1$.

(2) *If* $p \mid 2\ell + 1$ *and* $\alpha = 0$, *then* $\dim(V_u(1)) = \dim((\mathrm{S}^2(W))_u(1)) - 2$.

(3) *If* $p \mid 2\ell + 1$ *and* $\alpha > 0$, *then* $\dim(V_u(1)) = \dim((\mathrm{S}^2(W))_u(1))$.

Proposition 5.3.7. *Let* $V' = \mathrm{S}^2(W)$. *Then for all non-identity unipotent elements* $u \in G$ *we have*

$$\dim(V_u'(1)) \leq 2\ell^2 - \ell + 1.$$

Moreover, equality holds if and only if the Jordan form of u *on* W *is one of* $J_3 \oplus J_1^{2\ell-2}$ *and* $J_2^2 \oplus J_1^{2\ell-3}$.

Proof. Let u be a unipotent element of G as in hypothesis ($^\dagger H_u$). We first consider the case when u has Jordan form $J_{2\ell+1}$ on W. Then, as $p \neq 2$, we apply Lemma 2.9.4 and obtain:

$$\dim(V_u'(1)) = 2\ell + 1 - \left\lfloor \frac{2\ell+1}{2} \right\rfloor = \ell + 1 < 2\ell^2 - \ell + 1, \tag{5.30}$$

since the inequality $0 < 2\ell^2 - 2\ell$ holds for all $\ell \geq 3$. We can thus assume that the Jordan form of u on W consists of at least two blocks.

We now consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. Then u has Jordan form $J_{n_1} \oplus J_1^{2\ell+1-n_1}$, where n_1 is odd, since $r_1 = 1$, thus $3 \leq n_1 \leq 2\ell - 1$. We write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = 2\ell + 1 - n_1$ and u acts trivially on W_2. Then, as $k[u]$-modules, we have

$$V' \cong \mathrm{S}^2(W_1) \oplus (W_1 \otimes W_2) \oplus \mathrm{S}^2(W_2)$$

and so

$$\dim(V_u'(1)) = \dim((\mathrm{S}^2(W_1))_u(1)) + \dim((W_1 \otimes W_2)_u(1)) + \dim((\mathrm{S}^2(W_2))_u(1)). \tag{5.31}$$

Now, since $p \neq 2$, we apply Lemma 2.9.4, which gives $\dim((\mathrm{S}^2(W_1))_u(1)) = n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1+1}{2}$ and, moreover, as u acts as $J_{n_1} \otimes J_1^{2\ell+1-n_1}$ on $W_1 \otimes W_2$, we also have $\dim((W_1 \otimes W_2)_u(1)) = 2\ell + 1 - n_1$. Lastly, as u acts trivially on W_2, it also acts trivially on

$\mathrm{S}^2(W_2)$, and so $\dim((\mathrm{S}^2(W_2))_u(1)) = \dfrac{(2\ell - n_1 + 1)(2\ell - n_1 + 2)}{2}$. It follows that:

$$\begin{aligned}
\dim(V_u^{'}(1)) &= \frac{n_1+1}{2} + 2\ell + 1 - n_1 + \frac{(2\ell - n_1 + 1)(2\ell - n_1 + 2)}{2} \\
&= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 10\ell - 4n_1 + 5}{2} \\
&= 2\ell^2 - \ell + 1 + \frac{n_1^2 - 4\ell n_1 - 4n_1 + 12\ell + 3}{2} \\
&= 2\ell^2 - \ell + 1 + \frac{(n_1 - 3)(n_1 - 1 - 4\ell)}{2}.
\end{aligned} \tag{5.32}$$

Since $3 \leq n_1 \leq 2\ell - 1$, we have $(n_1 - 3)(n_1 - 1 - 4\ell) \leq 0$ for all $\ell \geq 3$, and, consequently, $\dim(V_u^{'}(1)) \leq 2\ell^2 - \ell + 1$ for all unipotent elements u of G with Jordan form $J_{n_1} \oplus J_1^{2\ell+1-n_1}$, where $3 \leq n_1 \leq 2\ell - 1$, on W. Moreover, equality holds if and only if $(n_1 - 3)(n_1 - 1 - 4\ell) = 0$, hence, if and only if the Jordan form of u on W is $J_3 \oplus J_1^{2\ell-2}$.

Lastly, we consider the case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell - 1$. We write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = n_1$ and u acts as J_{n_1} on $W_1^{'}$, and $\dim(W_2^{'}) = 2\ell + 1 - n_1$ and u acts as $J_{n_1}^{r_1 - 1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$. Now, by (5.31), in order to determine $\dim(V_u^{'}(1))$, we only need to know $\dim((\mathrm{S}^2(W_1^{'}))_u(1))$, $\dim((W_1^{'} \otimes W_2^{'})_u(1))$ and $\dim((\mathrm{S}^2(W_2^{'}))_u(1))$. As u acts as a single Jordan block on $W_1^{'}$, by Lemma 2.9.4, we have $\dim((\mathrm{S}^2(W_1^{'}))_u(1)) = n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 - \epsilon}{2}$, where $\epsilon = 0$ if n_1 is even, or $\epsilon = -1$ if n_1 is odd. Since u acts as $(J_{n_1} \otimes J_{n_1})^{r_1 - 1} \oplus \bigoplus_{i=2}^{m} (J_{n_1} \otimes J_{n_i})^{r_i}$ on $W_1^{'} \otimes W_2^{'}$, by (5.29), we have $\dim((W_1^{'} \otimes W_2^{'})_u(1)) = 2\ell + 1 - n_1$. Furthermore, since the Jordan form of u on W admits at least two nontrivial blocks, it follows that u acts nontrivially on $W_2^{'}$ and, by Proposition 3.3.5, we have $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) \leq \dfrac{(2\ell - n_1)(2\ell + 1 - n_1)}{2}$. Moreover, by the same result, equality holds if and only if u acts on $W_2^{'}$ as $J_2 \oplus J_1^{2\ell - n_1 - 1}$. Thus, by (5.31) we have:

$$\begin{aligned}
\dim(V_u^{'}(1)) &\leq \frac{n_1 - \epsilon}{2} + 2\ell + 1 - n_1 + \frac{(2\ell - n_1)(2\ell + 1 - n_1)}{2} \\
&= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 6\ell - 2n_1 + 2 - \epsilon}{2} \\
&= 2\ell^2 - \ell + 1 + \frac{n_1^2 - 4\ell n_1 - 2n_1 + 8\ell - \epsilon}{2}.
\end{aligned}$$

If $n_1 = 2$, then $\epsilon = 0$, $\dfrac{n_1^2 - 4\ell n_1 - 2n_1 + 8\ell}{2} = 0$ and, consequently, $\dim(V_u^{'}(1)) \leq 2\ell^2 - \ell + 1$. We have remarked earlier that equality holds if and only if u acts on $W_2^{'}$ as $J_2 \oplus J_1^{2\ell - n_1 - 1}$, hence, equality holds if and only if u has Jordan form $J_2^2 \oplus J_1^{2\ell - 3}$ on W.

We can now assume that $n_1 \geq 3$. One checks that the inequality

$$n_1^2 - 4\ell n_1 - 2n_1 + 8\ell - \epsilon < 0 \tag{5.33}$$

holds for all $n_1 \in (2\ell+1-\sqrt{(2\ell-1)^2+\epsilon},\ 2\ell+1+\sqrt{(2\ell-1)^2+\epsilon})$ and all $\ell \geq 1$. Since $2\ell+1-\sqrt{(2\ell-1)^2+\epsilon} < 3$, as $3+\epsilon < 4\ell$ for all $\ell \geq 3$, and $2\ell+1+\sqrt{(2\ell-1)^2+\epsilon} > 2\ell-1$, it follows that, in particular, Inequality (5.33) holds for all $3 \leq n_1 \leq 2\ell-1$ and all $\ell \geq 3$. Therefore $\dim(V_u^{'}(1)) < 2\ell^2-\ell+1$ for all $\ell \geq 3$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks and $n_1 \geq 3$.

Having considered all possible cases, we conclude that $\dim(V_u^{'}(1)) \leq 2\ell^2-\ell+1$ for all non-identity unipotent elements $u \in G$. Moreover, we have shown that equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$. □

The following result will be required for the proof of Proposition 5.3.10.

Proposition 5.3.8. *Let u be a non-identity unipotent element of G whose Jordan form on W is different than $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$. Let $V^{'} = \mathrm{S}^2(W)$, then:*

$$\dim(V_u^{'}(1)) < 2\ell^2-\ell-1.$$

Proof. Let u be a unipotent element of G as in hypothesis $({}^{\dagger}H_u)$ and assume that its Jordan form on W is different than $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$. We first consider the case when u has Jordan form $J_{2\ell+1}$. Then by (5.30), we have $\dim(V_u^{'}(1)) = \ell+1$ and therefore $\dim(V_u^{'}(1)) < 2\ell^2-\ell-1$, since $0 < \ell^2-\ell-1$ for all $\ell \geq 3$. We thus assume that the Jordan form of u on W consists of at least two blocks.

We now consider the case when exactly one block, J_{n_1}, appearing in the Jordan form of u on W, is nontrivial. We remark that since $r_1 = 1$, n_1 is odd and, since the Jordan form of u is different than $J_3 \oplus J_1^{2l-2}$, we have $5 \leq n_1 \leq 2\ell-1$. Furthermore, arguing as in the proof of Proposition 5.3.7, we see that (5.32) applies and we have

$$\dim(V_u^{'}(1)) = \frac{4\ell^2-4\ell n_1+n_1^2+10\ell-4n_1+5}{2} = 2\ell^2-\ell-1+\frac{n_1^2-4\ell n_1-4n_1+12\ell+7}{2}.$$

One checks that the inequality

$$n_1^2-4\ell n_1-4n_1+12\ell+7 < 0$$

holds for all $n_1 \in (2\ell+2-\sqrt{4\ell^2-4\ell-3},\ 2\ell+2+\sqrt{4\ell^2-4\ell-3})$ and all $\ell \geq 1$. Since $2\ell+2+\sqrt{4\ell^2-4\ell-3} > 2\ell-1$ and $2\ell+2-\sqrt{4\ell^2-4\ell-3} < 5$, as $3 < 4\ell$ for all $\ell \geq 3$, it follows that, in particular, the inequality holds for all $5 \leq n_1 \leq 2\ell-1$ and all $\ell \geq 3$. We conclude that $\dim(V_u^{'}(1)) < 2\ell^2-\ell-1$ for all unipotent elements u of G with Jordan form $J_{n_1} \oplus J_1^{2\ell+1-n_1}$ on W, where $5 \leq n_1 \leq 2\ell-1$.

Lastly, we consider the case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell-1$ and we distinguish two cases. First, if $n_1 = 2$, then the Jordan form of u on W is $J_2^{r_1} \oplus J_1^{2l+1-2r_1}$, where r_1 is even and so, by hypothesis, $r_1 \geq 4$. We remark that, in this case, we have $\ell \geq 4$. We write $W = W_1 \oplus W_2$, where

$$\dim(W_1) = 4,\ \dim(W_2) = 2\ell-3 \text{ and } u \text{ acts as } J_2^2 \text{ on } W_1 \text{ and as } J_2^{r_1-2} \oplus J_1^{2\ell+1-2r_1} \text{on } W_2.$$

Now, by Proposition 3.3.5, we have $\dim((\mathrm{S}^2(W_1))_u(1)) < 6$ and $\dim((\mathrm{S}^2(W_2))_u(1)) \leq \frac{(2\ell-4)(2\ell-3)}{2}$ $= 2\ell^2-7\ell+6$. Using Lemma 2.9.4 we determine that $\dim((W_1 \otimes W_2)_u(1)) = 2(2r_1-4)+4\ell+2-4r_1 = 4\ell-6$ and so, by (5.31), we have $\dim(V_u^{'}(1)) < 2\ell^2-3\ell+6$. Therefore,

since $\ell \geq 4$, we determine that $\dim(V_u^{'}(1)) < 2\ell^2 - \ell - 1$ for all unipotent elements u of G whose Jordan form on W is $J_2^{r_1} \oplus J_1^{2\ell+1-2r_1}$, where $r_1 \geq 4$. We now consider the case when $n_1 \geq 3$. We proceed as in the proof of Proposition 5.3.7, see the third-to-last paragraph, and write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = n_1$ and u acts as J_{n_1} on $W_1^{'}$, and $\dim(W_2^{'}) = 2\ell+1-n_1$ and u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$. Then $\dim((\mathrm{S}^2(W_1^{'}))_u(1)) = \dfrac{n_1-\epsilon}{2}$ and $\dim((W_1^{'} \otimes W_2^{'})_u(1)) = 2\ell+1-n_1$. Moreover, we remark that, since $n_1 \geq 3$ and r_i is even for even n_i, it follows that u does not act on $W_2^{'}$ as $J_2 \oplus J_1^{2\ell-3}$ and so, by Proposition 3.3.5, we have $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) < \dfrac{(2\ell-n_1)(2\ell+1-n_1)}{2}$. Therefore, by (5.31), we have

$$\begin{aligned}\dim(V_u^{'}(1)) &< \frac{n_1-\epsilon}{2} + 2\ell+1-n_1 + \frac{(2\ell-n_1)(2\ell+1-n_1)}{2}\\ &= \frac{4\ell^2-4\ell n_1+n_1^2+6\ell-2n_1+2-\epsilon}{2}\\ &= 2\ell^2-\ell-1+\frac{n_1^2-4\ell n_1-2n_1+8\ell+4-\epsilon}{2}.\end{aligned}$$

One checks that the inequality

$$n_1^2 - 4\ell n_1 - 2n_1 + 8\ell + 4 - \epsilon < 0 \tag{5.34}$$

holds for all $n_1 \in (2\ell+1-\sqrt{4\ell^2-4\ell-3+\epsilon},\ 2\ell+1+\sqrt{4\ell^2-4\ell-3+\epsilon})$ and all $\ell \geq 1$. Since $2\ell+1+\sqrt{4\ell^2-4\ell-3+\epsilon} > 2\ell+1$ and since $2\ell+1-\sqrt{4\ell^2-4\ell-3+\epsilon} < 3$, as $7-\epsilon < 4\ell$ for all $\ell \geq 3$, it follows that, in particular, Inequality (5.34) holds for all $3 \leq n_1 \leq 2\ell+1$ and all $\ell \geq 3$. Therefore $\dim(V_u^{'}(1)) < 2\ell^2-\ell-1$ for all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks and $n_1 \geq 3$.

Having considered all possible cases, we conclude that $\dim(V_u^{'}(1)) < 2\ell^2-\ell-1$ for all non-identity unipotent elements $u \in G$ whose Jordan form on W is different than $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$. □

Corollary 5.3.9. *Assume $p \nmid 2\ell+1$ and let $V = L_G(2\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - \ell,$$

where equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, set $V^{'} = \mathrm{S}^2(W)$. Then, as $p \neq 2$ and $p \nmid 2\ell+1$, by Lemma 2.8.4, we have the following kG-module isomorphism $V^{'} \cong V \oplus L_G(0)$. It follows that $\dim(V_u(1)) = \dim(V_u^{'}(1)) - 1$ and, consequently, by Proposition 5.3.7, $\dim(V_u(1)) \leq 2\ell^2 - \ell$ for all non-identity unipotent elements $u \in G$. Moreover, we have equality if and only if $\dim(V_u^{'}(1)) = 2\ell^2-\ell+1$, hence, by Proposition 5.3.7, if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$

In conclusion, we proved that $\dim(V_u(1)) \leq 2\ell^2 - \ell$ for all non-identity unipotent elements $u \in G$ and that there exist $u \in G$ for which the bound is attained. Lastly, we note that, as the inequality $14\ell^2 - 3\ell > 0$ holds for all $\ell \geq 3$, it follows that $2\ell^2 - \ell < 2\ell^2 + 3\ell - \sqrt{2\ell^2 + 3\ell}$ for all $\ell \geq 3$ and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 5.3.10. *Assume $p \mid 2\ell + 1$ and let $V = L_G(2\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - \ell - 1,$$

where equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, set $V' = \mathrm{S}^2(W)$ and let u be a unipotent element of G as in hypothesis $({}^{\dagger}H_u)$. If we denote by u', respectively by u_V, the action of u on V', respectively on V, then by Theorem 5.3.5 we know that we can determine the Jordan form of u_V from that of u'. Moreover, by Remark 5.3.6, we also know that $\dim(V_u(1)) \leq \dim(V_u'(1))$.

Set $\alpha = \nu_p(\gcd(n_1, \ldots, n_m))$. If $\alpha = 0$, then, since $p \mid 2\ell + 1$, we have $\dim(V_u(1)) = \dim(V_u'(1)) - 2$, see item (2) of Remark 5.3.6. Therefore, by Proposition 5.3.7, it follows that $\dim(V_u(1)) \leq 2\ell^2 - \ell - 1$, where equality holds if and only if $\dim(V_u'(1)) = 2\ell^2 - \ell + 1$, hence, if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$.

We can now assume that $\alpha > 0$. Then, by item (3) of Remark 5.3.6, as $p \mid 2\ell + 1$, we have $\dim(V_u(1)) = \dim(V_u'(1))$. Moreover, we note that since $\alpha > 0$, the Jordan form of u on W is different than $J_3 \oplus J_1^{2\ell-2}$ and $J_2^2 \oplus J_1^{2\ell-3}$. Therefore, we use Proposition 5.3.8 to deduce that $\dim(V_u'(1)) < 2\ell^2 - \ell - 1$ and, consequently, $\dim(V_u(1)) < 2\ell^2 - \ell - 1$ for all unipotent elements $u \in G$ with $\alpha > 0$

We have shown that $\dim(V_u(1)) \leq 2\ell^2 - \ell - 1$ for all non-identity unipotent elements $u \in G$. In particular, since the inequality $0 < 14\ell^2 - 3\ell + 1$ holds for all $\ell \geq 3$, it follows that $2\ell^2 - \ell - 1 < 2\ell^2 + 3\ell - 1 - \sqrt{2\ell^2 + 3\ell - 1}$ for all $\ell \geq 3$ and, consequently, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

To conclude this subsection, we remark that Lemma 5.3.3, Proposition 5.3.4 and Corollaries 5.3.9 and 5.3.10 give the proof of Theorems 5.3.1 and 5.3.2 for the families of kG-modules corresponding to p-restricted dominant weights $\lambda \in \{\omega_1, \omega_2, 2\omega_1\}$. Therefore, in view of Remark 5.1.1, they prove Theorems 5.3.1 and 5.3.2 for the families of $k\tilde{G}$-modules with p-restricted dominant weights $\tilde{\lambda} \in F^{B_\ell}$.

5.3.2 The particular modules

As previously mentioned, this subsection is dedicated to the proofs of Theorems 5.3.1 and 5.3.2 for the particular $k\tilde{G}$-modules, i.e. the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with corresponding p-restricted dominant highest weight $\tilde{\lambda}$ listed in Table 2.7.4. For each such $k\tilde{G}$-module V we will establish $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$, where $\tilde{G}_u$ is the set of unipotent elements of $\tilde{G}$, see

Propositions 5.3.11, 5.3.12, 5.3.13, 5.3.14 and 5.3.19. In order to achieve this goal, we will use the inductive algorithm of Subsection 2.4.4.

Now, let $u \in \tilde{G}$ be a unipotent element. We write $u = \prod_{\tilde{\alpha} \in \tilde{\Phi}^+} x_{\tilde{\alpha}}(c_{\tilde{\alpha}})$, where the product is taken with respect to the total order $\preceq$ on $\tilde{\Phi}$, see Section 1.3, and $c_{\tilde{\alpha}} \in k$ for all $\tilde{\alpha} \in \tilde{\Phi}^+$. To u we associate the subset $S_u \subseteq \tilde{\Phi}^+$ with the property that $u = \prod_{\tilde{\alpha} \in S_u} x_{\tilde{\alpha}}(c_{\tilde{\alpha}})$, where $c_{\tilde{\alpha}} \in k^*$ for all $\tilde{\alpha} \in S_u$. Now, as $p \neq 2$, by Lemma 2.9.1 and Theorem 2.9.2, we know that the unipotent conjugacy class of u in $\tilde{G}$ is completely determined by the Jordan form of u on W.

We first assume that $\ell = 3$. Let L_1 be the Levi subgroup of the maximal parabolic subgroup P_1 of $\tilde{G}$ given in Section 2.4. In Table 5.3.2 we list all the unipotent conjugacy classes in $\tilde{G}$ and give a class representative. Note that all the chosen non-identity class representatives u' have the property that $u'_{L_1} \neq 1$.

[LS12, Subsection 3.3.2], [MKT21, Table 10]

Class representative	Jordan normal form
1	J_1^7
$x_{\tilde{\alpha}_2}(1)$	$J_2^2 \oplus J_1^3$
$x_{\tilde{\alpha}_3}(1)$	$J_3 \oplus J_1^4$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_3}(1)$	$J_3 \oplus J_2^2$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)$	$J_3^2 \oplus J_1$
$x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)$	$J_5 \oplus J_1^2$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)$	J_7

Table 5.3.2: Unipotent class representatives in B_3 when $p \neq 2$.

Let $u \in \tilde{G}$ be a non-identity unipotent element and let u' be the class representative featured in Table 5.3.2 of the unipotent $\tilde{G}$-conjugacy class of u. Then, as $\dim(V_u(1)) = \dim(V_{u'}(1))$ and $\dim(V_{u'}(1)) \leq \dim(V_{u'_{L_1}}(1))$, by Inequality (2.7), it follows that $\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1))$ and, consequently:

$$\dim(V_u(1)) \leq \max\{\dim(V_{v'_{L_1}}(1)) \mid v' \text{ in Table 5.3.2}\} \tag{5.35}$$

for all non-identity unipotent elements $u \in \tilde{G}$. Note that, since all v' of Table 5.3.2 have been chosen such that $v'_{L_1} \neq 1$, we have that the bound in (5.35) is strictly less than $\dim(V)$. Now, let v'_{max} be the representative of Table 5.3.2 with the property that $\dim(V_{(v'_{max})_{L_1}}(1))$ realizes $\max\{\dim(V_{v'_{L_1}}(1)) \mid v' \text{ in Table 5.3.2}\}$. If v'_{max} is such that $(v'_{max})_{Q_1} = 1$, where $Q_1 = R_u(P_1)$, then, by Identity (2.8), we deduce that there exist unipotent elements $u \in \tilde{G}$ for which the bound in (5.35) is attained.

Proposition 5.3.11. *Assume that $\ell = 3$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 6,$$

where there exist $u \in \tilde{G}$ for which the bound is attained.

In particular, there exist non-identity unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. We begin by recalling the Decomposition (5.21) of Proposition 5.2.8, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_3).$$

Let $u \in \tilde{G}$ be a non-identity unipotent element. Then, by Inequality (5.35) and Decomposition (5.21), we have:

$$\dim(V_u(1)) \leq 2 \max_{v'}\{\dim((L_{L_1}(\tilde{\omega}_3))_{v'_{L_1}}(1))\},$$

where the maximum is taken over all non-identity unipotent class representatives v' in Table 5.3.2. Using Lemma 4.3.3, we determine that $\max_{v'} \dim((L_{L_1}(\tilde{\omega}_3))_{v'_{L_1}}(1)) = 3$ and so $\dim(V_u(1)) \leq 6$ for all non-identity unipotent elements $u \in \tilde{G}$.

We will now show that there exist $u \in \tilde{G}$ for which $\dim(V_u(1)) = 6$. For this, consider $x_{\tilde{\alpha}_2}(1) \in \tilde{G}$. First, we note that $(x_{\tilde{\alpha}_2}(1))_{Q_1} = 1$. Secondly, by Lemma 4.3.3 and Table 4.3.2, we have $\dim((L_{L_1}(\tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1)) = 3$ and so $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1)) = 6$.

In conclusion, we showed that $\dim(V_u(1)) \leq 6$ for all non-identity unipotent elements $u \in \tilde{G}$ and that there exist $u \in \tilde{G}$ for which the bound is attained, for example $u = x_{\tilde{\alpha}_2}(1)$. This proves that there exist non-identity unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. □

Proposition 5.3.12. *Assume $\ell = 3$ and let $V = L_{\tilde{G}}(2\tilde{\omega}_3)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 21,$$

where there exist $u \in \tilde{G}$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ *for all non-identity unipotent elements $u \in \tilde{G}$.*

Proof. To begin, we recall the Decomposition (5.23) from Proposition 5.2.10, which states:

$$V\mid_{[L_1,L_1]} = L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_2) \oplus L_{L_1}(2\tilde{\omega}_3).$$

Let $u \in \tilde{G}$ be a non-identity unipotent element. Then, by Inequality (5.35) and Decomposition (5.23), we have:

$$\dim(V_u(1)) \leq 3 \max_{v'}\{\dim((L_{L_1}(2\tilde{\omega}_3))_{v'_{L_1}}(1))\} + \max_{v'}\{\dim((L_{L_1}(\tilde{\omega}_2))_{v'_{L_1}}(1))\},$$

where the maximum is taken over all non-identity unipotent class representatives v' in Table 5.3.2. By Corollary 4.3.8, as $p \neq 2$, we have $\max_{v'} \dim((L_{L_1}(\tilde{\omega}_2))_{v'_{L_1}}(1)) = 3$, while, by Proposition 4.3.12, we have $\max_{v'} \dim((L_{L_1}(2\tilde{\omega}_3))_{v'_{L_1}}(1)) = 6$. It follows that $\dim(V_u(1)) \leq 21$ for all non-identity unipotent elements $u \in \tilde{G}$.

We will now show that there exist $u \in \tilde{G}$ for which $\dim(V_u(1)) = 21$. For this, consider $x_{\tilde{\alpha}_2}(1) \in G$. First, we note that $(x_{\tilde{\alpha}_2}(1))_{Q_1} = 1$. Secondly, by Table 4.3.2,

Corollary 4.3.8 and Proposition 4.3.12, respectively, we have $\dim((L_{L_1}(\tilde{\omega}_2))_{x_{\tilde{\alpha}_2}(1)}(1)) = 3$ and $\dim((L_{L_1}(2\tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1)) = 6$, therefore $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1)) = 21$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 21$ for all non-identity unipotent elements $u \in \tilde{G}$ and that there exist $u \in \tilde{G}$ for which the bound is attained, for example $x_{\tilde{\alpha}_2}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$. □

Proposition 5.3.13. *Let k be an algebraically closed field of characteristic $p = 7$. Assume $\ell = 3$ and let $V = L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_3)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 22,$$

where there exist $u \in \tilde{G}$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$.

Proof. To begin, we recall Decomposition (5.24) of Proposition 5.2.11, which states:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_3).$$

Let $u \in \tilde{G}$ be a non-identity unipotent element. Then, by Inequality (5.35) and Decomposition (5.24), we have:

$$\dim(V_u(1)) \leq 2 \max_{v'}\{\dim((L_{L_1}(\tilde{\omega}_3))_{v'_{L_1}}(1))\} + 2 \max_{v'}\{\dim((L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3))_{v'_{L_1}}(1))\},$$

where the maximum is taken over all non-identity unipotent class representatives v' in Table 5.3.2. By Lemma 4.3.3, we have $\max_{v'} \dim((L_{L_1}(\tilde{\omega}_3))_{v'_{L_1}}(1)) = 3$, while, by Proposition 4.3.14, as $p = 7$, we have $\max_{v'} \dim((L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3))_{v'_{L_1}}(1)) = 8$. It follows that $\dim(V_u(1)) \leq 22$ for all non-identity unipotent elements $u \in \tilde{G}$.

We will now show that there exist unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) = 22$. For this, we consider $x_{\tilde{\alpha}_2}(1) \in \tilde{G}$ and note that $(x_{\tilde{\alpha}_2}(1))_{Q_1} = 1$. Now, using Table 4.3.2, we determine that $\dim((L_{L_1}(\tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1)) = 3$, by Lemma 4.3.3, while, by the proof of Proposition 4.3.14, we have $\dim((L_{L_1}(\tilde{\omega}_2 + \tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1)) = 8$. Thus, $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1)) = 22$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 22$ for all non-identity unipotent elements $u \in \tilde{G}$ and that there exist $u \in \tilde{G}$ for which the bound is attained, for example $x_{\tilde{\alpha}_2}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$. □

Proposition 5.3.14. *Let k be an algebraically closed field of characteristic $p \neq 2, 7$. Assume $\ell = 3$ and let $V = L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_3)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 28,$$

where there exist $u \in \tilde{G}$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$.

Proof. To begin, we recall the decomposition of $V\mid_{[L_1,L_1]}$ from Proposition 5.2.12, which states:

$$V\mid_{[L_1,L_1]}= L_{L_1}(\tilde{\omega}_3)\oplus V^1\oplus V^2\oplus L_{L_1}(\tilde{\omega}_3),$$

where, if $p\neq 5$, V^1 and V^2 each have two composition factors: one isomorphic to $L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3)$ and one isomorphic to $L_{L_1}(\tilde{\omega}_3)$; while, if $p=5$, V^1 and V^2 each have three composition factors: one isomorphic to $L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3)$ and two isomorphic to $L_{L_1}(\tilde{\omega}_3)$.

First, assume that $p\neq 5$. Then, by [Jan07, II.2.14], we have $V^i\cong L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3)\oplus L_{L_1}(\tilde{\omega}_3)$, $i=1,2$, and so

$$V\mid_{[L_1,L_1]}= L_{L_1}(\tilde{\omega}_3)\oplus L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3)\oplus L_{L_1}(\tilde{\omega}_3)\oplus L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3)\oplus L_{L_1}(\tilde{\omega}_3)\oplus L_{L_1}(\tilde{\omega}_3). \quad (5.36)$$

Let $u\in\tilde{G}$ be a non-identity unipotent element. Then, by Inequality (5.35) and Decomposition (5.36), we have:

$$\dim(V_u(1))\leq 4\max_{v'}\{\dim((L_{L_1}(\tilde{\omega}_3))_{v'_{L_1}}(1))\}+2\max_{v'}\{\dim((L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3))_{v'_{L_1}}(1))\},$$

where the maximum is taken over all non-identity unipotent class representatives v' in Table 5.3.2. Now, by Lemma 4.3.3, we have $\max_{v'}\dim((L_{L_1}(\tilde{\omega}_3))_{v'_{L_1}}(1))=3$, while, by Proposition 4.3.14, as $p\neq 5$, we have $\max_{v'}\dim((L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3))_{v'_{L_1}}(1))=8$. It follows that $\dim(V_u(1))\leq 28$ for all non-identity unipotent elements $u\in\tilde{G}$.

We will now show that there exists $u\in\tilde{G}$ for which $\dim(V_u(1))=28$. For this consider $x_{\tilde{\alpha}_2}(1)\in\tilde{G}$. First, we note that $(x_{\tilde{\alpha}_2}(1))_{Q_1}=1$. Secondly, using Table 4.3.2, we determine that $\dim((L_{L_1}(\tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1))=3$, by Lemma 4.3.3, while, by the proof of Proposition 4.3.14, we have $\dim((L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1))=8$. Therefore, $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1))=28$.

We can now assume that $p=5$. We first determine an upper-bound for $\dim(V_{x_{\tilde{\alpha}_3}(1)}(1))$. Afterwards, we will assume that the unipotent element $u\in\tilde{G}$, $u\neq 1$, is not conjugate to $x_{\tilde{\alpha}_3}(1)$ and we will bound $\dim(V_u(1))$. Recall that, when $p=5$, V^1 and V^2 each have three composition factors: one isomorphic to $L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3)$ and two isomorphic to $L_{L_1}(\tilde{\omega}_3)$. Now, by Table 4.3.2, it follows that $\dim((L_{L_1}(\tilde{\omega}_3))_{x_{\tilde{\alpha}_3}(1)}(1))=2$, while, by Proposition 4.3.13, we have $\dim((L_{L_1}(\tilde{\omega}_2+\tilde{\omega}_3))_{x_{\tilde{\alpha}_3}(1)}(1))\leq 5$. Therefore, $\dim(V_{x_{\tilde{\alpha}_3}(1)}(1))\leq 22$, by Lemma 2.4.9.

Let $u\in\tilde{G}$ be a non-identity unipotent element that does not belong to the conjugacy class of $x_{\tilde{\alpha}_3}(1)$. In what follows, we will determine an upper-bound for $\dim(V_u(1))$. Let L_3 be the Levi subgroup of the maximal parabolic subgroup P_3 of $\tilde{G}$ given in Section 2.4. We note that $[L_3,L_3]$ is a simple simply connected linear algebraic group of type A_2 with maximal torus $T'=\tilde{T}\cap[L_3,L_3]$. Set $\tilde{\lambda}=\tilde{\omega}_1+\tilde{\omega}_3$ and note that $\dim(V)=48$, as $p\neq 7$. By Lemma 2.4.6, we have $e_3(\tilde{\lambda})=5$ and so:

$$V\mid_{[L_3,L_3]}= V^0\oplus\cdots\oplus V^5,$$

where $V^i=\bigoplus_{\tilde{\gamma}\in\mathbb{N}\tilde{\Delta}_3}V_{\tilde{\lambda}-i\tilde{\alpha}_3-\tilde{\gamma}}$ for all $0\leq i\leq 5$. Now, by [Smi82, Proposition], we have $V^0\cong L_{L_3}(\tilde{\omega}_1)$ and so, by Lemma 2.4.3, we also have $V^5\cong L_{L_3}(\tilde{\omega}_1)$. The weight $(\tilde{\lambda}-\tilde{\alpha}_3)\mid_{T'}=\tilde{\omega}_1+\tilde{\omega}_2$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2)$. Moreover, we note that the dominant weight $(\tilde{\lambda}-\tilde{\alpha}_1-\tilde{\alpha}_2-\tilde{\alpha}_3)\mid_{T'}=0$, which occurs with

multiplicity 3 in V^1, as $p \neq 3$, has multiplicity 2 in the composition factor of V^1 isomorphic to $L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2)$. Therefore, $\dim(V^1) \geq 9$ and so $\dim(V^4) \geq 9$, as $V^4 \cong (V^1)^*$, by Lemma 2.4.3. This gives $\dim(V^2) = \dim(V^3) \leq 12$. Now, in V^2, the dominant weight $(\tilde{\lambda}-\tilde{\alpha}_2-2\tilde{\alpha}_3)\mid_{T'}= 2\tilde{\omega}_1$ admits a maximal vector and so V^2 has a composition factor isomorphic to $L_{L_3}(2\tilde{\omega}_1)$. We also note that the dominant weight $(\tilde{\lambda}-\tilde{\alpha}_1-\tilde{\alpha}_2-2\tilde{\alpha}_3)\mid_{T'}= \tilde{\omega}_2$, which occurs with multiplicity 3 in V^2, as $p \neq 7$, is a sub-dominant weight in the composition factor of V^2 isomorphic to $L_{L_3}(2\tilde{\omega}_1)$, in which it has multiplicity 1. Thus, as $\dim(V^2) - \dim(L_{L_3}(2\tilde{\omega}_1)) \leq 6$ and $\dim(L_{L_3}(\tilde{\omega}_2)) = 3$, it follows that V^2 has exactly three composition factors: one isomorphic to $L_{L_3}(2\tilde{\omega}_1)$ and two isomorphic to $L_{L_3}(\tilde{\omega}_2)$. Moreover, as $p \neq 2$, using [Jan07, II.2.12 and 2.14], we determine that $V^2 \cong L_{L_3}(2\tilde{\omega}_1) \oplus L_{L_3}(\tilde{\omega}_2) \oplus L_{L_3}(\tilde{\omega}_2)$. Then, by Lemma 2.4.3, we also have $V^3 \cong L_{L_3}(2\tilde{\omega}_1) \oplus L_{L_3}(\tilde{\omega}_2) \oplus L_{L_3}(\tilde{\omega}_2)$, since $V^3 \cong (V^2)^*$. This gives $\dim(V^1) = 9$ and so V^1 has exactly two composition factors: one isomorphic to $L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2)$ and one isomorphic to $L_{L_3}(0)$. We use [Jan07, II.2.14] once more, to show that $V^1 \cong L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2) \oplus L_{L_3}(0)$. Lastly, by Lemma 2.4.3, we have $V^4 \cong L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2) \oplus L_{L_3}(0)$ and so:

$$\begin{aligned} V\mid_{[L_3,L_3]} \cong\ & L_{L_3}(\tilde{\omega}_1) \oplus L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2) \oplus L_{L_3}(0) \oplus L_{L_3}(2\tilde{\omega}_1) \oplus L_{L_3}(\tilde{\omega}_2) \oplus L_{L_3}(\tilde{\omega}_2) \oplus L_{L_3}(2\tilde{\omega}_1) \oplus \\ & \oplus L_{L_3}(\tilde{\omega}_2) \oplus L_{L_3}(\tilde{\omega}_2) \oplus L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2) \oplus L_{L_3}(0) \oplus L_{L_3}(\tilde{\omega}_1). \end{aligned} \tag{5.37}$$

Let u' be the representative listed in Table 5.3.2 of the unipotent $\tilde{G}$-conjugacy class of u. Now, as u and $x_{\tilde{\alpha}_3}(1)$ are not conjugate, it follows that u' and $x_{\tilde{\alpha}_3}(1)$ are not conjugate. Using Table 5.3.2, we determine that $u'_{L_3} \neq 1$. Then, by Decomposition (5.37) and Inequality (2.7), it follows that:

$$\begin{aligned} \dim(V_u(1)) \leq 2 + 2\dim((L_{L_3}(\tilde{\omega}_1))_{u'_{L_3}}(1)) + 2\dim((L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2))_{u'_{L_3}}(1)) + \\ + 4\dim((L_{L_3}(\tilde{\omega}_2))_{u'_{L_3}}(1)) + 2\dim((L_{L_3}(2\tilde{\omega}_1))_{u'_{L_3}}(1)). \end{aligned}$$

By Lemma 3.3.3, we have $\dim((L_{L_3}(\tilde{\omega}_1))_{u'_{L_3}}(1)) \leq 2$ and $\dim((L_{L_3}(\tilde{\omega}_2))_{u'_{L_3}}(1)) \leq 2$, while, by Proposition 3.3.5 and Corollary 3.3.9, we have $\dim((L_{L_3}(2\tilde{\omega}_1))_{u'_{L_3}}(1)) \leq 3$ and $\dim((L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2))_{u'_{L_3}}(1)) \leq 4$, respectively. It follows that $\dim(V_u(1)) \leq 28$.

We will now show that there exists $u \in \tilde{G}$ for which $\dim(V_u(1)) = 28$. For this, we consider $x_{\tilde{\alpha}_2}(1) \in \tilde{G}$. First, we note that $(x_{\tilde{\alpha}_2}(1))_{Q_3} = 1$, where $Q_3 = R_u(P_3)$. Secondly, by Table 4.3.2, we have $\dim((L_{L_3}(\tilde{\omega}_1))_{x_{\tilde{\alpha}_2}(1)}(1)) = 2$ and $\dim((L_{L_3}(\tilde{\omega}_2))_{x_{\tilde{\alpha}_2}(1)}(1)) = 2$, see Lemma 3.3.3; $\dim((L_{L_3}(2\tilde{\omega}_1))_{x_{\tilde{\alpha}_2}(1)}(1)) = 3$, see Proposition 3.3.5; and $\dim((L_{L_3}(\tilde{\omega}_1+\tilde{\omega}_2))_{x_{\tilde{\alpha}_2}(1)}(1)) = 4$, see Corollary 3.3.9. Therefore, $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1)) = 28$, by Identity (2.8).

In conclusion, we have shown that $\dim(V_u(1)) \leq 28$ for all non-identity unipotent elements $u \in \tilde{G}$ and that there exist $u \in \tilde{G}$ for which the bound is attained, for example $x_{\tilde{\alpha}_2}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$. $\square$

At this point, we have determined $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ for all the irreducible modules V of $\tilde{G}$ of type B_3 with highest weights featured in Table 2.7.4. In order to determine these maximums, we used the fact that all nontrivial unipotent $\tilde{G}$-conjugacy classes admit a representative u' such that $u'_{L_1} \neq 1$. We can now assume that $\ell \geq 4$. Let L_1 be the Levi subgroup of the maximal parabolic subgroup P_1 of $\tilde{G}$ given in Section 2.4. In Proposition

5.3.18 we will show that each non-identity unipotent $\tilde{G}$-conjugacy class has a representative $\tilde{u}'$ such that $\tilde{u}'_{L_1} \neq 1$. To achieve this, we will use the algorithm presented in [Kor18, Subsection 2.8.5], which constructs unipotent class representatives in G, and we will show that each unipotent G-conjugacy class admits a representatives u', $u' = \prod_{\alpha \in S_{u'}} x_\alpha(c_\alpha)$, where $S_{u'} \subseteq \Phi^+$, the product respects the order $\preceq$ on Φ, see Section 1.3, and $c_\alpha \in k^*$ for all $\alpha \in S_{u'}$, such that $S_{u'} \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$, see Proposition 5.3.17. We describe the algorithm below.

For the moment, we focus on $G = \mathrm{SO}(W)$, the simple adjoint group of type B_ℓ, $\ell \geq 4$. By Theorem 2.9.2, since $p \neq 2$, we know that the Jordan normal form completely determines unipotent conjugacy classes in G. We also know that even sized Jordan blocks occur with even multiplicity, thus, since $\dim(W)$ is odd, we deduce that the number of odd sized Jordan blocks is odd. Now, let u be a non-identity unipotent element in G and let V_i, $1 \leq i \leq \mathrm{ord}(u)$, be the unique, up to isomorphism, indecomposable $k[u]$-module with $\dim(V_i) = i$ and on which u acts as the full Jordan block of size i. To u we associate the (possibly empty) sequences $(e_i)_{1 \leq i \leq t}$, $(o_i)_{t+1 \leq i \leq t+s+1}$ and $(o'_i)_{t+1 \leq i \leq t+s}$ such that

$$W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} \left(V_{2o_j+1} \oplus V_{2o'_j+1} \right) \oplus V_{2o_{t+s+1}+1}, \tag{5.38}$$

where $2 \leq e_1 \leq \cdots \leq e_t$ are even and $0 \leq o_{t+1} \leq o'_{t+1} \leq \cdots \leq o_{t+s} \leq o'_{t+s} \leq o_{t+s+1}$ are such that $\sum_{i=1}^{t} e_i + \sum_{j=t+1}^{t+s} (o_j + o'_j + 1) + o_{t+s+1} = \ell$. We note that the above decomposition of $W\mid_{k[u]}$ completely determines the conjugacy class of u in G.

In [Kor18, p.46], it is explained how to construct subspaces W_1 and W_2 of W with the property that $W = W_1 \oplus W_2$ is an orthogonal direct sum. Furthermore, it is shown that $u_1 \cdot u_2$ is a representative of the unipotent conjugacy class determined by the Decomposition (5.38), where $u_1 \in \mathrm{SO}(W_1)$ and $W_1\mid_{k[u_1]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} (V_{2o_j+1} \oplus V_{2o'_j+1})$ and where $u_2 \in \mathrm{SO}(W_2)$ and $W_2\mid_{k[u_2]} = V_{2o_{t+s+1}+1}$. The construction of u_2 is given in [Kor18, p.47]. For u_1, if $\dim(W_1) = 2$, we choose $u_1 = 1$, while, if $\dim(W_1) > 2$, we use Lemma 6.3.17, a correction of [Kor18, Lemma 2.8.17], and [Kor18, Tables 2.7 and 2.8] to determine that u_1 is as in the lemma below.

Lemma 5.3.15. *Let $2 \leq e_1 \leq \cdots \leq e_t$ be even and let $0 \leq o_{t+1} \leq o'_{t+1} \leq \cdots \leq o_{t+s} \leq o'_{t+s} \leq o_{t+s+1}$ be such that $\sum_{i=1}^{t} e_i + \sum_{j=t+1}^{t+s} (o_j + o'_j + 1) + o_{t+s+1} = \ell$. Set $e_i = o_i + o'_i + 1$, for all $t+1 \leq i \leq t+s$, and, moreover, set $k_1 = 1$ and $k_i = 1 + e_1 + \cdots + e_{i-1}$, for all $2 \leq i \leq t+s+1$. For all $t+1 \leq i \leq t+s$ with $o_i > 0$, define:*

$$w_i = \prod_{j=k_i}^{k_i+o_i-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_\ell}(1).$$

For all $t+1 \leq i \leq t+s$ with $o_i' > 0$, define:

$$w_i' = \prod_{j=k_i+o_i}^{k_{i+1}-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_\ell}(-1).$$

For all $1 \leq i \leq t+s$, define:

$$v_i = \begin{cases} \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1), \text{ if } 1 \leq i \leq t; \\ 1, \text{ if } t+1 \leq i \leq t+s \text{ and } o_i = o_i' = 0; \\ \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_\ell}(1), \text{ if } t+1 \leq i \leq t+s \text{ and } o_i = 0,\ o_i' > 0; \\ w_i w_i', \text{ if } t+1 \leq i \leq t+s \text{ and } o_i, o_i' > 0. \end{cases}$$

Set $u_1 = v_1 \cdots v_{t+s}$ *and* $u_2 = \begin{cases} 1, \text{ if } o_{t+s+1} = 0; \\ \prod_{j=k_{s+1}}^{\ell} x_{\alpha_j}(1), \text{ if } o_{t+s+1} > 0. \end{cases}$

Then $u = u_1 u_2$ *satisfies* $W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} \left(V_{2o_j+1} \oplus V_{2o_j'+1}\right) \oplus V_{2o_{t+s+1}+1}$.

Remark 5.3.16. *Let $\mathcal{C}$ be a non-identity unipotent conjugacy class in G. The goal of this remark is to establish S_u for the representative $u \in \mathcal{C}$, $u = u_1 \cdot u_2$ with $u_1 = v_1 \cdots v_{t+s}$, constructed in Lemma 5.3.15. For this, let $W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} \left(V_{2o_j+1} \oplus V_{2o_j'+1}\right) \oplus V_{2o_{t+s+1}+1}$ be the corresponding decomposition of W as a $k[u]$-module. We distinguish the following cases:*

<u>Case 1</u>: $t = 0$. Then $\sum_{j=1}^{s}(o_j + o_j' + 1) + o_{s+1} = \ell$ with $0 \leq o_1 \leq o_1' \leq \cdots \leq o_s \leq o_s' \leq o_{s+1}$.

Now, as u is nontrivial, it follows that $o_{s+1} > 0$ and so $u_2 = \prod_{j=k_{s+1}}^{\ell} x_{\alpha_j}(1)$. With u_2 identified, we will now determine u_1. We have the following sub-cases:

<u>Sub-case 1.1</u>: If $o_s' = 0$, then $o_i = o_i' = 0$ for all $1 \leq i \leq s$ and, in this case, we have $u_1 = 1$, therefore $u = u_2$ and $S_u = \{\alpha_{k_{s+1}}, \ldots, \alpha_\ell\}$.

<u>Sub-case 1.2</u>: If $o_s' > 0$, let $1 \leq j \leq s$ be such that $o_{j-1}' = 0$ and $o_j' > 0$. Then, $o_i = o_i' = 0$ for all $0 \leq i \leq j-1$, hence $v_i = 1$, for all $0 \leq i \leq j-1$. Moreover, for all $j+1 \leq i \leq s$, since $o_i, o_i' > 0$, we have:

$$v_i = w_i w_i' = \prod_{j=k_i}^{k_i+o_i-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_\ell}(1) \cdot$$

$$\cdot \prod_{j=k_i+o_i}^{k_{i+1}-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_\ell}(-1)$$

Now, for all $j+1 \leq i \leq s$, set

$$\beta_i := \alpha_{k_i+o_i-1} + \cdots + \alpha_{k_{i+1}-2}$$

$$\delta_i := \alpha_{k_i+o_i-1} + \cdots + \alpha_{k_{i+1}-2} + 2\alpha_{k_{i+1}-1} + \cdots + 2\alpha_\ell$$

$$\gamma_i := \alpha_{k_{i+1}-2} + 2\alpha_{k_{i+1}-1} + \cdots + 2\alpha_\ell.$$

We apply the commutator relations [MT11, Theorem 11.8] *and use* [Cav10, Lemma 2.5.4] *to determine that:*

(1) $x_{\alpha_j}(1)$, $k_i + o_i \leq j \leq k_{i+1} - 3$, *commutes with both* $x_{\beta_i}(1)$ *and* $x_{\delta_i}(1)$;

(2) $[x_{\delta_i}(1), x_{\alpha_{k_{i+1}-2}}(1)] = x_{\xi_i}(-1)$, *where*

$$\xi_i := \alpha_{k_i+o_i-1} + \cdots + \alpha_{k_{i+1}-3} + 2\alpha_{k_{i+1}-2} + \cdots + 2\alpha_\ell.$$

Moreover, $x_{\xi_i}(-1)$ *commutes with both* $x_{\alpha_{k_{i+1}-2}}(1)$ *and* $x_{\delta_i}(1)$;

(3) $x_{\alpha_{k_{i+1}-2}}(1)$ *commutes with* $x_{\beta_i}(1)$;

(4) $x_{\gamma_i}(-1)$ *commutes with both* $x_{\delta_i}(1)$ *and* $x_{\xi_i}(-1)$.

Thus, for all $j+1 \leq i \leq s$, we have:

$$v_i = \prod_{j=k_i}^{k_i+o_i-2} x_{\alpha_j}(1) \cdot \prod_{j=k_i+o_i}^{k_{i+1}-2} x_{\alpha_j}(1) \cdot x_{\beta_i}(1) \cdot x_{\gamma_i}(-1) \cdot x_{\delta_i}(1) \cdot x_{\xi_i}(-1).$$

We use the commutator relations, [MT11, Theorem 11.8] *once more, and obtain:*

$$\begin{aligned} u_1 &= v_1 \cdots v_s = v_j \cdot v_{j+1} \cdots v_s \\ &= v_j \cdot \prod_{i=j+1}^{s} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \cdot x_{\beta_i}(1) \cdot x_{\gamma_i}(-1) \cdot x_{\delta_i}(1) \cdot x_{\xi_i}(-1) \right) \\ &= v_j \cdot \prod_{i=j+1}^{s} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{i=j+1}^{s} x_{\beta_i}(1) \cdot \prod_{i=j+1}^{s} x_{\gamma_i}(-1) \cdot \prod_{i=j+1}^{s} x_{\delta_i}(1) \cdot \prod_{i=j+1}^{s} x_{\xi_i}(-1). \end{aligned}$$

If $o_j = 0$, then, by [MT11, Theorem 11.8], *it follows that:*

$$\begin{aligned} u_1 = \prod_{r=k_j}^{k_{j+1}-2} x_{\alpha_r}(1) \cdot \prod_{i=j+1}^{s} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{i=j+1}^{s} x_{\beta_i}(1) \cdot x_{\gamma_j}(1) \cdot \prod_{i=j+1}^{s} x_{\gamma_i}(-1) \cdot \\ \cdot \prod_{i=j+1}^{s} x_{\delta_i}(1) \cdot \prod_{i=j+1}^{s} x_{\xi_i}(-1). \end{aligned}$$

Similarly, if $o_j > 0$, then

$$u_1 = \prod_{i=j}^{s} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{i=j}^{s} x_{\beta_i}(1) \cdot \prod_{i=j}^{s} x_{\gamma_i}(-1) \cdot \prod_{i=j}^{s} x_{\delta_i}(1) \cdot \prod_{i=j}^{s} x_{\xi_i}(-1).$$

Lastly, recalling that $u_2 = \prod_{j=k_{s+1}}^{\ell} x_{\alpha_j}(1)$, *by* [MT11, Theorem 11.8], *we determine that, if* $o_j = 0$, *then:*

$$u = \prod_{r=k_j}^{k_{j+1}-2} x_{\alpha_r}(1) \cdot \prod_{i=j+1}^{s} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{i=j+1}^{s} x_{\beta_i}(1) \cdot \prod_{j=k_{s+1}}^{\ell} x_{\alpha_j}(1) \cdot x_{\gamma_j}(1) \cdot$$
$$\cdot \prod_{i=j+1}^{s} x_{\gamma_i}(-1) \cdot \prod_{i=j+1}^{s} x_{\delta_i}(1) \cdot \prod_{i=j+1}^{s} x_{\xi_i}(-1). \tag{5.39}$$

Similarly, if $o_j > 0$, *then:*

$$u = \prod_{i=j}^{s} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{i=j}^{s} x_{\beta_i}(1) \cdot \prod_{j=k_{s+1}}^{\ell} x_{\alpha_j}(1) \cdot \prod_{i=j}^{s} x_{\gamma_i}(-1) \cdot \prod_{i=j}^{s} x_{\delta_i}(1) \cdot \prod_{i=j}^{s} x_{\xi_i}(-1). \tag{5.40}$$

In both cases we see that $\alpha_\ell \in S_u$.

<u>*Case 2*</u>: $t \geq 1$. *We distinguish the following sub-cases:*

<u>*Sub-case 2.1*</u>: $s = 0$. *If* $o_{t+1} = 0$, *then* $u_2 = 1$, $u_1 = v_1 \cdots v_t$ *and we have:*

$$u = \prod_{i=1}^{t} \left(\prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1) \right),$$

thereby $S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \alpha_{k_2}, \ldots, \alpha_{k_3-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}\}$, *as* $k_1 = 1$. *On the other hand, if* $o_{t+1} > 0$, *we have* $u_2 = \prod_{j=k_{t+1}}^{\ell} x_{\alpha_j}(1)$ *and so:*

$$u = \prod_{i=1}^{t} \left(\prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1) \right) \cdot \prod_{j=k_{t+1}}^{\ell} x_{\alpha_j}(1).$$

In this case, $S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}, \alpha_{k_{t+1}}, \ldots, \alpha_\ell\}$.

<u>*Sub-case 2.2*</u>: $s \geq 1$. *If* $o_{t+s+1} = 0$, *then* $o_{t+j} = o'_{t+j} = 0$, *for all* $1 \leq j \leq s$, *and so* $u_2 = 1$ *and* $v_{t+j} = 1$, *for all* $1 \leq j \leq s$. *In this case,* $u = v_1 \cdots v_t$ *and so* $S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \alpha_{k_2}, \ldots, \alpha_{k_3-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}\}$. *On the other hand, if* $o_{t+s+1} > 0$, *then* $u_2 = \prod_{j=k_{t+1}}^{\ell} x_{\alpha_j}(1)$. *It follows that*

$$u = v_1 \cdots v_t \cdot v_{t+1} \cdots v_{t+s} \cdot u_2 = \prod_{i=1}^{t} \left(\prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1) \right) \cdot v_{t+1} \cdots v_{t+s} \cdot u_2, \tag{5.41}$$

where $v_{t+1} \cdots v_{t+s} \cdot u_2$ *is as in* (5.39), *or* (5.40), *depending on whether* $o_j = 0$ *or* $o_j > 0$, *where* $1 \leq j \leq s+1$ *is such that* $o'_{j-1} = 0$ *and* $o'_j > 0$. *Now, since all* $\beta \in S_{v_1 \cdots v_t}$ *and all* $\gamma \in S_{v_{t+1} \cdots v_{t+s} \cdot u_2}$ *are such that* $\beta \preceq \gamma$, *we determine that the product in* (5.41) *respects the total order* $\preceq$ *on* Φ. *Lastly, we note that, in both cases,* $\alpha_\ell \in S_u$.

Proposition 5.3.17. *Each non-identity unipotent G-conjugacy class, admits a representative u' with the property that*

$$S_{u'} \cap \{\alpha_2, \dots, \alpha_\ell\} \neq \emptyset.$$

Proof. Let $\mathcal{C}$ be a non-identity unipotent G-conjugacy class and let $u \in \mathcal{C}$, $u = u_1 \cdot u_2$ with $u_1 = v_1 \cdots v_{t+s}$, be the representative of $\mathcal{C}$ constructed in Lemma 5.3.15. Let

$$W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} \left(V_{2o_j+1} \oplus V_{2o'_j+1} \right) \oplus V_{2o_{t+s+1}+1}$$

be the corresponding decomposition of W as a $k[u]$-module.

If $o_{t+s+1} > 0$, then, by Case 1 and (5.41) of Remark 5.3.16, we have that $\alpha_\ell \in S_u$. We can thus assume that $o_{t+s+1} = 0$, in which case $o_j = o'_j = 0$ for all $t+1 \leq j \leq t+s$. As u is nontrivial, it follows that $t \geq 1$. Then, by Case 2 of Remark 5.3.16, we have $S_u = \{\alpha_1, \dots, \alpha_{k_2-2}, \alpha_{k_2}, \dots, \alpha_{k_3-2}, \dots, \alpha_{k_t}, \dots, \alpha_{k_{t+1}-2}\}$. If $t \geq 2$, then $\alpha_{k_2} \in S_u$, where $k_2 = 1 + e_1 \geq 3$. We can thus assume that $t = 1$. If $e_1 \geq 4$, then $\alpha_{k_2-2} \in S_u$, where $k_2 - 2 = e_1 - 1 \geq 3$. Lastly, if $e_1 = 2$, then $u = x_{\alpha_1}(1)$ and $W\mid_{k[x_{\alpha_1}(1)]} = V_2^2 \oplus V_1^{2\ell-3}$. However, by [LS12, Subsection 3.3.2], $x_{\alpha_2}(1)$ is another unipotent element of G with corresponding decomposition $W\mid_{k[x_{\alpha_2}(1)]} = V_2^2 \oplus V_1^{2\ell-3}$. Therefore, $x_{\alpha_1}(1)$ and $x_{\alpha_2}(1)$ are G-conjugate and we choose $x_{\alpha_2}(1)$ as representative of the unipotent class of u.

Having considered all possible cases, we conclude that all nontrivial unipotent conjugacy classes of G admit a representative u' with the property that $S_{u'} \cap \{\alpha_2, \dots, \alpha_\ell\} \neq \emptyset$. □

We can now state the analog of Proposition 5.3.17 for the simple simply connected linear algebraic group $\tilde{G}$ of type B_ℓ, $\ell \geq 4$.

Proposition 5.3.18. *Each non-identity unipotent $\tilde{G}$-conjugacy class admits a representative $\tilde{u}'$ with the property that $S_{\tilde{u}'} \cap \{\tilde{\alpha}_2, \dots, \tilde{\alpha}_\ell\} \neq \emptyset$.*

Proof. Assume by contradiction that there exists a non-identity unipotent $\tilde{G}$-conjugacy class $\tilde{C}$ such that for all $\tilde{u}' \in \tilde{C}$ we have $S_{\tilde{u}'} \cap \{\tilde{\alpha}_2, \dots, \tilde{\alpha}_\ell\} = \emptyset$. Now, as $\phi : \tilde{G} \to G$ is a central isogeny and $\phi(U_{\tilde{\alpha}}) = U_\alpha$, for all $\tilde{\alpha} \in \tilde{\Phi}$, by Lemma 2.9.1, it follows that the unipotent conjugacy class C of G given by $C = \phi(\tilde{C})$ has the property that $S_{u'} \cap \{\alpha_2, \dots, \alpha_\ell\} = \emptyset$ for all $u' \in C$. However, this contradicts Proposition 5.3.17. We conclude that all non-identity unipotent $\tilde{G}$-conjugacy classes admit a representative $\tilde{u}'$ with the property that $S_{\tilde{u}'} \cap \{\tilde{\alpha}_2, \dots, \tilde{\alpha}_\ell\} \neq \emptyset$. □

Proposition 5.3.19. *Assume $\ell \geq 4$ and let $V = L_{\tilde{G}}(\tilde{\omega}_\ell)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 3 \cdot 2^{\ell-2},$$

where there exist $u \in \tilde{G}$ for which the bound is attained.

In particular, for $\ell = 4$ there exist non-identity unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. On the other hand, for $\ell \geq 5$, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$

Proof. To begin, we recall Decomposition (5.22) of Proposition 5.2.9, which states:

$$V\mid_{[L_1,L_1]}\cong L_{L_1}(\tilde{\omega}_\ell)\oplus L_{L_1}(\tilde{\omega}_\ell).$$

Let $u\in\tilde{G}$ be a non-identity unipotent element and let u' be a representative of the unipotent $\tilde{G}$-conjugacy class of u with $u'_{L_1}\neq 1$. Note that, by Proposition 5.3.18, such a representative always exists. Then, by Inequality (2.7) and Decomposition (5.22), we have:

$$\dim(V_u(1))\leq\dim(V_{u'_{L_1}}(1))=2\dim((L_{L_1}(\tilde{\omega}_\ell))_{u'_{L_1}}(1)).$$

Now, for $\ell=4$, by Proposition 5.3.11, we have $\dim((L_{L_1}(\tilde{\omega}_4))_{u'_{L_1}}(1)\leq 3\cdot 2$, therefore $\dim(V_u(1))\leq 3\cdot 2^2$ for all non-identity unipotent elements $u\in\tilde{G}$. Moreover, by [LS12, Subsection 3.3.2] and by the proof of Proposition 5.3.11, it follows that $\dim((L_{L_1}(\tilde{\omega}_4))_{x_{\tilde{\alpha}_3}(1)}(1))$ $=3\cdot 2$, therefore $\dim(V_{x_{\tilde{\alpha}_3}(1)}(1))=3\cdot 2^2$. Recursively, one shows that $\dim(V_u(1))\leq 3\cdot 2^{\ell-2}$ for all $\ell\geq 4$ and all non-identity unipotent elements $u\in\tilde{G}$ and that there exist $u\in\tilde{G}$ for which equality is attained, for example $x_{\tilde{\alpha}_{\ell-1}}(1)$.

In conclusion, we showed that $\dim(V_u(1))\leq 3\cdot 2^{\ell-2}$, for all $\ell\geq 4$ and all non-identity unipotent elements $u\in\tilde{G}$, and that there exist $u\in\tilde{G}$ for which equality holds, for example $x_{\tilde{\alpha}_{\ell-1}}(1)$. Now, if $\ell=4$, it follows that there exist non-identity $u\in\tilde{G}$ for which $\dim(V_u(1))\geq\dim(V)-\sqrt{\dim(V)}$. However, for $\ell\geq 5$, the inequality $1<2^{\ell-4}$ holds, therefore $3\cdot 2^{\ell-2}<2^\ell-\sqrt{2^\ell}$ for all $\ell\geq 5$, and so $\dim(V_u(1))<\dim(V)-\sqrt{\dim(V)}$ for all non-identity unipotent elements $u\in\tilde{G}$. □

We conclude this subsection by noting that Propositions 5.3.11, 5.3.12, 5.3.13, 5.3.14 and 5.3.19 complete the proof of Theorems 5.3.1 and 5.3.2, as they cover all the irreducible $k\tilde{G}$-modules correspon-ding to p-restricted dominant weights featured in Table 2.7.4.

5.4 Results

In this section, we collect the results proven in this chapter. In Proposition 5.4.1 we give the values of $\max\limits_{s\in\tilde{T}\setminus\mathrm{Z}(\tilde{G})}\{\dim(V_s(\mu))\mid\mu\in k^*\}$, $\max\limits_{u\in\tilde{G}_u\setminus\{1\}}\dim(V_u(1))$ and $\nu_{\tilde{G}}(V)$ for all $k\tilde{G}$-modules V belonging to one of the families we had to consider. Similarly, Proposition 5.4.2 records the same data for the particular $k\tilde{G}$-modules treated in this chapter.

Proposition 5.4.1. *Let k be an algebraically closed field of characteristic $p\neq 2$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type B_ℓ, $\ell\geq 3$. Let $\tilde{T}$ be a fixed maximal torus $\tilde{T}$ in $\tilde{G}$ and let $V=L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda}\in F^{B_\ell}$. Then the value of $\nu_{\tilde{G}}(V)$ is given in the table below:*

V	Char.	$\max\limits_{s\in\tilde{T}\setminus \mathrm{Z}(\tilde{G})}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$	$\max\limits_{u\in\tilde{G}_u\setminus\{1\}}\dim(V_u(1))$	$\nu_{\tilde{G}}(V)$
$L_{\tilde{G}}(\tilde{\omega}_1)$	$p\neq 2$	2ℓ	$2\ell-1$	1
$L_{\tilde{G}}(\tilde{\omega}_2)$	$p\neq 2$	$2\ell^2-\ell$	$2\ell^2-3\ell+4$	2ℓ
$L_{\tilde{G}}(2\tilde{\omega}_1)$	$p\neq 2,\ p\nmid 2\ell+1$	$2\ell^2+\ell$	$2\ell^2-\ell$	2ℓ
	$p\neq 2,\ p\mid 2\ell+1$	$2\ell^2+\ell-1$	$2\ell^2-\ell-1$	2ℓ

Table 5.4.1: The value of $\nu_{\tilde{G}}(V)$ for the families of modules of groups of type B_ℓ.

Proof. The result follows by Proposition 2.2.3, using Lemmas 5.2.3 and 5.3.3 for $V=L_{\tilde{G}}(\tilde{\omega}_1)$; Propositions 5.2.4 and 5.3.4 for $V=L_{\tilde{G}}(\tilde{\omega}_2)$; and Corollaries 5.2.6 and 5.3.9, in the case of $p\nmid 2\ell+1$, respectively Corollary 5.2.7 and Proposition 5.3.10, in the case of $p\mid 2\ell+1$, for $V=L_{\tilde{G}}(2\tilde{\omega}_1)$. □

Proposition 5.4.2. *Let k be an algebraically closed field of characteristic $p\neq 2$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type B_ℓ, $\ell\geq 3$. Let $\tilde{T}$ be a fixed maximal torus in $\tilde{G}$ and let $V=L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda}$ is featured in Table 2.7.4. The value of $\nu_{\tilde{G}}(V)$ is given in the table below:*

Rank	$\tilde{\lambda}$	Char.	$\max\limits_{s\in\tilde{T}\setminus \mathrm{Z}(\tilde{G})}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$	$\max\limits_{u\in\tilde{G}_u\setminus\{1\}}\dim(V_u(1))$	$\nu_{\tilde{G}}(V)$
$\ell=3$	$2\tilde{\omega}_3$	$p\neq 2$	20	21	14
$\ell=3$	$\tilde{\omega}_1+\tilde{\omega}_3$	$p=7$	20	22	18
$\ell=3$	$\tilde{\omega}_1+\tilde{\omega}_3$	$p\neq 2,7$	24	28	20
$3\leq\ell\leq 8$	$\tilde{\omega}_\ell$	$p\neq 2$	$2^{\ell-1}$	$3\cdot 2^{\ell-2}$	$2^{\ell-2}$

Table 5.4.2: The value of $\nu_{\tilde{G}}(V)$ for the particular modules of groups of type B_ℓ.

Proof. The result follows by Proposition 2.2.3, using the detailed results of Subsections 5.2.2 and 5.3.2. □

Chapter 6

Groups of type D_ℓ

In this chapter we prove Theorems 1.1.1 and 1.1.3 for the simple simply connected linear algebraic groups of type D_ℓ, $\ell \geq 4$. To begin, we fix k, an algebraically closed field of characteristic $p \geq 0$, and we let W be a 2ℓ-dimensional k-vector space, for some $\ell \geq 4$, equipped with a nondegenerate quadratic form Q. The structure of this chapter is as follows: in the first section we construct the simple linear algebraic group $G = \mathrm{SO}(W)$ of type D_ℓ and exhibit some properties of its semisimple and unipotent elements. In Section 6.2 we determine $\max\limits_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, where $\tilde{G}$ is a simple simply connected linear algebraic group of type D_ℓ with maximal torus $\tilde{T}$ and V runs through the list of $k\tilde{G}$-modules we identified in Subsection 2.7.4. Similarly, in Section 6.3, we determine $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ for the same $k\tilde{G}$-modules V. Lastly, Section 6.4 records all the results of this chapter.

We will now fix some notation which will be used throughout the chapter. We let G be a simple linear algebraic group of type D_ℓ, $\ell \geq 4$, and we let $\tilde{G}$ be the simple simply connected linear algebraic group of the same type as G. We also fix $\phi : \tilde{G} \to G$, a central isogeny with $d\phi \neq 0$ and $\ker(\phi) = \mathrm{Z}(\tilde{G})$. In G, we let T, $\mathrm{X}(T)$, Φ, B, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\omega_1, \ldots, \omega_\ell$ be as usual. We also let $\tilde{T}$, respectively $\tilde{B}$, be a preimage of T, respectively of B, in $\tilde{G}$, and note that $\tilde{T}$ is a maximal torus of $\tilde{G}$ contained in the Borel subgroup $\tilde{B}$ of $\tilde{G}$. As for G, we let $\mathrm{X}(\tilde{T})$, $\tilde{\Phi}$, $\tilde{\Delta} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_\ell\}$ and $\tilde{\omega}_1, \ldots, \tilde{\omega}_\ell$ be the rational character group of $\tilde{T}$, the root system of $\tilde{G}$ determined by $\tilde{T}$, the set of simple roots in $\tilde{\Phi}$ given by $\tilde{B}$, and the fundamental dominant weights of $\tilde{G}$ corresponding to $\tilde{\Delta}$.

6.1 Construction of linear algebraic groups of type D_ℓ

Let W be a 2ℓ-dimensional k-vector space, where $\ell \geq 4$, equipped with a nondegenerate quadratic form Q. We fix $B_W = \{u_1, u_2, \ldots, u_\ell, v_\ell, \ldots, v_2, v_1\}$ to be an ordered basis in W with the property that $W = \bigoplus\limits_{i=1}^{\ell} \langle u_i, v_i \rangle$ is an orthogonal direct sum, where $\{u_i, v_i\}$ is a hyperbolic pair for all $1 \leq i \leq \ell$, see Corollary 2.1.4, in the case of $p \neq 2$, or Theorem 2.1.1, in the case of $p = 2$. We denote by D the set of diagonal matrices and by U the set of upper-triangular matrices in $\mathrm{GL}(W)$. Set $G = \mathrm{SO}(W)$ and note that G is a simple linear

algebraic group of type D_ℓ, see [Car89, Theorem 11.3.2]. Set $T = D \cap G$ and $B = U \cap G$, and note that T is a maximal torus in G and B is a Borel subgroup of G with the property that $T \subseteq B$.

Remark 6.1.1. *We recall from Subsection* 2.7.4 *that F^{D_ℓ}, the set of p-restricted dominant weights $\tilde{\lambda} \in \mathrm{X}(\tilde{T})$ with the property that the associated irreducible $k\tilde{G}$-module $L_{\tilde{G}}(\tilde{\lambda})$ satisfies the dimensional criteria* (2.18) *for all $\ell \geq 4$, is given by $F^{D_\ell} = \{\tilde{\omega}_1, 2\tilde{\omega}_1, \tilde{\omega}_2\}$. As for groups of type B_ℓ, see Remark* 5.1.1*, we note that for all $\tilde{\lambda} \in \{\tilde{\omega}_1, 2\tilde{\omega}_1, \tilde{\omega}_2\}$, there exists $\lambda \in \mathrm{X}(T)$ such that $\tilde{\lambda}$ is the image of λ when viewed as an element of $\mathrm{X}(\tilde{T})$, see Subsection* 2.3.3*. In particular, for $\tilde{\omega}_1 \in \mathrm{X}(\tilde{T})$, we have $\omega_1 \in \mathrm{X}(T)$, for $2\tilde{\omega}_1 \in \mathrm{X}(\tilde{T})$, we have $2\omega_1 \in \mathrm{X}(T)$ and for $\tilde{\omega}_2 \in \mathrm{X}(\tilde{T})$, we have $\omega_2 \in \mathrm{X}(T)$. In all cases, by Lemma* 2.3.10*, we determine that:*

(1) $\tilde{M}_s = \max\limits_{\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(L_{\tilde{G}}(\tilde{\lambda})_{\tilde{s}}(\tilde{\mu})) \mid \tilde{\mu} \in k^*\} = \max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(L_G(\lambda)_s(\mu)) \mid \mu \in k^*\} = M_s$.

(2) $\tilde{M}_u = \max\limits_{\tilde{u} \in \tilde{G}_u \setminus \{1\}} \dim(L_{\tilde{G}}(\tilde{\lambda})_{\tilde{u}}(1)) = \max\limits_{u \in G_u \setminus \{1\}} \dim(L_G(\lambda)_u(1)) = M_u$, *where G_u is the set of unipotent elements in G.*

(3) $\nu_{\tilde{G}}(L_{\tilde{G}}(\tilde{\lambda})) = \nu_G(L_G(\lambda))$.

6.1.1 Semisimple elements

In the previous subsection, we took W to be a 2ℓ-dimensional k-vector space equipped with a nondegenerate quadratic form Q. We fixed the basis B_W in W and we built the subgroup $G = \mathrm{SO}(W, Q)$ of $\mathrm{GL}(W)$. We now take W and equip it with a nondegenerate alternating bilinear form a. We fix a basis B'_W, as in Theorem 2.1.1, in W and we set $H = \mathrm{Sp}(W, a)$. We note that $T_H = D \cap H$, where D is the set of diagonal matrices in $\mathrm{GL}(W)$, is a maximal torus in H. Moreover, recall from Subsection 4.1.1 that an element $s_H \in T_H$ has the form $s_H = \mathrm{diag}(\nu_1 \cdot \mathrm{I}_{m_1}, \nu_2 \cdot \mathrm{I}_{m_2}, \ldots, \nu_t \cdot \mathrm{I}_{m_t}, \nu_t^{-1} \cdot \mathrm{I}_{m_t}, \ldots, \nu_2^{-1} \cdot \mathrm{I}_{m_2}, \nu_1^{-1} \cdot \mathrm{I}_{m_1})$, where $\nu_i \neq \nu_j$ for all $1 \leq i < j \leq t$, $\sum\limits_{i=1}^{t} m_i = \ell$ and $\ell \geq m_1 \geq \cdots \geq m_t \geq 1$.

Let $s \in T$. Then $s = \mathrm{diag}(a_1, a_2, \ldots, a_\ell, a_\ell^{-1}, \ldots, a_2^{-1}, a_1^{-1})$ with $a_i \in k^*$ for all $1 \leq i \leq \ell$. Let $\{\mu_1, \mu_2, \ldots, \mu_m\}$, where $m \geq 1$, be the set of distinct a_j's and let n_i, $1 \leq i \leq m$, be the multiplicity of each μ_i in s. Then $\sum\limits_{i=1}^{m} n_i = \ell$ and we can assume, without loss of generality, that $\ell \geq n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$. By conjugating s by an element of $\mathrm{N}_G(T)$, we have that $s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \mu_2 \cdot \mathrm{I}_{n_2}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \mu_2^{-1} \cdot \mathrm{I}_{n_2}, \mu_1^{-1} \cdot \mathrm{I}_{n_1})$. Thus, any element $s \in T$ has the form $s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \mu_2 \cdot \mathrm{I}_{n_2}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \mu_2^{-1} \cdot \mathrm{I}_{n_2}, \mu_1^{-1} \cdot \mathrm{I}_{n_1})$, where $\mu_i \neq \mu_j$ for all $1 \leq i < j \leq m$, $\sum\limits_{i=1}^{m} n_i = \ell$ and $\ell \geq n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$. Consequently, we see that $T = T_H$, since every $s \in T$ is an element in T_H, see the previous paragraph, and, analogously, every $s_H \in T_H$ is an element in T. Moreover, we also have $\mathrm{Z}(G) = \mathrm{Z}(H)$. Therefore, if $s \in T \setminus \mathrm{Z}(G)$, then $s \in T_H \setminus \mathrm{Z}(H)$ and vice versa.

Lastly, let $s \in T$. Then, in particular $s \in T_H$ and, by Lemma 4.1.1, s is conjugate in H to an element $s_H \in T_H$ with the property that $s_H = \mathrm{diag}(\nu_1 \cdot \mathrm{I}_{m_1}, \nu_2 \cdot \mathrm{I}_{m_2}, \ldots, \nu_t \cdot$

$\mathrm{I}_{m_t}, \nu_t^{-1} \cdot \mathrm{I}_{m_t}, \ldots, \nu_2^{-1} \cdot \mathrm{I}_{m_2}, \nu_1^{-1} \cdot \mathrm{I}_{m_1})$ with $\nu_i \neq \nu_j^{\pm 1}$ for all $1 \leq i < j \leq t$, $\sum_{i=1}^{t} m_i = \ell$ and $\ell \geq m_1 \geq \cdots \geq m_t \geq 1$.

6.1.2 Unipotent elements

First, we suppose that the algebraically closed field k has characteristic $p \neq 2$. Now, by Theorem 2.9.2, we know that two unipotent elements $u, u' \in \mathrm{O}(W, Q)$ are $\mathrm{O}(W, Q)$-conjugate if and only if they have the same Jordan normal form on W. Furthermore, by the same result, we have that a unipotent element $u \in \mathrm{SL}(W)$ with Jordan form $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ is an element in $\mathrm{O}(W, Q)$ if and only if $r_i \geq 1$ is even for all even n_i. Lastly, we recall that the unipotent class $u^{\mathrm{O}(W,Q)}$ splits into two G-classes if and only if n_i is even for all i. Thus, if u is a unipotent element of G, then u has Jordan form $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ on W, where $m \geq 1$ and $r_i \geq 1$ is even for all even n_i. Furthermore, if there exists $1 \leq i \leq m$ such that n_i is odd, then the Jordan form of u on W completely characterizes its unipotent conjugacy class in G. However, if n_i is even for all $1 \leq i \leq m$, then there are two unipotent classes associated to that Jordan form. Lastly, for $u \in G$ unipotent with Jordan form $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$, we can assume without loss of generality that $2\ell - 1 \geq n_1 > n_2 > \cdots > n_m \geq 1$ and if $u \neq 1$, that $n_1 \geq 2$.

Having established a characterization of unipotent conjugacy classes in G over fields of characteristic $p \neq 2$, we now consider the case when $p = 2$. In this case, the classification of unipotent conjugacy classes in G is given by Proposition 2.9.20. To make this section more self-contained, we recall the aforementioned result. As $p = 2$, we have that $G < \mathrm{Sp}(W, a)$, where a is the nondegenerate alternating bilinear form on W with the property that $a(w_1, w_2) = Q(w_1) + Q(w_2) + Q(w_1 + w_2)$ for all $w_1, w_2 \in W$. Let $u \in G$ be a unipotent element. Then, in particular, u is a unipotent element of $\mathrm{Sp}(W, a)$ and, therefore, by Theorem 2.9.11, we know that the unipotent class of u in $\mathrm{Sp}(W, a)$ is completely determined by the Hesselink normal form of u. Let $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots n_{m_1}^{r_m})$ be the Hesselink normal form of u, see Theorem 2.9.15. Now, since $u \in G$, by Proposition 2.9.20, we have that $\sum_{i=t+1}^{m} r_i$ is even. Moreover, the conjugacy class of u in $\mathrm{Sp}(W, a)$ splits into two G-classes if and only if for all $1 \leq i \leq m$ we have that n_i is even and $t = m$. Therefore, the Hesselink normal form of a unipotent element, $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots n_{m_1}^{r_m})$, completely characterizes unipotent conjugacy classes in G over fields of characteristic $p = 2$, unless $t = m$ and n_i is even for all $1 \leq i \leq m$, in which case there exist two classes associated to that Hesselink normal form.

6.2 Eigenspace dimensions for semisimple elements

Theorem 6.2.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ, $\ell \geq 4$. Let $\tilde{T}$ be a fixed maximal*

torus in $\tilde{G}$ and let $V = L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda} \in F^{D_\ell}$, or $\tilde{\lambda}$ is given in Table 2.7.5. Then there exist $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and $\mu \in k^*$, an eigenvalue of s on V, such that

$$\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$$

if and only if $\tilde{\lambda} = \tilde{\omega}_1$.

Theorem 6.2.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ, $\ell \geq 4$. Let $\tilde{T}$ and V be as in Theorem 6.2.1. Then the value of $\max\limits_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ is given in the table below:*

V	Char.	Rank	$\max\limits_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$
$L_{\tilde{G}}(\tilde{\omega}_1)$	$p \geq 0$	$\ell \geq 4$	$2\ell - 2$
${}^\dagger L_{\tilde{G}}(\tilde{\omega}_2)$	$p \neq 2$	$\ell \geq 4$	$2\ell^2 - 5\ell + 4$
	$p = 2$	$\ell \geq 4$	$2\ell^2 - 5\ell + 4 - \gcd(2, \ell)$
${}^\dagger L_{\tilde{G}}(2\tilde{\omega}_1)$	$p \neq 2$ and $p \nmid \ell$	$\ell \geq 4$	$2\ell^2 - 3\ell + 3$
	$p \neq 2$ and $p \mid \ell$	$\ell \geq 4$	$2\ell^2 - 3\ell + 2$
${}^\dagger L_{\tilde{G}}(\tilde{\omega}_3 + \tilde{\omega}_4)$	$p \geq 0$	$\ell = 4$	$\leq 34 - 6\delta_{p,2}$
${}^\dagger L_{\tilde{G}}(\tilde{\omega}_3)$	$p = 2$	$\ell = 5$	≤ 58
${}^\dagger L_{\tilde{G}}(\tilde{\omega}_{\ell-1})$	$p \geq 0$	$5 \leq \ell \leq 9$	$\leq 5 \cdot 2^{\ell-4}$

Table 6.2.1: The value of $\max\limits_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$.

In particular, for each ${}^\dagger V$ in Table 6.2.1, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.

We will give the proofs of Theorems 6.2.1 and 6.2.2 in a series of results, each treating one of the candidate-modules. In Subsection 6.2.1, we determine $\max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, see Remark 6.1.1, where V belongs to one of the families of kG-modules we have to consider, i.e. V is an irreducible kG-module $L_G(\lambda)$ with p-restricted dominant highest weight $\lambda \in \{\omega_1, 2\omega_1, \omega_2\}$. In Subsection 6.2.2, we establish $\max\limits_{s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$ for the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with highest weight $\tilde{\lambda}$ featured in Table 2.7.5.

6.2.1 The families of modules

Lemma 6.2.3. *Let $V = L_G(\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell - 2,$$

where equality holds if and only if $\mu = \pm 1$ and, up to conjugation, $s = \mathrm{diag}(\pm 1, \ldots, \pm 1, \mu_2, \mu_2^{-1}, \pm 1, \ldots, \pm 1)$ with $\mu_2 \neq \pm 1$.

In particular, there exist $s \in T \setminus \mathrm{Z}(G)$ that afford an eigenvalue $\mu \in k^*$ on V for which $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. We first remark that $V \cong W$ as kG-modules. Now, since $T = T_H$ and $\mathrm{Z}(G) = \mathrm{Z}(H)$, by Lemma 4.2.3, we determine that $\dim(V_s(\mu)) \leq 2\ell - 2$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Moreover, by the same result, equality holds if and only if, up to conjugation, $s = \mathrm{diag}(\pm 1, \ldots, \pm 1, \mu_2, \mu_2^{-1}, \pm 1, \ldots, \pm 1)$, where $\mu_2 \neq \pm 1$, and $\mu = \pm 1$.

To conclude, we showed that $\dim(V_s(\mu)) \leq 2\ell - 2$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V and that there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained. In particular, this shows that there exist $s \in T \setminus \mathrm{Z}(G)$ with the property that $\dim(V_s(\mu)) \geq \dim(V) - \sqrt{\dim(V)}$ for some eigenvalue $\mu \in k^*$ on V. □

Proposition 6.2.4. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V' = \mathrm{S}^2(W)$. Then, for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim V_s'(\mu) \leq 2\ell^2 - 3\ell + 4,$$

where equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, -1, -1, 1, \ldots, 1)$.

Proof. Now, since $T = T_H$ and $\mathrm{Z}(G) = \mathrm{Z}(H)$, by Lemma 2.8.2 and by Proposition 4.2.4, we determine that $\dim(V_s(\mu)) \leq 2\ell^2 - 3\ell + 4$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Moreover, by the same result, we have equality if and only if $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, -1, -1, 1, \ldots, 1)$. □

Corollary 6.2.5. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V = L_G(2\omega_1)$. Then, for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim V_s(\mu) \leq 2\ell^2 - 3\ell + 3 - \varepsilon,$$

where $\varepsilon = 0$ if $p \nmid \ell$, and $\varepsilon = 1$ if $p \mid \ell$. Moreover, equality holds if and only if $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, -1, -1, 1, \ldots, 1)$.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Let $V' = \mathrm{S}^2(W)$. By Lemma 2.8.4, if $p \nmid \ell$, we have that $V' = V \oplus L_G(0)$, while, if $p \mid \ell$, then $V' = L_G(0) \mid V \mid L_G(0)$. Thus, $\dim(V) = \dim(V') - 1 - \varepsilon$, $\dim(V_s(\mu)) = \dim(V_s'(\mu))$, for all eigenvalues $\mu \neq 1$ of $s \in T \setminus \mathrm{Z}(G)$ on V, and $\dim(V_s(1)) = \dim(V_s'(1)) - 1 - \varepsilon$.

Let $s \in T \setminus \mathrm{Z}(G)$ and let $\mu \in k^*$ be an eigenvalue of s on V. If $\mu = 1$, then Proposition 6.2.4 gives the result. If $\mu \neq \mu^{-1}$, then, since, in particular, $s \in T_H \setminus \mathrm{Z}(H)$, we have $\dim(V_s(\mu)) \leq \ell^2$, by Inequality (4.2). Thus, since $\ell \geq 4$, we determine that $\dim(V_s(\mu)) \leq \ell^2 < 2\ell^2 - 3\ell + 3 - \varepsilon$, for all eigenvalues $\mu \neq \mu^{-1}$ of s on V. Therefore, to complete the proof, we only need to treat the case of $\mu = -1$.

Now, as $s \in T_H \setminus \mathrm{Z}(H)$, we have $s = \mathrm{diag}(\nu_1 \cdot \mathrm{I}_{m_1}, \ldots, \nu_t \cdot \mathrm{I}_{m_t}, \nu_t^{-1} \cdot \mathrm{I}_{m_t}, \ldots, \nu_1^{-1} \cdot \mathrm{I}_{m_1})$, where $\nu_i \neq \nu_j^{\pm 1}$ for all $1 \leq i < j \leq t$, $\sum_{i=1}^{t} m_i = \ell$ and $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$. If $\nu_i \nu_j \neq -1$ for all $1 \leq i < j \leq t$, then, by Inequality (4.5), we have that $\dim(V_s(-1)) \leq \ell^2 + \ell < 2\ell^2 - 3\ell + 3 - \varepsilon$, since $\ell \geq 4$. Similarly, if there exist $1 \leq i < j \leq t$ such that $\nu_i \nu_j = -1$, then, by Inequality (4.6), we have:

$$\dim(V_s(-1)) \leq 2\ell^2 + \ell - \sum_{r=1}^{t} m_r^2 - m_i(m_i + 1) - m_j(m_j + 1) - 2(m_i + m_j)(\ell - m_i - m_j).$$

Assume that $\dim(V_s(-1)) \geq 2\ell^2 - 3\ell + 3 - \varepsilon$. Then

$$4\ell - 3 + \varepsilon - \sum_{r=1}^{t} m_r^2 - m_i(m_i+1) - m_j(m_j+1) - 2(m_i+m_j)(\ell - m_i - m_j) \geq 0.$$

We proceed as in the proof for $V_s^{'}(-1)$, see (4.7), and arrive at

$$\ell(4 - m_i - m_j) - 3 + \varepsilon - \sum_{r \neq i,j} m_r^2 - (m_i - m_j)^2 - (m_i + m_j)(\ell + 1 - m_i - m_j) \geq 0. \quad (6.1)$$

As $\ell + 1 > m_i + m_j$, by (6.1), it follows that $m_i + m_j < 4$ and so, as $m_i \geq m_j$, we have $(m_i, m_j) \in \{(1,1),(2,1)\}$. If $(m_i, m_j) = (1,1)$, then, as $\ell \geq 4$, we have

$$-1 + \varepsilon - \sum_{r \neq i,j} m_r^2 \geq 0.$$

If $p \nmid \ell$, i.e. $\varepsilon = 0$, we see that the above inequality does not hold, while, if $p \mid \ell$, i.e. $\varepsilon = 1$, by the above, it follows that $t = 2$, contradicting $\ell \geq 4$. On the other hand, if $(m_i, m_j) = (2,1)$, then we have

$$-2\ell + 2 + \varepsilon - \sum_{r \neq i,j} m_r^2 \geq 0,$$

which clearly does not hold. Therefore, $\dim(V_s(-1)) < 2\ell^2 - 3\ell + 3 - \varepsilon$ for all $s \in T \setminus \mathrm{Z}(G)$.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 2\ell^2 - 3\ell + 3 - \varepsilon$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. As the inequality $0 < 14\ell^2 - 33\ell + 17 + \varepsilon$ holds for all $\ell \geq 4$, we have $2\ell^2 - 3\ell + 3 - \varepsilon < 2\ell^2 + \ell - 1 - \varepsilon - \sqrt{2\ell^2 + \ell - 1 - \varepsilon}$ for all $\ell \geq 4$, thus $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Corollary 6.2.6. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V = L_G(\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell^2 - 5\ell + 4,$$

where we have equality if and only if one of the following holds:

(1) *$\ell = 4$, $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq \pm 1$.*

(2) *$\ell = 4$, $\mu = -1$ and, up to conjugation, $s = \mathrm{diag}(1, 1, -1, -1, -1, -1, 1, 1)$.*

(3) *$\ell \geq 4$, $\mu = 1$ and, up to conjugation, $s = \pm\,\mathrm{diag}(1, \ldots, 1, \mu_2, \mu_2^{-1}, 1, \ldots, 1)$ with $\mu_2 \neq 1$.*

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. To begin, we remark that, as $p \neq 2$, by Lemma 2.8.4, we have $V \cong \wedge^2(W)$. Now, as $T = T_H$, the result follows by Proposition 4.2.5. □

Corollary 6.2.7. *Let k be an algebraically closed field of characteristic $p = 2$ and let $V = L_G(\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 2\ell^2 - 5\ell + 4 - \gcd(2, \ell),$$

where we have equality if and only if one of the following holds:

(1) *$\ell = 4$, $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_1, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1}, \mu_1^{-1})$ with $\mu_1 \neq 1$.*

(2) *$\ell \geq 4$, $\mu = 1$ and, up to conjugation, $s = \mathrm{diag}(1, \ldots, 1, \mu_2, \mu_2^{-1}, 1, \ldots, 1)$ with $\mu_2 \neq 1$.*

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $V' = \wedge^2(W)$. We first remark that, since $p = 2$, by Lemma 2.8.5, we have that either $V' \cong V \oplus L_G(0)$, or V' has three composition factors: one isomorphic to V and two isomorphic to $L_G(0)$, depending on whether $2 \nmid \ell$, or $2 \mid \ell$. Thus, we determine that $\dim(V) = \dim(V') - \gcd(2, \ell)$, $\dim(V_s(1)) = \dim(V'_s(1)) - \gcd(2, \ell)$ and $\dim(V_s(\mu)) = \dim(V'_s(\mu))$ for all eigenvalues $\mu \neq 1$ of $s \in T \setminus \mathrm{Z}(G)$ on V.

Now, let $s \in T \setminus \mathrm{Z}(G)$ and let $\mu \in k^*$ be an eigenvalue of s on V. Since, in particular, $s \in T_H \setminus \mathrm{Z}(H)$, by Proposition 4.2.5, we have $\dim(V_s(1)) \leq 2\ell^2 - 5\ell + 4 - \gcd(2, \ell)$, where equality holds if and only if s is as in the statement of the result. Having resolved the case of $\mu = 1$, we can now assume that the eigenvalue μ is such that $\mu \neq \mu^{-1}$. Then, we use Inequality (4.10) and the fact that $\ell \geq 4$, to determine that $\dim(V_s(\mu)) \leq \ell^2 - \ell < 2\ell^2 - 5\ell + 4 - \gcd(2, \ell)$.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 2\ell^2 - 5\ell + 4 - \gcd(2, \ell)$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. Therefore, as the inequality $0 < 14\ell^2 - 31\ell + 16 + \gcd(2, \ell)$ holds for all $\ell \geq 4$, it follows that $2\ell^2 - 5\ell + 4 - \gcd(2, \ell) < 2\ell^2 - \ell - \gcd(2, \ell) - \sqrt{2\ell^2 - \ell - \gcd(2, \ell)}$ and so $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

To conclude this subsection, we remark that Lemma 6.2.3 and Corollaries 6.2.5, 6.2.6 and 6.2.7 give the proof of Theorems 6.2.1 and 6.2.2 for the families of kG-modules corresponding to p-restricted dominant weights $\lambda \in \{\omega_1, \omega_2, 2\omega_1\}$. Therefore, in view of Remark 6.1.1, they prove Theorems 6.2.1 and 6.2.2 for the families of $k\tilde{G}$-modules with p-restricted dominant weights $\tilde{\lambda} \in F^{B_\ell}$.

6.2.2 The particular modules

As previously mentioned, in this subsection we will give an upper-bound for $\dim(V_s(\mu))$, where $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ and V is an irreducible $k\tilde{G}$-module with associated highest weight featured in Table 2.7.5. In order to achieve our goal, we will use the inductive algorithm of Subsection 2.4.3. To begin, let L_1 be the Levi subgroup of the maximal parabolic subgroup P_1 of $\tilde{G}$ constructed in Section 2.4. We recall that $L_1 = \mathrm{Z}(L_1)^\circ[L_1, L_1]$, where $\mathrm{Z}(L_1)^\circ$ is a one-dimensional torus and $[L_1, L_1]$ is a simply connected linear algebraic group of type $D_{\ell-1}$ with maximal torus $T' = \tilde{T} \cap [L_1, L_1]$. We note that, although we do not mention the result

explicitly, we make great use of the data in [Lü01b] when discussing weights and weight multiplicities in this subsection.

Let $s \in \tilde{T}$. Then $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. As $z \in \mathrm{Z}(L_1)^\circ$, we have $z = \prod_{i=1}^{\ell} h_{\tilde{\alpha}_i}(c^{k_i})$, where $c \in k^*$ and $k_i \in \mathbb{Z}$ for all $1 \leq i \leq \ell$. Moreover, as $\tilde{\alpha}_j(z(c)) = 1$ for all $2 \leq j \leq \ell$, it follows that $z = \left(\prod_{i=1}^{\ell-2} h_{\tilde{\alpha}_i}(c^2)\right) h_{\tilde{\alpha}_{\ell-1}}(c) h_{\tilde{\alpha}_\ell}(c)$ with $c \in k^*$. As $h \in [L_1, L_1]$, we have $h = \prod_{i=2}^{\ell} h_{\tilde{\alpha}_i}(a_i)$ with $a_i \in k^*$ for all $2 \leq i \leq \ell$, and therefore $s = h_{\tilde{\alpha}_1}(c^2)\left(\prod_{i=2}^{\ell-2} h_{\tilde{\alpha}_i}(c^2 a_i)\right) h_{\tilde{\alpha}_{\ell-1}}(ca_{\ell-1}) h_{\tilde{\alpha}_\ell}(ca_\ell)$ with $c \in k^*$ and $a_i \in k^*$ for all $2 \leq i \leq \ell$.

Let V be an irreducible $k\tilde{G}$-module with p-restricted dominant highest weight $\tilde{\lambda} \in \mathrm{X}(\tilde{T})$, $\tilde{\lambda} = \sum_{i=1}^{\ell} d_i \tilde{\omega}_i$ with $0 \leq d_i \leq p-1$ for all $1 \leq i \leq \ell$. We consider the decomposition:

$$V\mid_{[L_1,L_1]} = \bigoplus_{i=0}^{e_1(\tilde{\lambda})} V^i,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma}}$ for all $0 \leq i \leq e_1(\tilde{\lambda})$. Let $s \in \tilde{T}$ and write $s = z \cdot h$, as above. Then, by (2.5), we have:

$$s_z^i := (\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma})(z) = (\tilde{\lambda} - i\tilde{\alpha}_1)\left[\left(\prod_{j=1}^{\ell-2} h_{\tilde{\alpha}_j}(c^2)\right) h_{\tilde{\alpha}_{\ell-1}}(c) h_{\tilde{\alpha}_\ell}(c)\right] = \left(\prod_{j=1}^{\ell-2} c^{2d_j}\right) \cdot c^{d_{\ell-1}+d_\ell} \cdot c^{-2i}.$$

Therefore, z acts on V^i, $0 \leq i \leq e_1(\tilde{\lambda})$, as the scalar $s_z^i = c^{2d_1 + \cdots + 2d_{\ell-2} + d_{\ell-1} + d_\ell - 2i}$. Now, let $\mu_1^i, \ldots, \mu_{t_i}^i$, $t_i \geq 1$, be the distinct eigenvalues of h on V^i, $0 \leq i \leq e_1(\tilde{\lambda})$, and let $n_1^i, \ldots, n_{t_i}^i$ be their respective multiplicities. Then, by Lemma 2.4.8, it follows that the distinct eigenvalues of s on V^i are $s_z^i \mu_1^i, \ldots, s_z^i \mu_{t_i}^i$, with respective multiplicities $n_1^i, \ldots, n_{t_i}^i$.

Proposition 6.2.8. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 4$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3 + \tilde{\omega}_4)$. Then for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V we have*

$$\dim(V_{\tilde{s}}(\tilde{\mu})) \leq 28.$$

In particular, we have $\dim(V_{\tilde{s}}(\tilde{\mu})) < \dim(V) - \sqrt{\dim(V)}$ for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V.

Proof. First, we consider the weight $\lambda = \omega_3 + \omega_4 \in \mathrm{X}(T)$ and its associated irreducible kG-module $L_G(\lambda)$. We have seen in Subsection 2.3.3 that $L_G(\lambda)$ is also a simple $k\tilde{G}$-module and, as a $k\tilde{G}$-module, it is isomorphic to $L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda} = \tilde{\omega}_3 + \tilde{\omega}_4$ denotes λ when viewed as a weight in $\mathrm{X}(\tilde{T})$. Moreover, by Lemma 2.3.10, we have

$$\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{s}}(\tilde{\mu})) \leq \max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$$

for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V.

Secondly, let $H = \mathrm{Sp}(W, a)$, where a is the nondegenerate alternating bilinear form on W given by $a(w_1, w_2) = Q(w_1 + w_2) + Q(w_1) + Q(w_2)$, for all $w_1, w_2 \in W$. Note that H is a simple simply connected linear algebraic group of type C_4. Let T_H denote the maximal torus in H obtained by intersecting the set of diagonal matrices in $\mathrm{SL}(W)$ with H. Now, consider the irreducible kH-module $L_H(\omega_3^H)$ of highest weight ω_3^H, where ω_3^H is the fundamental dominant weight of H corresponding to the simple root α_3^H. As $p = 2$, G is a subgroup in H, and, by [Sei87, Table 1], the following isomorphism of kG-modules holds:

$$L_H(\omega_3^H) \mid_G \cong L_G(\omega_3 + \omega_4).$$

In particular, for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on $L_G(\omega_3 + \omega_4)$, we have

$$\dim((L_G(\omega_3 + \omega_4))_s(\mu)) \leq \max_{s_H \in T_H \setminus \mathrm{Z}(H)} \{\dim((L_H(\omega_3^H))_{s_H}(\mu_H)) \mid \mu_H \in k^*\}.$$

Now, $\max_{s_H \in T_H \setminus \mathrm{Z}(H)} \{\dim((L_H(\omega_3^H))_{s_H}(\mu_H)) \mid \mu_H \in k^*\} \leq 28$, by the second to last paragraph of the proof of Proposition 4.2.20. We determine that $\dim((L_G(\omega_3 + \omega_4))_s(\mu)) \leq 28$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on $L_G(\omega_3 + \omega_4)$. This gives $\dim(V_{\tilde{s}}(\tilde{\mu})) \leq 28$ for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V.

In conclusion, we have shown that $\dim(V_{\tilde{s}}(\tilde{\mu})) \leq 28 < \dim(V) - \sqrt{\dim(V)}$ for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V. □

Proposition 6.2.9. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 4$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3 + \tilde{\omega}_4)$. Then for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 34.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues μ of s on V.

Proof. Set $\tilde{\lambda} = \tilde{\omega}_3 + \tilde{\omega}_4$. Then, as $p \neq 2$, we have $\dim(V) = 56$ and, by Lemma 2.4.7, we have $e_1(\tilde{\lambda}) = 2$, therefore:

$$V \mid_{[L_1, L_1]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma}}$ for $i = 0, 1$ and $i = 2$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4)$ and so, by Lemma 2.4.3, we also have $V^2 \cong (L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4))^* \cong L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4)$. This gives $\dim(V^1) = 26$. Now, in V^1, both the weight $(\tilde{\lambda} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3) \mid_{T'} = 2\tilde{\omega}_4$ and the weight $(\tilde{\lambda} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_4) \mid_{T'} = 2\tilde{\omega}_3$ admit a maximal vector. Therefore V^1 has at least two compositions factors: one isomorphic to $L_{L_1}(2\tilde{\omega}_3)$ and one isomorphic to $L_{L_1}(2\tilde{\omega}_4)$. Moreover, since $p \neq 2$, the dominant weight $(\tilde{\lambda} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3 - \tilde{\alpha}_4) \mid_{T'} = \tilde{\omega}_2$, occurring with multiplicity 3 in V^1, is a sub-dominant weight in both the composition factor of V^1 isomorphic to $L_{L_1}(2\tilde{\omega}_3)$ and the one isomorphic to $L_{L_1}(2\tilde{\omega}_4)$, and it has multiplicity 1 in each. By comparing dimensions, we deduce that V^1 admits exactly three composition factors: the two previously mentioned and a third isomorphic to $L_{L_1}(\tilde{\omega}_2)$. Moreover, since $p \neq 2$, we use [Jan07, II.2.14] to determine that $V^1 \cong L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(2\tilde{\omega}_4) \oplus L_{L_1}(\tilde{\omega}_2)$, and so

$$V \mid_{[L_1, L_1]} = L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4) \oplus L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(2\tilde{\omega}_4) \oplus L_{L_1}(\tilde{\omega}_2) \oplus L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4). \tag{6.2}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$ and so $s = z$ with $c^2 \neq 1$. In this case, as s acts on each V^i, $0 \leq i \leq 2$, as the scalar c^{2-2i}, it follows that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^0) = 15; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1) = 26; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^2) = 15. \end{cases}$$

Since $c^2 \neq 1$, we have $\dim(V_s(\mu)) \leq 30$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. Recall that $V^0 \cong L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4)$, $V^2 \cong L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4)$ and $V^1 \cong L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(2\tilde{\omega}_4) \oplus L_{L_1}(\tilde{\omega}_2)$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1, L_1]$. Since z acts by scalar multiplication on each V^i, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where $\mu_h \in k^*$ is any eigenvalue of h on V^i. Now, by Corollary 3.2.7, as $p \neq 2$, we have $\dim(V_h^0(\mu_h)) \leq 9$ for all eigenvalues μ_h of h on V^0. Assume that $\dim(V_h^0(\mu_h)) = 9$. Then, by Corollary 3.2.7, we have $\mu_h = 1$ and $h = h_{\alpha_2}(\mu_1)h_{\alpha_3}(\mu_1^2)h_{\alpha_4}(\mu_1^3)$ with $\mu_1^4 \neq 1$. We will now determine the eigenvalues of h on V^1. Using (3.1), we determine that the eigenvalues of h, not necessarily distinct, on the composition factor of V^1 isomorphic to $L_{L_1}(2\tilde{\omega}_3)$ are μ_1^2 with multiplicity at least 6, μ_1^{-6} with multiplicity at least 1, and μ_1^{-2} with multiplicity at least 3. Therefore, as $L_{L_1}(2\tilde{\omega}_4) \cong (L_{L_1}(2\tilde{\omega}_3))^*$, it follows that the eigenvalues of h, not necessarily distinct, on the composition factor of V^1 isomorphic to $L_{L_1}(2\tilde{\omega}_4)$ are μ_1^{-2} with multiplicity at least 6, μ_1^6 with multiplicity at least 1, and μ_1^2 with multiplicity at least 3. Lastly, by (3.7), we determine that the distinct eigenvalues of h, as $\mu_1^4 \neq 1$, on the composition factor of V^1 isomorphic to $L_{L_1}(\tilde{\omega}_2)$ are μ_1^2 and μ_1^{-2}, both with multiplicity 3. Since $\mu_1^4 \neq 1$, it follows that $\dim(V_h^1(\nu_h)) \leq 14$ for all eigenvalues ν_h of h on V^1. Therefore, as $\dim(V_h^2(\nu_h)) \leq 9$ for all eigenvalues ν_h of h on V^2, see Corollary 3.2.7, it follows that $\dim(V_h(\nu_h)) \leq 32$ for all eigenvalues ν_h of h on V, thereby $\dim(V_s(\mu)) \leq 32$ for all eigenvalues $\mu \in k^*$ of s on V. We now consider the case when $\dim(V_h^0(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^0. Then, $\dim(V_h^2(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^2, as $V^2 \cong (V^0)^*$. Lastly, since $p \neq 2$, by Propositions 3.2.4 and 3.2.5, it follows that $\dim(V_h^1(\mu_h)) \leq 18$ for all eigenvalues μ_h of h on V^1. Therefore, $\dim(V_h(\mu_h)) \leq 34$ for all eigenvalues μ_h of h on V and so $\dim(V_s(\mu)) \leq 34$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 34 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 6.2.10. *Assume $\ell \geq 5$ and let $V = L_{\tilde{G}}(\tilde{\omega}_{\ell-1})$. Then for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 5 \cdot 2^{\ell-4}.$$

In particular, for all $\ell \geq 5$, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $\tilde{\lambda} = \tilde{\omega}_{\ell-1}$. We have $\dim(V) = 2^{\ell-1}$ and, by Lemma 2.4.7, we have $e_1(\tilde{\lambda}) = 1$, therefore:

$$V \mid_{[L_1, L_1]} = V^0 \oplus V^1,$$

where $V^i = \bigoplus_{\tilde{\gamma}\in\mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda}-i\tilde{\alpha}_1-\tilde{\gamma}}$ for $i=0$ and $i=1$. By [Smi82, Proposition], it follows that $V^0 \cong L_{L_1}(\omega_{\ell-1})$. Therefore, we have $\dim(V^1) = 2^{\ell-2}$. Now, since the weight $(\tilde{\lambda}-\tilde{\alpha}_1-\cdots-\tilde{\alpha}_{\ell-1})\mid_{T'} = \tilde{\omega}_\ell$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_{L_1}(\tilde{\omega}_\ell)$. We deduce $V^1 \cong L_{L_1}(\tilde{\omega}_\ell)$ and:

$$V\mid_{[L_1,L_1]} \cong L_{L_1}(\tilde{\omega}_{\ell-1}) \oplus L_{L_1}(\tilde{\omega}_\ell). \tag{6.3}$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i=0$ or $i=1$, then $s \in \mathrm{Z}(L_1)^\circ \setminus \mathrm{Z}(\tilde{G})$ and so $s = z$ with $c^2 \neq 1$. In this case, as s acts as scalar multiplication by c^{1-2i} on V^i, $i=0,1$, it follows that the distinct eigenvalues of s on V are:

$$\begin{cases} c \text{ with } \dim(V_s(c)) = \dim(V^0) = 2^{\ell-2}; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) = \dim(V^1) = 2^{\ell-2}. \end{cases}$$

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i=0$ and $i=1$. We write $s = z\cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1,L_1]$. Since z acts by scalar multiplication on each V^i, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for $i=0,1$, where $\mu_h \in k^*$ is any eigenvalue of h on V^i.

First, consider the case of $\ell = 5$. Let $\sigma' : \tilde{\Delta}_1 \to \tilde{\Delta}_1$ be the triality graph automorphism of $\tilde{\Delta}_1 = \{\tilde{\alpha}_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\tilde{\alpha}_5\}$ given by: $\tilde{\alpha}_2 \to \tilde{\alpha}_4$, $\tilde{\alpha}_3 \to \tilde{\alpha}_3$, $\tilde{\alpha}_4 \to \tilde{\alpha}_5$ and $\tilde{\alpha}_5 \to \tilde{\alpha}_2$. Then, by [Ste16, Corollary (b) of Theorem 29] and [Car89, Lemma 6.4.4 (ii),(iii)], there exists an automorphism $\sigma : [L_1,L_1] \to [L_1,L_1]$ such that $\sigma(h_{\tilde{\alpha}_i}(c)) = h_{\sigma'(\tilde{\alpha}_i)}(c)$, where $2 \le i \le 5$ and $c \in k^*$. Now, since $\tilde{\omega}_4 = \sigma'(\tilde{\omega}_2)$ and $\tilde{\omega}_5 = \sigma'(\tilde{\omega}_4)$, by Lemma 6.2.3, it follows that $\dim((L_{L_1}(\tilde{\omega}_4))_h(\mu_h)) \le 6$ and $\dim((L_{L_1}(\tilde{\omega}_5))_h(\mu_h)) \le 6$ for all eigenvalues μ_h of h on $L_{L_1}(\tilde{\omega}_4)$ and $L_{L_1}(\tilde{\omega}_5)$, respectively. Therefore, as $V^0 \cong L_{L_1}(\tilde{\omega}_4)$ and $V^1 \cong L_{L_1}(\tilde{\omega}_5)$, we determine that $\dim(V_h(\mu_h)) \le 12$ for all eigenvalues μ_h of h on V, hence $\dim(V_s(\mu)) \le 12$ for all eigenvalues $\mu \in k^*$ of s on V. However, we will show that $\dim(V_s(\mu)) \le 10$ for all eigenvalues $\mu \in k^*$ of s on V.

Assume there exist $(s,\mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which $\dim(V_s(\mu)) = 12$. We write $s = z\cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h = h_{\tilde{\alpha}_2}(a_2)h_{\tilde{\alpha}_3}(a_3)h_{\tilde{\alpha}_4}(a_4)h_{\tilde{\alpha}_5}(a_5) \in [L_1,L_1]$ with $a_j \in k^*$, $2 \le j \le 5$. Note that we have $\dim(V_h^i(\nu_h^i)) < \dim(V^i)$ for both $i=0$ and $i=1$ and all eigenvalues ν_h^i of h on V^i. Let μ_h^i, $i=0,1$, be the eigenvalue of h on V^i with the property that $\mu = c^{1-2i}\mu_h^i$. Since $\dim(V_s(\mu)) = 12$, we have $\dim(V_h^i(\mu_h^i)) = 6$ for both $i=0$ and $i=1$. Moreover, using Lemma 6.2.3, one can check that for any $h' \in [L_1,L_1]$, we have $\dim((L_{L_1}(\tilde{\omega}_2))_{h'}(\nu_{h'})) = 6$ if and only if either $\nu_{h'} = 1$ and, up to conjugacy, $h' = h_{\tilde{\alpha}_4}(a_4)h_{\tilde{\alpha}_5}(a_4^{-1})$ with $a_4^2 \neq 1$; or $\nu_{h'} = -1$ and, up to conjugacy, $h' = h_{\tilde{\alpha}_2}(-1)h_{\tilde{\alpha}_4}(a_4)h_{\tilde{\alpha}_5}(-a_4^{-1})$ with $a_4^2 \neq 1$. Therefore, $\dim(V_h^0(\mu_h^i)) = 6$ if and only if $h = \sigma'(h')$ and $\mu_h^i = \nu_{h'}$, i.e. $\dim(V_h^0(\mu_h^i)) = 6$ if and only if either $h = h_{\tilde{\alpha}_2}(a_4^{-1})h_{\tilde{\alpha}_5}(a_4)$, with $a_4^2 \neq 1$, and $\mu_h^i = 1$; or $h = h_{\tilde{\alpha}_2}(-a_4^{-1})h_{\tilde{\alpha}_4}(-1)h_{\tilde{\alpha}_5}(a_4)$, with $a_4^2 \neq 1$, and $\mu_h^i = -1$. However, since the weights in $L_{L_1}(\tilde{\omega}_5)$ are $\tilde{\omega}_5$, $\tilde{\omega}_5-\tilde{\alpha}_5$, $\tilde{\omega}_5-\tilde{\alpha}_3-\tilde{\alpha}_5$, $\tilde{\omega}_5-\tilde{\alpha}_2-\tilde{\alpha}_3-\tilde{\alpha}_5$, $\tilde{\omega}_5-\tilde{\alpha}_3-\tilde{\alpha}_4-\tilde{\alpha}_5$, $\tilde{\omega}_5-\tilde{\alpha}_2-\tilde{\alpha}_3-\tilde{\alpha}_4-\tilde{\alpha}_5$, $\tilde{\omega}_5-\tilde{\alpha}_2-2\tilde{\alpha}_3-\tilde{\alpha}_4-\tilde{\alpha}_5$ and $\tilde{\omega}_5-\tilde{\alpha}_2-2\tilde{\alpha}_3-\tilde{\alpha}_4-2\tilde{\alpha}_5$, we determine that, in both cases, the distinct eigenvalues of h on V^1 are a_4 and a_4^{-1}, each with multiplicity 4. We have arrived at a contradiction.

Similarly, assume there exist $(s,\mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$ for which $\dim(V_s(\mu)) = 11$. We write $s = z\cdot h$, where $z \in \mathrm{Z}(L_1)^\circ$ and $h \in [L_1,L_1]$. As in the previous case, we have that

$\dim(V_h^i(\nu_h^i)) < \dim(V^i)$, $i = 0, 1$, for all eigenvalues ν_h^i of h on V^i. Let μ_h^i, $i = 0, 1$, be the eigenvalue of h on V^i with the property that $\mu = c^{1-2i}\mu_h^i$. Now, since $\dim(V_s(\mu)) = 11$, it follows that either $\dim(V_h^0(\mu_h^0)) = 6$ and $\dim(V_h^1(\mu_h^1)) = 5$, or $\dim(V_h^0(\mu_h^0)) = 5$ and $\dim(V_h^1(\mu_h^1)) = 6$. We have seen that if $\dim(V_h^0(\mu_h^0)) = 6$, then $\dim(V_s(\nu)) \leq 10$ for all eigenvalues ν of s on V. Therefore, we must have $\dim(V_h^0(\mu_h^0)) = 5$ and $\dim(V_h^1(\mu_h^1)) = 6$. However, if $\dim(V_h^1(\mu_h^1)) = 6$, then, arguing exactly as in the case of $\dim(V_h^0(\mu_h^0)) = 6$, one shows that $\dim(V_s(\nu)) \leq 10$ for all eigenvalues ν of s on V. Once more, we have arrived at a contradiction.

In the case of $\ell = 5$, we have shown that $\dim(V_s(\mu)) \leq 10$ for all $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$. We now consider the case of $\ell \geq 6$. By recurrence and using the result for $\ell = 5$ as base case, one shows that $\dim(V_h^0(\mu_h)) \leq 5 \cdot 2^{\ell-5}$ and $\dim(V_h^1(\mu_h)) \leq 5 \cdot 2^{\ell-5}$ for all eigenvalues μ_h of h on V^0 and V^1, respectively. Therefore, $\dim(V_h(\mu_h)) \leq 5 \cdot 2^{\ell-4}$ for all eigenvalues μ_h of h on V and thus $\dim(V_s(\mu)) \leq 5 \cdot 2^{\ell-4}$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 5 \cdot 2^{\ell-4}$ for all $s \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\mu \in k^*$ of s on V. Moreover, as the inequality $1 < 9 \cdot 2^{\ell-7}$ holds for all $\ell \geq 5$, it follows that $5 \cdot 2^{\ell-4} < 2^{\ell-1} - \sqrt{2^{\ell-1}}$ for all $\ell \geq 5$, and so $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $(s, \mu) \in \tilde{T} \setminus \mathrm{Z}(\tilde{G}) \times k^*$. □

Proposition 6.2.11. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 5$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3)$. Then for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V we have*

$$\dim(V_{\tilde{s}}(\tilde{\mu})) \leq 58.$$

In particular, $\dim(V_{\tilde{s}}(\tilde{\mu})) < \dim(V) - \sqrt{\dim(V)}$ for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V.

Proof. First, we consider the weight $\omega_3 \in \mathrm{X}(T)$ and its associated irreducible kG-module $L_G(\omega_3)$. We have seen in Subsection 2.3.3 that $L_G(\omega_3)$ is also a simple $k\tilde{G}$-module and, as a $k\tilde{G}$-module, it is isomorphic to $L_{\tilde{G}}(\tilde{\omega}_3)$, where $\tilde{\omega}_3 \in \mathrm{X}(\tilde{T})$. Moreover, by Lemma 2.3.10, for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$, we have

$$\dim((L_{\tilde{G}}(\tilde{\omega}_3))_{\tilde{s}}(\tilde{\mu})) \leq \max_{s \in T \setminus \mathrm{Z}(G)} \{\dim((L_G(\omega_3))_s(\mu)) \mid \mu \in k^*\}.$$

Secondly, let $H = \mathrm{Sp}(W, a)$, where a is the nondegenerate alternating bilinear form on W given by $a(w_1, w_2) = Q(w_1 + w_2) + Q(w_1) + Q(w_2)$ for all $w_1, w_2 \in W$. Note that H is a simple simply connected linear algebraic group of type C_5. Let T_H denote the maximal torus in H obtained by intersecting the set of diagonal matrices in $\mathrm{SL}(W)$ with H. Now, consider the irreducible kH-module $L_H(\omega_3^H)$ of highest weight ω_3^H, where ω_3^H is the fundamental dominant weight of H corresponding to the simple root α_3^H. As $p = 2$, G is a subgroup in H, and, by [Sei87, Table 1], we have the following isomorphism of kG-modules:

$$L_H(\omega_3^H) \mid_G \cong L_G(\omega_3).$$

In particular, for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on $L_G(\omega_3)$, we have

$$\dim((L_G(\omega_3))_s(\mu)) \leq \max_{s_H \in T_H \setminus \mathrm{Z}(H)} \{\dim((L_H(\omega_3^H))_{s_H}(\mu_H)) \mid \mu_H \in k^*\}$$

and, as $\max_{s_H \in T_H \setminus \mathrm{Z}(H)} \{\dim((L_H(\omega_3^H))_{s_H}(\mu_H)) \mid \mu_H \in k^*\} \leq 58$, by Proposition 4.2.22, we determine that $\dim((L_G(\omega_3))_s(\mu)) \leq 58$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on $L_G(\omega_3)$. This gives $\dim(V_{\tilde{s}}(\tilde{\mu})) \leq 58$ for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V.

In conclusion, we have shown that $\dim(V_{\tilde{s}}(\tilde{\mu})) \leq 58 < \dim(V) - \sqrt{\dim(V)}$ for all $\tilde{s} \in \tilde{T} \setminus \mathrm{Z}(\tilde{G})$ and all eigenvalues $\tilde{\mu} \in k^*$ of $\tilde{s}$ on V. □

We conclude this subsection by noting that Propositions 6.2.8 through 6.2.11 complete the proofs of Theorems 6.2.1 and 6.2.2, as they cover all the particular $k\tilde{G}$-modules we had to consider, i.e. all the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with p-restricted dominant highest weight $\tilde{\lambda}$ listed in Table 2.7.5.

6.3 Eigenspace dimensions for unipotent elements

This section is dedicated to the proofs of the following two theorems, analogs of Theorems 6.2.1 and 6.2.2 in the case of the unipotent elements. Similar to the semisimple case, the proofs will be given in a series of results, each treating one of the candidate-modules. In Subsection 6.3.1, we determine $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, see Remark 6.1.1, where V belongs to one of the families of modules, i.e. V is an irreducible kG-module $L_G(\lambda)$ with $\lambda \in \{\omega_1, 2\omega_1, \omega_2\}$. We complete the proofs of these two results in Subsection 6.3.2, where we establish $\max_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ for the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with p-restricted dominant weight $\tilde{\lambda}$ listed in Table 2.7.5.

Theorem 6.3.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ, $\ell \geq 4$. Let $\tilde{T}$ be a fixed maximal torus in $\tilde{G}$ and let $V = L_{\tilde{G}}(\tilde{\lambda})$, where either $\tilde{\lambda} \in F^{D_\ell}$, or $\tilde{\lambda}$ is listed in Table 2.7.5. Then there exist non-identity unipotent elements $u \in \tilde{G}$ for which*

$$\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$$

if and only if ℓ and $\tilde{\lambda}$ appear in the following list:

(1) *$\ell \geq 4$ and $\tilde{\lambda} = \tilde{\omega}_1$;*

(2) *$\ell = 5$ and $V = L_{\tilde{G}}(\tilde{\omega}_{\ell-1})$.*

Theorem 6.3.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ, $\ell \geq 4$. Let $\tilde{T}$ and V be as in Theorem 6.3.1. Then the value of $\max_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ is given in the table below:*

V	Char.	Rank	$\max\limits_{u\in\tilde{G}_u\setminus\{1\}} \dim(V_u(1))$
$L_{\tilde{G}}(\tilde{\omega}_1)$	$p \geq 0$	$\ell \geq 4$	$2\ell - 2$
${}^{\dagger}L_{\tilde{G}}(\tilde{\omega}_2)$	$p \neq 2$	$\ell \geq 4$	$2\ell^2 - 5\ell + 6$
	$p = 2$	$\ell \geq 4$	$2\ell^2 - 5\ell + 6 - \gcd(2, \ell)$
${}^{\dagger}L_{\tilde{G}}(2\tilde{\omega}_1)$	$p \neq 2$ and $p \nmid \ell$	$\ell \geq 4$	$2\ell^2 - 3\ell + 1$
	$p \neq 2$ and $p \mid \ell$	$\ell \geq 4$	$2\ell^2 - 3\ell$
${}^{\dagger}L_{\tilde{G}}(\tilde{\omega}_3 + \tilde{\omega}_4)$	$p \geq 0$	$\ell = 4$	$34 - 6\delta_{p,2}$
${}^{\dagger}L_{\tilde{G}}(\tilde{\omega}_3)$	$p = 2$	$\ell = 5$	60
${}^{\ddagger}L_{\tilde{G}}(\tilde{\omega}_{\ell-1})$	$p \geq 0$	$5 \leq \ell \leq 9$	$3 \cdot 2^{\ell-3}$

Table 6.3.1: The value of $\max\limits_{u\in\tilde{G}_u\setminus\{1\}} \dim(V_u(1))$.

In particular, for each V in Table 6.3.1 labeled as ${}^{\dagger}V$, respectively as ${}^{\ddagger}V$ with $\ell \geq 6$, we have that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$.

6.3.1 The families of modules

Before we begin, we recall that over fields of characteristic $p \neq 2$, the Jordan normal form $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$ on W of a unipotent element $u \in G$ completely determines the unipotent conjugacy class of u in G, unless n_i is even for all i, in which case there exist two classes corresponding to that Jordan form. Thus, when the field k has characteristic $p \neq 2$, we fix the following hypothesis on unipotent elements $u \in G$:

$({}^{\dagger}H_u)$: every $u \in G_u \setminus \{1\}$ has Jordan normal form on W given by $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$, where

$$\sum_{i=1}^{m} n_i r_i = 2\ell,\ r_i \geq 1 \text{ is even for all even } n_i,\ 2\ell - 1 \geq n_1 > n_2 > \cdots > n_m \geq 1$$

and $n_1 > 1$.

Lemma 6.3.3. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V = L_G(\omega_1)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 2\ell - 2,$$

where equality holds if and only if the Jordan form of u on W is one of $J_2^2 \oplus J_1^{2\ell-4}$ and $J_3 \oplus J_1^{2\ell-3}$.

In particular, there exist non-identity unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. To begin, we note that $V \cong W$ as kG-modules. Now, let u be a unipotent element of G as in hypothesis $({}^{\dagger}H_u)$. Let u_W denote the action of u on W. Then:

$$\dim(V_u(1)) = \dim(W_{u_W}(1)) = \sum_{i=1}^{m} r_i = 2\ell - \sum_{i=1}^{m} (n_i - 1) r_i. \tag{6.4}$$

Assume that $\dim(V_u(1)) \geq 2\ell - 2$. Then, by (6.4), it follows that

$$2 \geq \sum_{i=1}^{m} (n_i - 1) r_i \tag{6.5}$$

and, in particular, that $2 \geq (n_1 - 1)r_1 \geq n_1 - 1$, hence $3 \geq n_1$.

If $n_1 = 3$, then by (6.5), we have $r_1 = 1$ and $\sum_{i=2}^{m} (n_i - 1) r_i = 0$, hence $2 \geq m$. Since $\ell \geq 4$, we deduce that $m = 2$, $n_2 = 1$ and $r_2 = 2\ell - 3$. Thus, u has Jordan form $J_3 \oplus J_1^{2l-3}$ on W. Conversely, let u be a unipotent element of G whose Jordan form on W is $J_3 \oplus J_1^{2\ell-3}$. Then, by (6.4), we have $\dim(V_u(1)) = 2\ell - 2$.

Similarly, if $n_1 = 2$, then r_1 is even and, by (6.5), it follows that $r_1 = 2$ and $\sum_{i=2}^{m} (n_i - 1) r_i = 0$. We argue as before to deduce that the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$. Conversely, let u be a unipotent element of G whose Jordan form on W is $J_2^2 \oplus J_1^{2\ell-4}$. Then, by (6.4), we have $\dim(V_u(1)) = 2\ell - 2$.

We conclude that $\dim(V_u(1)) \leq 2\ell - 2$ for all non-identity unipotent elements $u \in \tilde{G}$. Moreover, we have shown that equality holds if and only if the Jordan form of u on W is one of $J_2^2 \oplus J_1^{2\ell-4}$ and $J_3 \oplus J_1^{2\ell-3}$. Lastly, let u be such an element of G. Then, since the inequality $\sqrt{2\ell} \geq 2$ holds for all $\ell \geq 4$, it follows that $2\ell - 2 \geq 2\ell - \sqrt{2\ell}$ for all $\ell \geq 4$ and, consequently, $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. □

Lemma 6.3.4. *Let k be an algebraically closed field of characteristic $p = 2$ and let $V = L_G(\omega_1)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 2\ell - 2,$$

where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$, i.e. the Hesselink normal form of u is one of $(2_0^2, 1_0^{2\ell-4})$ and $(1_0^{2\ell-4}, 2_1^2)$.

In particular, there exist non-identity unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$.

Proof. First, we note that $V \cong W$ as kG-modules. Secondly, as $p = 2$, we have that $G < \mathrm{Sp}(W, a)$, where a is the nondegenerate alternating bilinear form on W given by $a(w_1, w_2) = Q(w_1 + w_2) + Q(w_1) + Q(w_2)$ for all $w_1, w_2 \in W$. Thus, in particular, we have $u \in \mathrm{Sp}(W, a)$ and, by Lemma 4.3.3, we determine that $\dim(V_u(1)) \leq 2\ell - 1$ for all non-identity unipotent elements $u \in \tilde{G}$.

Let $u \in \mathrm{Sp}(W, a)$ be a unipotent element whose Jordan form on W is $J_2 \oplus J_1^{2\ell-2}$. Then the Hesselink normal form of u is $(1_0^{2\ell-2}, 2_1)$ and, by Proposition 2.9.20, we deduce that $u \notin G$. Consequently, by Corollary 4.3.4, we determine $\dim(V_u(1)) \leq 2\ell - 2$ for all non-identity unipotent elements $u \in \tilde{G}$.

Now, let $u \in \mathrm{Sp}(W, a)$ be a unipotent element whose Jordan form on W is $J_2^2 \oplus J_1^{2\ell-4}$. Then the Hesselink normal form of u is one of $(2_0^2, 1_0^{2\ell-4})$ and $(1_0^{2\ell-4}, 2_1^2)$. In both cases we use Proposition 2.9.20 to determine that $u \in G$. We conclude, by Corollary 4.3.4, that $\dim(V_u(1)) \leq 2\ell - 2$ for all non-identity unipotent elements $u \in \tilde{G}$, where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$.

Lastly, as in the proof of Lemma 6.3.3, there exist non-identity unipotent elements $u \in \tilde{G}$, for example those whose Jordan form on W is given by $J_2^2 \oplus J_1^{2\ell-4}$, for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. $\square$

Proposition 6.3.5. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V = L_G(\omega_2)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 6,$$

where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$.

Proof. First, note that, by Lemma 2.8.4, as $p \neq 2$, we have the following isomorphism of kG-modules: $V \cong \wedge^2(W)$. Now, let u be a unipotent element of G as in hypothesis $({}^\dagger H_u)$. As r_i is even for all even n_i, it follows that the Jordan form of u on W consists of at least two blocks. We first consider the case when exactly one of these blocks, J_{n_1}, is nontrivial. Then the Jordan form of u on W is $J_{n_1} \oplus J_1^{2\ell-n_1}$, where, since $r_1 = 1$, n_1 is odd, thus $3 \leq n_1 \leq 2\ell - 1$. We write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = 2\ell - n_1$ and u acts trivially on W_2. Then, as $k[u]$-modules, we have

$$V \cong \wedge^2(W_1) \oplus (W_1 \otimes W_2) \oplus \wedge^2(W_2)$$

and so

$$\dim(V_u(1)) = \dim((\wedge^2(W_1))_u(1)) + \dim((W_1 \otimes W_2)_u(1)) + \dim((\wedge^2(W_2))_u(1)). \tag{6.6}$$

Now, by Lemma 2.9.4, as $p \neq 2$, we have $\dim((\wedge^2(W_1))_u(1)) = \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 - 1}{2}$ and, moreover, as u acts as $J_{n_1} \otimes J_1^{2\ell-n_1}$ on $W_1 \otimes W_2$, we also have $\dim((W_1 \otimes W_2)_u(1)) = 2\ell - n_1$. Lastly, as u acts trivially on W_2, it also acts trivially on $\wedge^2(W_2)$, and so $\dim((\wedge^2(W_2))_u(1)) = \frac{(2\ell - n_1 - 1)(2\ell - n_1)}{2}$. It follows that:

$$\begin{aligned}
\dim(V_u(1)) &= \frac{n_1 - 1}{2} + 2\ell - n_1 + \frac{(2\ell - n_1 - 1)(2\ell - n_1)}{2} \\
&= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 2\ell - 1}{2} \\
&= 2\ell^2 - 5\ell + 6 + \frac{n_1^2 - 4\ell n_1 + 12\ell - 13}{2}.
\end{aligned}$$

One checks that the inequality

$$n_1^2 - 4\ell n_1 + 12\ell - 13 < 0 \tag{6.7}$$

holds for all $n_1 \in (2\ell - \sqrt{(2\ell-3)^2 + 4},\ 2\ell + \sqrt{(2\ell-3)^2 + 4})$ and all $\ell \geq 1$. Since $2\ell - \sqrt{(2\ell-3)^2 + 4} < 2\ell - \sqrt{(2\ell-3)^2} = 3$ and since $2\ell + \sqrt{(2\ell-3)^2 + 4} > 2\ell - 1$, it follows that, in particular, Inequality (6.7) holds for all $3 \leq n_1 \leq 2\ell - 1$ and all $\ell \geq 4$. Therefore, $\dim(V_u(1)) < 2\ell^2 - 5\ell + 6$ for all $u \in G$ unipotent with Jordan form $J_{n_1} \oplus J_1^{2\ell-n_1}$ on W.

We now consider the second case, when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell - 3$. We write $W = W_1' \oplus W_2'$, where $\dim(W_1') = n_1$ and u acts as J_{n_1} on W_1', and $\dim(W_2') = 2\ell - n_1$ and u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on W_2'. Now, by (6.6), in order to determine $\dim(V_u(1))$, we only need to know $\dim((\wedge^2(W_1'))_u(1))$, $\dim((W_1' \otimes W_2')_u(1))$ and $\dim((\wedge^2(W_2'))_u(1))$. As u acts as a single Jordan block on W_1', by Lemma 2.9.4, we have $\dim((\wedge^2(W_1'))_u(1)) = \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 + \epsilon}{2}$, where $\epsilon = 0$ if n_1 is even, or $\epsilon = -1$ if n_1 is odd. Now, since u acts as $(J_{n_1} \otimes J_{n_1})^{r_1-1} \oplus \bigoplus_{i=2}^{m} (J_{n_1} \otimes J_{n_i})^{r_i}$ on $W_1' \otimes W_2'$, we again use Lemma 2.9.4 to deduce:

$$\dim((W_1' \otimes W_2')_u(1)) = (r_1 - 1)n_1 + \sum_{i=2}^{m} n_i r_i = 2\ell - n_1. \tag{6.8}$$

Furthermore, since the Jordan form of u on W admits at least two nontrivial blocks, it follows that u acts nontrivially on W_2' and so, by Proposition 3.3.4, we have $\dim((\wedge^2(W_2'))_u(1)) \leq \frac{(2\ell - n_1 - 1)^2 - (2\ell - n_1 - 1) + 2}{2}$, where equality holds if and only if u acts on W_2' as one of J_2^2 and $J_2 \oplus J_1^{2\ell - n_1 - 2}$. However, we note that u cannot act on W_2' as J_2^2, since if it did, then u would act on W as $J_{2\ell-4} \oplus J_2^2$, which contradicts the fact that even sized Jordan blocks occur with even multiplicity. Therefore $\dim((\wedge^2(W_2'))_u(1)) = \frac{(2\ell - n_1 - 1)^2 - (2\ell - n_1 - 1) + 2}{2}$ if and only if u acts on W_2' as $J_2 \oplus J_1^{2\ell - n_1 - 2}$. Now, by (6.6) and keeping in mind that $\epsilon \leq 0$, we have:

$$\begin{aligned}
\dim(V_u(1)) &\leq \frac{n_1 + \epsilon}{2} + 2\ell - n_1 + \frac{(2\ell - n_1 - 1)^2 - (2\ell - n_1 - 1) + 2}{2} \\
&= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 2n_1 - 2\ell + 4 + \epsilon}{2} \\
&\leq \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 2n_1 - 2\ell + 4}{2} \\
&= 2\ell^2 - 5\ell + 6 + \frac{n_1^2 - 4\ell n_1 + 2n_1 + 8\ell - 8}{2} \\
&= 2\ell^2 - 5\ell + 6 + \frac{(n_1 - 2)(n_1 + 4 - 4\ell)}{2}.
\end{aligned}$$

Since $2 \leq n_1 \leq 2\ell - 3$, it follows that $(n_1 - 2)(n_1 + 4 - 4\ell) \leq 0$ for all $\ell \geq 4$, and thus $\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 6$ for all $\ell \geq 4$ and all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks. Moreover, equality holds if and only if $(n_1 - 2)(n_1 + 4 - 4\ell) = 0$, $\epsilon = 0$ and $\dim((\wedge^2(W_2'))_u(1)) = \frac{(2\ell - n_1 - 1)^2 - (2\ell - n_1 - 1) + 2}{2}$, hence, if and only if $n_1 = 2$ and u acts as $J_2 \oplus J_1^{2\ell - n_1 - 2}$ on W_2'. We deduce that, in this case, u has Jordan form $J_2^2 \oplus J_1^{2\ell-4}$ on W.

Having considered all possible cases, we conclude that $\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 6$ for all non-identity unipotent elements $u \in \tilde{G}$. Moreover, we have shown that equality holds

if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$. In particular, as the inequality $0 < 14\ell^2 - 47\ell + 36$ holds for all $\ell \geq 4$, we have that $2\ell^2 - 5\ell + 6 < 2\ell^2 - \ell - \sqrt{2\ell^2 - \ell}$ for all $\ell \geq 4$ and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$. □

Proposition 6.3.6. *Let k be an algebraically closed field of characteristic $p = 2$ and let $V' = \wedge^2(W)$.*

(1) *Then, for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 6,$$

where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$, i.e. the Hesselink normal form is one of $(2_0^2, 1_0^{2\ell-4})$ and $(1_0^{2\ell-4}, 2_1^2)$.

(2) *If $u \in G$ is a non-identity unipotent element whose Jordan form on W is different than $J_2^2 \oplus J_1^{2\ell-4}$, then*

$$\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 4,$$

where equality holds if and only if $\ell = 4$ and the Hesselink normal form of u is one of (2_0^4) and (2_1^4).

Proof. (1) First, we note that, as $p = 2$, we have that $G < \mathrm{Sp}(W, a)$, where a is the nondegenerate alternating bilinear form on W that satisfies $a(w_1, w_2) = Q(w_1+w_2)+Q(w_1)+Q(w_2)$ for all $w_1, w_2 \in W$. Now, by Proposition 4.3.7, it follows that $\dim(V_u^{'}(1)) \leq 2\ell^2-3\ell+2$ for all non-identity unipotent elements $u \in \mathrm{Sp}(W, a)$, therefore $\dim(V_u^{'}(1)) \leq 2\ell^2 - 3\ell + 2$ for all non-identity unipotent elements $u \in \tilde{G}$. However, we have seen in the second paragraph of the proof of Lemma 6.3.4 that if $v \in \mathrm{Sp}(W, a)$ is unipotent with Jordan form $J_2 \oplus J_1^{2\ell-2}$ on W, then $v \notin G$. Thus, by Lemma 4.3.9, we deduce that $\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 6$ for all non-identity unipotent elements $u \in \tilde{G}$. Let $v \in \mathrm{Sp}(W, a)$ be such that its Jordan form on W is $J_2^2 \oplus J_1^{2\ell-4}$. Then the Hesselink normal form of v is either $(1_0^{2\ell-2}, 2_1^2)$ or $(2_0^2, 1_0^{2\ell-2})$. In both cases we apply Proposition 2.9.20 to determine that $v \in G$. Therefore, by Lemma 4.3.9, it follows that $\dim(V_u'(1)) \leq 2\ell^2-5\ell+6$ for all non-identity unipotent elements $u \in \tilde{G}$, where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$.

(2) We now assume that $u \in G_u \setminus \{1\}$, has Jordan form on W different than $J_2^2 \oplus J_1^{2\ell-4}$. As $p = 2$, we have that, in particular, $u \in \mathrm{Sp}(W, a)$ and thus, by Proposition 4.3.11, it follows that $\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 4$. Moreover, by the same result, we have that $\dim(V_v^{'}(1)) = 2\ell^2 - 5\ell + 4$ for $v \in \mathrm{Sp}(W, a)$ is unipotent, if and only if $\ell = 4$ and the Jordan form of v on W is one of J_2^4 and $J_2^3 \oplus J_1^2$.

Let $v \in \mathrm{Sp}(W, a)$ be unipotent with Jordan form J_2^4 on W. Then the Hesselink normal form of v is either (2_0^4) or (2_1^4). In both cases we apply Proposition 2.9.20 to deduce that $v \in G$. On the other hand, let $v^{'} \in \mathrm{Sp}(W, a)$ be unipotent with Jordan form $J_2^3 \oplus J_1^2$ on W. Then the Hesselink normal form of $v^{'}$ is $(1_0^2, 2_1^3)$, thus $v^{'} \notin G$, by Proposition 2.9.20.

By Proposition 4.3.11, we conclude that $\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 4$ for all non-identity unipotent elements $u \in G$ whose Jordan form on W is different than $J_2^2 \oplus J_1^{2\ell-4}$. Furthermore, equality holds if and only if the Hesselink normal form of u is one of (2_0^4) and (2_1^4). □

Corollary 6.3.7. *Let k be an algebraically closed field of characteristic $p = 2$. Assume that $2 \nmid \ell$ and let $V = L_G(\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 5,$$

where equality holds if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$, i.e. the Hesselink normal form of u is one of $(2_0^2, 1_0^{2\ell-4})$ and $(1_0^{2\ell-4}, 2_1^2)$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. Set $V' = \wedge^2(W)$ and note that, by Lemma 2.8.5, since $p = 2$ and $p \nmid \ell$, we have $V' \cong V \oplus L_G(0)$, as kG-modules. Therefore $\dim(V_u(1)) = \dim(V_u'(1)) - 1$. We apply Proposition 6.3.6.(1), to deduce that $\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 5$ for all non-identity unipotent elements $u \in G$, and that equality holds if and only if $\dim(V_u'(1)) = 2\ell^2 - 5\ell + 6$, hence, if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$. Lastly, we note that, as the inequality $0 < 14\ell^2 - 47\ell + 37$ holds for all $\ell \geq 4$, we have that $2\ell^2 - 5\ell + 5 < 2\ell^2 - \ell - 1 - \sqrt{2\ell^2 - \ell - 1}$ for all $\ell \geq 4$ and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 6.3.8. *Let k be an algebraically closed field of characteristic $p = 2$. Assume that $2 \mid \ell$ and let $V = L_G(\omega_2)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 4.$$

Furthermore, we have equality if and only if one of the following holds:

(1) *$\ell = 4$ and the Hesselink normal form of u is (2_0^4).*

(2) *$\ell \geq 4$ and the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$, i.e. the Hesselink normal form of u is one of $(2_0^2, 1_0^{2\ell-4})$ and $(1_0^{2\ell-4}, 2_1^2)$.*

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, we note that, as $p = 2$, we have $G < \mathrm{Sp}(W, a)$, where a is the nondegenerate alternating bilinear form on W that satisfies $a(w_1, w_2) = Q(w_1 + w_2) + Q(w_1) + Q(w_2)$ for all $w_1, w_2 \in W$. Set $V' = \wedge^2(W)$ and let $u \in G$ be a non-identity unipotent element. Then, in particular, u is a unipotent element of $\mathrm{Sp}(W, a)$. Let u' denote the action of u on V' and let u_V denote the action of u on V. Then, by Theorem 4.3.6, we know we can determine the Jordan form of u_V from that of u'.

Let $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, 2n_{t+1_1}^{r_{t+1}}, \ldots, 2n_{m_1}^{r_m})$ be the Hesselink normal form of u, where $m \geq 1$, $t \geq 0$ and $r_i \geq 1$ for all $1 \leq i \leq m$. Moreover, as $u \in G$, we have that $\sum_{i=t+1}^{m} r_i$ is even. Set $\alpha = \nu_2(\gcd(n_1, \ldots, n_t, n_{t+1}, \ldots, n_m))$.

First, assume that $\alpha = 0$. Then, by Theorem 4.3.6.(b), as $2 \mid \ell$, it follows that $\dim(V_u(1)) = \dim(V_u'(1)) - 2$. Now, by Proposition 6.3.6.(1), we deduce that $\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 4$ for all non-identity unipotent elements $u \in G$, and that equality holds if and only if $\dim(V_u'(1)) = 2\ell^2 - 5\ell + 6$, hence, if and only if the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell-4}$.

We can now assume that $\alpha > 0$. Then, by Theorem 4.3.6.(c), it follows that $\dim(V_u(1)) \leq \dim(V_u^{'}(1))$. As $\alpha > 0$, the Jordan form of u on W is different than $J_2^2 \oplus J_1^{2\ell-4}$ and thus, by Proposition 6.3.6.(2), we have $\dim(V_u^{'}(1)) \leq 2\ell^2 - 5\ell + 4$, hence $\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 4$ for all non-identity unipotent elements $u \in G$. Now, as $\alpha > 0$, by Proposition 6.3.6.(2), we have that $\dim(V_u^{'}(1)) = 2\ell^2 - 5\ell + 4$ if and only if $\ell = 4$ and the Hesselink normal form of u is (2_0^4). In this case, we have $\alpha = 1$, thus $2 \mid \frac{\ell}{2^\alpha}$ and, by Theorem 4.3.6.(c.1), it follows that $\dim(V_u(1)) = \dim(V_u^{'}(1)) = 2\ell^2 - 5\ell + 4$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 2\ell^2 - 5\ell + 4$ for all non-identity unipotent elements $u \in G$. In particular, as the inequality $0 < 14\ell^2 - 47\ell + 38$ holds for all $\ell \geq 4$, we have that $2\ell^2 - 5\ell + 4 < 2\ell^2 - \ell - 2 - \sqrt{2\ell^2 - \ell - 2}$ for all $\ell \geq 4$, and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. $\square$

We now consider the irreducible kG-module $L_G(2\omega_1)$. We have seen in Lemma 2.8.4 that $L_G(2\omega_1)$ is a composition factor of the kG-module $\mathrm{S}^2(W)$. As for groups of type B_ℓ, see Subsection 5.3, we first determine $\dim((\mathrm{S}^2(W))_u(1))$, where $u \in G$ is a unipotent element, and then apply Theorem 5.3.5 to deduce $\dim(L_G(2\omega_1)_u(1))$.

Proposition 6.3.9. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V^{'} = \mathrm{S}^2(W)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u^{'}(1)) \leq 2\ell^2 - 3\ell + 2,$$

where equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$.

Proof. Let u be a unipotent element of G as in hypothesis $({}^\dagger H_u)$. Since r_i is even for all even n_i, it follows that the Jordan form of u on W admits at least two blocks. We first consider the case when exactly one of these blocks, J_{n_1}, is nontrivial. Then u has Jordan form $J_{n_1} \oplus J_1^{2\ell-n_1}$, where n_1 is odd, since $r_1 = 1$, thus $3 \leq n_1 \leq 2\ell - 1$. We write $W = W_1 \oplus W_2$, where $\dim(W_1) = n_1$ and u acts as J_{n_1} on W_1, and $\dim(W_2) = 2\ell - n_1$ and u acts trivially on W_2. Then, as $k[u]$-modules, we have

$$V^{'} \cong \mathrm{S}^2(W_1) \oplus (W_1 \otimes W_2) \oplus \mathrm{S}^2(W_2)$$

and so

$$\dim(V_u^{'}(1)) = \dim((\mathrm{S}^2(W_1))_u(1)) + \dim((W_1 \otimes W_2)_u(1)) + \dim((\mathrm{S}^2(W_2))_u(1)). \tag{6.9}$$

Now, since $p \neq 2$, we apply Lemma 2.9.4, which gives $\dim((\mathrm{S}^2(W_1))_u(1)) = n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1+1}{2}$ and, moreover, as u acts as $J_{n_1} \otimes J_1^{2\ell-n_1}$ on $W_1 \otimes W_2$, we also have $\dim((W_1 \otimes W_2)_u(1)) = 2\ell - n_1$. Lastly, as u acts trivially on W_2, it also acts trivially on $\mathrm{S}^2(W_2)$, and so

$\dim((\mathrm{S}^2(W_2))_u(1)) = \dfrac{(2\ell - n_1)(2\ell - n_1 + 1)}{2}$. It follows that:

$$\begin{aligned}\dim(V_u^{'}(1)) &= \frac{n_1+1}{2} + 2\ell - n_1 + \frac{(2\ell - n_1)(2\ell - n_1 + 1)}{2} \\ &= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 6\ell - 2n_1 + 1}{2} \\ &= 2\ell^2 - 3\ell + 2 + \frac{n_1^2 - 4\ell n_1 - 2n_1 + 12\ell - 3}{2} \\ &= 2\ell^2 - 3\ell + 2 + \frac{(n_1 - 3)(n_1 + 1 - 4\ell)}{2}.\end{aligned} \tag{6.10}$$

Since $3 \leq n_1 \leq 2\ell - 1$, we have $(n_1 - 3)(n_1 + 1 - 4\ell) \leq 0$ for all $\ell \geq 4$, and therefore $\dim(V_u^{'}(1)) \leq 2\ell^2 - 3\ell + 2$ for all unipotent elements u of G with Jordan form $J_{n_1} \oplus J_1^{2\ell - n_1}$. Moreover, equality holds if and only if $(n_1 - 3)(n_1 + 1 - 4\ell) = 0$, hence, if and only if the Jordan form of u on W is $J_3 \oplus J_1^{2\ell - 3}$.

We now consider the second case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell - 3$. We write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = n_1$ and u acts as J_{n_1} on $W_1^{'}$, and $\dim(W_2^{'}) = 2\ell - n_1$ and u acts as $J_{n_1}^{r_1 - 1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$. Now, by (6.9), in order to determine $\dim(V_u^{'}(1))$, we only need to know $\dim((\mathrm{S}^2(W_1^{'}))_u(1))$, $\dim((W_1^{'} \otimes W_2^{'})_u(1))$ and $\dim((\mathrm{S}^2(W_2^{'}))_u(1))$. As u acts as a single Jordan block on $W_1^{'}$, by Lemma 2.9.4, as $p \neq 2$, we have $\dim((\mathrm{S}^2(W_1^{'}))_u(1)) = n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = \frac{n_1 - \epsilon}{2}$, where $\epsilon = 0$ if n_1 is even, or $\epsilon = -1$ if n_1 is odd. Since u acts as $(J_{n_1} \otimes J_{n_1})^{r_1 - 1} \oplus \bigoplus_{i=2}^{m} (J_{n_1} \otimes J_{n_i})^{r_i}$ on $W_1^{'} \otimes W_2^{'}$, by (6.8), we have $\dim((W_1^{'} \otimes W_2^{'})_u(1)) = 2\ell - n_1$. Furthermore, since the Jordan form of u on W admits at least two nontrivial blocks, it follows that u acts nontrivially on $W_2^{'}$. Thus, by Proposition 3.3.5, it follows that $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) \leq \dfrac{(2\ell - n_1 - 1)(2\ell - n_1)}{2}$, where equality holds if and only if u acts as $J_2 \oplus J_1^{2\ell - n_1 - 2}$ on $W_2^{'}$. Thus, by (6.9), we have:

$$\begin{aligned}\dim(V_u^{'}(1)) &\leq \frac{n_1 - \epsilon}{2} + 2\ell - n_1 + \frac{(2\ell - n_1 - 1)(2\ell - n_1)}{2} \\ &= \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 2\ell - \epsilon}{2} \\ &= 2\ell^2 - 3\ell + 2 + \frac{n_1^2 - 4\ell n_1 + 8\ell - 4 - \epsilon}{2}.\end{aligned}$$

If $n_1 = 2$, then $\epsilon = 0$, $\dfrac{n_1^2 - 4\ell n_1 + 8\ell - 4 - \epsilon}{2} = 0$ and, consequently, $\dim(V_u^{'}(1)) \leq 2\ell^2 - 3\ell + 2$. Now, equality holds if and only if $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) = \dfrac{(2\ell - 3)(2\ell - 2)}{2}$, hence if and only if u acts as $J_2 \oplus J_1^{2\ell - 4}$ on $W_2^{'}$. It follows that, in this case, the Jordan form of u on W is $J_2^2 \oplus J_1^{2\ell - 4}$. We now assume that $n_1 \geq 3$. One checks that the inequality

$$n_1^2 - 4\ell n_1 + 8\ell - 4 - \epsilon < 0 \tag{6.11}$$

holds for all $n_1 \in (2\ell - \sqrt{4\ell^2 - 8\ell + 4 + \epsilon},\ 2\ell + \sqrt{4\ell^2 - 8\ell + 4 + \epsilon})$ and all $\ell \geq 2$. Since $2\ell - \sqrt{4\ell^2 - 8\ell + 4 + \epsilon} < 3$, as $5 - \epsilon < 4\ell$ for all $\ell \geq 2$, and since $2\ell + \sqrt{4\ell^2 - 8\ell + 4 + \epsilon} > 2\ell - 3$, it follows that, in particular, Inequality (6.11) holds for all $3 \leq n_1 \leq 2\ell - 3$ and all $\ell \geq 4$. Therefore $\dim(V_u^{'}(1)) < 2\ell^2 - 3\ell + 2$ for all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks and $n_1 \geq 3$.

Having considered all possible cases, we conclude that $\dim(V_u^{'}(1)) \leq 2\ell^2 - 3\ell + 2$ for all non-identity unipotent elements $u \in G$. Moreover, we have shown that equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$. □

Corollary 6.3.10. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $p \nmid \ell$ and let $V = L_G(2\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - 3\ell + 1,$$

where equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, set $V^{'} = \mathrm{S}^2(W)$. As $p \neq 2$ and $p \nmid \ell$, by Lemma 2.8.4, it follows that $V^{'} \cong V \oplus L_G(0)$, as kG-modules, and therefore $\dim(V_u(1)) = \dim(V_u^{'}(1)) - 1$. We now apply Proposition 6.3.9, to see that $\dim(V_u(1)) \leq 2\ell^2 - 3\ell + 1$ for all non-identity unipotent elements $u \in G$. Moreover, we have equality if and only if $\dim(V_u^{'}(1)) = 2\ell^2 - 3\ell + 2$, hence, if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$.

Lastly, as the inequality $14\ell^2 - 17\ell + 5 > 0$ holds for all $\ell \geq 4$, it follows that $2\ell^2 - 3\ell + 1 < 2\ell^2 + \ell - 1 - \sqrt{2\ell^2 + \ell - 1}$ for all $\ell \geq 4$, and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

We will require the following result in the proof of Corollary 6.3.12.

Lemma 6.3.11. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V^{'} = \mathrm{S}^2(W)$. Let $u \in G$ be a non-identity unipotent element whose Jordan form on W is different than $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$. Then*

$$\dim(V_u^{'}(1)) < 2\ell^2 - 3\ell.$$

Proof. Let u be a unipotent element of G as in hypothesis ($^{\dagger}H_u$). As r_i is even for all even n_i, it follows that the Jordan form of u on W admits at least two blocks. We first consider the case when exactly one of these blocks, J_{n_1}, is nontrivial. We remark that since $r_1 = 1$, n_1 is odd and, since the Jordan form of u is different than $J_3 \oplus J_1^{2\ell-2}$, we have $5 \leq n_1 \leq 2\ell - 1$. Now, by (6.10), it follows that

$$\dim(V_u^{'}(1)) = \frac{4\ell^2 - 4\ell n_1 + n_1^2 + 6\ell - 2n_1 + 1}{2} = 2\ell^2 - 3\ell + \frac{n_1^2 - 4\ell n_1 - 2n_1 + 12\ell + 1}{2}.$$

One checks that the inequality

$$n_1^2 - 4\ell n_1 - 2n_1 + 12\ell + 1 < 0$$

holds for all $n_1 \in (2\ell+1-2\sqrt{\ell^2-2\ell},\ 2\ell+1+2\sqrt{\ell^2-2\ell})$ and all $\ell \geq 2$. Since $2\ell+1+2\sqrt{\ell^2-2\ell} > 2\ell-1$ and $2\ell+1-2\sqrt{\ell^2-2\ell} < 5$, as $4 < 2\ell$ for all $\ell \geq 4$, it follows that, in particular, the inequality holds for all $5 \leq n_1 \leq 2\ell-1$ and all $\ell \geq 4$. We deduce that $\dim(V_u^{'}(1)) < 2\ell^2-3\ell$ for all unipotent elements u of G with Jordan form $J_{n_1} \oplus J_1^{2\ell-n_1}$ on W, where $5 \leq n_1 \leq 2\ell-1$.

We now consider the second case when the Jordan form of u on W admits at least two nontrivial blocks. Then $2 \leq n_1 \leq 2\ell-3$ and, once again, we distinguish two cases.

First, if $n_1 = 2$, then the Jordan form of u on W is $J_2^{r_1} \oplus J_1^{2\ell-2r_1}$, where, by hypothesis, $r_1 \geq 4$ is even. For the moment, assume that $\ell = 4$. Then the Jordan form of u on W is J_2^4. Using Lemma 2.9.4, as $p \neq 2$, one determines that $\dim(V_u(1)) = 16 < 20$. We can now assume that $\ell \geq 5$ and we write $W = W_1 \oplus W_2$, where $\dim(W_1) = 4$ and u acts as J_2^2 on W_1, and $\dim(W_2) = 2\ell-4$ and u acts as $J_2^{r_1-2} \oplus J_1^{2\ell-2r_1}$ on W_2. Now, by Proposition 3.3.5, we have $\dim((\mathrm{S}^2(W_1))_u(1)) \leq 6$ and $\dim((\mathrm{S}^2(W_2))_u(1)) \leq 2\ell^2-9\ell+10$, respectively. Furthermore, as u acts on $W_1 \otimes W_2$ as $(J_2 \otimes J_2)^{2r_1-4} \oplus (J_2 \otimes J_1)^{4\ell-4r_1}$, by Lemma 2.9.4, we have $\dim((W_1 \otimes W_2)_u(1)) = 2(2r_1-4)+4\ell-4r_1 = 4\ell-8$. We use (6.9) to determine that $\dim(V_u^{'}(1)) \leq 2\ell^2-5\ell+8$ and therefore, as $8 < 2\ell$ for all $\ell \geq 5$, we showed that $\dim(V_u^{'}(1)) < 2\ell^2-3\ell$ for all unipotent elements u of G whose Jordan form on W is $J_2^{r_1} \oplus J_1^{2\ell-2r_1}$, where $r_1 \geq 4$.

Secondly, if $n_1 \geq 3$, then we proceed as in the proof of Proposition 6.3.9, see the second to last paragraph, and write $W = W_1^{'} \oplus W_2^{'}$, where $\dim(W_1^{'}) = n_1$ and u acts as J_{n_1} on $W_1^{'}$, and $\dim(W_2^{'}) = 2\ell-n_1$ and u acts as $J_{n_1}^{r_1-1} \oplus \bigoplus_{i=2}^{m} J_{n_i}^{r_i}$ on $W_2^{'}$. By Lemma 2.9.4, we have $\dim((\mathrm{S}^2(W_1^{'}))_u(1)) = \dfrac{n_1-\epsilon}{2}$, where $\epsilon = 0$ if n_1 is even, or $\epsilon = -1$ if n_1 is odd, and that $\dim((W_1^{'} \otimes W_2^{'})_u(1)) = 2\ell-n_1$. Moreover, as $n_1 \geq 3$ and as r_i is even for even n_i, it follows that u does not act on $W_2^{'}$ as $J_2 \oplus J_1^{2\ell-4}$ and so, by Proposition 3.3.5, we have $\dim((\mathrm{S}^2(W_2^{'}))_u(1)) < \dfrac{(2\ell-n_1-1)(2\ell-n_1)}{2}$. Therefore, by (6.9), it follows that

$$\begin{aligned}\dim(V_u^{'}(1)) &< \frac{n_1-\epsilon}{2} + 2\ell - n_1 + \frac{(2\ell-n_1-1)(2\ell-n_1)}{2} \\ &= \frac{4\ell^2-4\ell n_1+n_1^2+2\ell-\epsilon}{2} \\ &= 2\ell^2-3\ell+\frac{n_1^2-4\ell n_1+8\ell-\epsilon}{2}.\end{aligned}$$

One checks that the inequality

$$n_1^2-4\ell n_1+8\ell-\epsilon < 0 \tag{6.12}$$

holds for all $n_1 \in (2\ell-\sqrt{4\ell^2-8\ell+\epsilon},\ 2\ell+\sqrt{4\ell^2-8\ell+\epsilon})$ and all $\ell \geq 3$. Since $2\ell+\sqrt{4\ell^2-8\ell+\epsilon} > 2\ell-3$ and since $2\ell-\sqrt{4\ell^2-8\ell+\epsilon} < 3$, as $9-\epsilon < 4\ell$ for all $\ell \geq 4$, it follows that, in particular, Inequality (6.12) holds for all $3 \leq n_1 \leq 2\ell-3$ and all $\ell \geq 4$. Therefore $\dim(V_u^{'}(1)) < 2\ell^2-3\ell$ for all unipotent elements u of G whose Jordan form on W admits at least two nontrivial blocks and $n_1 \geq 3$.

Having considered all possible cases, we conclude that $\dim(V_u^{'}(1)) < 2\ell^2 - 3\ell$ for all non-identity unipotent elements $u \in G$ whose Jordan form on W is different than $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$. $\square$

Corollary 6.3.12. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $p \mid \ell$ and let $V = L_G(2\omega_1)$. Then for all non-identity unipotent elements $u \in G$ we have*

$$\dim(V_u(1)) \leq 2\ell^2 - 3\ell,$$

where equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. To begin, set $V^{'} = \mathrm{S}^2(W)$ and let u be a unipotent element of G as in hypothesis $({}^{\dagger}H_u)$. If we denote by $u^{'}$, respectively by u_V, the action of u on $V^{'}$, respectively on V, then, as $p \neq 2$, using Theorem 5.3.5 we can determine the Jordan form of u_V from that of $u^{'}$.

Set $\alpha = \nu_p(\gcd(n_1, \ldots, n_m))$. If $\alpha = 0$, we apply Theorem 5.3.5.(b) to deduce that $\dim(V_u(1)) = \dim(V_u^{'}(1)) - 2$. By Proposition 6.3.9, it follows that $\dim(V_u(1)) \leq 2\ell^2 - 3\ell$ for all non-identity unipotent elements $u \in G$, where equality holds if and only if $\dim(V_u^{'}(1)) = 2\ell^2 - 3\ell + 2$, hence, if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$.

If $\alpha > 0$, we apply Theorem 5.3.5.(c) to deduce that $\dim(V_u(1)) = \dim(V_u^{'}(1))$. Now, since $\alpha > 0$, it follows that the Jordan form of u on W is different than $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$. We use Lemma 6.3.11 to deduce that $\dim(V_u^{'}(1)) < 2\ell^2 - 3\ell$ and, consequently, $\dim(V_u(1)) < 2\ell^2 - 3\ell$ for all unipotent elements $u \in G$ with $\alpha > 0$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 2\ell^2 - 3\ell$ for all non-identity unipotent elements $u \in G$, and that equality holds if and only if the Jordan form of u on W is one of $J_3 \oplus J_1^{2\ell-3}$ and $J_2^2 \oplus J_1^{2\ell-4}$. In particular, since the inequality $0 < 14\ell^2 - 17\ell + 6$ holds for all $\ell \geq 4$, it follows that $2\ell^2 - 3\ell < 2\ell^2 + \ell - 2 - \sqrt{2\ell^2 + \ell - 2}$ for all $\ell \geq 4$ and, consequently, $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. $\square$

To conclude this subsection, we remark that Lemmas 6.3.3 and 6.3.4, Propositions 6.3.5 and 6.3.8, and Corollaries 6.3.7, 6.3.10 and 6.3.12 give the proof of Theorems 6.3.1 and 6.3.2 for the families of kG-modules corresponding to p-restricted dominant weights $\lambda \in \{\omega_1, \omega_2, 2\omega_1\}$. Therefore, in view of Remark 6.1.1, they prove Theorems 6.3.1 and 6.3.2 for the families of $k\tilde{G}$-modules with p-restricted dominant weights $\tilde{\lambda} \in F^{B_\ell}$.

6.3.2 The particular modules

As previously mentioned, this subsection is devoted to the proofs of Theorems 6.3.1 and 6.3.2 for the particular $k\tilde{G}$-modules, i.e. the irreducible $k\tilde{G}$-modules $L_{\tilde{G}}(\tilde{\lambda})$ with corresponding p-restricted dominant highest weight $\tilde{\lambda}$ listed in Table 2.7.5. For each such $k\tilde{G}$-module V we will establish $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$, see Propositions 6.3.13, 6.3.14, 6.3.26 and 6.3.27. In order to achieve this goal, we will use the same inductive algorithm we used for groups of type A_ℓ, C_ℓ and B_ℓ. For a description of this algorithm, we refer the reader to Subsection 2.4.4.

Now, let $\tilde{u} \in \tilde{G}$ be a unipotent element. We write $\tilde{u} = \prod_{\tilde{\alpha} \in \tilde{\Phi}^+} x_{\tilde{\alpha}}(c_{\tilde{\alpha}})$, where the product respects the total order $\preceq$ on $\tilde{\Phi}$, see Section 1.3, and $c_{\tilde{\alpha}} \in k$ for all $\tilde{\alpha} \in \tilde{\Phi}^+$. To $\tilde{u}$ we associate the subset $S_{\tilde{u}} \subseteq \tilde{\Phi}^+$ with the property that $\tilde{u} = \prod_{\tilde{\alpha} \in S_{\tilde{u}}} x_{\tilde{\alpha}}(c_{\tilde{\alpha}})$, where the product respects $\preceq$ and $c_{\tilde{\alpha}} \in k^*$ for all $\tilde{\alpha} \in S_{\tilde{u}}$. Similarly, to the unipotent element $u \in G$ we associate the subset $S_u \subseteq \Phi^+$ with the property that $u = \prod_{\alpha \in S_u} x_\alpha(c_\alpha)$, where the product respects the total order $\preceq$ on Φ and $c_\alpha \in k^*$ for all $\alpha \in S_u$.

When $p \neq 2$, Theorem 2.9.2 and Lemma 2.9.1 tell us that unipotent conjugacy classes in $\tilde{G}$ are completely determined by the Jordan form, $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$, on W of a class representative, unless n_i is even for all i, in which case there exists two classes corresponding to that Jordan form. Similarly, when $p = 2$, we have seen that unipotent conjugacy classes in $\tilde{G}$ are completely determined by the Hesselink normal form, $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots, n_{m_1}^{r_m})$, of a class representative, see Theorem 2.9.11, Proposition 2.9.20 and Lemma 2.9.1, unless n_i is even for all $1 \leq i \leq m$ and $t = m$, in which case there exist two classes corresponding to that Hesselink form.

We end this introductory part by recalling some notation from Section 2.4. Let P_1 be the maximal parabolic subgroup of $\tilde{G}$ corresponding to $\tilde{\Delta}_1 = \{\tilde{\alpha}_2, \ldots, \tilde{\alpha}_\ell\}$ and let L_1 be a Levi subgroup of P_1. We have $L_1 = \mathrm{Z}(L_1)^\circ[L_1, L_1]$, where $\mathrm{Z}(L_1)^\circ$ is a one-dimensional torus and $[L_1, L_1]$ is a simply connected linear algebraic group of type $D_{\ell-1}$ with maximal torus $T' = \tilde{T} \cap [L_1, L_1]$.

We first consider the case of $\ell = 4$. In Table 6.3.2, respectively in Table 6.3.3, we list all unipotent conjugacy classes in $\tilde{G}$ when $p \neq 2$, respectively when $p = 2$, and we give a representative for each class. Note that each chosen non-identity class representative u' has the property that $u'_{L_1} \neq 1$.

[MKT21, Table 12][LS12, Subsection 3.3.2]

Class representative	Jordan form
1	J_1^8
$x_{\tilde{\alpha}_2}(1)$	$J_2^2 \oplus J_1^4$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_3}(1)$	J_2^4
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_4}(1)$	J_2^4
$x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	$J_3 \oplus J_1^5$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	$J_3 \oplus J_2^2 \oplus J_1$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)$	$J_3^2 \oplus J_1^2$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)$	J_4^2
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_4}(1)$	J_4^2
$x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	$J_5 \oplus J_1^3$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_4}(1)$	$J_5 \oplus J_3$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	$J_7 \oplus J_1$

Table 6.3.2: Unipotent class representatives in D_4 when $p \neq 2$.

[Remark 2.9.19][MKT21, Table 12]

Class representative	Hesselink normal form
1	(1_0^8)
$x_{\tilde{\alpha}_2}(1)$	$(2_0^2, 1_0^4)$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_3}(1)$	(2_0^4)
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_4}(1)$	(2_0^4)
$x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	$(1_0^4, 2_1^2)$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	(2_1^4)
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)$	$(3_0^2, 1_0^2)$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)$	(4_0^2)
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_4}(1)$	(4_0^2)
$x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	$(1_0^2, 4_1, 2_1)$
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_4}(1)$	(4_1^2)
$x_{\tilde{\alpha}_1}(1)x_{\tilde{\alpha}_2}(1)x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$	$(6_1, 2_1)$

Table 6.3.3: Unipotent class representatives in D_4 when $p = 2$.

Proposition 6.3.13. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 4$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3 + \tilde{\omega}_4)$. Then for all non-identity unipotent elements $\tilde{u} \in \tilde{G}$ we have*

$$\dim(V_{\tilde{u}}(1)) \leq 28,$$

where there exist $\tilde{u} \in \tilde{G}$ for which the bound is attained.

In particular, we have $\dim(V_{\tilde{u}}(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $\tilde{u} \in \tilde{G}$.

Proof. To begin, we consider the dominant weight $\lambda = \omega_3 + \omega_4 \in \mathrm{X}(T)$ of G and its associated irreducible kG-module $L_G(\lambda)$. We have seen in Subsection 2.3.3 that $L_G(\lambda)$ is also a simple $k\tilde{G}$-module and, as a $k\tilde{G}$-module, it is isomorphic to $L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda} = \tilde{\omega}_3 + \tilde{\omega}_4 \in \mathrm{X}(\tilde{T})$. Now, by Lemma 2.3.10, for all non-identity unipotent elements $\tilde{u} \in \tilde{G}$, we have that:

$$\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{u}}(1)) \leq \max_{u \in G_u \setminus \{1\}} \dim((L_G(\lambda))_u(1)),$$

where G_u is the set of unipotent elements in G.

Secondly, we recall that over fields of characteristic 2, G is a subgroup of $H := \mathrm{Sp}(W, a)$, where a is the nondegenerate alternating bilinear form on W which satisfies $a(w_1, w_2) = Q(w_1 + w_2) + Q(w_1) + Q(w_2)$ for all $w_1, w_2 \in W$, see Subsection 2.9.3. Note that H is a simple simply connected linear algebraic group of type C_4, [Car89, p.184]. Now, consider the irreducible kH-module, $L_H(\omega_3^H)$, of highest weight $\omega_3^H \in \mathrm{X}(T_H)$, where T_H is the maximal torus of H constructed in Section 4.1 and ω_3^H is the fundamental dominant weight of H corresponding to the simple root α_3^H. By [Sei87, Table 1], we have the following isomorphism of kG-modules:

$$L_H(\omega_3^H) \mid_G \cong L_G(\omega_3 + \omega_4). \tag{6.13}$$

Furthermore, by [McN98, Lemma 4.8.2], as $p \neq 3$, it follows that

$$\wedge^3(W) \cong L_H(\omega_3^H) \oplus L_H(\omega_1^H) \cong L_H(\omega_3^H) \oplus W, \tag{6.14}$$

as kH-modules. Therefore, since $G < H$, restricting (6.14) to G and using (6.13) gives the following isomorphism of kG-modules:

$$\wedge^3(W) \cong L_H(\omega_3^H) \mid_G \oplus W \mid_G \cong L_G(\omega_3 + \omega_4) \oplus W,$$

where we used the fact that $\wedge^3(W) \mid_G \cong \wedge^3(W)$, since $W \mid_G \cong W$ as kG-modules. We deduce that, in order to determine $\dim((L_G(\omega_3+\omega_4))_u(1))$, where $u \in G$ is a unipotent element, we need to determine $\dim((\wedge^3(W))_u(1))$ and $\dim(W_u(1))$.

Let $\{\tilde{u}_1', \ldots, \tilde{u}_{11}'\}$ be the set of non-identity unipotent $\tilde{G}$-conjugacy class representatives listed in Table 6.3.3. Recall that $\phi : \tilde{G} \to G$ is a fixed central isogeny with $d\phi \neq 0$ and $\ker(\phi) \leq \mathrm{Z}(\tilde{G})$. For each $1 \leq i \leq 11$, let $u_i' = \phi(\tilde{u}_i')$. Then, by Lemma 2.9.1, $\{u_1', \ldots, u_{11}'\}$ is a set of representatives for the non-identity unipotent conjugacy classes in G. Moreover, if $\tilde{u}_i = \prod_{j=1}^{t_i} x_{\tilde{\beta}_j}(1)$, where $t_i \geq 1$ for all $1 \leq i \leq 11$, then $u_i = \prod_{j=1}^{t_i} x_{\beta_j}(1)$ for all $1 \leq i \leq 11$. Now, using Table 6.3.3, we establish the following:

u'	Jordan form	Action on $\wedge^3(W)$	$\dim(L_G(\omega_3+\omega_4)_{u'}(1))$
$x_{\alpha_2}(1)$	$J_2^2 \oplus J_1^4$	$J_2^{22} \oplus J_1^{12}$	28
$x_{\alpha_1}(1)x_{\alpha_3}(1)$	J_2^4	J_2^{28}	24
$x_{\alpha_1}(1)x_{\alpha_4}(1)$	J_2^4	J_2^{28}	24
$x_{\alpha_3}(1)x_{\alpha_4}(1)$	$J_2^2 \oplus J_1^4$	$J_2^{22} \oplus J_1^{12}$	28
$x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$	J_2^4	J_2^{28}	24
$x_{\alpha_1}(1)x_{\alpha_2}(1)$	$J_3^2 \oplus J_1^2$	$J_4^8 \oplus J_3^6 \oplus J_1^6$	16
$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)$	J_4^2	J_4^{14}	12
$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_4}(1)$	J_4^2	J_4^{14}	12
$x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$	$J_4 \oplus J_2 \oplus J_1^2$	$J_4^{11} \oplus J_2^5 \oplus J_1^2$	14
$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_1+\alpha_2+\alpha_4}(1)$	J_4^2	J_4^{14}	12
$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$	$J_6 \oplus J_2$	$J_8^4 \oplus J_6^3 \oplus J_4 \oplus J_2$	7

We outline the algorithm used to obtain the data in the above table. Let $v' \in G$ be a non-identity unipotent element and let V_i, $1 \leq i \leq \mathrm{ord}(v')$, denote the unique, up to isomorphism, indecomposable $k[v']$-module with $\dim(V_i) = i$ and on which v' acts as the full Jordan block of size i. Let $\bigoplus_{i=1}^{m} J_{n_i}^{r_i}$, where $m \geq 1$, be the Jordan form of v' on W. Note that $\dim(W_{v'}(1)) = \sum_{i=1}^{m} r_i$, see (6.4). We obtain the following decomposition of W as a $k[v']$-module:

$$W \cong V_{n_1}^{r_1} \oplus \cdots \oplus V_{n_m}^{r_m}.$$

This gives rise to the following isomorphism of $k[v']$-modules:

$$\wedge^3(W) \cong \bigoplus_{i=1}^{m} \wedge^3(V_{n_i}^{r_i}) \oplus \bigoplus_{i=1}^{m-1} \left[\wedge^2 (V_{n_i}^{r_i}) \otimes \bigoplus_{j=i+1}^{m} V_{n_j}^{r_j} \right] \oplus \bigoplus_{i=1}^{m-1} \left[V_{n_i}^{r_i} \otimes \wedge^2 (\bigoplus_{j=i+1}^{m} V_{n_j}^{r_j}) \right]. \quad (6.15)$$

Recursively, since, for all $1 \leq i \leq j \leq \mathrm{ord}(v')$, we have $\wedge^2(V_i \oplus V_j) \cong \wedge^2(V_i) \oplus V_i \otimes V_j \oplus \wedge^2(V_j)$ as $k[v']$-modules, one can show that, for all $1 \leq i \leq j \leq m$, we have

$$\wedge^2(\bigoplus_{j=i+1}^{m} V_{n_j}^{r_j}) \cong \bigoplus_{j=i+1}^{m} \wedge^2(V_{n_j}^{r_j}) \oplus \bigoplus_{j=i+1}^{m-1} \bigoplus_{r=j+1}^{m} \left[V_{n_j}^{r_j} \otimes V_{n_r}^{r_r} \right],$$

as $k[v']$-modules. Moreover, for all $1 \leq i \leq m$, we also have the $k[v']$-modules isomorphism:

$$\wedge^2(V_{n_i}^{r_i}) \cong \wedge^2(V_{n_i})^{r_i} \oplus (V_{n_i} \otimes V_{n_i})^{\frac{r_i(r_i-1)}{2}}. \tag{6.16}$$

Now, using (6.16), recursively, for all $1 \leq i \leq m$, one can show that

$$\wedge^3(V_{n_i}^{r_i}) \cong (\wedge^3(V_{n_i}))^{r_i} \oplus (\wedge^2(V_{n_i}) \otimes V_{n_i})^{r_i(r_i-1)} \oplus (V_{n_i} \otimes V_{n_i} \otimes V_{n_i})^{\frac{r_i(r_i-1)(r_i-2)}{2}},$$

as $k[v']$-modules. At this point, keeping in mind that $p = 2$, we compute the Jordan form of the action of v' on each $\wedge^3(V_i)$, $\wedge^2(V_i)$, $\wedge^2(V_i) \otimes V_j$ and $V_i \otimes V_i \otimes V_i$, $1 \leq i \leq j \leq m$, appearing in (6.15). This calculation can be done either by hand or using a computer. Lastly, having established the Jordan form of the action of v' on $\wedge^3(W)$ in (6.15), we calculate $\dim((\wedge^3(W))_{v'}(1))$ and, consequently, we determine $\dim((L_G(\omega_3 + \omega_4))_{v'}(1))$.

Coming back to the table, we see that $\dim((L_G(\omega_3 + \omega_4))_u(1)) \leq 28$, for all non-identity unipotent elements $u \in G$. Moreover, there exist elements $u \in G$ for which $\dim((L_G(\omega_3 + \omega_4))_u(1)) = 28$, for example $x_{\alpha_2}(1)$ or $x_{\alpha_3}(1)x_{\alpha_4}(1)$.

In conclusion, we showed that $\dim(V_{\tilde{u}}(1)) \leq 28$ for all non-identity unipotent elements $\tilde{u} \in \tilde{G}$ and that there exist $\tilde{u} \in \tilde{G}$ for which the bound is attained, for example $x_{\tilde{\alpha}_2}(1)$ or $x_{\tilde{\alpha}_3}(1)x_{\tilde{\alpha}_4}(1)$, see Lemma 2.3.10. Lastly, we note that $\dim(V_{\tilde{u}}(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $\tilde{u} \in \tilde{G}$. □

Proposition 6.3.14. *Let k be an algebraically closed field of characteristic $p \neq 2$. Assume $\ell = 4$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3 + \tilde{\omega}_4)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 34,$$

where there exist $u \in \tilde{G}$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$.

Proof. To begin, recall Decomposition (6.2) of Proposition 6.2.9 which states:

$$V\mid_{[L_1,L_1]} = L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4) \oplus L_{L_1}(2\tilde{\omega}_3) \oplus L_{L_1}(2\tilde{\omega}_4) \oplus L_{L_1}(\tilde{\omega}_2) \oplus L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4).$$

Let $u \in \tilde{G}$ be a non-identity unipotent element and let u' be the representative listed in Table 6.3.2 of the unipotent $\tilde{G}$-conjugacy class of u. Then, by Decomposition (6.2) and, by Inequality (2.7), we deduce that:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}(1)) = 2\dim((L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4))_{u'_{L_1}}(1)) + \dim((L_{L_1}(2\tilde{\omega}_3))_{u'_{L_1}}(1))) + \\ + \dim((L_{L_1}(2\tilde{\omega}_4))_{u'_{L_1}}(1))) + \dim((L_{L_1}(\tilde{\omega}_2))_{u'_{L_1}}(1)).$$

Now, by Corollary 3.3.9, as $p \neq 2$, we have $\dim((L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4))_{u'_{L_1}}(1)) \leq 9$, while, by Proposition 3.3.4, respectively by Proposition 3.3.5, we have $\dim((L_{L_1}(\tilde{\omega}_2))_{u'_{L_1}}(1) \leq 4$, respectively $\dim((L_{L_1}(2\tilde{\omega}_3))_{u'_{L_1}}(1)) \leq 6$ and $\dim((L_{L_1}(2\tilde{\omega}_4))_{u'_{L_1}}(1)) \leq 6$. It follows that $\dim(V_u(1)) \leq 34$ for all non-identity unipotent elements $u \in \tilde{G}$.

Lastly, assume that u belongs to the unipotent conjugacy class with representative $x_{\tilde{\alpha}_2}(1)$. First, we note that $(x_{\tilde{\alpha}_2}(1))_{Q_1} = 1$. Secondly, using Table 6.3.2, we see that $\dim((L_{L_1}(\tilde{\omega}_3 + \tilde{\omega}_4))_{x_{\tilde{\alpha}_2}(1)}(1)) = 9$, by Corollary 3.3.9, while by Proposition 3.3.5, respectively by Proposition 3.3.4, we have $\dim((L_{L_1}(2\tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1)) = 6$ and $\dim((L_{L_1}(2\tilde{\omega}_4))_{x_{\tilde{\alpha}_2}(1)}(1)) = 6$, respectively $\dim((L_{L_1}(\tilde{\omega}_2))_{x_{\tilde{\alpha}_2}(1)}(1)) = 4$. Therefore, by Identity (2.8), it follows that $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1)) = 34$. This shows that there exist unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) = 34$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 34$ for all non-identity unipotent elements $u \in \tilde{G}$ and that there exist $u \in \tilde{G}$ for which the bound is attained, for example $x_{\tilde{\alpha}_2}(1)$. Lastly, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$. $\square$

At this point, we have determined $\max\limits_{u \in \tilde{G}_u \setminus \{1\}} \dim(V_u(1))$ for all the irreducible modules V of $\tilde{G}$ of type D_4 with highest weights featured in Table 2.7.5. In order to determine these maximums, we used the fact that all nontrivial unipotent $\tilde{G}$-conjugacy classes admit a representative u' such that $u'_{L_1} \neq 1$. We now consider the case when $\ell \geq 5$. We will show that each non-identity unipotent $\tilde{G}$-conjugacy class admits a representative $\tilde{u}'$ with the property that $\tilde{u}'_{L_1} \neq 1$, see Proposition 6.3.25. Now, to prove such a result, we first need to use the algorithm described in [Kor18, Subsection 2.8.3] to construct class representatives in G. Afterwards, we will show that the each non-identity unipotent G-conjugacy class admits a representatives u' with the property that $S_{u'} \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$, see Propositions 6.3.19 and 6.3.24.

For the moment, we will assume $p \neq 2$. By Theorem 2.9.2, we know that the Jordan normal form completely determines unipotent conjugacy classes in G, unless the form consists of only even sized blocks, in which case there are two unipotent classes associated to that Jordan form. Now, let $u \in G$ be a non-identity unipotent element and let V_i, $1 \leq i \leq \mathrm{ord}(u)$, be the unique, up to isomorphism, indecomposable $k[u]$-module with $\dim(V_i) = i$ and on which u acts as the full Jordan block of size i. Following [Kor18, Subsection 2.8.3], to u we associate the (possible empty) sequences $(e_i)_{1 \leq i \leq t}$, $(o_i)_{t+1 \leq i \leq t+s}$ and $(o'_i)_{t+1 \leq i \leq t+s}$ with the property that:

$$W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} (V_{2o_j+1} \oplus V_{2o'_j+1}), \tag{6.17}$$

where $2 \leq e_1 \leq \cdots \leq e_t$ are even and $0 \leq o_{t+1} \leq o'_{t+1} \leq \cdots \leq o_{t+s} \leq o'_{t+s}$ are such that $\sum\limits_{i=1}^{t} e_i + \sum\limits_{j=t+1}^{t+s} (o_j + o'_j + 1) = \ell$. We note that the decomposition in (6.17) completely determines the unipotent conjugacy class of u in G, unless $s = 0$, in which case u belongs to a split-class from $\mathrm{O}(W, Q)$ to G. Lemma 6.3.15, for the split case, and Lemma 6.3.17, for the non-split case, give an algorithm for identifying unipotent elements $u \in G$ as products of root elements which correspond to given decompositions $W\mid_{k[u]}$ as in (6.17).

Lemma 6.3.15. [Kor18, Lemma 2.8.19] *Let k be an algebraically closed field of characteristic $p \neq 2$. Let $2 \leq e_1 \leq \cdots \leq e_t$ be such that e_i is even, for all $1 \leq i \leq t$, and $\sum_{i=1}^{t} e_t = \ell$. Set $k_1 = 1$ and $k_i = 1 + e_1 + \cdots + e_{i-1}$, for all $2 \leq i \leq t+1$. For all $1 \leq i \leq t$, define $u_i = \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1)$. Set $u'_t = x_{\alpha_\ell}(1)$ if $e_t = 2$, and $u'_t = \prod_{j=k_t}^{\ell-2} x_{\alpha_j}(1)x_{\alpha_\ell}(1)$ if $e_t \geq 4$. Then, the unipotent elements $u_1 \cdots u_{t-1}u_t$ and $u_1 \cdots u_{t-1}u'_t$ are representatives for the two split-classes of unipotent elements $u \in G$ with $W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2$.*

Remark 6.3.16. *Let $u \in G$ be a non-identity unipotent element and let $W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2$ be the corresponding decomposition of W as a $k[u]$-module. Then, by Lemma 6.3.15, we can assume, without loss of generality, that $u = u_1 \cdots u_{t-1} \cdot u_t$, or $u = u_1 \cdots u_{t-1} \cdot u'_t$. In the first case, since $S_{u_i} = \{\alpha_{k_i}, \ldots, \alpha_{k_{i+1}-2}\}$, it follows that*

$$S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \alpha_{k_2}, \ldots, \alpha_{k_3-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{\ell-1}\},$$

since $k_1 = 1$ and $k_{t+1} - 2 = \ell - 1$. In the second case, since $S_{u'_t} = \{\alpha_\ell\}$, or $S_{u'_t} = \{\alpha_{k_t}, \ldots, \alpha_{\ell-2}, \alpha_\ell\}$, depending on whether $e_t = 2$ or $e_t \geq 4$, it follows that

$$S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \alpha_{k_2}, \ldots, \alpha_{k_3-2}, \ldots, \alpha_{k_{t-1}}, \ldots, \alpha_{k_t-2}, \alpha_\ell\}, \text{ or}$$

$$S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \ldots, \alpha_{k_{t-1}}, \ldots, \alpha_{k_t-2}, \alpha_{k_t}, \ldots, \alpha_{\ell-2}, \alpha_\ell\}.$$

The following lemma is a correction of [Kor18, Lemma 2.8.17], which had a mistake when expressing the element u_i, $t+1 \leq i \leq t+s$, in the case of $o_i = 0$ and $o'_i > 0$. We obtain the result by following the argument in [Kor18, p.41-42] and using [Kor18, Lemma 2.8.1, 2.8.14 and 2.8.15].

Lemma 6.3.17. [Kor18, Lemma 2.8.1, 2.8.14 and 2.8.15] *Let k be an algebraically closed field of characteristic $p \neq 2$. Let $2 \leq e_1 \leq \cdots \leq e_t$ be even and let $0 \leq o_{t+1} \leq o'_{t+1} \leq \cdots \leq o_{t+s} \leq o'_{t+s}$, where $s \geq 1$, be such that $\sum_{i=1}^{t} e_i + \sum_{j=t+1}^{t+s} (o_j + o'_j + 1) = \ell$. For all $t+1 \leq i \leq t+s$, set $e_i = o_i + o'_i + 1$. Moreover, set $k_1 = 1$ and $k_i = 1 + e_1 + \cdots + e_{i-1}$, for $2 \leq i \leq t+s$. For $t+1 \leq i \leq t+s-1$ with $o_i, o'_i > 0$, set:*

$$v_i = \prod_{j=k_i}^{k_i+o_i-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1),$$

while, for $i = t+s$ with $o_{t+s}, o'_{t+s} > 0$, set:

$$v_{t+s} = \prod_{j=k_{t+s}}^{k_{t+s}+o_{t+s}-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_{t+s}+o_{t+s}-1}+\cdots+\alpha_{\ell-1}}(1) \cdot x_{\alpha_{k_{t+s}+o_{t+s}-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1).$$

For all $t+1 \leq i \leq t+s-1$ with $o'_i > 0$, set:

$$v'_i = \prod_{j=k_i+o_i}^{k_{i+1}-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(-1),$$

while, for $i = t+s$ with $o'_{t+s} > 0$, set:

$$v'_{t+s} = \prod_{j=k_{t+s}+o_{t+s}}^{\ell-1} x_{\alpha_j}(1) \cdot x_{\alpha_\ell}(-1),$$

For all $1 \leq i \leq t+s-1$, define:

$$u_i = \begin{cases} \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1), \text{ if } 1 \leq i \leq t; \\ 1, \text{ if } t+1 \leq i \leq t+s-1 \text{ and } o_i = o'_i = 0; \\ \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1), \text{ if } t+1 \leq i \leq t+s-1 \text{ and } o_i = 0, o'_i > 0; \\ v_i v'_i, \text{ if } t+1 \leq i \leq t+s-1 \text{ and } o_i, o'_i > 0; \end{cases}$$

$$\text{and } u_{t+s} = \begin{cases} \prod_{j=k_{t+s}}^{\ell} x_{\alpha_j}(1), \text{ if } o_{t+s} = 0, o'_{t+s} > 0; \\ v_{t+s} v'_{t+s}, \text{ if } o_{t+s}, o'_{t+s} > 0. \end{cases}$$

Then $u = u_1 \cdots u_{t+s}$ satisfies $W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} (V_{2o_j+1} \oplus V_{2o'_j+1})$.

Remark 6.3.18. *Let $u \in G$ be a non-identity unipotent element and let $W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} (V_{2o_j+1} \oplus V_{2o'_j+1})$, where $s \geq 1$, be the corresponding decomposition of W as a $k[u]$-module. Then, by Lemma 6.3.17, we can assume, without loss of generality, that $u = u_1 \cdots u_{t+s}$. In what follows we identify S_u. For this, we distinguish the following cases:*

<u>Case 1</u>: $t = 0$. As u is nontrivial, it follows that $o'_s > 1$. Let $1 \leq j \leq s$ be such that $o'_{j-1} = 0$ and $o'_j > 0$. Then $u_i = 1$, for all $1 \leq i \leq j-1$, as $o_i = o'_i = 1$.

<u>Sub-case 1.1</u>: Assume $j = s$. Then $u = u_s$, $k_s = s$ and $\ell = s + o_s + o'_s$. If $o_s = 0$, then $u = \prod_{r=s}^{\ell} x_{\alpha_r}(1)$. Similarly, if $o_s > 0$, then

$$u = \prod_{r=s}^{s+o_s-2} x_{\alpha_r}(1) \cdot x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-1}}(1) \cdot x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1) \cdot \prod_{r=s+o_s}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\alpha_\ell}(-1).$$

Using the commutator relations [MT11, Theorem 11.8] *with the structure constants in* [Cav10, Lemma 2.5.5], *we determine that:*

(1) $x_{\alpha_r}(1)$, $s+o_s \leq r \leq \ell-2$, *commutes with both* $x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-1}}(1)$ *and* $x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1)$*;*

(2) $[x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1), x_{\alpha_{\ell-1}}(1)] = x_{\alpha_{s+o_s-1}+\cdots+\alpha_\ell}(-1)$*;*

(3) $x_{\alpha_\ell}(-1)$ *commutes with* $x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1)$.

Therefore, we have:

$$u = \prod_{r=s}^{s+o_s-2} x_{\alpha_r}(1) \cdot \prod_{r=s+o_s}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-1}}(1) \cdot x_{\alpha_\ell}(-1) \cdot x_{\alpha_{s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1). \quad (6.18)$$

In both cases we see that $\alpha_\ell \in S_u$.

<u>Sub-case 1.2</u>*: Assume that* $j \leq s-1$. *Then, for all* $j+1 \leq i \leq s-1$, *we have* $o_i, o_i' > 0$ *and so*

$$u_i = \prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}}(1) \cdot x_{\alpha_{k_i+o_i-1}+\cdots+\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1) \cdot$$
$$\cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(-1).$$

For all $j \leq i \leq s$, *set*

$$\beta_i := \alpha_{k_i+o_i-1} + \cdots + \alpha_{k_{i+1}-2}$$
$$\gamma_i := \alpha_{k_{i+1}-2} + 2\alpha_{k_{i+1}-1} + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$$
$$\delta_i := \alpha_{k_i+o_i-1} + \cdots + \alpha_{k_{i+1}-2} + 2\alpha_{k_{i+1}-1} + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell.$$

Using the commutator relations [MT11, Theorem 11.8] *with the structure constants in* [Cav10, Lemma 2.5.5], *we determine that:*

(1) $x_{\alpha_r}(1)$, $k_i + o_i \leq r \leq k_{i+1} - 3$, *commutes with both* $x_{\beta_i}(1)$ *and* $x_{\delta_i}(1)$*;*

(2) $[x_{\delta_i}(1), x_{\alpha_{k_{i+1}-2}}(1)] = x_{\xi_i}(-1)$, *where*

$$\xi_i := \alpha_{k_i+o_i-1} + \cdots + \alpha_{k_{i+1}-3} + 2\alpha_{k_{i+1}-2} + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell.$$

Moreover, $x_{\xi_i}(-1)$ *commutes with* $x_{\alpha_{k_{i+1}-2}}(1)$, $x_{\delta_i}(1)$ *and* $x_{\gamma_i}(-1)$*;*

(3) $x_{\alpha_{k_{i+1}-2}}(1)$ *commutes with* $x_{\beta_i}(1)$.

Thus, for all $j+1 \leq i \leq s-1$, *we have*

$$u_i = \prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \cdot x_{\beta_i}(1) \cdot x_{\gamma_i}(-1) \cdot x_{\delta_i}(1) \cdot x_{\xi_i}(-1).$$

Moreover, by (6.18), *we have*

$$u_s = \prod_{r=k_s}^{k_s+o_s-2} x_{\alpha_r}(1) \cdot \prod_{r=k_s+o_s}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\beta_s}(1) \cdot x_{\alpha_\ell}(-1) \cdot x_{\alpha_{k_s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1).$$

Lastly, to determine u_j, we note that, if $o_j > 0$, then u_j has the same form as u_i with $j+1 \leq i \leq s-1$, while, if $o_j = 0$, then $u_j = \prod_{r=k_j}^{k_{j+1}-2} x_{\alpha_r}(1) \cdot x_{\gamma_j}(1)$. Therefore, by the commutator relations [MT11, Theorem 11.8], *we have*

$$\begin{aligned}
u &= u_1 \cdots u_s = u_j \cdot u_{j+1} \cdots u_s \\
&= u_j \cdot \prod_{i=j+1}^{s-1} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \cdot x_{\beta_i}(1) \cdot x_{\gamma_i}(-1) \cdot x_{\delta_i}(1) \cdot x_{\xi_i}(-1) \right) \cdot \\
&\cdot \prod_{r=k_s}^{k_s+o_s-2} x_{\alpha_r}(1) \cdot \prod_{r=k_s+o_s}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\beta_s}(1) \cdot x_{\alpha_\ell}(-1) \cdot x_{\alpha_{k_s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1) \\
&= u_j \cdot \prod_{i=j+1}^{s-1} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{r=k_s}^{k_s+o_s-2} x_{\alpha_r}(1) \cdot \prod_{i=j+1}^{s-1} x_{\beta_i}(1) \cdot \prod_{r=k_s+o_s}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\beta_s}(1) \cdot \\
&\cdot x_{\alpha_\ell}(-1) \cdot x_{\alpha_{k_s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1) \cdot \prod_{i=j+1}^{s-1} x_{\gamma_i}(-1) \cdot \prod_{i=j+1}^{s-1} x_{\delta_i}(1) \cdot \prod_{i=j+1}^{s-1} x_{\xi_i}(-1).
\end{aligned}$$

If $o_j > 0$, then:

$$\begin{aligned}
u = &\prod_{i=j}^{s-1} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{r=k_s}^{k_s+o_s-2} x_{\alpha_r}(1) \cdot \prod_{i=j}^{s-1} x_{\beta_i}(1) \cdot \prod_{r=k_s+o_s}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\beta_s}(1) \cdot \\
&\cdot x_{\alpha_\ell}(-1) \cdot x_{\alpha_{k_s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1) \cdot \prod_{i=j}^{s-1} x_{\gamma_i}(-1) \cdot \prod_{i=j}^{s-1} x_{\delta_i}(1) \cdot \prod_{i=j}^{s-1} x_{\xi_i}(-1),
\end{aligned} \tag{6.19}$$

while, if $o_j = 0$, then

$$\begin{aligned}
u = &\prod_{r=k_j}^{k_{j+1}-2} x_{\alpha_r}(1) \cdot \prod_{i=j+1}^{s-1} \left(\prod_{r=k_i}^{k_i+o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_i+o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{r=k_s}^{k_s+o_s-2} x_{\alpha_r}(1) \cdot \prod_{i=j+1}^{s-1} x_{\beta_i}(1) \cdot \prod_{r=k_s+o_s}^{\ell-1} x_{\alpha_r}(1) \cdot \\
&\cdot x_{\beta_s}(1) \cdot x_{\alpha_\ell}(-1) \cdot x_{\alpha_{k_s+o_s-1}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1) \cdot x_{\gamma_j}(1) \cdot \prod_{i=j+1}^{s-1} x_{\gamma_i}(-1) \cdot \prod_{i=j+1}^{s-1} x_{\delta_i}(1) \cdot \prod_{i=j+1}^{s-1} x_{\xi_i}(-1).
\end{aligned} \tag{6.20}$$

In both cases we see that $\alpha_\ell \in S_u$.

<u>Case 2</u>: $t \geq 1$. Then $u = u_1 \cdots u_t \cdot u_{t+1} \cdots u_{t+s}$, where

$$u_1 \cdots u_t = \prod_{i=1}^{t} \left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) = x_{\alpha_1}(1) \cdots x_{\alpha_{k_2-2}}(1) \cdot x_{\alpha_{k_2}}(1) \cdots x_{\alpha_{k_3-2}}(1) \cdots x_{\alpha_{k_t}}(1) \cdots x_{\alpha_{k_{t+1}-2}}(1).$$

Now, if $o'_{t+s} = 0$, then $o_j = o'_j = 0$ for all $t+1 \leq j \leq s$, and we have

$$S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \alpha_{k_2}, \ldots, \alpha_{k_3-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}\}.$$

On the other hand, if $o'_{t+s} > 0$, let $t+1 \leq j \leq t+s$ be such that $o'_{j-1} = 0$ and $o'_j > 0$. Then $u_{t+1} \cdots u_{t+s}$ is as in (6.19), or (6.20), depending on whether $o_j > 0$, or $o_j = 0$. Since all $\beta \in S_{u_1 \cdots u_t}$ and all $\gamma \in S_{u_{t+1} \cdots u_{t+s}}$ are such that $\beta \preceq \gamma$, it follows that the product in $u_1 \cdots u_t \cdot u_{t+1} \cdots u_{t+s}$, with $u_{t+1} \cdots u_{t+s}$ as in (6.19), or (6.20), respects the total order $\preceq$. Moreover, in this case, we have $\alpha_\ell \in S_u$.

Proposition 6.3.19. *Let k be an algebraically closed field of characteristic $p \neq 2$. Then each non-identity unipotent conjugacy class in G admits a representative v' with the property that $S_{v'} \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$.*

Proof. Let $u \in G$ be a non-identity unipotent element and let

$$W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2 \oplus \bigoplus_{j=t+1}^{t+s} (V_{2o_j+1} \oplus V_{2o'_j+1}), \tag{6.21}$$

where $2 \leq e_1 \leq \cdots \leq e_t$ are even and $0 \leq o_{t+1} \leq o'_{t+1} \leq \cdots \leq o_{t+s} \leq o'_{t+s}$ are such that $\sum_{i=1}^{t} e_i + \sum_{j=t+1}^{t+s} (o_j + o'_j + 1) = \ell$, be the corresponding decomposition of $W\mid_{k[u]}$. We will use Lemmas 6.3.15 and 6.3.17 and their respective remarks to prove that the unipotent conjugacy class of u in G admits a representative v' with the property that $S_{v'} \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$.

First, we consider the case when u belongs to a split unipotent conjugacy class, i.e. $W\mid_{k[u]} = \bigoplus_{i=1}^{t} V_{e_i}^2$. Then, by Lemma 6.3.15, we can assume, without loss of generality, that $u = u_1 \cdots u_t$, or $u = u_1 \cdots u_{t-1} u'_t$. By Remark 6.3.16, we see that, in the first case we have $\alpha_{\ell-1} \in S_u$, while, in the second, we have $\alpha_\ell \in S_u$. It follows that $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$.

We can now consider the case when u belongs to a non-split conjugacy class, i.e. $s \geq 1$. Then, the decomposition in (6.21) completely determines the conjugacy class of u. If $o'_{t+s} > 0$, then, by <u>Case 1</u> and <u>Case 2</u> of Remark 6.3.18, we have $\alpha_\ell \in S_u$. We can thus assume that $o'_{t+s} = 0$. In this case $o_j = o'_j = 0$, for all $t+1 \leq j \leq t+s$, and so $u_{t+1} \cdots u_{t+s} = 1$. Now, as u is nontrivial, it follows that $t \geq 1$ and, by <u>Case 2</u> of Remark 6.3.18, we have

$$S_u = \{\alpha_1, \ldots, \alpha_{k_2-2}, \alpha_{k_2}, \ldots, \alpha_{k_3-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}\}.$$

If $t \geq 2$, then $\alpha_{k_2} \in S_u$, where $k_2 = 1 + e_1 \geq 3$. If $t = 1$ and $e_1 \geq 4$, then $\alpha_{k_2-2} \in S_u$, where $k_2 - 2 = e_1 - 1 \geq 3$. Lastly, if $t = 1$ and $e_1 = 2$, then $u = x_{\alpha_1}(1)$ and $W\mid_{k[x_{\alpha_1}(1)]} = V_2^2 \oplus V_1^{2\ell-4}$. However, by [LS12, Subsection 3.3.2], $x_{\alpha_2}(1)$ is another unipotent element of G with $W\mid_{k[x_{\alpha_2}(1)]} = V_2^2 \oplus V_1^{2\ell-4}$. Therefore, $x_{\alpha_1}(1)$ and $x_{\alpha_2}(1)$ are G-conjugate and we choose $x_{\alpha_2}(1)$ as representative of the unipotent class of u. This completes the proof of the proposition. □

We will now consider the case of $p = 2$. Let $u \in G$ be a unipotent element. Theorem 2.9.11 together with Proposition 2.9.20 tell us that the unipotent conjugacy class of u in G is completely determined by the Hesselink normal form of u, except for the case when $W\mid_{k[u]} = \bigoplus_{i=1}^{t} W(n_i)$ with n_i even for all $1 \leq i \leq t$, when we have two classes corresponding

to this orthogonal decomposition. As in the $p \neq 2$ case, we refer to these classes as split. Now, following [Kor18, Subsection 2.8.3], to u we associate the (possibly empty) sequences $(e_i)_{1\leq i\leq t}$, $(o_i)_{t+1\leq i\leq t+s}$ and $(o'_i)_{t+1\leq i\leq t+s}$ with the property that:

$$W\mid_{k[u]}=\bigoplus_{i=1}^{t} W(e_i)\oplus\bigoplus_{j=t+1}^{t+s}(V(2o_j)\oplus V(2o'_j)), \tag{6.22}$$

where $1\leq e_1\leq\cdots\leq e_t$ and $1\leq o_{t+1}\leq o'_{t+1}\leq\cdots\leq o_{t+s}\leq o'_{t+s}$ are such that $\sum_{i=1}^{t} e_i+\sum_{j=t+1}^{t+s}(o_j+o'_j)=\ell$. Lemma 6.3.20, for the split case, and Lemma 6.3.22, for the non-split case, give an algorithm on identifying unipotent elements $u\in G$ corresponding to given orthogonal decompositions $W\mid_{k[u]}$ as in (6.22).

Lemma 6.3.20. [Kor18, Lemma 2.8.19] *Let k be an algebraically closed field of characteristic $p=2$. Then, the unipotent elements $u_1\cdots u_{t-1}u_t$ and $u_1\cdots u_{t-1}u'_t$ of G constructed in Lemma* 6.3.15 *are representatives for the two split conjugacy classes of unipotent elements $u\in G$ with $W\mid_{k[u]}=\bigoplus_{i=1}^{t} W(e_i)$.*

Remark 6.3.21. *Let $u\in G$ be a unipotent element and let $W\mid_{k[u]}=\bigoplus_{i=1}^{t} W(e_i)$ be the corresponding orthogonal decomposition of W. Then, by Lemma* 6.3.20*, we can assume without loss of generality that $u=u_1\cdots u_{t-1}u_t$, or $u=u_1\cdots u_{t-1}u'_t$. Then, in the first case, by Remark* 6.3.16*, we have*

$$S_u=\{\alpha_1,\ldots,\alpha_{k_2-2},\alpha_{k_2},\ldots,\alpha_{k_3-2},\ldots,\alpha_{k_t},\ldots,\alpha_{\ell-1}\},$$

while, in the second, depending on whether $e_t=2$, or $e_t=4$, we have

$$S_u=\{\alpha_1,\ldots,\alpha_{k_2-2},\alpha_{k_2},\ldots,\alpha_{k_3-2},\ldots,\alpha_{k_{t-1}},\ldots,\alpha_{k_t-2},\alpha_\ell\},\ \text{or}$$

$$S_u=\{\alpha_1,\ldots,\alpha_{k_2-2},\ldots,\alpha_{k_{t-1}},\ldots,\alpha_{k_t-2},\alpha_{k_t},\ldots,\alpha_{\ell-2},\alpha_\ell\}.$$

The following lemma is a correction of Lemma 2.8.18 of [Kor18]. Its proof follows the discussion in [Kor18, p.41-42] and uses [Kor18, Lemmas 2.8.1, 2.8.14, 2.8.16 and 2.8.18].

Lemma 6.3.22. [Kor18, Lemmas 2.8.1, 2.8.14, 2.8.16 and 2.8.18] *Let k be an algebraically closed field of characteristic $p=2$. Let $1\leq e_1\leq\cdots\leq e_t$ and $1\leq o_{t+1}\leq o'_{t+1}\leq\cdots\leq o_{t+s}\leq o'_{t+s}$, where $s\geq 1$, be such that $\sum_{i=1}^{t} e_i+\sum_{j=t+1}^{t+s}(o_j+o'_j)=\ell$. Set $e_i=o_i+o'_i$, for all $t+1\leq i\leq t+s$. Set $k_1=1$ and $k_i=1+e_1+\cdots+e_{i-1}$, for all $2\leq i\leq t+s$. For $t+1\leq i\leq t+s-1$ with $o_i=1$, set:*

$$v_i=\prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1)\cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1),$$

while, for $i = t+s$ with $o_{t+s} = 1$, set:

$$v_{t+s} = \prod_{j=k_{t+s}}^{\ell} x_{\alpha_j}(1).$$

Similarly, for $t+1 \leq i \leq t+s-1$ with $o_i > 1$, set:

$$v_i' = \prod_{j=k_i}^{k_{i+1}-o_i-2} x_{\alpha_j}(1) \cdot x_{\alpha_{k_{i+1}-o_i-1}+\cdots+\alpha_{k_{i+1}-2}}(1) \cdot$$
$$\cdot\, x_{\alpha_{k_{i+1}-o_i-1}+\cdots+\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1) \cdot \prod_{j=k_{i+1}-o_i}^{k_{i+1}-2} x_{\alpha_j}(1),$$

while, for $i = t+s$ with $o_{t+s} > 1$, set

$$v_{t+s}' = \prod_{j=k_{t+s}}^{\ell-o_{t+s}-1} x_{\alpha_j}(1) \cdot x_{\alpha_{\ell-o_{t+s}}+\cdots+\alpha_{\ell-1}}(1) \cdot x_{\alpha_{\ell-o_{t+s}}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1) \cdot \prod_{j=\ell-o_{t+s}+1}^{\ell-1} x_{\alpha_j}(1).$$

Lastly, define:

$$u_i = \begin{cases} 1, \text{ if } 1 \leq i \leq t \text{ and } e_i = 1; \\ \prod_{j=k_i}^{k_{i+1}-2} x_{\alpha_j}(1), \text{ if } 1 \leq i \leq t \text{ and } e_i \geq 2; \\ v_i, \text{ if } t+1 \leq i \leq t+s \text{ and } o_i = 1; \\ v_i', \text{ if } t+1 \leq i \leq t+s \text{ and } o_i > 1. \end{cases}$$

Then $u = u_1 \cdots u_{t+s}$ satisfies $W\,|_{k[u]} = \bigoplus_{i=1}^{t} W(e_i) \oplus \bigoplus_{j=t+1}^{t+s} (V(2o_j) \oplus V(2o_j'))$.

Remark 6.3.23. *Let $u \in G$ be a non-identity unipotent element and let $W\,|_{k[u]} = \bigoplus_{i=1}^{t} W(e_i) \oplus \bigoplus_{j=t+1}^{t+s} (V(2o_j) \oplus V(2o_j'))$, where $s \geq 1$, be the corresponding decomposition of W. Then, by Lemma 6.3.22, we can assume, without loss of generality, that $u = u_1 \cdots u_{t+s}$. We set out to determine S_u. For this, we consider the following cases:*

Case 1: $t = 0$. If $o_s = 1$, then $o_i = 1$, for all $1 \leq i \leq s$, and, using the commutator relations [MT11, Theorem 11.8]*, we get:*

$$u = \prod_{i=1}^{s-1} \left[\left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1) \right] \cdot \prod_{r=k_s}^{\ell} x_{\alpha_r}(1)$$
$$= \prod_{i=1}^{s-1} \left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \right) \cdot \prod_{r=k_s}^{\ell} x_{\alpha_r}(1) \cdot \prod_{i=1}^{s-1} x_{\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1).$$

For all $1 \le i \le s-1$, *set* $\beta_i := \alpha_{k_{i+1}-2} + 2\alpha_{k_{i+1}-1} + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$. *Then*

$$u = \prod_{i=1}^{s-1}\left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1)\right) \cdot \prod_{r=k_s}^{\ell} x_{\alpha_r}(1) \cdot \prod_{i=1}^{s-1} x_{\beta_i}(1),$$

therefore $\alpha_\ell \in S_u$.

If $o_s > 1$, *let* $1 \le j \le s$ *be such that* $o_{j-1} = 1$ *and* $o_j > 1$. *Then:*

$$\begin{aligned} u = &\prod_{i=1}^{j-1}\left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1) \cdot x_{\beta_i}(1)\right) \cdot \prod_{i=j}^{s-1}\Bigg[\left(\prod_{r=k_i}^{k_{i+1}-o_i-2} x_{\alpha_r}(1)\right) \cdot x_{\alpha_{k_{i+1}-o_i-1}+\cdots+\alpha_{k_{i+1}-2}}(1) \cdot \\ &\cdot x_{\alpha_{k_{i+1}-o_i-1}+\cdots+\alpha_{k_{i+1}-2}+2\alpha_{k_{i+1}-1}+\cdots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell}(1) \cdot \prod_{r=k_{i+1}-o_i}^{k_{i+1}-2} x_{\alpha_r}(1)\Bigg] \cdot \\ &\cdot \prod_{r=k_s}^{\ell-o_s-1} x_{\alpha_r}(1) \cdot x_{\alpha_{\ell-o_s}+\cdots+\alpha_{\ell-1}}(1) \cdot x_{\alpha_{\ell-o_s}+\cdots+\alpha_{\ell-2}+\alpha_\ell}(1) \cdot \prod_{r=\ell-o_s+1}^{\ell-1} x_{\alpha_r}(1). \end{aligned}$$

For all $j \le i \le s-1$, *set*

$$\gamma_i := \alpha_{k_{i+1}-o_i-1} + \cdots + \alpha_{k_{i+1}-2} \text{ and}$$

$$\delta_i := \alpha_{k_{i+1}-o_i-1} + \cdots + \alpha_{k_{i+1}-2} + 2\alpha_{k_{i+1}-1} + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell.$$

Moreover, set

$$\gamma_s := \alpha_{\ell-o_s} + \cdots + \alpha_{\ell-1} \text{ and } \delta_s := \alpha_{\ell-o_s} + \cdots + \alpha_{\ell-2} + \alpha_\ell.$$

Then, using the commutator relations [MT11, Theorem 11.8] *with the structure constants in* [Cav10, Lemma 2.5.5], *we determine that the only noncommuting products in the above are:*

(1) $[x_{\delta_i}(1), x_{\alpha_{k_{i+1}-2}}(1)] = x_{\xi_i}(-1)$, *where*

$$\xi_i := \alpha_{k_{i+1}-o_i-1} + \cdots + \alpha_{k_{i+1}-3} + 2\alpha_{k_{i+1}-2} + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell;$$

(2) $[x_{\delta_s}(1), x_{\ell-1}(1)] = x_{\xi_s}(-1)$, *where* $\xi_s := \alpha_{\ell-o_s} + \cdots + \alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$.

Thus, it follows that

$$\begin{aligned} u = &\prod_{i=1}^{j-1}\left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1)\right) \cdot \prod_{i=1}^{j-1} x_{\beta_i}(1) \cdot \prod_{i=j}^{s-1}\left(\prod_{r=k_i}^{k_{i+1}-o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_{i+1}-o_i}^{k_{i+1}-2} x_{\alpha_r}(1) \cdot x_{\gamma_i}(1) \cdot x_{\delta_i}(1) \cdot x_{\xi_i}(-1)\right) \\ &\cdot \prod_{r=k_s}^{\ell-o_s-1} x_{\alpha_r}(1) \cdot \prod_{r=\ell-o_s+1}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\gamma_s}(1) \cdot x_{\delta_s}(1) \cdot x_{\xi_s}(-1) \\ = &\prod_{i=1}^{j-1}\left(\prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1)\right) \cdot \prod_{i=j}^{s-1}\left(\prod_{r=k_i}^{k_{i+1}-o_i-2} x_{\alpha_r}(1) \cdot \prod_{r=k_{i+1}-o_i}^{k_{i+1}-2} x_{\alpha_r}(1)\right) \cdot \prod_{i=j}^{s-1} x_{\gamma_i}(1) \cdot \prod_{r=k_s}^{\ell-o_s-1} x_{\alpha_r}(1) \cdot \\ &\cdot \prod_{r=\ell-o_s+1}^{\ell-1} x_{\alpha_r}(1) \cdot x_{\gamma_s}(1) \cdot \prod_{i=j}^{s} x_{\delta_i}(1) \cdot x_{\xi_s}(-1) \cdot \prod_{i=1}^{j-1} x_{\beta_i}(1) \cdot \prod_{i=j}^{s-1} x_{\xi_i}(-1). \end{aligned}$$

We note that $\alpha_{\ell-1} \in S_u$.

<u>Case 2</u>: $t \geq 1$. If $e_t = 1$, then $u = u_{t+1} \cdots u_{t+s}$ and, by <u>Case 1</u>, we have $\alpha_\ell \in S_u$, if $o_{t+s} = 1$, or $\alpha_{\ell-1} \in S_u$, if $o_{t+s} > 1$. We can thus assume that $e_t > 1$. Let $1 \leq j_1 \leq t$ be such that $e_{j_1-1} = 1$ and $e_{j_1} > 1$. Then $u_i = 1$, for all $1 \leq i \leq j_1 - 1$ and

$$u_{j_1} \cdots u_t = \prod_{i=j_1}^{t} \prod_{r=k_i}^{k_{i+1}-2} x_{\alpha_r}(1).$$

Therefore $S_{u_{j_1} \cdots u_t} = \{\alpha_{k_{j_1}}, \ldots, \alpha_{k_{j_1+1}-2}, \ldots, \alpha_{k_t}, \ldots, \alpha_{k_{t+1}-2}\}$. Furthermore, $u_{t+1} \cdots u_{t+s}$ is as in one of the situations of <u>Case 1</u> and, again, we have that $\alpha_\ell \in S_u$, if $o_{t+s} = 1$, or $\alpha_{\ell-1} \in S_u$, if $o_{t+s} > 1$.

Proposition 6.3.24. *Let k be an algebraically closed field of characteristic $p = 2$. Then each non-identity unipotent conjugacy class in G admits a representative v' with the property that $S_{v'} \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$.*

Proof. Let $u \in G$ be a non-identity unipotent element and let

$$W \mid_{k[u]} = \bigoplus_{i=1}^{t} W(e_i) \oplus \bigoplus_{j=t+1}^{t+s} (V(2o_j) \oplus V(2o'_j)), \tag{6.23}$$

where $1 \leq e_1 \leq \cdots \leq e_t$ and $1 \leq o_{t+1} \leq o'_{t+1} \leq \cdots \leq o_{t+s} \leq o'_{t+s}$ are such that $\sum_{i=1}^{t} e_i + \sum_{j=t+1}^{t+s} (o_j + o'_j) = \ell$, be the corresponding orthogonal decomposition of $W \mid_{k[u]}$. We will use Lemmas 6.3.20, 6.3.22 and their corresponding remarks to show that $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$.

We first consider the case when u belongs to a split class, i.e. $s = 0$. Then, by Lemma 6.3.20, we can assume, without loss of generality, that $u = u_1 \cdots u_t$, or $u = u_1 \cdots u_{t-1} \cdot u'_t$. In the first case, by Remark 6.3.21, we have $\alpha_{\ell-1} \in S_u$, while, in the second case, by the same result, we have $\alpha_\ell \in S_u$. Therefore, in both cases, $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$.

We can now assume that u belongs to a non-split conjugacy class, i.e. $s \geq 1$. Then the decomposition of $W \mid_{k[u]}$ in (6.23) completely determines the conjugacy class of u in G. In this case, by Remark 6.3.23, we have that $\alpha_\ell \in S_u$, if $o_{t+s} = 1$, or $\alpha_{\ell-1} \in S_u$, if $o_{t+s} > 1$. We conclude that $S_u \cap \{\alpha_2, \ldots, \alpha_\ell\} \neq \emptyset$. This completes the proof of the proposition. □

Proposition 6.3.25. *Let k be an algebraically closed field of characteristic $p \geq 0$. Then, each non-identity unipotent $\tilde{G}$-conjugacy class admits a representative $\tilde{u}'$ with the property that $S_{\tilde{u}'} \cap \{\tilde{\alpha}_2, \ldots, \tilde{\alpha}_\ell\} \neq \emptyset$.*

Proof. Assume by contradiction that there exists a non-identity unipotent $\tilde{G}$-conjugacy class $\tilde{C}$ with the property for all $\tilde{u}' \in \tilde{C}$ we have $S_{\tilde{u}'} \cap \{\tilde{\alpha}_2, \ldots, \tilde{\alpha}_\ell\} = \emptyset$. Now, as $\phi : \tilde{G} \to G$ is a central isogeny and $\phi(U_{\tilde{\alpha}}) = U_\alpha$, for all $\tilde{\alpha} \in \tilde{\Phi}$, it follows that the unipotent conjugacy class C of G given by $C = \phi(\tilde{C})$ has the property that $S_{u'} \cap \{\alpha_2, \ldots, \alpha_\ell\} = \emptyset$ for all $u' \in C$. However, if $p \neq 2$, this contradicts Proposition 6.3.19, while, if $p = 2$, it contradicts Proposition 6.3.24. We conclude that all non-identity unipotent $\tilde{G}$-conjugacy classes admit a representative $\tilde{u}'$ with the property that $S_{\tilde{u}'} \cap \{\tilde{\alpha}_2, \ldots, \tilde{\alpha}_\ell\} \neq \emptyset$. □

Proposition 6.3.26. *Let k be an algebraically closed field of characteristic $p = 2$. Assume $\ell = 5$ and let $V = L_{\tilde{G}}(\tilde{\omega}_3)$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 60,$$

where there exist $u \in \tilde{G}$ for which the bound is attained.

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$

Proof. Set $\tilde{\lambda} = \tilde{\omega}_3$. We first note that the $k\tilde{G}$-module V is self-dual. Secondly, using Lemma 2.4.7, we determine that $e_1(\tilde{\lambda}) = 2$, therefore:

$$V \mid_{[L_1,L_1]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\tilde{\gamma} \in \mathbb{N}\tilde{\Delta}_1} V_{\tilde{\lambda} - i\tilde{\alpha}_1 - \tilde{\gamma}}$ for all $0 \leq i \leq 2$. Now, by [Smi82, Proposition], we have $V^0 \cong L_{L_1}(\tilde{\omega}_3)$, therefore $V^2 \cong L_{L_1}(\tilde{\omega}_3)$, as $V^2 \cong (V^0)^*$ by Lemma 2.4.3. Recall that T' denotes the maximal torus $\tilde{T} \cap [L_1, L_1]$ of $[L_1, L_1]$. In V^1, the weight $(\tilde{\lambda} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3) \mid_{T'} = \tilde{\omega}_4 + \tilde{\omega}_5$ admits a maximal vector, therefore V^1 has a composition factor isomorphic to $L_{L_1}(\tilde{\omega}_4 + \tilde{\omega}_5)$. By comparing dimensions, we determine that:

$$V \mid_{[L_1,L_1]} \cong L_{L_1}(\tilde{\omega}_3) \oplus L_{L_1}(\tilde{\omega}_4 + \tilde{\omega}_5) \oplus L_{L_1}(\tilde{\omega}_3). \tag{6.24}$$

Let $u \in \tilde{G}$ be a non-identity unipotent element and let u' be a representative of the unipotent $\tilde{G}$-conjugacy class of u with the property that $u'_{L_1} \neq 1$. Note that, by Proposition 6.3.25 such a representative always exists. Then, by (2.7) and (6.24), we have:

$$\dim(V_u(1)) \leq \dim(V_{u'_{L_1}}) = 2\dim((L_{L_1}(\tilde{\omega}_3))_{u'_{L_1}}(1)) + \dim((L_{L_1}(\tilde{\omega}_4 + \tilde{\omega}_5))_{u'_{L_1}}(1)).$$

Now, by Proposition 6.3.13, we have $\dim((L_{L_1}(\tilde{\omega}_4 + \tilde{\omega}_5))_{u'_{L_1}}(1)) \leq 28$, while, by Proposition 6.3.8, we have $\dim((L_{L_1}(\tilde{\omega}_3))_{u'_{L_1}}(1)) \leq 16$. It follows that $\dim(V_u(1)) \leq 60$ for all non-identity unipotent elements $u \in \tilde{G}$. Lastly, consider the element $x_{\tilde{\alpha}_2}(1) \in \tilde{G}$. We note that $(x_{\tilde{\alpha}_2}(1))_{Q_1} = 1$. Therefore, by Table 6.3.3, Identity (2.8), Decomposition (6.24) and by the proofs of Propositions 6.3.8 and 6.3.13, it follows that $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1)) = 2\dim((L_{L_1}(\tilde{\omega}_3))_{x_{\tilde{\alpha}_2}(1)}(1)) + \dim((L_{L_1}(\tilde{\omega}_4 + \tilde{\omega}_5))_{x_{\tilde{\alpha}_2}(1)}(1)) = 32 + 28 = 60$.

In conclusion, we have shown that $\dim(V_u(1)) \leq 60$ for all non-identity unipotent elements $u \in \tilde{G}$ and that there exist $u \in \tilde{G}$, for example $x_{\tilde{\alpha}_2}(1)$, for which the bound in attained. Lastly, we note that we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$. □

Proposition 6.3.27. *Assume $\ell \geq 5$ and let $V = L_{\tilde{G}}(\tilde{\omega}_{\ell-1})$. Then for all non-identity unipotent elements $u \in \tilde{G}$ we have*

$$\dim(V_u(1)) \leq 3 \cdot 2^{\ell-3},$$

where there exist unipotent elements $u \in \tilde{G}$ for which the bound is attained.

In particular, in the case of $\ell = 5$, there exist non-identity unipotent elements $u \in \tilde{G}$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$. However, if $\ell \geq 6$, then we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in \tilde{G}$.

Proof. To begin, recall the Decomposition (6.3) from Proposition 6.2.10 which states:

$$V\mid_{[L_1,L_1]}\cong L_{L_1}(\tilde{\omega}_{\ell-1})\oplus L_{L_1}(\tilde{\omega}_\ell).$$

Let $u\in\tilde{G}$ be a non-identity unipotent element and let u' be a representative of the unipotent $\tilde{G}$-conjugacy class of u with the property that $u'_{L_1}\neq 1$, see Proposition 6.3.25. Then, by (6.3) and (2.7), it follows that:

$$\dim(V_u(1))\leq\dim(V_{u'_{L_1}}(1))=\dim((L_{L_1}(\tilde{\omega}_{\ell-1}))_{u'_{L_1}}(1))+\dim((L_{L_1}(\tilde{\omega}_\ell))_{u'_{L_1}}(1)).$$

We first consider the case of $\ell=5$. Then, either by Lemma 6.3.3, or by Lemma 6.3.4, depending on whether $p\neq 2$, or $p=2$, we determine that $\dim((L_{L_1}(\tilde{\omega}_4))_{u'_{L_1}}(1))\leq 6$ and $\dim((L_{L_1}(\tilde{\omega}_5))_{u'_{L_1}}(1))\leq 6$, respectively. It follows that $\dim(V_u(1))\leq 3\cdot 2^2$ for all non-identity unipotent elements $u\in\tilde{G}$. Moreover, if we consider the unipotent element $x_{\tilde{\alpha}_2}(1)\in\tilde{G}$, we have that $(x_{\tilde{\alpha}_2}(1))_{Q_1}=1$ and $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1))=3\cdot 2^2$, by either Table 6.3.2 and Lemma 6.3.3, or by Table 6.3.3 and Lemma 6.3.4. This shows that there exist unipotent elements $u\in\tilde{G}$ for which $\dim(V_u(1))=3\cdot 2^2$. We can now assume that $\ell\geq 6$. Recursively and using the result for $\ell=5$ as base case, one shows that $\dim((L_{L_1}(\tilde{\omega}_{\ell-1}))_{u'_{L_1}}(1))\leq 3\cdot 2^{\ell-4}$ and $\dim((L_{L_1}(\tilde{\omega}_\ell))_{u'_{L_1}}(1))\leq 3\cdot 2^{\ell-4}$. Therefore, $\dim(V_u(1))\leq 3\cdot 2^{\ell-3}$ for all non-identity unipotent elements $u\in\tilde{G}$. Moreover, by [LS12, Subsection 3.3.2 and Section 6.1], recursively and using the result for $\ell=5$ as base case, one can also show that for the unipotent element $x_{\tilde{\alpha}_2}(1)\in\tilde{G}$ we have $\dim(V_{x_{\tilde{\alpha}_2}(1)}(1))=3\cdot 2^{\ell-3}$.

In conclusion, we have shown that $\dim(V_u(1))\leq 3\cdot 2^{\ell-3}$ for all non-identity unipotent elements $u\in\tilde{G}$ and that there exist $u\in\tilde{G}$ for which the bound is attained, for example $x_{\tilde{\alpha}_2}(1)$. Thus, in the case of $\ell=5$, there exist unipotent elements $u\in\tilde{G}$ for which $\dim(V_u(1))\geq\dim(V)-\sqrt{\dim(V)}$. On the other hand, for $\ell\geq 6$, we see that the inequality $1<2^{\ell-5}$ holds, therefore $3\cdot 2^{\ell-3}<2^{\ell-1}-\sqrt{2^{\ell-1}}$ also holds for all $\ell\geq 6$, thus $\dim(V_u(1))<\dim(V)-\sqrt{\dim(V)}$ for all non-identity unipotent elements $u\in\tilde{G}$ □

We conclude this subsection by noting that Propositions 6.3.13, 6.3.14, 6.3.26 and 6.3.27 complete the proof of Theorems 6.3.1 and 6.3.2, as they cover all the irreducible $k\tilde{G}$-modules corresponding to p-restricted dominant weights featured in Table 2.7.5.

6.4 Results

In this section, we collect the results proven in this chapter. In Proposition 6.4.1 we give the values of $\max_{s\in\tilde{T}\setminus Z(\tilde{G})}\{\dim(V_s(\mu))\mid\mu\in k^*\}$, $\max_{u\in\tilde{G}_u\setminus\{1\}}\dim(V_u(1))$ and $\nu_{\tilde{G}}(V)$ for all $k\tilde{G}$-modules V belonging to one of the families we had to consider. Similarly, Proposition 6.4.2 records the same data for the particular $k\tilde{G}$-modules treated in this chapter.

Proposition 6.4.1. *Let k be an algebraically closed field of characteristic $p\geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ, $\ell\geq 4$. Let $\tilde{T}$ be a fixed maximal torus in $\tilde{G}$ and let $V=L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda}\in F^{D_\ell}$. Then the value of $\nu_{\tilde{G}}(V)$ is given in the table below:*

V	Char.	$\max\limits_{s\in\tilde{T}\setminus \mathrm{Z}(\tilde{G})}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$	$\max\limits_{u\in\tilde{G}_u\setminus\{1\}}\dim(V_u(1))$	$\nu_{\tilde{G}}(V)$
$L_{\tilde{G}}(\tilde{\omega}_1)$	$p\geq 0$	$2\ell-2$	$2\ell-2$	2
$L_{\tilde{G}}(\tilde{\omega}_2)$	$p\neq 2$	$2\ell^2-5\ell+4$	$2\ell^2-5\ell+6$	$4\ell-6$
	$p=2$	$2\ell^2-5\ell+4-\gcd(2,\ell)$	$2\ell^2-5\ell+6-\gcd(2,\ell)$	$4\ell-6$
$L_{\tilde{G}}(2\tilde{\omega}_1)$	$p\neq 2,\, p\nmid \ell$	$2\ell^2-3\ell+3$	$2\ell^2-3\ell+1$	$4\ell-4$
	$p\neq 2,\, p\mid \ell$	$2\ell^2-3\ell+2$	$2\ell^2-3\ell$	$4\ell-4$

Table 6.4.1: The value of $\nu_{\tilde{G}}(V)$ for the families of modules of groups of type D_ℓ.

Proof. The result for $V=L_{\tilde{G}}(\tilde{\omega}_1)$ follows from Lemmas 6.2.3, 6.3.3 and 6.3.4; the result for $L_{\tilde{G}}(\tilde{\omega}_2)$ follows from either Corollary 6.2.6 and Proposition 6.3.5, or from Corollaries 6.2.7, 6.3.7 and 6.3.8, depending on whether $p\neq 2$, or $p=2$; and the result for $L_{\tilde{G}}(2\tilde{\omega}_1)$, follows from Corollaries 6.2.5, 6.3.10 and 6.3.12. □

Proposition 6.4.2. *Let k be an algebraically closed field of characteristic $p\geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ. Let $\tilde{T}$ be a fixed maximal torus in $\tilde{G}$ and let $V=L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda}$ is featured in Table 2.7.5. The value of $\nu_{\tilde{G}}(V)$ is given in the table below:*

Rank	$\tilde{\lambda}$	Char.	$\max\limits_{s\in\tilde{T}\setminus \mathrm{Z}(\tilde{G})}\{\dim(V_s(\mu)) \mid \mu\in k^*\}$	$\max\limits_{u\in\tilde{G}_u\setminus\{1\}}\dim(V_u(1))$	$\nu_{\tilde{G}}(V)$
$\ell=4$	$\tilde{\omega}_3+\tilde{\omega}_4$	$p=2$	≤ 28	28	20
$\ell=4$	$\tilde{\omega}_3+\tilde{\omega}_4$	$p\neq 2$	≤ 34	34	22
$\ell=5$	$\tilde{\omega}_3$	$p=2$	≤ 58	60	40
$5\leq\ell\leq 9$	$\tilde{\omega}_{\ell-1}$	$p\geq 0$	$\leq 5\cdot 2^{\ell-4}$	$3\cdot 2^{\ell-3}$	$2^{\ell-3}$

Table 6.4.2: The value of $\nu_{\tilde{G}}(V)$ for the particular modules of groups of type D_ℓ.

Proof. The result follows by Proposition 2.2.3, using the detailed results of Subsections 6.2.2 and 6.3.2. □

Chapter 7

Exceptional Groups

7.1 Main theorems

In this chapter, we shift the focus to the linear algebraic groups of exceptional type. We will study them and their modules in the same manner as for the classical linear algebraic groups. We will use Guralnick and Saxl's generation result, Theorem 2.5.4, together with the dimensional criteria (2.12) of Section 2.6, to determine a bound for the dimension of the candidate-modules. Afterwards, we will use Tables 6.49 - 6.53 of [Lü01a] to provide a finite list, see Table 7.1.1, of nontrivial kG-modules whose dimensions respect the newly established bound, for each type of exceptional group G.

With this approach in mind, we fix an algebraically closed field k of characteristic $p \geq 0$ and a simple simply connected linear algebraic group of exceptional type G of rank $\ell \geq 2$. We recall that T is a maximal torus in G with rational character group $\mathrm{X}(T)$. Let $\mathrm{X}^{\mathbb{Q}}(T) = \mathbb{Q} \otimes \mathrm{X}(T)$. Now, the root system of G determined by T is Φ and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subseteq \Phi$ is a set of simple roots. Here, we use the standard ordering of simple roots, as given in [Bou08, p.275, 280, 283, 287, 288]. Further, B is the Borel subgroup of G generated by T and the T-root subgroups corresponding to the simple roots in Δ; $\mathcal{W} = \mathrm{N}_G(T)/T$ is the Weyl group of G associated to T with $w_0 \in \mathcal{W}$ the longest word; and, lastly, $\omega_1, \ldots, \omega_\ell$ are the fundamental dominant weights of G corresponding to Δ. Having all the necessary notation is place, we state the main result of this chapter:

Theorem 7.1.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of exceptional type. Let T be a maximal torus in G and let $\mathrm{X}(T)$ be its rational character group. Let $\lambda \in \mathrm{X}(T)$ be a nonzero p-restricted dominant weight and let $V = L_G(\lambda)$. Then there exist elements $g \in G \setminus \mathrm{Z}(G)$ which admit an eigenvalue $\mu \in k^*$ on V for which*

$$\dim(V_g(\mu)) \geq \dim(V) - \sqrt{\dim(V)},$$

if and only if G is of type G_2 and $(\lambda, p) \in \{(\omega_1, \mathrm{all}), (\omega_2, 3)\}$.

We have seen in Theorem 2.2.2 that any $g \in G$ admits a unique decomposi-tion $g = g_s g_u$, where $g_s \in G_s$ is the semisimple part of g and $g_u \in G_u$ is the unipotent part of g. Now, let

V be an irreducible tensor-indecomposable kG-module. By Proposition 2.2.3, we know that in order to determine $\nu_G(V)$, we have to calculate

$$M_V = \max\{\max_{s\in T\setminus \mathrm{Z}(G)}\{\dim(V_s(\mu)) \mid \mu \in k^*\}, \max_{u\in G_u\setminus\{1\}} \dim(V_u(1))\}.$$

Therefore, one sees that the proofs of the following two results will also provide a proof of Theorem 7.1.1.

Theorem 7.1.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of exceptional type. Let T be a maximal torus in G and let $\mathrm{X}(T)$ be its rational character group. Let $\lambda \in \mathrm{X}(T)$ be a nonzero p-restricted dominant weight and let $V = L_G(\lambda)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}.$$

Theorem 7.1.3. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of exceptional type. Let T be a maximal torus in G and let $\mathrm{X}(T)$ be its rational character group. Let $\lambda \in \mathrm{X}(T)$ be a nonzero p-restricted dominant weight and let $V = L_G(\lambda)$. Then there exist non-identity unipotent elements $u \in G$ for which*

$$\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)},$$

if and only if G is of type G_2 and $(\lambda, p) \in \{(\omega_1, \mathrm{all}), (\omega_2, 3)\}$.

In Section 7.2, we will give the proof of Theorem 7.1.2 in a series of results, each treating one of the candidate-modules, whereas Section 7.3 is dedicated to the proof of Theorem 7.1.3, which will be split over several results, each covering one of the exceptional group types. Now, in order to establish the list of irreducible tensor-indecomposable kG-modules V that need to be treated in both theorems, as mentioned in the beginning of this section, we use Theorem 2.5.4 and Inequality (2.12), to determine that

$$\dim(V) \leq (\ell + \varepsilon)^2, \tag{7.1}$$

where $\varepsilon = 4$ if G is of type F_4 and $\varepsilon = 3$ in all other cases. Having established the bound in (7.1), we now use [Lü01a, Tables 6.49 - 6.53] to determine that the only nontrivial irreducible tensor-indecomposable kG-modules V that need to be treated in order to give a complete proof of Theorems 7.1.2 and 7.1.3 are the irreducible kG-modules $L_G(\lambda)$ whose corresponding p-restricted dominant highest weight λ is listed in the table below:

Group	Highest Weight	Characteristic of k	Dimension
E_6	ω_1	all	27
	ω_6	all	27
	ω_2	$p = 3$	77
	ω_2	$p \neq 3$	78
E_7	ω_7	all	56
F_4	ω_4	$p = 3$	25
	ω_4	$p \neq 3$	26
	ω_1	$p = 2$	26
	ω_1	$p \neq 2$	52
G_2	ω_1	$p = 2$	6
	ω_1	$p \neq 2$	7
	ω_2	$p = 3$	7
	ω_2	$p \neq 3$	14

Table 7.1.1: Nontrivial irreducible tensor-indecomposable kG-modules satisfying (7.1) for simple simply connected linear algebraic groups of exceptional type.

Remark 7.1.4. (a) *Note that all the modules listed in Table* 7.1.1 *are self-dual, except when G is of type E_6 and $\lambda \in \{\omega_1, \omega_6\}$. In that case, we have $L_G(\omega_6) = L_G(-w_0(\omega_1)) \cong (L_G(\omega_1))^*$, therefore, we will only treat $L_G(\omega_1)$, as the result for $L_G(\omega_6)$ will follow.*

(b) *Let G be of type E_6. If $p \neq 3$, then, by* [MT11, Theorem 15.20], $\mathrm{Lie}(G)$ *is an irreducible kG-module of highest weight ω_2. On the other hand, if $p = 3$, then there exists M, a proper kG-submodule of* $\mathrm{Lie}(G)$, *with* $0 \neq M \subseteq \mathrm{Z}(\mathrm{Lie}(G))$, *see* [MT11, Theorem 15.20]. *Now* $\mathrm{Z}(\mathrm{Lie}(G))$ *is* 1*-dimensional, see* [Die57, Theorem], *therefore* $M = \mathrm{Z}(\mathrm{Lie}(G))$. *Lastly, we note that ω_2 is the highest root of G, see* [Hum72, Table 2 of p.66 and Table 1 of p.69] *and so $L_G(\omega_2)$ is a composition factor of* $\mathrm{Lie}(G)$ *and, moreover, it has codimension* 1. *Therefore, we have* $L_G(\omega_2) \cong \mathrm{Lie}(G)/\,\mathrm{Z}(\mathrm{Lie}(G))$.

(c) *We give the following definition from* [CCGL05, Section 18.2].

Definition 7.1.5. *Let k be an algebraically closed field of characteristic $p > 0$, let G and G' be two semisimple linear algebraic groups with respective maximal tori T and T', and let* $\mathrm{X}(T)$ *and* $\mathrm{X}(T')$ *be the rational character groups of T and T', respectively. An isomorphism $\phi : \mathrm{X}^{\mathbb{Q}}(T') \to \mathrm{X}^{\mathbb{Q}}(T)$ is called* <u>*special*</u> *if the following conditions are satisfied:*

- *We have* $\phi(\mathrm{X}(T')) \subseteq \mathrm{X}(T)$.
- *Let Φ, respectively Φ', denote the set of roots of G determined by T, respectively of G' determined by T'. There exists a bijection $\psi : \Phi \to \Phi'$ with the property that, for all $\alpha \in \Phi$, we have $\phi(\psi(\alpha)) = q(\alpha)\alpha$, where $q(\alpha)$ is a power of p. The $q(\alpha)$'s are called the* <u>*radical exponents*</u> *of ϕ.*

Note that the isomorphism attached to an isogeny is always special and that every special isomorphism is attached to an isogeny, see [CCGL05, Section 18.2, p.194]. *Having set up the notion of a special isomorphism, we make the following two remarks:*

(c.1) *Let G be of type G_2 and assume that $p = 3$. Then, there exists an exceptional isogeny of G onto itself whose corresponding special isomorphism $\phi : \mathrm{X}^{\mathbb{Q}}(T) \to \mathrm{X}^{\mathbb{Q}}(T)$ has the property that $\phi(\mathrm{X}(T)) \subseteq X(T)$ and $\phi(\alpha_1) = \alpha_2$ and $\phi(\alpha_2) = 3\alpha_1$, see* [CCGL05, Proposition 1 of Section 21.4]. *It follows that $\phi(\omega_1) = \phi(2\alpha_1+\alpha_2) = 2\alpha_2 + 3\alpha_1 = \omega_2$. Therefore when $p = 3$ we have $L_G(\omega_2) = L_G(\phi(\omega_1))$ and thus the result for $L_G(\omega_2)$ will follow from that of $L_G(\omega_1)$. Consequently, we will only treat $L_G(\omega_2)$ when $p \neq 3$.*

(c.2) *If G is of type F_4 and $p = 2$, there exists an exceptional isogeny of G onto itself such that the corresponding special isomorphism $\phi : \mathrm{X}^{\mathbb{Q}}(T) \to \mathrm{X}^{\mathbb{Q}}(T)$ has the property that $\phi(\mathrm{X}(T)) \subseteq X(T)$ and $\phi(\alpha_1) = 2\alpha_4$, $\phi(\alpha_2) = 2\alpha_3$, $\phi(\alpha_3) = \alpha_2$ and $\phi(\alpha_4) = \alpha_1$, see* [CCGL05, p.260]. *It follows that $\phi(\omega_4) = \phi(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4) = 2\alpha_4 + 4\alpha_3 + 3\alpha_2 + 2\alpha_1 = \omega_1$. Therefore when $p = 2$ we have $L_G(\omega_1) = L_G(\phi(\omega_4))$ and thus the result for $L_G(\omega_1)$ will follow from that of $L_G(\omega_4)$. Consequently, we will only treat $L_G(\omega_1)$ when $p \neq 2$.*

(d) *Recall that we use the notation $V_G(\lambda)$ to denote the Weyl kG-module of highest weight $\lambda \in \mathrm{X}(T)$. If G is of type E_6, or E_7, we will denote the Weyl kG-module $V_G(\omega_\ell)$ of dimension d by V_d. If G is not of type E, we will denote by V_d the nontrivial Weyl kG-module whose dimension d is less than that of* $\mathrm{Lie}(G)$.

(d.1) *Let G be of type G_2. In this case, we have $V_7 = V_G(\omega_1)$. If $p \neq 2$, then V_7 is irreducible as a kG-module and we have $V_7 \cong L_G(\omega_1)$. On the other hand, if $p = 2$, then $V_7/\mathrm{Rad}(V_7) \cong L_G(\omega_1)$ and $\dim(\mathrm{Rad}(V_7)) = 1$.*

(d.2) *Let G be of type F_4. In this case, we have $V_{26} = V_G(\omega_4)$. If $p \neq 3$, then V_{26} is irreducible as a kG-module and we have $V_{26} \cong L_G(\omega_4)$. On the other hand, if $p = 3$, then $V_{26}/\mathrm{Rad}(V_{26}) \cong L_G(\omega_4)$ and $\dim(\mathrm{Rad}(V_{26})) = 1$.*

7.2 The proof of Theorem 7.1.2

7.2.1 Simple simply connected groups of type E_6

Let k be an algebraically closed field of characteristic $p \geq 0$, let G be a simple simply connected linear algebraic group of type E_6 and let T be a maximal torus in G. In this subsection, we will determine an upper-bound for $\dim(V_s(\mu))$, where $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ and V is an irreducible kG-module with corresponding highest weight featured in Table 7.1.1. In order to achieve this, we will use the inductive algorithm described in Subsection 2.4.3. For this, we refer the reader to the construction of the maximal parabolic subgroup P_6 of G given in Section 2.4. Let L_6 be a Levi subgroup of P_6. Then $L_6 = \mathrm{Z}(L_6)^\circ[L_6, L_6]$, where $\mathrm{Z}(L_6)^\circ$ is a one-dimensional torus and $[L_6, L_6]$ is a simple simply connected linear algebraic group of type D_5 with maximal torus $T' = [L_6, L_6] \cap T$. Recall that we abuse notation and denote by $\omega_1, \cdots, \omega_5$ the fundamental dominant weights of L_6 corresponding to $\alpha_1, \ldots, \alpha_5$.

Let $s \in T$. Then $s = z \cdot h$, where $z \in \mathrm{Z}(L_6)^\circ$ and $h \in [L_6, L_6]$. Since, $\alpha_j(z) = 1$ for all $1 \leq j \leq 5$, it follows that $z = h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^4)h_{\alpha_4}(c^6)h_{\alpha_5}(c^5)h_{\alpha_6}(c^4)$ with $c \in k^*$. Moreover, if $z \notin \mathrm{Z}(G)$, then $c^3 \neq 1$. Lastly, as $h \in [L_6, L_6]$, we determine that $s = h_{\alpha_1}(a_1c^2)h_{\alpha_2}(a_2c^3)h_{\alpha_3}(a_3c^4)h_{\alpha_4}(a_4c^6)h_{\alpha_5}(a_5c^5)h_{\alpha_6}(c^4)$ with $a_i, c \in k^*$ for all $1 \leq i \leq 5$.

Set $V = L_G(\lambda)$, where $\lambda \in \mathrm{X}(T)$ is a nonzero p-restricted dominant weight. In what follows, we determine $e_6(\lambda)$, the maximum α_6-level of weights in V, see Definition 2.4.1. Now, by Remark 2.4.2, we have that $e_6(\lambda)$ is equal to the α_6-level of the weight $w_0(\lambda)$. For $\lambda = \omega_1$, we have that the lowest weight in $V = L_G(\omega_1)$ is $w_0(\omega_1) = -\omega_6$ and so $e_6(\omega_1)$ is equal to the α_6-level of $-\omega_6$. Using [Hum72, Table 1, p.69], we determine that:

$$-\omega_6 = \omega_1 - (\omega_1 + \omega_6) = \omega_1 - 2\alpha_6 - \sum_{j=1}^{5} c_j\alpha_j,$$

where $c_j \in \mathbb{Z}_{\geq 0}$, for all $1 \leq j \leq 5$. Therefore $e_6(\omega_1) = 2$. For $\lambda = \omega_2$, we have $w_0(\omega_2) = -\omega_2$, and, using [Hum72, Table 1, p.69] for a second time, we determine that:

$$-\omega_2 = \omega_2 - 2\omega_2 = \omega_2 - 2\alpha_6 - \sum_{j=1}^{5} c_j\alpha_j,$$

where $c_j \in \mathbb{Z}_{\geq 0}$, for all $1 \leq j \leq 5$. Therefore $e_6(\omega_2) = 2$.

We return to the general case and let $\lambda = \sum_{i=1}^{6} d_i\omega_i$ with $0 \leq d_i \leq p-1$ for all $1 \leq i \leq 6$. We consider the decomposition:

$$V\mid_{[L_6,L_6]} = \bigoplus_{i=0}^{e_6(\lambda)} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_6} V_{\lambda - i\alpha_6 - \gamma}$, for all $0 \leq i \leq e_6(\lambda)$. Let $s \in T$ and write $s = z \cdot h$, as above. By (2.5), we have

$$\begin{aligned} s_z^i := (\lambda - i\alpha_6 - \gamma)(z) &= (\lambda - i\alpha_6)(h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^4)h_{\alpha_4}(c^6)h_{\alpha_5}(c^5)h_{\alpha_6}(c^4)) \\ &= c^{2d_1+3d_2+4d_3+6d_4+5d_5+4d_6} \cdot c^{-3i}. \end{aligned}$$

Therefore, z acts on V^i, $0 \leq i \leq e_6(\lambda)$, as the scalar $s_z^i = c^{2d_1+3d_2+4d_3+6d_4+5d_5+4d_6-3i}$. Now, let $\mu_1^i, \ldots, \mu_{t_i}^i$, $t_i \geq 1$, be the distinct eigenvalues of h on V^i, $0 \leq i \leq e_6(\lambda)$, and let $n_1^i, \ldots, n_{t_i}^i$ be their respective multiplicities. Then, by Lemma 2.4.8, it follows that the distinct eigenvalues of s on V^i are $s_z^i\mu_1^i, \ldots, s_z^i\mu_{t_i}^i$, with respective multiplicities $n_1^i, \ldots, n_{t_i}^i$.

Proposition 7.2.1. *Let $V = L_G(\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 19.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $\lambda = \omega_1$ and $L = L_6$. Then $\dim(V) = 27$ and, as $e_6(\lambda) = 2$, we have:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_6} V_{\lambda - i\alpha_6 - \gamma}$ and $V^i \neq 0$, by Proposition 2.3.6, for all $i = 0, 1, 2$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_1)$, therefore $\dim(V^0) = 10$. Since the weight $(\lambda - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6) \mid_{T'} = \omega_2$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_2)$ and so $\dim(V^1) \geq \dim(L_L(\omega_2)) = 16$. By dimensional considerations, we determine that $\dim(V^2) \leq 1$, hence $V^2 \cong L_L(0)$, as $V^2 \neq 0$, and, consequently, $\dim(V^1) = 16$. We deduce that $V^1 \cong L_L(\omega_2)$ and:

$$V \mid_{[L,L]} \cong L_L(\omega_1) \oplus L_L(\omega_2) \oplus L_L(0).$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0$ or $i = 1$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$, with $c^3 \neq 1$. In this case, since s acts on V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{2-3i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^0) = 10; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) = \dim(V^1) = 16; \\ c^{-4} \text{ with } \dim(V_s(c^{-4})) \geq \dim(V^2) = 1. \end{cases}$$

Since $c^3 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 16$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 0$ and $i = 1$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on each V^i, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for both $i = 0$ and $i = 1$, where $\mu_h \in k^*$ is any eigenvalue of h on V^i. Now, by Lemma 6.2.3, we have $\dim(V_h^0(\mu_h)) \leq 8$ for all eigenvalues μ_h of h on V^0. Similarly, by Proposition 6.2.10, we have $\dim(V_h^1(\mu_h)) \leq 10$ for all eigenvalues μ_h of h on V^1. It follows that $\dim(V_s(\mu)) \leq 19$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 19 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 7.2.2. *Let k be an algebraically closed field of characteristic $p = 3$ and let $V = L_G(\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 49.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $\lambda = \omega_2$ and $L = L_6$. Then $\dim(V) = 77$, as $p = 3$, and moreover, as $e_6(\lambda) = 2$, we have that:

$$V \mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_6} V_{\lambda - i\alpha_6 - \gamma}$ and $V^i \neq 0$, by Proposition 2.3.6, for all $i = 0, 1, 2$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_2)$ with $\dim(V^0) = 16$. Moreover, by Lemma 2.4.3, it follows that $V^2 \cong (L_L(\omega_2))^* \cong L_L(\omega_5)$, hence $\dim(V^2) = 16$. As the weight $(\lambda - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_6) \mid_{T'} = \omega_3$ admits a maximal vector in V^1, it follows that V^1 has a composition factor

isomorphic to $L_L(\omega_3)$, hence $\dim(V^1) \geq \dim(L_L(\omega_3)) = 45$, since $p = 3$. By dimensional considerations, we deduce that $V^1 \cong L_L(\omega_3)$ and:

$$V\mid_{[L,L]} \cong L_L(\omega_2) \oplus L_L(\omega_3) \oplus L_L(\omega_5).$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$, with $c^3 \neq 1$. In this case, since z acts on V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{3-3i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 16; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1) = 45; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^2) = 16. \end{cases}$$

Since $c^3 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 45$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on each V^i, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where $\mu_h \in k^*$ is any eigenvalue of h on V^i. Now, by Proposition 6.2.10, we have $\dim(V_h^0(\mu_h)) \leq 10$ and $\dim(V_h^2(\mu_h)) \leq 10$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. Similarly, by Proposition 6.2.6, as $p = 3$, we have $\dim(V_h^1(\mu_h)) \leq 29$ for all eigenvalues μ_h of h on V^1. Therefore, $\dim(V_s(\mu)) \leq 49$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 49 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 7.2.3. *Let k be an algebraically closed field of characteristic $p \neq 3$ and let $V = L_G(\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 50.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $\lambda = \omega_2$ and $L = L_6$. Then $\dim(V) = 78$, as $p \neq 3$, and moreover, as $e_6(\lambda) = 2$, we have that:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_6} V_{\lambda - i\alpha_6 - \gamma}$ and $V^i \neq 0$, by Proposition 2.3.6, for all $i = 0, 1, 2$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_2)$ with $\dim(V^0) = 16$. Moreover, by Lemma 2.4.3, we have $V^2 \cong (L_L(\omega_2))^* \cong L_L(\omega_5)$, hence $\dim(V^2) = 16$. It follows that $\dim(V^1) = 46$. Now, the weight $(\lambda - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_6)\mid_{T'} = \omega_3$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_L(\omega_3)$. We also note that the weight $(\lambda - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6)\mid_{T'} = 0$, which appears with multiplicity 6 in V^1, is a sub-dominant weight in the composition factor of V^1 isomorphic to $L_L(\omega_3)$, where it has multiplicity $5 - \delta_{p,2}$. Lastly, as $\dim(L_L(\omega_3)) = 45 - \delta_{p,2}$, we deduce that V^1 admits $2 + \delta_{p,2}$ composition factors: one isomorphic to $L_L(\omega_3)$ and $1 + \delta_{p,2}$ isomorphic to $L_L(0)$.

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$ with $c^3 \neq 1$. In this case, since z acts on V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{3-3i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 16; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1) = 46; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^2) = 16. \end{cases}$$

Since $c^3 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 46$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on each V^i, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for all $0 \leq i \leq 2$, where $\mu_h \in k^*$ is any eigenvalue of h on V^i. Now, by Proposition 6.2.10, we have $\dim(V_h^0(\mu_h)) \leq 10$ and $\dim(V_h^2(\mu_h)) \leq 10$ for all eigenvalues μ_h of h on V^0 and V^2, respectively. Similarly, by Corollary 6.2.6 when $p \neq 2$, or by Corollary 6.2.7 when $p = 2$, to determine that $\dim(V_h^1(\mu_h)) \leq 30$ for all eigenvalues μ_h of h on V^1. It follows that $\dim(V_s(\mu)) \leq 50$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 50 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

We conclude this subsection by noting that Propositions 7.2.1, 7.2.2 and 7.2.3 give the proof of Theorem 7.1.2 for simple simply connected linear algebraic groups of type E_6.

7.2.2 Simple simply connected groups of type E_7

Let k be an algebraically closed field of characteristic $p \geq 0$, let G be a simple simply connected linear algebraic group of type E_7 and let T be a maximal torus in G. In this subsection, we will determine an upper-bound for $\dim(V_s(\mu))$, where $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ and $V = L_G(\omega_7)$, see Table 7.1.1. In order to achieve this, we will use the inductive algorithm of Subsection 2.4.3. For this, let L_7 be the Levi subgroup of the maximal parabolic subgroup P_7 of G constructed in Section 2.4. We have that $L_7 = \mathrm{Z}(L_7)^\circ[L_7, L_7]$, where $\mathrm{Z}(L_7)^\circ$ is a one-dimensional torus and $[L_7, L_7]$ is a simply connected linear algebraic group of type E_6 with maximal torus $T' = [L_7, L_7] \cap T$. Recall that we abuse notation and denote by $\omega_1, \cdots, \omega_6$ the fundamental dominant weights of L_7 corresponding to $\alpha_1, \ldots, \alpha_6$.

Set $V = L_G(\omega_7)$. We will now determine $e_7(\omega_7)$, the maximum α_7-level of weights in V. We recall that $w_0(\omega_7)$, where w_0 is the longest word in $\mathcal{W}$, is the lowest weight in V. Now, we have that $w_0(\omega_7) = -\omega_7$, therefore $e_7(\omega_7)$ will equal the α_7-level of $-\omega_7$, see Remark 2.4.2. Using [Hum72, Table 1, p.69], we determine that:

$$-\omega_7 = \omega_7 - 2\omega_7 = \omega_7 - 3\alpha_7 - \sum_{j=1}^{6} c_j \alpha_j,$$

where $c_j \in \mathbb{Z}_{\geq 0}$, for all $1 \leq j \leq 6$. Therefore $e_7(\omega_7) = 3$.

Let $s \in T$. Then $s = z \cdot h$, where $z \in \mathrm{Z}(L_7)^\circ$ and $h \in [L_7, L_7]$. Since $\alpha_j(z) = 1$ for all $1 \leq j \leq 6$, it follows that $z = h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^4)h_{\alpha_4}(c^6)h_{\alpha_5}(c^5)h_{\alpha_6}(c^4)h_{\alpha_7}(c^3)$

with $c \in k^*$. Moreover, if $z \notin \mathrm{Z}(G)$, then $c^2 \neq 1$. Lastly, as $h \in [L_7, L_7]$, we determine that $s = h_{\alpha_1}(a_1c^2)h_{\alpha_2}(a_2c^3)h_{\alpha_3}(a_3c^4)h_{\alpha_4}(a_4c^6)h_{\alpha_5}(a_5c^5)h_{\alpha_6}(a_6c^4)h_{\alpha_7}(c^3)$ with $a_i, c \in k^*$ for all $1 \leq i \leq 6$.

We now consider the decomposition:

$$V\mid_{[L_7,L_7]} = \bigoplus_{i=0}^{3} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_7} V_{\lambda - i\alpha_7 - \gamma}$ for all $0 \leq i \leq 3$. Let $s \in T$ and write $s = z \cdot h$, as above. By (2.5), we have that:

$$\begin{aligned} s_z^i = (\omega_7 - i\alpha_7 - \gamma)(z) &= (\omega_7 - i\alpha_7)(h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^4)h_{\alpha_4}(c^6)h_{\alpha_5}(c^5)h_{\alpha_6}(c^4)h_{\alpha_7}(c^3)) \\ &= c^{2d_1+3d_2+4d_3+6d_4+5d_5+4d_6+3d_7} \cdot c^{-2i}. \end{aligned}$$

Therefore, z acts on V^i, $0 \leq i \leq 3$, as the scalar $s_z^i = c^{2d_1+3d_2+4d_3+6d_4+5d_5+4d_6+3d_7-2i}$. Now, let $\mu_1^i, \ldots, \mu_{t_i}^i$, $t_i \geq 1$, be the distinct eigenvalues of h on V^i, $0 \leq i \leq 3$, and let $n_1^i, \ldots, n_{t_i}^i$ be their respective multiplicities. Then, by Lemma 2.4.8, it follows that the distinct eigenvalues of s on V^i are $s_z^i\mu_1^i, \ldots, s_z^i\mu_{t_i}^i$, with respective multiplicities $n_1^i, \ldots, n_{t_i}^i$.

Proposition 7.2.4. *Let $V = L_G(\omega_7)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 40.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $\lambda = \omega_7$ and $L = L_7$. Then $\dim(V) = 56$ and, as $e_7(\omega_7) = 3$, we have:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_7} V_{\lambda - i\alpha_7 - \gamma}$ and $V^i \neq 0$, by Proposition 2.3.6, for all $0 \leq i \leq 3$. By [Smi82, Proposition], we have $V^0 \cong L_L(0)$ and so, by Lemma 2.4.3, we also have $V^3 \cong L_L(0)$. Now, since the weight $(\lambda - \alpha_7)\mid_{T'} = \omega_6$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_6)$, hence $\dim(V^1) \geq \dim(L_L(\omega_6)) = 27$. We apply Lemma 2.4.3 once more to determine that $\dim(V^2) \geq 27$ and thus, by comparing dimensions, we deduce that $V^1 \cong L_L(\omega_6)$ and $V^2 \cong (L_L(\omega_6))^* \cong L_L(\omega_1)$. We have showed that:

$$V\mid_{[L,L]} \cong L_L(0) \oplus L_L(\omega_6) \oplus L_L(\omega_1) \oplus L_L(0).$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z$ with $c^2 \neq 1$. In this case, since s acts on V^i, $0 \leq i \leq 3$, as scalar multiplication by c^{3-2i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 1; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 27; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 27; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^3) = 1. \end{cases}$$

Since $c^2 \neq 1$, it follows that $\dim(V_s(\mu)) \leq 28$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and for both $i = 1$ and $i = 2$. We write $s = z \cdot h$, where $z \in \mathrm{Z}(L)^\circ$ and $h \in [L, L]$. Since z acts by scalar multiplication on each V^i, it follows that $\dim(V_h^i(\mu_h)) < \dim(V^i)$ for both $i = 1$ and $i = 2$, where $\mu_h \in k^*$ is any eigenvalue of h on V^i. Now, by Proposition 7.2.1, we have $\dim(V_h^1(\mu_h)) \leq 19$ and $\dim(V_h^2(\mu_h)) \leq 19$ for all eigenvalues μ_h of h on V^1 and V^2, respectively. It follows that $\dim(V_s(\mu)) \leq 40$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 40 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

We conclude this subsection by noting that Proposition 7.2.4 gives the proof of Theorem 7.1.2 for simple simply connected linear algebraic groups of type E_7.

7.2.3 Simple simply connected groups of type F_4

Let k be an algebraically closed field of characteristic $p \geq 0$, let G be a simple simply connected linear algebraic group of type F_4 and let T be a maximal torus in G. In this subsection, we will determine an upper-bound for $\dim(V_s(\mu))$, where $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ and V is a rational irreducible kG-module featured in Table 7.1.1. To achieve this, we use the inductive algorithm of Subsection 2.4.3. We let L_i be the Levi subgroup of the maximal parabolic subgroup P_i, $i = 1, 4$, of G constructed in Section 2.4. We have that $L_i = \mathrm{Z}(L_i)^\circ[L_i, L_i]$, where $\mathrm{Z}(L_i)^\circ$ is a one-dimensional torus and $[L_1, L_1]$ is a simply connected linear algebraic group of type C_3, while $[L_4, L_4]$ is a simply connected linear algebraic group of type B_3. Moreover, recall that we denote the maximal torus $[L_i, L_i] \cap T$ of $[L_i, L_i]$ by T' and that we abuse notation and denote by ω_2, ω_3 and ω_4, respectively by ω_1, ω_2 and ω_3, the fundamental dominant weights of L_1, respectively of L_4, corresponding to α_2, α_3 and α_4, respectively to α_1, α_2 and α_3.

We set $V = L_G(\lambda)$ for some nonzero p-restricted dominant weight $\lambda \in \mathrm{X}(T)$. Now, as V is self-dual and as $w_0(\lambda)$ is the lowest weight in V, it follows that $e_1(\lambda)$, the maximal α_1-level of weight in V, respectively $e_4(\lambda)$, the maximal α_4-level of weights in V, will be equal to the α_1-level, respectively the α_4-level, of the weight $w_0(\lambda) = -\lambda$, see Remark 2.4.2. In what follows, we will compute $e_1(\lambda)$ for $\lambda = \omega_4$ and $e_4(\lambda)$ for $\lambda = \omega_1$. We will do this by writing the fundamental dominant weights ω_1 and ω_4 in terms of the simple roots α_j, $1 \leq j \leq 4$, see [Hum72, Table 1, p.69]. First, we have that:

$$-\omega_4 = \omega_4 - 2\omega_4 = \omega_4 - 2\alpha_1 - \sum_{j=2}^{4} c_j \alpha_j,$$

where $c_j \in \mathbb{Z}_{\geq 0}$, for all $2 \leq j \leq 4$. Therefore $e_1(\omega_4) = 2$. Secondly, we have that:

$$-\omega_1 = \omega_1 - 2\omega_1 = \omega_1 - 4\alpha_4 - \sum_{j=1}^{3} c_j \alpha_j,$$

where $c_j \in \mathbb{Z}_{\geq 0}$, for all $1 \leq j \leq 3$. Therefore $e_4(\omega_1) = 4$.

Let $s_1 \in T$. Then $s_1 = z_1 \cdot h_1$, where $z_1 \in \mathrm{Z}(L_1)^\circ$ and $h_1 \in [L_1, L_1]$. As $\alpha_j(z_1) = 1$ for all $2 \leq j \leq 4$, we determine that $z_1 = h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^2)h_{\alpha_4}(c)$ with $c \in k^*$. Furthermore, if $z_1 \notin \mathrm{Z}(G)$, then $c \neq 1$. As $h_1 \in [L_1, L_1]$, we determine that $s_1 = h_{\alpha_1}(c^2)h_{\alpha_2}(a_2c^3)h_{\alpha_3}(a_3c^2)h_{\alpha_4}(a_4c)$ with $a_j, c \in k^*$, $2 \leq j \leq 4$. Similarly, let $s_4 \in T$. Then $s_4 = z_4 \cdot h_4$, where $z_4 \in \mathrm{Z}(L_4)^\circ$ and $h_4 \in [L_4, L_4]$. Since $\alpha_j(z_4) = 1$ for all $1 \leq j \leq 3$, one determines that $z_4 = h_{\alpha_1}(c^2)h_{\alpha_2}(c^4)h_{\alpha_3}(c^3)h_{\alpha_4}(c^2)$ with $c \in k^*$. Moreover, if $z_4 \notin \mathrm{Z}(G)$, we have $c \neq 1$. As $h_4 \in [L_4, L_4]$, we have $s_4 = h_{\alpha_1}(a_1c^2)h_{\alpha_2}(a_2c^4)h_{\alpha_3}(a_3c^3)h_{\alpha_4}(c^2)$ with $a_j, c \in k^*$, $1 \leq j \leq 3$.

Now, let $V = L_G(\omega_4)$. Then:

$$V\mid_{[L_1,L_1]} = \bigoplus_{i=0}^{2} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. Let $s \in T$ and write $s = z_1 \cdot h_1$, as above. By (2.5), it follows that:

$$s^i_{z_1} = (\lambda - i\alpha_1 - \gamma)(z_1) = (\lambda - i\alpha_1)(h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^2)h_{\alpha_4}(c)) = c^{2d_1+3d_2+2d_3+d_4} \cdot c^{-i}.$$

Therefore, z_1 acts on V^i, $0 \leq i \leq 2$, as the scalar $s^i_{z_1} = c^{2d_1+3d_2+2d_3+d_4-i}$. Now, let $\mu^i_1, \ldots, \mu^i_{t_i}$, $t_i \geq 1$, be the distinct eigenvalues of h_1 on V^i, $0 \leq i \leq 2$, and let $n^i_1, \ldots, n^i_{t_i}$ be their respective multiplicities. Then, by Lemma 2.4.8, it follows that the distinct eigenvalues of s on V^i are $s^i_{z_1}\mu^i_1, \ldots, s^i_{z_1}\mu^i_{t_i}$, with respective multiplicities $n_1, \ldots, n_{t_i}$.

Similarly, let $V = L_G(\omega_1)$. Then:

$$V\mid_{[L_4,L_4]} = \bigoplus_{i=0}^{4} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_4} V_{\lambda - i\alpha_4 - \gamma}$ for all $0 \leq i \leq 4$. Let $s \in T$ and write $s = z_4 \cdot h_4$, as above. By (2.5), we have:

$$s^i_{z_4} = (\lambda - i\alpha_4 - \gamma)(z_4) = (\lambda - i\alpha_4)(h_{\alpha_1}(c^2)h_{\alpha_2}(c^4)h_{\alpha_3}(c^3)h_{\alpha_4}(c^2)) = c^{2d_1+4d_2+3d_3+2d_4} \cdot c^{-i}.$$

Therefore, z_4 acts on V^i, $0 \leq i \leq 4$, as the scalar $s^i_{z_4} = c^{2d_1+4d_2+3d_3+2d_4-i}$. Now, let $\mu^i_1, \ldots, \mu^i_{t_i}$, $t_i \geq 1$, be the distinct eigenvalues of h_4 on V^i, $0 \leq i \leq 4$, and let $n^i_1, \ldots, n^i_{t_i}$ be their respective multiplicities. Then, we apply Lemma 2.4.8 once more to determine that the distinct eigenvalues of s on V^i are $s^i_{z_4}\mu^i_1, \ldots, s^i_{z_4}\mu^i_{t_i}$, with respective multiplicities $n^i_1, \ldots, n^i_{t_i}$.

Proposition 7.2.5. *Let $V = L_G(\omega_4)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 16.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $L = L_1$ and set $\lambda = \omega_4$. Then $e_1(\lambda) = 2$ and we have:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_4)$ and so $\dim(V^0) = 6$. Moreover, by Lemma 2.4.3, we also have $V^2 \cong (L_L(\omega_4))^* \cong L_L(\omega_4)$, hence $\dim(V^2) = 6$. Now, since the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)\mid_{T'} = \omega_3$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_3)$, hence $\dim(V^1) \geq \dim(L_L(\omega_3))$. By comparing dimensions, we deduce that $V^1 \cong L_L(\omega_3)$ and

$$V\mid_{[L,L]} \cong L_L(\omega_4) \oplus L_L(\omega_3) \oplus L_L(\omega_4).$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z_1$ with $c \neq 1$. In this case, since s acts on V^1, $0 \leq i \leq 2$, as scalar multiplication by c^{1-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c \text{ with } \dim(V_s(c)) \geq \dim(V^0) = 6; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1); \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 6. \end{cases}$$

Since $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 14 - \delta_{p,3}$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z_1 \cdot h_1$, where $z_1 \in \mathrm{Z}(L)^\circ$ and $h_1 \in [L, L]$. Since z_1 acts by scalar multiplication on each V^i, it follows that $\dim(V_{h_1}^i(\mu_{h_1})) < \dim(V^i)$ for all $0 \leq i \leq 2$, where $\mu_{h_1} \in k^*$ is any eigenvalue of h_1 on V^i. Now, by Lemma 4.2.3, we have $\dim(V_{h_1}^0(\mu_{h_1})) \leq 4$ and $\dim(V_{h_1}^2(\mu_{h_1})) \leq 4$ for all eigenvalues μ_{h_1} of h_1 on V^0 and V^2, respectively. We apply Corollary 4.2.6 when $p \neq 3$, or Corollary 4.2.7 when $p = 3$, to determine that $\dim(V_{h_1}^1(\mu_{h_1})) \leq 8$ for all eigenvalues μ_{h_1} of h_1 on V^1. It follows that $\dim(V_s(\mu)) \leq 16$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 16 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

Proposition 7.2.6. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $V = L_G(\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 36,$$

where there exist pairs $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $L = L_4$ and set $\lambda = \omega_1$. Then $\dim(V) = 52$, as $p \neq 2$, and, as $e_4(\lambda) = 4$, we have:

$$V\mid_{[L,L]} = V^0 \oplus \cdots \oplus V^4,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_4} V_{\lambda - i\alpha_4 - \gamma}$ for all $0 \leq i \leq 4$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_1)$ and so $\dim(V^0) = 7$. Moreover, by Lemma 2.4.3, we also have $V^4 \cong (L_L(\omega_1))^* \cong L_L(\omega_1)$,

hence $\dim(V^4) = 7$. Now, since the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \mid_{T'} = \omega_3$ admits a maximal vector in V^1, it follows that V^1 has a composition factor isomorphic to $L_L(\omega_3)$, hence $\dim(V^1) \geq \dim(L_L(\omega_3)) = 8$. We apply Lemma 2.4.3 once more, this time to $V^3 \cong (V^1)^*$, to determine that $\dim(V^3) \geq 8$. Consequently, we have $\dim(V^2) \leq 22$. As the weight $(\lambda - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4) \mid_{T'} = \omega_2$ admits a maximal vector in V^2, it follows that V^2 has a composition factor isomorphic to $L_L(\omega_2)$. Moreover, the weight $(\lambda - 2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4) \mid_{T'} = 0$, which appears with multiplicity 4 in V^2, is a sub-dominant weight in the composition factor of V^2 isomorphic to $L_L(\omega_2)$, in which it has multiplicity 3. Lastly, as $\dim(L_L(\omega_2)) = 21$, we deduce that V^2 consists of two composition factors: one isomorphic to $L_L(\omega_2)$ and one isomorphic to $L_L(0)$, and so, by [Jan07, II.2.14], we have $V^2 \cong L_L(\omega_2) \oplus L_L(0)$. Furthermore, we also determine that $V^1 \cong L_L(\omega_3)$, hence $V^3 \cong (L_L(\omega_3))^* \cong L_L(\omega_3)$, and so:

$$V \mid_{[L,L]} \cong L_L(\omega_1) \oplus L_L(\omega_3) \oplus L_L(\omega_2) \oplus L_L(0) \oplus L_L(\omega_3) \oplus L_L(\omega_1).$$

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $0 \leq i \leq 4$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = z_4$ with $c \neq 1$. In this case, since s acts on V^i, $0 \leq i \leq 4$, as scalar multiplication by c^{2-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^0) = 7; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^1) = 8; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^2) = 22; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^3) = 8; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^4) = 7. \end{cases}$$

Since $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 36$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_3}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(-1)) = 36$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 4$. We write $s = z_4 \cdot h_4$, where $z_4 \in \mathrm{Z}(L)^\circ$ and $h_4 \in [L, L]$. Since z_4 acts by scalar multiplication on each V^i, it follows that $\dim(V_{h_4}^i(\mu_{h_4})) < \dim(V^i)$ for all $0 \leq i \leq 4$, where $\mu_{h_4} \in k^*$ is any eigenvalue of h_4 on V^i. Now, since $p \neq 2$, by Lemma 5.2.3, we have $\dim(V_{h_4}^0(\mu_{h_4})) \leq 6$ and $\dim(V_{h_4}^4(\mu_{h_4})) \leq 6$ for all eigenvalues μ_{h_4} of h_4 on V^0 and V^2, respectively. By Proposition 5.2.8, we have $\dim(V_{h_4}^1(\mu_{h_4})) \leq 4$ and $\dim(V_{h_4}^3(\mu_{h_4})) \leq 4$ for all eigenvalues μ_{h_4} of h_4 on V^1 and V^3, respectively. Lastly, by Proposition 5.2.4, we have $\dim(V_{h_4}^2(\mu_{h_4})) \leq 16$ for all eigenvalues μ_{h_4} of h_4 on V^2. It follows that $\dim(V_s(\mu)) \leq 36$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 36 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

We conclude this section by noting that Propositions 7.2.5 and 7.2.6 give the proof of Theorem 7.1.2 for simple simply connected linear algebraic groups of type F_4.

7.2.4 Simple simply connected groups of type G_2

Let k be an algebraically closed field of characteristic $p \geq 0$, let G be a simple simply connected linear algebraic group of type G_2 and let T be a maximal torus in G. In this

subsection, we will determine an upper-bound for $\dim(V_s(\mu))$, where $(s,\mu) \in T \setminus \mathrm{Z}(G) \times k^*$ and V is an irreducible kG-module featured in Table 7.1.1. For this, we will use the inductive algorithm of Subsection 2.4.3. To begin, let L_i be the Levi subgroup of the maximal parabolic subgroup P_i, $i = 1, 2$, of G constructed in Section 2.4. We have that $L_i = \mathrm{Z}(L_i)^\circ[L_i, L_i]$, where $\mathrm{Z}(L_i)^\circ$ is a one-dimensional torus and $[L_i, L_i]$ is a simply connected linear algebraic group of type A_1 with maximal torus $T' = [L_i, L_i] \cap T$. Recall that we abuse notation and denote by ω_{3-i} the fundamental dominant weight of L_i corresponding to α_{3-i}.

Set $V = L_G(\lambda)$ for some nonzero p-restricted dominant weight $\lambda \in \mathrm{X}(T)$, $\lambda = d_1\omega_1 + d_2\omega_2$, where $0 \leq d_1, d_2 \leq p-1$ not both 0. Let w_0 be the longest word in $\mathcal{W}$. As $w_0(\lambda)$ is the lowest weight in V, it follows that $e_1(\lambda)$, the maximal α_1-level of weights in V, respectively $e_2(\lambda)$, the maximal α_2-level of weights in V, will be equal to the α_1-level, respectively the α_2-level, of the weight $w_0(\lambda) = -\lambda$, see Remark 2.4.2. By writing the fundamental dominant weights ω_1 and ω_2 in terms of the simple roots α_1 and α_2, see [Hum72, Table 1, p.69], we determine that:

$$-\lambda = \lambda - 2\lambda = \lambda - 2(2d_1 + 3d_2)\alpha_1 - 2(d_1 + 2d_2)\alpha_2.$$

Therefore, $e_1(\lambda) = 4d_1 + 6d_2$ and $e_2(\lambda) = 2d_1 + 4d_2$.

Let $s_1 \in T$. Then $s_1 = z_1 \cdot h_1$, where $z_1 \in \mathrm{Z}(L_1)^\circ$ and $h_1 \in [L_1, L_1]$. Now, as $\alpha_2(z_1) = 1$, it follows that $z_1 = h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)$, where $c \in k^*$. Moreover, if $z_1 \notin \mathrm{Z}(G)$, then $c \neq 1$. As $h_1 \in [L_1, L_1]$, we have $h_1 = h_{\alpha_2}(a)$, where $a \in k^*$, and so $s_1 = h_{\alpha_1}(c^2)h_{\alpha_2}(ac^3)$, where $a, c \in k^*$. Similarly, let $s_2 \in T$. Then $s_2 = z_2 \cdot h_2$, where $z_2 \in \mathrm{Z}(L_2)^\circ$ and $h_2 \in [L_2, L_2]$. As $\alpha_1(z_2) = 1$, it follows that $z_2 = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$, where $c \in k^*$. Moreover, if $z_2 \notin \mathrm{Z}(G)$, then $c \neq 1$. As $h_2 \in [L_2, L_2]$, we have $h_2 = h_{\alpha_1}(a)$, where $a \in k^*$, and so $s_2 = h_{\alpha_1}(ac)h_{\alpha_2}(c^2)$ with $a, c \in k^*$.

Recall that V is an irreducible kG-module of highest weight $\lambda \in \mathrm{X}(T)$, $\lambda = d_1\omega_1 + d_2\omega_2$ with $0 \leq d_1, d_2 \leq p-1$ not both 0. First, we consider the decomposition:

$$V\mid_{[L_1,L_1]} = \bigoplus_{i=0}^{e_1(\lambda)} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq e_1(\lambda)$. Let $s \in T$ and write $s = z_1 \cdot h_1$, as above. By (2.5), we have that:

$$s_{z_1}^i = (\lambda - i\alpha_1 - \gamma)(z_1) = (\lambda - i\alpha_1)(h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)) = c^{2d_1+3d_2} \cdot c^{-i}.$$

Therefore, z_1 acts on V^i, $0 \leq i \leq e_1(\lambda)$, as the scalar $s_{z_1}^i = c^{2d_1+3d_2-i}$. Now, let $\mu_1^i, \ldots, \mu_{t_i}^i$, $t_i \geq 1$, be the distinct eigenvalues of h_1 on V^i, $0 \leq i \leq e_1(\lambda)$, and let $n_1^i, \ldots, n_{t_i}^i$ be their respective multiplicities. Then, by Lemma 2.4.8, it follows that the distinct eigenvalues of s on V^i are $s_{z_1}^i\mu_1^i, \ldots, s_{z_1}^i\mu_{t_i}^i$, with respective multiplicities $n_1^i, \ldots, n_{t_i}^i$.

We now consider the decomposition:

$$V\mid_{[L_2,L_2]} = \bigoplus_{i=0}^{e_2(\lambda)} V^i,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$, for all $0 \leq i \leq e_2(\lambda)$. Let $s \in T$ and write $s = z_2 \cdot h_2$, as above. Then, by (2.5), we have:

$$s^i_{z_2} = (\lambda - i\alpha_2 - \gamma)(z_2) = (\lambda - i\alpha_2)(h_{\alpha_1}(c)h_{\alpha_2}(c^2)) = c^{d_1+2d_2} \cdot c^{-i}.$$

Therefore, z_2 acts on V^i, $0 \leq i \leq e_2(\lambda)$, as the scalar $s^i_{z_2} = c^{d_1+2d_2-i}$. Now, let $\mu^i_1, \ldots, \mu^i_{t_i}$, $t_i \geq 1$, be the distinct eigenvalues of h_2 on V^i, $0 \leq i \leq e_2(\lambda)$, and let $n^i_1, \ldots, n^i_{t_i}$ be their respective multiplicities. We apply Lemma 2.4.8 once more, to determine that the distinct eigenvalues of s on V^i are $s^i_{z_2}\mu^i_1, \ldots, s^i_{z_2}\mu^i_{t_i}$, with respective multiplicities $n^i_1, \ldots, n^i_{t_i}$.

Proposition 7.2.7. *Let $V = L_G(\omega_1)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 4 - \delta_{p,2}.$$

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $L = L_2$ and set $\lambda = \omega_1$. Then $e_2(\lambda) = 2$ and we have:

$$V\mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_2} V_{\lambda - i\alpha_2 - \gamma}$ for all $0 \leq i \leq 2$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_1)$ and so, by Lemma 2.4.3, we also have $V^2 \cong L_L(\omega_1)$. Now, the weight $(\lambda - \alpha_1 - \alpha_2)\mid_{T'} = 2\omega_1$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_L(2\omega_1)$ and $\dim(V^1) \geq \dim(L_L(2\omega_1))$. We deduce that $V^1 \cong L_L(2\omega_1)$ and:

$$V\mid_{[L,L]} \cong L_L(\omega_1) \oplus L_L(2\omega_1) \oplus L_L(\omega_1).$$

If $\dim(V^i_s(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $i = 0, 1$ or $i = 2$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$ with $c \neq 1$. In this case, as s acts on V^i, $0 \leq i \leq 2$, as scalar multiplication by c^{1-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c \text{ with } \dim(V_s(c)) \geq \dim(V^0) = 2; \\ 1 \text{ with } \dim(V_s(1)) = \dim(V^1); \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^2) = 2. \end{cases}$$

It follows that $\dim(V_s(\mu)) \leq 4 - \delta_{p,2}$ for all eigenvalues $\mu \in k^*$ of s on V.

We can now assume that $\dim(V^i_s(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 2$. We write $s = z_2 \cdot h_2$, where $z_2 \in \mathrm{Z}(L)^\circ$ and $h_2 \in [L, L]$. Since z_2 acts by scalar multiplication on each V^i, it follows that $\dim(V^i_{h_2}(\mu_{h_2})) < \dim(V^i)$ for all $0 \leq i \leq 2$, where $\mu_{h_2} \in k^*$ is any eigenvalue of h_2 on V^i. Now, by Lemma 3.2.3, we have $\dim(V^0_{h_2}(\mu_{h_2})) \leq 1$ and $\dim(V^2_{h_2}(\mu_{h_2})) \leq 1$ for all eigenvalues μ_{h_2} of h_2 on V^0 and V^2, respectively. We apply Lemma 3.2.3 when $p = 2$, or Proposition 3.2.4 when $p \neq 2$, to determine that $\dim(V^1_{h_2}(\mu_{h_2})) \leq 2 - \delta_{p,2}$ for all eigenvalues μ_{h_2} of h_2 on V^1. It follows that $\dim(V_s(\mu)) \leq 4 - \delta_{p,2}$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 4 - \delta_{p,2} < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. $\square$

Proposition 7.2.8. *Let k be an algebraically closed field of characteristic $p \neq 3$ and let $V = L_G(\omega_2)$. Then for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V we have*

$$\dim(V_s(\mu)) \leq 8,$$

where there exist $(s, \mu) \in T \setminus \mathrm{Z}(G) \times k^*$ for which the bound is attained.

In particular, we have $\dim(V_s(\mu)) < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V.

Proof. Set $L = L_1$ and set $\lambda = \omega_2$. Then $\dim(V) = 14$, as $p \neq 3$, and moreover, as $e_1(\lambda) = 6$, we have that:

$$V \mid_{[L,L]} = V^0 \oplus V^1 \oplus \cdots \oplus V^6,$$

where $V^i = \bigoplus_{\gamma \in \mathbb{N}\Delta_1} V_{\lambda - i\alpha_1 - \gamma}$ for all $0 \leq i \leq 6$. By [Smi82, Proposition], we have $V^0 \cong L_L(\omega_2)$ and so, by Lemma 2.4.3, we also have $V^6 \cong L_L(\omega_2)$. The weight $(\lambda - \alpha_1 - \alpha_2) \mid_{T'} = 0$ admits a maximal vector in V^1, therefore V^1 has a composition factor isomorphic to $L_L(0)$ and $\dim(V^1) \geq 1$. Once again, we apply Lemma 2.4.3 to determine that $\dim(V^5) \geq 1$. The weight $(\lambda - 2\alpha_1 - \alpha_2) \mid_{T'} = \omega_2$ admits a maximal vector in V^2 and so V^2 has a composition factor isomorphic to $L_L(\omega_2)$ and $\dim(V^2) \geq \dim(L_L(\omega_2)) = 2$. Consequently, by Lemma 2.4.3, we also have $\dim(V^4) \geq 2$. From all of the above, it follows that $\dim(V^3) \leq 4$. Now, in V^3, the weight $(\lambda - 3\alpha_1 - \alpha_2) \mid_{T'} = 2\omega_2$ admits a maximal vector, therefore V^3 has a composition factor isomorphic to $L_L(2\omega_2)$. We also note that the dominant weight $(\lambda - 3\alpha_1 - 2\alpha_2) \mid_{T'} = 0$ has multiplicity 2 in V^3. When $p \neq 2$, the weight 0 is a sub-dominant weight of multiplicity 1 in the composition factor of V^3 isomorphic to $L_L(2\omega_2)$. However, when $p = 2$, this is no longer the case. Therefore, as $\dim(V^3) \leq 4$, we determine that when $p \neq 2$, V^3 consists of 2 composition factors: one isomorphic to $L_L(2\omega_2)$ and one isomorphic to $L_L(0)$, while, when $p = 2$, then V^3 consists of 3 composition factors: one isomorphic to $L_L(2\omega_2) \cong L_L(\omega_2)^{(2)}$ and two isomorphic to $L_L(0)$. Lastly, by dimensional considerations, we determine that $V^1 \cong L_L(0)$ and $V^2 \cong L_L(\omega_2)$, hence $V^5 \cong L_L(0)$ and $V^4 \cong L_L(\omega_2)$, respectively, by Lemma 2.4.3.

If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue $\mu \in k^*$ of s on V, where $0 \leq i \leq 6$, $i \neq 1, 5$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$, and so $s = h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)$ with $c \neq 1$. In this case, as s acts on V^i, $0 \leq i \leq 6$, as scalar multiplication by c^{3-i}, we determine that the eigenvalues of s on V, not necessarily distinct, are:

$$\begin{cases} c^3 \text{ with } \dim(V_s(c^3)) \geq \dim(V^0) = 2; \\ c^2 \text{ with } \dim(V_s(c^2)) \geq \dim(V^1) = 1; \\ c \text{ with } \dim(V_s(c)) \geq \dim(V^2) = 2; \\ 1 \text{ with } \dim(V_s(1)) \geq \dim(V^3) = 4; \\ c^{-1} \text{ with } \dim(V_s(c^{-1})) \geq \dim(V^4) = 2; \\ c^{-2} \text{ with } \dim(V_s(c^{-2})) \geq \dim(V^5) = 1; \\ c^{-3} \text{ with } \dim(V_s(c^{-3})) \geq \dim(V^6) = 2. \end{cases}$$

Since $c \neq 1$, it follows that $\dim(V_s(\mu)) \leq 8$ for all eigenvalues $\mu \in k^*$ of s on V. Moreover, for $s = h_{\alpha_2}(-1) \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ we have $\dim(V_s(-1)) = 8$.

We can now assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues $\mu \in k^*$ of s on V and all $0 \leq i \leq 6$, $i \neq 1, 5$. We write $s = z_1 \cdot h_1$, where $z_1 \in \mathrm{Z}(L)^\circ$ and $h_1 \in [L, L]$. Since z_1 acts by scalar multiplication on each V^i, it follows that $\dim(V_{h_1}^i(\mu_{h_1})) < \dim(V^i)$ for all $0 \leq i \leq 6$, $i \neq 1, 5$, where $\mu_{h_1} \in k^*$ is any eigenvalue of h_1 on V^i. Since V^3 has $2 + \delta_{p,2}$ composition factors, one isomorphic to $L_L(2\omega_2)$ and $1+\delta_{p,2}$ isomorphic to $L_L(0)$, using (3.1), we determine that the eigenvalues of h_1 on V^3 are 1 with multiplicity 2, d and d^{-1}, where $d \in k^*$. Now, if $d = 1$, then $\dim(V_{h_1}^3(1)) = 4$, hence $\dim(V_s^3(1)) = 4$, contradicting our assumption. It follows that $\dim(V_{h_1}^3(\mu_{h_1})) \leq 2$ for all eigenvalues μ_{h_1} of h_1 on V^3. Furthermore, by Lemma 3.2.3, we have $\dim(V_{h_1}^i(\mu_{h_1})) \leq 1$ for all eigenvalues μ_{h_1} of h_1 on V^i, where $i = 0, 2, 4$ and $i = 6$. It follows that $\dim(V_s(\mu)) \leq 8$ for all eigenvalues $\mu \in k^*$ of s on V.

In conclusion, we have shown that $\dim(V_s(\mu)) \leq 8 < \dim(V) - \sqrt{\dim(V)}$ for all $s \in T \setminus \mathrm{Z}(G)$ and all eigenvalues $\mu \in k^*$ of s on V. □

We conclude this section by noting that Propositions 7.2.7 and 7.2.8 prove Theorem 7.1.2 for simple simply connected linear algebraic groups of type G_2. As this was the last type of exceptional algebraic groups we had to consider, we can conclude, by Remark 7.1.4, that the proof of Theorem 7.1.2 is now complete.

7.3 The proof of Theorem 7.1.3

Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of exceptional type. Before we begin to investigate $\dim(V_u(1))$, where V is given in Table 7.1.1 and u is a nontrivial unipotent element of G, we need to set up additional theoretical background which will aid us along the way.

Recall that, for exceptional linear algebraic groups G of type E_6, or E_7, we have denoted by V_d the Weyl kG-module $V_G(\omega_\ell)$ of dimension d, while, for groups G of type different than E, we have denoted by V_d the nontrivial Weyl kG-module whose dimension d is less than that of $\mathrm{Lie}(G)$. Now, let u be a unipotent element of G. Lawther observed that, in good characteristic, the Jordan block structure of the action of u on V_d is sufficient to determine the unipotent conjugacy class of u in G, see [Law95, Theorem 2]. Because of this, instead of introducing additional notation for the conjugacy classes of unipotent elements in exceptional algebraic groups, we will use the Jordan form of their action on V_d to identify their classes. We will also use [Law95, Tables A and B] and [Sim13, Tables 3.4 - 3.6] to identify representatives for each class. However, we have to mention that, in bad characteristic, there are a handful of examples of non-conjugate unipotent elements whose action on V_d have the same block structure. These are listed in [Law95, Table C].

Let V be a finite-dimensional vector space over k and let $u \in \mathrm{GL}(V)$ be a unipotent element. Recall that $r_i(u)$ is the number of Jordan blocks of size $i \geq 1$ occurring in the Jordan form of u on V. In order to treat the modules $L_G(\omega_1)$ for G of type G_2 in characteristic $p = 2$, $L_G(\omega_4)$ for G of type F_4 in characteristic $p = 3$, and $L_G(\omega_2)$ for G of type E_6 in characteristic $p = 3$; we need the following lemma:

Lemma 7.3.1. [Kor20, Lemma 3.4] *Let $u \in \mathrm{GL}(V)$ be a unipotent element and set $X = u-1$. Suppose that $W \subseteq V$ is a subspace invariant under u such that $\dim(W) = 1$. Let $m \geq 1$ be such that $W \subseteq \mathrm{im}(X^{m-1})$ and $W \nsubseteq \mathrm{im}(X^m)$. Then:*

(1) *if* $m = 1$ *we have* $\begin{cases} r_1(u_{V/W}) = r_1(u) - 1; \\ r_i(u_{V/W}) = r_i(u) \text{ for all } i \neq 1. \end{cases}$

In particular, $\dim((V/W)_u(1)) = \dim(V_u(1)) - 1$.

(2) *if* $m \geq 2$ *we have* $\begin{cases} r_m(u_{V/W}) = r_m(u) - 1; \\ r_{m-1}(u_{V/W}) = r_{m-1}(u) + 1; \\ r_i(u_{V/W}) = r_i(u) \text{ for all } i \neq m, m-1. \end{cases}$

In particular, $\dim((V/W)_u(1)) = \dim(V_u(1))$.

We will now give the proof of Theorem 7.1.3. Each type of exceptional algebraic group G will have its own result in which all the corresponding kG-modules of Table 7.1.1 will be investigated.

Proposition 7.3.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type G_2.*

(1) *If $V = L_G(\omega_1)$ and $p \neq 2$, then $\dim(V_u(1)) \leq 5$, for all non-identity unipotent elements $u \in G$. Equality holds if and only if u is conjugate to $x_{\alpha_2}(1)$. Moreover, for $V = L_G(\omega_2)$ and $p = 3$, we also have $\dim(V_u(1)) \leq 5$ for all non-identity unipotent elements $u \in G$, and there exist unipotent elements $u \in G$ for which the bound is attained.*

(2) *If $V = L_G(\omega_1)$ and $p = 2$, then $\dim(V_u(1)) \leq 4$ for all non-identity unipotent elements $u \in G$. Moreover, equality holds if and only if u is conjugate to $x_{\alpha_2}(1)$.*

(3) *If $V = L_G(\omega_2)$ and $p \neq 3$, then $\dim(V_u(1)) \leq 8$ for all non-identity unipotent elements $u \in G$. Moreover, equality holds if and only if u is conjugate to $x_{\alpha_2}(1)$.*

In particular, for $V = L_G(\omega_1)$, respectively for $V = L_G(\omega_2)$ when $p = 3$, there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$, while, for $V = L_G(\omega_2)$ when $p \neq 3$, we have that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. (1) Since $p \neq 2$, we have that $V_7 \cong V$, see item (d.1) of Remark 7.1.4. By [Law95, Table 1], it follows that $\dim(V_u(1)) \leq 5$ for all non-identity unipotent elements $u \in G$. Moreover, by [Law95, Tables 1 and B], equality holds if and only if u is conjugate to $x_{\alpha_2}(1)$. In particular, for $x_{\alpha_2}(1)$ we have $\dim(V_{x_{\alpha_2}(1)}(1)) > \dim(V) - \sqrt{\dim(V)}$. Lastly, assume that $p = 3$ and let $V = L_G(\omega_2)$. We know by item (c.1) of Remark 7.1.4 that $V = L_G(\phi(\omega_1))$, where $\phi : \mathrm{X}(T)^{\mathbb{Q}} \to \mathrm{X}(T)^{\mathbb{Q}}$ is the special isomorphism associated to the exceptional isogeny of G. Therefore, $\dim(V_u(1)) \leq 5$ for all non-identity unipotent elements $u \in G$. Moreover, since there exist $u \in G$ for which $\dim((L_G(\phi(\omega_1)))_u(1)) = 5$, it follows that there exist $u \in G$ such that $\dim(V_u(1)) = 5$. In particular, this shows that there exist non-identity unipotent elements $u \in G$ for which $\dim(V_u(1)) \geq \dim(V) - \sqrt{\dim(V)}$.

(2) We denote by u_V the action of u on V and by u' the action of u on V_7. Now, by item (d.1) of Remark 7.1.4 and by Lemma 7.3.1, it follows that the Jordan block structure of u_V can be determined from that of u' and, moreover, we have $\dim(V_u(1)) \leq \dim((V_7)_u(1))$.

As $p = 2$, we can assume that $u \in S$, where $S = \{x_{\alpha_2}(1), x_{\alpha_1}(1), x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(1), x_{\alpha_1}(1)x_{\alpha_2}(1)\}$ is a set of class representatives for the four non-identity unipotent conjugacy classes in G, see [Law95, Table B]. By [Law95, Table 1], if $u = x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(1)$, then $\dim(V_u(1)) \leq \dim((V_7)_u(1)) = 3$, while if $u = x_{\alpha_1}(1)x_{\alpha_2}(1)$, then $\dim(V_u(1)) \leq 2$. Let $\{e_{\alpha_i}, f_{\alpha_i}, e_{n\alpha_1+\alpha_2}, f_{n\alpha_1+\alpha_2}, e_{3\alpha_1+2\alpha_2}, f_{3\alpha_1+2\alpha_2}, h_{\alpha_i} \mid i = 1, 2,\ n = 1, 2, 3\}$ be a Chevalley basis of $\mathrm{Lie}(G)$. Choose $v^+ \in V_7$ such that $e_\alpha v^+ = 0$, for all positive roots $\alpha \in \Phi$, and fix the following Kostant basis of V_7:

$$\begin{aligned}
&u_1 = v^+, && u_2 = f_{\alpha_1}v^+, && u_3 = f_{\alpha_1+\alpha_2}v^+,\\
&u_4 = f_{2\alpha_1+\alpha_2}v^+, && u_5 = f_{3\alpha_1+\alpha_2}v^+, && u_6 = f_{3\alpha_1+2\alpha_2}v^+,\\
&u_7 = f_{\alpha_1}f_{3\alpha_1+2\alpha_2}v^+.
\end{aligned}$$

One checks that $\mathrm{Rad}(V_7) = \langle u_4 \rangle$. Now, if $u = x_{\alpha_1}(1)$, then, by [Tes88, p.43], we have that $u' = \mathrm{I}_7 + \mathrm{E}_{1,2} + \mathrm{E}_{3,5} + \mathrm{E}_{4,5} + \mathrm{E}_{6,7}$, where $\mathrm{E}_{i,j}$ denotes the matrix whose (q, r) entry is $\delta_{iq}\delta_{jr}$ and I_7 is the identity matrix. Let $X = u - 1$. Then $\mathrm{im}(X) = \langle u_1, u_3 + u_4, u_6 \rangle$ and, in the notation of Lemma 7.3.1, we have $m = 1$. Now, by [Law95, Table 1], we have $r_1(u') = 1$ and $r_2(u') = 3$. Therefore, as $m = 1$, by Lemma 7.3.1, we have $r_1(u_V) = 0$ and $r_2(u_V) = 3$, thereby $\dim(V_u(1)) = 3$. Similarly, if $u = x_{\alpha_2}(1)$, then, by [Tes88, p.43], we have that $u' = \mathrm{I}_7 + \mathrm{E}_{2,3} + \mathrm{E}_{5,6}$. Again, let $X = u - 1$. Then $\mathrm{im}(X) = \langle u_2, u_5 \rangle$ and, in the notation of Lemma 7.3.1, we have $m = 1$. By [Law95, Table 1], we have $r_1(u') = 3$ and $r_2(u') = 2$, therefore, by Lemma 7.3.1, $r_1(u_V) = 2$ and $r_2(u_V) = 2$, as $m = 1$, and so $\dim(V_u(1)) = 4$.

We conclude that $\dim(V_u(1)) \leq 4$ for all non-identity unipotent elements $u \in G$. Moreover, we showed that equality holds if and only if u is conjugate to $x_{\alpha_2}(1)$. In particular, for $x_{\alpha_2}(1)$ we have $\dim(V_{x_{\alpha_2}(1)}(1)) > \dim(V) - \sqrt{\dim(V)}$.

(3) Since $p \neq 3$, by [MT11, Theorem 15.20], we have that $\mathrm{Lie}(G)$ is an irreducible kG-module and we identify it with $L_G(\omega_2)$, as ω_2 is the highest root of G. We now apply [Law95, Table 2] to determine that $\dim(V_u(1)) \leq 8$ for all non-identity unipotent elements $u \in G$. Moreover, using [Law95, Table 2 and B], we determine that equality holds if and only if u belongs to the unipotent conjugacy class with representative $x_{\alpha_2}(1)$. Lastly, we have that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 7.3.3. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type F_4.*

(1) *If $V = L_G(\omega_4)$ and $p \neq 3$, then $\dim(V_u(1)) \leq 20$ for all non-identity unipotent elements $u \in G$. Equality holds if and only if u and $x_{\alpha_1}(1)$ are conjugate. Moreover, for $V = L_G(\omega_1)$ and $p = 2$, we also have $\dim(V_u(1)) \leq 20$ for all non-identity unipotent elements $u \in G$.*

(2) *If $V = L_G(\omega_4)$ and $p = 3$, then $\dim(V_u(1)) \leq 19$ for all non-identity unipotent elements $u \in G$. Moreover, equality holds if and only if u and $x_{\alpha_1}(1)$ are conjugate.*

(3) *If $V = L_G(\omega_1)$ and $p \neq 2$, then $\dim(V_u(1)) \leq 36$ for all non-identity unipotent elements $u \in G$. Moreover, equality holds if and only if u and $x_{\alpha_1}(1)$ are conjugate.*

In particular, in all cases, we have that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

Proof. (1) Since $p \neq 3$, we have that $V \cong V_{26}$, see item (d.2) of Remark 7.1.4, and, by [Law95, Table 3], it follows that $\dim(V_u(1)) \leq 20$ for all non-identity unipotent elements $u \in G$. Moreover, by [Law95, Tables A and 3], we determine that equality holds if and only if u belongs to the unipotent conjugacy class with representative $x_{\alpha_1}(1)$. Lastly, assume that $p = 2$ and let $V = L_G(\omega_1)$. We know, by item (c.2) of Remark 7.1.4, that $V = L_G(\phi(\omega_4))$, where $\phi : \mathrm{X}(T)^{\mathbb{Q}} \to \mathrm{X}(T)^{\mathbb{Q}}$ is the special isomorphism associated to the exceptional isogeny of G. Therefore, $\dim(V_u(1)) \leq 20$ for all non-identity unipotent elements $u \in G$. Moreover, in both cases, we have that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

(2) For u, a unipotent element of G, let u_V, respectively $u^{'}$, denote the action of u on $V = L_G(\omega_4)$, respectively on V_{26}. As in case (2) of Proposition 7.3.2, by item (d.2) of Remark 7.1.4 and by Lemma 7.3.1, it follows that the Jordan block structure of u_V can be determined from that of $u^{'}$ and that $\dim(V_u(1)) \leq \dim((V_{26})_u(1))$. Assume that u does not belong to the unipotent conjugacy class with representative $x_{\alpha_1}(1)$, see [Law95, Table A]. Then, by [Law95, Table 3], it follows that $\dim((V_{26})_u(1)) \leq 16$ and, consequently, $\dim(V_u(1)) \leq 16$.

We can thus focus on the unipotent conjugacy class with representative $x_{\alpha_1}(1)$. Let $H = \langle x_{\alpha_1}(c), x_{-\alpha_1}(c) \mid c \in k\rangle$. We note that $H = [L_I, L_I]$, where L_I is a Levi subgroup of the parabolic subgroup P_I of G corresponding to $\Delta_I = \{\alpha_1\}$. Therefore, H is a simple simply connected linear algebraic subgroup of type A_1 of G. Now $T_H = T \cap H$ is a maximal torus of H, $B_H = B \cap H$ is a Borel subgroup of H with the property that $T_H \subseteq B_H$, and $\Delta_H = \{\alpha_1\}$ is a base of the root system $\Phi_H = \Phi \cap \mathbb{Z}\alpha_1$ of H. We abuse notation and let ω_1 denote the fundamental dominant weight of H corresponding to α_1.

One calculates and determines that $V|_{[H,H]}$ affords 6 composition factors isomorphic to $L_H(\omega_1)$ and 13 composition factors isomorphic to $L_H(0)$. Since ω_1 is p-restricted and since $H \cong \mathrm{SL}_2(k)$, it follows that $V_H(\omega_1) \cong L_H(\omega_1)$. We use [Jan07, II.2.12 and 2.14] to show that the kH-module V is completely reducible with $V|_{[H,H]} \cong (L_H(\omega_1))^6 \oplus (L_H(0))^{13}$. Thus, the Jordan form of the action of $x_{\alpha_1}(1)$ on V is $J_2^6 \oplus J_1^{13}$, hence $\dim(V_{x_{\alpha_1}(1)}(1)) = 19$.

We conclude that $\dim(V_u(1)) \leq 19$ for all non-identity unipotent elements $u \in G$. Moreover, we showed that equality holds if and only if u belongs to the unipotent conjugacy class with representative $x_{\alpha_1}(1)$. In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$.

(3) Since $p \neq 2$, by [MT11, Theorem 15.20], we have that $\mathrm{Lie}(G)$ is an irreducible kG-module and we identify it with $L_G(\omega_1)$, as ω_1 is the highest root of G. We now apply [Law95, Table 4] and determine that $\dim(V_u(1)) \leq 36$ for all non-identity unipotent elements $u \in G$. Moreover, by [Law95, Tables A and 4], equality holds if and only if u belongs to the unipotent conjugacy class with representative $x_{\alpha_1}(1)$. Lastly, we note that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 7.3.4. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type E_6.*

(1) *If $V = L_G(\omega_1)$, or $V = L_G(\omega_6)$, then $\dim(V_u(1)) \leq 21$ for all non-identity unipotent elements $u \in G$. Moreover, equality holds if and only if u is conjugate to $x_{\alpha_1}(1)$.*

(2) *If $V = L_G(\omega_2)$ and $p \neq 3$, then $\dim(V_u(1)) \leq 56$ for all non-identity unipotent elements $u \in G$. Moreover, equality holds if and only if u is conjugate to $x_{\alpha_1}(1)$.*

(3) *If* $V = L_G(\omega_2)$ *and* $p = 3$*, then* $\dim(V_u(1)) \leq 55$ *for all non-identity unipotent elements* $u \in G$*. Moreover, equality holds if and only if* u *is conjugate to* $x_{\alpha_1}(1)$*.*

In particular, in all cases, we have that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ *for all non-identity unipotent elements* $u \in G$*.*

Proof. (1) First, we note that, by item (a) of Remark 7.1.4, we can assume without loss of generality that $V = L_G(\omega_1)$. We have that $V \cong V_{27}$ and thus, by [Law95, Table 5], it follows that $\dim(V_u(1)) \leq 21$ for all non-identity unipotent elements $u \in G$. Moreover, by [Sim13, Table 3.4] and [Law95, Table 5], we determine that equality holds if and only if u is conjugate to $x_{\alpha_1}(1)$.

(2) As $p \neq 3$, we have $V \cong \mathrm{Lie}(G)$, see item (b) of Remark 7.1.4, and thus, by [Law95, Table 6], it follows that $\dim(V_u(1)) \leq 56$ for all non-identity unipotent elements $u \in G$. Now, as in the previous case, using [Sim13, Table 3.4] and [Law95, Table 6], we determine that equality holds if and only if u is conjugate to $x_{\alpha_1}(1)$.

(3) As $p = 3$, we have $V \cong \mathrm{Lie}(G)/\,\mathrm{Z}(\mathrm{Lie}(G))$ and $\dim(\mathrm{Z}(\mathrm{Lie}(G))) = 1$, see item (b) of Remark 7.1.4. Therefore, by Lemma 7.3.1, it follows that $\dim(V_u(1)) \leq \dim((\mathrm{Lie}(G))_u(1))$. Now, if u does not belong to the unipotent conjugacy class of $x_{\alpha_1}(1)$, then, using [Law95, Table 6], we determine that $\dim(V_u(1)) \leq 46$. Thus, we can focus on the unipotent conjugacy class of $x_{\alpha_1}(1)$. Let $H = \langle x_{\alpha_1}(c), x_{-\alpha_1}(c) \mid c \in k\rangle$. Then, as in the proof of item (2) of Proposition 7.3.3, we have that H is simply connected of type A_1. Furthermore, $T_H = T \cap H$ is a maximal torus in H, $\Delta_H = \{\alpha_1\}$ is a base of the root system $\Phi_H = \Phi \cap \mathbb{Z}\alpha_1$ of H, and, abusing notation, ω_1 is the fundamental dominant weight of H corresponding to α_1.

Keeping in mind that the set of weights of V consists of the roots of G together with the weight 0 with multiplicity 5, one determines that $V|_{[H,H]}$ affords 1 composition factor isomorphic to $L_H(2\omega_1)$, 20 composition factors isomorphic to $L_H(\omega_1)$ and 34 composition factors isomorphic to $L_H(0)$. We now use [Jan07, II.2.12 and 2.14] to show that the kH-module V is completely reducible with $V|_{[H,H]} \cong L_H(2\omega_1) \oplus (L_H(\omega_1))^{20} \oplus (L_H(0))^{34}$. Therefore, the Jordan form of the action of $x_{\alpha_1}(1)$ on V is $J_3 \oplus J_2^{20} \oplus J_1^{34}$, thus $\dim(V_{x_{\alpha_1}(1)}(1)) = 55$. We conclude that $\dim(V_u(1)) \leq 55$ for all non-identity unipotent elements $u \in G$. Moreover, we showed that equality holds if and only if u belongs to the unipotent conjugacy class with representative $x_{\alpha_1}(1)$.

Lastly, in all cases, we note that $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. □

Proposition 7.3.5. *Let* k *be an algebraically closed field of characteristic* $p \geq 0$ *and let* G *be a simple simply connected linear algebraic group of type* E_7*. Let* $V = L_G(\omega_7)$*. Then for all non-identity unipotent elements* $u \in G$ *we have*

$$\dim(V_u(1)) \leq 44,$$

where equality holds if and only if u *and* $x_{\alpha_1}(1)$ *are conjugate.*

In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ *for all non-identity unipotent elements* $u \in G$*.*

Proof. We have that $V \cong V_{56}$ and thus, by [Law95, Table 7], it follows that $\dim(V_u(1)) \leq 44$ for all non-identity unipotent elements $u \in G$. Moreover, by [Sim13, Table 3.6] and [Law95,

Table 7], we determine that equality holds if and only if u is conjugate to $x_{\alpha_1}(1)$. In particular, we have $\dim(V_u(1)) < \dim(V) - \sqrt{\dim(V)}$ for all non-identity unipotent elements $u \in G$. $\square$

We end this section by noting that Propositions 7.3.2 through 7.3.5 prove Theorem 7.1.3. Consequently, as Theorem 7.1.2 has already been proven in the previous section, we have also completed the proof of Theorem 7.1.1. Lastly, we collect all the results of this chapter in the following table, which records the values of $\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$ and $\nu_G(V)$ for all the irreducible kG-modules we discussed. Note that in the construction of the table, we used Remark 7.1.4, Table 7.1.1 and Proposition 2.2.3.

Group	λ	Char.	$\max_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$	$\max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$	$\nu_G(V)$
E_6	ω_1	$p \geq 0$	≤ 19	21	6
	ω_6	$p \geq 0$	≤ 19	21	6
	ω_2	$p = 3$	≤ 49	55	22
	ω_2	$p \neq 3$	≤ 50	56	22
E_7	ω_7	$p \geq 0$	≤ 40	44	12
F_4	ω_4	$p = 3$	≤ 16	19	6
	ω_4	$p \neq 3$	≤ 16	20	6
	ω_1	$p = 2$	≤ 16	20	6
	ω_1	$p \neq 2$	36	36	16
G_2	ω_1	$p = 2$	≤ 3	4	2
	ω_1	$p \neq 2$	≤ 4	5	2
	ω_2	$p = 3$	≤ 4	5	2
	ω_2	$p \neq 3$	8	8	6

Table 7.3.1: The value of $\nu_G(V)$ for exceptional groups.

Index of notation

- $R_u(G)$ is the *unipotent radical* of G and is the maximal closed, connected, normal, unipotent subgroup of G.
- G° is the connected component of the identity.
- The notation $V = W_1 \mid W_2 \mid \cdots \mid W_m$ means that the kG-module V has a composition series $V = V_1 \supset V_2 \supset \cdots \supset V_m \supset V_{m+1} = 0$ with composition factors $W_i \cong V_i/V_{i+1}$, $1 \leq i \leq m$.
- $V^m = \underbrace{V \oplus V \oplus \cdots \oplus V}_{m}$.
- For a prime $p > 0$, we set $\nu_p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ to be the p-adic valuation on $\mathbb{Z}_{\geq 0}$, i.e. for $n \in \mathbb{Z}_{\geq 0}$ we have $\nu_p(n) = \max\{k \in \mathbb{Z}_{\geq 0} \mid p^k \mid n\}$.
- For a unipotent element $u \in \mathrm{GL}(V)$, we let $r_t(u)$ be the number of Jordan blocks of size $t \geq 1$ appearing in the Jordan form of u on V.
- $V_g(\mu)$, the eigenspace corresponding to the eigenvalue $\mu \in k^*$ of $g \in G$ on V, p.2
- $\nu_G(V) := \min\{\dim(V) - \dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in k^*\}$, p.2
- $\boldsymbol{G_a}$, the additive group of k, and $\boldsymbol{G_m}$, the multiplicative group of k, p.6
- $\mathrm{X}(T)$, the character group of T, and $\mathrm{Y}(T)$, the cocharacter group of T, p.6
- $\mathcal{W}$, the Weyl group of G, $w_0 \in \mathcal{W}$, the longest word, p.6
- $\preceq$ a total order on Φ, p.6
- $\mathrm{GL}(V)$, the general linear group of V, and $\mathrm{SL}(V)$, the special linear group of V, p.10.
- $\mathrm{Rad}(V)$, the radical of V, p.10
- the $\ell \times \ell$ matrix K_ℓ, p.11
- $\mathrm{Sp}(V)$, the symplectic group of V, p.11
- $\mathrm{CS_p}(V)$ the conformal symplectic group of V and $\mathrm{PCS_p}(V)$, the projective conformal symplectic group of V, p.12

- $\mathrm{O}(V)$, the orthogonal group of V, p.14,16
- $\mathrm{SO}(V)$, the special orthogonal group of V, p.14,16
- (V, Q) a quadratic space, p.15
- G_s, the set of semisimple elements in G, and G_u, the set of unipotent elements in G, p.18
- $M_V = \max\{\dim(V_g(\mu)) \mid g \in G \setminus \mathrm{Z}(G),\ \mu \in k^*\}$, p.18
- $M_s = \max\limits_{s \in T \setminus \mathrm{Z}(G)} \{\dim(V_s(\mu)) \mid \mu \in k^*\}$, p.19
- $M_u = \max\limits_{u \in G_u \setminus \{1\}} \dim(V_u(1))$, p.19
- $V = \bigoplus\limits_{\lambda \in \mathrm{X}(T)} V_\lambda$ and the T-weight spaces V_λ, p.20
- $\Lambda(V)$, the set of weights in V, p.20
- $v^+ \in V$ a maximal vector, p.21
- $V_G(\lambda)$, the Weyl kG-module of highest weight λ, p.21
- $L_G(\lambda)$ the irreducible kG-module of highest weight λ, p.21
- $F_p : k \to k$, the Frobenius automorphism, and $V^{(p^i)}$ p.22
- Δ_I, Φ_I, P_I, Q_I and L_I, p.25
- $\Delta_i = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell\}$, $1 \leq i \leq \ell$, p.25
- the α_i-level of a weight and $e_i(\lambda)$ the maximum α_i-level of a weight, p.26
- the scalar s_z^i, $1 \leq i \leq \ell$, p.33
- u_{L_i} and u_{Q_i}, p.35
- $\alpha(g)$, $g \in G$, and $\alpha(G) = \max\{\alpha(g) \mid g \in G \setminus \mathrm{Z}(G)\}$, p.37
- F^{A_ℓ} and F^{C_ℓ}, p.40; F^{B_ℓ} and F^{D_ℓ}, p.41
- $k[u]$, the group algebra of $\langle u \rangle$ over k, p.45
- V_i, $0 \leq i \leq \mathrm{ord}(u)$, the nonisomorphic indecomposable $k[u]$-modules, p.45
- J_i, $i \geq 0$, the Jordan block of size i, p.45
- $V(d)$, $d \geq 1$, and $W(d)$, $d \geq 2$ even, p.49
- $(n_{1_0}^{r_1}, \ldots, n_{t_0}^{r_t}, n_{t+1_1}^{r_{t+1}}, \ldots, n_{m_1}^{r_m})$, the Hesselink normal form, p.51
- $V_D(d)$, $d \geq 2$ even, and $W_D(d)$, $d \geq 1$, p.52

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