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Generic direct summands of tensor products for simple algebraic groups and quantum groups at roots of unity

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Abstract

Let $\mathbf{G}$ be either a simple linear algebraic group over an algebraically closed field of characteristic $\ell > 0$ or a quantum group at an ℓ-th root of unity. The category $\mathrm{Rep}(\mathbf{G})$ of finite-dimensional $\mathbf{G}$-modules is non-semisimple. In this thesis, we develop new techniques for studying Krull-Schmidt decompositions of tensor products of $\mathbf{G}$-modules.

More specifically, we use minimal complexes of tilting modules to define a tensor ideal of singular $\mathbf{G}$-modules, and we show that, up to singular direct summands, taking tensor products of $\mathbf{G}$-modules respects the decomposition of $\mathrm{Rep}(\mathbf{G})$ into linkage classes. In analogy with the classical translation principle, this allows us to reduce questions about tensor products of $\mathbf{G}$-modules in arbitrary ℓ-regular linkage classes to questions about tensor products of $\mathbf{G}$-modules in the principal block of $\mathbf{G}$. We then identify a particular non-singular indecomposable direct summand of the tensor product of two simple $\mathbf{G}$-modules in the principal block (with highest weights in two given ℓ-alcoves), which we call the *generic direct summand* because it appears generically in Krull-Schmidt decompositions of tensor products of simple $\mathbf{G}$-modules (with highest weights in the given ℓ-alcoves). We initiate the study of generic direct summands, and we use them to prove a necessary condition for the complete reducibility of tensor products of simple $\mathbf{G}$-modules, when $\mathbf{G}$ is a simple algebraic group of type A_n.

Keywords: algebraic groups, quantum groups, representation theory, tensor products, Krull-Schmidt decomposition, tensor ideals, linkage principle, translation principle, complete reducibility

Zusammenfassung

Sei $\mathbf{G}$ entweder eine einfache lineare algebraische Gruppe über einem algebraisch abgeschlossenen Körper von Charakteristik $\ell > 0$ oder eine Quantengruppe an einer ℓ-ten Einheitswurzel. Die Kategorie $\mathrm{Rep}(\mathbf{G})$ der endlichdimensionalen $\mathbf{G}$-Moduln ist nicht halbeinfach. In dieser Arbeit entwickeln wir neue Techniken um Krull-Schmidt Zerlegungen von Tensorprodukten von $\mathbf{G}$-Moduln zu untersuchen.

Genauer gesagt werden wir minimale Komplexe von Kippmoduln nutzen um ein Tensorideal von singulären $\mathbf{G}$-Moduln zu definieren und wir zeigen, dass Tensorprodukte von $\mathbf{G}$-Moduln – abgesehen von singulären direkten Summanden – die Zerlegung von $\mathrm{Rep}(\mathbf{G})$ in Verkettungsklassen respektieren. In Analogie mit dem klassischen Translationsprinzip können wir so Fragen über Tensorprodukte von $\mathbf{G}$-Moduln in beliebigen ℓ-regulären Verkettungsklassen auf Fragen über Tensorprodukte von $\mathbf{G}$-Moduln im Hauptblock von $\mathbf{G}$ reduzieren. Weiterhin identifizieren wir einen bestimmten nicht singulären unzerlegbaren direkten Summanden im Tensorprodukt zweier einfacher $\mathbf{G}$-Moduln im Hauptblock (mit höchsten Gewichten in zwei gegebenen ℓ-Alkoven), und wir bezeichnen diesen als *generischen direkten Summanden*, da er generisch in Krull-Schmidt Zerlegungen von Tensorprodukten einfacher $\mathbf{G}$-Moduln (mit höchsten Gewichten in den gegebenen ℓ-Alkoven) vorkommt. Wir beginnen das Studium generischer direkter Summanden und benutzen diese, um eine notwendige Bedingung für die vollständige Reduzibilität von Tensorprodukten einfacher $\mathbf{G}$-Moduln zu beweisen, wenn $\mathbf{G}$ eine einfache algebraische Gruppe vom Typ A_n ist.

Stichwörter: algebraische Gruppen, Quantengruppen, Darstellungstheorie, Tensorprodukte, Krull-Schmidt Zerlegung, Tensorideale, Verkettungsprinzip, Translationsprinzip, vollständige Reduzibilität

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Introduction

Tensor products are ubiquitous in representation theory, and the problem of finding direct sum decompositions of tensor products of representations has been studied by many mathematicians and in many different settings. A prime example is the well-known *Clebsch-Gordan formula*, which describes the decomposition of a tensor product of two irreducible representations of the algebraic group $\mathrm{SL}_2(\mathbb{C})$ as a direct sum of irreducible representations. More generally, one has the *Littlewood-Richardson rule*, which gives a combinatorial description of the multiplicity of an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ in a tensor product of two irreducible representations. The situation becomes more complicated when one replaces the field $\mathbb{C}$ of complex numbers by a field of positive characteristic. Then, the corresponding categories of representations are no longer semisimple, i.e. not every representation can be written as a direct sum of irreducible representations. Nevertheless, a finite-dimensional representation usually still admits an essentially unique decomposition as a direct sum of indecomposable representations, called a *Krull-Schmidt decomposition*. One of the main objectives of this thesis is to develop new techniques for studying Krull-Schmidt decompositions of tensor products of representations in certain non-semisimple categories.

When studying representations in a non-semisimple setting, it is often helpful to decompose the category under consideration as a direct product of subcategories called *blocks*, in such a way that every indecomposable representation belongs to a unique block. One can then hope to obtain stronger results by considering one block of the category at a time. However, this strategy is generally not well suited for understanding the monoidal structure[1] of the category, because a tensor product of two representations, each belonging to a given block, may have indecomposable direct summands in many different blocks. Our results provide a way of partially overcoming this obstacle, for categories of representations of simple algebraic groups (over fields of positive characteristic) and quantum groups (at roots of unity). More precisely, we will use the theory of *tilting modules* and *minimal complexes* to define a tensor ideal of *singular modules* in the representation categories. When considering representations modulo this tensor ideal, it turns out that the so-called *principal block* is closed under tensor products and that the monoidal structure of the entire category is governed to a large extent by the resulting monoidal structure on the principal block. We refer to these results as a *linkage principle* and a *translation principle* for tensor products, in analogy with the classical results describing the block decomposition of the categories in question (as recalled below).

The categories of (finite-dimensional) representations of simple algebraic groups and of quantum groups have many structural properties in common. For instance, the isomorphism classes of irreducible representations in either of these categories are in bijection with a certain set of *dominant weights*, which also parametrizes the classes of *Weyl modules*, *induced modules* and *indecomposable tilting modules*. Furthermore, the decomposition of the representation category into blocks is governed

[1] A category is called *monoidal* if it has a tensor product bifunctor that satisfies certain natural axioms.

by the *alcove geometry* associated with the so-called *dot action* of an *affine Weyl group*, for simple algebraic groups and quantum groups alike.

In the following, we refer to the representation theory of simple algebraic groups as the *modular case* and to the representation theory of quantum groups as the *quantum case*. The aforementioned similarities often make it possible to treat the two cases simultaneously. In order to do this in a consistent way, we fix the following notational conventions:

The modular case Here $\mathbf{G}$ is a simply connected simple linear algebraic group over an algebraically closed field of characteristic $\ell > 0$. We write $\mathrm{Rep}(\mathbf{G})$ for the category of finite-dimensional rational $\mathbf{G}$-modules.

The quantum case Here $\mathbf{G} = U_\zeta(\mathfrak{g})$ is the specialization at a complex ℓ-th root of unity ζ of Lusztig's divided powers version of the quantum group corresponding to a complex simple Lie algebra $\mathfrak{g}$. We write $\mathrm{Rep}(\mathbf{G})$ for the category of finite-dimensional $\mathbf{G}$-modules of type 1.

In either of the two cases, $\mathbf{G}$ comes equipped with a simple root system Φ and a weight lattice X. For this introduction (and for most of this document), we suppose that $\ell \geq h$, the Coxeter number of Φ. In the quantum case, we further assume that ℓ is odd (and not divisible by 3 if Φ is of type G_2). From now on, we use the term $\mathbf{G}$*-module* to refer to the objects of $\mathrm{Rep}(\mathbf{G})$; in particular, all $\mathbf{G}$-modules that we consider are implicitly assumed to be finite-dimensional.

The isomorphism classes of simple $\mathbf{G}$-modules are parametrized by the set X^+ of dominant weights in X, with respect to a fixed positive system $\Phi^+ \subseteq \Phi$, and we write $L(\lambda)$ for the simple $\mathbf{G}$-module corresponding to a dominant weight $\lambda \in X^+$. Furthermore, we denote by $\Delta(\lambda)$, $\nabla(\lambda)$ and $T(\lambda)$ the Weyl module, the induced module and the indecomposable tilting module of highest weight $\lambda \in X^+$. Let us write W_{fin} and $W_{\mathrm{aff}} = \mathbb{Z}\Phi \rtimes W_{\mathrm{fin}}$ for the finite Weyl group and the affine Weyl group of Φ, respectively, and denote the natural embedding $\mathbb{Z}\Phi \to W_{\mathrm{aff}}$ by $\gamma \mapsto t_\gamma$. We consider the *dot action* of W_{aff} on X, defined by

$$t_\gamma w \cdot \lambda = w(\lambda + \rho) - \rho + \ell\gamma$$

for $\gamma \in \mathbb{Z}\Phi$, $w \in W_{\mathrm{fin}}$ and $\lambda \in X$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. Finally, let us write C_{fund} for the *fundamental ℓ-alcove* in $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$, with respect to the dot action (see Section I.6), and note that $C_{\mathrm{fund}} \cap X$ is non-empty because $\ell \geq h$. The *linkage principle* asserts that $\mathrm{Rep}(\mathbf{G})$ admits a decomposition

$$\mathrm{Rep}(\mathbf{G}) = \bigoplus_{\lambda \in \overline{C}_{\mathrm{fund}} \cap X} \mathrm{Rep}_\lambda(\mathbf{G}),$$

where the *linkage class* $\mathrm{Rep}_\lambda(\mathbf{G})$ of $\lambda \in \overline{C}_{\mathrm{fund}} \cap X$ is defined as the full subcategory of $\mathrm{Rep}(\mathbf{G})$ whose objects are the $\mathbf{G}$-modules all of whose composition factors have highest weight in $W_{\mathrm{aff}} \cdot \lambda$.[2] The linkage class $\mathrm{Rep}_0(\mathbf{G})$ is called the *principal block* of $\mathbf{G}$. Furthermore, for $\lambda, \mu \in C_{\mathrm{fund}} \cap X$, the linkage classes of λ and μ are equivalent, via a so-called *translation functor* $T_\lambda^\mu \colon \mathrm{Rep}_\lambda(\mathbf{G}) \to \mathrm{Rep}_\mu(\mathbf{G})$ with quasi-inverse T_μ^λ. With this notation in place, we can start giving a more detailed summary of our results.

[2]Not all of the linkage classes are blocks (in the usual sense that they can not be decomposed any further), but those corresponding to weights in $C_{\mathrm{fund}} \cap X$ are. A precise description of the blocks of $\mathrm{Rep}(\mathbf{G})$ can be found in Section II.7.2 of [Jan03].

Minimal tilting complexes

Tilting modules were introduced to the representation theory of algebraic groups by S. Donkin [Don93], following earlier work of C. Ringel in the setting of quasi-hereditary algebras [Rin91], and they have been at the heart of some exciting new developments in this area in recent years [RW18]. The full subcategory $\mathrm{Tilt}(\mathbf{G})$ of tilting modules in $\mathrm{Rep}(\mathbf{G})$ is closed under tensor products, and the canonical functor

$$\mathfrak{T}\colon K^b\big(\mathrm{Tilt}(\mathbf{G})\big) \longrightarrow D^b\big(\mathrm{Rep}(\mathbf{G})\big)$$

from the bounded homotopy category of $\mathrm{Tilt}(\mathbf{G})$ to the bounded derived category of $\mathrm{Rep}(\mathbf{G})$ is an equivalence of triangulated monoidal categories, which we call the *tilting equivalence*. In some respects, the monoidal structure of $\mathrm{Tilt}(\mathbf{G})$ is better understood than that of $\mathrm{Rep}(\mathbf{G})$ (see for instance [AP95, Ost97]), and we will follow the strategy of transporting information about tensor products from $\mathrm{Tilt}(\mathbf{G})$ to $\mathrm{Rep}(\mathbf{G})$ via the tilting equivalence and the notion of *minimal complexes*.

As a consequence of the tilting equivalence, there exists for every $\mathbf{G}$-module M a bounded complex of tilting modules that is isomorphic to M when considered as an object of $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$, and this complex is unique up to homotopy equivalence. Furthermore, as $\mathrm{Tilt}(\mathbf{G})$ is a Krull-Schmidt category, every homotopy class in $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$ contains a complex that is *minimal* in a suitable sense, and the latter is unique up to isomorphism of complexes. Combining these two observations, it follows that there exists a unique bounded minimal complex $C_{\min}(M)$ of tilting modules that is isomorphic to M when considered as an object of $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$, and we call $C_{\min}(M)$ the *minimal tilting complex* of M. The minimal tilting complex $C_{\min}(M)$ is a powerful invariant of M, and it encodes important information, such as the *good filtration dimension* $\mathrm{gfd}(M)$ and the *Weyl filtration dimension* $\mathrm{wfd}(M)$. Furthermore, minimal tilting complexes are well-behaved with respect to direct sums, short exact sequences and tensor products of $\mathbf{G}$-modules.

Singular G-modules

The definition of *singular* $\mathbf{G}$*-modules*, which we will give below, serves as a good example of how minimal tilting complexes allow us to use results about tensor products in $\mathrm{Tilt}(\mathbf{G})$ to study the monoidal structure of $\mathrm{Rep}(\mathbf{G})$. A tilting module in $\mathrm{Rep}(\mathbf{G})$ is called *negligible* if it has no direct summands of the form $T(\lambda)$ with $\lambda \in C_{\mathrm{fund}} \cap X$. It is well-known that the negligible tilting modules are a *tensor ideal* in $\mathrm{Tilt}(\mathbf{G})$, that is, that they form an isomorphism-closed set which is stable under direct sums, retracts and tensor products with arbitrary tilting modules in $\mathrm{Rep}(\mathbf{G})$ (see [GM94, AP95]).

Definition. A $\mathbf{G}$-module M is called *singular* if all terms of $C_{\min}(M)$ are negligible. Otherwise, we say that M is *regular*.[3]

Using elementary properties of minimal tilting complexes, we can show that the singular $\mathbf{G}$-modules form a tensor ideal in $\mathrm{Rep}(\mathbf{G})$. For every $\mathbf{G}$-module M, we can now write

$$M \cong M_{\mathrm{sing}} \oplus M_{\mathrm{reg}},$$

where M_{sing} is the direct sum of all singular indecomposable direct summands of M and where M_{reg} is the direct sum of all regular indecomposable direct summands of M. We call M_{sing} and M_{reg} the *singular part* and the *regular part* of M, respectively.

[3] Our terminology is justified by the fact that, for $\lambda \in X^+$, the simple $\mathbf{G}$-module $L(\lambda)$ is regular if and only if its highest weight λ is *ℓ-regular*, i.e. if $\lambda \in W_{\mathrm{aff}} \cdot \lambda'$ for some $\lambda' \in C_{\mathrm{fund}} \cap X$ (see Lemma II.4.3 below).

We can now state our *linkage principle for tensor products*, which asserts that the monoidal structure of $\mathrm{Rep}(\mathbf{G})$ is compatible with the decomposition into linkage classes, when we consider $\mathrm{Rep}(\mathbf{G})$ modulo the tensor ideal of singular $\mathbf{G}$-modules.

Theorem A. *Let $\lambda \in C_{\mathrm{fund}} \cap X$ and let M and N be $\mathbf{G}$-modules in the linkage classes of 0 and λ, respectively. Then $(M \otimes N)_{\mathrm{reg}}$ belongs to the linkage class of λ.*

As a consequence of Theorem A, the essential image of the principal block $\mathrm{Rep}_0(\mathbf{G})$ in the quotient category of $\mathrm{Rep}(\mathbf{G})$ by the ideal of singular $\mathbf{G}$-modules is closed under tensor products. The next result is our *translation principle for tensor products.*

Theorem B. *Let M and N be $\mathbf{G}$-modules in $\mathrm{Rep}_0(\mathbf{G})$. For $\lambda, \mu \in C_{\mathrm{fund}} \cap X$, we have*

$$\big(T_0^\lambda M \otimes T_0^\mu N\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T_0^\nu (M \otimes N)_{\mathrm{reg}}^{\oplus c_{\lambda,\mu}^\nu},$$

where $c_{\lambda,\mu}^\nu = [T(\lambda) \otimes T(\mu) : T(\nu)]_\oplus$ for $\nu \in C_{\mathrm{fund}} \cap X$.

The translation functors T_0^λ and T_0^μ are equivalences for $\lambda, \mu \in C_{\mathrm{fund}} \cap X$, so Theorem B implies that the monoidal structure of $\mathrm{Rep}(\mathbf{G})$ modulo singular $\mathbf{G}$-modules is completely determined by the monoidal structure of $\mathrm{Rep}_0(\mathbf{G})$ modulo singular $\mathbf{G}$-modules. We point out that the coefficients $c_{\lambda,\mu}^\nu$ are the structure constants of the so-called *Verlinde algebra* (i.e. the split Grothendieck group of the quotient of $\mathrm{Tilt}(\mathbf{G})$ by the tensor ideal of negligible tilting modules) and that they can be computed as an alternating sum of dimensions of weight spaces of the Weyl module $\Delta(\lambda)$ (see Section I.9).

Generic direct summands

We now consider the regular parts of tensor products of specific $\mathbf{G}$-modules, such as Weyl modules and simple $\mathbf{G}$-modules. By Theorems A and B, we can restrict our attention to tensor products of $\mathbf{G}$-modules in the principal block. Let us write $W_{\mathrm{aff}}^+ = \{x \in W_{\mathrm{aff}} \mid x \cdot 0 \in X^+\}$. Then, for $x, y \in W_{\mathrm{aff}}^+$, we show that the tensor product $\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)$ has a unique regular indecomposable direct summand, which we denote by $G_\Delta(x, y)$ and call the *generic direct summand* of $\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)$. This name reflects the fact that, for $\lambda, \mu \in C_{\mathrm{fund}} \cap X$, we have

$$\big(\Delta(x \cdot \lambda) \otimes \Delta(y \cdot \mu)\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T_0^\nu G_\Delta(x, y)^{\oplus c_{\lambda,\mu}^\nu}$$

by Theorem B; hence the $\mathbf{G}$-modules $T_0^\nu G_\Delta(x, y)$ appear *generically* in Krull-Schmidt decompositions of tensor products of Weyl modules with highest weights in the alcoves $x \cdot C_{\mathrm{fund}}$ and $y \cdot C_{\mathrm{fund}}$.

A tensor product of simple $\mathbf{G}$-modules in the principal block may generally have more than one regular indecomposable direct summand, but there is still a unique one with maximal good filtration dimension: For $x, y \in W_{\mathrm{aff}}^+$, the tensor product $L(x \cdot 0) \otimes L(y \cdot 0)$ has a unique regular indecomposable direct summand $G(x, y)$ with good filtration dimension $\ell(x) + \ell(y)$, where $\ell \colon W_{\mathrm{aff}} \to \mathbb{Z}_{\geq 0}$ denotes the length function with respect to the reflections in the walls of C_{fund}. We call $G(x, y)$ the *generic direct summand* of $L(x \cdot 0) \otimes L(y \cdot 0)$.

We believe that describing the structure of generic direct summands of tensor products (of Weyl modules and of simple $\mathbf{G}$-modules) is a key problem for better understanding the monoidal structure of $\mathrm{Rep}(\mathbf{G})$. We give such descriptions for $\mathbf{G}$ of type A_1 and A_2 in Chapter III, but the problem seems elusive in its full generality. In order to get a grasp on it, one can try to understand families of generic direct summands $G(x_0, y)$, where $x_0 \in W_{\mathrm{aff}}^+$ is fixed and $y \in W_{\mathrm{aff}}^+$ is arbitrary. The most basic example, where $x_0 = e$ is the neutral element in W_{aff}, leads to the family of generic direct summands

$$G(e, y) \cong L(y \cdot 0)$$

for $y \in W_{\mathrm{aff}}^+$ because $L(e \cdot 0) = L(0)$ is the trivial $\mathbf{G}$-module. Next we consider the case where $x_0 = s_0$ is the *affine simple reflection* in W_{aff}. As s_0 is the unique element of length one in W_{aff}^+, this may be considered as the smallest non-trivial instance of our problem. For $\mathbf{G}$ of type A_n, we can give a detailed description of the functor $L(s_0 \cdot \lambda) \otimes -$, for certain weights $\lambda \in C_{\mathrm{fund}} \cap X$, and use this description to study the generic direct summands $G(s_0, y)$ of $L(s_0 \cdot 0) \otimes L(y \cdot 0)$, for $y \in W_{\mathrm{aff}}^+$. We will prove the following necessary condition for the simplicity of $G(s_0, y)$:

Theorem C. *Suppose that* $\mathbf{G}$ *is of type* A_n *and let* $y \in W_{\mathrm{aff}}^+$. *If the generic direct summand* $G(s_0, y)$ *of the tensor product* $L(s_0 \cdot 0) \otimes L(y \cdot 0)$ *is simple then* $y \cdot C_{\mathrm{fund}} = C_{\mathrm{fund}} + \ell\gamma$ *for some* $\gamma \in X^+$.

Let us also mention that the converse of Theorem C is true for $n \geq 2$ in the modular case, and for all n in the quantum case.

Complete reducibility of tensor products

Our original motivation for developing the theory of generic direct summands was to find necessary conditions for the complete reducibility of tensor products of simple $\mathbf{G}$-modules (in the modular case). We have initiated our investigation of this problem in [Gru21], where the main result was a *reduction theorem* that allows us to restrict our attention to tensor products of simple $\mathbf{G}$-modules with ℓ-restricted highest weights. In Chapter V, we will demonstrate how our theory of generic direct summands can be used to make further progress on this problem. More specifically, we will prove the following *complete reducibility theorem*:

Theorem D. *Suppose that we are in the modular case and that* $\mathbf{G}$ *is of type* A_n *for some* $n \geq 1$. *Let* λ *and* μ *be* ℓ*-regular* ℓ*-restricted weights. If the tensor product* $L(\lambda) \otimes L(\mu)$ *is completely reducible then either* $\lambda \in C_{\mathrm{fund}}$ *or* $\mu \in C_{\mathrm{fund}}$.

Let us briefly explain how generic direct summands can be used to prove Theorem D. For ℓ-regular weights $\lambda, \mu \in X^+$, there exist $x, y \in W_{\mathrm{aff}}^+$ and $\lambda', \mu' \in C_{\mathrm{fund}} \cap X$ such that $\lambda = x \cdot \lambda'$ and $\mu = y \cdot \mu'$. Then we have $L(\lambda) \cong T_0^{\lambda'} L(x \cdot 0)$ and $L(\mu) \cong T_0^{\mu'} L(x \cdot 0)$, and using Theorem B, one sees that there exists a weight $\nu \in C_{\mathrm{fund}} \cap X$ such that $T_0^{\nu} G(x, y)$ is a direct summand of $L(\lambda) \otimes L(\mu)$. In particular, if $L(\lambda) \otimes L(\mu)$ is completely reducible then $G(x, y)$ is simple. Therefore, we can prove Theorem D by establishing the non-simplicity of certain generic direct summands.

In addition to the argument given above, we will use truncation to Levi subgroups and proceed by induction on the rank n of $\mathbf{G}$. The base case is given by our examples for $\mathbf{G}$ of type A_1 and A_2. The proof then essentially splits in two cases, one of which can be resolved using the necessary condition for the simplicity of $G(s_0, y)$, for $y \in W_{\mathrm{aff}}^+$, from Theorem C. The second case requires a detailed study of the composition multiplicities and the Loewy structure of certain Weyl modules, which we will compute via the *Jantzen sum formula*. The results of these computations may well be of independent interest, beyond their applications to the study of generic direct summands.

Outlook

We conclude this introduction by discussing our results in the context of other important developments in the representation theory of reductive algebraic groups and quantum groups.

One of the key results in modular representation theory in the past decade was G. Williamson's discovery of counterexamples to *Lusztig's conjecture* in characteristics that are much larger than the previously expected bound $\ell \geq h$ [Wil17]. The conjecture proposes a character formula for simple $\mathbf{G}$-modules in terms of so-called *Kazhdan-Lusztig polynomials* for the affine Weyl group and was proven in the quantum case by combining results of D. Kazhdan, G. Lusztig, M. Kashiwara and T. Tanisaki [KL93, KL94, Lus94, KT95, KT96]. While it had long been believed that the conjecture should still be true for $\ell \geq h$ in the modular case (under certain assumptions on the highest weight of the simple $\mathbf{G}$-module), it had only been possible to establish its validity in the case where ℓ is larger than some non-explicit bound depending on the root system of $\mathbf{G}$, by a reduction to the quantum case due to H.H. Andersen, J.C. Jantzen and W. Soergel [AJS94]. An explicit (but enormous) bound was later found by P. Fiebig [Fie12]. Using geometric methods and the theory of *diagrammatic Soergel bimodules* that was pioneered by M. Khovanov, B. Elias and G. Williamson in [EK10, Eli16, EW16], G. Williamson was able to exhibit a sequence of counterexamples to Lusztig's conjecture where the characteristic ℓ grows at least exponentially in the Coxeter number h of $\mathbf{G}$, thus also demonstrating that the quantum case and the modular case are less similar than one might previously have expected.

Diagrammatic Soergel bimodules play an increasingly important role in representation theory, especially when it comes to understanding tilting modules. In their landmark monograph [RW18], S. Riche and G. Williamson conjectured that the category of diagrammatic Soergel bimodules should admit a 'categorical action' on the principal block $\mathrm{Rep}_0(\mathbf{G})$ of $\mathbf{G}$ in the modular case. Furthermore, they explained how their conjecture leads to a character formula for indecomposable tilting modules in terms of *ℓ-Kazhdan-Lusztig polynomials*, which is analogous to a well-known character formula that was found by W. Soergel in the quantum case [Soe97]. The conjecture from [RW18] was proven by S. Riche and R. Bezrukavnikov in [BR20] and independently by J. Ciappara in [Cia21]. In view of these results, it would be very interesting to try to use diagrammatic Soergel bimodules in order to further study minimal tilting complexes and generic direct summands. This approach may be helpful for finding combinatorial descriptions of generic direct summands in terms of the affine Weyl group.

Beyond the theory of generic direct summands (which only works for $\ell \geq h$ and for ℓ-regular weights), it may be worthwhile to use minimal tilting complexes to study tensor products of $\mathbf{G}$-modules in a broader sense. An important open conjecture (in the modular case) that may be amenable to these techniques proposes that the tensor product of the *Steinberg module* with any simple $\mathbf{G}$-module of ℓ-restricted highest weight should be a tilting module in any characteristic $\ell > 0$. The latter conjecture has important connections with *Donkin's tilting module conjecture* and the theory of *ℓ-Weyl filtrations*. These topics present another important way of understanding the differences between the modular case and the quantum case, and they were explored in detail in work of C.P. Bendel, T. Kildetoft, D.K. Nakano, C. Pillen and P. Sobaje [KN15, Sob18, BNPS20b]. Notably, counterexamples to Donkin's conjecture in small characteristics were recently found in [BNPS20a, BNPS21]. It is possible that our techniques for studying minimal tilting complexes of tensor products can be used to gain a better understanding of tensor products of the Steinberg module with simple $\mathbf{G}$-modules of ℓ-restricted highest weight. In relation with this, it would be very interesting to further investigate the tensor ideals in $\mathrm{Tilt}(\mathbf{G})$ and the minimal tilting complexes of simple $\mathbf{G}$-modules in small characteristics.

Finally, our results about tensor ideals (in Section II.3) may be interesting from the point of view of *support theory* and *tensor triangular geometry* (as introduced by P. Balmer in [Bal05]). In the quantum case, tensor ideals in the *stable module category* were studied in detail by B.D. Boe, J.R. Kujawa and D.K. Nakano in [BKN19]. Using our techniques, it may be possible to obtain similar results for the tensor triangulated category $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$. In the modular case, a new support theory and a stable module category were recently introduced by E.M. Friedlander in [Fri21]. It would be interesting to explore connections with our construction of tensor ideals in $\mathrm{Rep}(\mathbf{G})$ using minimal tilting complexes.

Structure of the thesis

In the following, we give a brief outline of the content of the different chapters.

Chapter I. We recall some important results about representations of algebraic groups and quantum groups, and we fix the notation that will be used for the rest of the thesis.

Chapter II. We prove our main results about regular parts and generic direct summands of tensor products, including Theorems A and B (see Lemma II.4.12 and Theorem II.4.14). The existence of generic direct summands is established in Propositions II.5.1 and II.5.7.

Chapter III. We determine the regular parts and the generic direct summands of tensor products of simple $\mathbf{G}$-modules and of induced $\mathbf{G}$-modules for $\mathbf{G}$ or type A_1 and A_2.

Chapter IV. We study generic direct summands of the form $G(s_0, y)$, for $y \in W_{\mathrm{aff}}^+$. The proof of Theorem C is given in Theorem IV.6.3.

Chapter V. We use generic direct summands to give a necessary condition for the complete reducibility of tensor products of simple $\mathbf{G}$-modules, for $\mathbf{G}$ of type A_n. Sections V.3 and V.4 are devoted to a detailed study of the composition multiplicities and the Loewy structure of certain Weyl modules, and the proof of Theorem D is given in Theorem V.6.3.

Numbering conventions. Before we start with the actual content, a few remarks are in order about the numbering of results in this document. Our chapters are numbered with Roman numerals. Every chapter consists of a number of sections, which are numbered with Arabic numerals, starting at 1 whenever we begin a new chapter. The theorems, lemmas, definitions etc. are numbered consecutively within each section, so the third element in Section III.4 would get referred to as Theorem / Lemma / Definition III.4.3. However, to avoid excessive numbering, we **suppress the number of the chapter** from the notation, when referencing a result that belongs to the same chapter as the position of the reference in the text. For example, when referring to Lemma II.4.12 from *within* Chapter II, we will omit the Roman numeral and simply write Lemma 4.12.

I. Foundations

In the first chapter, we present some background material, mostly about the representation theory of simply connected simple algebraic groups and of quantum groups at roots of unity. Let us briefly discuss the contents of this chapter. In Section 1, we set up our notation for root systems and weight lattices, and in Section 2, we discuss some preliminary results about affine Weyl groups and the associated alcove geometry. We will introduce simply connected simple algebraic groups and quantum groups at roots of unity in Section 3, and we start recalling some classical results about the categories of finite-dimensional representations of these objects in Section 4. The remaining sections will be devoted to giving brief expositions to different aspects of the representation theory of algebraic groups and quantum groups. Specifically, we will discuss *good filtrations* and *tilting modules* in Section 5, the *linkage principle* and the *translation principle* in Section 6 and the notion of *good filtration dimension* in Section 7. In Section 8, we recall some results from the representation theory of *Frobenius kernels* and the *small quantum group*, and in Section 9, we introduce the tensor ideal of *negligible tilting modules*, which will play an important role in the following chapters.

1 Roots and weights

Let Φ be a simple root system in a euclidean space $X_\mathbb{R}$ with scalar product $(-\,,-)$. For $\alpha \in \Phi$, we denote by $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ the *coroot* of α. The *weight lattice* of Φ is

$$X \coloneqq \{\lambda \in X_\mathbb{R} \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\},$$

and the *Weyl group* of Φ is the (finite) subgroup $W_{\mathrm{fin}} = \langle s_\alpha \mid \alpha \in \Phi \rangle$ of $\mathrm{GL}(X_\mathbb{R})$ generated by the reflections s_α, where $s_\alpha(x) = x - (x, \alpha^\vee) \cdot \alpha$ for $x \in X_\mathbb{R}$. The index of the root lattice $\mathbb{Z}\Phi$ in the weight lattice X is finite, and the quotient $X/\mathbb{Z}\Phi$ is is called the *fundamental group* of Φ. Now fix a positive system $\Phi^+ \subseteq \Phi$ corresponding to a base Π of Φ, and let

$$X^+ \coloneqq \{\lambda \in X \mid (\lambda, \alpha^\vee) \geq 0 \text{ for all } \alpha \in \Phi^+\}$$

be the set of *dominant weights* with respect to Φ^+. We consider the partial order on X that is defined by $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a non-negative integer linear combination of positive roots. Furthermore, we write $\tilde{\alpha}_\mathrm{h}$ and α_h for the highest root and the highest short root in Φ^+, respectively, with the convention that $\tilde{\alpha}_\mathrm{h} = \alpha_\mathrm{h}$ (and that all roots are short) if Φ is simply laced. The *height* of an element $\gamma = \sum_{\alpha\in\Pi} c_\alpha \cdot \alpha$ of the root lattice is $\mathrm{ht}(\gamma) = \sum_{\alpha\in\Pi} c_\alpha$. We let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be the half sum of all positive roots and write $h = (\rho, \alpha_\mathrm{h}^\vee) + 1$ for the *Coxeter number* of Φ. The *dot action* of W_{fin} on $X_\mathbb{R}$ is defined by

$$w \cdot x = w(x + \rho) - \rho$$

for $w \in W_{\text{fin}}$ and $x \in X_{\mathbb{R}}$. The set of *simple reflections* $S_{\text{fin}} = \{s_\alpha \mid \alpha \in \Pi\}$ with respect to Π is a minimal generating set of W_{fin}, and $(W_{\text{fin}}, S_{\text{fin}})$ is a Coxeter system. As W_{fin} is finite, there exists a unique longest element $w_0 \in W_{\text{fin}}$ with respect to S_{fin}.

The root system Φ is determined up to isomorphism by its *Dynkin diagram*; a graph with vertex set Π, where the number of edges between $\alpha \in \Pi$ and $\beta \in \Pi$ is $(\alpha, \beta^\vee)(\beta, \alpha^\vee)$. If $(\alpha, \alpha) > (\beta, \beta)$ then we decorate the edge between α and β with an arrow pointing from α to β. Simple root systems come in four infinite families of classical root systems denoted by

$$\mathrm{A}_n \quad (n \geq 1), \qquad \mathrm{B}_n \quad (n \geq 3), \qquad \mathrm{C}_n \quad (n \geq 2) \qquad \text{and} \qquad \mathrm{D}_n \quad (n \geq 4)$$

and five exceptional types

$$\mathrm{E}_6, \qquad \mathrm{E}_7, \qquad \mathrm{E}_8, \qquad \mathrm{F}_4 \qquad \text{and} \qquad \mathrm{G}_2.$$

In Figures 1.1 and 1.2, we give the Dynkin diagrams of the irreducible root systems. Whenever we choose a numbering of the simple roots Π, it will be in accordance with the labeling of these Dynkin diagrams (which also coincides with the standard labeling from [Bou02]). For every simple root $\alpha \in \Pi$, there is a *fundamental dominant weight* $\varpi_\alpha \in X$ with $(\varpi_\alpha, \alpha^\vee) = 1$ and $(\varpi_\alpha, \beta^\vee) = 0$ for $\alpha \neq \beta \in \Pi$. The fundamental dominant weights form a basis of $X_{\mathbb{R}}$ and a $\mathbb{Z}$-basis of X. To be more precise, we have

$$\lambda = \sum_{\alpha \in \Pi} (\lambda, \alpha^\vee) \cdot \varpi_\alpha$$

for all $\lambda \in X_{\mathbb{R}}$. Whenever a numbering of the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is given, we number the fundamental dominant weights accordingly (that is $\varpi_i = \varpi_{\alpha_i}$ for $i = 1, \ldots, n$).

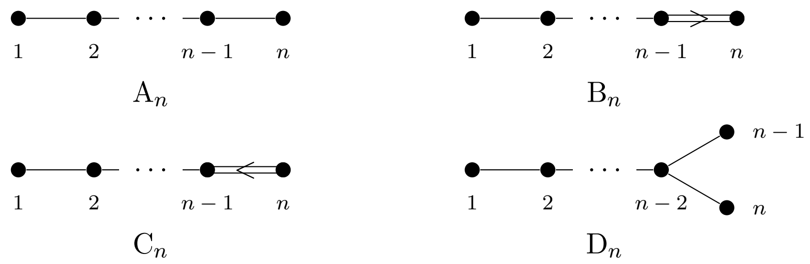

Figure 1.1: Dynkin diagrams of classical root systems

2 Alcove geometry and the affine Weyl group

The affine Weyl group (to be defined below) is a subgroup of the group $\mathrm{AGL}(X_{\mathbb{R}}) = X_{\mathbb{R}} \rtimes \mathrm{GL}(X_{\mathbb{R}})$ of invertible affine linear transformations of $X_{\mathbb{R}}$. In order to distinguish elements of $X_{\mathbb{R}}$ from elements of the translation subgroup of $\mathrm{AGL}(X_{\mathbb{R}})$, we write the canonical embedding $X_{\mathbb{R}} \to \mathrm{AGL}(X_{\mathbb{R}})$ as $x \mapsto t_x$. The *standard action* of $\mathrm{AGL}(X_{\mathbb{R}})$ on $X_{\mathbb{R}}$ is given by

$$(t_x g)(y) = g(y) + x$$

for $x, y \in X_{\mathbb{R}}$ and $g \in \mathrm{GL}(X_{\mathbb{R}})$.

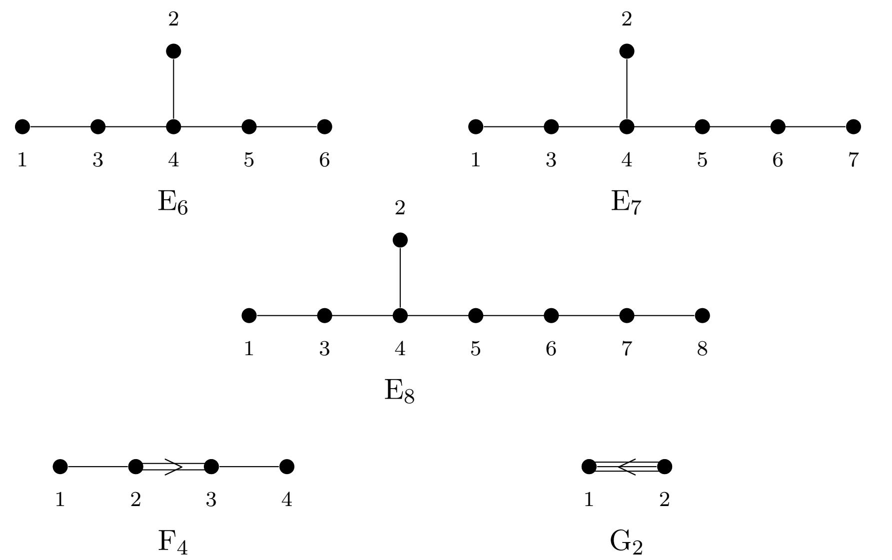

Figure 1.2: Dynkin diagrams of exceptional root systems

type	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
h	$n+1$	$2n$	$2n$	$2n-2$	12	18	30	12	6

Figure 1.3: Coxeter numbers of irreducible root systems

Definition 2.1. The *affine Weyl group* of Φ is $W_{\text{aff}} \coloneqq \mathbb{Z}\Phi \rtimes W_{\text{fin}}$, and the *extended affine Weyl group* of Φ is $W_{\text{ext}} \coloneqq X \rtimes W_{\text{fin}}$.[1]

We can restrict the standard action of $\mathrm{AGL}(X_{\mathbb{R}})$ on $X_{\mathbb{R}}$ to an action of W_{ext} that preserves the weight lattice $X \subseteq X_{\mathbb{R}}$. Analogously, the action of W_{aff} on $X_{\mathbb{R}}$ preserves the root lattice $\mathbb{Z}\Phi \subseteq X_{\mathbb{R}}$. One easily verifies that the induced action of W_{aff} on $X/\mathbb{Z}\Phi$ is trivial, that W_{aff} is a normal subgroup of W_{ext} and that $W_{\text{ext}}/W_{\text{aff}} \cong X/\mathbb{Z}\Phi$.

For $\beta \in \Phi^+$ and $m \in \mathbb{Z}$, consider the affine reflection $s_{\beta,m} \coloneqq t_{m\beta} s_\beta$ with

$$s_{\beta,m}(x) = s_\beta(x) + m\beta = x - \big((x, \beta^\vee) - m \big) \cdot \beta$$

for $x \in X_{\mathbb{R}}$. The set of fixed points of $s_{\beta,m}$ in $X_{\mathbb{R}}$ is the hyperplane

$$H_{\beta,m} \coloneqq \{ x \in X_{\mathbb{R}} \mid (x, \beta^\vee) = m \};$$

we call $H_s = H_{\beta,m}$ the *reflection hyperplane* of $s = s_{\beta,m}$.

Definition 2.2. An *alcove* is a connected component of $X_{\mathbb{R}} \setminus \big(\bigcup_{\beta,m} H_{\beta,m} \big)$.

The action of W_{ext} (and of W_{aff}) on $X_{\mathbb{R}}$ permutes the reflection hyperplanes, so it also permutes the alcoves. Every reflection hyperplane $H_{\beta,m}$ divides $X_{\mathbb{R}}$ into two half spaces

$$H_{\beta,m}^+ = \{ x \in X_{\mathbb{R}} \mid (x, \beta^\vee) > m \} \qquad \text{and} \qquad H_{\beta,m}^- = \{ x \in X_{\mathbb{R}} \mid (x, \beta^\vee) < m \}$$

[1]Some autors define the affine Weyl group as the semidirect product of W_{fin} with the coroot lattice (rather than the root lattice). The reader should note that this is the case in the book [Hum90], which we use as a reference here.

with $X_{\mathbb{R}} \setminus H_{\beta,m} = H^+_{\beta,m} \sqcup H^-_{\beta,m}$, and every alcove is contained in one of these half spaces. We say that two alcoves $A, A' \subseteq X_{\mathbb{R}}$ are *separated* by $H_{\beta,m}$ if they do not belong to the same half space with respect to $H_{\beta,m}$. Furthermore, we call A and A' *adjacent* if they are separated by a unique reflection hyperplane, which is then called a *wall* of A.

For a fixed alcove $A \subseteq X_{\mathbb{R}}$ and $\beta \in \Phi^+$, we set

$$n_\beta = n_\beta(A) \coloneqq \max\{m \in \mathbb{Z} \mid A \subseteq H^+_{\beta,m}\},$$

so that $A \nsubseteq H^+_{\beta,n_\beta+1}$ and therefore $A \subseteq H^-_{\beta,n_\beta+1}$. As $H^+_{\beta,n_\beta} \cap H^-_{\beta,n_\beta+1}$ is convex for all $\beta \in \Phi^+$, it follows that

$$A = \bigcap_{\beta\in\Phi^+} H^+_{\beta,n_\beta} \cap H^-_{\beta,n_\beta+1} = \big\{x \in X_{\mathbb{R}} \bigm| n_\beta < (x,\beta^\vee) < n_\beta + 1 \text{ for all } \beta \in \Phi^+\big\}.$$

Let $X^+_{\mathbb{R}} = \{x \in X_{\mathbb{R}} \mid (x,\alpha^\vee) > 0 \text{ for all } \alpha \in \Phi^+\}$ be the dominant Weyl chamber. As

$$X_{\mathbb{R}} \setminus \Big(\bigcup\nolimits_{\beta,m} H_{\beta,m}\Big) \subseteq X_{\mathbb{R}} \setminus \Big(\bigcup\nolimits_{\beta} H_{\beta,0}\Big) = \bigsqcup_{w\in W_{\mathrm{fin}}} w(X^+_{\mathbb{R}}),$$

every alcove is contained in a unique Weyl chamber $w(X^+_{\mathbb{R}})$ with $w \in W_{\mathrm{fin}}$. The alcoves that are contained in the dominant Weyl chamber are called *dominant alcoves*; an alcove $A \subseteq X_{\mathbb{R}}$ is dominant if and only if $n_\beta(A) \geq 0$ for all $\beta \in \Phi^+$. Note that a hyperplane $H_{\beta,m}$ separates two alcoves A and A' if and only if $n_\beta(A) < m \leq n_\beta(A')$ or $n_\beta(A') < m \leq n_\beta(A)$. In particular, we have $m > 0$ for every hyperplane $H_{\beta,m}$ separating two dominant alcoves.

Example 2.3. The set $A_{\mathrm{fund}} \coloneqq \{x \in X_{\mathbb{R}} \mid 0 < (x,\beta^\vee) < 1 \text{ for all } \beta \in \Phi^+\}$ is an alcove, called the *fundamental alcove*. We have $n_\beta(A_{\mathrm{fund}}) = 0$ for all $\beta \in \Phi^+$, and A_{fund} is the unique dominant alcove whose closure contains 0. Note that $\alpha_{\mathrm{h}}^\vee$ is the highest root in the dual root system $\Phi^\vee$, and that, for every positive root $\beta \in \Phi^+$, there exists a simple root $\alpha \in \Pi$ with $\alpha^\vee \leq \beta^\vee$. Then, for all $x \in X^+_{\mathbb{R}}$, we have $(x,\alpha^\vee) \leq (x,\beta^\vee) \leq (x,\alpha_{\mathrm{h}}^\vee)$, and it follows that

$$A_{\mathrm{fund}} = \{x \in X_{\mathbb{R}} \mid 0 < (x,\alpha^\vee) \text{ for all } \alpha \in \Pi \text{ and } (x,\alpha_{\mathrm{h}}^\vee) < 1\}.$$

Definition 2.4. We call $S \coloneqq \{s_\alpha \mid \alpha \in \Pi\} \cup \{s_{\alpha_{\mathrm{h}},1}\}$ the set of *simple reflections* in W_{aff}.

In view of Example 2.3, S is the set of reflections in the hyperplanes bounding A_{fund}. We now recall some important results from Chapter 4 in [Hum90]:

Theorem 2.5. *(1) The action of W_{aff} on $X_{\mathbb{R}}$ permutes the set of alcoves simply transitively.*

(2) For every alcove $A \subseteq X_{\mathbb{R}}$, the closure $\overline{A}$ is a fundamental domain for the action of W_{aff} on $X_{\mathbb{R}}$.

(3) (W_{aff}, S) is a Coxeter system.

Example 2.6. Let Φ be of type A_n and fix a numbering $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of the simple roots, in accordance with the Dynkin diagram in Figure 1.1. Denote by $\varpi_i = \varpi_{\alpha_i}$ and $s_i = s_{\alpha_i}$ the fundamental dominant weight and the simple reflection corresponding to α_i, for $i = 1, \ldots, n$. For $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have

$$s_i(\alpha_j) = \begin{cases} \alpha_i + \alpha_j & \text{if } |i-j| = 1, \\ \alpha_j & \text{otherwise }, \end{cases}$$

and it follows that $s_i s_j s_i = s_{s_i(\alpha_j)} = s_{s_j(\alpha_i)} = s_j s_i s_j$ when $|i-j|=1$, and $s_i s_j = s_j s_i$ otherwise. Now let us write $s_0 \coloneqq s_{\alpha_{\mathrm{h}},1}$ and note that $\alpha_{\mathrm{h}} = \alpha_1 + \ldots + \alpha_n = \varpi_1 + \varpi_n$. Using this observation, it is straightforward to see that $s_0 s_i = s_i s_0$ for $1 < i < n$ and that $s_0 s_1 s_0 = s_1 s_0 s_1$ and $s_0 s_n s_0 = s_n s_0 s_n$. We conclude that the Coxeter system (W_{aff}, S) has the Coxeter diagram given in Figure 2.1.[2]

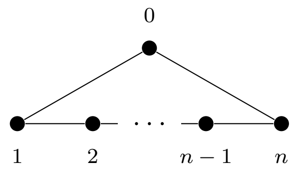

Figure 2.1: The Coxeter diagram of (W_{aff}, S) for Φ of type A_n

Let us write $\ell\colon W_{\mathrm{aff}} \to \mathbb{Z}_{\geq 0}$ for the *length function* with respect to S. (Recall that $\ell(w)$ is the minimal length of an expression $w = s_1 \cdots s_m$ with $s_1, \ldots, s_m \in S$. Such an expression of minimal length is also called a *reduced expression*.) As for any Coxeter group, the map $w \mapsto (-1)^{\ell(w)}$ is a group homomorphism, called the *sign representation* and denoted by $\mathrm{sign}\colon W_{\mathrm{aff}} \to \{1,-1\}$. In particular, we have $\ell(xs) \in \{\ell(x)-1, \ell(x)+1\}$ for all $x \in W_{\mathrm{aff}}$ and $s \in S$. For $w \in W_{\mathrm{ext}}$, we denote by $\mathcal{L}(w)$ the set of reflection hyperplanes separating A_{fund} and $w(A_{\mathrm{fund}})$. By Section 4.5 in [Hum90], the length function is also given by $\ell(w) = |\mathcal{L}(w)|$ for $w \in W_{\mathrm{aff}}$, so we can extend it to a function $\ell\colon W_{\mathrm{ext}} \to \mathbb{Z}_{\geq 0}$ by setting

$$\ell(w) \coloneqq |\mathcal{L}(w)|$$

for $w \in W_{\mathrm{ext}}$. An element $w \in W_{\mathrm{ext}}$ satisfies $\ell(w) = 0$ if and only if $w(A_{\mathrm{fund}}) = A_{\mathrm{fund}}$, so we have

$$\Omega \coloneqq \mathrm{Stab}_{W_{\mathrm{ext}}}(A_{\mathrm{fund}}) = \{w \in W_{\mathrm{ext}} \mid \ell(w) = 0\}.$$

As W_{aff} acts simply transitively on the set of alcoves, there is a natural decomposition $W_{\mathrm{ext}} = W_{\mathrm{aff}} \rtimes \Omega$, and it follows that $\Omega \cong W_{\mathrm{ext}}/W_{\mathrm{aff}} \cong X/\mathbb{Z}\Phi$ is finite and abelian. We write $x \mapsto \omega_x$ for the canonical epimorphism $W_{\mathrm{ext}} \to \Omega$ with kernel W_{aff}.

Remark 2.7. Note that for $w \in W_{\mathrm{aff}}$, the alcove $w(A_{\mathrm{fund}})$ is adjacent to A_{fund} if and only if

$$1 = |\mathcal{L}(w)| = \ell(w),$$

or equivalently, if $w \in S$ is a simple reflection. As W_{aff} acts transitively on the set of alcoves, this implies that an alcove $A \subseteq X_{\mathbb{R}}$ is adjacent to A_{fund} if and only if $A = s(A_{\mathrm{fund}})$ for some $s \in S$, and that the walls of A_{fund} are precisely the hyperplanes $\{H_{\alpha,0} \mid \alpha \in \Pi\} \cup \{H_{\alpha_{\mathrm{h}},1}\}$ corresponding to the simple reflections. The action of Ω on $X_{\mathbb{R}}$ permutes the walls of A_{fund} because Ω stabilizes A_{fund}; therefore the action of Ω on W_{aff} by conjugation permutes the set of simple reflections.

Next we study the set of elements of W_{ext} (or W_{aff}) that send A_{fund} to a dominant alcove.

Definition 2.8. We write $W_{\mathrm{ext}}^+ \coloneqq \{w \in W_{\mathrm{ext}} \mid w(A_{\mathrm{fund}}) \text{ is dominant}\}$ and $W_{\mathrm{aff}}^+ \coloneqq W_{\mathrm{ext}}^+ \cap W_{\mathrm{aff}}$.

Remark 2.9. As Ω stabilizes A_{fund}, we have $W_{\mathrm{ext}}^+ = W_{\mathrm{aff}}^+ \Omega$.

Lemma 2.10. *Let $w \in W_{\mathrm{fin}}$ and $x \in W_{\mathrm{ext}}^+$. Then $\mathcal{L}(wx) = \mathcal{L}(w) \sqcup w\big(\mathcal{L}(x)\big)$ and $\ell(wx) = \ell(w) + \ell(x)$.*

[2] The *Coxeter diagram* of a Coxeter system (W,S) has vertices labeled by S. Two simple reflections $s,t \in S$ are joined by an edge if s and t do not commute, and the edge is labeled by the order of st if that order is greater than 3.

Proof. As $\overline{A}_{\text{fund}}$ and $w(\overline{A}_{\text{fund}})$ both contain 0, so does every hyperplane separating A_{fund} and $w(A_{\text{fund}})$, and we conclude that $\mathcal{L}(w) \subseteq \{H_{\beta,0} \mid \beta \in \Phi^+\}$. Furthermore, we have $\mathcal{L}(x) \subseteq \{H_{\beta,m} \mid \beta \in \Phi^+, m > 0\}$ because A_{fund} and $x(A_{\text{fund}})$ are dominant, hence $w\big(\mathcal{L}(x)\big) \subseteq \{H_{\beta,m} \mid \beta \in \Phi^+, m \neq 0\}$ and $\mathcal{L}(w)$ is disjoint from $w\big(\mathcal{L}(x)\big)$. For any hyperplane $H \in \mathcal{L}(wx)$, the half space with respect to H that contains the alcove $w(A_{\text{fund}})$ cannot contain both A_{fund} and $wx(A_{\text{fund}})$, so either H separates A_{fund} and $w(A_{\text{fund}})$ or H separates $w(A_{\text{fund}})$ and $wx(A_{\text{fund}})$. As $w\big(\mathcal{L}(x)\big)$ is the set of hyperplanes separating the alcoves $w(A_{\text{fund}})$ and $wx(A_{\text{fund}})$, we conclude that $\mathcal{L}(wx) \subseteq \mathcal{L}(w) \sqcup w\big(\mathcal{L}(x)\big)$. Now let $H \in \mathcal{L}(w)$. By the disjointness of $\mathcal{L}(w)$ and $w\big(\mathcal{L}(x)\big)$, the alcoves $w(A_{\text{fund}})$ and $wx(A_{\text{fund}})$ belong to the same half space with respect to H, and it follows that $H \in \mathcal{L}(wx)$. Similarly, if $H \in \mathcal{L}(x)$ then the alcoves A_{fund} and $w(A_{\text{fund}})$ belong to the same half space with respect to $w(H)$, and it follows that $w(H) \in \mathcal{L}(wx)$. We conclude that $\mathcal{L}(wx) = \mathcal{L}(w) \sqcup w\big(\mathcal{L}(x)\big)$, and the second claim is immediate. □

Corollary 2.11. *For all $w \in W_{\text{ext}}$, the coset $W_{\text{fin}}w$ has a unique element of minimal length. Furthermore, we have*

$$W_{\text{ext}}^+ = \{w \in W_{\text{ext}} \mid w \textit{ has minimal length in } W_{\text{fin}}w\}.$$

Proof. As W_{fin} acts simply transitively on the set of Weyl chambers, every coset $W_{\text{fin}}w$ with $w \in W_{\text{ext}}$ contains a unique element of W_{ext}^+. Now the claim is immediate from Lemma 2.10. □

The next two results will be important for proofs by induction on the length of an element $w \in W_{\text{aff}}^+$ in the following sections.

Lemma 2.12. *Let $x \in W_{\text{aff}}^+$ and let $x = s_1 \cdots s_m$ be a reduced expression. Then $s_1 \cdots s_i \in W_{\text{aff}}^+$ for $1 \leq i \leq m$ and $s_1 = s_{\alpha_{\text{h}},1}$.*

Proof. We prove the first claim for $i = m-1$, the general case then follows by descending induction. Let us write $x' = s_1 \cdots s_{m-1}$, so that $x = x's_m$ and $\ell(x') = m-1$. For $w \in W_{\text{fin}}$ with $\ell(wx') \leq \ell(x')$, we have

$$\ell(wx) = \ell(wx's_m) \leq \ell(wx') + 1 \leq \ell(x') + 1 = m = \ell(x),$$

and Corollary 2.11 forces $w = e$. Hence x' is the unique element of minimal length in the coset $W_{\text{fin}}x'$, and therefore $x' \in W_{\text{aff}}^+$, again by Corollary 2.11. The second claim follows from the first because $s_{\alpha_{\text{h}},1}$ is the only simple reflection that does not belong to W_{fin}. □

Corollary 2.13. *For $x \in W_{\text{aff}}^+$ and $s \in S$ such that $\ell(xs) < \ell(x)$, we have $xs \in W_{\text{aff}}^+$.*

Proof. Recall that $\ell(xs) \in \{\ell(x)-1, \ell(x)+1\}$ for all $x \in W_{\text{aff}}$ and $s \in S$, so the assumption implies that $\ell(xs) = \ell(x) - 1$. Hence, if $xs = s_1 \cdots s_m$ is a reduced expression then so is $x = s_1 \cdots s_m s$, and the claim follows from Lemma 2.12. □

For applications in representation theory, two partial orders, one on W_{aff} and the other on the set of alcoves, play an important role. The first one is the usual *Bruhat order* $\leq$ on W_{aff}, that can be defined as the reflexive and transitive closure of the relation that is given by $x \leq y$ if $\ell(x) < \ell(y)$ and $y = sx$, for a reflection $s \in W_{\text{aff}}$. Note that we could equally well ask that $y = xs$, because the conjugate of a reflection is a reflection. The *linkage order* $\uparrow$ on the set of alcoves is the reflexive and transitive closure of the relation that is given by $A \uparrow A'$ if there exists a reflection $s \in W_{\text{aff}}$ such that $A \subseteq H_s^-$, $A' \subseteq H_s^+$ and $A' = s(A)$. These partial orders are equivalent on the set of dominant alcoves, in the following sense:

Theorem 2.14. *For $x, y \in W_{\mathrm{aff}}^{+}$, we have $x \leq y$ if and only if $x(A_{\mathrm{fund}}) \uparrow y(A_{\mathrm{fund}})$.*

The proof of the theorem, based on results of J. Wang [Wan87], is postponed to Section IV.1, where we will study the alcove geometry associated with W_{aff} in more detail.

3 Algebraic groups and quantum groups

The root system Φ is at the heart of the structure of two kinds of Lie theoretic objects whose finite-dimensional simple modules are canonically indexed by X^{+}: *simple algebraic groups* (over a field of positive characteristic) and *quantum groups* (at a root of unity). The representation theory of quantum groups parallels that of algebraic groups to a large extent, so we will often treat the two cases simultaneously. When a distinction becomes necessary, we refer to the representation theory of the algebraic group as the *modular case* and to the representation theory of the quantum group as the *quantum case*.

The modular case

We follow the notational conventions from Section II.1 in [Jan03]. Let $\mathbf{G}_{\mathbb{Z}}$ be a split simply-connected simple algebraic group scheme over $\mathbb{Z}$ with split maximal torus $\mathbf{T}_{\mathbb{Z}}$, such that the root system of $\mathbf{G}_{\mathbb{Z}}$ with respect to $\mathbf{T}_{\mathbb{Z}}$ is isomorphic to Φ. For every root $\alpha \in \Phi$, there is a *root subgroup* $\mathbf{U}_{\alpha,\mathbb{Z}}$ of $\mathbf{G}_{\mathbb{Z}}$ and a *root homomorphism* $x_\alpha \colon \mathbb{Z} \to \mathbf{U}_{\alpha,\mathbb{Z}}$ (where by abuse of notation, we write $\mathbb{Z}$ for the additive group scheme over $\mathbb{Z}$), and the latter is unique up to a sign change. The positive system $\Phi^{+} \subseteq \Phi$ determines a unipotent subgroup $\mathbf{U}_{\mathbb{Z}}^{+} = \prod_{\alpha \in \Phi^{+}} \mathbf{U}_{\alpha,\mathbb{Z}}$ and a Borel subgroup $\mathbf{B}_{\mathbb{Z}}^{+} = \mathbf{U}_{\mathbb{Z}}^{+} \rtimes \mathbf{T}_{\mathbb{Z}}$. Analogously, the negative roots $-\Phi^{+}$ determine a Borel subgroup $\mathbf{B}_{\mathbb{Z}} = \mathbf{U}_{\mathbb{Z}} \rtimes \mathbf{T}_{\mathbb{Z}}$. We fix an algebraically closed field $\Bbbk$ of characteristic $\ell > 0$ and denote by $\mathbf{G} = \mathbf{G}_{\Bbbk}$ the simply-connected simple algebraic group scheme over $\Bbbk$ corresponding to $\mathbf{G}_{\mathbb{Z}}$, with maximal torus $\mathbf{T} = \mathbf{T}_{\Bbbk}$, Borel subgroup $\mathbf{B} = \mathbf{B}_{\Bbbk}$ with unipotent radical $\mathbf{U} = \mathbf{U}_{\Bbbk}$, and root subgroups $\mathbf{U}_\beta$ for $\beta \in \Phi$.

The quantum case

The term *quantum group* is broadly used to refer to a class of Hopf algebras that are obtained by deforming the universal enveloping algebra of a Lie algebra over the field of rational functions $\mathbb{Q}(q)$. The quantum groups that we will be interested in admit an integral version over the ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials over the integers (due to G. Lusztig). For any ring R and unit $\zeta \in R^{\times}$, there is a unique ring homomorphism $\mathbb{Z}[q, q^{-1}] \to R$ with $q \mapsto \zeta$, and we can extend scalars along this homomorphism to obtain a *specialization* of the quantum group at the parameter ζ. The most interesting cases arise when either R has positive characteristic or ζ is a root of unity. In the first case (and when $\zeta = 1$), one essentially recovers the *distribution algebra* of a simply-connected simple algebraic group. When $\zeta \in \mathbb{C}^{\times}$ is a primitive ℓ-th root of unity (for some $\ell > 0$) and $R = \mathbb{Q}(\zeta)$ then one obtains a Hopf algebra whose representation theory is very similar to that of the corresponding algebraic group 'in characteristic ℓ'. In the discussion below, we closely follow Chapter II.H in [Jan03].

Suppose that the short roots $\alpha \in \Phi$ satisfy $(\alpha, \alpha) = 2$, so that $d_\beta \coloneqq \frac{(\beta,\beta)}{2} \in \{1, 2, 3\}$ and

$$(\lambda, \beta) = (\lambda, \beta^\vee) \cdot d_\beta \in \mathbb{Z}$$

for all $\lambda \in X$ and $\beta \in \Phi$. Furthermore, let $c_{\beta,\alpha} = (\beta, \alpha^\vee)$, and write $q_\alpha = q^{d_\alpha} \in \mathbb{Q}(q)$, for $\alpha, \beta \in \Pi$.

The *quantum integer* associated with $m \in \mathbb{Z}_{>0}$ and q_α is

$$[m]_\alpha = \frac{q_\alpha^m - q_\alpha^{-m}}{q_\alpha - q_\alpha^{-1}}$$

and we can define quantum factorials by $[m]_\alpha! = [m]_\alpha \cdot [m-1]_\alpha!$ and $[0]_\alpha = 1$.

Now let $\mathfrak{g}$ be the complex simple Lie algebra with root system Φ. The quantum group $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra with generators E_α, F_α, K_α and K_α^{-1}, for $\alpha \in \Pi$, subject to the relations

$$\begin{gathered}
K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha, \qquad\qquad K_\alpha K_\beta = K_\beta K_\alpha, \\
K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} \cdot E_\beta, \qquad\qquad K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} \cdot F_\beta, \\
E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \cdot \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}, \\
\sum_{i+j=1-c_{\beta,\alpha}} (-1)^i \cdot E_\alpha^{(i)} E_\beta E_\alpha^{(j)} = 0, \\
\sum_{i+j=1-c_{\beta,\alpha}} (-1)^i \cdot F_\alpha^{(i)} F_\beta F_\alpha^{(j)} = 0
\end{gathered}$$

for $\alpha, \beta \in \Pi$, where $\delta_{\alpha\beta}$ denotes the Kronecker delta and

$$E_\alpha^{(i)} = \frac{E_\alpha^i}{[i]_\alpha!} \qquad \text{and} \qquad F_\alpha^{(i)} = \frac{F_\alpha^i}{[i]_\alpha!}$$

are the *quantum divided powers*. There is a Hopf algebra structure on $U_q(\mathfrak{g})$ with comultiplication Δ, antipode σ and counit ϵ defined on the generators by

$$\begin{aligned}
\Delta(E_\alpha) &= E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, & \sigma(E_\alpha) &= -K_\alpha^{-1} E_\alpha, & \epsilon(E_\alpha) &= 0, \\
\Delta(F_\alpha) &= F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, & \sigma(F_\alpha) &= -F_\alpha K_\alpha, & \epsilon(F_\alpha) &= 0, \\
\Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, & \sigma(K_\alpha) &= K_\alpha^{-1}, & \epsilon(K_\alpha) &= 1.
\end{aligned} \tag{3.1}$$

The *Lusztig integral form* of $U_q(\mathfrak{g})$ is the $\mathbb{Z}[q,q^{-1}]$-subalgebra $U_q^{\mathbb{Z}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ that is generated by the elements $K_\alpha^{\pm 1}$ along with the quantum divided powers defined above for $\alpha \in \Pi$ and $i > 0$. By taking commutators of suitable divided powers, one sees that $U_q^{\mathbb{Z}}(\mathfrak{g})$ contains the elements

$$\binom{K_\alpha; m}{k} = \prod_{i=1}^{k} \frac{K_\alpha q_\alpha^{m-i+1} - K_\alpha^{-1} q_\alpha^{-m+i-1}}{q_\alpha^i - q_\alpha^{-i}}$$

for all $\alpha \in \Pi$, $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. We write $U_q^0(\mathfrak{g})$ for the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements $K_\alpha^{\pm 1}$ and $U_q^{\mathbb{Z},0}(\mathfrak{g})$ for the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_q^{\mathbb{Z}}(\mathfrak{g})$ generated by the elements $K_\alpha^{\pm 1}$ and $\binom{K_\alpha; m}{k}$, for $\alpha \in \Pi$, $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.

Let $\ell > 1$ be an odd integer, and further assume that ℓ is not divisible by 3 if Φ is of type G_2. We fix a primitive ℓ-th root of unity $\zeta \in \mathbb{C}$ and set $\Bbbk = \mathbb{C}$. (One reason for the assumptions on ℓ is to ensure that ζ^{d_α} is a primitive ℓ-th root of unity for all $\alpha \in \Pi$.) Then there is a unique ring homomorphism $\mathbb{Z}[q,q^{-1}] \to \Bbbk$ with $q \mapsto \zeta$, and we define

$$U_\zeta'(\mathfrak{g}) \coloneqq U_q^{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \Bbbk$$

to be the specialization of $U_q^{\mathbb{Z}}(\mathfrak{g})$ along this homomorphism. In $U'_\zeta(\mathfrak{g})$, the images of the ℓ-th powers of the generators K_α are central. We will only be interested in 'type 1' representations where these central elements act by the scalar 1, so we define

$$U_\zeta(\mathfrak{g}) \coloneqq U'_\zeta(\mathfrak{g}) / \big\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 \bigm| \alpha \in \Pi \big\rangle.$$

The quantum group $U_\zeta(\mathfrak{g})$ inherits a Hopf algebra structure from $U_q(\mathfrak{g})$, and by abuse of notation, we denote the images of the generators of $U_q^{\mathbb{Z}}(\mathfrak{g})$ in $U'_\zeta(\mathfrak{g})$ or $U_\zeta(\mathfrak{g})$ by the same symbols. We define two subalgebras $U_\zeta^0(\mathfrak{g})$ and $U_\zeta^-(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$ by

$$U_\zeta^0(\mathfrak{g}) = \Big\langle K_\alpha, \tbinom{K_\alpha;m}{k} \Bigm| \alpha \in \Pi, m \in \mathbb{Z}, k > 0 \Big\rangle \qquad \text{and} \qquad U_\zeta^-(\mathfrak{g}) = \big\langle U_\zeta^0(\mathfrak{g}), F_\alpha^{(i)} \bigm| \alpha \in \Pi, i > 0 \big\rangle.$$

These subalgebras will later play roles similar to those that the maximal torus $\mathbf{T}$ and the Borel subgroup $\mathbf{B}$ play in the modular case.

4 Representation categories

Throughout this thesis, we will only consider finite-dimensional representations, so whenever we talk about modules over a group scheme or an algebra, they are implicitly assumed to be finite-dimensional.

In the modular case, we write $\mathrm{Rep}(\mathbf{H})$ for the category of (finite-dimensional) modules over a $\Bbbk$-group scheme $\mathbf{H}$, in the sense of Section I.2.7 in [Jan03]. By definition of $\mathbf{G}$ and $\mathbf{T}$, there is an isomorphism between the weight lattice X and the character group $X(\mathbf{T})$ of $\mathbf{T}$ (i.e. the group of $\Bbbk$-group scheme homomorphisms from $\mathbf{T}$ to the multiplicative group scheme). As $\mathbf{T}$ is a diagonalizable group scheme, every $\mathbf{T}$-module M admits a weight space decomposition

$$M = \bigoplus_{\lambda \in X} M_\lambda.$$

As $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$, every weight $\lambda \in X \cong X(\mathbf{T})$ gives rise to a one-dimensional $\mathbf{B}$-module $\Bbbk_\lambda$, where $\mathbf{T}$ acts via λ and $\mathbf{U}$ acts trivially.

In many cases, it will be useful to replace $\mathbf{G}$-modules by modules over the *distribution algebra* of $\mathbf{G}$, which we will define next. By Section I.2.3 in [Jan03], the group scheme structure of $\mathbf{G}$ gives rise to a Hopf algebra structure on the coordinate algebra $\Bbbk[\mathbf{G}]$ of $\mathbf{G}$, with counit $\varepsilon_{\mathbf{G}} \colon \Bbbk[\mathbf{G}] \to \Bbbk$, comultiplication $\Delta_{\mathbf{G}} \colon \Bbbk[\mathbf{G}] \to \Bbbk[\mathbf{G}] \otimes \Bbbk[\mathbf{G}]$ and antipode $\sigma_{\mathbf{G}} \colon \Bbbk[\mathbf{G}] \to \Bbbk[\mathbf{G}]$ coming from the neutral element, the multiplication morphism and the inversion morphism of $\mathbf{G}$, respectively. The *augmentation ideal* of $\Bbbk[\mathbf{G}]$ is $I_{\mathbf{G}} \coloneqq \ker(\varepsilon_{\mathbf{G}})$, and we define the *distribution algebra* of $\mathbf{G}$ as

$$\mathrm{Dist}(\mathbf{G}) \coloneqq \big\{ \vartheta \in \Bbbk[\mathbf{G}]^* \bigm| \vartheta(I_{\mathbf{G}}^n) = 0 \text{ for some } n > 0 \big\},$$

with multiplication given by $\vartheta \cdot \vartheta' \coloneqq (\vartheta \otimes \vartheta') \circ \Delta_{\mathbf{G}}$, for $\vartheta, \vartheta' \in \mathrm{Dist}(\mathbf{G})$, and neutral element $\varepsilon_{\mathbf{G}}$; see Section I.7.7 in [Jan03]. One can show that the image of $\mathrm{Dist}(\mathbf{G}) \subseteq \Bbbk[\mathbf{G}]^*$ under the dual of the multiplication map $\Bbbk[\mathbf{G}] \otimes \Bbbk[\mathbf{G}] \to \Bbbk[\mathbf{G}]$ is contained in the naturally embedded subspace

$$\mathrm{Dist}(\mathbf{G}) \otimes \mathrm{Dist}(\mathbf{G}) \subseteq \Bbbk[\mathbf{G}]^* \otimes \Bbbk[\mathbf{G}]^* \subseteq \big(\Bbbk[\mathbf{G}] \otimes \Bbbk[\mathbf{G}] \big)^*;$$

thus the Hopf algebra structure on $\Bbbk[\mathbf{G}]$ gives rise to a Hopf algebra structure on $\mathrm{Dist}(\mathbf{G})$, with counit, comultiplication and antipode denoted by ϵ, Δ and σ, respectively. Every $\mathbf{G}$-module M is naturally a $\Bbbk[\mathbf{G}]$-comodule, and the comodule map $\Delta_M \colon M \to M \otimes \Bbbk[\mathbf{G}]$ defines a $\mathrm{Dist}(\mathbf{G})$-module structure on

M via $\vartheta \otimes m \mapsto (\mathrm{id}_M \otimes \vartheta) \circ \Delta_M(m)$ for $\vartheta \in \mathrm{Dist}(\mathbf{G})$ and $m \in M$; see Sections I.2.8 and I.7.11 in [Jan03]. Under our assumptions on $\mathbf{G}$, the resulting functor from $\mathrm{Rep}(\mathbf{G})$ to the category of finite-dimensional $\mathrm{Dist}(\mathbf{G})$-modules is an equivalence of categories by Section II.1.20 in [Jan03].

Let us give a more explicit description of $\mathrm{Dist}(\mathbf{G})$, following Sections II.1.11 and II.1.12 in [Jan03]. The root homomorphisms $x_\beta \colon \mathbb{Z} \to \mathbf{U}_{\beta,\mathbb{Z}}$, for $\beta \in \Phi$, give rise to a Chevalley basis

$$\{X_\beta, H_\alpha \mid \beta \in \Phi, \alpha \in \Pi\}$$

of the complex simple Lie algebra $\mathfrak{g}$ with root system Φ, and the divided powers

$$X_{\beta,r} \coloneqq \frac{X_\beta^r}{r!} \qquad \text{and} \qquad H_{\alpha,m} \coloneqq \frac{H_\alpha \cdot (H_\alpha - 1) \cdots (H_\alpha - m)}{m!}$$

generate a $\mathbb{Z}$-subalgebra $U_\mathbb{Z}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ which admits a PBW-type basis, consisting of products of the form

$$\prod_{\beta \in -\Phi^+} X_{\beta,r_\beta} \cdot \prod_{\alpha \in \Pi} H_{\alpha,m_\alpha} \cdot \prod_{\beta \in \Phi^+} X_{\beta,r_\beta}$$

with $r_\beta, m_\alpha \in \mathbb{Z}_{\geq 0}$ for $\beta \in \Phi$ and $\alpha \in \Pi$, for any fixed ordering of the roots in the product. There is a canonical Hopf algebra structure on $U(\mathfrak{g})$, with comultiplication, counit and antipode given by

$$x \mapsto x \otimes 1 + 1 \otimes x, \qquad x \mapsto 0 \qquad \text{and} \qquad x \mapsto -x$$

for all $x \in \mathfrak{g}$, and one can show that $U_\mathbb{Z}(\mathfrak{g})$ is a $\mathbb{Z}$-Hopf subalgebra. Furthermore, there are isomorphisms of Hopf algebras

$$\mathrm{Dist}(\mathbf{G}_\mathbb{Z}) \cong U_\mathbb{Z}(\mathfrak{g}) \qquad \text{and} \qquad \mathrm{Dist}(\mathbf{G}) \cong U_\mathbb{Z}(\mathfrak{g}) \otimes \Bbbk \eqqcolon U_\Bbbk(\mathfrak{g}),$$

over $\mathbb{Z}$ and $\Bbbk$, respectively. We write $X_{\beta,r}$ instead of $X_{\beta,r} \otimes 1$ and $H_{\alpha,m}$ instead of $H_{\alpha,m} \otimes 1$ for the images of the divided powers in $U_\Bbbk(\mathfrak{g})$.

In the quantum case, the role of the maximal torus $\mathbf{T}$ is played by the subalgebra $U_\zeta^0(\mathfrak{g})$, and the role of the Borel subgroup $\mathbf{B}$ is played by the subalgebra $U_\zeta^-(\mathfrak{g})$. For every weight $\lambda \in X$, there is a $\mathbb{Q}(q)$-algebra homomorphism $\varepsilon_{\lambda,q} \colon U_q^0(\mathfrak{g}) \longrightarrow \mathbb{Q}(q)$ with $K_\alpha \mapsto q^{(\lambda,\alpha)}$ for all $\alpha \in \Pi$, which restricts to a $\mathbb{Z}[q,q^{-1}]$-algebra homomorphism $\varepsilon_\lambda^\mathbb{Z} \colon U_q^{\mathbb{Z},0}(\mathfrak{g}) \to \mathbb{Z}[q,q^{-1}]$. By specialization of q at ζ, we obtain a $\Bbbk$-algebra homomorphism $\varepsilon_\lambda \colon U_\zeta^0(\mathfrak{g}) \to \Bbbk$. We say that a $U_\zeta^0(\mathfrak{g})$-module M has a *weight space decomposition* if

$$M = \bigoplus_{\lambda \in X} M_\lambda,$$

where the *weight spaces* are defined by

$$M_\lambda \coloneqq \big\{ m \in M \bigm| u \cdot m = \varepsilon_\lambda(u) \cdot m \text{ for all } u \in U_\zeta^0(\mathfrak{g}) \big\}.$$

By Theorem 9.12 in [AKP91], every finite-dimensional $U_\zeta(\mathfrak{g})$-module has a weight space decomposition. We write $\mathrm{Rep}\big(U_\zeta(\mathfrak{g})\big)$ for the category of finite-dimensional $U_\zeta(\mathfrak{g})$-modules and $\mathrm{Rep}\big(U_\zeta^-(\mathfrak{g})\big)$ for the category of finite-dimensional $U_\zeta^-(\mathfrak{g})$-modules that have a weight space decomposition. As in the modular case, a weight $\lambda \in X$ gives rise to a one-dimensional $U_\zeta^-(\mathfrak{g})$-module $\Bbbk_\lambda$, where $U_\zeta^0(\mathfrak{g})$ acts via the homomorphism ε_λ and all divided powers act by zero. Keeping these analogies in mind, we introduce a notation that will allow as to treat the modular case and the quantum case simultaneously.

Notation. In the quantum case, we write $\mathbf{G} = U_\zeta(\mathfrak{g})$, $\mathbf{B} = U_\zeta^-(\mathfrak{g})$ and $\mathbf{T} = U_\zeta^0(\mathfrak{g})$.

From now on, we will not distinguish between the modular case and the quantum case any more, except when there are significant differences. In both cases, the objects of the category $\mathrm{Rep}(\mathbf{G})$ will be called $\mathbf{G}$-modules, and we write $\mathrm{Hom}_{\mathbf{G}}(M,N)$ and $\mathrm{Ext}^i_{\mathbf{G}}(M,N)$ for the space of homomorphisms and the Ext-groups between $\mathbf{G}$-modules M and N, respectively. A $\mathbf{G}$-module is called *completely reducible* if it is isomorphic to a direct sum of simple $\mathbf{G}$-modules, and we define the *socle* $\mathrm{soc}_{\mathbf{G}} M$ to be the largest completely reducible $\mathbf{G}$-submodule of M. Similarly, we define the *radical* $\mathrm{rad}_{\mathbf{G}} M$ to be the smallest $\mathbf{G}$-submodule of M such that $M/\mathrm{rad}_{\mathbf{G}} M$ is completely reducible, and we call $\mathrm{head}_{\mathbf{G}} M \coloneqq M/\mathrm{rad}_{\mathbf{G}} M$ the *head* of M. We say that a $\mathbf{G}$-module is *uniserial* if its submodules are totally ordered by inclusion, or equivalently, if it has a unique composition series. Recall that every $\mathbf{G}$-module M admits a weight space decomposition

$$M = \bigoplus_{\lambda \in X} M_\lambda;$$

we call $\lambda \in X$ a *weight of* M if $M_\lambda \neq 0$. The *character* of M is defined as the element

$$\mathrm{ch}\, M = \sum_{\lambda \in X} \dim(M_\lambda) \cdot e^\lambda$$

of the group ring $\mathbb{Z}[X]$ (with a basis of formal exponentials $\{e^\lambda \mid \lambda \in X\}$, where $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ for $\lambda, \mu \in X$). The standard action of W_{fin} on X induces an action of W_{fin} on $\mathbb{Z}[X]$ by ring automorphisms, and it turns out that the characters of all $\mathbf{G}$-modules belong to the ring $\mathbb{Z}[X]^{W_{\mathrm{fin}}}$ of W_{fin}-fixed points in $\mathbb{Z}[X]$. For $\mathbf{G}$-modules M and N, the tensor product $M \otimes N$ (over $\Bbbk$) has a canonical $\mathbf{G}$-module structure, defined in the usual way in the modular case and via the comultiplication of $U_\zeta(\mathfrak{g})$ in the quantum case. This endows $\mathrm{Rep}(\mathbf{G})$ with the structure of a *braided monoidal category*, i.e. a category with a tensor product bifunctor and a natural *braiding* isomorphism $M \otimes N \cong N \otimes M$ that commutes with the associativity isomorphisms between triple tensor products in a suitable sense. In the modular case, the braiding is the standard one and $\mathrm{Rep}(\mathbf{G})$ is *symmetric* (i.e. the square of the braiding is the identity); in the quantum case, it is constructed in Chapter 32 in [Lus10]. The dual space $M^* = \mathrm{Hom}_\Bbbk(M,\Bbbk)$ of a $\mathbf{G}$-module M also carries a natural $\mathbf{G}$-module structure, defined in the usual way in the modular case and via the antipode of $U_\zeta(\mathfrak{g})$ in the quantum case. Taking duals is a contravariant autoequivalence of $\mathrm{Rep}(\mathbf{G})$, and we have $\mathrm{ch}\, M^* = \sum_{\lambda \in X} \dim(M_\lambda) \cdot e^{-\lambda}$. The natural evaluation map and coevaluation map

$$\mathrm{ev}_M \colon M \otimes M^* \longrightarrow \Bbbk \qquad \text{and} \qquad \mathrm{coev}_M \colon \Bbbk \longrightarrow M^* \otimes M$$

are homomorphisms of $\mathbf{G}$-modules, where $\Bbbk$ denotes the trivial $\mathbf{G}$-module. For $\mathbf{G}$-modules N and N', there are natural isomorphisms $(M \otimes N)^* \cong N^* \otimes M^*$ and $\mathrm{Hom}_{\mathbf{G}}(N \otimes M, N') \cong \mathrm{Hom}_{\mathbf{G}}(N, N' \otimes M^*)$. As explained in Section II.2.13 in [Jan03] (for the modular case), there is a second duality on $\mathrm{Rep}(\mathbf{G})$ which we call *contravariant duality* and denote by $M \mapsto M^\tau$. On the level of characters, we have $\mathrm{ch}\, M = \mathrm{ch}\, M^\tau$, and we call a $\mathbf{G}$-module M *contravariantly self-dual* if $M \cong M^\tau$. The quantum analogue of this duality is constructed in Section 9.20 in [Jan96] for the quantum group $U_q(\mathfrak{g})$ (not at a root of unity); one can check that a similar construction works for $U_\zeta(\mathfrak{g})$. For $\mathbf{G}$-modules M and N, there are natural isomorphisms $(M \otimes N)^\tau \cong N^\tau \otimes M^\tau$ and $\mathrm{Hom}_{\mathbf{G}}(M,N) \cong \mathrm{Hom}_{\mathbf{G}}(N^\tau, M^\tau)$.

Next we recall the definitions of some important $\mathbf{G}$-modules, following Chapters II.2 and II.H in [Jan03]. The restriction functor

$$\mathrm{res}^{\mathbf{G}}_{\mathbf{B}} \colon \mathrm{Rep}(\mathbf{G}) \longrightarrow \mathrm{Rep}(\mathbf{B})$$

has a right adjoint *induction functor*

$$\mathrm{ind}_{\mathbf{B}}^{\mathbf{G}} \colon \mathrm{Rep}(\mathbf{B}) \longrightarrow \mathrm{Rep}(\mathbf{G}),$$

and the induction of a simple $\mathbf{B}$-module $\Bbbk_\lambda$ is non-zero if and only if λ is dominant. For $\lambda \in X^+$, we call

$$\nabla(\lambda) \coloneqq \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_\lambda)$$

the *costandard module* (or *induced module*) of highest weight λ. The terminology *highest weight* λ here refers to the fact that the weight spaces $\nabla(\lambda)_\mu$ are zero unless $\mu \leq \lambda$, and that $\dim \nabla(\lambda)_\lambda = 1$. The characters $\chi(\lambda) \coloneqq \mathrm{ch}\, \nabla(\lambda)$ of the costandard modules are given by *Weyl's character formula*, that is

$$\chi(\lambda) = \frac{\sum_{w \in W_{\mathrm{fin}}} \det(w) \cdot e^{w(\lambda+\rho)}}{\sum_{w \in W_{\mathrm{fin}}} \det(w) \cdot e^{w\rho}},$$

and they form a basis of $\mathbb{Z}[X]^{W_{\mathrm{fin}}}$. In fact, the formula above can be used to define $\chi(\lambda) \in \mathbb{Z}[X]^{W_{\mathrm{fin}}}$ for any $\lambda \in X$ (and not just for dominant weights) and one easily checks that $\chi(w \cdot \lambda) = \det(w) \cdot \chi(\lambda)$ for all $\lambda \in X$ and $w \in W_{\mathrm{fin}}$, and that $\chi(\lambda) = 0$ if $(\lambda, \alpha^\vee) = -1$ for some $\alpha \in \Pi$. The costandard module $\nabla(\lambda)$ has a unique simple submodule

$$L(\lambda) \coloneqq \mathrm{soc}_{\mathbf{G}} \nabla(\lambda),$$

and the $\mathbf{G}$-modules $L(\lambda)$ with $\lambda \in X^+$ form a set of representatives for the isomorphism classes of simple objects in $\mathrm{Rep}(\mathbf{G})$; see Sections II.2.3–6 and II.H.11 in [Jan03]. Every $\mathbf{G}$-module M has a finite composition series, and we write $[M : L(\lambda)]$ for the multiplicity of the simple module $L(\lambda)$ as a composition factor of M. The existence of finite composition series also implies that $\mathrm{Rep}(\mathbf{G})$ is a Krull-Schmidt category (in the sense of Appendix A). For a $\mathbf{G}$-module M and an indecomposable $\mathbf{G}$-module N, we write $[M : N]_\oplus$ for the multiplicity of N in a Krull-Schmidt decomposition of M. The dual of a simple $\mathbf{G}$-module is simple, and as $\mathrm{ch}\, L(\lambda)^* = \sum_{\mu \in X} \dim(L(\lambda)_\mu) \cdot e^{-\mu}$ and $-w_0\lambda$ is the unique dominant weight in the W_{fin}-orbit of $-\lambda$, for all $\lambda \in X^+$, we have $L(\lambda)^* \cong L(-w_0\lambda)$. Similarly, we see that $L(\lambda)^\tau \cong L(\lambda)$, for all $\lambda \in X^+$. The *standard module* (or *Weyl module*) of highest weight λ is defined as

$$\Delta(\lambda) \coloneqq \nabla(-w_0\lambda)^* \cong \nabla(\lambda)^\tau;$$

it has a unique maximal submodule $\mathrm{rad}_{\mathbf{G}} \Delta(\lambda)$ and $\Delta(\lambda)/\mathrm{rad}_{\mathbf{G}}\Delta(\lambda) \cong L(\lambda)$.

Remark 4.1. The standard modules and costandard modules satisfy the following important Ext-vanishing property:

$$\mathrm{Ext}^i_{\mathbf{G}}\big(\Delta(\lambda), \nabla(\mu)\big) \cong \begin{cases} \Bbbk & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

More generally, if M is a $\mathbf{G}$-module and $\lambda \in X^+$ such that $\mathrm{Ext}^i_{\mathbf{G}}\big(M, \nabla(\lambda)\big) \neq 0$ for some $i \geq 0$ then M has a weight μ with $\mu \geq \lambda$ and $\mathrm{ht}(\mu - \lambda) \geq i$. In the modular case, this is derived as a consequence of Kempf's vanishing theorem; see Propositions II.4.5 and II.4.13 in [Jan03]. The quantum analogue of Kempf's vanishing theorem was proven (in its most general form) in [RH03b], and (4.1) then follows as in the modular case.

Using the preceding remark about Ext-vanishing, we can prove the following well-known universal property of Weyl modules:

Lemma 4.2. *Let $\lambda \in X^+$ and let M be a $\mathbf{G}$-module with* $\mathrm{head}_{\mathbf{G}} M \cong L(\lambda)$ *and such that λ is maximal among the weights of M. Then there is a surjective homomorphism $\Delta(\lambda) \to M$.*

Proof. First observe that the maximality of λ among the weights of M implies that

$$\mathrm{Ext}^1_{\mathbf{G}}\big(\Delta(\lambda), \mathrm{rad}_{\mathbf{G}} M\big) \cong \mathrm{Ext}^1_{\mathbf{G}}\big((\mathrm{rad}_{\mathbf{G}} M)^\tau, \nabla(\lambda)\big) = 0,$$

by Remark 4.1. Thus, the short exact sequence

$$0 \longrightarrow \mathrm{rad}_{\mathbf{G}} M \longrightarrow M \longrightarrow L(\lambda) \longrightarrow 0$$

gives rise to a short exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{G}}\big(\Delta(\lambda), \mathrm{rad}_{\mathbf{G}} M\big) \to \mathrm{Hom}_{\mathbf{G}}\big(\Delta(\lambda), M\big) \to \mathrm{Hom}_{\mathbf{G}}\big(\Delta(\lambda), L(\lambda)\big) \to 0.$$

It follows that there exists a homomorphism $\varphi\colon \Delta(\lambda) \to M$ such that the composition of φ with the epimorphism $M \to L(\lambda)$ with kernel $\mathrm{rad}_{\mathbf{G}} M$ is non-zero. In particular, the image of φ is not contained in the unique maximal submodule $\mathrm{rad}_{\mathbf{G}} M$ of M, and we conclude that φ is surjective. □

Let us conclude this section with some remarks about *truncation to Levi subgroups*.

Remark 4.3. Suppose that we are in the modular case. For a subset $I \subseteq \Pi$, we consider the root system $\Phi_I = \mathbb{Z}I \cap \Phi$, and write $\mathbf{L}_I$ for the derived subgroup of the Levi subgroup $\langle \mathbf{T}, \mathbf{U}_\beta \mid \beta \in \Phi_I \rangle$ of $\mathbf{G}$ corresponding to I. The weight lattice of $\mathbf{L}_I$ can be identified with $X_I \coloneqq \bigoplus_{\alpha \in I} \mathbb{Z}\varpi_\alpha$, and for $\lambda \in X$, we call

$$\lambda_I \coloneqq \sum_{\alpha \in I} (\lambda, \alpha^\vee) \cdot \varpi_\alpha$$

the *truncation* of λ to X_I. For $\mu \in X_I \cap X^+$, let us write $L_I(\mu)$ and $\nabla_I(\mu)$ for the simple $\mathbf{L}_I$-module and the costandard $\mathbf{L}_I$-module of highest weight μ, respectively. The *truncation* of a $\mathbf{G}$-module M to $\mathbf{L}_I$ at a weight $\lambda \in X$ is defined as the direct sum of weight spaces

$$\mathrm{Tr}_I^\lambda M \coloneqq \bigoplus_{\gamma \in \mathbb{Z}\Phi_I} M_{\lambda - \gamma}.$$

By Sections II.2.10 and II.2.11 in [Jan03], we have

$$L_I(\lambda_I) \cong \mathrm{Tr}_I^\lambda L(\lambda) \qquad \text{and} \qquad \nabla_I(\lambda_I) \cong \mathrm{Tr}_I^\lambda \nabla(\lambda)$$

for all $\lambda \in X^+$. For $\lambda, \mu \in X^+$ such that $\lambda - \mu \in \mathbb{Z}\Phi_I$, it follows that $[\nabla(\lambda) : L(\mu)] = [\nabla_I(\lambda_I) : L_I(\mu_I)]$. Furthermore, we have

$$L_I(\lambda_I) \otimes L_I(\mu_I) \cong \mathrm{Tr}_I^\lambda L(\lambda) \otimes \mathrm{Tr}_I^\mu L(\mu) = \mathrm{Tr}_I^{\lambda+\mu}\big(L(\lambda) \otimes L(\mu)\big)$$

for all $\lambda, \mu \in X^+$; in particular, $L_I(\lambda_I) \otimes L_I(\mu_I)$ is completely reducible whenever $L(\lambda) \otimes L(\mu)$ is completely reducible. (This observation will be important in Chapter V.) Similar results apply in the quantum case (see for instance Section 4.2 in [GGN18]), but they will not be needed here.

5 Good filtrations and tilting modules

A *good filtration* of a $\mathbf{G}$-module M is a sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$$

such that either $M_i = M_{i-1}$ or $M_i/M_{i-1} \cong \nabla(\lambda_i)$ for some $\lambda_i \in X^+$, for $i = 1, \ldots, r$. The following proposition is sometimes referred to as *Donkin's cohomological criterion* for the existence of good filtrations. See Proposition 4.16 in [Jan03] for a proof in the modular case, the quantum case is analogous.

Proposition 5.1. *For a $\mathbf{G}$-module M, the following are equivalent:*

(1) M has a good filtration;

(2) $\mathrm{Ext}^i_{\mathbf{G}}\big(\Delta(\lambda), M\big) = 0$ for all $\lambda \in X^+$ and $i > 0$;

(3) $\mathrm{Ext}^1_{\mathbf{G}}\big(\Delta(\lambda), M\big) = 0$ for all $\lambda \in X^+$.

If $0 = M_0 \subseteq \cdots \subseteq M_r = M$ is a good filtration with $M_i/M_{i-1} \cong \nabla(\lambda_i)$ for $i = 1, \ldots, r$ then

$$\big|\{i \mid \lambda_i = \lambda\}\big| = \dim \mathrm{Hom}_{\mathbf{G}}\big(\Delta(\lambda), M\big)$$

for all $\lambda \in X^+$.

The last statement of the proposition tells us that the multiplicity of a costandard module in a good filtration is independent of the chosen filtration. For a $\mathbf{G}$-module M admitting a good filtration and for $\lambda \in X^+$, we write

$$[M : \nabla(\lambda)]_\nabla = \dim \mathrm{Hom}_{\mathbf{G}}\big(\Delta(\lambda), M\big)$$

for this multiplicity. The direct sum $M \oplus N$ of two $\mathbf{G}$-modules M and N admits a good filtration if and only if M and N do, since

$$\mathrm{Ext}^i_{\mathbf{G}}\big(\Delta(\lambda), M \oplus N\big) \cong \mathrm{Ext}^i_{\mathbf{G}}\big(\Delta(\lambda), M\big) \oplus \mathrm{Ext}^i_{\mathbf{G}}\big(\Delta(\lambda), N\big)$$

for all $\lambda \in X^+$ and $i > 0$. As is pointed out in Remark 4 after Proposition II.4.16 in [Jan03], a $\mathbf{G}$-module M admitting a good filtration always admits a good filtration $0 \subseteq M_0 \subseteq \cdots \subseteq M_r = M$ with quotients $M_i/M_{i-1} \cong \nabla(\lambda_i)$, such that $i < j$ whenever $\lambda_i < \lambda_j$.

A key property of good filtrations is that they are preserved under tensor products. In the modular case, this was proven in type A_n by J. Wang [Wan82] and, for almost all primes and root systems, by S. Donkin [Don85]. A uniform proof for all primes and root systems was given by O. Mathieu in [Mat90] using Frobenius splitting; the quantum analogue is treated in [Par94] using crystal bases.

Theorem 5.2. *If M and N are $\mathbf{G}$-modules admitting a good filtration then so is $M \otimes N$.*

A *Weyl filtration* of a $\mathbf{G}$-module M is a sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$$

such that either $M_i = M_{i-1}$ or $M_i/M_{i-1} \cong \Delta(\lambda_i)$ for some $\lambda_i \in X^+$, for $i = 1, \ldots, r$. As the dual (or the contravariant dual) of a standard module is a costandard module, we see that M has a Weyl filtration if and only if M^* (or M^τ) has a good filtration. By taking duals, it is straightforward to obtain analogues of Proposition 5.1 and Theorem 5.2 for modules that admit a Weyl filtration. We refrain from spelling this out in detail and only mention that, for a $\mathbf{G}$-module M admitting a Weyl filtration $0 = M_0 \subseteq \cdots \subseteq M_r = M$ with $M_i/M_{i-1} \cong \Delta(\lambda_i)$, we have

$$[M : \Delta(\lambda)]_\Delta \coloneqq \dim \operatorname{Hom}_{\mathbf{G}}\big(M, \nabla(\lambda)\big) = |\{i \mid \lambda_i = \lambda\}|$$

for $\lambda \in X^+$. Here, one may assume that $i < j$ whenever $\lambda_i > \lambda_j$

A $\mathbf{G}$-module is called a *tilting module* if it admits a good filtration and a Weyl filtration, and we write $\mathrm{Tilt}(\mathbf{G})$ for the full subcategory of tilting modules in $\mathrm{Rep}(\mathbf{G})$. By the results about $\mathbf{G}$-modules admitting a good filtration, the category $\mathrm{Tilt}(\mathbf{G})$ is closed under forming direct sums and tensor products, and under taking direct summands. In particular, every tilting module is isomorphic to a direct sum of indecomposable tilting modules. As was first pointed out by C.M. Ringel [Rin91] and S. Donkin [Don93], the indecomposable tilting modules are classified by their highest weight in X^+.

Proposition 5.3. *For every $\lambda \in X^+$, there exists a tilting module $T(\lambda)$, unique up to isomorphism, with $\dim T(\lambda)_\lambda = 1$ and $\dim T(\lambda)_\mu = 0$ unless $\mu \leq \lambda$. Every indecomposable tilting module is isomorphic to $T(\lambda)$ for some $\lambda \in X^+$.*

The characterization of the tilting module $T(\lambda)$ in terms of its weight spaces implies that

$$[T(\lambda) : \nabla(\lambda)]_\nabla = 1 = [T(\lambda) : \Delta(\lambda)]_\Delta,$$

and that

$$[T(\lambda) : \nabla(\mu)]_\nabla = 0 = [T(\lambda) : \Delta(\mu)]_\Delta$$

unless $\mu \leq \lambda$. By the above discussion, a good filtration $0 = M_0 \subseteq \cdots \subseteq M_r = T(\lambda)$ of $T(\lambda)$ can be chosen in such a way that $M_r/M_{r-1} \cong \nabla(\lambda)$, so there exists an epimorphism $T(\lambda) \to \nabla(\lambda)$ whose kernel has a good filtration. Similarly, we can find a Weyl filtration of $T(\lambda)$ that starts with $\Delta(\lambda)$, giving rise to a monomorphism $\Delta(\lambda) \to T(\lambda)$ whose cokernel has a Weyl filtration. Further weight considerations show that $T(\lambda)^* \cong T(-w_0\lambda)$ and $T(\lambda)^\tau \cong T(\lambda)$, and it follows that

$$[T(\lambda) : \nabla(\mu)]_\nabla = [T(\lambda) : \Delta(\mu)]_\Delta$$

for all $\mu \in X^+$.

The Ext-vanishing property (4.1) implies that $\operatorname{Ext}^i_{\mathbf{G}}(M, N) = 0$, for all tilting modules M and N and all $i > 0$. Combining this observation with the fact that every simple $\mathbf{G}$-module can be realized as a subquotient of a tilting module, one can prove that the canonical functor from the bounded homotopy category $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$ of $\mathrm{Tilt}(\mathbf{G})$ to the bounded derived category $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$ of $\mathrm{Rep}(\mathbf{G})$ is an equivalence of (triangulated monoidal) categories. This statement is well-known to experts in the field; we refer to it as the *tilting equivalence*. It was proven (in a different context) in [BBM04], some variations of the result were certainly known earlier (see Lemma III.2.1 in [Hap88]). We include a sketch of a proof here, for the reader's convenience. For some background on homotopy categories and derived categories, see Appendix B.

Proposition 5.4. *The canonical functor*

$$\mathfrak{T} \colon K^b\big(\mathrm{Tilt}(\mathbf{G})\big) \longrightarrow D^b\big(\mathrm{Rep}(\mathbf{G})\big)$$

is an equivalence of categories.

Proof. For the sake of convenience, let us write $\mathcal{K} = K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$ and $\mathcal{D} = D^b\big(\mathrm{Rep}(\mathbf{G})\big)$. The functor $\mathfrak{T}$ sends a complex of tilting modules to itself, and a homotopy class of chain maps to its equivalence class in the derived category. By a standard result from category theory, it suffices to prove that $\mathfrak{T}$ is fully faithful and essentially surjective.[3] First let M and N be tilting $\mathbf{G}$-modules and consider the corresponding one-term complexes. For $i < 0$, we have

$$\mathrm{Hom}_{\mathcal{D}}(M, N[i]) = 0 = \mathrm{Hom}_{\mathcal{K}}(M, N[i]),$$

and for $i > 0$, we get

$$\mathrm{Hom}_{\mathcal{D}}(M, N[i]) = \mathrm{Ext}^i_{\mathbf{G}}(M, N) = 0 = \mathrm{Hom}_{\mathcal{K}}(M, N[i])$$

by the above discussion. Finally, we have

$$\mathrm{Hom}_{\mathcal{D}}(M, N) \cong \mathrm{Hom}_{\mathbf{G}}(M, N) \cong \mathrm{Hom}_{\mathcal{K}}(M, N),$$

so $\mathrm{Hom}_{\mathcal{D}}(M, N[i]) \cong \mathrm{Hom}_{\mathcal{K}}(M, N[i])$ for all $i \in \mathbb{Z}$, and the isomorphism is induced by $\mathfrak{T}$. For bounded non-zero complexes $M = (M_\bullet, d_\bullet^M)$ and $N = (N_\bullet, d_\bullet^N)$ of tilting modules, we can choose $r \in \mathbb{Z}$ maximal with the property that $M_r \neq 0$. With

$$M' \coloneqq (\cdots \to M_{r-2} \to M_{r-1} \to 0 \to \cdots) \qquad \text{and} \qquad M'' \coloneqq (\cdots \to 0 \to M_r \to 0 \to \cdots),$$

there is a distinguished triangle

$$M' \longrightarrow M'' \longrightarrow M \longrightarrow M'[1]$$

(in both $\mathcal{K}$ and $\mathcal{D}$). Applying the cohomological functors

$$\mathrm{Hom}_{\mathcal{K}}(-, N) \qquad \text{and} \qquad \mathrm{Hom}_{\mathcal{D}}(-, N)$$

yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{K}}(M''[1], N) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{K}}(M'[1], N) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{K}}(M, N) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{K}}(M'', N) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{K}}(M', N) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(M''[1], N) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(M'[1], N) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(M, N) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(M'', N) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(M', N)
\end{tikzcd}$$

with exact rows, and where the vertical arrows are induced by $\mathfrak{T}$. Now the five lemma implies that the third vertical arrow is an isomorphism if all the other vertical arrows are isomorphisms, so we can use induction on the number of non-zero terms in M to reduce to the case where M is a (possibly shifted) one-term complex. Similarly, we can use induction on the number of non-zero terms in N to reduce to the case where N is also a (shifted) one-term complex. In this case, we have already shown that $\mathfrak{T}$ induces isomorphisms between the Hom-spaces in $\mathcal{K}$ and $\mathcal{D}$, so it follows that $\mathfrak{T}$ is fully faithful.

It remains to show that $\mathfrak{T}$ is essentially surjective, i.e. that every bounded complex $M = (M_\bullet, d_\bullet^M)$ of $\mathbf{G}$-modules is isomorphic to a bounded complex of tilting $\mathbf{G}$-modules in $\mathcal{D}$. If M fits into a distinguished triangle

$$M' \xrightarrow{f} M'' \xrightarrow{g} M \longrightarrow M'[1]$$

[3] This means that $\mathfrak{T}$ induces bijections between the Hom-spaces in the categories $\mathcal{K}$ and $\mathcal{D}$ and that every object in $\mathcal{D}$ is isomorphic to $\mathfrak{T}X$ for some object X of $\mathcal{K}$. See Theorem IV.4.1 in [ML98].

then $M \cong \mathrm{cone}(f)$, and using the fact that $\mathfrak{T}$ preserves distinguished triangles, we can once again reduce to the case where M is a one-term complex. As every short exact sequence in $\mathrm{Rep}(\mathbf{G})$ gives rise to a distinguished triangle, we can further reduce to the case where $M = L(\lambda)$ is a simple $\mathbf{G}$-module, viewed as a complex with a single non-zero term in degree zero. Then the claim follows by induction on the number of dominant weights below λ: If there are no dominant weights below λ then $L(\lambda) = T(\lambda)$, and the one-term complex with $T(\lambda)$ in degree zero certainly belongs to the essential image of $\mathfrak{T}$. In general, we can write $L(\lambda)$ as a subquotient of $T(\lambda)$, and all the other composition factors of $T(\lambda)$ are of the form $L(\mu)$ with $\mu < \lambda$, so the claim follows by induction. □

6 Linkage and translation

The *(ℓ-dilated) dot action* of the extended affine Weyl group W_{ext} on $X_{\mathbb{R}}$ is defined by

$$t_\gamma w \cdot x = w(x+\rho) + \ell\gamma - \rho,$$

for $x \in X_{\mathbb{R}}$, $\gamma \in X$ and $w \in W_{\mathrm{fin}}$. The *linkage principle* describes the decomposition of $\mathrm{Rep}(\mathbf{G})$ into *linkage classes* that arise from this action, and the *translation principle* relates the different linkage classes via *translation functors*. Before recalling these results, we need to introduce some more notation, describing the alcove geometry with respect to the dot action.

The set of fixed points of a reflection $s = s_{\beta,m}$ with respect to the dot action is the affine hyperplane

$$H_s^\ell = H_{\beta,m}^\ell := \{x \in X_{\mathbb{R}} \mid (x+\rho, \beta^\vee) = \ell m\},$$

and the *ℓ-alcoves* are the connected components of $X_{\mathbb{R}} \setminus \bigcup_{\beta,m} H_{\beta,m}^\ell$. A weight $\lambda \in X$ is called *ℓ-singular* if it lies on at least one of the hyperplanes $H_{\beta,m}^\ell$, and *ℓ-regular* if it lies in an ℓ-alcove. Recall that we write $H_{\beta,m}$ for the hyperplane of fixed points of the affine reflection $s_{\beta,m}$ with respect to the standard action. We have

$$H_{\beta,m}^\ell = \{\ell \cdot x - \rho \mid x \in H_{\beta,m}\},$$

so the map $A \mapsto \ell \cdot A - \rho$ is a bijection between the set of alcoves (in the sense of Section 2) and the set of ℓ-alcoves. Using this bijection, we can translate the results from Section 2 into results about ℓ-alcoves, and we generally use the notation which was introduced in Section 2 for ℓ-alcoves as well. For instance, we call

$$C_{\mathrm{fund}} := \ell \cdot A_{\mathrm{fund}} - \rho = \{x \in X_{\mathbb{R}} \mid 0 < (x+\rho, \beta^\vee) < \ell \text{ for all } \beta \in \Phi^+\}$$

the *fundamental ℓ-alcove*; its closure is a fundamental domain for the dot action of W_{aff} on $X_{\mathbb{R}}$. In order to distinguish between ℓ-alcoves and alcoves, we usually label the former by the letter C and the latter by the letter A (as we did with C_{fund} and A_{fund}). As in Section 2, an ℓ-alcove $C \subseteq X_{\mathbb{R}}$ is determined by a collection of integers $n_\beta(C)$, for $\beta \in \Phi^+$, such that

$$C = \left\{x \in X_{\mathbb{R}} \;\middle|\; n_\beta(C) \cdot \ell < (x+\rho, \beta^\vee) < (n_\beta(C)+1) \cdot \ell \text{ for all } \beta \in \Phi^+\right\},$$

and we set $d(C) := \sum_\beta n_\beta(C)$. For all $\lambda \in X$ and $\beta \in \Phi^+$, we can choose $n_\beta(\lambda) \in \mathbb{Z}$ such that

$$n_\beta(\lambda) \cdot \ell \leq (\lambda+\rho, \beta^\vee) < (n_\beta(\lambda)+1) \cdot \ell,$$

and we set $d(\lambda) := \sum_\beta n_\beta(\lambda)$. The *linkage order* $\uparrow_\ell$ on X is the reflexive and transitive closure of the relation given by $\mu \uparrow_\ell \lambda$ if $\mu \leq \lambda$ and there exists a reflection $s \in W_{\mathrm{aff}}$ with $\lambda = s \cdot \mu$. Using the bijection between alcoves and ℓ-alcoves, the linkage order $\uparrow$ from Section 2 induces a partial order $\uparrow_\ell$ on the set of ℓ-alcoves. For ℓ-alcoves C_1 and C_2 with $C_1 \uparrow_\ell C_2$ and a weight $\lambda \in \overline{C}_1$, there is a unique weight $\lambda' \in W_{\mathrm{aff}} \cdot \lambda \cap \overline{C}_2$, and we have $\lambda \uparrow_\ell \lambda'$.

Now we are ready to give the key results establishing the linkage principle; see Sections II.6.13–20 in [Jan03] for the modular case. The quantum analogues were first established in [AKP91] under the assumption that ℓ is an odd prime power, but this restriction was subsequently removed, as is pointed out in [And94].

Proposition 6.1 (The strong linkage principle)**.** *If $\lambda, \mu \in X^+$ such that*

$$[\nabla(\lambda) : L(\mu)] \neq 0$$

then $\mu \uparrow_\ell \lambda$.[4]

Corollary 6.2 (The weak linkage principle)**.** *If $\lambda, \mu \in X^+$ such that*

$$\mathrm{Ext}^i_{\mathbf{G}}\big(L(\lambda), L(\mu)\big) \neq 0$$

for some $i \geq 0$ then $\mu \in W_{\mathrm{aff}} \cdot \lambda$.

Proposition 6.3. *If $\lambda, \mu \in X^+$ and $i \geq 0$ such that*

$$\mathrm{Ext}^i_{\mathbf{G}}\big(L(\lambda), \nabla(\mu)\big) \neq 0 \qquad \text{or} \qquad \mathrm{Ext}^i_{\mathbf{G}}\big(\nabla(\lambda), \nabla(\mu)\big) \neq 0$$

then $\mu \uparrow_\ell \lambda$ and $i \leq d(\lambda) - d(\mu)$.

The strong linkage principle has an analogue for Weyl filtration multiplicities in indecomposable tilting modules; see the remarks after Lemma II.E.3 in [Jan03].

Proposition 6.4. *If $\lambda, \mu \in X^+$ such that*

$$[T(\lambda) : \Delta(\mu)]_\Delta \neq 0$$

then $\mu \uparrow_\ell \lambda$.

As an immediate consequence of Propositions 6.1 and 6.4, we obtain that

$$L(\lambda) \cong \Delta(\lambda) \cong \nabla(\lambda) \cong T(\lambda)$$

for all $\lambda \in \overline{C}_{\mathrm{fund}} \cap X^+$.

For $\mu \in \overline{C}_{\mathrm{fund}} \cap X$, the *linkage class* $\mathrm{Rep}_\mu(\mathbf{G})$ of μ is the full subcategory of $\mathrm{Rep}(\mathbf{G})$ whose objects are the $\mathbf{G}$-modules all of whose composition factors are of the form $L(x \cdot \mu)$, for some $x \in W_{\mathrm{aff}}$. We call the linkage class $\mathrm{Rep}_\mu(\mathbf{G})$ *ℓ-regular* if $\mu \in C_{\mathrm{fund}}$, and *$\ell$-singular* if $\mu \in \overline{C}_{\mathrm{fund}} \setminus C_{\mathrm{fund}}$. According to Corollary 6.2, every $\mathbf{G}$-module M admits a decomposition

$$M = \bigoplus_{\mu \in \overline{C}_{\mathrm{fund}} \cap X} \mathrm{pr}_\mu M,$$

[4] This is a slightly weaker version of what is called the *strong linkage principle* in [Jan03]. The version that is given there also takes into account the $\mathbf{G}$-modules $R^i \operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_\lambda)$, arising from the derived functors of $\operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}$.

where $\mathrm{pr}_\mu M$ denotes the unique largest submodule of M that belongs to $\mathrm{Rep}_\mu(\mathbf{G})$. Furthermore, there cannot be any non-zero homomorphisms between $\mathbf{G}$-modules that belong to different linkage classes, so we obtain a decomposition

$$\mathrm{Rep}(\mathbf{G}) = \bigoplus_{\mu\in\overline{C}_{\mathrm{fund}}\cap X} \mathrm{Rep}_\mu(\mathbf{G})$$

with projection functors $\mathrm{pr}_\mu \colon \mathrm{Rep}(\mathbf{G}) \to \mathrm{Rep}_\mu(\mathbf{G})$. The linkage class $\mathrm{Rep}_0(\mathbf{G})$ containing the trivial $\mathbf{G}$-module $L(0) \cong \Bbbk$ is called the *principal block* of $\mathbf{G}$, and we call

$$\mathrm{Rep}_{\Omega\cdot 0}(\mathbf{G}) \coloneqq \bigoplus_{\lambda\in\Omega\cdot 0} \mathrm{Rep}_\lambda(\mathbf{G})$$

the *extended principal block*, where $\Omega = \mathrm{Stab}_{W_{\mathrm{ext}}}(C_{\mathrm{fund}})$ as in Section 2.

Now fix $\lambda,\mu \in \overline{C}_{\mathrm{fund}} \cap X$ and let ν be the unique dominant weight in the W_{fin}-orbit of $\mu-\lambda$. The *translation functor* from $\mathrm{Rep}_\lambda(\mathbf{G})$ to $\mathrm{Rep}_\mu(\mathbf{G})$ is defined as

$$T_\lambda^\mu \coloneqq \mathrm{pr}_\mu\big(L(\nu)\otimes -\big) \colon \mathrm{Rep}_\lambda(\mathbf{G}) \longrightarrow \mathrm{Rep}_\mu(\mathbf{G}).$$

The results about translation functors that we list below can all be found in Chapter II.7 in [Jan03] for the modular case, the proofs in the quantum case are analogous.

First note that T_λ^μ is an exact functor. As $-w_0\nu$ is the unique dominant weight in the W_{fin}-orbit of $\lambda-\mu$ and as $L(-w_0\nu) \cong L(\nu)^*$, the functor T_μ^λ is both left and right adjoint to T_λ^μ. Furthermore, we have

$$(T_\lambda^\mu M)^\tau \cong T_\lambda^\mu M^\tau \qquad \text{and} \qquad (T_\lambda^\mu M)^* \cong T_{-w_0\lambda}^{-w_0\mu} M^*$$

for every $\mathbf{G}$-module M in $\mathrm{Rep}_\lambda(\mathbf{G})$. In the remarks in Sections II.7.6–7 in [Jan03], it is explained that the simple module $L(\nu)$ in the definition of T_λ^μ can be replaced by any $\mathbf{G}$-module of highest weight ν, such as $\nabla(\nu)$, $\Delta(\nu)$ or $T(\nu)$, without changing T_λ^μ (up to a natural isomorphism). In particular, translation functors preserve the subcategories of modules with good filtrations or Weyl filtrations, and the subcategory of tilting modules. On the level of characters, the action of translation functors is described by the following proposition:

Proposition 6.5. *Let M be a $\mathbf{G}$-module in $\mathrm{Rep}_\lambda(\mathbf{G})$ and write $\mathrm{ch}\, M = \sum_{x\in W_{\mathrm{aff}}} a_x \cdot \chi(x\cdot\lambda)$, for certain $a_x \in \mathbb{Z}$ such that $a_x = 0$ for all but finitely many $x \in W_{\mathrm{aff}}$. Then*

$$\mathrm{ch}\,\big(T_\lambda^\mu M\big) = \sum_{x\in W_{\mathrm{aff}}} a_x \sum_y \chi(xy\cdot\mu),$$

where y runs over a system of representatives for $\mathrm{Stab}_{W_{\mathrm{aff}}}(\lambda)/\big(\mathrm{Stab}_{W_{\mathrm{aff}}}(\lambda)\cap\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)\big)$.

As a consequence, we can determine the multiplicities in a good filtration of $T_\lambda^\mu\nabla(x\cdot\lambda)$.

Proposition 6.6. *Let $x \in W_{\mathrm{aff}}$ such that $x\cdot\lambda \in X^+$. Then $T_\lambda^\mu\nabla(x\cdot\lambda)$ has a good filtration with subquotients $\nabla(xw\cdot\mu)$ for $w \in \mathrm{Stab}_{W_{\mathrm{aff}}}(\lambda)$ such that $xw\cdot\mu \in X^+$, with each weight $xw\cdot\mu$ appearing precisely once.*

In order to discuss the properties of translation functors in more detail, we need a refinement of the notion of ℓ-alcoves: A subset $F \subseteq X_\mathbb{R}$ is called an *ℓ-facet* if there exist a decomposition $\Phi^+ = \Phi_0^+ \sqcup \Phi_1^+$ and integers n_β for $\beta\in\Phi^+$ such that

$$F = \big\{ x\in X_\mathbb{R} \bigm| (x+\rho,\beta^\vee) = n_\beta\cdot\ell \text{ for } \beta\in\Phi_0^+ \text{ and } n_\beta\cdot\ell < (x+\rho,\beta^\vee) < (n_\beta+1)\cdot\ell \text{ for } \beta\in\Phi_1^+ \big\}.$$

The *upper closure* of F is defined as

$$\widehat{F} := \big\{ x \in X_{\mathbb{R}} \bigm| (x+\rho, \beta^\vee) = n_\beta \cdot \ell \text{ for } \beta \in \Phi_0^+ \text{ and } n_\beta \cdot \ell < (x+\rho, \beta^\vee) \leq (n_\beta + 1) \cdot \ell \text{ for } \beta \in \Phi_1^+ \big\}.$$

We will sometimes write $\Phi_0^+ = \Phi_0^+(F)$, $\Phi_1^+ = \Phi_1^+(F)$ and $n_\beta = n_\beta(F)$, for $\beta \in \Phi^+$. Note that the ℓ-alcoves are precisely the ℓ-facets F with $\Phi_0^+(F) = \varnothing$. Every element $x \in X_{\mathbb{R}}$ belongs to a unique ℓ-facet, which we denote by F_x.

The translation functor T_λ^μ is particularly well-behaved when λ and μ belong to the same facet.

Proposition 6.7. *Suppose that $F_\lambda = F_\mu$. Then*

$$T_\lambda^\mu \colon \mathrm{Rep}_\lambda(\mathbf{G}) \longrightarrow \mathrm{Rep}_\mu(\mathbf{G})$$

is an equivalence of categories, with quasi-inverse T_μ^λ.

Under slightly weaker assumtions, it is still possible to describe the action of T_λ^μ on costandard modules and on simple $\mathbf{G}$-modules.

Proposition 6.8. *Suppose that $\mu \in \overline{F}_\lambda$ and let $x \in W_{\mathrm{aff}}$ such that $x \cdot \lambda \in X^+$. Then*

$$T_\lambda^\mu \nabla(x \cdot \lambda) \cong \begin{cases} \nabla(x \cdot \mu) & \text{if } x \cdot \mu \in X^+, \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_\lambda^\mu L(x \cdot \lambda) \cong \begin{cases} L(x \cdot \mu) & \text{if } x \cdot \mu \in \widehat{F}_{x \cdot \lambda}, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude this section with some results about *translation from a wall*, that is, about the translation functor T_μ^λ when $\lambda \in C_{\mathrm{fund}}$ and $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s\}$ for some $s \in S$. By Section II.6.2 in [Jan03], we have $C_{\mathrm{fund}} \cap X \subseteq X^+$, and $C_{\mathrm{fund}} \cap X$ is non-empty if and only if $\ell \geq h$ (the Coxeter number of Φ). Observe that for $\lambda \in C_{\mathrm{fund}} \cap X$ and $x \in W_{\mathrm{aff}}$, we have $x \cdot \lambda \in X^+$ if and only if $x \in W_{\mathrm{aff}}^+$. Furthermore, if $\ell \geq h$ then there exists, for every $s \in S$, a weight $\mu_s \in \overline{C}_{\mathrm{fund}} \cap X$ such that

$$\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_s) = \{e, s\};$$

see Section II.6.3 in [Jan03]. The following result, describing the translation from a wall of a costandard module, is taken from Proposition II.7.19 in [Jan03].

Proposition 6.9. *Suppose that $\lambda \in C_{\mathrm{fund}}$ and $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s\}$ for some $s \in S$, and let $x \in W_{\mathrm{aff}}$ with $x \cdot \lambda \in X^+$ and $x \cdot \lambda < xs \cdot \lambda$. Then $x \cdot \mu \in X^+$ and $T_\mu^\lambda \nabla(x \cdot \mu)$ is indecomposable, with simple socle*

$$\mathrm{soc}_{\mathbf{G}}\big(T_\mu^\lambda \nabla(x \cdot \mu)\big) \cong L(x \cdot \lambda).$$

Furthermore, there is a (non-split) short exact sequence

$$0 \longrightarrow \nabla(x \cdot \lambda) \longrightarrow T_\mu^\lambda \nabla(x \cdot \mu) \longrightarrow \nabla(xs \cdot \lambda) \longrightarrow 0.$$

As usual, there is an analogue of Proposition 6.9 where costandard modules are replaced by standard modules; we leave the details to the reader. By Lemma II.7.20 in [Jan03], the translation from a wall of a simple $\mathbf{G}$-module admits the following description:

Proposition 6.10. *Suppose that* $\lambda \in C_{\mathrm{fund}}$ *and* $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s\}$ *for some* $s \in S$*, and let* $x \in W_{\mathrm{aff}}$ *with* $x \cdot \lambda \in X^+$ *and* $x \cdot \lambda < xs \cdot \lambda$*. Then* $x \cdot \mu \in X^+$ *and* $T_\mu^\lambda L(x \cdot \mu)$ *is indecomposable, with simple head and socle*

$$\mathrm{head}_{\mathbf{G}}\big(T_\mu^\lambda L(x \cdot \mu)\big) \cong \mathrm{soc}_{\mathbf{G}}\big(T_\mu^\lambda L(x \cdot \mu)\big) \cong L(x \cdot \lambda).$$

Furthermore, we have

$$[T_\mu^\lambda L(x \cdot \mu) : L(xs \cdot \lambda)] = 1 \qquad \textit{and} \qquad [T_\mu^\lambda L(x \cdot \mu) : L(x \cdot \lambda)] = 2.$$

For any $y \in W_{\mathrm{aff}}^+$ *with* $y \neq x$ *and* $[T_\mu^\lambda L(x \cdot \mu) : L(y \cdot \lambda)] \neq 0$*, we have* $ys \cdot \lambda \uparrow_\ell y \cdot \lambda \uparrow_\ell xs \cdot \lambda$*.*

Proof. Most of the statements of the proposition are proven in Lemma II.7.20 in [Jan03]; it only remains to show that $y \cdot \lambda \uparrow_\ell xs \cdot \lambda$ for all $y \in W_{\mathrm{aff}}^+$ with $[T_\mu^\lambda L(x \cdot \mu) : L(y \cdot \lambda)] \neq 0$. This follows from Proposition 6.9, the strong linkage principle and the fact that $T_\mu^\lambda L(x \cdot \mu)$ embeds into $T_\mu^\lambda \nabla(x \cdot \mu)$. $\square$

Under the assumptions of Propositions 6.9 and 6.10, it is also shown in Proposition II.7.19 in [Jan03] that

$$\dim \mathrm{Ext}_{\mathbf{G}}^1\big(L(xs \cdot \lambda), \nabla(x \cdot \lambda)\big) = \dim \mathrm{Ext}_{\mathbf{G}}^1\big(\nabla(xs \cdot \lambda), \nabla(x \cdot \lambda)\big) = 1.$$

In particular $T_\mu^\lambda \nabla(x \cdot \mu)$ is the unique non-split extension of $\nabla(xs \cdot \lambda)$ by $\nabla(x \cdot \lambda)$. Furthermore, we have the following result, which is very useful for the computation of composition multiplicities in costandard modules (see Proposition II.7.18 in [Jan03]).

Proposition 6.11. *Suppose that* $\lambda \in C_{\mathrm{fund}}$ *and let* $x, y \in W_{\mathrm{aff}}^+$ *and* $s \in S$ *such that* $y \cdot \lambda < ys \cdot \lambda$*. If* $xs \in W_{\mathrm{aff}}^+$ *then*

$$[\nabla(x \cdot \lambda) : L(y \cdot \lambda)] = [\nabla(xs \cdot \lambda) : L(y \cdot \lambda)].$$

7 Good filtration dimension

The good filtration dimension of a $\mathbf{G}$-module M is an invariant which was introduced by E.M. Friedlander and B.J. Parshall in order to study the cohomology of Lie algebras and algebraic groups [FP86]. Their results can easily be generalized to the quantum case.

Definition 7.1. The *good filtration dimension* of a $\mathbf{G}$-module M is

$$\mathrm{gfd}(M) \coloneqq \max\big\{d \;\big|\; \mathrm{Ext}_{\mathbf{G}}^d(\Delta(\mu), M) \neq 0 \text{ for some } \mu \in X^+\big\}.$$

The *Weyl filtration dimension* of M is

$$\mathrm{wfd}(M) \coloneqq \max\big\{d \;\big|\; \mathrm{Ext}_{\mathbf{G}}^d(M, \nabla(\mu)) \neq 0 \text{ for some } \mu \in X^+\big\}.$$

The good filtration dimension and Weyl filtration dimension are well-defined by the Ext-vanishing property in Remark 4.1. By Donkin's cohomological criterion, a $\mathbf{G}$-module M satisfies $\mathrm{gfd}(M) = 0$ if and only if M has a good filtration. More generally, E.M. Friedlander and B.J. Parshall showed in Proposition 3.4 in [FP86] that M satisfies $\mathrm{gfd}(M) \leq d$ if and only if there exists a coresolution

$$0 \to M \to M_0 \to \cdots \to M_d \to 0,$$

where $M_0, \ldots, M_d$ are $\mathbf{G}$-modules admitting good filtrations. We call such a coresolution a *costandard coresolution*. Similarly, we have $\mathrm{wfd}(M) \leq d$ if and only if there exists a resolution

$$0 \to M_d \to \cdots \to M_0 \to M \to 0,$$

where $M_0, \ldots, M_d$ admit Weyl filtrations, and we call such a resolution a *standard resolution*. The following lemma is also due to E.M. Friedlander and B.J. Parshall, see Proposition 3.4 in [FP86]. The restrictions on the characteristic ℓ in [FP86] can be removed in view of [Mat90]. The third part of the lemma will play an important role later, so we include a proof for the sake of completeness.

Lemma 7.2. *Let M and M' be $\mathbf{G}$-modules. Then*

$$\begin{gathered}\mathrm{gfd}(M) = \mathrm{wfd}(M^*) = \mathrm{wfd}(M^\tau),\\ \mathrm{gfd}(M \oplus M') = \max\{\mathrm{gfd}(M), \mathrm{gfd}(M')\},\\ \mathrm{gfd}(M \otimes M') \leq \mathrm{gfd}(M) + \mathrm{gfd}(M').\end{gathered}$$

Proof. The first and the second equality are straightforward from the definitions. Now let $d = \mathrm{gfd}(M)$ and $d' = \mathrm{gfd}(M')$, and fix costandard coresolutions

$$0 \to M \to M_0 \to \cdots \to M_d \to 0 \qquad \text{and} \qquad 0 \to M' \to M'_0 \to \cdots \to M'_{d'} \to 0.$$

Note that the tensor product $M_i \otimes M'_j$ admits a good filtration for all $0 \leq i \leq d$ and $0 \leq j \leq d'$ by Theorem 5.2. Using the tensor product of complexes from Appendix B, we obtain a costandard coresolution

$$0 \to M \otimes M' \to N_0 \to \cdots \to N_{d+d'} \to 0$$

with $N_k = \bigoplus_{i+j=k} M_i \otimes M'_j$ for $k = 0, \ldots, d + d'$, and it follows that $\mathrm{gfd}(M \otimes M') \leq d + d'$. □

Remark 7.3. Let $\lambda, \mu \in \overline{C}_{\mathrm{fund}} \cap X$ and let M be a $\mathbf{G}$-module in $\mathrm{Rep}_\lambda(\mathbf{G})$. As $T_\lambda^\mu M$ is a direct summand of $M \otimes T$ for some tilting module T, we have $\mathrm{gfd}\big(T_\lambda^\mu M\big) \leq \mathrm{gfd}(M)$ and $\mathrm{wfd}\big(T_\lambda^\mu M\big) \leq \mathrm{wfd}(M)$ by Lemma 7.2.

Next we discuss how the good filtration dimension and Weyl filtration dimension interact with short exact sequences of $\mathbf{G}$-modules.

Lemma 7.4. *Consider a short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbf{G}$-modules. We have*

(1) $\mathrm{gfd}(A) \leq \max\{\mathrm{gfd}(B), \mathrm{gfd}(C) + 1\}$ *with equality if* $\mathrm{gfd}(B) \neq \mathrm{gfd}(C)$*;*

(2) $\mathrm{gfd}(B) \leq \max\{\mathrm{gfd}(A), \mathrm{gfd}(C)\}$ *with equality if* $\mathrm{gfd}(A) \neq \mathrm{gfd}(C) + 1$*;*

(3) $\mathrm{gfd}(C) \leq \max\{\mathrm{gfd}(A) - 1, \mathrm{gfd}(B)\}$ *with equality if* $\mathrm{gfd}(A) \neq \mathrm{gfd}(B)$*;*

(4) $\mathrm{wfd}(A) \leq \max\{\mathrm{wfd}(B), \mathrm{wfd}(C) - 1\}$ *with equality if* $\mathrm{wfd}(B) \neq \mathrm{wfd}(C)$*;*

(5) $\mathrm{wfd}(B) \leq \max\{\mathrm{wfd}(A), \mathrm{wfd}(C)\}$ *with equality if* $\mathrm{wfd}(C) \neq \mathrm{wfd}(A) + 1$*;*

(6) $\mathrm{wfd}(C) \leq \max\{\mathrm{wfd}(B), \mathrm{wfd}(A) + 1\}$ *with equality if* $\mathrm{wfd}(A) \neq \mathrm{wfd}(B)$*.*

Proof. We only prove (1), the proofs of (2) and (3) are completely analogous and (4)–(6) follow by taking duals. For $i \geq 0$ and $\mu \in X^+$, the short exact sequence $0 \to A \to B \to C \to 0$ gives rise to an exact sequence

$$\mathrm{Ext}^i_{\mathbf{G}}\big(\Delta(\mu), B\big) \to \mathrm{Ext}^i_{\mathbf{G}}\big(\Delta(\mu), C\big) \to \mathrm{Ext}^{i+1}_{\mathbf{G}}\big(\Delta(\mu), A\big) \to \mathrm{Ext}^{i+1}_{\mathbf{G}}\big(\Delta(\mu), B\big) \to \mathrm{Ext}^{i+1}_{\mathbf{G}}\big(\Delta(\mu), C\big).$$

If $i + 1 > \max\{\mathrm{gfd}(B), \mathrm{gfd}(C) + 1\}$ then $\mathrm{Ext}^i_{\mathbf{G}}(\Delta(\mu), C) = 0$ and $\mathrm{Ext}^{i+1}_{\mathbf{G}}(\Delta(\mu), B) = 0$ for all $\mu \in X^+$, and we conclude that $\mathrm{Ext}^{i+1}_{\mathbf{G}}(\Delta(\mu), A) = 0$ for all $\mu \in X^+$, hence

$$\mathrm{gfd}(A) \leq \max\{\mathrm{gfd}(B), \mathrm{gfd}(C) + 1\}.$$

If $\mathrm{gfd}(B) < \mathrm{gfd}(C) =: d$ then $\mathrm{Ext}^d_{\mathbf{G}}(\Delta(\mu), C) \neq 0$ for some $\mu \in X^+$, and as $\mathrm{Ext}^d_{\mathbf{G}}(\Delta(\mu), B) = 0$, the Ext-group $\mathrm{Ext}^d_{\mathbf{G}}(\Delta(\mu), C)$ embeds into $\mathrm{Ext}^{d+1}_{\mathbf{G}}(\Delta(\mu), A)$. This implies that $\mathrm{Ext}^{d+1}_{\mathbf{G}}(\Delta(\mu), A) \neq 0$ and therefore $\mathrm{gfd}(A) = d+1$. Analogously, if $\mathrm{gfd}(C) < \mathrm{gfd}(B) =: d'$ then $\mathrm{Ext}^{d'}_{\mathbf{G}}(\Delta(\mu), B) \neq 0$ for some weight $\mu \in X^+$, and $\mathrm{Ext}^{d'}_{\mathbf{G}}(\Delta(\mu), A)$ surjects onto $\mathrm{Ext}^{d'}_{\mathbf{G}}(\Delta(\mu), B)$ because $\mathrm{Ext}^{d'}_{\mathbf{G}}(\Delta(\mu), C) = 0$. As before, we conclude that $\mathrm{Ext}^{d'}_{\mathbf{G}}(\Delta(\mu), A) \neq 0$ and $\mathrm{gfd}(A) = d'$. □

Corollary 7.5. *Let M be a $\mathbf{G}$-module. Then*

$$\mathrm{gfd}(M) \leq \max\big\{\mathrm{gfd}\big(L(\delta)\big) \bigm| \delta \in X^+ \textit{ with } [M : L(\delta)] \neq 0\big\}$$

and

$$\mathrm{wfd}(M) \leq \max\big\{\mathrm{wfd}\big(L(\delta)\big) \bigm| \delta \in X^+ \textit{ with } [M : L(\delta)] \neq 0\big\}.$$

Proof. This follows from parts (2) and (5) of Lemma 7.4, by induction on the composition length. □

By Proposition 6.3, we have $\mathrm{gfd}\big(L(\lambda)\big) \leq d(\lambda)$ and $\mathrm{gfd}\big(\Delta(\lambda)\big) \leq d(\lambda)$ for all $\lambda \in X^+$. These inequalities become equalities when λ is an ℓ-regular weight, as was shown by A. Parker in [Par03]. We will rederive her results in Section II.2 using different methods.

8 Infinitesimal theory

In the modular case, the group scheme $\mathbf{G}$ admits a *Frobenius endomorphism* $\mathrm{Fr}\colon \mathbf{G} \to \mathbf{G}$ that fixes the Borel subgroup $\mathbf{B}$ and the maximal torus $\mathbf{T}$; see Section II.3.1 in [Jan03]. The *Frobenius kernels* $\mathbf{G}_r := \ker(\mathrm{Fr}^r)$ for $r > 0$ are infinitesimal subgroup schemes (in the sense of Section I.8.1 in [Jan03]) and play an important role in the representation theory of $\mathbf{G}$. In this section, we will discuss some results related to the representation theory of the subgroup schemes $\mathbf{G}_r$ and $\mathbf{G}_r\mathbf{T}$ for $r > 0$, and to the *Frobenius twist functors* $M \mapsto M^{[r]}$ on $\mathrm{Rep}(\mathbf{G})$, which arise by composing the action of $\mathbf{G}$ on a $\mathbf{G}$-module M with the powers Fr^r of the Frobenius endomorphism.

In the quantum case, the analogue of the Frobenius morphism was constructed by G. Lusztig in [Lus89], but it is no longer an endomorphism of $\mathbf{G} = U_\zeta(\mathbf{G})$. Instead, G. Lusztig defined a surjective Hopf algebra homomorphism $\mathrm{Fr}\colon U_\zeta(\mathfrak{g}) \to U(\mathfrak{g})$ from $U_\zeta(\mathfrak{g})$ to the universal enveloping algebra $U(\mathfrak{g})$ of the complex simple Lie algebra $\mathfrak{g}$. Again, this gives rise to an exact and monoidal Frobenius twist functor $M \mapsto M^{[1]}$, this time from the (semisimple) category $\mathrm{Rep}(\mathfrak{g})$ of finite-dimensional $\mathfrak{g}$-modules to $\mathrm{Rep}(\mathbf{G})$. The kernel of Fr is generated by a finite-dimensional normal Hopf subalgebra $u_\zeta(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$, called the *small quantum group*. The representation theory of the small quantum group is similar to that of the first Frobenius kernel $\mathbf{G}_1$ in the modular case, but there are no quantum analogues of the higher Frobenius kernels and Frobenius twist functors (because it does not make sense to take powers of the quantum Frobenius morphism). Because of these (and other) differences between the infinitesimal representation theory in the modular case and in the quantum case, we temporarily deviate from our strategy of treating the two cases simultaneously.

The modular case

For $r > 0$, we write $\mathrm{Rep}(\mathbf{G}_r)$ for the category of (finite-dimensional) $\mathbf{G}_r$-modules, and $\mathrm{Hom}_{\mathbf{G}_r}(M, N)$ for the space of homomorphisms between $\mathbf{G}_r$-modules M and N. As $\mathbf{G}_r$ is a normal subgroup scheme of $\mathbf{G}$, there is a natural restriction functor $\mathrm{res}^{\mathbf{G}}_{\mathbf{G}_r}\colon \mathrm{Rep}(\mathbf{G}) \to \mathrm{Rep}(\mathbf{G}_r)$, and for every $\mathbf{G}$-module M, the $\mathbf{G}_r$-fixed points $M^{\mathbf{G}_r}$ form a $\mathbf{G}$-submodule of M. By Lemma II.3.3 in [Jan03], the distribution algebra $\mathrm{Dist}(\mathbf{G}_r)$ of $\mathbf{G}_r$ can be identified with the finite-dimensional subalgebra of $U_{\Bbbk}(\mathfrak{g})$ with basis given by the products of the form

$$\prod_{\beta\in-\Phi^+} X_{\beta,r_\beta}\cdot\prod_{\alpha\in\Pi} H_{\alpha,m_\alpha}\cdot\prod_{\beta\in\Phi^+} X_{\beta,r_\beta},$$

with $0\leq r_\beta, m_\alpha < \ell^r$ for $\beta\in\Phi$ and $\alpha\in\Pi$, and the category $\mathrm{Rep}(\mathbf{G}_r)$ is equivalent to the category of $\mathrm{Dist}(\mathbf{G}_r)$-modules because $\mathbf{G}_r$ is infinitesimal; see Sections I.8.4, I.8.6 and I.9.6 in [Jan03]. The Frobenius endomorphism $\mathrm{Fr}\colon \mathbf{G}\to\mathbf{G}$ induces a Hopf algebra endomorphism

$$\mathrm{Dist}(\mathrm{Fr})\colon \mathrm{Dist}(\mathbf{G})\to\mathrm{Dist}(\mathbf{G})$$

with

$$X_{\beta,r}\mapsto\begin{cases}X_{\beta,r/\ell} & \text{if } \ell\mid r,\\ 0 & \text{otherwise}\end{cases}\qquad\text{and}\qquad H_{\alpha,m}\mapsto\begin{cases}H_{\alpha,m/\ell} & \text{if } \ell\mid m,\\ 0 & \text{otherwise,}\end{cases}$$

for $r,m\geq 0$, and the action of $\mathrm{Dist}(\mathbf{G})$ on the Frobenius twist $M^{[1]}$ of a $\mathbf{G}$-module M is obtained by composing the $\mathrm{Dist}(\mathbf{G})$-action on M with $\mathrm{Dist}(\mathrm{Fr})$. Let us denote by

$$X_r \coloneqq \{\lambda\in X^+ \mid (\lambda,\alpha^\vee)<\ell^r \text{ for all } \alpha\in\Pi\}$$

the set of *ℓ^r-restricted weights*, and observe that every weight $\lambda\in X$ can be uniquely written in the form $\lambda=\lambda_0+\ell^r\cdot\lambda_1$, with $\lambda_0\in X_r$ and $\lambda_1\in X$ (where $\lambda\in X^+$ if and only if $\lambda_1\in X^+$). According to Section II.3.15 in [Jan03], the restriction to $\mathbf{G}_r$ of a simple $\mathbf{G}$-module $L(\lambda)$ with $\lambda\in X_r$ affords a simple $\mathbf{G}_r$-module, which we denote by $L_r(\lambda)$, and the different $L_r(\lambda)$, for $\lambda\in X_r$, form a set of representatives for the isomorphism classes of simple $\mathbf{G}_r$-modules.

Note that, for a $\mathbf{G}$-module M, the restriction to $\mathbf{G}_r$ of the Frobenius twist $M^{[r]}$ is a direct sum of copies of the trivial one-dimensional $\mathbf{G}_r$-module. Conversely, if N is a $\mathbf{G}$-module whose restriction to $\mathbf{G}_r$ is a direct sum of copies of the trivial one-dimensional $\mathbf{G}_r$-module then there exists a $\mathbf{G}$-module M, uniquely determined by N, with $N=M^{[r]}$, and we write $M=N^{[-r]}$. For $\mathbf{G}$-modules M and N, the identification $\mathrm{Hom}_{\mathbf{G}_r}(M,N)\cong(N\otimes M^*)^{\mathbf{G}_r}$ gives rise to a $\mathbf{G}$-module structure on $\mathrm{Hom}_{\mathbf{G}_r}(M,N)$. The $\mathbf{G}_r$-socle $\mathrm{soc}_{\mathbf{G}_r}M$ of M is a $\mathbf{G}$-submodule of M, and there is an isomorphism of $\mathbf{G}$-modules

$$\mathrm{soc}_{\mathbf{G}_r}M\cong\bigoplus_{\lambda\in X_r} L(\lambda)\otimes\mathrm{Hom}_{\mathbf{G}_r}\big(L(\lambda),M\big).$$

Applying these observations to a simple $\mathbf{G}$-module yields the following important result; see Section II.3.16 in [Jan03].

Theorem 8.1 (Steinberg's tensor product theorem)**.** *Let $\lambda\in X^+$ and write $\lambda=\lambda_0+\ell^r\cdot\lambda_1$ with $\lambda_0\in X_r$ and $\lambda_1\in X^+$. Then*

$$L(\lambda)\cong L(\lambda_0)\otimes L(\lambda_1)^{[r]}.$$

In Section II.6 below, it will be important to have an indecomposability criterion for twisted tensor products $M\otimes N^{[1]}$ of $\mathbf{G}$-modules. The following result of S. Donkin will be very useful; see the lemma in Section 2 of [Don80].

Lemma 8.2. *Let V and W be $\mathbf{G}$-modules such that V is indecomposable as a $\mathbf{G}_1$-module, W is indecomposable as a $\mathbf{G}_r$-module for some $r>0$ and $\mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1}W$ is a direct sum of copies of the trivial one-dimensional $\mathbf{G}_1$-module. Then $V\otimes W$ is indecomposable as a $\mathbf{G}_r$-module.*

Some remarks are in order about the preceding lemma. First, note that the condition that $\mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1} W$ is a direct sum of copies of the trivial one-dimensional $\mathbf{G}_1$-module implies that $W = M^{[1]}$ for the $\mathbf{G}$-module $M = W^{[-1]}$ defined above. Furthermore, W being indecomposable as a $\mathbf{G}_r$-module is equivalent to M being indecomposable as a $\mathbf{G}_{r-1}$-module, because $\mathrm{Dist}(\mathrm{Fr})$ restricts to a surjective algebra homomorphism $\mathrm{Dist}(\mathbf{G}_r) \to \mathrm{Dist}(\mathbf{G}_{r-1})$. Therefore, we can reformulate the lemma as follows:

Corollary 8.3. *Let V and M be $\mathbf{G}$-modules such that V is indecomposable as a $\mathbf{G}_1$-module and M is indecomposable as a $\mathbf{G}_r$-module, for some $r > 0$. Then $V \otimes M^{[1]}$ is indecomposable as a $\mathbf{G}_{r+1}$-module.*

Recall that by assumption, all $\mathbf{G}$-modules (and $\mathbf{G}_r$-modules) we consider are finite-dimensional. Therefore, one can use the Fitting lemma to show that a $\mathbf{G}$-module (or $\mathbf{G}_r$-module) is indecomposable if and only if its endomorphism algebra is local.

Lemma 8.4. *Let M be a $\mathbf{G}$-module. Then M is indecomposable as a $\mathbf{G}$-module if and only if M is indecomposable as a $\mathbf{G}_r$ module for some $r > 0$.*

Proof. As $\mathbf{G}_r$ is a subgroup scheme of $\mathbf{G}$ for all $r > 0$, every $\mathbf{G}$-module that is indecomposable as a $\mathbf{G}_r$-module is also indecomposable as a $\mathbf{G}$-module. Now suppose that M is indecomposable as a $\mathbf{G}$-module. By point (6) in Section I.9.8 in [Jan03], there is an $n \geq 0$ with $\mathrm{End}_{\mathbf{G}}(M) = \mathrm{End}_{\mathbf{G}_r}(M)$ for all $r > n$. Thus $\mathrm{End}_{\mathbf{G}_r}(M)$ is local and M is indecomposable as a $\mathbf{G}_r$-module for all such r. $\square$

Equipped with the preceding lemma, we can prove two more corollaries of S. Donkin's lemma.

Corollary 8.5. *Let V and M be $\mathbf{G}$-modules such that V is indecomposable as a $\mathbf{G}_1$-module and M is indecomposable as a $\mathbf{G}$-module. Then $V \otimes M^{[1]}$ is indecomposable as a $\mathbf{G}$-module.*

Proof. By Lemma 8.4, M is indecomposable as a $\mathbf{G}_r$-module for some $r > 0$, and by Corollary 8.3, this implies that $V \otimes M^{[1]}$ is indecomposable as a $\mathbf{G}_{r+1}$-module. Again by Lemma 8.4, we conclude that $V \otimes M^{[1]}$ is indecomposable as a $\mathbf{G}$-module. $\square$

Corollary 8.6. *Let $M_0, \ldots, M_r$ be $\mathbf{G}$-modules such that $M_0, \ldots, M_{r-1}$ are indecomposable as $\mathbf{G}_1$-modules and M_r is indecomposable as a $\mathbf{G}$-module. Then the tensor product $M_0 \otimes M_1^{[1]} \otimes \cdots \otimes M_r^{[r]}$ is indecomposable as a $\mathbf{G}$-module.*

Proof. Note that $M_0 \otimes M_1^{[1]} \otimes \cdots \otimes M^{[r]} \cong M_0 \otimes \big(M_1 \otimes \cdots \otimes M_r^{[r-1]} \big)^{[1]}$. Using this observation, the claim follows from Corollary 8.5, by induction on r. $\square$

For applications in Chapter IV, we also need to discuss some results about representations of the subgroup schemes $\mathbf{G}_r\mathbf{T}$ of $\mathbf{G}$ for $r > 0$; see Chapter II.9 in [Jan03] for references. As for $\mathbf{G}$ and $\mathbf{G}_r$, we denote the category of (finite-dimensional) $\mathbf{G}_r\mathbf{T}$-modules by $\mathrm{Rep}(\mathbf{G}_r\mathbf{T})$ and write $\mathrm{Hom}_{\mathbf{G}_r\mathbf{T}}(M,N)$ for the space of $\mathbf{G}_r\mathbf{T}$-module homomorphisms between $\mathbf{G}_r\mathbf{T}$-modules M and N. By Proposition II.9.6 in [Jan03], the isomorphism classes of simple $\mathbf{G}_r\mathbf{T}$-modules are naturally in bijection with the weight lattice X, and we write $\widehat{L}_r(\lambda)$ for the simple $\mathbf{G}_r\mathbf{T}$-module corresponding to $\lambda \in X$. The simple $\mathbf{G}_r\mathbf{T}$-modules of the form $\widehat{L}_r(\ell^r\lambda)$ are one-dimensional, with $\mathbf{T}$ acting via $\ell^r\lambda$ and $\mathbf{G}_r$ acting trivially, and we simplify notation by writing $\widehat{L}_r(\ell^r\lambda) = \ell^r\lambda$. For a $\mathbf{G}$-module M, the restriction to $\mathbf{G}_r\mathbf{T}$ of the Frobenius twist $M^{[r]}$ decomposes as a direct sum of one-dimensional simple $\mathbf{G}_r\mathbf{T}$-modules $\ell^r\mu$ for the different weights μ of M, each occurring $\dim M_\mu$ times. For $\lambda = \lambda_0 + \ell^r\lambda_1$ with $\lambda_0 \in X_r$, we have

$$\widehat{L}_r(\lambda) \cong \widehat{L}_r(\lambda_0) \otimes \ell^r\lambda_1 \qquad \text{and} \qquad \widehat{L}_r(\lambda_0) \cong \mathrm{res}^{\mathbf{G}}_{\mathbf{G}_r\mathbf{T}} L(\lambda_0).$$

The quantum case

Recall from the beginning of this section that the quantum Frobenius morphism, constructed by G. Lusztig in [Lus89], is a surjective Hopf algebra homomorphism $\mathrm{Fr}\colon \mathbf{G} = U_\zeta(\mathfrak{g}) \to U(\mathfrak{g})$, which gives rise to an exact and monoidal Frobenius twist functor $M \mapsto M^{[1]}$ from $\mathrm{Rep}(\mathfrak{g})$ to $\mathrm{Rep}(\mathbf{G})$. For $\lambda \in X^+$, let us write $L_\mathbb{C}(\lambda)$ for the simple $\mathfrak{g}$-module of highest weight λ. Recall from above the notation

$$X_1 = \{\lambda \in X^+ \mid (\lambda, \alpha^\vee) < \ell \text{ for all } \alpha \in \Pi\}$$

for the set of ℓ-restricted weights. We have the following quantum analogue, due to G. Lusztig, of Steinberg's tensor product theorem; see Section II.H.10 in [Jan03].

Theorem 8.7 (Lusztig's tensor product theorem)**.** *Let $\lambda \in X^+$ and write $\lambda = \lambda_0 + \ell\lambda_1$ with $\lambda_0 \in X_1$ and $\lambda_1 \in X^+$. Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(\ell\lambda_1)$$

and $L(\ell\lambda_1) \cong L_\mathbb{C}(\lambda_1)^{[1]}$.

Let us write $\mathbf{G}_1 = u_\zeta(\mathfrak{g})$ for the *small quantum group*, that is, the subalgebra of $\mathbf{G} = U_\zeta(\mathfrak{g})$ generated by the elements E_α, F_α and $K_\alpha^{\pm 1}$ for $\alpha \in \Pi$. (We will mostly use the notation $\mathbf{G}_1$ when we want to emphasize analogies with the representation theory of Frobenius kernels, and $u_\zeta(\mathfrak{g})$ when we discuss the algebra structure.) As before, we write $\mathrm{Rep}(\mathbf{G}_1)$ for the category of (finite-dimensional) $\mathbf{G}_1$-modules and $\mathrm{Hom}_{\mathbf{G}_1}(M,N)$ for the space of $\mathbf{G}_1$-module homomorphisms between $\mathbf{G}_1$-modules M and N. Similarly to the modular case, the restriction to $\mathbf{G}_1$ of a simple $\mathbf{G}$-module $L(\lambda)$ with ℓ-restricted highest weight $\lambda \in X_1$ affords a simple $\mathbf{G}_1$-module which we denote by $L_1(\lambda)$, and the different $L_1(\lambda)$, with $\lambda \in X_1$, form a set of representatives for the isomorphism classes of simple $\mathbf{G}_1$-modules; see Section II.H.13 in [Jan03]. The algebra $u_\zeta(\mathfrak{g})$ is a finite-dimensional Hopf subalgebra of $U_\zeta(\mathfrak{g})$, and we write u_+ for the augmentation ideal of $u_\zeta(\mathfrak{g})$ (i.e. the kernel of the restriction to $u_\zeta(\mathfrak{g})$ of the counit ϵ of $U_\zeta(\mathfrak{g})$). It was shown by G. Lusztig that the two-sided ideal $U_\zeta(\mathfrak{g}) \cdot u_+ = u_+ \cdot U_\zeta(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$ is precisely the kernel of Fr; see Section 8.16 of [Lus90]. This implies that every $\mathbf{G}$-module M whose restriction to $\mathbf{G}_1$ is isomorphic to a direct sum of copies of the trivial one-dimensional $\mathbf{G}_1$-module (or equivalently, that is annihilated by u_+) is also annihilated by ker(Fr), and it follows that $M = N^{[1]}$ for a (uniquely determined) $\mathfrak{g}$-module $N = M^{[-1]}$. As in the modular case, the $\mathbf{G}_1$-fixed points $M^{\mathbf{G}_1}$ of a $\mathbf{G}$-module M, defined as the subspace of M of elements that are annihilated by the augmentation ideal u_+, form a $\mathbf{G}$-submodule of M. Similarly, the $\mathbf{G}_1$-socle $\mathrm{soc}_{\mathbf{G}_1} M$ is a $\mathbf{G}$-submodule of M, and as before, there is an isomorphism of $\mathbf{G}$-modules

$$\mathrm{soc}_{\mathbf{G}_1} M \cong \bigoplus_{\lambda \in X_1} L(\lambda) \otimes \mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), M\big)$$

(see Section 3.4 in [AKP92]), where the $\mathbf{G}$-module structure on $\mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), M\big)$ comes from the identification

$$\mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), M\big) \cong \big(M \otimes L(\lambda)^*\big)^{\mathbf{G}_1}.$$

Note that Lusztig's tensor product theorem, together with the above observation about 'untwisting' $\mathbf{G}$-modules on which $\mathbf{G}_1$ acts trivially and the fact that $\mathrm{Rep}(\mathfrak{g})$ is a semisimple category, implies that $\mathrm{soc}_{\mathbf{G}_1} M$ is completely reducible as a $\mathbf{G}$-module, whence $\mathrm{soc}_{\mathbf{G}_1} M = \mathrm{soc}_{\mathbf{G}} M$.

Next we establish a quantum analogue of the indecomposability criterion for twisted tensor products from Corollary 8.5. Note that the hypotheses in the following lemma are stronger than those that we imposed in the modular case. We do not know if a direct analogue of Corollary 8.5 holds in the quantum case.

Lemma 8.8. *Let V be a $\mathbf{G}$-module that has simple socle as a $\mathbf{G}_1$-module, and let L be a simple $\mathfrak{g}$-module. Then $V \otimes L^{[1]}$ is an indecomposable $\mathbf{G}$-module.*

Proof. By the assumption, there exists a weight $\lambda \in X_1$ such that $\dim \mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), V\big) = 1$, and as $\mathbf{G}_1$ acts trivially on $L^{[1]}$, there are isomorphisms of $\mathbf{G}$-modules

$$\mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), V \otimes L^{[1]}\big) \cong \mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), V\big) \otimes L^{[1]} \cong L^{[1]}.$$

Suppose for a contradiction that there is a non-trivial direct sum decomposition $V \otimes L^{[1]} \cong M_1 \oplus M_2$. As $\mathbf{G}_1$-modules, both M_1 and M_2 are isomorphic to (non-empty) direct sums of copies of V, so we obtain a non-trivial direct sum decomposition (as $\mathbf{G}$-modules)

$$L^{[1]} \cong \mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), V \otimes L^{[1]}\big) \cong \mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), M_1\big) \oplus \mathrm{Hom}_{\mathbf{G}_1}\big(L(\lambda), M_2\big),$$

contradicting the simplicity of $L^{[1]}$. □

To conclude this section, let us discuss some topics that are analogous to the theory of $\mathbf{G}_r\mathbf{T}$-modules which we discussed in the modular case. See Section II.H.13 in [Jan03] for an overview with further references. We set $\mathbf{G}_1\mathbf{T} = u_\zeta(\mathfrak{g}) \cdot U_\zeta^0(\mathfrak{g})$ (which is consistent with the notations $\mathbf{G}_1 = u_\zeta(\mathfrak{g})$ and $\mathbf{T} = U_\zeta^0(\mathfrak{g})$ defined earlier), denote by $\mathrm{Rep}(\mathbf{G}_1\mathbf{T})$ the category of (finite-dimensional) $\mathbf{G}_1\mathbf{T}$-modules and write $\mathrm{Hom}_{\mathbf{G}_1\mathbf{T}}(M, N)$ for the space of $\mathbf{G}_1\mathbf{T}$-module homomorphisms between $\mathbf{G}_1\mathbf{T}$-modules M and N. As in the modular case, the isomorphism classes of simple $\mathbf{G}_1\mathbf{T}$-modules are naturally in bijection with X, and we write $\widehat{L}_1(\lambda)$ for the irreducible $\mathbf{G}_1\mathbf{T}$-module corresponding to $\lambda \in X$. Still as in the modular case, the simple $\mathbf{G}_1\mathbf{T}$-modules of the form $\widehat{L}_1(\ell\lambda)$ are one-dimensional, and we write $\widehat{L}_1(\ell\lambda) = \ell\lambda$. For a $\mathbf{G}$-module M, the restriction to $\mathbf{G}_1\mathbf{T}$ of the Frobenius twist $M^{[1]}$ decomposes as a direct sum of one-dimensional simple $\mathbf{G}_1\mathbf{T}$-modules $\ell\mu$ for the different weights μ of M, each occurring $\dim M_\mu$ times. Furthermore, for $\lambda = \lambda_0 + \ell\lambda_1$ with $\lambda_0 \in X_1$, we have

$$\widehat{L}_1(\lambda) \cong \widehat{L}_1(\lambda_0) \otimes \ell\lambda_1 \qquad \text{and} \qquad \widehat{L}_1(\lambda_0) \cong \mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1\mathbf{T}} L(\lambda_0).$$

Linkage and translation for $\mathbf{G}_r\mathbf{T}$-modules

For applications in Chapter IV, we briefly discuss some 'infinitesimal analogues' of the results about linkage classes and translation functors from Section 6. We return to our strategy of treating the modular case and the quantum case simultaneously. The results that we will outline below can be found in Section 9.22 of [Jan03] in the modular case; the proofs in the quantum case are analogous.

Let us fix $r > 0$ in the modular case and $r = 1$ in the quantum case. For $\mu \in \overline{C}_{\mathrm{fund}} \cap X$, we define the *linkage class* $\mathrm{Rep}_\mu(\mathbf{G}_r\mathbf{T})$ of μ as the full subcategory of $\mathrm{Rep}(\mathbf{G}_r\mathbf{T})$ whose objects are the $\mathbf{G}_r\mathbf{T}$-modules all of whose composition factors are of the form $\widehat{L}_r(x \cdot \mu)$, for some $x \in W_{\mathrm{aff}}$. In analogy with the situation in Section 6, there is a projection functor $\mathrm{pr}_\mu \colon \mathrm{Rep}(\mathbf{G}_r\mathbf{T}) \to \mathrm{Rep}_\mu(\mathbf{G}_r\mathbf{T})$ and we obtain a decomposition

$$\mathrm{Rep}(\mathbf{G}_r\mathbf{T}) = \bigoplus_{\mu \in \overline{C}_{\mathrm{fund}} \cap X} \mathrm{Rep}_\mu(\mathbf{G}_r\mathbf{T}).$$

Furthermore, the functors pr_μ on $\mathrm{Rep}(\mathbf{G})$ (from Section 6) and on $\mathrm{Rep}(\mathbf{G}_1\mathbf{T})$ (described above) are intertwined by the restriction functor $\mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1\mathbf{T}} \colon \mathrm{Rep}(\mathbf{G}) \to \mathrm{Rep}(\mathbf{G}_1\mathbf{T})$, that is

$$\mathrm{pr}_\mu \circ \mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1\mathbf{T}} = \mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1\mathbf{T}} \circ \mathrm{pr}_\mu.$$

This justifies using the same symbol for these functors. For all $\gamma \in \mathbb{Z}\Phi$ and $\lambda \in \overline{C}_{\mathrm{fund}} \cap X$, the weights λ and $\lambda + \ell\gamma$ lie in the same W_{aff}-orbit, and it follows that $\mathrm{pr}_\lambda \circ (\ell\gamma \otimes -) = (\ell\gamma \otimes -) \circ \mathrm{pr}_\lambda$. Using the same definition as in Section 6, we can define translation functors T_λ^μ from $\mathrm{Rep}_\lambda(\mathbf{G}_1\mathbf{T})$ to $\mathrm{Rep}_\mu(\mathbf{G}_1\mathbf{T})$, for $\lambda, \mu \in \overline{C}_{\mathrm{fund}} \cap X$, such that

$$\mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1\mathbf{T}} \circ T_\lambda^\mu = T_\lambda^\mu \circ \mathrm{res}^{\mathbf{G}}_{\mathbf{G}_1\mathbf{T}}.$$

If λ and μ belong to the same ℓ-facet then T_λ^μ is an equivalence of categories, with quasi-inverse T_μ^λ, and we have $T_\lambda^\mu \widehat{L}_1(x \cdot \lambda) \cong \widehat{L}_1(x \cdot \mu)$ for all $x \in W_{\mathrm{aff}}$.

9 Negligible modules and the fusion category

Tensor ideals are a natural generalization of the notion of ideals in rings to the setting of monoidal categories. We will be particularly interested in one specific tensor ideal in $\mathrm{Tilt}(\mathbf{G})$, namely the ideal of *negligible tilting modules*, which will be defined below. We start with some general definitions.

Definition 9.1. Let $\mathcal{A}$ be an additive braided monoidal category. A *thick tensor ideal* in $\mathcal{A}$ is an isomorphism-closed collection $\mathcal{J}$ of objects of $\mathcal{A}$ that is stable under direct sums and retracts, and under tensor products with arbitrary objects of $\mathcal{A}$.

More specifically, this means that, for any pair of objects M and N of $\mathcal{A}$, we have

(1) if $M \in \mathcal{J}$ and $M \cong N$ then $N \in \mathcal{J}$;

(2) if $M \in \mathcal{J}$ and $N \in \mathcal{J}$ then $M \oplus N \in \mathcal{J}$;

(3) if $M \oplus N \in \mathcal{J}$ then $M \in \mathcal{J}$ and $N \in \mathcal{J}$;

(4) if $M \in \mathcal{J}$ then $M \otimes N \in \mathcal{J}$.

Given a thick tensor ideal $\mathcal{J}$ in $\mathcal{A}$ and objects M and N of $\mathcal{A}$, we define

$$\mathcal{J}(M,N) \coloneqq \big\{ f \in \mathrm{Hom}_{\mathcal{A}}(M,N) \bigm| f \text{ factors through an object in } \mathcal{J} \big\}.$$

The subgroups $\mathcal{J}(M,N) \subseteq \mathrm{Hom}_{\mathcal{A}}(M,N)$ form a tensor ideal of morphisms in $\mathcal{A}$, i.e. they are stable under composition from the left and the right and under tensor products with arbitrary morphisms in $\mathcal{A}$. Thus the quotient category $\mathcal{A}/\mathcal{J}$, with the same objects as $\mathcal{A}$ and homomorphisms given by

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{J}}(A,B) \coloneqq \mathrm{Hom}_{\mathcal{A}}(A,B)/\mathcal{J}(A,B),$$

has a natural monoidal structure, and the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{J}$, that sends an object to itself and a homomorphism to its residue class, is monoidal. If $\mathcal{A}$ has split idempotents then an object M of $\mathcal{A}$ belongs to $\mathcal{J}$ if and only if its image under the quotient functor is isomorphic to the zero object in $\mathcal{A}/\mathcal{J}$ (see Appendix A).

By Theorem 5.2, the category $\mathrm{Tilt}(\mathbf{G})$ is a braided monoidal subcategory of $\mathrm{Rep}(\mathbf{G})$, so it makes sense to ask about its thick tensor ideals. Under the assumption that $\ell > h$, these thick tensor ideals were classified by V. Ostrik in the quantum case, using anti-spherical Kazhdan-Lusztig cells [Ost97]. His results were extended to the modular case by P. Achar, W. Hardesty and S. Riche in Section 7 of [AHR19], using ℓ-cells instead of Kazhdan-Lusztig cells. We are only interested in one particular tensor ideal, which had already been studied (in both cases) by H.H. Andersen and J. Pardowski [AP95] before the work of V. Ostrik: Even under the slightly weaker assumption that $\ell \geq h$, the set

$\mathcal{N}$ of tilting $\mathbf{G}$-modules T with $[T : T(\lambda)]_{\oplus} = 0$ for all $\lambda \in C_{\text{fund}} \cap X$ forms a thick tensor ideal, which we call the ideal of *negligible tilting modules*.[5] The quotient category $\mathcal{F} = \text{Tilt}(\mathbf{G})/\mathcal{N}$ is a semisimple tensor category; it is called the *fusion category* or the *semisimplification* of $\text{Tilt}(\mathbf{G})$.[6] The tilting modules $T(\lambda)$ with $\lambda \in C_{\text{fund}} \cap X$ are called *(indecomposable) fusion modules*; they are precisely the indecomposable tilting modules whose image under the quotient functor $\text{Tilt}(G) \to \mathcal{F}$ is non-zero. We also call *fusion module* any tilting module that is isomorphic to a direct sum of indecomposable fusion modules. The split Grothendieck group $[\mathcal{F}]_{\oplus}$ of the fusion category is called the *Verlinde algebra*; it is a free $\mathbb{Z}$-algebra with basis $\{[T(\lambda)] \mid \lambda \in C_{\text{fund}} \cap X\}$ the classes of the indecomposable fusion modules and multiplication given by

$$[T(\lambda)] \cdot [T(\mu)] = [T(\lambda) \otimes T(\mu)] = \sum_{\nu \in C_{\text{fund}} \cap X} c_{\lambda,\mu}^{\nu} \cdot [T(\nu)],$$

where $c_{\lambda,\mu}^{\nu} \coloneqq [T(\lambda) \otimes T(\mu) : T(\nu)]_{\oplus}$ for $\lambda, \mu, \nu \in C_{\text{fund}} \cap X$. The structure constants $c_{\lambda,\mu}^{\nu}$ of the Verlinde algebra can also be computed as

$$\begin{aligned} c_{\lambda,\mu}^{\nu} &= [T(\lambda) \otimes T(\mu) : T(\nu)]_{\oplus} \\ &= \sum_{x \in W_{\text{aff}}^{+}} (-1)^{\ell(x)} \cdot [\Delta(\lambda) \otimes \Delta(\mu) : \Delta(x \cdot \nu)]_{\Delta} \\ &= \sum_{x \in W_{\text{aff}}} (-1)^{\ell(x)} \cdot \dim \Delta(\lambda)_{x \cdot \nu - \mu}; \end{aligned}$$

see Proposition II.E.12 in [Jan03]. For later use, we need to establish two elementary properties of these structure constants. We first prove that they are invariant under the action of $\Omega = \text{Stab}_{W_{\text{ext}}}(C_{\text{fund}})$ on C_{fund}, in the following sense:

Lemma 9.2. *Let $\lambda, \mu, \nu \in C_{\text{fund}} \cap X$ and $\omega \in \Omega$. Then*

$$[T(\lambda) \otimes T(\omega \cdot \mu) : T(\omega \cdot \nu)]_{\oplus} = [T(\lambda) \otimes T(\mu) : T(\nu)]_{\oplus}.$$

In particular, we have $T(\lambda) \otimes T(\omega \cdot 0) \cong T(\omega \cdot \lambda)$ in $\mathcal{F}$.

Proof. As conjugation by ω is an automorphism of W_{aff}, we have

$$[T(\lambda) \otimes T(\omega \cdot \mu) : T(\omega \cdot \nu)]_{\oplus} = \sum_{x \in W_{\text{aff}}} \dim \Delta(\lambda)_{x\omega \cdot \nu - \omega \cdot \mu} = \sum_{x \in W_{\text{aff}}} \dim \Delta(\lambda)_{\omega x \cdot \nu - \omega \cdot \mu}.$$

Writing $\omega = t_{\gamma} w$ with $\gamma \in X$ and $w \in W_{\text{fin}}$, it is straightforward to see that

$$\omega x \cdot \nu - \omega \cdot \mu = w(x \cdot \nu - \mu)$$

and therefore $\dim \Delta(\lambda)_{\omega x \cdot \nu - \omega \cdot \mu} = \dim \Delta(\lambda)_{x \cdot \nu - \mu}$. We conclude that

$$[T(\lambda) \otimes T(\omega \cdot \mu) : T(\omega \cdot \nu)]_{\oplus} = \sum_{x \in W_{\text{aff}}} \dim \Delta(\lambda)_{x \cdot \nu - \mu} = [T(\lambda) \otimes T(\mu) : T(\nu)]_{\oplus},$$

as claimed. ∎

[5] One can use the braided monoidal structure of $\text{Tilt}(\mathbf{G})$ to define a notion of 'dimension' that takes values in the ground field $\Bbbk$, and it turns out that the negligible tilting modules are precisely those where the 'dimension' of all indecomposable direct summands is zero. In the modular case, the 'dimension' is just the usual dimension modulo ℓ. In the quantum case, it is the so-called *quantum dimension*.

[6] The semisimplification of monoidal categories has been studied in detail by P. Etingof and V. Ostrik in [EO18]. Similar ideas can already be found in D. Benson and J. Carlson's article [BC86].

Lemma 9.3. *Let $\lambda, \mu \in C_{\mathrm{fund}} \cap X$ and denote by ν be the unique dominant weight in the W_{fin}-orbit of $w_0\lambda + \mu$. Then $\nu \in C_{\mathrm{fund}} \cap X$ and $c_{\lambda,\mu}^{\nu} \neq 0$.*

Proof. Recall that $T(\delta) \cong \nabla(\delta) \cong L(\delta)$ for all weights $\delta \in C_{\mathrm{fund}} \cap X$. As $-w_0\nu$ is the unique dominant weight in the W_{fin}-orbit of $-w_0\lambda - \mu$, the remarks above Proposition 6.5 imply that the translation functor $T_{\mu}^{-w_0\lambda}$ is naturally isomorphic to $\mathrm{pr}_{-w_0\lambda}\big(T(-w_0\nu) \otimes -\big)$. By Proposition 6.8, we have

$$T(-w_0\lambda) \cong \nabla(-w_0\lambda) \cong T_{\mu}^{-w_0\lambda}\nabla(\mu) \cong \mathrm{pr}_{-w_0\lambda}\big(T(-w_0\nu) \otimes \nabla(\mu)\big) \cong \mathrm{pr}_{-w_0\lambda}\big(T(-w_0\nu) \otimes T(\mu)\big),$$

so $T(-w_0\lambda)$ is a direct summand of $T(-w_0\nu) \otimes T(\mu)$. Analogously, we see that

$$T(0) \cong \nabla(0) \cong T_{\lambda}^{0}\nabla(\lambda) \cong \mathrm{pr}_0\big(T(-w_0\lambda) \otimes \nabla(\lambda)\big) \cong \mathrm{pr}_0\big(T(-w_0\lambda) \otimes T(\lambda)\big),$$

whence $T(0)$ is a direct summand of $T(-w_0\lambda) \otimes T(\lambda)$ and of $T(-w_0\nu) \otimes T(\mu) \otimes T(\lambda)$. Hence there exists a weight $\nu' \in X^+$ such that $T(\nu')$ is a direct summand of $T(\mu) \otimes T(\lambda)$ and $T(0)$ is a direct summand of $T(-w_0\nu) \otimes T(\nu')$. Now $T(0)$ is non-negligible, and as the negligible tilting modules form a thick tensor ideal in $\mathrm{Tilt}(\mathbf{G})$, it follows that $T(\nu')$ is non-negligible and $\nu' \in C_{\mathrm{fund}} \cap X$. Furthermore, the existence of a non-zero homomorphism from the trivial $\mathbf{G}$-module $T(0)$ to the tensor product $T(-w_0\nu) \otimes T(\nu') \cong L(-w_0\nu) \otimes L(\nu')$ implies that $L(\nu') \cong L(-w_0\nu)^* \cong L(\nu)$ by Schur's lemma. We conclude that $\nu = \nu'$, so $T(\nu)$ is a direct summand of $T(\lambda) \otimes T(\mu)$, as claimed. $\square$

II. Generic direct summands

In this chapter, we develop the theory of *singular* $\mathbf{G}$*-modules* and of *generic direct summands* of tensor products. The main idea is to use the tilting equivalence

$$\mathfrak{T}\colon K^b\big(\mathrm{Tilt}(\mathbf{G})\big) \longrightarrow D^b\big(\mathrm{Rep}(\mathbf{G})\big)$$

from Proposition I.5.4 to associate to every $\mathbf{G}$-module a *minimal tilting complex* and to study tensor products of $\mathbf{G}$-modules via these complexes. For instance, we can use minimal tilting complexes to define a map from the set of thick tensor ideals in $\mathrm{Tilt}(\mathbf{G})$ to the set of thick tensor ideals in $\mathrm{Rep}(\mathbf{G})$, and the tensor ideal of singular $\mathbf{G}$-modules in $\mathrm{Rep}(\mathbf{G})$ arises as the image of the tensor ideal of negligible tilting modules from Section I.9 under this map.

A large part of this chapter is devoted to a detailed investigation of the tensor ideal of singular $\mathbf{G}$-modules and of the corresponding quotient category. Among other things, we prove two results that we consider as a 'linkage principle' and a 'translation principle' for tensor products of $\mathbf{G}$-modules. The linkage principle for tensor products asserts that the monoidal structure of $\mathrm{Rep}(\mathbf{G})$ is compatible with the decomposition into linkage classes when considering $\mathrm{Rep}(\mathbf{G})$ modulo the tensor ideal of singular $\mathbf{G}$-modules, in the sense that the essential images of the principal block $\mathrm{Rep}_0(\mathbf{G})$ and the extended principal block $\mathrm{Rep}_{\Omega\cdot 0}(\mathbf{G})$ are closed under tensor products in the quotient category. The translation principle for tensor products allows us to describe the monoidal structure of the entire quotient category in terms of the monoidal structure on the principal block (modulo singular $\mathbf{G}$-modules), via translation functors. It also shows that the multiplicities in Krull-Schmidt decompositions of tensor products of $\mathbf{G}$-modules are governed to a large extent by the Verlinde algebra from Section I.9. After establishing these general results, we turn to tensor products of specific $\mathbf{G}$-modules, such as Weyl modules and simple $\mathbf{G}$-modules. We will show that a tensor product of two Weyl modules in the extended principal block has a unique regular indecomposable direct summand and that a tensor product of two simple $\mathbf{G}$-modules in the extended principal block has a unique regular indecomposable direct summand of maximal good filtration dimension. These are the *generic direct summands* from the title.

The content of this chapter is organized as follows: We start by discussing the theory of *minimal complexes* over an arbitrary Krull-Schmidt category in Section 1. In Section 2, we define *minimal tilting complexes* and study their properties, and in Section 3, we use minimal tilting complexes to construct tensor ideals in $\mathrm{Rep}(\mathbf{G})$ from tensor ideals in $\mathrm{Tilt}(\mathbf{G})$. Our results about singular $\mathbf{G}$-modules, including the 'linkage principle' and the 'translation principle' for tensor products, will be proven in Section 4, and the existence of generic direct summands is established in Section 5. Finally, in Section 6, we explain how the Steinberg-Lusztig tensor product theorem can be used to describe generic direct summands of tensor products of arbitrary simple $\mathbf{G}$-modules in terms of generic direct summands of tensor products of simple $\mathbf{G}$-modules with ℓ-restricted highest weights.

1 Minimal complexes

In order to transport information from $\operatorname{Tilt}(\mathbf{G})$ to $\operatorname{Rep}(\mathbf{G})$ using the tilting equivalence

$$\mathfrak{T}\colon K^b\big(\operatorname{Tilt}(\mathbf{G})\big) \longrightarrow D^b\big(\operatorname{Rep}(\mathbf{G})\big)$$

from Proposition I.5.4, it will be helpful to choose a unique representative from every homotopy class in $K^b\big(\operatorname{Tilt}(\mathbf{G})\big)$ that is minimal in a suitable sense. One way to achieve this is via the theory of *minimal complexes* which we explain below. Some of the ideas in this section stem from Lemma 4.2 in [BN07] and from Section 2.10 in [EH17].

Let $\mathcal{A}$ be an additive category. As in Appendix B, we write $C^b(\mathcal{A})$ for the category of bounded (cochain) complexes over $\mathcal{A}$ and $K^b(\mathcal{A})$ for the bounded homotopy category of $\mathcal{A}$. The unbounded versions of these categories are denoted by $C(\mathcal{A})$ and $K(\mathcal{A})$.

Definition 1.1. The *radical* of $\mathcal{A}$ is the ideal $\operatorname{rad}_{\mathcal{A}}$ with

$$\operatorname{rad}_{\mathcal{A}}(A,B) = \big\{ f \in \operatorname{Hom}_{\mathcal{A}}(A,B) \,\big|\, b \circ f \circ a \in J\big(\operatorname{End}_{\mathcal{A}}(C)\big) \text{ for all } a\colon C \to A \text{ and } b\colon B \to C \big\},$$

for all objects A and B of $\mathcal{A}$, where $J\big(\operatorname{End}_{\mathcal{A}}(C)\big)$ denotes the Jacobson radical of the ring $\operatorname{End}_{\mathcal{A}}(C)$.

Definition 1.2. A complex

$$\cdots \xrightarrow{d_{-2}} A_{-1} \xrightarrow{d_{-1}} A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} \cdots$$

over $\mathcal{A}$ is called *minimal* if $d_i \in \operatorname{rad}_{\mathcal{A}}(A_i, A_{i+1})$ for all $i \in \mathbb{Z}$.

We first prove two results that show that every homotopy class in $K(A)$ contains at most one minimal complex.

Lemma 1.3. *Let C and C' be complexes over $\mathcal{A}$, and let $f\colon C \to C'$ be a morphism of complexes.*

(1) If C is a minimal complex and f is a split monomorphism in $K(\mathcal{A})$ then f is also a split monomorphism in $C(\mathcal{A})$.

(2) If C' is a minimal complex and f is a split epimorphism in $K(\mathcal{A})$ then f is also a split epimorphism in $C(\mathcal{A})$.

Proof. Let us write $C = (A_\bullet, d_\bullet)$ and suppose that $d_i \in \operatorname{rad}_{\mathcal{A}}(A_i, A_{i+1})$ for all $i \in \mathbb{Z}$. If f is a split monomorphism in the homotopy category $K(\mathcal{A})$ then there exist a morphism of complexes $g\colon C' \to C$ and a homotopy equivalence $h = (h_i)_{i\in\mathbb{Z}}$ from $g \circ f$ to id_C, so

$$\mathrm{id}_{A_i} - g_i \circ f_i = h_{i+1} \circ d_i + d_{i-1} \circ h_i$$

for all $i \in \mathbb{Z}$. Now $h_{i+1} \circ d_i + d_{i-1} \circ h_i \in \operatorname{rad}_{\mathcal{A}}(A_i, A_i) \subseteq J\big(\operatorname{End}_{\mathcal{A}}(A_i)\big)$, and it follows that $\varphi_i \coloneqq g_i \circ f_i$ is invertible. Then $g' \coloneqq (\varphi_i^{-1} \circ g_i)_{i\in\mathbb{Z}}$ is a morphism of complexes, and $g' \circ f = \mathrm{id}_C$, so f is a split monomorphism in $C(\mathcal{A})$. The second claim can be proven analogously. □

Corollary 1.4. *Let C and C' be minimal complexes over $\mathcal{A}$, and let $f\colon C \to C'$ be a morphism of complexes. If f is an isomorphism in $K(\mathcal{A})$ then f is an isomorphism in $C(\mathcal{A})$.*

Proof. By Lemma 1.3, f has a left inverse and a right inverse in $C(\mathcal{A})$. It is straightforward to check that these must coincide, so f is invertible, as claimed. □

From now on, suppose that $\mathcal{A}$ is a Krull-Schmidt category (as defined in Appendix A). Then we can give an alternative characterization of the radical of $\mathcal{A}$ as follows:

Lemma 1.5. *Let A and B be objects of $\mathcal{A}$ and $f \in \operatorname{Hom}_{\mathcal{A}}(A, B)$. The following are equivalent:*

(1) $f \in \operatorname{rad}_{\mathcal{A}}(A, B)$;

(2) no isomorphism between non-zero objects of $\mathcal{A}$ factors through f;

(3) no isomorphism between indecomposable objects of $\mathcal{A}$ factors through f.

Proof. If some isomorphism $g\colon C \to D$ between non-zero objects of $\mathcal{A}$ factors through f then so does $\mathrm{id}_C = g^{-1} \circ g$, so we can write $\mathrm{id}_C = b \circ f \circ a$ for certain morphisms $a\colon C \to A$ and $b\colon B \to C$. As id_C does not belong to the Jacobson radical of $\operatorname{End}_{\mathcal{A}}(C)$, we conclude that $f \notin \operatorname{rad}_{\mathcal{A}}(A, B)$ and that (1) implies (2). It is obvious that (2) implies (3).

Now assume (3), let C be an object of $\mathcal{A}$ with homomorphisms $a\colon C \to A$ and $b\colon B \to C$ and write $f' \coloneqq b \circ f \circ a$. We need to show that $\mathrm{id}_C - x \circ f' \circ y$ is invertible for all $x, y \in \operatorname{End}_{\mathcal{A}}(C)$. Note that $x \circ f' \circ y$ factors through f, hence no isomorphism between indecomposable objects of $\mathcal{A}$ factors through $x \circ f' \circ y$. Therefore, it suffices to show that $\mathrm{id}_C - g$ is invertible for all $g \in \operatorname{End}_{\mathcal{A}}(C)$ with the property that no isomorphism between indecomposable objects of $\mathcal{A}$ factors through g. If C is indecomposable then this is clear from the fact that $\operatorname{End}_{\mathcal{A}}(C)$ is local, so now suppose that we have $C = C_1 \oplus C_2$, for certain objects $C_1 \neq 0$ and $C_2 \neq 0$ of $\mathcal{A}$, and write

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \qquad \text{with} \quad g_{ij} \in \operatorname{Hom}_{\mathcal{A}}(C_j, C_i).$$

Then g_{ij} factors through g for all $i, j \in \{1, 2\}$, so no isomorphism between indecomposable objects of $\mathcal{A}$ can factor through g_{ij}. By induction on the number of indecomposable direct summands in a Krull-Schmidt decomposition, the endomorphism $\varphi \coloneqq \mathrm{id}_{C_2} - g_{22}$ of C_2 is invertible, and we can write

$$\mathrm{id}_C - g = \begin{pmatrix} \mathrm{id}_{C_1} - g_{11} & -g_{12} \\ -g_{21} & \varphi \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{C_1} & -g_{12} \circ \varphi^{-1} \\ 0 & \mathrm{id}_{C_2} \end{pmatrix} \circ \begin{pmatrix} \psi & 0 \\ 0 & \varphi \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_{C_1} & 0 \\ -\varphi^{-1} \circ g_{21} & \mathrm{id}_{C_2} \end{pmatrix},$$

where $\psi = \mathrm{id}_{C_1} - g_{11} - g_{12} \circ \varphi^{-1} \circ g_{21}$. Again using the fact that endomorphism rings of indecomposable objects are local, we see that no isomorphism between indecomposable objects of $\mathcal{A}$ factors through the endomorphism $g_{11} + g_{12} \circ \varphi^{-1} \circ g_{21}$ of C_1, and again by induction on the number of indecomposable direct summands in a Krull-Schmidt decomposition, we get that ψ is invertible. Hence $\mathrm{id}_C - g$ is invertible, as required. $\square$

Next we show that every bounded complex over a Krull-Schmidt category is homotopy equivalent to a unique minimal complex. The proof is based on ideas from Lemma 4.2 in [BN07].

Lemma 1.6. *Every bounded complex C over $\mathcal{A}$ is homotopy equivalent to a minimal complex $C_{\min}$, and $C_{\min}$ is unique up to isomorphism in the category of complexes $C^b(\mathcal{A})$. Furthermore, $C_{\min}$ is a direct summand of C in $C^b(\mathcal{A})$.*

Proof. The uniqueness statement is clear from Corollary 1.4. If C is minimal then there is nothing to show, so now write $C = (A_\bullet, d_\bullet)$ and suppose that $d_i \notin \operatorname{rad}_{\mathcal{A}}(A_i, A_{i+1})$ for some $i \in \mathbb{Z}$. By Lemma 1.5, there exists an indecomposable object M of $\mathcal{A}$ such that id_M factors through d_i, so $\mathrm{id}_M = b \circ d_i \circ a$ for some $a\colon M \to A_i$ and $b\colon A_{i+1} \to M$. Then a is a split monomorphism and b is a split epimorphism, so $A_i \cong B_i \oplus M$ and $A_{i+1} \cong B_{i+1} \oplus M$ for certain objects B_i and B_{i+1} of $\mathcal{A}$. Consequently, we can write C as

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} B_i \oplus M \xrightarrow{\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & \mathrm{id}_M \end{pmatrix}} B_{i+1} \oplus M \xrightarrow{\begin{pmatrix} g_1 & g_2 \end{pmatrix}} A_{i+2} \longrightarrow \cdots,$$

and arguing as in the proof of Lemma 1.5, we see that C is isomorphic to a complex of the form

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\begin{pmatrix} f_1 \\ 0 \end{pmatrix}} B_i \oplus M \xrightarrow{\begin{pmatrix} \psi & 0 \\ 0 & \mathrm{id}_M \end{pmatrix}} B_{i+1} \oplus M \xrightarrow{\begin{pmatrix} g_1 & 0 \end{pmatrix}} A_{i+2} \longrightarrow \cdots. \tag{1.1}$$

Indeed, with $\psi = d_{11} - d_{12} \circ d_{21}$, an isomorphism of complexes is given by

$$\begin{tikzcd}[ampersand replacement=\&]
\cdots \arrow[r] \& A_{i-1} \arrow[r, "\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}"] \arrow[d, "\mathrm{id}_{A_{i-1}}"'] \& B_i \oplus M \arrow[r, "\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & \mathrm{id}_M \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \mathrm{id}_{B_i} & 0 \\ d_{21} & \mathrm{id}_M \end{pmatrix}"'] \& B_{i+1} \oplus M \arrow[r, "\begin{pmatrix} g_1 & g_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \mathrm{id}_{B_{i+1}} & -d_{12} \\ 0 & \mathrm{id}_M \end{pmatrix}"'] \& A_{i+2} \arrow[r] \arrow[d, "\mathrm{id}_{A_{i+2}}"'] \& \cdots \\
\cdots \arrow[r] \& A_{i-1} \arrow[r, "\begin{pmatrix} f_1 \\ 0 \end{pmatrix}"'] \& B_i \oplus M \arrow[r, "\begin{pmatrix} \psi & 0 \\ 0 & \mathrm{id}_M \end{pmatrix}"'] \& B_{i+1} \oplus M \arrow[r, "\begin{pmatrix} g_1 & 0 \end{pmatrix}"'] \& A_{i+2} \arrow[r] \& \cdots,
\end{tikzcd}$$

where the left square and the right square commute because C is a complex and the middle square commutes by direct computation. Now the complex in (1.1) is isomorphic to the direct sum of the complexes

$$C'\colon\ \cdots \to A_{i-1} \xrightarrow{f_1} B_i \xrightarrow{\psi} B_{i+1} \xrightarrow{g_1} A_{i+2} \to \cdots \qquad \text{and} \qquad C''\colon\ \cdots \to 0 \to M \to M \to 0 \to \cdots$$

(with M in degrees i and $i+1$). As C'' is homotopy equivalent to the zero complex, we conclude that C is homotopy equivalent to C'. Now the existence of $C_{\min}$ easily follows by induction on the sum of the numbers of indecomposable direct summands of the terms in C. The final claim is a consequence of the construction, since C' is a direct summand of C in $C^b(\mathcal{A})$. Alternatively, we can just note that a homotopy equivalence between $C_{\min}$ and C is a split monomorphism in $K^b(\mathcal{A})$, and thus also a split monomorphism in $C^b(\mathcal{A})$ by Lemma 1.3. □

Definition 1.7. Let C be a bounded complex over $\mathcal{A}$ and let $C_{\min}$ be the unique minimal complex in the homotopy class of C. We say that $C_{\min}$ is the *minimal complex of* C.

Corollary 1.8. *Let C be a bounded complex over $\mathcal{A}$, with minimal complex $C_{\min}$, and let M be an indecomposable object of $\mathcal{A}$. Write C and $C_{\min}$ as*

$$\cdots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots \qquad \textit{and} \qquad \cdots \longrightarrow B_i \longrightarrow B_{i+1} \longrightarrow \cdots,$$

respectively. Then

$$[A_i : M]_\oplus \geq [B_i : M]_\oplus \geq [A_i : M]_\oplus - [A_{i-1} : M]_\oplus - [A_{i+1} : M]_\oplus$$

for all $i \in \mathbb{Z}$.

Proof. By the uniqueness of minimal complexes and by the proof of Lemma 1.6, we can obtain $C_{\min}$ from C by successively removing pairs of isomorphic indecomposable direct summands from two adjacent terms A_j and A_{j+1} of C. The maximum number of times that an indecomposable direct summand isomorphic to M can be removed from A_i in this fashion is $[A_{i-1} : M]_\oplus + [A_{i+1} : M]_\oplus$, and the claim follows. $\square$

2 Minimal complexes of tilting modules

As $\mathrm{Tilt}(\mathbf{G})$ is a Krull-Schmidt category, the theory of minimal complexes explained in the previous section can be applied. Let M be a $\mathbf{G}$-module. We can view M as a one-term complex, concentrated in degree 0, in the derived category $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$, and the latter corresponds to a unique homotopy class in $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$ under the tilting equivalence (see Proposition I.5.4). By Lemma 1.6, this homotopy class contains a unique minimal complex, up to isomorphism in $C^b\big(\mathrm{Tilt}(\mathbf{G})\big)$, which we denote by $C_{\min}(M)$ and call the *minimal tilting complex of M*. By construction, $C_{\min}(M)$ is the unique bounded minimal complex of tilting modules with $C_{\min}(M) \cong M$ in $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$, or equivalently, with

$$H^i\big(C_{\min}(M)\big) \cong \begin{cases} M & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that taking the cohomology of a complex over $\mathrm{Tilt}(\mathbf{G})$ makes sense because $\mathrm{Tilt}(\mathbf{G})$ is a subcategory of the abelian category $\mathrm{Rep}(\mathbf{G})$. We start by listing some elementary properties of minimal tilting complexes.

Lemma 2.1. *Let M, M_1 and M_2 be $\mathbf{G}$-modules.*

(1) If M is a tilting module then $C_{\min}(M) = M$, viewed as a one-term complex with M in degree 0.

(2) We have $C_{\min}(M_1 \oplus M_2) \cong C_{\min}(M_1) \oplus C_{\min}(M_2)$ in $C^b\big(\mathrm{Tilt}(\mathbf{G})\big)$.

(3) If C is a bounded complex of tilting modules with $C \cong M$ in $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$ then $C_{\min}(M)$ is the minimal complex of C and there is a split monomorphism $C_{\min}(M) \to C$ in $C^b\big(\mathrm{Tilt}(\mathbf{G})\big)$.

(4) $C_{\min}(M_1 \otimes M_2)$ is the minimal complex of $C_{\min}(M_1) \otimes C_{\min}(M_2)$. In particular, there is a split monomorphism $C_{\min}(M_1 \otimes M_2) \to C_{\min}(M_1) \otimes C_{\min}(M_2)$ in $C^b\big(\mathrm{Tilt}(\mathbf{G})\big)$.

Proof. The first claim is obvious since M (viewed as a complex with M in degree 0) is a minimal complex, and the second claim follows from the observation that a direct sum of minimal complexes is minimal. If C is a bounded complex of tilting modules with $C \cong M$ in $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$ then we also have $C \cong C_{\min}(M)$ in $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$. Using the tilting equivalence from Proposition I.5.4, it follows that $C \cong C_{\min}(M)$ in $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$, whence $C_{\min}(M)$ is the minimal complex of C. By Lemma 1.3, any homotopy equivalence from $C_{\min}(M)$ to C is a split monomorphism in $C^b\big(\mathrm{Tilt}(\mathbf{G})\big)$. Finally, we have

$$H^i\big(C_{\min}(M_1) \otimes C_{\min}(M_2)\big) \cong \begin{cases} M_1 \otimes M_2 & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

by the Künneth formula (see Appendix B), hence the tensor product complex $C_{\min}(M_1) \otimes C_{\min}(M_2)$ is isomorphic to $M_1 \otimes M_2$ in $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$, and the fourth claim follows from the third. $\square$

Lemma 2.2. *Let* $\lambda \in \overline{C}_{\mathrm{fund}} \cap X$ *and let* M *be a* $\mathbf{G}$*-module in* $\mathrm{Rep}_\lambda(\mathbf{G})$*. Then all terms of* $C_{\min}(M)$ *belong to* $\mathrm{Rep}_\lambda(\mathbf{G})$*.*

Proof. As M belongs to $\mathrm{Rep}_\lambda(\mathbf{G})$ and the projection functor $\mathrm{pr}_\lambda \colon \mathrm{Rep}(\mathbf{G}) \to \mathrm{Rep}_\lambda(\mathbf{G})$ is exact, we have $M \cong \mathrm{pr}_\lambda M \cong \mathrm{pr}_\lambda C_{\min}(M)$ in $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$. By part (3) of Lemma 2.1, $C_{\min}(M)$ admits a split monomorphism into $\mathrm{pr}_\lambda C_{\min}(M)$ in $C^b\big(\mathrm{Tilt}(\mathbf{G})\big)$, and the claim follows. □

Next we observe that the good filtration dimension and the Weyl filtration dimension of a $\mathbf{G}$-module can be read off from its minimal tilting complex.

Lemma 2.3. *Let* M *be a* $\mathbf{G}$*-module and write* $C_{\min}(M)$ *as*

$$\cdots \xrightarrow{d_{-2}} T_{-1} \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots .$$

Then $\mathrm{gfd}(M) = \max\{i \mid T_i \neq 0\}$ *and* $\mathrm{wfd}(M) = -\min\{i \mid T_i \neq 0\}$*.*

Proof. As $M \cong H^0\big(C_{\min}(M)\big) = \ker(d_0)/\mathrm{im}(d_{-1})$, there is a short exact sequence

$$0 \longrightarrow \mathrm{im}(d_{-1}) \longrightarrow \ker(d_0) \longrightarrow M \longrightarrow 0,$$

and we claim that $\mathrm{gfd}\big(\mathrm{im}(d_{-1})\big) = 0$. Indeed, as $C_{\min}(M)$ is exact in all non-zero degrees, there are short exact sequences

$$0 \longrightarrow \mathrm{im}(d_{i-1}) \longrightarrow T_i \longrightarrow \mathrm{im}(d_i) \longrightarrow 0$$

for all $i \leq -1$, and using part (3) of Lemma I.7.4, it follows that

$$\mathrm{gfd}\big(\mathrm{im}(d_i)\big) \leq \max\big\{\mathrm{gfd}\big(\mathrm{im}(d_{i-1})\big) - 1, 0\big\} \leq \mathrm{gfd}\big(\mathrm{im}(d_{i-1})\big).$$

Furthermore, as $C_{\min}(M)$ is bounded, we have $\mathrm{im}(d_j) \cong T_j$ for some $j \leq -1$, and we conclude that

$$0 = \mathrm{gfd}\big(\mathrm{im}(d_j)\big) = \mathrm{gfd}\big(\mathrm{im}(d_{j+1})\big) = \cdots = \mathrm{gfd}\big(\mathrm{im}(d_{-1})\big),$$

as claimed. By applying part (3) of Lemma I.7.4 to our first short exact sequence, it now follows that

$$\mathrm{gfd}(M) = \mathrm{gfd}\big(\ker(d_0)\big),$$

so it suffices to prove that $\ker(d_0)$ has good filtration dimension $r \coloneqq \max\{i \mid T_i \neq 0\}$.

If $r = 0$ then $\ker(d_0) = T_0$ is a tilting module and $\mathrm{gfd}\big(\ker(d_0)\big) = 0$, so now suppose that $r > 0$. For all $i \geq 0$, there is a short exact sequence

$$0 \longrightarrow \ker(d_i) \longrightarrow T_i \longrightarrow \ker(d_{i+1}) \longrightarrow 0,$$

and part (1) of Lemma I.7.4 yields

$$\mathrm{gfd}\big(\ker(d_i)\big) \leq \mathrm{gfd}\big(\ker(d_{i+1})\big) + 1,$$

with equality whenever $\mathrm{gfd}\big(\ker(d_{i+1})\big) > 0$. Observe that $\ker(d_r) = T_r$ is a tilting module and that the minimal tilting complex of $\ker(d_{r-1})$ is given by

$$0 \longrightarrow T_{r-1} \longrightarrow T_r \longrightarrow 0,$$

with T_{r-1} in homological degree zero. By part (1) of Lemma 2.1, $\ker(d_{r-1})$ is not a tilting module, and as $\mathrm{wfd}\big(\ker(d_{r-1})\big) = 0$ by part (4) of Lemma I.7.4, it follows that $\mathrm{gfd}\big(\ker(d_{r-1})\big) = 1$. Now induction on i yields $\mathrm{gfd}\big(\ker(d_{r-i})\big) = i$ for $i = 0, \ldots, r$ and therefore $\mathrm{gfd}(M) = \mathrm{gfd}\big(\ker(d_0)\big) = r$, as required. The claim about $\mathrm{wfd}(M)$ follows by taking duals. □

Let us introduce an additional piece of notation: For $\mathbf{G}$-modules M and N, we write $M \overset{\oplus}{\subseteq} N$ if there exists a split monomorphism from M into N.

Lemma 2.4. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathbf{G}$-modules. Then*

$$\begin{aligned} C_{\min}(A)_i &\overset{\oplus}{\subseteq} C_{\min}(B)_i \oplus C_{\min}(C)_{i-1}, \\ C_{\min}(B)_i &\overset{\oplus}{\subseteq} C_{\min}(A)_i \oplus C_{\min}(C)_i, \\ C_{\min}(C)_i &\overset{\oplus}{\subseteq} C_{\min}(A)_{i+1} \oplus C_{\min}(B)_i \end{aligned}$$

for all $i \in \mathbb{Z}$. For an indecomposable tilting module M and $i \in \mathbb{Z}$, we have

$$\begin{aligned} \big[C_{\min}(B)_i : M\big]_\oplus + \big[C_{\min}(C)_{i-1} : M\big]_\oplus &\geq \big[C_{\min}(A)_i : M\big]_\oplus \\ &\geq \big[C_{\min}(B)_i : M\big]_\oplus - \big[C_{\min}(B)_{i-1} : M\big]_\oplus - \big[C_{\min}(B)_{i+1} : M\big]_\oplus \\ &\qquad + \big[C_{\min}(C)_{i-1} : M\big]_\oplus - \big[C_{\min}(C)_{i-2} : M\big]_\oplus - \big[C_{\min}(C)_i : M\big]_\oplus, \\ \big[C_{\min}(A)_i : M\big]_\oplus + \big[C_{\min}(C)_i : M\big]_\oplus &\geq \big[C_{\min}(B)_i : M\big]_\oplus \\ &\geq \big[C_{\min}(A)_i : M\big]_\oplus - \big[C_{\min}(A)_{i-1} : M\big]_\oplus - \big[C_{\min}(A)_{i+1} : M\big]_\oplus \\ &\qquad + \big[C_{\min}(C)_i : M\big]_\oplus - \big[C_{\min}(C)_{i-1} : M\big]_\oplus - \big[C_{\min}(C)_{i+1} : M\big]_\oplus, \\ \big[C_{\min}(A)_{i+1} : M\big]_\oplus + \big[C_{\min}(B)_i : M\big]_\oplus &\geq \big[C_{\min}(C)_i : M\big]_\oplus \\ &\geq \big[C_{\min}(A)_{i+1} : M\big]_\oplus - \big[C_{\min}(A)_i : M\big]_\oplus - \big[C_{\min}(A)_{i+2} : M\big]_\oplus \\ &\qquad + \big[C_{\min}(B)_i : M\big]_\oplus - \big[C_{\min}(B)_{i-1} : M\big]_\oplus - \big[C_{\min}(B)_{i+1} : M\big]_\oplus. \end{aligned}$$

Proof. The short exact sequence gives rise to a distinguished triangle $A \to B \to C \to A[1]$ in the derived category $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$, and via the tilting equivalence, to a distinguished triangle

$$C_{\min}(A) \longrightarrow C_{\min}(B) \longrightarrow C_{\min}(C) \longrightarrow C_{\min}(A)[1]$$

in the homotopy category $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$. Let us write $f\colon C_{\min}(A) \to C_{\min}(B)$ for the leftmost chain map in this distinguished triangle. By the definition of distinguished triangles in $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$ (see Appendix B), the complexes $C_{\min}(C)$ and $\mathrm{cone}(f)$ are homotopy equivalent, whence $C_{\min}(C)$ is the minimal complex of $\mathrm{cone}(f)$. Now Lemma 1.3, applied to a homotopy equivalence between $C_{\min}(C)$ and $\mathrm{cone}(f)$, implies that $C_{\min}(C)$ admits a split monomorphism into $\mathrm{cone}(f)$; in particular

$$C_{\min}(C)_i \overset{\oplus}{\subseteq} \mathrm{cone}(f)_i = C_{\min}(A)_{i+1} \oplus C_{\min}(B)_i$$

for all $i \in \mathbb{Z}$. Furthermore, we have

$$\begin{aligned} \big[C_{\min}(A)_{i+1} : M\big]_\oplus + \big[C_{\min}(B)_i : M\big]_\oplus = \big[\mathrm{cone}(f)_i : M\big]_\oplus &\geq \big[C_{\min}(C)_i : M\big]_\oplus \\ &\geq \big[\mathrm{cone}(f)_i : M\big]_\oplus - \big[\mathrm{cone}(f)_{i-1} : M\big]_\oplus - \big[\mathrm{cone}(f)_{i+1} : M\big]_\oplus \\ &= \big[C_{\min}(A)_{i+1} : M\big]_\oplus - \big[C_{\min}(A)_i : M\big]_\oplus - \big[C_{\min}(A)_{i+2} : M\big]_\oplus \\ &\qquad + \big[C_{\min}(B)_i : M\big]_\oplus - \big[C_{\min}(B)_{i-1} : M\big]_\oplus - \big[C_{\min}(B)_{i+1} : M\big]_\oplus \end{aligned}$$

for all $i \in \mathbb{Z}$, by Corollary 1.8. By triangle rotation, there are also distinguished triangles

$$C_{\min}(C)[-1] \longrightarrow C_{\min}(A) \longrightarrow C_{\min}(B) \longrightarrow C_{\min}(C)$$

and

$$C_{\min}(B)[-1] \longrightarrow C_{\min}(C)[-1] \longrightarrow C_{\min}(A) \longrightarrow C_{\min}(B)$$

in $K^b\big(\mathrm{Tilt}(\mathbf{G})\big)$, and the remaining claims follow from Lemma 1.3 and Corollary 1.8 as before. □

Now we proceed to study the minimal complexes of some specific $\mathbf{G}$-modules. Let us assume until the end of the section that $\ell \geq h$, the Coxeter number of $\mathbf{G}$, and recall that we write $x \mapsto \omega_x$ for the canonical epimorphism $W_{\mathrm{ext}} = W_{\mathrm{aff}} \rtimes \Omega \to \Omega$, where $\Omega = \mathrm{Stab}_{W_{\mathrm{ext}}}(A_{\mathrm{fund}})$.

Proposition 2.5. *Let $x \in W_{\mathrm{ext}}^+$ and $\lambda \in C_{\mathrm{fund}} \cap X$, and write $C_{\min}\big(\Delta(x \cdot \lambda)\big)$ as*

$$\cdots \xrightarrow{d_{-2}} T_{-1} \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots .$$

Then

(1) $T_i = 0$ for all $i < 0$ and all $i > \ell(x)$;

(2) if $\nu \in X^+$ and $i \in \mathbb{Z}$ such that $[T_i : T(\nu)]_\oplus \neq 0$ then $\nu = y\omega_x \cdot \lambda$ for some $y \in W_{\mathrm{aff}}^+$ with

$$0 \leq i \leq \ell(x) - \ell(y);$$

(3) $T_0 \cong T(x \cdot \lambda)$ and $T_{\ell(x)} \cong T(\omega_x \cdot \lambda)$;

(4) T_i is negligible for all $i \neq \ell(x)$.

Proof. For $x' := x\omega_x^{-1}$, we have $x' \in W_{\mathrm{aff}}^+$ and $\ell(x') = \ell(x)$ because $x(A_{\mathrm{fund}}) = x'(A_{\mathrm{fund}})$. Hence, after replacing x by x' and λ by $\omega_x \cdot \lambda \in C_{\mathrm{fund}} \cap X$, we may (and shall) assume that $x \in W_{\mathrm{aff}}^+$ and $\omega_x = e$.

We prove the claims by induction on $\ell(x)$. If $\ell(x) = 0$ then $x = e$ and $\Delta(\lambda) \cong T(\lambda)$, so $\Delta(\lambda)$ has minimal tilting complex $0 \to T(\lambda) \to 0$ and all claims are satisfied. Now suppose that $\ell(x) > 0$ and that the proposition holds for all $y \in W_{\mathrm{aff}}^+$ with $\ell(y) < \ell(x)$. For a simple reflection $s \in S$ such that $xs < x$, we have $xs \in W_{\mathrm{aff}}^+$ by Corollary I.2.13 and $xs \cdot \lambda < x \cdot \lambda$ by Theorem I.2.14. Let $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s\}$, and consider the short exact sequence

$$0 \longrightarrow \Delta(x \cdot \lambda) \longrightarrow T_\mu^\lambda \Delta(x \cdot \mu) \longrightarrow \Delta(xs \cdot \lambda) \longrightarrow 0,$$

which is obtained from the short exact sequence in Proposition I.6.9 by taking duals. Furthermore, let us write

$$C_{\min}\big(T_\mu^\lambda \Delta(x \cdot \mu)\big) = (A_\bullet, d_\bullet^A) \qquad \text{and} \qquad C_{\min}\big(\Delta(xs \cdot \lambda)\big) = (B_\bullet, d_\bullet^B),$$

and observe that T_i is a direct summand of $C_i := A_i \oplus B_{i-1}$ for all $i \in \mathbb{Z}$, by Lemma 2.4. By the induction hypothesis, we may assume that $B_i = 0$ for $i < 0$ and $i > \ell(xs) = \ell(x) - 1$, that B_i is negligible for $i \neq \ell(x) - 1$, that $B_{\ell(x)-1} \cong T(\lambda)$ and that all weights $\nu \in X^+$ with $[B_i : T(\nu)]_\oplus \neq 0$ for some $i \in \mathbb{Z}$ are of the form $y \cdot \lambda$ for some $y \in W_{\mathrm{aff}}^+$ with $0 \leq i \leq \ell(xs) - \ell(y)$. By Proposition I.6.8, we have

$$\Delta(x \cdot \mu) = \Delta(xs \cdot \mu) \cong T_\lambda^\mu \Delta(xs \cdot \lambda),$$

and it follows that $T_\mu^\lambda \Delta(x \cdot \mu)$ is isomorphic to the complex $T_\mu^\lambda T_\lambda^\mu C_{\min}\big(\Delta(xs \cdot \lambda)\big)$ in $D^b\big(\mathrm{Rep}(\mathbf{G})\big)$. Using Lemma 2.1, we conclude that A_i is a direct summand of $T_\mu^\lambda T_\lambda^\mu B_i$ for all $i \in \mathbb{Z}$, and it follows that $A_i = 0$ for $i < 0$ and $i > \ell(x) - 1$. Further note that all tilting modules in $\mathrm{Rep}_\mu(\mathbf{G})$ are negligible because $\mu \notin C_{\mathrm{fund}}$ and that the translation functor T_μ^λ sends negligible tilting modules to negligible tilting modules, because negligible tilting modules form a thick tensor ideal in $\mathrm{Tilt}(\mathbf{G})$. It follows that the functor $T_\mu^\lambda \circ T_\lambda^\mu$ sends all tilting modules to negligible tilting modules, so $T_\mu^\lambda T_\lambda^\mu B_i$ and A_i are negligible for all $i \in \mathbb{Z}$. We conclude that $C_i = A_i \oplus B_{i-1} = 0$ for $i < 0$ and $i > \ell(x)$, that C_i is negligible for all $i \neq \ell(x)$ and that

$$C_{\ell(x)} \cong A_{\ell(x)} \oplus B_{\ell(x)-1} \cong T(\lambda).$$

As T_i is a direct summand of C_i for all $i \in \mathbb{Z}$, this implies that $T_i = 0$ for $i < 0$ and $i > \ell(x)$ and that T_i is negligible for all $i \neq \ell(x)$. Furthermore, Lemma 2.4 yields

$$1 = [C_{\ell(x)} : T(\lambda)]_\oplus \geq [T_{\ell(x)} : T(\lambda)]_\oplus \geq [C_{\ell(x)} : T(\lambda)]_\oplus - [C_{\ell(x)-1} : T(\lambda)]_\oplus - [C_{\ell(x)+1} : T(\lambda)]_\oplus = 1$$

because $C_{\ell(x)-1}$ is negligible and $C_{\ell(x)+1} = 0$, and we conclude that $T_{\ell(x)} \cong T(\lambda)$.

Now suppose that $\nu \in X^+$ such that $[A_i : T(\nu)]_\oplus \neq 0$ for some $i \in \mathbb{Z}$. Then $\mathrm{Hom}_\mathbf{G}\big(\Delta(\nu), A_i\big) \neq 0$ and therefore $\mathrm{Hom}_\mathbf{G}\big(\Delta(\nu), T_\mu^\lambda T_\lambda^\mu B_i\big) \neq 0$. This implies that $\nu = y \cdot \lambda$ for some $y \in W_{\mathrm{aff}}^+$ by the linkage principle, and as T_λ^μ and T_μ^λ are mutually left and right adjoint, we have

$$0 \neq \mathrm{Hom}_\mathbf{G}\big(\Delta(y \cdot \lambda), T_\mu^\lambda T_\lambda^\mu B_i\big) \cong \mathrm{Hom}_\mathbf{G}\big(T_\mu^\lambda T_\lambda^\mu \Delta(y \cdot \lambda), B_i\big).$$

By Proposition I.6.8, this implies that $y \cdot \mu \in X^+$ and $T_\lambda^\mu \Delta(y \cdot \lambda) \cong \Delta(y \cdot \mu)$, and by Proposition I.6.6, the $\mathbf{G}$-module $T_\mu^\lambda T_\lambda^\mu \Delta(y \cdot \lambda)$ has a Weyl filtration with subquotients $\Delta(y \cdot \lambda)$ and $\Delta(ys \cdot \lambda)$. Hence, at least one of the Hom-spaces $\mathrm{Hom}_\mathbf{G}\big(\Delta(y \cdot \lambda), B_i\big)$ and $\mathrm{Hom}_\mathbf{G}\big(\Delta(ys \cdot \lambda), B_i\big)$ is non-zero, and it follows that there exists a weight $\nu' \in X^+$ such that $[B_i : T(\nu')]_\oplus \neq 0$ and at least one of the multiplicities $[T(\nu') : \nabla(y \cdot \lambda)]_\nabla$ and $[T(\nu') : \nabla(ys \cdot \lambda)]_\nabla$ is non-zero. By Proposition I.6.4, we have either $y \cdot \lambda \uparrow_\ell \nu'$ or $ys \cdot \lambda \uparrow_\ell \nu'$, and it follows that $\nu' = y' \cdot \lambda$ for some $y' \in W_{\mathrm{aff}}^+$ with $\ell(y') \geq \min\{\ell(y), \ell(ys)\} \geq \ell(y) - 1$. By our initial observations about $C_{\min}\big(\Delta(xs \cdot \lambda)\big)$, the condition $[B_i : T(y' \cdot \lambda)]_\oplus \neq 0$ implies that

$$0 \leq i \leq \ell(xs) - \ell(y') = \ell(x) - 1 - \ell(y') \leq \ell(x) - \ell(y).$$

We conclude that any weight $\nu \in X^+$ with $[A_i : T(\nu)]_\oplus \neq 0$ for some $i \in \mathbb{Z}$ is of the form $y \cdot \lambda$ for some $y \in W_{\mathrm{aff}}^+$ with $0 \leq i \leq \ell(x) - \ell(y)$. As T_i is a direct summand of $C_i = A_i \oplus B_{i-1}$ for all $i \in \mathbb{Z}$, it is straightforward to see that this statement remains true when we replace A_i by T_i.

It remains to show that $T_0 \cong T(x \cdot \lambda)$. Recall that there is a short exact sequence

$$0 \longrightarrow \Delta(x \cdot \lambda) \longrightarrow T(x \cdot \lambda) \longrightarrow M \longrightarrow 0,$$

where M is a $\mathbf{G}$-module admitting a Weyl filtration. By Lemma 2.3, we have $C_{\min}(M)_i = 0$ for $i < 0$, and Lemma 2.4 yields

$$T_0 \overset{\oplus}{\subseteq} C_{\min}\big(T(x \cdot \lambda)\big)_0 \oplus C_{\min}(M)_{-1} \cong T(x \cdot \lambda),$$

whence $T_0 \cong T(x \cdot \lambda)$, as required. □

Proposition 2.6. *Let $x \in W_{\mathrm{ext}}^+$ and $\lambda \in C_{\mathrm{fund}} \cap X$, and write $C_{\min}\big(L(x \cdot \lambda)\big)$ as*

$$\cdots \xrightarrow{d_{-2}} T_{-1} \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots .$$

Then

(1) $T_i \cong T_{-i}$ for all $i \in \mathbb{Z}$;

(2) $T_i = 0$ *for all* $i \in \mathbb{Z}$ *with* $|i| > \ell(x)$*;*

(3) *if* $\mu \in X^+$ *and* $i \in \mathbb{Z}$ *with* $[T_i : T(\mu)]_\oplus \neq 0$ *then* $\mu = y\omega_x \boldsymbol{\cdot} \lambda$ *for some* $y \in W_{\mathrm{aff}}^+$ *with* $|i| \leq \ell(x) - \ell(y)$*;*

(4) $[T_0 : T(x \boldsymbol{\cdot} \lambda)]_\oplus = 1$ *and* $T_{\ell(x)} \cong T_{-\ell(x)} \cong T(\omega_x \boldsymbol{\cdot} \lambda)$*;*

(5) $T_{\ell(x)-1} \cong T_{1-\ell(x)}$ *is negligible.*

Proof. As in the proof of Proposition 2.5, we can replace x by $x\omega_x^{-1} \in W_{\mathrm{aff}}^+$ and λ by $\omega_x \boldsymbol{\cdot} \lambda \in C_{\mathrm{fund}} \cap X$, so we will henceforth assume that $x \in W_{\mathrm{aff}}^+$ and $\omega_x = e$. As $L(x \boldsymbol{\cdot} \lambda)$ is contravariantly self-dual, the complex

$$\cdots \xrightarrow{d_1^\tau} T_1^\tau \xrightarrow{d_0^\tau} T_0^\tau \xrightarrow{d_{-1}^\tau} T_{-1}^\tau \xrightarrow{d_{-2}^\tau} \cdots$$

is a minimal tilting complex of $L(x \boldsymbol{\cdot} \lambda)$, and by uniqueness, we have $T_i \cong T_{-i}^\tau \cong T_{-i}$ for all i. (Recall that all tilting modules are contravariantly self-dual.) We prove the remaining claims by induction on $\ell(x)$. If $\ell(x) = 0$ then $x = e$ and $L(\lambda) \cong T(\lambda)$, so $L(\lambda)$ has minimal tilting complex $0 \to T(\lambda) \to 0$ and all claims are satisfied. Now suppose that $\ell(x) > 0$ and that the proposition holds for all $y \in W_{\mathrm{aff}}^+$ with $\ell(y) < \ell(x)$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{rad}_{\mathbf{G}} \Delta(x \boldsymbol{\cdot} \lambda) \longrightarrow \Delta(x \boldsymbol{\cdot} \lambda) \longrightarrow L(x \boldsymbol{\cdot} \lambda) \longrightarrow 0$$

and the minimal tilting complexes

$$C_{\min}\big(\mathrm{rad}_{\mathbf{G}} \Delta(x \boldsymbol{\cdot} \lambda)\big) = (A_\bullet, d_\bullet^A) \qquad \text{and} \qquad C_{\min}\big(\Delta(x \boldsymbol{\cdot} \lambda)\big) = (B_\bullet, d_\bullet^B),$$

and observe that T_i is a direct summand of $C_i \coloneqq A_{i+1} \oplus B_i$ for all $i \in \mathbb{Z}$, by Lemma 2.4. By the induction hypothesis and the linkage principle, we may assume that (1)–(5) are satisfied for the minimal tilting complexes of all composition factors of $\mathrm{rad}_{\mathbf{G}} \Delta(x \boldsymbol{\cdot} \lambda)$. Using Lemma 2.4 and induction on the length of a composition series of $\mathrm{rad}_{\mathbf{G}} \Delta(x \boldsymbol{\cdot} \lambda)$, we see that every weight $\mu \in X^+$ with $[A_i : T(\mu)]_\oplus \neq 0$ for some $i \in \mathbb{Z}$ is of the form $y \boldsymbol{\cdot} \lambda$, for some $y \in W_{\mathrm{aff}}^+$ with $|i| \leq \ell(x) - \ell(y) - 1$. In particular, we have $A_i = 0$ for all $i \in \mathbb{Z}$ with $|i| \geq \ell(x)$. Now recall from Proposition 2.5 that B_i is negligible for all $i \neq \ell(x)$, that $B_{\ell(x)} \cong T(\lambda)$ and that every weight $\mu \in X^+$ with $[B_i : T(\mu)]_\oplus \neq 0$ for some $i \in \mathbb{Z}$ is of the form $y \boldsymbol{\cdot} \lambda$, for some $y \in W_{\mathrm{aff}}^+$ with $|i| \leq \ell(x) - \ell(y)$. As $C_i = A_{i+1} \oplus B_i$ for all $i \in \mathbb{Z}$, we conclude that every weight $\mu \in X^+$ with $[C_i : T(\mu)]_\oplus \neq 0$ for some $i \in \mathbb{Z}$ is of the form $y \boldsymbol{\cdot} \lambda$, for some $y \in W_{\mathrm{aff}}^+$ with $|i| \leq \ell(x) - \ell(y)$. Furthermore, we have $A_{i+1} = 0$ for $i \geq \ell(x) - 1$, so $C_{\ell(x)-1} = B_{\ell(x)-1}$ is negligible and $C_{\ell(x)} = B_{\ell(x)} \cong T(\lambda)$. The claims (2), (3) and (5) are now immediate because T_i is a direct summand of C_i for all $i \in \mathbb{Z}$. The first part of claim (4) follows from Lemma 2.4 because

$$[C_0 : T(x \boldsymbol{\cdot} \lambda)]_\oplus = [A_1 : T(x \boldsymbol{\cdot} \lambda)]_\oplus + [B_0 : T(x \boldsymbol{\cdot} \lambda)]_\oplus = 1$$

and $[C_{-1} : T(x \boldsymbol{\cdot} \lambda)]_\oplus = 0 = [C_1 : T(x \boldsymbol{\cdot} \lambda)]_\oplus$, and therefore

$$1 = [C_0 : T(x \boldsymbol{\cdot} \lambda)]_\oplus \geq [T_0 : T(x \boldsymbol{\cdot} \lambda)]_\oplus \geq [C_0 : T(x \boldsymbol{\cdot} \lambda)]_\oplus - [C_{-1} : T(x \boldsymbol{\cdot} \lambda)]_\oplus - [C_1 : T(x \boldsymbol{\cdot} \lambda)]_\oplus = 1.$$

Analogously, we have $C_{\ell(x)} \cong T(\lambda)$ and $[C_{\ell(x)-1} : T(\lambda)] = 0 = [C_{\ell(x)+1} : T(\lambda)]$ because $C_{\ell(x)-1}$ is negligible and $C_{\ell(x)+1} = 0$. Using Lemma 2.4 again, it follows that $T_{\ell(x)} \cong T(\lambda)$. □

As an immediate consequence of Propositions 2.5 and 2.6, we can reprove A. Parker's results about the good filtration dimension of Weyl modules and simple $\mathbf{G}$-modules from [Par03].

Corollary 2.7. *Let $x \in W_{\mathrm{ext}}^{+}$ and $\lambda \in C_{\mathrm{fund}} \cap X$. Then*

$$\mathrm{gfd}\big(\Delta(x \cdot \lambda)\big) = \ell(x) \qquad \text{and} \qquad \mathrm{gfd}\big(L(x \cdot \lambda)\big) = \mathrm{wfd}\big(L(x \cdot \lambda)\big) = \ell(x).$$

Proof. By Lemma 2.3, we have

$$\mathrm{gfd}(M) = \max\{i \mid C_{\min}(M)_i \neq 0\} \qquad \text{and} \qquad \mathrm{wfd}(M) = -\min\{i \mid C_{\min}(M)_i \neq 0\}$$

for every **G**-module M, and the claim follows from the description of the minimal tilting complexes of $\Delta(x \cdot \lambda)$ and $L(x \cdot \lambda)$ in Propositions 2.5 and 2.6. □

3 Tensor ideals in Rep(G)

In this section, we explain how minimal tilting complexes can be used to construct thick tensor ideals in Rep(**G**) from thick tensor ideals in Tilt(**G**). Later, we will mainly be interested in the thick tensor ideal in Rep(**G**) that corresponds to the ideal $\mathcal{N}$ of negligible tilting modules from Section I.9, but our construction works in greater generality. The thick tensor ideals in Rep(**G**) which are obtained from arbitrary thick tensor ideals in Tilt(**G**) may be useful for other applications than those that are considered here. All of the thick tensor ideals in Rep(**G**) that can be obtained by our construction will have the following property:

Definition 3.1. Let $\mathcal{J}$ be a thick tensor ideal in Rep(**G**). We say that $\mathcal{J}$ has the *2/3-property* (or *two-out-of-three property*) if for any short exact sequence $0 \to A \to B \to C \to 0$ of **G**-modules such that two of the **G**-modules A, B and C belong to $\mathcal{J}$, the third also belongs to $\mathcal{J}$.

Definition 3.2. For any thick tensor ideal $\mathcal{I}$ in Tilt(**G**), we call

$$\langle \mathcal{I} \rangle \coloneqq \big\{ M \in \mathrm{Rep}(\mathbf{G}) \,\big|\, \text{all terms of } C_{\min}(M) \text{ belong to } \mathcal{I} \big\}$$

the *extension* of $\mathcal{I}$ to Rep(**G**).

Lemma 3.3. *Let $\mathcal{I}$ be a thick tensor ideal in* Tilt(**G**)*. Then $\langle \mathcal{I} \rangle$ is a thick tensor ideal in* Rep(**G**) *and $\langle \mathcal{I} \rangle$ has the 2/3-property.*

Proof. First note that $\langle \mathcal{I} \rangle$ is closed under direct sums and retracts because

$$C_{\min}(M_1 \oplus M_2) = C_{\min}(M_1) \oplus C_{\min}(M_2)$$

for all **G**-modules M_1 and M_2, by part (2) of Lemma 2.1. If $M_2 \in \langle \mathcal{I} \rangle$ then all terms of the tensor product complex $C_{\min}(M_1) \otimes C_{\min}(M_2)$ belong to $\mathcal{I}$ because $\mathcal{I}$ is a tensor ideal. As $C_{\min}(M_1 \otimes M_2)$ is a direct summand of $C_{\min}(M_1) \otimes C_{\min}(M_2)$ in $C^b\big(\mathrm{Tilt}(\mathbf{G})\big)$ by part (4) of Lemma 2.1 and as $\mathcal{I}$ is closed under retracts, we conclude that $M_1 \otimes M_2 \in \langle \mathcal{I} \rangle$. Finally, for a short exact sequence

$$0 \to A \to B \to C \to 0$$

of **G**-modules, we have

$$\begin{aligned} C_{\min}(A)_i &\overset{\oplus}{\subseteq} C_{\min}(B)_i \oplus C_{\min}(C)_{i-1}, \\ C_{\min}(B)_i &\overset{\oplus}{\subseteq} C_{\min}(A)_i \oplus C_{\min}(C)_i, \\ C_{\min}(C)_i &\overset{\oplus}{\subseteq} C_{\min}(A)_{i+1} \oplus C_{\min}(B)_i \end{aligned}$$

for all $i \in \mathbb{Z}$, by Lemma 2.4. As $\mathcal{I}$ is closed under retracts, we conclude that $\langle \mathcal{I} \rangle$ has the 2/3-property. □

The following Lemma justifies the notation $\langle\mathcal{I}\rangle$ for the extension of a thick tensor ideal $\mathcal{I}$ in $\mathrm{Tilt}(\mathbf{G})$ to $\mathrm{Rep}(\mathbf{G})$.

Lemma 3.4. *Let $\mathcal{I}$ be a thick tensor ideal in $\mathrm{Tilt}(\mathbf{G})$. Then $\langle\mathcal{I}\rangle$ is the smallest thick tensor ideal with the 2/3-property in $\mathrm{Rep}(\mathbf{G})$ that contains $\mathcal{I}$.*

Proof. The inclusion $\mathcal{I}\subseteq\langle\mathcal{I}\rangle$ follows from the fact that $C_{\min}(M)=M$ for every tilting module M; see part (1) of the Lemma 2.1. Now let $\mathcal{J}$ be a thick tensor ideal with the 2/3-property in $\mathrm{Rep}(\mathbf{G})$ such that $\mathcal{I}\subseteq\mathcal{J}$, and let M be a $\mathbf{G}$-module in $\langle\mathcal{I}\rangle$. We claim that M belongs to $\mathcal{J}$. Writing $C_{\min}(M)$ as

$$\cdots\xrightarrow{d_{-2}}T_{-1}\xrightarrow{d_{-1}}T_0\xrightarrow{d_0}T_1\xrightarrow{d_1}\cdots,$$

we have $M\cong\ker(d_0)/\mathrm{im}(d_{-1})$, so there is a short exact sequence

$$0\longrightarrow\mathrm{im}(d_{-1})\longrightarrow\ker(d_0)\longrightarrow M\longrightarrow 0.$$

As $\mathcal{J}$ has the 2/3-property, it suffices to show that $\mathrm{im}(d_{-1})$ and $\ker(d_0)$ belong to $\mathcal{J}$. As $C_{\min}(M)$ is exact in all degrees except zero, there are short exact sequences

$$0\longrightarrow\ker(d_i)\longrightarrow T_i\longrightarrow\ker(d_{i+1})\longrightarrow 0$$

for all $i\geq 0$, where $T_i\in\mathcal{I}\subseteq\mathcal{J}$, and using the 2/3-property, we see that $\ker(d_i)$ belongs to $\mathcal{J}$ if and only if $\ker(d_{i+1})$ belongs to $\mathcal{J}$. Now $C_{\min}(M)$ is bounded, so $T_i=0$ for some $i>0$, and we conclude that $\ker(d_0)$ belongs to $\mathcal{J}$. Analogously, we see that $\mathrm{im}(d_{-1})$ belongs to $\mathcal{J}$, and the claim follows. □

For a thick tensor ideal $\mathcal{J}$ in $\mathrm{Rep}(\mathbf{G})$, it is straightforward to see that $\mathcal{J}\cap\mathrm{Tilt}(\mathbf{G})$ (the set of tilting modules in $\mathcal{J}$) is a thick tensor ideal in $\mathrm{Tilt}(\mathbf{G})$. The next result shows that the map $\mathcal{J}\mapsto\mathcal{J}\cap\mathrm{Tilt}(\mathbf{G})$ is a section to the map $\mathcal{I}\mapsto\langle\mathcal{I}\rangle$ from the set of thick tensor ideals in $\mathrm{Tilt}(\mathbf{G})$ to the set of thick tensor ideals with the 2/3-property in $\mathrm{Rep}(\mathbf{G})$.

Lemma 3.5. *Let $\mathcal{I}$ be a thick tensor ideal in $\mathrm{Tilt}(\mathbf{G})$. Then $\langle\mathcal{I}\rangle\cap\mathrm{Tilt}(\mathbf{G})=\mathcal{I}$.*

Proof. For a tilting $\mathbf{G}$-module M, we have $C_{\min}(M)=M$ by part (1) of Lemma 2.1, and it follows that M belongs to $\mathcal{I}$ if and only if all terms of $C_{\min}(M)$ belong to $\mathcal{I}$. □

Remark 3.6. In the quantum case, the map $\mathcal{I}\mapsto\langle\mathcal{I}\rangle$ from the set of thick tensor ideals in $\mathrm{Tilt}(\mathbf{G})$ to the set of thick tensor ideals in $\mathrm{Rep}(\mathbf{G})$ with the 2/3-property is actually a bijection, when $\ell>h$. We give an outline of a proof of this fact; a more detailed account will appear elsewhere.

Let us call a proper thick tensor ideal $\mathcal{P}$ in $\mathrm{Rep}(\mathbf{G})$ a *prime ideal* if $M\otimes N\in\mathcal{P}$ implies that either $M\in\mathcal{P}$ or $N\in\mathcal{P}$, for $\mathbf{G}$-modules M and N. As in Lemma 4.2 in [Bal05], one can adapt standard techniques from commutative algebra to show that the intersection of all prime thick tensor ideals with the 2/3-property in $\mathrm{Rep}(\mathbf{G})$ containing a given thick tensor ideal $\mathcal{J}$ with the 2/3-property in $\mathrm{Rep}(\mathbf{G})$ is the radical of $\mathcal{J}$, i.e. the set of $\mathbf{G}$-modules M such that $M^{\otimes n}\in\mathcal{J}$ for some $n>0$. Now as every $\mathbf{G}$-module M admits a dual M^* and as M is a direct summand of $M\otimes M^*\otimes M$, we can show that the ideal $\mathcal{J}$ coincides with its radical (see Remark 4.3 and Proposition 4.4 in [Bal05]). In particular, we have $\mathcal{J}=\bigcap_{\mathcal{J}\subseteq\mathcal{P}}\mathcal{P}$. By Theorem 8.2.1 in [BKN19], every prime thick tensor ideal $\mathcal{P}$ with the 2/3-property in $\mathrm{Rep}(\mathbf{G})$ is generated by some tilting module, and it follows that $\mathcal{P}=\langle\mathcal{P}\cap\mathrm{Tilt}(\mathbf{G})\rangle$. We conclude that

$$\mathcal{J}=\bigcap_{\mathcal{J}\subseteq\mathcal{P}}\mathcal{P}=\bigcap_{\mathcal{J}\subseteq\mathcal{P}}\langle\mathcal{P}\cap\mathrm{Tilt}(\mathbf{G})\rangle=\Big\langle\bigcap_{\mathcal{J}\subseteq\mathcal{P}}\mathcal{P}\cap\mathrm{Tilt}(\mathbf{G})\Big\rangle,$$

where the last equality follows from the definition of the extension of a thick tensor ideal from $\mathrm{Tilt}(\mathbf{G})$ to $\mathrm{Rep}(\mathbf{G})$. This implies that the map $\mathcal{I} \mapsto \langle \mathcal{I} \rangle$ is surjective, and it is also injective by Lemma 3.5.

We remark that the existence of a bijection between the set of thick tensor ideals in $\mathrm{Tilt}(\mathbf{G})$ and the set of thick tensor ideals in $\mathrm{Rep}(\mathbf{G})$ with the 2/3-property can also be deduced by combining Corollary 7.7.2 and Theorem 8.1.1 in [BKN19]. However, it is not clear from the results in [BKN19] that this bijection can be described in terms of minimal tilting complexes, as we have done here.

Let us also point out that the analogous statement is not true in the modular case: Using Proposition 2.6, one sees that there is no proper thick tensor ideal $\mathcal{I}$ in $\mathrm{Tilt}(\mathbf{G})$ such that the simple $\mathbf{G}$-module

$$L \coloneqq L\big(\ell \cdot (\ell-1) \cdot \rho\big) \cong L\big((\ell-1) \cdot \rho\big)^{[1]}$$

belongs to $\langle \mathcal{I} \rangle$ (because $C_{\min}(L)$ has a non-negligible term in degree $\ell(t_{(\ell-1)\cdot\rho})$ and $\mathcal{N}$ is maximal among the thick tensor ideals in $\mathrm{Tilt}(\mathbf{G})$). On the other hand, Theorem 2.4 in [Nak95] provides an upper bound on the *complexity* of L over the second Frobenius kernel $\mathbf{G}_2$, which can be used to show that L generates a proper thick tensor ideal with the 2/3-property in $\mathrm{Rep}(\mathbf{G})$. In particular, this tensor ideal is not of the form $\langle \mathcal{I} \rangle$ for any thick tensor ideal $\mathcal{I}$ in $\mathrm{Tilt}(\mathbf{G})$. Again, the details will be presented elsewhere.

4 The ideal of singular G-modules

From now on and for the rest of this chapter, we suppose that $\ell \geq h$, the Coxeter number of $\mathbf{G}$. Recall from Section I.9 that we write $\mathcal{N}$ for the thick tensor ideal of negligible tilting modules. In this section, we study the extension $\langle \mathcal{N} \rangle$ of $\mathcal{N}$ to $\mathrm{Rep}(\mathbf{G})$, as defined in the Section 3.

Definition 4.1. We call $\langle \mathcal{N} \rangle$ the ideal of *singular* $\mathbf{G}$*-modules* and say that a $\mathbf{G}$-module is *regular* if it does not belong to $\langle \mathcal{N} \rangle$. We refer to the quotient category $\underline{\mathrm{Rep}}(\mathbf{G}) \coloneqq \mathrm{Rep}(\mathbf{G})/\langle \mathcal{N} \rangle$ as the *regular quotient* of $\mathrm{Rep}(\mathbf{G})$ and write $q \colon \mathrm{Rep}(\mathbf{G}) \to \underline{\mathrm{Rep}}(\mathbf{G})$ for the quotient functor.

Note that a $\mathbf{G}$-module M is regular if and only if $q(M)$ is non-zero in the regular quotient $\underline{\mathrm{Rep}}(\mathbf{G})$; see the material on quotient categories in Appendix A. We first prove two results that justify our terminology.

Lemma 4.2. *The ideal $\langle \mathcal{N} \rangle$ of singular $\mathbf{G}$-modules is the smallest thick tensor ideal in* $\mathrm{Rep}(\mathbf{G})$ *with the 2/3-property that contains all ℓ-singular linkage classes.*

Proof. Recall that a linkage class $\mathrm{Rep}_\mu(\mathbf{G})$ is called ℓ-singular if $\mu \in \overline{C}_{\mathrm{fund}} \setminus C_{\mathrm{fund}}$. For a $\mathbf{G}$-module M in an ℓ-singular linkage class $\mathrm{Rep}_\mu(\mathbf{G})$, all terms of the minimal complex $C_{\min}(M)$ are negligible because they belong to $\mathrm{Rep}_\mu(\mathbf{G})$ by Lemma 2.2, so $M \in \langle \mathcal{N} \rangle$.

Now let $\mathcal{I}$ be a thick tensor ideal with the 2/3-property that contains all ℓ-singular linkage classes. In order to show that $\mathcal{I}$ contains $\langle \mathcal{N} \rangle$, it suffices to verify that $\mathcal{I}$ contains $\mathcal{N}$, by Lemma 3.4. All indecomposable tilting modules of ℓ-singular highest weight belong to $\mathcal{I}$ by assumption, so now consider a negligible tilting module $T(x \cdot \lambda)$ of ℓ-regular highest weight, where $\lambda \in C_{\mathrm{fund}} \cap X$ and $x \in W_{\mathrm{aff}}^+$ with $x \neq e$. For a simple reflection $s \in S$ with $xs < x$, we have $xs \in W_{\mathrm{aff}}^+$ by Corollary I.2.13 and $xs \cdot \lambda < x \cdot \lambda$ by Theorem I.2.14. We can choose a weight $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s\}$, and using Proposition I.6.9 and weight considerations, it is straightforward to see that $T(x \cdot \lambda)$ is a direct summand of $T_\mu^\lambda T(x \cdot \mu)$. (In fact, we have $T_\mu^\lambda T(x \cdot \mu) \cong T(x \cdot \lambda)$ by Section II.E.11 in [Jan03].) As $T(x \cdot \mu)$ belongs to $\mathcal{I}$ and as $T_\mu^\lambda T(x \cdot \mu)$ is a direct summand of $T(x \cdot \mu) \otimes T(\nu)$, for ν the unique dominant weight in the W_{fin}-orbit of $\lambda - \mu$, we conclude that $T(x \cdot \lambda)$ belongs to $\mathcal{I}$, as required. $\square$

Lemma 4.3. *For $\lambda \in X^+$, the following are equivalent:*

(1) $\Delta(\lambda)$ *is regular;*

(2) $L(\lambda)$ *is regular;*

(3) λ *is ℓ-regular.*

Proof. Suppose first that λ is ℓ-regular and write $\lambda = x \cdot \lambda'$ for some $x \in W_{\mathrm{aff}}^+$ and $\lambda' \in C_{\mathrm{fund}} \cap X$. By Propositions 2.5 and 2.6, the minimal tilting complexes of both $\Delta(\lambda)$ and $L(\lambda)$ have the non-negligible tilting module $T(\lambda')$ as their term in degree $\ell(x)$, and it follows that $\Delta(\lambda) \notin \langle \mathcal{N} \rangle$ and $L(\lambda) \notin \langle \mathcal{N} \rangle$. Conversely, if λ is ℓ-singular then the linkage class containing $\Delta(\lambda)$ and $L(\lambda)$ is contained in $\langle \mathcal{N} \rangle$ by Lemma 4.2, and it follows that $\Delta(\lambda) \in \langle \mathcal{N} \rangle$ and $L(\lambda) \in \langle \mathcal{N} \rangle$. □

Our next goal is to prove two results that we consider as a 'linkage principle' and a 'translation principle' for tensor products. (See Remark 4.9 below for an explanation of this terminology.) The first one (Corollary 4.5) asserts that the principal block (and the extended principal block) are closed under tensor products in the regular quotient. The second one (Theorem 4.7) shows that the Krull-Schmidt decomposition of any tensor product in $\underline{\mathrm{Rep}}(\mathbf{G})$ can be determined by looking at the Krull-Schmidt decomposition of (the projection to $\mathrm{Rep}_0(\mathbf{G})$ of) a tensor product of $\mathbf{G}$-modules in $\mathrm{Rep}_0(\mathbf{G})$ and that the multiplicities of indecomposable direct summands are governed by the Verlinde algebra. Our main tool for proving these results will be the following lemma.

Lemma 4.4. *Let $\lambda \in C_{\mathrm{fund}} \cap X$ and $\omega \in \Omega$. For $\mathbf{G}$-modules M and N in the linkage classes $\mathrm{Rep}_\lambda(\mathbf{G})$ and $\mathrm{Rep}_{\omega \cdot 0}(\mathbf{G})$, respectively, the canonical embedding*

$$\mathrm{pr}_{\omega \cdot \lambda}\big(M \otimes N\big) \longrightarrow M \otimes N$$

and the canonical projection

$$M \otimes N \longrightarrow \mathrm{pr}_{\omega \cdot \lambda}\big(M \otimes N\big)$$

descend to isomorphisms in $\underline{\mathrm{Rep}}(\mathbf{G})$.

Proof. By the linkage principle, we have

$$M \otimes N \cong \bigoplus_{\nu \in \overline{C}_{\mathrm{fund}} \cap X} \mathrm{pr}_\nu\big(M \otimes N\big),$$

and the lemma is equivalent to the statement that $\mathrm{pr}_\nu\big(M \otimes N\big) \cong 0$ in the regular quotient $\underline{\mathrm{Rep}}(\mathbf{G})$, for all weights $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\nu \neq \omega \cdot \lambda$. Observe that all terms of $C_{\min}(M)$ belong to $\mathrm{Rep}_\lambda(\mathbf{G})$ and all terms of $C_{\min}(N)$ belong to $\mathrm{Rep}_{\omega \cdot 0}(\mathbf{G})$ by Lemma 2.2. As $C_{\min}\big(\mathrm{pr}_\nu(M \otimes N)\big)$ admits a split monomorphism into the complex $\mathrm{pr}_\nu\big(C_{\min}(M) \otimes C_{\min}(N)\big)$ by part (4) of Lemma 2.1, it suffices to prove that $\mathrm{pr}_\nu\big(T(x \cdot \lambda) \otimes T(y\omega \cdot 0)\big)$ is negligible for all $x, y \in W_{\mathrm{aff}}^+$ and $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\nu \neq \omega \cdot \lambda$. If $x \neq e$ or $y \neq e$ then $T(x \cdot \lambda) \otimes T(y\omega \cdot 0)$ is negligible, because the negligible tilting modules form a thick tensor ideal in $\mathrm{Tilt}(\mathbf{G})$. For $x = y = e$, we have $T(\lambda) \otimes T(\omega \cdot 0) \cong T(\omega \cdot \lambda)$ in the fusion category $\mathrm{Tilt}(\mathbf{G})/\mathcal{N}$ by Lemma I.9.2, and it follows that $\mathrm{pr}_\nu\big(T(\lambda) \otimes T(\omega \cdot 0)\big)$ is negligible for all $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\nu \neq \omega \cdot \lambda$, as required. □

For $\lambda \in C_{\mathrm{fund}} \cap X$, let us write $\underline{\mathrm{Rep}}_\lambda(\mathbf{G})$ for the essential image of the linkage class $\mathrm{Rep}_\lambda(\mathbf{G})$ under the quotient functor $q\colon \mathrm{Rep}(\mathbf{G}) \to \underline{\mathrm{Rep}}(\mathbf{G})$, i.e. the full subcategory of $\underline{\mathrm{Rep}}(\mathbf{G})$ whose objects are the $\mathbf{G}$-modules that are isomorphic to a $\mathbf{G}$-module in $\mathrm{Rep}_\lambda(\mathbf{G})$, when considered as objects in $\underline{\mathrm{Rep}}(\mathbf{G})$. We also write $\underline{\mathrm{Rep}}_{\Omega \cdot 0}(\mathbf{G})$ for the essential image of the extended principal block $\mathrm{Rep}_{\Omega \cdot 0}(\mathbf{G})$ in $\underline{\mathrm{Rep}}(\mathbf{G})$. As a consequence of Lemma 4.4, we obtain our 'linkage principle' for tensor products.

Corollary 4.5. *The subcategories* $\underline{\mathrm{Rep}}_0(\mathbf{G})$ *and* $\underline{\mathrm{Rep}}_{\Omega\cdot 0}(\mathbf{G})$ *are closed under tensor products.*

Proof. For $\omega, \omega' \in \Omega$ and **G**-modules M and N in the linkage classes of $\omega \cdot 0$ and $\omega' \cdot 0$, respectively, we have $M \otimes N \cong \mathrm{pr}_{\omega\omega'\cdot 0}(M \otimes N)$ in $\underline{\mathrm{Rep}}(\mathbf{G})$ by Lemma 4.4, so $M \otimes N$ belongs to $\underline{\mathrm{Rep}}_{\omega\omega'\cdot 0}(\mathbf{G})$. The claim about $\underline{\mathrm{Rep}}_0(\mathbf{G})$ follows by setting $\omega = \omega' = e$. □

As a further consequence of Lemma 4.4, we prove that a translation functor with source in the extended principal block descends in the regular quotient to tensoring with a tilting module.

Corollary 4.6. *Let* $\lambda \in C_{\mathrm{fund}} \cap X$ *and* $\omega \in \Omega$. *Then* λ *is the unique dominant weight in the* W_{fin}*-orbit of* $\omega \cdot \lambda - \omega \cdot 0$, *and the canonical natural transformations*

$$T_{\omega\cdot 0}^{\omega\cdot\lambda} = \mathrm{pr}_{\omega\cdot\lambda}\big(T(\lambda) \otimes -\big) \implies \big(T(\lambda) \otimes -\big) \implies \mathrm{pr}_{\omega\cdot\lambda}\big(T(\lambda) \otimes -\big) = T_{\omega\cdot 0}^{\omega\cdot\lambda}$$

of functors from $\mathrm{Rep}_{\omega\cdot 0}(\mathbf{G})$ *to* $\mathrm{Rep}(\mathbf{G})$ *give rise to an isomorphism of functors*

$$q \circ T_{\omega\cdot 0}^{\omega\cdot\lambda} \cong q \circ \big(T(\lambda) \otimes -\big).$$

Proof. Writing $\omega = t_\gamma w$ with $\gamma \in X$ and $w \in W_{\mathrm{fin}}$, it is straightforward to see that $\omega \cdot \lambda - \omega \cdot 0 = w(\lambda)$, so λ is indeed the unique dominant weight in the W_{fin}-orbit of $\omega \cdot \lambda - \omega \cdot 0$. By Lemma 4.4, the component at a **G**-module N in $\mathrm{Rep}_{\omega\cdot 0}(\mathbf{G})$ of either of the two natural transformations descends to an isomorphism in $\underline{\mathrm{Rep}}(\mathbf{G})$, and the claim follows. □

We are now ready to establish our 'translation principle' for tensor products.

Theorem 4.7. *For* $\lambda, \mu \in C_{\mathrm{fund}} \cap X$ *and* $\omega, \omega' \in \Omega$, *there is a natural transformation of bifunctors*

$$\Psi \colon \big(T_{\omega\cdot 0}^{\omega\cdot\lambda} - \big) \otimes \big(T_{\omega'\cdot 0}^{\omega'\cdot\mu} - \big) \implies \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_{\omega\omega'\cdot 0}^{\omega\omega'\cdot\nu} \circ \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big)^{\oplus c_{\lambda,\mu}^{\nu}}$$

from $\mathrm{Rep}_{\omega\cdot 0}(\mathbf{G}) \times \mathrm{Rep}_{\omega'\cdot 0}(\mathbf{G})$ *to* $\mathrm{Rep}(\mathbf{G})$, *where* $c_{\lambda,\mu}^{\nu} = [T(\lambda) \otimes T(\mu) : T(\nu)]_{\oplus}$, *such that* $q\Psi$ *is an isomorphism of bifunctors.*

Proof. We construct the natural transformation in several steps.

(1) By Corollary 4.6, the natural embedding

$$\big(T_{\omega\cdot 0}^{\omega\cdot\lambda} - \big) \otimes \big(T_{\omega'\cdot 0}^{\omega'\cdot\mu} - \big) = \mathrm{pr}_{\omega\cdot\lambda}\big(T(\lambda) \otimes -\big) \otimes \mathrm{pr}_{\omega'\cdot\mu}\big(T(\mu) \otimes -\big) \implies \big(T(\lambda) \otimes -\big) \otimes \big(T(\mu) \otimes -\big)$$

induces an isomorphism of functors upon passage to the regular quotient $\underline{\mathrm{Rep}}(\mathbf{G})$.

(2) The braiding on $\mathrm{Rep}(\mathbf{G})$ gives rise to a natural isomorphism

$$\big(T(\lambda) \otimes -\big) \otimes \big(T(\mu) \otimes -\big) \cong \big(T(\lambda) \otimes T(\mu)\big) \otimes (- \otimes -).$$

(3) The canonical projection to the linkage class of $\omega\omega' \cdot 0$ gives rise to a natural transformation

$$(- \otimes -) \implies \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)$$

of bifunctors from $\mathrm{Rep}_{\omega\cdot 0}(\mathbf{G}) \times \mathrm{Rep}_{\omega'\cdot 0}(\mathbf{G})$ to $\mathrm{Rep}(\mathbf{G})$, which descends to a natural isomorphism in $\underline{\mathrm{Rep}}(\mathbf{G})$ by Lemma 4.4. Tensoring with $T(\lambda) \otimes T(\mu)$ yields a natural transformation

$$\big(T(\lambda) \otimes T(\mu)\big) \otimes (- \otimes -) \implies \big(T(\lambda) \otimes T(\mu)\big) \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -),$$

which again descends to a natural isomorphism in $\underline{\mathrm{Rep}}(\mathbf{G})$.

(4) The tensor product $T(\lambda) \otimes T(\mu)$ can be decomposed as a direct sum

$$T(\lambda) \otimes T(\mu) \cong N \oplus \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T(\nu)^{\oplus c_{\lambda,\mu}^{\nu}}$$

of its fusion part and a negligible tilting module N. This decomposition gives rise to a natural isomorphism

$$\big(T(\lambda) \otimes T(\mu)\big) \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -) \cong \big(N \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big) \oplus \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T(\nu) \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big)^{\oplus c_{\lambda,\mu}^{\nu}}.$$

As N is negligible, the essential image of the bifunctor $N \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)$ is contained in $\langle \mathcal{N} \rangle$, and it follows that $q \circ \big(N \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big) = 0$. Therefore, the projection onto the fusion part gives rise to a natural transformation

$$\big(T(\lambda) \otimes T(\mu)\big) \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -) \Longrightarrow \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T(\nu) \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big)^{\oplus c_{\lambda,\mu}^{\nu}},$$

which descends to an isomorphism of functors in $\underline{\mathrm{Rep}}(\mathbf{G})$.

(5) Again by Corollary 4.6, the canonical natural transformation

$$\bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T(\nu) \otimes \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big)^{\oplus c_{\lambda,\mu}^{\nu}} \Longrightarrow \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_{\omega\omega'\cdot 0}^{\omega\omega'\cdot \nu} \circ \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big)^{\oplus c_{\lambda,\mu}^{\nu}}$$

induces an isomorphism of functors upon passage to the regular quotient.

All of the natural transformations in (1)–(5) give rise to natural isomorphisms upon passage to the regular quotient $\underline{\mathrm{Rep}}(\mathbf{G})$. Therefore, their composition is a natural transformation

$$\Psi \colon \big(T_{\omega\cdot 0}^{\omega\cdot\lambda} - \big) \otimes \big(T_{\omega'\cdot 0}^{\omega'\cdot\mu} - \big) \Longrightarrow \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_{\omega\omega'\cdot 0}^{\omega\omega'\cdot \nu} \circ \mathrm{pr}_{\omega\omega'\cdot 0}(- \otimes -)\big)^{\oplus c_{\lambda,\mu}^{\nu}}$$

such that $q\Psi$ is a natural isomorphism. □

Remark 4.8. The statement of Theorem 4.7 becomes more readable (but also slightly less general) if we set $\omega = \omega' = e$: For $\lambda, \mu \in C_{\mathrm{fund}} \cap X$, there is a natural transformation of bifunctors

$$\Psi \colon \big(T_0^{\lambda} - \big) \otimes \big(T_0^{\mu} - \big) \Longrightarrow \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_0^{\nu} \circ \mathrm{pr}_0(- \otimes -)\big)^{\oplus c_{\lambda,\mu}^{\nu}}$$

from $\mathrm{Rep}_0(\mathbf{G}) \times \mathrm{Rep}_0(\mathbf{G})$ to $\mathrm{Rep}(\mathbf{G})$, such that $q\Psi$ is an isomorphism of bifunctors. Taking the action of Ω into account complicates our notation here, but it will be very useful in the following chapters.

Remark 4.9. Let us briefly explain why we think of Corollary 4.5 and Theorem 4.7 as a 'linkage principle' and a 'translation principle' for tensor products. The usual linkage principle asserts that the category $\mathrm{Rep}(\mathbf{G})$ decomposes into linkage classes, and the usual translation principle establishes equivalences between the different ℓ-regular linkage classes. Thus, many questions about the structure of the category $\mathrm{Rep}(\mathbf{G})$ can be reduced to questions about the principal block $\mathrm{Rep}_0(\mathbf{G})$. However, this strategy fails for two reasons when one tries to take the monoidal structure of $\mathrm{Rep}(\mathbf{G})$ into account. Firstly, the principal block is not closed under tensor products. In fact, the tensor product of two $\mathbf{G}$-modules in $\mathrm{Rep}_0(\mathbf{G})$ can have non-zero indecomposable direct summands in many different linkage classes, including ℓ-singular ones. Secondly, it is a priori not clear how structural information about tensor products of $\mathbf{G}$-modules in the principal block can be used to deduce (precise) structural information about tensor products of $\mathbf{G}$-modules in arbitrary ℓ-regular linkage classes.

The preceding results show that both of these obstacles can be partially resolved by passing to the regular quotient. Indeed, Corollary 4.5 tells us that the essential image $\underline{\mathrm{Rep}}_0(\mathbf{G})$ of the principal block in the regular quotient is closed under tensor products; hence, the decomposition of $\mathrm{Rep}(\mathbf{G})$ into linkage classes is, to some extent, compatible with the monoidal strucure of $\mathrm{Rep}(\mathbf{G})$. Furthermore, Theorem 4.7 enables us to describe (the regular parts of) tensor products of $\mathbf{G}$-modules in arbitrary ℓ-regular linkage classes, once we know the structure of (the components in $\mathrm{Rep}_0(\mathbf{G})$ of) tensor products of $\mathbf{G}$-modules in $\mathrm{Rep}_0(\mathbf{G})$. The reader should note, however, that all information about singular direct summands is lost in the process.

In the following, we present a second approach to the 'linkage principle' and the 'translation principle' for tensor products, which largely bypasses the quotient category $\underline{\mathrm{Rep}}(\mathbf{G})$, but also loses the functoriality of Theorem 4.7. When studying tensor product of specific $\mathbf{G}$-modules, rather than categorical properties of $\mathrm{Rep}(\mathbf{G})$, this second approach will turn out to be more convenient.

Definition 4.10. For a $\mathbf{G}$-module M, we write $M \cong M_{\mathrm{sing}} \oplus M_{\mathrm{reg}}$, where for a fixed Krull-Schmidt decomposition of M, we define M_{sing} to be the direct sum of the singular indecomposable direct summands of M and M_{reg} to be the direct sum of the regular indecomposable direct summands of M. We call M_{sing} the *singular part* of M and M_{reg} the *regular part* of M.

Note that the decomposition $M \cong M_{\mathrm{sing}} \oplus M_{\mathrm{reg}}$ in the previous definition is neither canonical nor functorial. Nevertheless, the singular part and the regular part are uniquely determined up to isomorphism by the Krull-Schmidt decomposition of M.

Lemma 4.11. *For $\mathbf{G}$-modules M and N, we have*

$$M_{\mathrm{reg}} \oplus N_{\mathrm{reg}} \cong (M \oplus N)_{\mathrm{reg}} \qquad \text{and} \qquad (M \otimes N)_{\mathrm{reg}} \cong \big(M_{\mathrm{reg}} \otimes N_{\mathrm{reg}}\big)_{\mathrm{reg}}.$$

Proof. The first isomorphism is straightforward to see from the definition. The second one follows from the direct sum decomposition

$$\begin{aligned} M \otimes N &\cong (M_{\mathrm{reg}} \oplus M_{\mathrm{sing}}) \otimes (N_{\mathrm{reg}} \oplus N_{\mathrm{sing}}) \\ &\cong (M_{\mathrm{reg}} \otimes N_{\mathrm{reg}}) \oplus (M_{\mathrm{reg}} \otimes N_{\mathrm{sing}}) \oplus (M_{\mathrm{sing}} \otimes N_{\mathrm{reg}}) \oplus (M_{\mathrm{sing}} \otimes N_{\mathrm{sing}}) \end{aligned}$$

and the fact that singular $\mathbf{G}$-modules form a thick tensor ideal. □

The following lemma can be seen as another version of the 'linkage principle' for tensor products.

Lemma 4.12. *Let $\lambda \in C_{\mathrm{fund}} \cap X$ and $\omega \in \Omega$, and let M and N be $\mathbf{G}$-modules such that M_{reg} belongs to $\mathrm{Rep}_\lambda(\mathbf{G})$ and N_{reg} belongs to $\mathrm{Rep}_{\omega\cdot 0}(\mathbf{G})$. Then $(M \otimes N)_{\mathrm{reg}}$ belongs to $\mathrm{Rep}_{\omega\cdot\lambda}(\mathbf{G})$.*

Proof. By Lemma 4.11 and the linkage principle, we have

$$(M \otimes N)_{\mathrm{reg}} = \big(M_{\mathrm{reg}} \otimes N_{\mathrm{reg}}\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in \overline{C}_{\mathrm{fund}} \cap X} \big(\mathrm{pr}_\nu(M_{\mathrm{reg}} \otimes N_{\mathrm{reg}})\big)_{\mathrm{reg}},$$

and it suffices to show that $\mathrm{pr}_\nu(M_{\mathrm{reg}} \otimes N_{\mathrm{reg}})$ is singular for all $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\nu \neq \omega \cdot \lambda$. This was already observed in the proof of Lemma 4.4, for arbitrary $\mathbf{G}$-modules in the linkage classes $\mathrm{Rep}_\lambda(\mathbf{G})$ and $\mathrm{Rep}_{\omega\cdot 0}(\mathbf{G})$. □

Next we give a reformulation of Corollary 4.6 in terms of regular parts of $\mathbf{G}$-modules.

Corollary 4.13. *Let $\lambda \in C_{\mathrm{fund}} \cap X$ and $\omega \in \Omega$, and let M be a $\mathbf{G}$-module in $\mathrm{Rep}_{\omega \cdot 0}(\mathbf{G})$. Then*

$$\big(T(\lambda) \otimes M\big)_{\mathrm{reg}} \cong \big(T_{\omega \cdot 0}^{\omega \cdot \lambda} M\big)_{\mathrm{reg}}.$$

Proof. Recall from Corollary 4.6 that λ is the unique dominant weight in the W_{fin}-orbit of $\omega \cdot \lambda - \omega \cdot 0$, so $T_{\omega \cdot 0}^{\omega \cdot \lambda} = \mathrm{pr}_{\omega \cdot \lambda}\big(T(\lambda) \otimes -\big)$. Using Lemma 4.12, we obtain

$$\big(T(\lambda) \otimes M\big)_{\mathrm{reg}} \cong \mathrm{pr}_{\omega \cdot \lambda}\Big(\big(T(\lambda) \otimes M\big)_{\mathrm{reg}}\Big) \cong \Big(\mathrm{pr}_{\omega \cdot \lambda}\big(T(\lambda) \otimes M\big)\Big)_{\mathrm{reg}} \cong \big(T_{\omega \cdot 0}^{\omega \cdot \lambda} M\big)_{\mathrm{reg}},$$

as required. □

The following result is a non-functorial version of the 'translation principle' for tensor products from Theorem 4.7.

Theorem 4.14. *Let $\lambda, \mu \in C_{\mathrm{fund}} \cap X$ and let M and N be $\mathbf{G}$-modules in the linkage classes $\mathrm{Rep}_{\omega \cdot 0}(\mathbf{G})$ and $\mathrm{Rep}_{\omega' \cdot 0}(\mathbf{G})$, respectively, for certain $\omega, \omega' \in \Omega$. Then*

$$\big(T_{\omega \cdot 0}^{\omega \cdot \lambda} M \otimes T_{\omega' \cdot 0}^{\omega' \cdot \mu} N\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_{\omega\omega' \cdot 0}^{\omega\omega' \cdot \nu} (M \otimes N)_{\mathrm{reg}}\big)^{\oplus c_{\lambda,\mu}^{\nu}}$$

Proof. By Lemma 4.11 and Corollary 4.13, we have

$$\begin{aligned}
\big(T_{\omega \cdot 0}^{\omega \cdot \lambda} M \otimes T_{\omega' \cdot 0}^{\omega' \cdot \mu} N\big)_{\mathrm{reg}} &\cong \Big(\big(T_{\omega \cdot 0}^{\omega \cdot \lambda} M\big)_{\mathrm{reg}} \otimes \big(T_{\omega' \cdot 0}^{\omega' \cdot \mu} N\big)_{\mathrm{reg}}\Big)_{\mathrm{reg}} \\
&\cong \Big(\big(T(\lambda) \otimes M\big)_{\mathrm{reg}} \otimes \big(T(\mu) \otimes N\big)_{\mathrm{reg}}\Big)_{\mathrm{reg}} \\
&\cong \big(T(\lambda) \otimes M \otimes T(\mu) \otimes N\big)_{\mathrm{reg}} \\
&\cong \Big(\big(T(\lambda) \otimes T(\mu)\big)_{\mathrm{reg}} \otimes \big(M \otimes N\big)_{\mathrm{reg}}\Big)_{\mathrm{reg}}.
\end{aligned}$$

Now

$$\big(T(\lambda) \otimes T(\mu)\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T(\nu)^{\oplus c_{\lambda,\mu}^{\nu}}$$

and $(M \otimes N)_{\mathrm{reg}}$ belongs to the linkage class $\mathrm{Rep}_{\omega\omega' \cdot 0}(\mathbf{G})$ by Lemma 4.12. Again using Corollary 4.13, we obtain

$$\big(T(\nu) \otimes (M \otimes N)_{\mathrm{reg}}\big)_{\mathrm{reg}} \cong \big(T_{\omega\omega' \cdot 0}^{\omega\omega' \cdot \nu} (M \otimes N)_{\mathrm{reg}}\big)_{\mathrm{reg}} \cong T_{\omega\omega' \cdot 0}^{\omega\omega' \cdot \nu} (M \otimes N)_{\mathrm{reg}}$$

for all $\nu \in C_{\mathrm{fund}} \cap X$, and we conclude that

$$\begin{aligned}
\big(T_{\omega \cdot 0}^{\omega \cdot \lambda} M \otimes T_{\omega' \cdot 0}^{\omega' \cdot \mu} N\big)_{\mathrm{reg}} &\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T(\nu) \otimes (M \otimes N)_{\mathrm{reg}}\big)_{\mathrm{reg}}^{\oplus c_{\lambda,\mu}^{\nu}} \\
&\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_{\omega\omega' \cdot 0}^{\omega\omega' \cdot \nu} (M \otimes N)_{\mathrm{reg}}\big)^{\oplus c_{\lambda,\mu}^{\nu}},
\end{aligned}$$

as claimed. □

For applications in the following chapters, let us briefly explain how the action of Ω can be used to compare the regular parts of tensor products of $\mathbf{G}$-modules with constituents belonging to different linkage classes in the extended principal block $\mathrm{Rep}_{\Omega \cdot 0}(\mathbf{G})$. For $\omega \in \Omega$, consider the auto-equivalence

$$T^{\omega} := \bigoplus_{\lambda \in \Omega \cdot 0} T_{\lambda}^{\omega \cdot \lambda}$$

of $\mathrm{Rep}_{\Omega \cdot 0}(\mathbf{G})$.

Lemma 4.15. *Let M and N be* **G**-*modules in* $\mathrm{Rep}_{\Omega\cdot 0}(\mathbf{G})$ *and let* $\omega, \omega' \in \Omega$. *Then*

$$\big(T^{\omega} M \otimes T^{\omega'} N\big)_{\mathrm{reg}} \cong T^{\omega\omega'}(M \otimes N)_{\mathrm{reg}}.$$

Proof. We could deduce this as a special case of Theorem 4.14 where λ and μ belong to $\Omega \cdot 0$, but to avoid excessive indexing, we prefer to prove the claim directly. (The reader will note that the proof is also just a special case of the proof of Theorem 4.14.) By Lemma 4.11 and Corollary 4.13, we have

$$\begin{aligned}
\big(T^{\omega} M \otimes T^{\omega'} N\big)_{\mathrm{reg}} &\cong \big((T^{\omega} M)_{\mathrm{reg}} \otimes (T^{\omega'} N)_{\mathrm{reg}}\big)_{\mathrm{reg}} \\
&\cong \Big(\big(T(\omega\cdot 0) \otimes M\big)_{\mathrm{reg}} \otimes \big(T(\omega'\cdot 0) \otimes N\big)_{\mathrm{reg}}\Big)_{\mathrm{reg}} \\
&\cong \big(T(\omega\cdot 0) \otimes M \otimes T(\omega'\cdot 0) \otimes N\big)_{\mathrm{reg}} \\
&\cong \Big(\big(T(\omega\cdot 0) \otimes T(\omega'\cdot 0)\big)_{\mathrm{reg}} \otimes \big(M \otimes N\big)_{\mathrm{reg}}\Big)_{\mathrm{reg}},
\end{aligned}$$

where $\big(T(\omega\cdot 0) \otimes T(\omega'\cdot 0)\big)_{\mathrm{reg}} \cong T(\omega\omega'\cdot 0)$ by Lemma I.9.2. Again using Corollary 4.13, we obtain

$$\big(T(\omega\omega'\cdot 0) \otimes (M \otimes N)_{\mathrm{reg}}\big)_{\mathrm{reg}} \cong \big(T^{\omega\omega'}(M \otimes N)_{\mathrm{reg}}\big)_{\mathrm{reg}} \cong T^{\omega\omega'}(M \otimes N)_{\mathrm{reg}},$$

and the claim follows. □

For certain applications, it will be important to decide if the tensor product $M \otimes N$ of two regular **G**-modules M and N is regular. Though we are not aware of any examples where this is not the case, we were not able to prove it in general (but see Remark 4.19 below). To overcome this problem, we introduce the notion of *strong regularity.*

Definition 4.16. A **G**-module M with $\mathrm{gfd}(M) = d$ is called *strongly regular* if $C_{\min}(M)_{d-1}$ is negligible and $C_{\min}(M)_d$ is non-negligible.

Remark 4.17. Observe that, for $x \in W_{\mathrm{ext}}^{+}$ and $\lambda \in C_{\mathrm{fund}} \cap X$, the Weyl module $\Delta(x\cdot\lambda)$ and the simple **G**-module $L(x\cdot\lambda)$ are both strongly regular. Indeed, by the description of the minimal tilting complexes of $\Delta(x\cdot\lambda)$ and $L(x\cdot\lambda)$ in Propositions 2.5 and 2.6, the tilting modules $C_{\min}\big(\Delta(x\cdot\lambda)\big)_{\ell(x)-1}$ and $C_{\min}\big(L(x\cdot\lambda)\big)_{\ell(x)-1}$ are negligible and we have

$$C_{\min}\big(\Delta(x\cdot\lambda)\big)_{\ell(x)} \cong C_{\min}\big(L(x\cdot\lambda)\big)_{\ell(x)} \cong T(\omega_x\cdot\lambda),$$

where $\ell(x) = \mathrm{gfd}\big(\Delta(x\cdot\lambda)\big) = \mathrm{gfd}\big(L(x\cdot\lambda)\big)$ by Corollary 2.7.

Our interest in strongly regular **G**-modules is founded in the following result:

Lemma 4.18. *Let M and N be strongly regular* **G**-*modules. Then $M \otimes N$ is strongly regular and*

$$\mathrm{gfd}(M \otimes N) = \mathrm{gfd}(M) + \mathrm{gfd}(N).$$

Proof. Set $d = \mathrm{gfd}(M)$ and $d' = \mathrm{gfd}(N)$ and note that by Lemma 2.3, we have $C_{\min}(M)_i = 0$ for $i > d$ and $C_{\min}(N)_i = 0$ for $i > d'$. By the definition of strong regularity, there exist $\nu, \nu' \in C_{\mathrm{fund}} \cap X$ such that

$$\big[C_{\min}(M)_d : T(\nu)\big]_{\oplus} \neq 0 \qquad \text{and} \qquad \big[C_{\min}(N)_{d'} : T(\nu')\big]_{\oplus} \neq 0.$$

By Lemma I.9.3, there exists $\delta \in C_{\text{fund}} \cap X$ with $\big[T(\nu) \otimes T(\nu') : T(\delta)\big]_\oplus \neq 0$, so $T(\delta)$ appears as a direct summand of the tensor product $C_{\min}(M)_d \otimes C_{\min}(N)_{d'}$, which is the degree $d + d'$ term of the tensor product complex $C_{\min}(M) \otimes C_{\min}(N)$. Furthermore, the degree $d + d' - 1$ term

$$\big(C_{\min}(M)_{d-1} \otimes C_{\min}(N)_{d'}\big) \oplus \big(C_{\min}(M)_d \otimes C_{\min}(N)_{d'-1}\big)$$

of the tensor product complex is negligible, and the terms in degree $i > d + d'$ of the tensor product complex are zero. Now $C_{\min}(M \otimes N)$ is the minimal complex of $C_{\min}(M) \otimes C_{\min}(N)$ by part(4) of Lemma 2.1, hence $C_{\min}(M \otimes N)_{d+d'-1}$ is negligible and $C_{\min}(M \otimes N)_i = 0$ for $i > d+d'$. As the terms of the tensor product complex $C_{\min}(M) \otimes C_{\min}(N)$ in degrees $d + d' - 1$ and $d + d' + 1$ are negligible or zero, respectively, Corollary 1.8 implies that

$$0 \neq \big[C_{\min}(M)_d \otimes C_{\min}(N)_{d'} : T(\delta)\big]_\oplus = \big[C_{\min}(M \otimes N)_{d+d'} : T(\delta)\big]_\oplus .$$

Finally, Lemma 2.3 yields $\mathrm{gfd}(M \otimes N) = d + d'$, and it follows that $M \otimes N$ is strongly regular. □

Remark 4.19. In the quantum case, we claim that the tensor product $M \otimes N$ of two regular $\mathbf{G}$-modules M and N is always regular, if $\ell > h$. Observe that the claim is equivalent to the statement that singular $\mathbf{G}$-modules form a *prime ideal*, i.e. that the tensor product $M \otimes N$ of two $\mathbf{G}$-modules M and N is singular only if at least one of M and N is singular. As in Remark 3.6, we have

$$\langle \mathcal{N} \rangle = \bigcap_{\langle \mathcal{N} \rangle \subseteq \mathcal{P}} \mathcal{P} = \bigcap_{\langle \mathcal{N} \rangle \subseteq \mathcal{P}} \langle \mathcal{P} \cap \mathrm{Tilt}(\mathbf{G}) \rangle,$$

where the intersection runs over the prime thick tensor ideals $\mathcal{P}$ in $\mathrm{Rep}(\mathbf{G})$ with the 2/3-property such that $\langle \mathcal{N} \rangle \subseteq \mathcal{P}$. For any such tensor ideal $\mathcal{P}$, we have $\mathcal{N} \subseteq \mathcal{P} \cap \mathrm{Tilt}(\mathbf{G})$, and as $\mathcal{N}$ is maximal among the proper thick tensor ideals in $\mathrm{Tilt}(\mathbf{G})$ (by Lemma I.9.3), it follows that $\mathcal{N} = \mathcal{P} \cap \mathrm{Tilt}(\mathbf{G})$ and

$$\langle \mathcal{N} \rangle = \langle \mathcal{P} \cap \mathrm{Tilt}(\mathbf{G}) \rangle = \mathcal{P}$$

is prime, as required. We do not know if the analogous statement is true in the modular case.

The fact that strongly regular $\mathbf{G}$-modules M and N satisfy $\mathrm{gfd}(M \otimes N) = \mathrm{gfd}(M) + \mathrm{gfd}(N)$ (rather than just the inequality from Lemma I.7.2) is a very convenient additional feature of strong regularity. It allows us to prove the following generalization to tensor products of A. Parker's results from [Par03] about the good filtration dimension of Weyl modules and simple $\mathbf{G}$-modules (see Corollary 2.7).

Theorem 4.20. *Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in W_{\mathrm{ext}}^+$ and $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n \in C_{\text{fund}} \cap X$. Then the tensor product*

$$\Delta(x_1 \cdot \lambda_1) \otimes \cdots \otimes \Delta(x_m \cdot \lambda_m) \otimes L(y_1 \cdot \mu_1) \otimes \cdots \otimes L(y_n \cdot \mu_n)$$

is strongly regular and has good filtration dimension $\ell(x_1) + \cdots + \ell(x_m) + \ell(y_1) + \cdots + \ell(y_n)$.

Proof. For $1 \leq i \leq m$ and $1 \leq j \leq n$, the $\mathbf{G}$-modules $\Delta(x_i \cdot \lambda_i)$ and $L(y_j \cdot \mu_j)$ are strongly regular by Remark 4.17, and their good filtration dimensions are given by

$$\mathrm{gfd}\big(\Delta(x_i \cdot \lambda_i)\big) = \ell(x_i) \qquad \text{and} \qquad \mathrm{gfd}\big(L(y_j \cdot \lambda_j)\big) = \ell(y_j)$$

according to Corollary 2.7. The claim follows from Lemma 4.18, by induction on $m + n$. □

We conclude this section by giving a further application of the tensor ideal $\langle \mathcal{N} \rangle$ of singular $\mathbf{G}$-modules, which may be of independent interest. We prove that the composition of two translation functors between ℓ-regular linkage classes is naturally isomorphic to a translation functor. This statement should not be very surprising to experts in the field, but the author is not aware of a proof in the literature.

Proposition 4.21. *Let $\lambda, \mu, \delta \in C_{\mathrm{fund}} \cap X$. Then there is an isomorphism of functors $T_\delta^\mu \cong T_\lambda^\mu \circ T_\delta^\lambda$.*

Proof. First suppose that $\delta = 0$ and recall that

$$T_0^\lambda = \mathrm{pr}_\lambda\big(\nabla(\lambda) \otimes -\big), \qquad T_0^\mu = \mathrm{pr}_\mu\big(\nabla(\mu) \otimes -\big) \qquad \text{and} \qquad T_\lambda^\mu = \mathrm{pr}_\mu\big(\nabla(\nu) \otimes -\big),$$

where ν is the unique dominant weight in the W_{fin}-orbit of $\mu - \lambda$. Consider the functor

$$\Psi \coloneqq \mathrm{pr}_\mu\big(\nabla(\nu) \otimes \nabla(\lambda) \otimes -\big),$$

and note that the canonical embedding of functors $T_0^\lambda \Longrightarrow \big(\nabla(\lambda) \otimes -\big)$ gives rise to a natural transformation $T_\lambda^\mu \circ T_0^\lambda \Longrightarrow \Psi$. Furthermore, we have

$$\nabla(\mu) \cong T_\lambda^\mu \nabla(\lambda) = \mathrm{pr}_\mu\big(\nabla(\nu) \otimes \nabla(\lambda)\big)$$

by Proposition I.6.8, and the canonical projection $\nabla(\nu) \otimes \nabla(\lambda) \to \nabla(\mu)$ affords a natural transformation $\Psi \Longrightarrow T_0^\mu$. We claim that the composition of these natural transformations

$$\vartheta \colon T_\lambda^\mu \circ T_0^\lambda \Longrightarrow \Psi \Longrightarrow T_0^\mu$$

is a natural isomorphism.

Let N be a complement of $\nabla(\mu) \cong \mathrm{pr}_\mu\big(\nabla(\nu) \otimes \nabla(\lambda)\big)$ in $\nabla(\lambda) \otimes \nabla(\nu)$, and observe that $\mathrm{pr}_\mu N = 0$. For a $\mathbf{G}$-module M in $\mathrm{Rep}_0(\mathbf{G})$, we have

$$(M \otimes N)_{\mathrm{reg}} \cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} (M \otimes \mathrm{pr}_\nu N)_{\mathrm{reg}}$$

by the linkage principle and Lemma 4.2, and as $(M \otimes \mathrm{pr}_\nu N)_{\mathrm{reg}}$ belongs to $\mathrm{Rep}_\nu(\mathbf{G})$ by Lemma 4.12, we conclude that

$$\mathrm{pr}_\nu (M \otimes N)_{\mathrm{reg}} \cong (M \otimes \mathrm{pr}_\nu N)_{\mathrm{reg}}$$

for all $\nu \in C_{\mathrm{fund}} \cap X$. In particular, the functor $\mathrm{pr}_\mu(N \otimes -)$ maps every $\mathbf{G}$-module in $\mathrm{Rep}_0(\mathbf{G})$ into the tensor ideal $\langle \mathcal{N} \rangle$ of singular $\mathbf{G}$-modules. As Ψ decomposes as the direct sum of the functors T_0^μ and $\mathrm{pr}_\mu(N \otimes -)$, this implies that all components of the natural transformation $\Psi \Longrightarrow T_0^\mu$ descend to isomorphisms in the regular quotient $\underline{\mathrm{Rep}}(\mathbf{G})$. Similarly, the embedding of functors

$$T_0^\lambda \Longrightarrow \big(\nabla(\lambda) \otimes -\big) \cong \big(T(\lambda) \otimes -\big)$$

descends to a natural isomorphism in $\underline{\mathrm{Rep}}(\mathbf{G})$ by Corollary 4.6, and it follows that the same is true for the natural transformation

$$T_\lambda^\mu \circ T_0^\lambda \Longrightarrow \mathrm{pr}_\mu\big(\nabla(\nu) \otimes -\big) \circ \big(\nabla(\lambda) \otimes -\big) = \Psi.$$

In particular, the component of ϑ at any simple $\mathbf{G}$-module $L(x \cdot 0)$ with $x \in W_{\mathrm{aff}}^+$ affords an isomorphism in $\underline{\mathrm{Rep}}(\mathbf{G})$. Now $L(x \cdot 0)$ is non-zero in $\underline{\mathrm{Rep}}(\mathbf{G})$ by Lemma 4.3, whence the endomorphism algebra of $L(x \cdot 0)$ in $\underline{\mathrm{Rep}}(\mathbf{G})$ is also non-zero. Since the latter endomorphism algebra is a quotient of the endomorphism algebra of $L(x \cdot 0)$ in $\mathrm{Rep}(\mathbf{G})$, we conclude that the component of ϑ at $L(x \cdot 0)$ is non-zero; hence it affords an isomorphism between $T_\lambda^\mu T_0^\lambda L(x \cdot 0) \cong L(x \cdot \mu)$ and $T_0^\mu L(x \cdot 0) \cong L(x \cdot \mu)$, by Schur's Lemma. Using the snake Lemma and induction on the length of a composition series, one easily deduces that the component of ϑ at every $\mathbf{G}$-module in $\mathrm{Rep}_0(\mathbf{G})$ is an isomorphism, so ϑ is a natural isomorphism, as claimed.

Now since $T_0^\lambda \circ T_\lambda^0$ is isomorphic to the identity functor on $\mathrm{Rep}_\lambda(\mathbf{G})$, we further obtain isomorphisms of functors

$$T_\lambda^\mu \cong T_\lambda^\mu \circ T_0^\lambda \circ T_\lambda^0 \cong T_0^\mu \circ T_\lambda^0.$$

For arbitrary $\delta \in C_{\mathrm{fund}} \cap X$, we conclude that

$$T_\mu^\delta \circ T_\lambda^\mu \cong T_0^\delta \circ T_\mu^0 \circ T_0^\mu \circ T_\lambda^0 \cong T_0^\delta \circ T_\lambda^0 \cong T_\lambda^\delta,$$

as required. □

5 Generic direct summands

In this section, we study the regular indecomposable direct summands of tensor products of specific $\mathbf{G}$-modules (such as Weyl modules and simple modules). Our knowledge about the minimal tilting complexes of these $\mathbf{G}$-modules allows us to show that (certain) regular indecomposable direct summands of the corresponding tensor products are essentially unique. Recall that we write $x \mapsto \omega_x$ for the canonical epimorphism $W_{\mathrm{ext}} = W_{\mathrm{aff}} \rtimes \Omega \to \Omega$ with kernel W_{aff} and that we assume $\ell \geq h$.

Proposition 5.1. *Let $x, y \in W_{\mathrm{ext}}^+$. Then the tensor product $\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)$ has a unique regular indecomposable direct summand $G_\Delta(x, y)$. Furthermore, $G_\Delta(x, y)$ belongs to the linkage class of $\omega_{xy} \cdot 0$ and satisfies*

$$\mathrm{gfd}\big(G_\Delta(x, y)\big) = \ell(x) + \ell(y).$$

Proof. Recall from Lemma 2.1 that $C_{\min}\big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big)$ is the minimal complex of the tensor product complex $C_{\min}\big(\Delta(x \cdot 0)\big) \otimes C_{\min}\big(\Delta(y \cdot 0)\big)$. By Proposition 2.5, the terms $C_{\min}\big(\Delta(x \cdot 0)\big)_i$ of the minimal tilting complex of $\Delta(x \cdot 0)$ are negligible for $i < \ell(x)$ and zero for $i > \ell(x)$, and we have

$$C_{\min}\big(\Delta(x \cdot 0)\big)_{\ell(x)} \cong T(\omega_x \cdot 0).$$

Analogously, we have

$$C_{\min}\big(\Delta(y \cdot 0)\big)_{\ell(y)} \cong T(\omega_y \cdot 0),$$

and $C_{\min}\big(\Delta(y \cdot 0)\big)_i$ is negligible for $i < \ell(y)$ and zero for $i > \ell(y)$.

Combining the above observations, we see that the terms $C_{\min}\big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big)_i$ of the minimal tilting complex of $\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)$ are negligible for $i < \ell(x) + \ell(y)$ and zero for $i > \ell(x) + \ell(y)$. Furthermore, the term in degree $\ell(x) + \ell(y)$ is a direct summand of the tensor product of tilting modules

$$C_{\min}\big(\Delta(x \cdot 0)\big)_{\ell(x)} \otimes C_{\min}\big(\Delta(y \cdot 0)\big)_{\ell(y)} \cong T(\omega_x \cdot 0) \otimes T(\omega_y \cdot 0),$$

which is isomorphic to $T(\omega_x \omega_y \cdot 0) = T(\omega_{xy} \cdot 0)$ in the fusion category $\mathrm{Tilt}(\mathbf{G})/\mathcal{N}$ by Lemma I.9.2. As the terms of the tensor product complex $C_{\min}\big(\Delta(x \cdot 0)\big) \otimes C_{\min}\big(\Delta(y \cdot 0)\big)$ in degrees $\ell(x) + \ell(y) - 1$ and $\ell(x) + \ell(y) + 1$ are negligible or zero, respectively, Corollary 1.8 implies that

$$\big[C_{\min}\big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big)_{\ell(x)+\ell(y)} : T(\omega_{xy} \cdot 0)\big]_\oplus = 1,$$

and $T(\omega_{xy} \cdot 0)$ is the unique non-negligible indecomposable direct summand of the degree $\ell(x) + \ell(y)$ term of $C_{\min}\big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big)$.

Now fix a Krull-Schmidt decomposition

$$\Delta(x \cdot 0) \otimes \Delta(y \cdot 0) \cong M_1 \oplus \cdots \oplus M_r$$

and note that

$$C_{\min}\big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big) \cong C_{\min}(M_1) \oplus \cdots \oplus C_{\min}(M_r).$$

Hence there exists a unique $k \in \{1, \ldots, r\}$ with $[C_{\min}(M_k)_{\ell(x)+\ell(y)} : T(\omega_{xy} \cdot 0)]_\oplus \neq 0$, and all of the terms $C_{\min}(M_i)_j$ of the minimal tilting complexes $C_{\min}(M_i)$ are negligible for $i \neq k$ (or $i = k$ and $j < \ell(x) + \ell(y)$) and zero for $j > \ell(x) + \ell(y)$. In particular, M_k is the unique regular indecomposable direct summand of $\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)$, M_k belongs to the linkage class of $\omega_{xy} \cdot 0$ by Lemma 2.2 and we have $\mathrm{gfd}(M_k) = \ell(x) + \ell(y)$ by Lemma 2.3. □

Remark 5.2. For $x, y \in W^+_{\mathrm{ext}}$, one can show as in Proposition 5.1 that $\nabla(x \cdot 0) \otimes \nabla(y \cdot 0)$ has a unique regular indecomposable direct summand $G_\nabla(x, y)$. Furthermore, $G_\nabla(x, y)$ satisfies

$$\mathrm{wfd}\big(G_\nabla(x, y)\big) = \ell(x) + \ell(y)$$

and belongs to the linkage class of $\omega_{xy} \cdot 0$. In the following, we will mostly restrict our attention to one of the classes of modules $G_\Delta(x, y)$ or $G_\nabla(x, y)$, which is justified by the fact that $G_\nabla(x, y) \cong G_\Delta(x, y)^\tau$. Indeed, $G_\Delta(x, y)^\tau$ is a direct summand of

$$\big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big)^\tau \cong \nabla(x \cdot 0) \otimes \nabla(y \cdot 0),$$

and $G_\Delta(x, y)^\tau$ is regular since $C_{\min}\big(G_\Delta(x, y)^\tau\big)_i \cong C_{\min}\big(G_\Delta(x, y)\big)_{-i}$ for all $i \in \mathbb{Z}$.

Definition 5.3. For $x, y \in W^+_{\mathrm{ext}}$, we call the $\mathbf{G}$-module $G_\Delta(x, y)$ from Proposition 5.1 the *generic direct summand* of $\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)$. Analogously, we call the $\mathbf{G}$-module $G_\nabla(x, y)$ from Remark 5.2 the *generic direct summand* of $\nabla(x \cdot 0) \otimes \nabla(y \cdot 0)$.

Remark 5.4. The term *generic direct summand* is justified by the fact that translates of $G_\Delta(x, y)$ appear 'generically' in Krull-Schmidt decompositions of tensor products of Weyl modules with highest weights in the alcoves $x \cdot C_{\mathrm{fund}}$ and $y \cdot C_{\mathrm{fund}}$: For $\lambda, \mu \in C_{\mathrm{fund}}$ and $x, y \in W^+_{\mathrm{aff}}$, we have

$$\begin{aligned}
\big(\Delta(x \cdot \lambda) \otimes \Delta(y \cdot \mu)\big)_{\mathrm{reg}} &\cong \big(T_0^\lambda \Delta(x \cdot 0) \otimes T_0^\mu \Delta(y \cdot 0)\big)_{\mathrm{reg}} \\
&\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T_0^\nu \big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big)_{\mathrm{reg}}^{\oplus c_{\lambda,\mu}^\nu} \\
&\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T_0^\nu G_\Delta(x, y)^{\oplus c_{\lambda,\mu}^\nu}
\end{aligned}$$

by Theorem 4.14 and Proposition 5.1.

The following elementary lemma is an immediate consequence of Lemma 4.15.

Lemma 5.5. *Let $x, y \in W^+_{\mathrm{ext}}$ and $\omega, \omega' \in \Omega$. Then*

$$G_\Delta(x\omega, y\omega') \cong T^{\omega\omega'} G_\Delta(x, y) \qquad \textit{and} \qquad G_\nabla(x\omega, y\omega') \cong T^{\omega\omega'} G_\nabla(x, y).$$

Proof. First note that $\Delta(x\omega \cdot 0) \cong T^{\omega}\Delta(x \cdot 0)$ and $\Delta(y\omega' \cdot 0) \cong T^{\omega'}\Delta(y \cdot 0)$. By Proposition 5.1 and Lemma 4.15, we have

$$\begin{aligned} G_{\Delta}(x\omega, y\omega') &\cong \big(\Delta(x\omega \cdot 0) \otimes \Delta(y\omega' \cdot 0)\big)_{\mathrm{reg}} \\ &\cong \big(T^{\omega}\Delta(x \cdot 0) \otimes T^{\omega'}\Delta(y \cdot 0)\big)_{\mathrm{reg}} \\ &\cong T^{\omega\omega'}\big(\Delta(x \cdot 0) \otimes \Delta(y' \cdot 0)\big)_{\mathrm{reg}} \\ &\cong T^{\omega\omega'} G_{\Delta}(x, y), \end{aligned}$$

as claimed. The isomorphism $G_{\nabla}(x\omega, y\omega') \cong T^{\omega\omega'} G_{\nabla}(x, y)$ can be proven analogously. □

Remark 5.6. Observe that the proof of Proposition 5.1 implies that

$$\big[C_{\min}\big(G_{\Delta}(x, y)\big)_{\ell(x)+\ell(y)} : T(\omega_{xy} \cdot 0)\big]_{\oplus} = 1$$

for all $x, y \in W_{\mathrm{ext}}^{+}$. Now suppose that $x, y \in W_{\mathrm{aff}}^{+}$, so that $\Delta(x \cdot 0)$ and $\Delta(y \cdot 0)$ belong to $\mathrm{Rep}_0(\mathbf{G})$. Then we have $C_{\min}\big(\Delta(x \cdot 0)\big)_{\ell(x)} \cong T(0)$ and $C_{\min}\big(\Delta(y \cdot 0)\big)_{\ell(y)} \cong T(0)$ by Proposition 2.5. Arguing as in the proof of Proposition 5.1, we see that

$$C_{\min}\big(\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)\big)_{\ell(x)+\ell(y)} \cong T(0) \otimes T(0) \cong T(0),$$

not only in the fusion category but as actual $\mathbf{G}$-modules, and that

$$C_{\min}\big(G_{\Delta}(x, y)\big)_{\ell(x)+\ell(y)} \cong T(0).$$

Note that this argument also shows that $G_{\Delta}(x, y)$ is the unique indecomposable direct summand of the tensor product $\Delta(x \cdot 0) \otimes \Delta(y \cdot 0)$ with good filtration dimension $\ell(x) + \ell(y)$. For $x, y \in W_{\mathrm{ext}}^{+}$, it follows from Lemma 5.5 that

$$C_{\min}\big(G_{\Delta}(x, y)\big)_{\ell(x)+\ell(y)} \cong T(\omega_{xy} \cdot 0).$$

The regular indecomposable direct summands of tensor products of simple $\mathbf{G}$-modules in the extended principal block are in general not unique. To get uniqueness, we need to impose a condition on the good filtration dimension.

Proposition 5.7. *Let $x, y \in W_{\mathrm{ext}}^{+}$. Then the tensor product $L(x \cdot 0) \otimes L(y \cdot 0)$ has a unique regular indecomposable direct summand $G(x, y)$ with $\mathrm{gfd}\big(G(x, y)\big) = \ell(x) + \ell(y)$. Furthermore, $G(x, y)$ belongs to the linkage class of $\omega_{xy} \cdot 0$.*

Proof. First suppose that $x, y \in W_{\mathrm{aff}}^{+}$. Recall from Proposition 2.6 that the terms $C_{\min}\big(L(x \cdot 0)\big)_i$ of the minimal tilting complex of $L(x \cdot 0)$ are zero for $i > \ell(x)$ and negligible for $i = \ell(x) - 1$, and that

$$C_{\min}\big(L(x \cdot 0)\big)_{\ell(x)} \cong T(0).$$

Analogously, we have

$$C_{\min}\big(L(y \cdot 0)\big)_{\ell(y)} \cong T(0),$$

and $C_{\min}\big(L(y \cdot 0)\big)_i$ is zero for $i > \ell(y)$ and negligible for $i = \ell(x) - 1$. Hence the degree $\ell(x) + \ell(y) - 1$ term of the tensor product complex $C_{\min}\big(L(x \cdot 0)\big) \otimes C_{\min}\big(L(y \cdot 0)\big)$ is the negligible tilting module

$$\begin{aligned} &\Big(C_{\min}\big(L(x \cdot 0)\big) \otimes C_{\min}\big(L(y \cdot 0)\big)\Big)_{\ell(x)+\ell(y)-1} \\ &\qquad = \Big(C_{\min}\big(L(x \cdot 0)\big)_{\ell(x)-1} \otimes C_{\min}\big(L(y \cdot 0)\big)_{\ell(y)}\Big) \oplus \Big(C_{\min}\big(L(x \cdot 0)\big)_{\ell(x)} \otimes C_{\min}\big(L(y \cdot 0)\big)_{\ell(y)-1}\Big), \end{aligned}$$

the degree $\ell(x)+\ell(y)$ term of $C_{\min}\big(L(x\cdot 0)\big)\otimes C_{\min}\big(L(y\cdot 0)\big)$ is

$$C_{\min}\big(L(x\cdot 0)\big)_{\ell(x)}\otimes C_{\min}\big(L(y\cdot 0)\big)_{\ell(y)}\cong T(0)\otimes T(0)\cong T(0),$$

and the terms of $C_{\min}\big(L(x\cdot 0)\big)\otimes C_{\min}\big(L(y\cdot 0)\big)$ in degree $i>\ell(x)+\ell(y)$ are all zero. By part (4) of Lemma 2.1, $C_{\min}\big(L(x\cdot 0)\otimes L(y\cdot 0)\big)$ is the minimal complex of $C_{\min}\big(L(x\cdot 0)\big)\otimes C_{\min}\big(L(y\cdot 0)\big)$, and using Corollary 1.8, we conclude that $C_{\min}\big(L(x\cdot 0)\otimes L(y\cdot 0)\big)_{\ell(x)+\ell(y)}\cong T(0)$. This implies that there is a unique indecomposable direct summand $G(x,y)$ of $L(x\cdot 0)\otimes L(y\cdot 0)$ whose minimal tilting complex has a non-zero term in degree $\ell(x)+\ell(y)$, and the latter satisfies

$$C_{\min}\big(G(x,y)\big)_{\ell(x)+\ell(y)}\cong T(0).$$

In particular, $G(x,y)$ is regular and belongs to the linkage class of 0 by Lemma 2.2, and $G(x,y)$ is the unique indecomposable direct summand of $L(x\cdot 0)\otimes L(y\cdot 0)$ with good filtration dimension $\ell(x)+\ell(y)$ by Lemma 2.3.

For arbitrary elements $x,y\in W_{\mathrm{ext}}^{+}$, we can write $x=x'\omega_x$ and $y=y'\omega_y$ with $x',y'\in W_{\mathrm{aff}}^{+}$, and the claim follows from the previous case because

$$\big(L(x\cdot 0)\otimes L(y\cdot 0)\big)_{\mathrm{reg}}\cong\big(T^{\omega_x}L(x'\cdot 0)\otimes T^{\omega_y}L(y'\cdot 0)\big)_{\mathrm{reg}}\cong T^{\omega_{xy}}\big(L(x'\cdot 0)\otimes L(y'\cdot 0)\big)_{\mathrm{reg}}$$

by Lemma 4.15 (and because the translation functor $T^{\omega_{xy}}$ preserves the good filtration dimension). □

Remark 5.8. For $x,y\in W_{\mathrm{ext}}^{+}$, one can show as in Proposition 5.7 that $L(x\cdot 0)\otimes L(y\cdot 0)$ has a unique regular indecomposable direct summand $G'(x,y)$ with $\mathrm{wfd}\big(G'(x,y)\big)=\ell(x)+\ell(y)$. In the following, we will only study the modules $G(x,y)$, which is justified by the fact that $G'(x,y)\cong G(x,y)^{\tau}$. Indeed, $G(x,y)^{\tau}$ is a direct summand of $\big(L(x\cdot 0)\otimes L(y\cdot 0)\big)^{\tau}\cong L(x\cdot 0)\otimes L(y\cdot 0)$ with

$$\mathrm{wfd}\big(G(x,y)^{\tau}\big)=\mathrm{gfd}\big(G(x,y)\big)=\ell(x)+\ell(y),$$

and $G(x,y)^{\tau}$ is regular since $C_{\min}\big(G(x,y)^{\tau}\big)_i\cong C_{\min}\big(G(x,y)\big)_{-i}$ for all $i\in\mathbb{Z}$.

Definition 5.9. For $x,y\in W_{\mathrm{ext}}^{+}$, we call the indecomposable $\mathbf{G}$-module $G(x,y)$ from Proposition 5.7 the *generic direct summand* of $L(x\cdot 0)\otimes L(y\cdot 0)$.

Lemma 5.5 has an obvious analogue for generic direct summands of tensor products of simple $\mathbf{G}$-modules.

Lemma 5.10. *Let $x,y\in W_{\mathrm{ext}}^{+}$ and $\omega,\omega'\in\Omega$. Then*

$$G(x\omega,y\omega')\cong T^{\omega\omega'}G(x,y).$$

Proof. We can essentially copy the proof of Lemma 5.5, replacing Weyl modules by simple modules. The only additional fact that one needs to use is that the translation functor $T^{\omega\omega'}$ preserves the good filtration dimension. □

Remark 5.11. Observe that the proof of Proposition 5.7 implies that

$$C_{\min}\big(G(x,y)\big)_{\ell(x)+\ell(y)}\cong T(\omega_{xy}\cdot 0)$$

for all $x,y\in W_{\mathrm{ext}}^{+}$. Furthermore, if $x,y\in W_{\mathrm{aff}}^{+}$ then $G(x,y)$ is the unique indecomposable direct summand of the tensor product $L(x\cdot 0)\otimes L(y\cdot 0)$ with good filtration dimension $\ell(x)+\ell(y)$.

Remark 5.12. The $\mathbf{G}$-modules $G(x,y)$ and $G_\Delta(x,y)$, for $x,y \in W_{\mathrm{ext}}^+$, are not only regular but strongly regular. Indeed, by Remark 4.17 and Lemma 4.18, the tensor products $L(x\cdot 0)\otimes L(y\cdot 0)$ and $\Delta(x\cdot 0)\otimes\Delta(y\cdot 0)$ are both strongly regular of good filtration dimension $\ell(x)+\ell(y)$. Therefore, there exist strongly regular indecomposable direct summands M of $L(x\cdot 0)\otimes L(y\cdot 0)$ and N of $\Delta(x\cdot 0)\otimes\Delta(y\cdot 0)$ with $\mathrm{gfd}(M)=\mathrm{gfd}(N)=\ell(x)+\ell(y)$, and the uniqueness statements in Propositions 5.1 and 5.7 imply that $G(x,y)\cong M$ and $G_\Delta(x,y)\cong N$ are strongly regular. We point out that the strong regularity of the $\mathbf{G}$-modules $G(x,y)$ and $G_\Delta(x,y)$ is also implicit in the proofs of Propositions 5.1 and 5.7.

Remark 5.13. In principle, there is no reason why one would need to restrict one's attention to the study of regular indecomposable direct summands of tensor products of $\mathbf{G}$-modules that belong to the same 'class' of modules (such as Weyl modules or simple modules). For $x,y\in W_{\mathrm{ext}}^+$, the proof of Proposition 5.1 can easily be adapted to show that a tensor product of the form $\Delta(x\cdot 0)\otimes\nabla(y\cdot 0)$ has a unique regular indecomposable direct summand $G_{\Delta,\nabla}(x,y)$, and the latter satisfies

$$\big[C_{\min}\big(G_{\Delta,\nabla}(x,y)\big)_{\ell(x)-\ell(y)} : T(\omega_{xy}\cdot 0)\big]_\oplus = 1.$$

Similarly, we can adapt the proof of Proposition 5.7 to show that $L(x\cdot 0)\otimes\Delta(y\cdot 0)$ has a unique regular indecomposable direct summand with good filtration dimension $\ell(x)+\ell(y)$ and that $L(x\cdot 0)\otimes\nabla(y\cdot 0)$ has a unique regular indecomposable direct summand with Weyl filtration dimension $\ell(x)+\ell(y)$.

Remark 5.14. For the most part of this manuscript, we have been (and will be) restricting our attention to tensor products of two $\mathbf{G}$-modules, but one may also ask about regular indecomposable direct summands of iterated tensor products with more than two constituents. For $x_1,\ldots,x_n\in W_{\mathrm{ext}}^+$, one can use the techniques from the proofs of Propositions 5.1 and 5.7 to show that the iterated tensor product

$$\Delta(x_1\cdot 0)\otimes\cdots\otimes\Delta(x_n\cdot 0)$$

has a unique regular indecomposable direct summand $G_\Delta(x_1,\ldots,x_n)$ (which has good filtration dimension $\ell(x_1)+\cdots+\ell(x_n)$) and that the iterated tensor product

$$L(x_1\cdot 0)\otimes\cdots\otimes L(x_n\cdot 0)$$

has a unique regular indecomposable direct summand $G(x_1,\ldots,x_n)$ that satisfies

$$\mathrm{gfd}\big(G(x_1,\ldots,x_n)\big)=\ell(x_1)+\cdots+\ell(x_n).$$

In analogy with the results from Section 4, we can also prove a translation principle for iterated tensor products, where the structure constants of the Verlinde algebra are replaced by multiplicities of basis elements in iterated products.

6 The Steinberg-Lusztig tensor product theorem

Recall from Section I.8 that every dominant weight $\lambda\in X^+$ can be uniquely written as $\lambda=\lambda_0+\ell\lambda_1$ with $\lambda_0\in X_1$ an ℓ-restricted weight and $\lambda_1\in X^+$. Then, by Steinberg's and Lusztig's tensor product theorems, the simple module $L(\lambda)$ admits a tensor product decomposition

$$L(\lambda)\cong L(\lambda_0)\otimes L(\ell\lambda_1),$$

where $L(\ell\lambda_1)\cong L(\lambda_1)^{[1]}$ in the modular case and $L(\ell\lambda_1)\cong L_{\mathbb{C}}(\lambda_1)^{[1]}$ in the quantum case. Further recall that we assume that $\ell\geq h$, the Coxeter number of $\mathbf{G}$. It is straightforward to see that the weight $\lambda = \lambda_0 + \ell\lambda_1$ is ℓ-regular if and only if λ_0 is ℓ-regular, and that $\ell\lambda_1$ is always ℓ-regular. For ℓ-regular dominant weights $\lambda = \lambda_0 + \ell\lambda_1$ and $\mu = \mu_0 + \ell\mu_1$, our goal in this section is to describe the regular indecomposable direct summands of the tensor product $L(\lambda) \otimes L(\mu)$ in terms of the regular indecomposable direct summands of $L(\lambda_0) \otimes L(\mu_0)$ and $L(\lambda_1)^{[1]} \otimes L(\mu_1)^{[1]}$ (in the modular case) or of $L(\lambda_0) \otimes L(\mu_0)$ and $L(\lambda_1)_{\mathbb{C}}^{[1]} \otimes L_{\mathbb{C}}(\mu_1)^{[1]}$ (in the quantum case). Note that by Theorem 4.14, we can restrict our attention to the case where λ and μ belong to $W_{\mathrm{ext}}^+ \cdot 0$. The main results from this section will be crucial for the description of the regular indecomposable direct summands of tensor products of simple **G**-modules in small rank cases, which will be given in Chapter III.

Because of the connection between the good filtration dimension of simple modules and the length function on the (extended) affine Weyl group (see Corollary 2.7), we start by proving some elementary properties of this length function. The following result is taken from Proposition 1.23 in [IM65]:

Proposition 6.1. *Let $\gamma \in X$ and $w \in W_{\mathrm{fin}}$. Then*

$$\ell(t_\gamma w) = \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\in\Phi^+}} |(\gamma,\alpha^\vee)| + \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\notin\Phi^+}} |(\gamma,\alpha^\vee)-1|.$$

Recall that we write ρ for the half-sum of all positive roots in Φ. Analogously, we define $\rho^\vee$ to be the half-sum of all positive coroots, that is

$$\rho^\vee := \frac{1}{2} \cdot \sum_{\alpha\in\Phi^+} \alpha^\vee.$$

Using this notation, the length function on W_{ext}^+ takes a very simple form.

Corollary 6.2. *For $\gamma \in X$ and $w \in W_{\mathrm{fin}}$ such that $t_\gamma w \in W_{\mathrm{ext}}^+$, we have*

$$\ell(t_\gamma w) = 2\cdot(\gamma,\rho^\vee) - \ell(w).$$

Proof. For $\alpha \in \Phi^+$ and $x \in A_{\mathrm{fund}}$, we have

$$0 < \big(t_\gamma w(x),\alpha^\vee\big) = \big(\gamma,\alpha^\vee\big) + \big(x, w^{-1}(\alpha)^\vee\big),$$

where $0 < \big(x, w^{-1}(\alpha)^\vee\big) < 1$ if $w^{-1}(\alpha) \in \Phi^+$ and $-1 < \big(x, w^{-1}(\alpha)^\vee\big) < 0$ if $w^{-1}(\alpha) \notin \Phi^+$, and where $(\gamma,\alpha^\vee)$ is an integer. This implies that $(\gamma,\alpha^\vee) \geq 0$ for all $\alpha \in \Phi^+$ with $w^{-1}(\alpha) \in \Phi^+$ and that $(\gamma,\alpha^\vee) \geq 1$ for all $\alpha \in \Phi^+$ with $w^{-1}(\alpha) \notin \Phi^+$, whence we can omit the absolute values from the length formula in Proposition 6.1. More precisely, we have

$$\begin{aligned}
\ell(t_\gamma w) &= \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\in\Phi^+}} |(\gamma,\alpha^\vee)| + \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\notin\Phi^+}} |(\gamma,\alpha^\vee)-1| \\
&= \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\in\Phi^+}} (\gamma,\alpha^\vee) + \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\notin\Phi^+}} \big((\gamma,\alpha^\vee)-1\big) \\
&= 2\cdot(\gamma,\rho^\vee) - \big|\big\{\alpha\in\Phi^+ \bigm| w^{-1}(\alpha)\notin\Phi^+\big\}\big|,
\end{aligned}$$

and the claim follows from the well-known fact that $\ell(w) = \ell(w^{-1})$ equals the number of positive roots that are sent to negative roots by w^{-1}, again by Proposition 6.1. □

Corollary 6.3. *For $x \in W_{\mathrm{ext}}^+$ and $\lambda \in X^+$, we have $\ell(t_\lambda x) = \ell(t_\lambda) + \ell(x)$.*

Proof. Let us write $x = t_\gamma w$ with $\gamma \in X$ and $w \in W_{\mathrm{fin}}$, so $\ell(x) = 2 \cdot (\gamma, \rho^\vee) - \ell(w)$ by Corollary 6.2. Analogously, we have $\ell(t_\lambda) = 2 \cdot (\lambda, \rho^\vee)$ and

$$\ell(t_\lambda x) = \ell(t_{\lambda+\gamma} w) = 2 \cdot (\lambda + \gamma, \rho^\vee) - \ell(w) = 2 \cdot (\lambda, \rho^\vee) + 2 \cdot (\gamma, \rho^\vee) - \ell(w) = \ell(t_\lambda) + \ell(x),$$

as claimed. □

Next we prove two results that compare the canonical order on X induced by Φ^+ with the linkage order and the Bruhat order.

Lemma 6.4. *For $\lambda, \mu \in X$ with $\lambda \leq \mu$, we have $t_\lambda(A_{\mathrm{fund}}) \uparrow t_\mu(A_{\mathrm{fund}})$.*

Proof. Recall that $\lambda < \mu$ means that $\mu - \lambda$ is a non-zero non-negative integral linear combination of simple roots. Hence it suffices to prove that $t_\lambda(A_{\mathrm{fund}}) \uparrow t_{\lambda+\alpha}(A_{\mathrm{fund}})$ for $\alpha \in \Phi^+$. We set $r = (\lambda, \alpha^\vee) + 1$ and claim that

$$t_\lambda(A_{\mathrm{fund}}) \uparrow s_{\alpha,r} t_\lambda(A_{\mathrm{fund}}) = s_{\alpha,r+1} t_{\lambda+\alpha}(A_{\mathrm{fund}}) \uparrow t_{\lambda+\alpha}(A_{\mathrm{fund}}).$$

Indeed, it is straightforward to see that for $x \in A_{\mathrm{fund}}$, we have $(\lambda+x, \alpha^\vee) < r$ and $(\lambda+\alpha+x, \alpha^\vee) > r+1$, so $t_\lambda(A_{\mathrm{fund}}) \subseteq H^-_{\alpha,r}$ and $t_{\lambda+\alpha}(A_{\mathrm{fund}}) \subseteq H^+_{\alpha,r+1}$, as required. □

Corollary 6.5. *For $\lambda, \mu \in X^+$ with $\lambda \leq \mu$, we have $t_\lambda \leq t_\mu$ and $\ell(t_\lambda) \leq \ell(t_\mu)$.*

Proof. This follows from Lemma 6.4 and the fact that the linkage order is equivalent to the Bruhat order on W^+_{aff}; see Theorem I.2.14. □

Equipped with the above results about the length function, we can prove our results relating generic direct summands with the tensor product theorems of R. Steinberg and G. Lusztig. We start with the quantum case and then proceed to study the modular case.

The quantum case

Suppose until otherwise stated that we are in the quantum case. Until the end of the section, we fix two elements $x, y \in W^+_{\mathrm{ext}}$ and write

$$x \cdot 0 = \lambda' + \ell\lambda \qquad \text{and} \qquad y \cdot 0 = \mu' + \ell\mu$$

with $\lambda', \mu' \in X_1$ and $\lambda, \mu \in X^+$. Furthermore, we set

$$x_0 \coloneqq t_{-\lambda} x \qquad \text{and} \qquad y_0 = t_{-\mu} y.$$

Observe that $x_0 \cdot 0 = x \cdot 0 - \ell\lambda = \lambda'$, whence $x_0 \in W^+_{\mathrm{ext}}$ and

$$\ell(x) = \ell(t_\lambda x_0) = \ell(t_\lambda) + \ell(x_0)$$

by Corollary 6.3, and similarly $\ell(y) = \ell(t_\mu) + \ell(y_0)$. As $\mathrm{Rep}(\mathfrak{g})$ is a semisimple category, the tensor product of the simple $\mathfrak{g}$-modules $L_{\mathbb{C}}(\lambda)$ and $L_{\mathbb{C}}(\mu)$ decomposes as a direct sum

$$L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu) \cong \bigoplus_{\nu \in X^+} L_{\mathbb{C}}(\nu)^{\oplus d^\nu_{\lambda,\mu}}$$

of simple $\mathfrak{g}$-modules, for certain multiplicities $d^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$.

Lemma 6.6. *There is an isomorphism*

$$\big(L(x\cdot 0)\otimes L(y\cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{\nu\in X^+} \Big(\big(L(x_0\cdot 0)\otimes L(y_0\cdot 0)\big)_{\mathrm{reg}}\otimes L_{\mathbb{C}}(\nu)^{[1]}\Big)_{\mathrm{reg}}^{\oplus d_{\lambda,\mu}^{\nu}}$$

Proof. By Lusztig's tensor product theorem, we have

$$L(x\cdot 0)\cong L(x_0\cdot 0)\otimes L_{\mathbb{C}}(\lambda)^{[1]} \qquad \text{and} \qquad L(y\cdot 0)\cong L(y_0\cdot 0)\otimes L_{\mathbb{C}}(\mu)^{[1]},$$

and using Lemma 4.11, we obtain

$$\big(L(x\cdot 0)\otimes L(y\cdot 0)\big)_{\mathrm{reg}} \cong \Big(\big(L(x_0\cdot 0)\otimes L(y_0\cdot 0)\big)_{\mathrm{reg}}\otimes \big(L_{\mathbb{C}}(\lambda)^{[1]}\otimes L_{\mathbb{C}}(\mu)^{[1]}\big)_{\mathrm{reg}}\Big)_{\mathrm{reg}}.$$

Now

$$L_{\mathbb{C}}(\lambda)^{[1]}\otimes L_{\mathbb{C}}(\mu)^{[1]} \cong \big(L_{\mathbb{C}}(\lambda)\otimes L_{\mathbb{C}}(\mu)\big)^{[1]} \cong \bigoplus_{\nu\in X^+}\big(L_{\mathbb{C}}(\nu)^{[1]}\big)^{\oplus d_{\lambda,\mu}^{\nu}},$$

and the claim follows since $L_{\mathbb{C}}(\nu)^{[1]}\cong L(\ell\nu)$ is regular for all $\nu\in X^+$, by Lemma 4.3. □

Lemma 6.7. *The* $\mathbf{G}$*-module* $G(x,y)$ *is a direct summand of* $G(x_0,y_0)\otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]}$.

Proof. Recall that by definition, $G(x,y)$ is the unique regular indecomposable direct summand of the tensor product $L(x\cdot 0)\otimes L(y\cdot 0)$ that attains the maximal good filtration dimension

$$\mathrm{gfd}\big(G(x,y)\big)=\ell(x)+\ell(y)=\ell(x_0)+\ell(y_0)+\ell(t_\lambda)+\ell(t_\mu).$$

For $\nu\in X^+$ with $d_{\lambda,\mu}^{\nu}>0$, we have $\nu\leq\lambda+\mu$ and by Corollaries 2.7, 6.3 and 6.5, we get

$$\mathrm{gfd}\big(L_{\mathbb{C}}(\nu)^{[1]}\big)=\mathrm{gfd}\big(L(\ell\nu)\big)=\ell(t_\nu)\leq\ell(t_{\lambda+\mu})=\ell(t_\lambda)+\ell(t_\mu),$$

with equality precisely when $\nu=\lambda+\mu$. As $G(x_0,y_0)$ is the unique regular indecomposable direct summand of $L(x_0\cdot 0)\otimes L(y_0\cdot 0)$ that attains the maximal good filtration dimension $\ell(x_0)+\ell(y_0)$, we conclude (using Lemmas I.7.2 and 6.6) that $G(x,y)$ is a direct summand of $G(x_0,y_0)\otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]}$, as claimed. □

Corollary 6.8. *If* $G(x_0,y_0)$ *has simple socle as a* $\mathbf{G}_1$*-module then* $G(x,y)\cong G(x_0,y_0)\otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]}$.

Proof. The assumption implies that $G(x_0,y_0)\otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]}$ is indecomposable by Lemma I.8.8, and the claim is immediate since $G(x,y)$ is a direct summand of $G(x_0,y_0)\otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]}$ by Lemma 6.7. □

In order to discuss other regular indecomposable direct summands of tensor products of simple modules, we will use the following lemma:

Lemma 6.9. *Let* M *be a regular* $\mathbf{G}$*-module. Then* $M\otimes L_{\mathbb{C}}(\nu)^{[1]}$ *is regular for all* $\nu\in X^+$.

Proof. Note that $L_{\mathbb{C}}(-w_0\nu)$ is dual to $L_{\mathbb{C}}(\nu)$, hence the trivial one-dimensional $\mathfrak{g}$-module $L_{\mathbb{C}}(0)$ is a direct summand of $L_{\mathbb{C}}(\nu)\otimes L_{\mathbb{C}}(-w_0\nu)$, and the trivial one-dimensional $\mathbf{G}$-module $L(0)\cong L_{\mathbb{C}}(0)^{[1]}$ is a direct summand of $L_{\mathbb{C}}(\nu)^{[1]}\otimes L_{\mathbb{C}}(-w_0\nu)^{[1]}$. This implies that $M\cong M\otimes L(0)$ is a direct summand of the tensor product $M\otimes L_{\mathbb{C}}(\nu)^{[1]}\otimes L_{\mathbb{C}}(-w_0\nu)^{[1]}$, and the claim follows because singular $\mathbf{G}$-modules form a thick tensor ideal (see Lemma 3.3). □

Corollary 6.10. *Let M be a regular indecomposable direct summand of $L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)$ and let $\nu \in X^+$ with $d^\nu_{\lambda,\mu} > 0$. Then $M' \coloneqq M \otimes L_{\mathbb{C}}(\nu)^{[1]}$ is regular, and every regular indecomposable direct summand of M' is a regular indecomposable direct summand of $L(x \cdot 0) \otimes L(y \cdot 0)$. If M has simple socle as a $\mathbf{G}_1$-module then M' is a regular indecomposable direct summand of $L(x \cdot 0) \otimes L(y \cdot 0)$.*

Proof. The tensor product $M' = M \otimes L(\nu)^{[1]}$ is regular by Lemma 6.9. Every regular indecomposable direct summand of M' is also a regular indecomposable direct summand of $L(x \cdot 0) \otimes L(y \cdot 0)$ by Lemma 6.6, whence the first claim. If M has simple socle as a $\mathbf{G}_1$-module then M' is indecomposable as a $\mathbf{G}$-module by Lemma I.8.8. □

Corollary 6.11. *Suppose that all regular indecomposable direct summands of $L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)$ have simple socle as $\mathbf{G}_1$-modules. Then*

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in X^+} \Big(\big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big)_{\mathrm{reg}} \otimes L_{\mathbb{C}}(\nu)^{[1]} \Big)^{\oplus d^\nu_{\lambda,\mu}}.$$

Proof. By Lemma 6.6, we have

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in X^+} \Big(\big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big)_{\mathrm{reg}} \otimes L_{\mathbb{C}}(\nu)^{[1]} \Big)^{\oplus d^\nu_{\lambda,\mu}}_{\mathrm{reg}}$$

and the claim follows because all indecomposable direct summands of $\big(L(x_0 \cdot \lambda) \otimes L(y_0 \cdot \lambda)\big)_{\mathrm{reg}} \otimes L_{\mathbb{C}}(\nu)^{[1]}$ are regular by Corollary 6.10. □

The modular case

Suppose until the end of the section that we are in the modular case. We would like to establish modular analogues of the results that were proven above for the quantum case. One key argument in Lemma 6.6 was that the (quantum) Frobenius twist of a $\mathfrak{g}$-module is always regular, because it is a direct sum of simple $U_\zeta(\mathfrak{g})$-modules of ℓ-regular highest weight. Since in the modular case, the domain of the Frobenius twist functor is the non-semisimple category $\mathrm{Rep}(\mathbf{G})$, we will need to replace this argument by the following result:

Proposition 6.12. *Let M be a $\mathbf{G}$-module in $\mathrm{Rep}_\nu(\mathbf{G})$, for some $\nu \in \overline{C}_{\mathrm{fund}} \cap X$. Then the Frobenius twist $M^{[1]}$ is strongly regular. More precisely, define*

$$d = d_M \coloneqq \max\big\{ 2 \cdot (\gamma, \rho^\vee) \bigm| \gamma \in X^+ \text{ with } [M : L(\gamma)] \neq 0 \big\},$$

and let $\gamma \in X^+$ with $[M : L(\gamma)] \neq 0$ and $2 \cdot (\gamma, \rho^\vee) = d$. Then $C_{\min}\big(M^{[1]}\big)_d$ is a fusion module with

$$\big[C_{\min}\big(M^{[1]}\big)_d : T(\omega_{t_\gamma} \cdot 0) \big]_\oplus \neq 0.$$

Furthermore, we have $C_{\min}\big(M^{[1]}\big)_i = 0$ for $i > d$ and $C_{\min}\big(M^{[1]}\big)_{d-1}$ is negligible.

Proof. Note that the composition factors of $M^{[1]}$ are of the form $L(\gamma')^{[1]} \cong L(\ell\gamma')$, for $\gamma' \in X^+$ such that $[M : L(\gamma')] \neq 0$. By Corollaries 2.7 and 6.3, we have

$$\mathrm{gfd}\big(L(\ell\gamma')\big) = \ell(t_{\gamma'}) = 2 \cdot (\gamma', \rho^\vee) \leq d,$$

so $\mathrm{gfd}\big(M^{[1]}\big) \leq d$ by Corollary I.7.5 and $C_{\min}\big(M^{[1]}\big)_i = 0$ for all $i > d$ by Lemma 2.3. We prove the remaining claims by induction on the composition length of M.

If $M \cong L(\gamma)$ is simple then $M^{[1]} \cong L(\ell\gamma) = L(t_\gamma \cdot 0)$ and $\ell(t_\gamma) = 2 \cdot (\gamma, \rho^\vee) = d$ by Corollary 6.2, so the claim follows from Proposition 2.6. Now suppose that M has at least two composition factors and fix a short exact sequence

$$0 \to L \to M \to N \to 0$$

with L and N non-zero. By induction, we may assume that the proposition holds for L and N.

Note that by assumption, both L and N belong to the linkage class $\mathrm{Rep}_\nu(\mathbf{G})$. In particular, all highest weights of composition factors of L and of N belong to the same $\mathbb{Z}\Phi$-coset in X. As $(\beta, \rho^\vee) \in \mathbb{Z}$ for all $\beta \in \Phi$, it follows that d_L and d_N have the same parity. Furthermore, by applying Lemma 2.4 to the short exact sequence

$$0 \to L^{[1]} \to M^{[1]} \to N^{[1]} \to 0,$$

we see that

$$C_{\min}\big(M^{[1]}\big)_i \overset{\oplus}{\subseteq} C_{\min}\big(L^{[1]}\big)_i \oplus C_{\min}\big(N^{[1]}\big)_i =: C_i$$

for all $i \in \mathbb{Z}$. It is straightforward to see that $d = d_M = \max\{d_L, d_N\}$, and we distinguish three cases:

(1) Suppose that $d = d_L > d_N$. Then $L(\gamma)$ is a composition factor of L with $d_L = 2 \cdot (\gamma, \rho^\vee)$. Furthermore, we have $d_N \leq d_L - 2 = d - 2$ because d_L and d_N have the same parity, and it follows that $C_{\min}\big(N^{[1]}\big)_i = 0$ for $i > d - 2$. In particular, we have

$$C_{\min}\big(M^{[1]}\big)_{d-1} \overset{\oplus}{\subseteq} C_{\min}\big(L^{[1]}\big)_{d-1} \qquad \text{and} \qquad C_{\min}\big(M^{[1]}\big)_d \overset{\oplus}{\subseteq} C_{\min}\big(L^{[1]}\big)_d,$$

whence $C_{\min}\big(M^{[1]}\big)_{d-1}$ is negligible and $C_{\min}\big(M^{[1]}\big)_d$ is a fusion module. Finally, the assumption on L and Lemma 2.4 imply that

$$0 \neq \big[C_{\min}\big(L^{[1]}\big)_d : T(\omega_{t_\gamma} \cdot 0)\big]_\oplus = \big[C_{\min}\big(M^{[1]}\big)_d : T(\omega_{t_\gamma} \cdot 0)\big]_\oplus$$

because $C_{d-1} = C_{\min}\big(L^{[1]}\big)_{d-1}$ is negligible and $C_{d+1} = 0$.

(2) Suppose that $d = d_N > d_L$. Then $L(\gamma)$ is a composition factor of N with $d_N = 2 \cdot (\gamma, \rho^\vee)$. Furthermore, we have $d_L \leq d_N - 2 = d - 2$ because d_L and d_N have the same parity. Now the claim follows precisely as in case (1), with the roles of L and N interchanged.

(3) Suppose that $d = d_L = d_N$. Then $C_{\min}\big(M^{[1]}\big)_{d-1}$ is negligible because

$$C_{\min}\big(M^{[1]}\big)_{d-1} \overset{\oplus}{\subseteq} C_{\min}\big(L^{[1]}\big)_{d-1} \oplus C_{\min}\big(N^{[1]}\big)_{d-1},$$

and $C_{\min}\big(M^{[1]}\big)_d$ is a fusion module because

$$C_{\min}\big(M^{[1]}\big)_d \overset{\oplus}{\subseteq} C_{\min}\big(L^{[1]}\big)_d \oplus C_{\min}\big(N^{[1]}\big)_d.$$

Furthermore, the simple $\mathbf{G}$-module $L(\gamma)$ is a composition factor of at least one of the $\mathbf{G}$-modules L and N, and we have $2 \cdot (\gamma, \rho^\vee) = d_L = d_N$. Using again the assumptions on L and N together with Lemma 2.4, we obtain

$$0 \neq \big[C_{\min}\big(L^{[1]}\big)_d : T(\omega_{t_\gamma} \cdot 0)\big]_\oplus + \big[C_{\min}\big(N^{[1]}\big)_d : T(\omega_{t_\gamma} \cdot 0)\big]_\oplus = \big[C_{\min}\big(M^{[1]}\big)_d : T(\omega_{t_\gamma} \cdot 0)\big]_\oplus$$

because the $C_{d-1} = C_{\min}\big(L^{[1]}\big)_{d-1} \oplus C_{\min}\big(N^{[1]}\big)_{d-1}$ is negligible and $C_{d+1} = 0$. $\square$

Now we are ready to give a modular analogue of Lemma 6.6. Recall that we fix $x, y \in W_{\mathrm{ext}}^{+}$ and write $x \cdot 0 = \lambda' + \ell\lambda$ and $y \cdot 0 = \mu' + \ell\mu$ with $\lambda', \mu' \in X_1$ and $\lambda, \mu \in X^{+}$. As before, we set $x_0 \coloneqq t_{-\lambda}x$ and $y_0 \coloneqq t_{-\mu}y$.

Lemma 6.13. *Fix a Krull-Schmidt decomposition $L(\lambda) \otimes L(\mu) \cong M_1 \oplus \cdots \oplus M_r$. Then there is an isomorphism*

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{i=1}^{r} \Big(\big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big)_{\mathrm{reg}} \otimes M_i^{[1]}\Big)_{\mathrm{reg}}$$

Proof. As in the proof of Lemma 6.6 (replacing Lusztig's tensor product theorem by Steinberg's tensor product theorem), we see that

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \Big(\big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big)_{\mathrm{reg}} \otimes \big(L(\lambda)^{[1]} \otimes L(\mu)^{[1]}\big)_{\mathrm{reg}}\Big)_{\mathrm{reg}}.$$

Now

$$L(\lambda)^{[1]} \otimes L(\mu)^{[1]} \cong \big(L(\lambda) \otimes L(\mu)\big)^{[1]} \cong M_1^{[1]} \oplus \cdots \oplus M_r^{[1]},$$

and the claim follows since $M_i^{[1]}$ is regular for $i = 1, \ldots, r$ by Proposition 6.12. □

In order to formulate an analogue of Lemma 6.7, we need the following definition:

Definition 6.14. Let $M(\lambda, \mu)$ be the unique indecomposable direct summand of $L(\lambda) \otimes L(\mu)$ with a non-zero $\lambda + \mu$-weight space.

Note that $M(\lambda, \mu)$ is well-defined since the $\lambda + \mu$-weight space of $L(\lambda) \otimes L(\mu)$ is one-dimensional. We could alternatively define $M(\lambda, \mu)$ as the unique indecomposable direct summand of the tensor product $L(\lambda) \otimes L(\mu)$ that has $L(\lambda + \mu)$ as a composition factor.

Lemma 6.15. *The $\mathbf{G}$-module $G(x, y)$ is a direct summand of $G(x_0, y_0) \otimes M(\lambda, \mu)^{[1]}$.*

Proof. Recall that $\ell(x) = \ell(x_0) + \ell(t_\lambda)$ and $\ell(y) = \ell(y_0) + \ell(t_\mu)$ by Corollary 6.3, and that $G(x, y)$ is the unique regular indecomposable direct summand of $L(x \cdot 0) \otimes L(y \cdot 0)$ that attains the maximal good filtration dimension

$$\mathrm{gfd}\big(G(x, y)\big) = \ell(x) + \ell(y) = \ell(x_0) + \ell(y_0) + \ell(t_\lambda) + \ell(t_\mu).$$

By Lemma 6.13, $G(x, y)$ is a direct summand of $M \otimes N^{[1]}$, for some regular indecomposable direct summand M of $L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)$ and some indecomposable direct summand N of $L(\lambda) \otimes L(\mu)$, and it suffices to show that $M \cong G(x_0, y_0)$ and $N \cong M(\lambda, \mu)$.

By Lemma I.7.2 and Corollary 2.7, we have

$$\mathrm{gfd}(M) \leq \mathrm{gfd}\big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big) \leq \mathrm{gfd}\big(L(x_0 \cdot 0)\big) + \mathrm{gfd}\big(L(y_0 \cdot 0)\big) = \ell(x_0) + \ell(y_0)$$

and similarly

$$\mathrm{gfd}\big(N^{[1]}\big) \leq \mathrm{gfd}\big(L(\lambda)^{[1]} \otimes L(\mu)^{[1]}\big) = \mathrm{gfd}\big(L(\ell\lambda) \otimes L(\ell\mu)\big) \leq \ell(t_\lambda) + \ell(t_\mu).$$

Again using Lemma I.7.2, we see that

$$\ell(x_0) + \ell(y_0) + \ell(t_\lambda) + \ell(t_\mu) = \mathrm{gfd}\big(G(x, y)\big) \leq \mathrm{gfd}\big(M \otimes N^{[1]}\big) \leq \mathrm{gfd}\big(M\big) + \mathrm{gfd}\big(N^{[1]}\big),$$

and it follows that $\mathrm{gfd}(M) = \ell(x_0) + \ell(y_0)$ and $\mathrm{gfd}\big(N^{[1]}\big) = \ell(t_\lambda) + \ell(t_\mu)$. As $G(x_0, y_0)$ is the unique regular indecomposable direct summand of $L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)$ that has good filtration dimension $\ell(x_0) + \ell(y_0)$, we conclude that $M \cong G(x_0, y_0)$.

Next note that $N^{[1]}$ has a composition factor $L(\gamma)$ with

$$\mathrm{gfd}\big(L(\gamma)\big) \geq \mathrm{gfd}\big(N^{[1]}\big) = \ell(t_\lambda) + \ell(t_\mu) = \ell(t_{\lambda+\mu}),$$

by Corollaries I.7.5 and 6.3, and that $\gamma = \ell\gamma'$ for some $\gamma' \in X^+$ with $[N : L(\gamma')] \neq 0$. As N is a direct summand of $L(\lambda) \otimes L(\mu)$, it follows that $\gamma' \leq \lambda + \mu$, whence $t_{\gamma'} \leq t_{\lambda+\mu}$ by Corollary 6.5. Furthermore, we have

$$\ell(t_{\lambda+\mu}) \leq \mathrm{gfd}\big(L(\gamma)\big) = \mathrm{gfd}\big(L(\ell\gamma')\big) = \ell(t_{\gamma'})$$

by Corollary 2.7, whence $t_{\gamma'} = t_{\lambda+\mu}$ and $\gamma' = \lambda + \mu$. We conclude that $L(\lambda + \mu)$ is a composition factor of N, hence $N \cong M(\lambda, \mu)$, as required. □

Corollary 6.16. *If $G(x_0, y_0)$ is indecomposable as a $\mathbf{G}_1$-module then $G(x, y) \cong G(x_0, y_0) \otimes M(\lambda, \mu)^{[1]}$.*

Proof. The assumption implies that $G(x_0, y_0) \otimes M(\lambda, \mu)^{[1]}$ is indecomposable by Lemma I.8.5, and the claim is immediate since $G(x, y)$ is a direct summand of $G(x_0, y_0) \otimes M(\lambda, \mu)^{[1]}$ by Lemma 6.15. □

The weights λ and μ can be written as $\lambda = \sum_{i \geq 0} \ell^i \cdot \lambda_i$ and $\mu = \sum_{i \geq 0} \ell^i \cdot \mu_i$ with $\lambda_i, \mu_i \in X_1$ for all $i \geq 0$. By iterating Steinberg's tensor product theorem, we obtain tensor product decompositions

$$L(\lambda) \cong \bigotimes_{i \geq 0} L(\lambda_i)^{[i]} \qquad \text{and} \qquad L(\mu) \cong \bigotimes_{i \geq 0} L(\mu_i)^{[i]}.$$

In many cases, this allows us to describe $M(\lambda, \mu)$ as a tensor product of Frobenius twists of the different $M(\lambda_i, \mu_i)$.

Lemma 6.17. *Suppose that $M(\lambda_i, \mu_i)$ is indecomposable as a $\mathbf{G}_1$-module for all $i \geq 0$. Then*

$$M(\lambda, \mu) \cong \bigotimes_{i \geq 0} M(\lambda_i, \mu_i)^{[i]}.$$

Proof. Since $M(\lambda_i, \mu_i)$ is a direct summand of $L(\lambda_i) \otimes L(\mu_i)$, the tensor product $M \coloneqq \bigotimes_i M(\lambda_i, \mu_i)^{[i]}$ is a direct summand of

$$L(\lambda) \otimes L(\mu) \cong \bigotimes_{i \geq 0} \big(L(\lambda_i) \otimes L(\mu_i)\big)^{[i]}.$$

Furthermore, the $\lambda + \mu$-weight space of M is non-zero because the $\lambda_i + \mu_i$ weight space of $M(\lambda_i, \mu_i)$ is non-zero for all i, so it remains to show that M is indecomposable. This follows from Corollary I.8.6, by our assumption on the $\mathbf{G}$-modules $M(\lambda_i, \mu_i)$. □

As in the quantum case, we would like to discuss other regular indecomposable direct summands of tensor products of simple modules as well. Unfortunately, there is no modular analogue of Lemma 6.9 that suits our purpose. We will solve this problem using the notion of *strong regularity* which was introduced in Section 4.

Lemma 6.18. *Let M and N be $\mathbf{G}$-modules such that M is strongly regular and N is indecomposable as a $\mathbf{G}$-module. Then $M \otimes N^{[1]}$ is strongly regular. If M is indecomposable as a $\mathbf{G}_1$-module then $M \otimes N^{[1]}$ is indecomposable as a $\mathbf{G}$-module.*

Proof. The Frobenius twist $N^{[1]}$ is strongly regular by Proposition 6.12, so Lemma 4.18 implies that the tensor product $M \otimes N^{[1]}$ is strongly regular. If M is indecomposable as a $\mathbf{G}_1$-module then $M \otimes N^{[1]}$ is indecomposable as a $\mathbf{G}$-module by Corollary I.8.5. □

Corollary 6.19. *Let M be a strongly regular indecomposable direct summand of $L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)$, and let N be an indecomposable direct summand of $L(\lambda) \otimes L(\mu)$. Then $M' := M \otimes N^{[1]}$ is strongly regular, and every regular indecomposable direct summand of M' is a regular indecomposable direct summand of $L(x \cdot 0) \otimes L(y \cdot 0)$. If M is indecomposable as a $\mathbf{G}_1$-module then M' is a regular indecomposable direct summand of $L(x \cdot 0) \otimes L(y \cdot 0)$.*

Proof. The tensor product $M' = M \otimes N^{[1]}$ is strongly regular by Lemma 6.18, and every regular indecomposable direct summand of M' is also a regular indecomposable direct summand of the tensor product $L(x \cdot 0) \otimes L(y \cdot 0)$ by Lemma 6.13, whence the first claim. If M is indecomposable as a $\mathbf{G}_1$-module then M' is indecomposable as a $\mathbf{G}$-module, again by Lemma 6.18. □

Corollary 6.20. *Fix a Krull-Schmidt decomposition $L(\lambda) \otimes L(\mu) \cong M_1 \oplus \cdots \oplus M_r$, and suppose that all regular indecomposable direct summands of $L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)$ are strongly regular and indecomposable as $\mathbf{G}_1$-modules. Then*

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{i=1}^{r} \big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big)_{\mathrm{reg}} \otimes M_i^{[1]}.$$

Proof. By Lemma 6.13, we have

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{i=1}^{r} \Big(\big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big)_{\mathrm{reg}} \otimes M_i^{[1]}\Big)_{\mathrm{reg}},$$

and the claim follows because all indecomposable direct summands of $\big(L(x_0 \cdot \lambda) \otimes L(y_0 \cdot \lambda)\big)_{\mathrm{reg}} \otimes M_i^{[1]}$ are regular for $i = 1, \ldots, r$, by Corollary 6.19. □

III. Results in small rank

In this chapter, we compute examples of generic direct summands for $\mathbf{G}$ of type A_1 and A_2. In each of the two cases, we consider first the generic direct summands $G(x,y)$ of tensor products $L(x\cdot 0)\otimes L(y\cdot 0)$ of simple $\mathbf{G}$-modules, and then turn our attention to the generic direct summands $G_\nabla(x,y)$ of tensor products $\nabla(x\cdot 0)\otimes\nabla(y\cdot 0)$ of induced modules, for $x,y\in W_{\mathrm{ext}}^+$.

1 Type A_1

In this section, we suppose (unless otherwise stated) that $\mathbf{G}$ is of type A_1. Then $\Phi^+=\Pi=\{\alpha_{\mathrm{h}}\}$ and we write $\alpha\coloneqq\alpha_{\mathrm{h}}$ for the unique positive root. The weight lattice X is the free $\mathbb{Z}$-module of rank 1, spanned by the fundamental dominant weight ϖ_α, and we can identify X with $\mathbb{Z}$ via $\varpi_\alpha\mapsto 1$. Under this identification, the unique positive root α is mapped to 2, $\rho=\frac{1}{2}\cdot\alpha$ is mapped to 1 and the scalar product $(-\,,-)$ on the euclidean space $X_{\mathbb{R}}=X\otimes_{\mathbb{Z}}\mathbb{R}\cong\mathbb{R}$ corresponds to the multiplication of real numbers. Furthermore, the set X^+ of dominant weights is identified with the set $\mathbb{Z}_{\geq 0}$ of non-negative integers. Accordingly, we denote the $\mathbf{G}$-modules $L(a\varpi_\alpha)$, $\nabla(a\varpi_\alpha)$, $\Delta(a\varpi_\alpha)$ and $T(a\varpi_\alpha)$ by $L(a)$, $\nabla(a)$, $\Delta(a)$ and $T(a)$, respectively, for $a\in\mathbb{Z}_{\geq 0}$.

The representation theory of $\mathbf{G}$ in type A_1 has been studied extensively, including results about tensor products of simple modules [DH05] and of induced modules [Cav11] (both in the modular case). Nevertheless, we choose to present an approach that is largely self-contained, allowing us to treat the quantum case and the modular case simultaneously.

Simple modules

Our first aim is to determine the regular indecomposable direct summands of tensor products of simple $\mathbf{G}$-modules. In view of the results from Section II.6, we should start by trying to understand tensor products of simple $\mathbf{G}$-modules with ℓ-restricted highest weights. In a second step, we can describe the regular indecomposable direct summands of a tensor product of two simple $\mathbf{G}$-modules with arbitrary highest weights as tensor products of different Frobenius twists of the indecomposable direct summands of restricted tensor products.

Note that under our identification of X with $\mathbb{Z}$, the set of ℓ-restricted weights $X_1=\{0,\dots,\ell-1\}$ is contained in $\overline{C}_{\mathrm{fund}}\cap X=\{-1,0,\dots,\ell-1\}$. In particular, the fundamental ℓ-alcove C_{fund} is the unique ℓ-alcove containing ℓ-restricted weights. By Theorem II.4.14, the observation that $L(0)\otimes L(0)\cong L(0)$, and therefore $G(e,e)\cong\big(L(0)\otimes L(0)\big)_{\mathrm{reg}}\cong L(0)$, completely determines the regular parts of all tensor products of simple $\mathbf{G}$-modules with ℓ-restricted ℓ-regular highest weight. However, in order to determine the regular indecomposable direct summands of tensor products of simple $\mathbf{G}$-modules with arbitrary ℓ-regular highest weights (in the modular case), we also need some information about possibly non-regular indecomposable direct summands of tensor products of simple modules with possibly ℓ-

singular highest weights (see Lemma II.6.13). Therefore, we will compute in Lemma 1.6 below the complete Krull-Schmidt decomposition of a tensor product of simple **G**-modules with ℓ-restricted highest weights. We first need to establish some preliminary (and mostly well-known) results.

Note that since $X_1 \subseteq \overline{C}_{\mathrm{fund}}$, we have $L(a) \cong \nabla(a) \cong T(a)$ for all $a \in X_1$, by the linkage principle. Using Theorem I.5.2 and weight considerations, we obtain the following elementary lemma:

Lemma 1.1. *All indecomposable direct summands of a tensor product $L(a) \otimes L(b)$, with $a, b \in X_1$, are of the form $T(c)$ for some $c \leq 2\ell - 2$.*

As explained above, we have $T(c) \cong \nabla(c) \cong L(c)$ for all $c \leq \ell - 1$. In the following well-known lemma, we describe the submodule structure of the tilting modules $T(c)$ with $\ell \leq c \leq 2\ell - 2$.

Lemma 1.2. *Let $\ell \leq c \leq 2\ell - 2$ and set $c' := 2\ell - 2 - c$. Then there is a short exact sequence*

$$0 \to \nabla(c') \to T(c) \to \nabla(c) \to 0.$$

Furthermore, $T(c)$ is uniserial and of composition length 3, with

$$\mathrm{soc}_{\mathbf{G}} T(c) \cong L(c'), \qquad \mathrm{head}_{\mathbf{G}} T(c) \cong L(c') \qquad \text{and} \qquad \mathrm{rad}_{\mathbf{G}} T(c)/\mathrm{soc}_{\mathbf{G}} T(c) \cong L(c).$$

Proof. Let $t = s_{\alpha,1} \in W_{\mathrm{aff}}$ and note that $t \boldsymbol{\cdot} x = 2\ell - 2 - x$ for all $x \in \mathbb{Z}$, so $t \boldsymbol{\cdot} c = c'$. Furthermore, it is straightforward to see that $\mathrm{Stab}_{W_{\mathrm{aff}}}(\ell - 1) = \{e, t\}$ and $T(\ell - 1) \cong \nabla(\ell - 1)$. Now Proposition I.6.9 shows that there is a short exact sequence

$$0 \to \nabla(c') \to T_{\ell-1}^{c'} T(\ell - 1) \to \nabla(c) \to 0$$

and that $\mathrm{soc}_{\mathbf{G}} T_{\ell-1}^{c'} T(\ell - 1) \cong L(c')$. In particular, $T_{\ell-1}^{c'} T(\ell - 1)$ is indecomposable, and as translation functors preserve tilting modules, we conclude that $T_{\ell-1}^{c'} T(\ell - 1) \cong T(c)$. The final claim follows from Proposition I.6.10 because $T(\ell - 1) \cong L(\ell - 1)$. □

The final result that we need in order to determine the Krull-Schmidt decomposition of a tensor product of simple **G**-modules with ℓ-restricted highest weight is the following 'Clebsch-Gordan rule':

Lemma 1.3. *Let $a, b \in \mathbb{Z}_{\geq 0}$ and suppose that $a \geq b$. Then $\nabla(a) \otimes \nabla(b)$ has a good filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_b \subseteq M_{b+1} = \nabla(a) \otimes \nabla(b)$$

with $M_{i+1}/M_i \cong \nabla(a - b + 2i)$ for $i = 0, \ldots, b$.

Proof. Recall from Theorem I.5.2 that the tensor product $\nabla(a) \otimes \nabla(b)$ has a good filtration. By Weyl's character formula, we have

$$\mathrm{ch}\, \nabla(c) = \frac{e^{c+1} - e^{-c-1}}{e - e^{-1}} = e^c + e^{c-2} + \cdots + e^{2-c} + e^{-c}$$

for all $c \in \mathbb{Z}_{\geq 0}$, and it is straightforward to verify that

$$\mathrm{ch}\big(\nabla(a) \otimes \nabla(b)\big) = \mathrm{ch}\, \nabla(a) \cdot \mathrm{ch}\, \nabla(b) = \sum_{i=0}^{b} \mathrm{ch}\, \nabla(a - b + 2i).$$

As the characters of the induced modules $\nabla(c)$, for $c \in \mathbb{Z}_{\geq 0}$, form a basis of $\mathbb{Z}[X]^{W_{\mathrm{fin}}}$, it follows that $\nabla(a) \otimes \nabla(b)$ has a good filtration with subquotients $\nabla(a - b + 2i)$, for $i = 0, \ldots, b$, each appearing with multiplicity one. By the remarks after Proposition I.5.1, we can choose a good filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_b \subseteq M_{b+1} = \nabla(a) \otimes \nabla(b)$$

with $M_{i+1}/M_i \cong \nabla(a - b + 2i)$ for $i = 0, \ldots, b$, as claimed. □

Remark 1.4. The 'classical' Clebsch-Gordan rule for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ states that

$$L_{\mathbb{C}}(a) \otimes L_{\mathbb{C}}(b) \cong \bigoplus_{i=0}^{b} L_{\mathbb{C}}(a-b+2i)$$

for all $a, b \in \mathbb{Z}_{\geq 0}$ with $a \geq b$. Just as Lemma 1.3, this can be proven by comparing characters.

Definition 1.5. For $a, b \in \mathbb{Z}_{\geq 0}$, we set

$$\mathrm{CG}(a,b) = \big\{ |a-b| + 2i \bigm| i = 0, \ldots, \min\{a,b\} \big\}$$

and

$$\mathrm{CG}_\ell(a,b) \coloneqq \mathrm{CG}(a,b) \setminus \big\{ 2\ell - 2 - c \bigm| c \in \mathrm{CG}(a,b) \text{ with } c \geq \ell \big\}.$$

The following result also appears as Lemma 1.3 in [DH05] in the modular case; the proof in the quantum case is completely analogous. Note that the set $\mathrm{CG}_\ell(a,b)$ is denoted by $W(a,b)$ in [DH05].

Lemma 1.6. *For $a, b \in X_1$, we have*

$$L(a) \otimes L(b) \cong \bigoplus_{c \in \mathrm{CG}_\ell(a,b)} T(c).$$

Proof. Recall from Lemma 1.1 that $L(a) \otimes L(b)$ is a direct sum of indecomposable tilting modules. As the characters $\operatorname{ch} T(c)$, for $c \in \mathbb{Z}_{\geq 0}$, form a basis of $\mathbb{Z}[X]^{W_{\mathrm{fin}}}$, it suffices to prove that the tensor product on the left hand side has the same good filtration multiplicities as the direct sum on the right hand side. This is straightforward to see, using Lemmas 1.2 and 1.3. □

Before we return to generic direct sumands, let us say some words about the (extended) affine Weyl group of $\mathbf{G}$ and its (dot) action on $X \cong \mathbb{Z}$. The finite Weyl group W_{fin} is cyclic of order 2, generated by the reflection $s \coloneqq s_\alpha$, which acts on $X_{\mathbb{R}} \cong \mathbb{R}$ via $s(z) = -z$ for $z \in \mathbb{R}$. The root lattice $\mathbb{Z}\Phi \subseteq X$ identifies with the set $2\mathbb{Z}$ of even integers, and it follows that $\Omega \cong X/\mathbb{Z}\Phi \cong \mathbb{Z}/2\mathbb{Z}$ is also cyclic of order 2. The non trivial element of Ω is $w \coloneqq t_1 s$ (where t_1 denotes the translation $z \mapsto z+1$ on $\mathbb{R}$, according to our identification of $X_{\mathbb{R}}$ with $\mathbb{R}$), and its (dot) action on $\mathbb{R}$ is given by $w \cdot z = \ell - 2 - z$, for $z \in \mathbb{R}$. For $x = t_a s^\varepsilon \in W_{\mathrm{ext}}$, with $a \in \mathbb{Z}$ and $\varepsilon \in \{0,1\}$, we have

$$x \cdot 0 = t_a s^\varepsilon \cdot 0 = \ell a - 2\varepsilon = \ell \cdot (a - \varepsilon) + \varepsilon \cdot (\ell - 2) = t_{a-\varepsilon} w^\varepsilon \cdot 0,$$

and it follows that, for all $x \in W_{\mathrm{ext}}$, we can choose an integer $b \in \mathbb{Z}$ and and element $\omega \in \Omega$ such that $x \cdot 0 = t_b \omega \cdot 0$. Furthermore, b and ω are unique with this property if $\ell > 2$, and we have $b \geq 0$ whenever $x \in W_{\mathrm{ext}}^+$.

In the quantum case, the classical Clebsch-Gordan rule already allows us to determine all regular indecomposable direct summands of tensor products of simple $\mathbf{G}$-modules with highest weights in arbitrary ℓ-alcoves.

Lemma 1.7. *Suppose that we are in the quantum case. For $x, y \in W_{\mathrm{ext}}^+$, let $a, b \in \mathbb{Z}_{\geq 0}$ and $\omega, \omega' \in \Omega$ such that $x \cdot 0 = t_a \omega \cdot 0$ and $y \cdot 0 = t_b \omega' \cdot 0$. Then*

$$\big(L(x \cdot 0) \otimes L(y \cdot 0) \big)_{\mathrm{reg}} \cong \bigoplus_{c \in \mathrm{CG}(a,b)} T^{\omega\omega'} L(\ell c)$$

and $G(x,y) \cong T^{\omega\omega'} L\big(\ell \cdot (a+b)\big)$.

Proof. As $L(x \cdot 0) \cong T^{\omega} L(t_a \cdot 0)$ and $L(y \cdot 0) \cong T^{\omega'} L(t_b \cdot 0)$ by Proposition I.6.8 and

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \big(T^{\omega} L(t_a \cdot 0) \otimes T^{\omega'} L(t_b \cdot 0)\big)_{\mathrm{reg}} \cong T^{\omega\omega'} \big(L(t_a \cdot 0) \otimes L(t_b \cdot 0)\big)_{\mathrm{reg}}$$

by Theorem II.4.14, it suffices to prove that

$$\big(L(t_a \cdot 0) \otimes L(t_b \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{c \in \mathrm{CG}(a,b)} L(\ell c).$$

Observe that $L(t_a \cdot 0) = L(\ell a) \cong L_{\mathbb{C}}(a)^{[1]}$ and $L(t_b \cdot 0) = L(\ell b) \cong L_{\mathbb{C}}(b)^{[1]}$, and therefore

$$L(t_a \cdot 0) \otimes L(t_b \cdot 0) \cong \big(L_{\mathbb{C}}(a) \otimes L_{\mathbb{C}}(b)\big)^{[1]} \cong \bigoplus_{c \in \mathrm{CG}(a,b)} L_{\mathbb{C}}(c)^{[1]}$$

by Remark 1.4. Now the first claim follows since $L_{\mathbb{C}}(c)^{[1]} \cong L(\ell c)$ is regular for all $c \in \mathrm{CG}(a,b)$, by Lemma II.4.3. Using Corollary II.2.7, we see that $T^{\omega\omega'} L\big(\ell \cdot (a+b)\big)$ is the unique regular indecomposable direct summand of $L(x \cdot 0) \otimes L(y \cdot 0)$ of good filtration dimension $\ell(t_{a+b}) = a + b = \ell(x) + \ell(y)$, and we conclude that $G(x,y) \cong T^{\omega\omega'} L\big(\ell \cdot (a+b)\big)$. □

From now on until Lemma 1.11 (included), suppose that we are in the modular case.

Definition 1.8. For a sequence $\mathbf{u} = (u_0, u_1, \ldots) \in \mathbb{Z}^{\mathbb{N}}$ with $0 \leq u_i \leq 2\ell - 2$ for all i and $u_i = 0$ for all but finitely many i, let

$$J(\mathbf{u}) := \bigotimes_{i \geq 0} T(u_i)^{[i]}.$$

Lemma 1.9. *For any sequence $\mathbf{u} = (u_0, u_1, \ldots) \in \mathbb{Z}^{\mathbb{N}}$ with $0 \leq u_i \leq 2\ell - 2$ for all i and $u_i = 0$ for all but finitely many i, the $\mathbf{G}$-module $J(\mathbf{u})$ is indecomposable.*

Proof. By Lemma I.8.6, it suffices to prove that the tilting module $T(a)$ is indecomposable as a $\mathbf{G}_1$-module for all $0 \leq a \leq 2\ell - 2$. If $a < \ell$ then $T(a) \cong L(a)$ affords the simple $\mathbf{G}_1$-module $L_1(a)$, so now suppose that $a \geq \ell$, and recall from Lemma 1.2 that $T(a)$ (considered as a $\mathbf{G}$-module) is uniserial and of composition length 3, with simple head and socle isomorphic to $L(2\ell - 2 - a)$. As explained in Section I.8, there is an isomorphism of $\mathbf{G}$-modules

$$\mathrm{soc}_{\mathbf{G}_1} T(a) \cong \bigoplus_{b=0}^{\ell-1} L(b) \otimes \mathrm{Hom}_{\mathbf{G}_1}\big(L(b), T(a)\big),$$

and as $\mathrm{soc}_{\mathbf{G}_1} T(a)$ is a $\mathbf{G}$-submodule of $T(a)$, the fact that $\mathrm{soc}_{\mathbf{G}} T(a) \cong L(2\ell - 2 - a)$ implies that

$$\mathrm{soc}_{\mathbf{G}_1} T(a) \cong L(2\ell - 2 - a) \otimes \mathrm{Hom}_{\mathbf{G}_1}\big(L(2\ell - 2 - a), T(a)\big).$$

Now weight considerations show that $\mathrm{Hom}_{\mathbf{G}_1}\big(L(2\ell - 2 - a), T(a)\big) \cong L(0)$ is the trivial $\mathbf{G}$-module, whence $\mathrm{soc}_{\mathbf{G}_1} T(a) \cong L(2\ell - 2 - a)$ and $T(a)$ is indecomposable as a $\mathbf{G}_1$-module. □

Lemma 1.10. *Let $a, b \in \mathbb{Z}_{\geq 0}$ and write $a = \sum_i a_i \ell^i$ and $b = \sum_i b_i \ell^i$ with $0 \leq a_i, b_i < \ell$ for all i. Then the Krull-Schmidt decomposition of $L(a) \otimes L(b)$ is given by*

$$L(a) \otimes L(b) \cong \bigoplus_{\mathbf{u}} J(\mathbf{u}),$$

where the direct sum runs over all sequences $\mathbf{u} = (u_0, u_1, \ldots)$ with $u_i \in \mathrm{CG}_\ell(a_i, b_i)$ for all i.

Proof. By Steinberg's tensor product theorem, we have $L(a) \cong \bigotimes_i L(a_i)^{[i]}$ and $L(b) \cong \bigotimes_i L(b_i)^{[i]}$, and it follows that

$$L(a) \otimes L(b) \cong \bigotimes_{i \geq 0} \big(L(a_i) \otimes L(b_i) \big)^{[i]}.$$

Furthermore, we have

$$L(a_i) \otimes L(b_i) \cong \bigoplus_{u \in \mathrm{CG}_\ell(a_i,b_i)} T(u)$$

for all $i \geq 0$ by Lemma 1.6, and by distributivity of tensor products and direct sums, we obtain the claimed direct sum decomposition

$$L(a) \otimes L(b) \cong \bigoplus_{\mathbf{u}} J(\mathbf{u}).$$

This is a Krull-Schmidt decomposition because $J(\mathbf{u})$ is indecomposable for all $\mathbf{u}$, by Lemma 1.9 $\square$

Lemma 1.11. *Suppose that we are in the modular case. For $x, y \in W_{\mathrm{ext}}^+$, let $a, b \in \mathbb{Z}_{\geq 0}$ and $\omega, \omega' \in \Omega$ such that $x \cdot 0 = t_a \omega \cdot 0$ and $y \cdot 0 = t_b \omega' \cdot 0$. Furthermore, write $a = \sum_i a_i \ell^i$ and $b = \sum_i b_i \ell^i$ with $0 \leq a_i < \ell$ and $0 \leq b_i < \ell$ for all i. Then*

$$\big(L(x \cdot 0) \otimes L(y \cdot 0) \big)_{\mathrm{reg}} \cong \bigoplus_{\mathbf{u}} T^{\omega\omega'} J(\mathbf{u})^{[1]},$$

where the direct sum runs over all sequences $\mathbf{u} = (u_0, u_1, \ldots)$ with $u_i \in \mathrm{CG}_\ell(a_i, b_i)$ for all i. Furthermore, with $\mathbf{a} = (a_0 + b_0, a_1 + b_1, \ldots)$, we have and $G(x,y) \cong T^{\omega\omega'} J(\mathbf{a})^{[1]}$.

Proof. As in the proof of Lemma 1.7, we have

$$\big(L(x \cdot 0) \otimes L(y \cdot 0) \big)_{\mathrm{reg}} \cong \big(T^{\omega} L(t_a \cdot 0) \otimes T^{\omega'} L(t_b \cdot 0) \big)_{\mathrm{reg}} \cong T^{\omega\omega'} \big(L(t_a \cdot 0) \otimes L(t_b \cdot 0) \big)_{\mathrm{reg}},$$

and it suffices to prove that

$$\big(L(t_a \cdot 0) \otimes L(t_b \cdot 0) \big)_{\mathrm{reg}} \cong \bigoplus_{\mathbf{u}} J(\mathbf{u})^{[1]}.$$

Observe that $L(t_a \cdot 0) = L(\ell a) \cong L(a)^{[1]}$ and $L(t_b \cdot 0) = L(\ell b) \cong L(b)^{[1]}$, and therefore

$$L(t_a \cdot 0) \otimes L(t_b \cdot 0) \cong \big(L(a) \otimes L(b) \big)^{[1]} \cong \bigoplus_{\mathbf{u}} J(\mathbf{u})^{[1]}$$

by Lemma 1.10. Now the first claim follows since $J(\mathbf{u})^{[1]}$ is regular for all $\mathbf{u}$ by Proposition II.6.12.

Furthermore, it is straightforward to see that $J(\mathbf{a}) \cong M(a,b)$ is the unique indecomposable direct summand of $L(a) \otimes L(b)$ with a non-zero $a+b$-weight space. As $G(e,e) \cong L(0)$ is indecomposable as a $\mathbf{G}_1$-module, Lemmas II.5.10 and II.6.16 yield

$$G(x,y) = G(t_a \omega, t_b \omega') \cong T^{\omega\omega'} G(t_a, t_b) \cong T^{\omega\omega'} M(a,b)^{[1]} \cong T^{\omega\omega'} J(\mathbf{a})^{[1]},$$

as claimed. $\square$

Costandard modules

Instead of computing the generic direct summands of tensor products of costandard modules directly for $\mathbf{G}$ of type A_1, we are going to prove a much more general result about tensor products of costandard modules whose highest weights are multiples of the first fundamental dominant weight ϖ_1, for $\mathbf{G}$ of type A_n and $n \geq 1$. It will be necessary to temporarily leave the realm of simple algebraic group schemes (or quantum groups corresponding to simple Lie algebras) and work with $\hat{\mathbf{G}} = \mathrm{GL}_{n+1}(\Bbbk)$ or $\hat{\mathbf{G}} = U_\zeta\big(\mathfrak{gl}_{n+1}(\mathbb{C})\big)$ instead of $\mathrm{SL}_{n+1}(\Bbbk)$ or $U_\zeta\big(\mathfrak{sl}_{n+1}(\mathbb{C})\big)$, in the modular case or the quantum case, respectively. Furthermore, the Schur algebras corresponding to the (quantum) general linear group will play an important role. More specifically, we will show that the unique indecomposable direct summand of the tensor product $\nabla(a\varpi_1) \otimes \nabla(b\varpi_1)$ with a non-zero $(a+b)\cdot\varpi_1$-weight space is the injective hull of a simple module of the form $L\big((a+b)\cdot\varpi_1 - c\alpha_1\big)$ over a Schur algebra $S\big(n+1, a+b\big)$, for $a, b \geq 0$. We start with the modular case and then discuss the changes one needs to make to obtain a quantum analogue.

The modular case. Suppose that we are in the modular case and that Φ is of type A_n for some $n \geq 1$. Then $\mathbf{G} = \mathrm{SL}_{n+1}(\Bbbk)$ and we can choose $\mathbf{T}$ as the subgroup of diagonal matrices in $\mathbf{G}$ and $\mathbf{B}$ as the subgroup of lower triangular matrices in $\mathbf{G}$. We fix a numbering of the simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ according to the Dynkin diagram in Figure I.1.1 and define $\hat{\mathbf{G}} = \mathrm{GL}_{n+1}(\Bbbk)$.

Following Sections II.1.21 and II.2.15 in [Jan03] (up to an index shift), the weight lattice $\hat{X}$ of $\hat{\mathbf{G}}$, with respect to the maximal torus of diagonal matrices in $\hat{\mathbf{G}}$, is a free $\mathbb{Z}$-module of rank $n+1$ with basis $\varepsilon_0, \dots, \varepsilon_n$, given by the weights of the canonical basis vectors $e_0, \dots, e_n$ of the natural $\hat{\mathbf{G}}$-module $E = \Bbbk^{n+1}$. A base of the root system $\hat{\Phi} = \{\varepsilon_i - \varepsilon_j \mid 0 \leq i, j \leq n, i \neq j\}$ of $\hat{\mathbf{G}}$, corresponding to the Borel subgroup of upper triangular matrices, is given by $\hat{\Pi} = \{\varepsilon_{i-1} - \varepsilon_i \mid 1 \leq i \leq n\}$, and the set of dominant weights in $\hat{X}$ with respect to $\hat{\Pi}$ is

$$\hat{X}^+ = \Big\{ \sum_{i=0}^n \lambda_i \varepsilon_i \ \Big|\ \lambda_0 \geq \cdots \geq \lambda_n \Big\}.$$

For $\lambda \in \hat{X}^+$, we write $\hat{L}(\lambda)$, $\hat{\Delta}(\lambda)$, $\hat{\nabla}(\lambda)$ and $\hat{T}(\lambda)$ for the simple module, the Weyl module, the induced module and the indecomposable tilting module of highest weight λ over $\hat{\mathbf{G}}$, respectively. It is straightforward to see that

$$E \cong \hat{\nabla}(\varepsilon_0) \cong \hat{L}(\varepsilon_0),$$

and as in Section II.2.16 in [Jan03], one can show that $S^aE \cong \hat{\nabla}(a\varepsilon_0)$ for all $a \geq 0$, where S^aE denotes the a-th symmetric power of the natural $\hat{\mathbf{G}}$-module E.

There is a surjective homomorphism of $\mathbb{Z}$-modules $\hat{X} \to X$ (coming from restriction to the maximal torus $\mathbf{T}$ of $\mathbf{G}$) with

$$\lambda = \sum_{i=0}^n \lambda_i \varepsilon_i \longmapsto \lambda' \coloneqq \sum_{i=1}^n (\lambda_{i-1} - \lambda_i) \cdot \varpi_i, \tag{1.1}$$

and it is straightforward to verify that the latter maps $\hat{\Pi}$ to Π and $\hat{X}^+$ onto X^+. By Section II.2.10 in [Jan03], we have $\mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \hat{L}(\lambda) \cong L(\lambda')$, for all $\lambda \in \hat{X}^+$, and similarly, one sees that $\mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \hat{\nabla}(\lambda) \cong \nabla(\lambda')$.

As for simply-connected simple algebraic groups, let us write $\mathrm{Rep}(\hat{\mathbf{G}})$ for the category of (finite-dimensional) $\hat{\mathbf{G}}$-modules. For any subset $\pi \subseteq X^+$, we denote by $\mathrm{Rep}(\mathbf{G}, \pi)$ the *truncated subcategory* of $\mathrm{Rep}(\mathbf{G})$ corresponding to π, that is, the full subcategory whose objects are the $\mathbf{G}$-modules M such that for $\lambda \in X^+$, we have $[M : L(\lambda)] = 0$ unless $\lambda \in \pi$. We say that π is *saturated* if $\mu \uparrow_\ell \lambda$ and $\lambda \in \pi$ implies that $\mu \in \pi$, for all $\lambda, \mu \in X^+$. Analogously, we write $\mathrm{Rep}(\hat{\mathbf{G}}, \hat{\pi})$ for the truncated subcategory of $\mathrm{Rep}(\hat{\mathbf{G}})$ corresponding to $\hat{\pi} \subseteq \hat{X}^+$, and we say that $\hat{\pi}$ is saturated if $\mu \uparrow_\ell \lambda$ and $\lambda \in \hat{\pi}$ implies that $\mu \in \hat{\pi}$, for all $\lambda, \mu \in \hat{X}^+$. Note that for $\hat{\pi} \subseteq \hat{X}^+$ and $\pi \subseteq X^+$ with $\lambda' \in \pi$ for all $\lambda \in \hat{\pi}$, the restriction functor $\mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \colon \mathrm{Rep}(\hat{\mathbf{G}}) \to \mathrm{Rep}(\mathbf{G})$ gives rise to a restriction functor $\mathrm{Rep}(\hat{\mathbf{G}}, \hat{\pi}) \to \mathrm{Rep}(\mathbf{G}, \pi)$.

Now let $\hat{\pi} \subseteq \hat{X}^+$ be finite and saturated, and note that every $\hat{\mathbf{G}}$-module has a natural $\mathrm{Dist}(\hat{\mathbf{G}})$-module structure, as for simply-connected simple algebraic groups. We define the *Schur algebra* of $\hat{\mathbf{G}}$ with respect to $\hat{\pi}$ as the quotient

$$S_{\hat{\mathbf{G}}}(\hat{\pi}) \coloneqq \mathrm{Dist}(\hat{\mathbf{G}})/I_{\hat{\mathbf{G}}}(\hat{\pi})$$

of $\mathrm{Dist}(\hat{\mathbf{G}})$ by the two-sided ideal $I_{\hat{\mathbf{G}}}(\hat{\pi})$ of elements of $\mathrm{Dist}(\hat{\mathbf{G}})$ that annihilate all $\hat{\mathbf{G}}$-modules in the truncated category $\mathrm{Rep}(\hat{\mathbf{G}}, \hat{\pi})$. Analogously, we define the Schur algebra of $\mathbf{G}$ with respect to $\pi \subseteq X^+$ (finite and saturated) as the quotient

$$S_{\mathbf{G}}(\pi) \coloneqq \mathrm{Dist}(\mathbf{G})/I_{\mathbf{G}}(\pi)$$

of $\mathrm{Dist}(\mathbf{G})$ by the two-sided ideal $I_{\mathbf{G}}(\pi)$ of elements of $\mathrm{Dist}(\mathbf{G})$ that annihilate all $\mathbf{G}$-modules in the truncated category $\mathrm{Rep}(\mathbf{G}, \pi)$.[1] By Sections II.A.15 and II.A.16 in [Jan03], the Schur algebras $S_{\hat{\mathbf{G}}}(\hat{\pi})$ and $S_{\mathbf{G}}(\pi)$ are finite-dimensional. More precisely, we have

$$\dim S_{\hat{\mathbf{G}}}(\hat{\pi}) = \sum_{\lambda \in \hat{\pi}} \big(\dim \hat{\Delta}(\lambda) \big)^2 \qquad \text{and} \qquad \dim S_{\mathbf{G}}(\pi) = \sum_{\lambda \in \pi} \big(\dim \Delta(\lambda) \big)^2.$$

Note that by definition, every $\hat{\mathbf{G}}$-module in $\mathrm{Rep}(\hat{\mathbf{G}}, \hat{\pi})$ has a natural $S_{\hat{\mathbf{G}}}(\hat{\pi})$-module structure and every $\mathbf{G}$-module in $\mathrm{Rep}(\mathbf{G}, \pi)$ has a natural $S_{\mathbf{G}}(\pi)$-module structure. According to Section II.A.17 in [Jan03], this gives rise to an equivalence between $\mathrm{Rep}(\hat{\mathbf{G}}, \hat{\pi})$ and the category of (finite-dimensional) $S_{\hat{\mathbf{G}}}(\hat{\pi})$-modules, and between $\mathrm{Rep}(\mathbf{G}, \pi)$ and the category of (finite-dimensional) $S_{\mathbf{G}}(\pi)$-modules.

By point (3) in Section I.7.2 in [Jan03], the embedding of $\mathbf{G}$ into $\hat{\mathbf{G}}$ gives rise to an embedding of $\mathrm{Dist}(\mathbf{G})$ into $\mathrm{Dist}(\hat{\mathbf{G}})$. For $\hat{\pi} \subseteq \hat{X}^+$ and $\pi \subseteq X^+$ with $\lambda' \in \pi$ for all $\lambda \in \hat{\pi}$, it is straightforward to see that the latter affords a homomorphism $S_{\mathbf{G}}(\pi) \to S_{\hat{\mathbf{G}}}(\hat{\pi})$.

Lemma 1.12. *Let $\hat{\pi} \subseteq \hat{X}^+$ and $\pi \subseteq X^+$ be finite and saturated, and suppose that the map $\lambda \mapsto \lambda'$ induces a bijection between $\hat{\pi}$ and π. Then the canonical homomorphism $S_{\mathbf{G}}(\pi) \to S_{\hat{\mathbf{G}}}(\hat{\pi})$ is an isomorphism.*

Proof. By Weyl's character formula, we have $\dim \hat{\Delta}(\lambda) = \dim \Delta(\lambda')$ for all $\lambda \in \hat{X}^+$, and it follows that

$$\dim S_{\hat{\mathbf{G}}}(\hat{\pi}) = \dim S_{\mathbf{G}}(\pi).$$

Hence it suffices to prove that the homomorphism is injective, or in other words, that an element of $\mathrm{Dist}(\mathbf{G})$ which annihilates the restriction to $\mathbf{G}$ of every $\hat{\mathbf{G}}$-module in $\mathrm{Rep}(\hat{\mathbf{G}}, \hat{\pi})$ also annihilates all $\mathbf{G}$-modules in $\mathrm{Rep}(\mathbf{G}, \pi)$. So let $\vartheta \in \mathrm{Dist}(\mathbf{G})$ and suppose that ϑ annihilates the restriction to $\mathbf{G}$ of all $\hat{\mathbf{G}}$-modules in $\mathrm{Rep}(\hat{\mathbf{G}}, \hat{\pi})$. By Section II.E.7 in [Jan03], the restriction to $\mathbf{G}$ of an indecomposable tilting $\hat{\mathbf{G}}$-module $\hat{T}(\lambda)$ with $\lambda \in \hat{\pi}$ is isomorphic to $T(\lambda')$, hence ϑ annihilates all indecomposable tilting $\mathbf{G}$-modules $T(\mu)$ with $\mu \in \pi$.

[1] The definition of Schur algebras in Section II.A of [Jan96] is a different one, but the two definitions are equivalent by Proposition II.A.16 in [Jan03].

For an arbitrary $\mathbf{G}$-module M in $\mathrm{Rep}(\mathbf{G},\pi)$, we claim that all terms of $C_{\min}(M)$, the minimal tilting complex of M, belong to $\mathrm{Rep}(\mathbf{G},\pi)$. Indeed, as

$$\mathrm{Ext}^i_{\mathrm{Rep}(\mathbf{G},\pi)}(V,W) \cong \mathrm{Ext}^i_{\mathbf{G}}(V,W)$$

for all $\mathbf{G}$-modules V and W in $\mathrm{Rep}(\mathbf{G},\pi)$ (see Proposition II.A.10 in [Jan03]), we can show as in Proposition I.5.4 that the natural functor

$$\mathfrak{T}_\pi\colon K^b\big(\mathrm{Tilt}(\mathbf{G},\pi)\big) \longrightarrow D^b\big(\mathrm{Rep}(\mathbf{G},\pi)\big),$$

from the bounded homotopy category of the full subcategory $\mathrm{Tilt}(\mathbf{G},\pi)$ of tilting modules in $\mathrm{Rep}(\mathbf{G},\pi)$ to the bounded derived category of $\mathrm{Rep}(\mathbf{G},\pi)$, is an equivalence of categories. Therefore, we can choose the minimal tilting complex of M with terms in $\mathrm{Tilt}(\mathbf{G},\pi)$, as claimed. As ϑ annihilates all tilting modules in $\mathrm{Rep}(\mathbf{G},\pi)$, this implies that ϑ annihilates M, as required. $\square$

Now for $d \geq 0$, let us fix

$$\begin{aligned}\hat{\pi}(d) &= \{\lambda \in \hat{X}^+ \mid \lambda \leq d\varepsilon_0\} \\ &= \big\{a_0\varepsilon_0 + \cdots + a_n\varepsilon_n \;\big|\; a_0 \geq \cdots \geq a_n \geq 0 \text{ and } a_0 + \cdots + a_n = d\big\}\end{aligned}$$

and

$$\begin{aligned}\pi(d) &= \{\lambda \in X^+ \mid \lambda \leq d\varpi_1\} \\ &= \Big\{\sum\nolimits_i b_i\varpi_i \;\Big|\; b_1,\ldots,b_n \geq 0 \text{ and } d - \sum\nolimits_i ib_i = (n+1)\cdot a \text{ for some } a \in \mathbb{Z}_{\geq 0}\Big\}.\end{aligned}$$

Note that $\hat{\pi}(d)$ and $\pi(d)$ satisfy the hypothesis of Lemma 1.12, so

$$S_{\mathbf{G}}\big(\pi(d)\big) \cong S_{\hat{\mathbf{G}}}\big(\hat{\pi}(d)\big) =: S(n+1,d).$$

By Section II.A.18 in [Jan03], the algebra $S(n+1,d)$ coincides with the *classical Schur algebra* (as introduced in Section 2.3 of [Gre07]), defined as the dual algebra of a certain finite-dimensional subcoalgebra $A(n+1,d)$ of the coordinate ring $\Bbbk[\hat{\mathbf{G}}]$. The main reason for our interest in Schur algebras is the following result of S. Donkin:

Proposition 1.13. *Let $\alpha = a_0\varepsilon_0 + \cdots + a_n\varepsilon_n \in \hat{X}$ with $a_0,\ldots,a_n \geq 0$ and set $r = a_0 + \cdots + a_n$. The tensor product of symmetric powers*

$$S^\alpha E := S^{a_0}E \otimes \cdots \otimes S^{a_n}E$$

belongs to $\mathrm{Rep}\big(\hat{\mathbf{G}},\hat{\pi}(r)\big)$. Furthermore, for every $\hat{\mathbf{G}}$-module M in $\mathrm{Rep}\big(\hat{\mathbf{G}},\hat{\pi}(r)\big)$, we have

$$\mathrm{Hom}_{S(n+1,r)}(M,S^\alpha E) \cong M_\alpha,$$

the α-weight space of M, and $S^\alpha E$ is an injective $S(n+1,r)$-module.

Proof. The first claim is straightforward to verify by weight considerations. The remaining claims are proven in Section 2.1(8) of [Don98] for the so-called *q-Schur algebra* $S_q(n+1,r)$. For $q=1$, this is just the classical Schur algebra $S(n+1,r)$. $\square$

Let again $\pi \subseteq X^+$ be saturated. By Section II.A.6 in [Jan03], every simple $\mathbf{G}$-module $L(\lambda)$ with highest weight $\lambda \in \pi$ has an injective hull $I_\pi(\lambda)$ in $\mathrm{Rep}(\mathbf{G}, \pi)$, and the latter has a good filtration with

$$\big[I_\pi(\lambda) : \nabla(\mu)\big]_\nabla = \big[\nabla(\mu) : L(\lambda)\big] \tag{1.2}$$

for all $\mu \in \pi$. In view of Proposition 1.13, a tensor product of the form

$$\nabla(a\varpi_1) \otimes \nabla(b\varpi_1) \cong S^a E \otimes S^b E,$$

for $a, b \geq 0$, (where we omit the restriction functor from $\hat{\mathbf{G}}$ to $\mathbf{G}$) decomposes as a direct sum of injective indecomposable $S(n+1, a+b) = S_{\mathbf{G}}\big(\pi(a+b)\big)$-modules, and for $\mu \in \pi(a+b)$, the multiplicity of the injective hull $I_{\pi(a+b)}(\mu)$ of $L(\mu)$ in such a direct sum decomposition is

$$\begin{aligned}\big[\nabla(a\varpi_1) \otimes \nabla(b\varpi_1) : I_{\pi(a+b)}(\mu)\big]_\oplus &= \dim \mathrm{Hom}_{S(n+1,a+b)}\big(L(\mu), S^a E \otimes S^b E\big)\\ &= \dim L(\mu)_{(a\varepsilon_0 + b\varepsilon_1)'}\\ &= \dim L(\mu)_{(a-b)\cdot\varpi_1 + b\varpi_2}.\end{aligned} \tag{1.3}$$

In addition to this observation, we will need the following well-known lemma about composition multiplicities in induced modules for $\mathbf{G}$ of type A_1:

Lemma 1.14. *Suppose that $\mathbf{G}$ is of type A_1. Let $a, b \in \mathbb{Z}_{\geq 0}$ and write $b = \sum_{i\geq 0} b_i \ell^i$ with $0 \leq b_i < \ell$ for all $i \geq 0$. Then*

$$[\nabla(a) : L(b)] \leq 1,$$

and $[\nabla(a) : L(b)] = 1$ if and only if there exist $a_i \in \mathbb{Z}_{\geq 0}$, with $a_i \in \{b_i, 2\ell - b_i - 2\}$ for all $i \geq 0$, such that $a = \sum_{i\geq 0} a_i \ell^i$.

Proof. The first statement follows from the fact that $\nabla(a)$ has one-dimensional weight spaces. The second one can be found in Theorem 2.1 in [Hen01]. □

Now let us return to $\mathbf{G}$ of type A_n for some $n \geq 1$. Before discussing generic direct summands, we determine the unique indecomposable direct summand with a non-zero $(a+b)\cdot\varpi_1$-weight space in a tensor product of the form $\nabla(a\varpi_1) \otimes \nabla(b\varpi_1)$.

Proposition 1.15. *Let $a, b \in \mathbb{Z}_{\geq 0}$ and write $a = \sum_{i\geq 0} a_i \ell^i$ and $b = \sum_{i\geq 0} b_i \ell^i$ with $0 \leq a_i, b_i < \ell$ for all $i \geq 0$. Furthermore, define*

$$c_i := \begin{cases} a_i + b_i - (\ell - 1) & \text{if } a_i + b_i \geq \ell - 1,\\ 0 & \text{otherwise,}\end{cases}$$

for all $i \geq 0$, and $c = c(a,b) := \sum_{i\geq 0} c_i \ell^i$. Then $I_{\pi(a+b)}\big((a+b)\cdot\varpi_1 - c\alpha_1\big)$ is the unique indecomposable direct summand of $\nabla(a\varpi_1) \otimes \nabla(b\varpi_1)$ with a non-zero $(a+b)\cdot\varpi_1$-weight space.

Proof. By weight considerations, it is straightforward to see that $\nabla(a\varpi_1) \otimes \nabla(b\varpi_1)$ has a unique indecomposable direct summand that has a non zero $(a+b)\cdot\varpi_1$-weight space. Note that

$$a + b - 2c = \sum_{i\geq 0} (a_i + b_i - 2c_i) \cdot \ell^i$$

with $0 \leq a_i + b_i - 2c_i < \ell$ and

$$a_i + b_i \in \{a_i + b_i - 2c_i, 2\ell - (a_i + b_i - 2c_i) - 2\}$$

for all $i \geq 0$, by definition of the c_i. For $\mathbf{G}$ of type A_1, we have $\big[\nabla(a+b) : L(a+b-2c)\big] = 1$ by Lemma 1.14. Using truncation to the Levi subgroup corresponding to the subset $\{\alpha_1\} \subseteq \Pi$ and the reciprocity formula (1.2), we conclude (for $\mathbf{G}$ of type A_n with $n \geq 1$) that

$$\begin{aligned} 1 &= \big[\nabla(a+b) : L(a+b-2c)\big] \\ &= \big[\nabla\big((a+b)\cdot\varpi_1\big) : L\big((a+b)\cdot\varpi_1 - c\alpha_1\big)\big] \\ &= \big[I_{\pi(a+b)}\big((a+b)\cdot\varpi_1 - c\alpha_1\big) : \nabla\big((a+b)\cdot\varpi_1\big)\big]_\nabla. \end{aligned}$$

In particular, $I_{\pi(a+b)}\big((a+b)\cdot\varpi_1 - c\alpha_1\big)$ has a non-zero $(a+b)\cdot\varpi_1$-weight space.

By equation (1.3), it now suffices to prove that the weight space $L\big((a+b)\cdot\varpi_1 - c\alpha_1\big)_{(a-b)\cdot\varpi_1+b\varpi_2}$ is non-zero. As before, we can truncate to the Levi subgroup corresponding to $\{\alpha_1\} \subseteq \Pi$ and consider the weight space $L(a+b-2c)_{a-b}$ of the simple $\mathrm{SL}_2(\Bbbk)$-module $L(a+b-2c)$ instead. Recall that

$$a+b-2c = \sum_{i\geq 0}(a_i+b_i-2c_i)\cdot\ell^i,$$

where $0 \leq a_i + b_i - 2c_i < \ell$ for all $i \geq 0$, and note that

$$a-b = \sum_{i\geq 0}(a_i-b_i)\cdot\ell^i.$$

By Steinberg's tensor product theorem, it suffices to prove that $a_i - b_i$ is a weight of $L(a_i+b_i-2c_i)$ for all i. If $a_i + b_i \leq \ell - 1$ then $c_i = 0$ and $a_i - b_i$ belongs to the set $\{a_i+b_i, a_i+b_i-2, \ldots, -a_i-b_i\}$ of weights of $L(a_i+b_i)$. Otherwise, we have

$$a_i + b_i - 2c_i = 2\cdot(\ell-1) - (a_i+b_i) \geq |a_i - b_i|$$

since $a_i, b_i \leq \ell - 1$, and again, it follows that $a_i - b_i$ is a weight of $L(a_i+b_i-2c_i)$. ☐

The following corollary is also proved in Proposition 4.8.(12) in [Don98].

Corollary 1.16. *Let $a, b \in \mathbb{Z}_{\geq 0}$ and write $a = \sum_{i\geq 0} a_i\ell^i$ and $b = \sum_{i\geq 0} b_i\ell^i$ with $0 \leq a_i, b_i < \ell$ for all $i \geq 0$. Then $\nabla\big((a+b)\cdot\varpi_1\big)$ is a direct summand of $\nabla(a\varpi_1)\otimes\nabla(b\varpi_1)$ if and only if $a_i + b_i \leq \ell - 1$ for all $i \geq 0$.*

Proof. With $c = c(a,b)$ as in Proposition 1.15, the unique indecomposable direct summand of the tensor product $\nabla(a\varpi_1)\otimes\nabla(b\varpi_1)$ with a non-zero $(a+b)\cdot\varpi_1$-weight space is the injective indecomposable $S(n+1, a+b)$-module $I_{\pi(a+b)}\big((a+b)\cdot\varpi_1 - c\alpha_1\big)$. In particular, $\nabla\big((a+b)\cdot\varpi_1\big)$ is a direct summand of $\nabla(a\varpi_1)\otimes\nabla(b\varpi_1)$ if and only if $I_{\pi(a+b)}\big((a+b)\cdot\varpi_1 - c\alpha_1\big) \cong \nabla\big((a+b)\cdot\varpi_1\big)$. By the reciprocity formula (1.2) and the fact that $(a+b)\cdot\varpi_1$ is maximal in $\pi(a+b)$, we have

$$\nabla\big((a+b)\cdot\varpi_1\big) \cong I_{\pi(a+b)}\big((a+b)\cdot\varpi_1\big),$$

and it follows that $\nabla\big((a+b)\cdot\varpi_1\big)$ is a direct summand of $\nabla(a\varpi_1)\otimes\nabla(b\varpi_1)$ if and only if $c = 0$, or equivalently, if $a_i + b_i \leq \ell - 1$ for all $i \geq 0$. ☐

From the preceding corollary, we can also get some information about generic direct summands.

Corollary 1.17. *Suppose that $\ell \geq h = n+1$. Let $a, b \in \mathbb{Z}_{\geq 0}$ and write $a = \sum_{i \geq 0} a_i \ell^i$ and $b = \sum_{i \geq 0} b_i \ell^i$ with $0 \leq a_i, b_i < \ell$ for all $i \geq 0$. If $a_i + b_i \leq \ell - 1$ for all $i \geq 0$ then $G_\nabla(t_{a\varpi_1}, t_{b\varpi_1}) \cong \nabla\big(\ell \cdot (a+b) \cdot \varpi_1\big)$.*

Proof. By Corollary 1.16, the assumption implies that $\nabla\big(\ell \cdot (a+b) \cdot \varpi_1\big)$ is a direct summand of

$$\nabla(\ell a \varpi_1) \otimes \nabla(\ell b \varpi_1) = \nabla(t_{a\varpi_1} \boldsymbol{\cdot} 0) \otimes \nabla(t_{b\varpi_1} \boldsymbol{\cdot} 0),$$

and the claim is immediate because $\nabla\big(\ell \cdot (a+b) \cdot \varpi_1\big)$ is regular (see Lemma II.4.3). $\square$

Now let us return to $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$. Using Proposition 1.15, we can determine the generic direct summands of tensor products of induced modules.

Lemma 1.18. *For $x, y \in W_{\mathrm{ext}}^+$, let $a, b \in \mathbb{Z}_{\geq 0}$ and $\omega, \omega' \in \Omega$ such that $x \boldsymbol{\cdot} 0 = t_a \omega \boldsymbol{\cdot} 0$ and $y \boldsymbol{\cdot} 0 = t_b \omega' \boldsymbol{\cdot} 0$. Define $c = c(a,b)$ as in Proposition 1.15. Then*

$$G_\nabla(x,y) \cong T^{\omega\omega'} I_{\pi(\ell \cdot (a+b))}\big(\ell \cdot (a+b-2c)\big).$$

Proof. By Lemma II.5.5, we have $G_\nabla(x,y) \cong T^{\omega\omega'} G_\nabla(t_a, t_b)$, so it suffices to prove that

$$G_\nabla(t_a, t_b) \cong I_{\pi(\ell \cdot (a+b))}\big(\ell \cdot (a+b-2c)\big).$$

Let us write $M \coloneqq I_{\pi(\ell \cdot (a+b))}\big(\ell \cdot (a+b-2c)\big)$, and recall from Proposition 1.15 that M is the unique indecomposable direct summand of $\nabla(t_a \boldsymbol{\cdot} 0) \otimes \nabla(t_b \boldsymbol{\cdot} 0) = \nabla(\ell a) \otimes \nabla(\ell b)$ with a non-zero $\ell \cdot (a+b)$-weight space. As $G_\nabla(t_a, t_b)$ has Weyl filtration dimension $\ell(t_a) + \ell(t_b) = a + b$ and belongs to the linkage class of $\omega_{t_a t_b} \boldsymbol{\cdot} 0 = \omega_{t_{a+b}} \boldsymbol{\cdot} 0$, it suffices to prove that every indecomposable direct summand $M' \not\cong M$ of $\nabla(\ell a) \otimes \nabla(\ell b)$ that belongs to the linkage class of $\omega_{t_{a+b}} \boldsymbol{\cdot} 0$ satisfies $\mathrm{wfd}(M') < a + b$. By weight considerations, every composition factor of M' has highest weight in an ℓ-alcove $t_d \boldsymbol{\cdot} C_{\mathrm{fund}}$ for some $d \in \mathbb{Z}_{\geq 0}$ with $d < a + b$, and using Corollaries I.7.5 and II.2.7, we conclude that $\mathrm{wfd}(M') < a + b$. $\square$

We complete our discussion of the modular case by the determination of the good filtration mutiplicities of the generic direct summands $G_\nabla(x,y)$ for $x, y \in W_{\mathrm{ext}}^+$ (and $\mathbf{G}$ of type A_1).

Corollary 1.19. *Let $a, b, d \in \mathbb{Z}_{\geq 0}$ and write $a = \sum_{i \geq 0} a_i \ell^i$ and $b = \sum_{i \geq 0} b_i \ell^i$ with $0 \leq a_i, b_i < \ell$ for all $i \geq 0$. Furthermore, define $a_{-1} = b_{-1} = 0$. Then*

$$[G_\nabla(t_a, t_b) : \nabla(d)]_\nabla \leq 1,$$

and $[G_\nabla(t_a, t_b) : \nabla(d)]_\nabla = 1$ if and only if there exist

$$d_i \in \{a_{i-1} + b_{i-1}, 2\ell - (a_{i-1} + b_{i-1}) - 2\},$$

for $i \in \mathbb{Z}_{\geq 0}$, such that $d = \sum_{i \geq 0} d_i \ell^i$ and $d_j < a_{j-1} + b_{j-1}$ if j is maximal with $d_j \neq a_{j-1} + b_{j-1}$.

Proof. Recall from Lemma 1.18 that $G_\nabla(t_a, t_b) \cong I_{\pi(\ell \cdot (a+b))}\big(\ell \cdot (a+b-2c)\big)$, where $c = \sum_{i \geq 0} c_i \ell^i$ and

$$c_i \coloneqq \begin{cases} a_i + b_i - (\ell - 1) & \text{if } a_i + b_i \geq \ell - 1, \\ 0 & \text{otherwise} \end{cases}$$

for all $i \geq 0$. Let us also set $c_{-1} = 0$, and note that we can write

$$\ell \cdot (a+b-2c) = \sum_{i \geq 0} (a_{i-1} + b_{i-1} - 2c_{i-1}) \cdot \ell^i,$$

where $0 \leq a_{i-1} + b_{i-1} - 2c_{i-1} < \ell$ for all $i \geq 0$. By Lemma 1.14 and the reciprocity formula (1.2), the good filtration multiplicities in $G_\nabla(t_a, t_b)$ are bounded by 1. Furthermore, an induced module $\nabla(d)$, for $d \geq 0$, appears in a good filtration of $G_\nabla(t_a, t_b)$ if and only if $d \in \pi(\ell \cdot (a+b))$ and $d = \sum_{i \geq 0} d_i \ell^i$, where

$$d_i \in \big\{ a_{i-1} + b_{i-1} - 2c_{i-1}, 2\ell - (a_{i-1} + b_{i-1} - 2c_{i-1}) - 2 \big\} = \big\{ a_{i-1} + b_{i-1}, 2\ell - (a_{i-1} + b_{i-1}) - 2 \big\}$$

for all $i \geq 0$. It is straightforward to see that the condition $d \in \pi(\ell \cdot (a+b))$ is equivalent to the requirement that $d_j < a_{j-1} + b_{j-1}$ if j is maximal with $d_j \neq a_{j-1} + b_{j-1}$. $\square$

The quantum case. Let us start by introducing the 'quantum version' of $\mathrm{GL}_{n+1}(\Bbbk)$ that is suitable for our purpose. As before, we denote by $\hat{X}$ the free $\mathbb{Z}$-module with basis $\varepsilon_0, \ldots, \varepsilon_n$. We consider the scalar product $(-\,,-)$ on $\hat{X}_\mathbb{R} = \hat{X} \otimes_\mathbb{Z} \mathbb{R}$ with $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, for $0 \leq i, j \leq n$, and the root system

$$\hat{\Phi} = \{ \varepsilon_i - \varepsilon_j \mid 0 \leq i, j \leq n, i \neq j \}$$

with base

$$\hat{\Pi} = \{ \varepsilon_{i-1} - \varepsilon_i \mid 1 \leq i \leq n \}.$$

The quantized enveloping algebra of the reductive Lie algebra $\hat{\mathfrak{g}} = \mathfrak{gl}_{n+1}(\mathbb{C})$ is the $\mathbb{Q}(q)$-algebra $U_q(\hat{\mathfrak{g}})$ with generators E_α, F_α and K_i for $\alpha \in \hat{\Pi}$ and $0 \leq i \leq n$, subject to the relations

$$\begin{gathered}
K_i K_i^{-1} = 1 = K_i^{-1} K_i, \qquad\qquad K_i K_{i'} = K_i K_{i'}, \\
K_i E_\alpha K_i^{-1} = q^{(\varepsilon_i, \alpha)} \cdot E_\alpha, \qquad\qquad K_i F_\alpha K_i^{-1} = q^{-(\varepsilon_i, \alpha)} \cdot F_\alpha, \\
E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \cdot \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}}, \\
\sum_{j+k = 1 - c_{\alpha,\beta}} (-1)^j \cdot E_\alpha^{(j)} E_\beta E_\alpha^{(k)} = 0, \\
\sum_{j+k = 1 - c_{\beta,\alpha}} (-1)^j \cdot F_\alpha^{(j)} F_\beta F_\alpha^{(k)} = 0
\end{gathered}$$

for $\alpha, \beta \in \hat{\Pi}$ and $0 \leq i, i' \leq n$, where $c_{\alpha,\beta} = (\alpha, \beta)$ and $K_\alpha \coloneqq K_{r-1} K_r^{-1}$ if $\alpha = \varepsilon_{r-1} - \varepsilon_r$, and where

$$E_\alpha^{(j)} = \frac{E_\alpha^j}{[j]_\alpha!} \qquad \text{and} \qquad F_\alpha^{(j)} = \frac{F_\alpha^j}{[j]_\alpha!}$$

are the quantum divided powers, as in Section I.3. There is a Hopf algebra structure on $U_q(\hat{\mathfrak{g}})$ with comultiplication, antipode and counit defined as in (I.3.1), with K_α replaced by K_i in the third line (but not in the first and the second line). The *Lusztig integral form* $U_q^\mathbb{Z}(\hat{\mathfrak{g}})$ of $U_q(\hat{\mathfrak{g}})$ is the $\mathbb{Z}[q, q^{-1}]$-subalgebra generated by the quantum divided powers along with the elements $K_i^{\pm 1}$ and

$$\binom{K_i; m}{r} = \prod_{j=1}^{r} \frac{K_i q^{m-j+1} - K_i^{-1} q^{-m+j-1}}{q^j - q^{-j}},$$

for $0 \leq i \leq n$ and $m, r \geq 0$. We define

$$U'_\zeta(\hat{\mathfrak{g}}) \coloneqq U_q^\mathbb{Z}(\hat{\mathfrak{g}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$$

to be the specialization of $U_q^{\mathbb{Z}}(\hat{\mathfrak{g}})$ along the ring homomorphism $\mathbb{Z}[q,q^{-1}] \to \mathbb{C}$ with $q \mapsto \zeta$. As before, we will only be interested in 'type 1' representations on which the central elements $K_i^\ell \otimes 1$, for $0 \leq i \leq n$, act trivially, so we further define

$$\hat{\mathbf{G}} = U_\zeta(\hat{\mathfrak{g}}) \coloneqq U'_\zeta(\hat{\mathfrak{g}}) / \langle K_i^\ell \otimes 1 - 1 \otimes 1 \bigm| 0 \leq i \leq n \rangle.$$

By abuse of notation, we denote the images of the generators of $U_q^{\mathbb{Z}}(\hat{\mathfrak{g}})$ in $U'_\zeta(\hat{\mathfrak{g}})$ or $U_\zeta(\hat{\mathfrak{g}})$ by the same symbols. With these conventions, the quantum group $U'_\zeta(\mathfrak{g})$ corresponding to the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ is isomorphic to the subalgebra of $U'_\zeta(\hat{\mathfrak{g}})$ generated by the elements $K_\alpha^{\pm 1}$ and the divided powers of the elements E_α and F_α, for $\alpha \in \hat{\Pi}$, and there is a natural homomorphism from $\mathbf{G} = U_\zeta(\mathfrak{g})$ to $\hat{\mathbf{G}} = U_\zeta(\hat{\mathfrak{g}})$. For all $\lambda \in \hat{X}^+$, we can define the simple module $\hat{L}(\lambda)$, the costandard module $\hat{\nabla}(\lambda)$ and the Weyl module $\hat{\Delta}(\lambda)$ over $\hat{\mathbf{G}}$, just as we did for $\mathbf{G}$. As in the modular case, we have

$$\mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \hat{L}(\lambda) \cong L(\lambda'), \qquad \mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \hat{\Delta}(\lambda) \cong \Delta(\lambda') \qquad \text{and} \qquad \mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \hat{\nabla}(\lambda) \cong \nabla(\lambda'),$$

where $\lambda \mapsto \lambda'$ is as in (1.1). One way of seeing this for the Weyl modules is to note that $\mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \hat{\Delta}(\lambda)$ and $\Delta(\lambda')$ have the same character and that $\mathrm{res}_{\mathbf{G}}^{\hat{\mathbf{G}}} \hat{\Delta}(\lambda)$ is generated by a maximal vector of weight λ'. The claim for induced modules follows by taking duals. Finally, any non-zero homomorphism of $\hat{\mathbf{G}}$-modules $\hat{\Delta}(\lambda) \to \hat{\nabla}(\lambda)$ affords a non-zero homomorphism of $\mathbf{G}$-modules $\Delta(\lambda') \to \nabla(\lambda')$, and the statement about simple modules follows because $\hat{L}(\lambda)$ is the image of the former homomorphism, while $L(\lambda')$ is the image of the latter homomorphism.

The *natural* $U_q(\hat{\mathfrak{g}})$*-module* is the $\mathbb{Q}(q)$-vector space E_q with basis $e_0, \dots, e_n$, on which the action of the generators is given, for $\alpha = \varepsilon_{i-1} - \varepsilon_i \in \hat{\Pi}$ and $0 \leq j, k \leq n$, by

$$E_\alpha \cdot e_k = \delta_{i,k} e_{k-1}, \qquad F_\alpha \cdot e_k = \delta_{i-1,k} e_{k+1} \qquad \text{and} \qquad K_j \cdot e_k = q^{\delta_{j,k}} \cdot e_k.$$

The $\mathbb{Z}[q,q^{-1}]$-submodule $E_q^{\mathbb{Z}}$ spanned by $e_0, \dots, e_n$ is stable under $U_q^{\mathbb{Z}}(\hat{\mathfrak{g}})$, so it specializes to a $U'_\zeta(\hat{\mathfrak{g}})$-module $E \coloneqq E_q^{\mathbb{Z}} \otimes_{\mathbb{Z}[q,q^{-1}]} \otimes \mathbb{C}$, which naturally descends to a $U_\zeta(\hat{\mathfrak{g}})$-module (denoted again by E), because K_i^ℓ acts trivially for $0 \leq i \leq n$. We have $E \cong \hat{L}(\varepsilon_0) \cong \hat{\nabla}(\varepsilon_0)$.

For $r > 0$, the tensor space $E^{\otimes r}$ is naturally a $U'_\zeta(\hat{\mathfrak{g}})$-module and a $U_\zeta(\hat{\mathfrak{g}})$-module (via the Hopf algebra structure inherited from $U_q(\hat{\mathfrak{g}})$) and we define the ζ*-Schur algebra* $S_\zeta(n+1,r)$ as the image of either of $U'_\zeta(\hat{\mathfrak{g}})$ or $U_\zeta(\hat{\mathfrak{g}})$ in $\mathrm{End}_{\mathbb{C}}(E^{\otimes r})$. Alternatively, as in the modular case, $S_\zeta(n+1,r)$ can be defined as the dual algebra of a coalgebra $A_\zeta(n+1,r)$ as in Section 0.20 of [Don98]. The two definitions coincide because in both cases, one finds that $S_\zeta(n+1,r)$ can be identified with the full centralizer in $\mathrm{End}_{\mathbb{C}}(E^{\otimes r})$ of the action of the *Hecke algebra* of the symmetric group S_r; see Theorem 3.6 in [Du95] and Section 4.1.3 in [Don98]. The category of finite-dimensional $S_\zeta(n+1,r)$-modules is naturally equivalent to the category of finite-dimensional $\hat{\mathbf{G}}$-modules that are annihilated by the kernel of the representation $\hat{\mathbf{G}} \to \mathrm{End}_{\mathbb{C}}(E^{\otimes r})$. Recall from the modular case that we write

$$\begin{aligned} \hat{\pi}(r) &= \{\lambda \in \hat{X}^+ \mid \lambda \leq r\varepsilon_0\} \\ &= \{a_0\varepsilon_0 + \dots + a_n\varepsilon_n \bigm| a_0 \geq \dots \geq a_n \geq 0 \text{ and } a_0 + \dots + a_n = r\}. \end{aligned}$$

By Sections 0.22 and 2.1.(7) in [Don98], every $S_\zeta(n+1,r)$-module M admits a 'weight space decomposition' $M = \bigoplus_\lambda M_\lambda$ with weights in the set

$$W_{\mathrm{fin}}\hat{\pi}(r) = \{a_0\varepsilon_0 + \dots + a_n\varepsilon_n \mid a_0, \dots, a_n \geq 0 \text{ and } a_0 + \dots + a_n = r\}.$$

Furthermore, again by Section 0.22 in [Don98], the isomorphism classes of simple $S_\zeta(n+1,r)$-modules are in bijection with the set $\hat\pi(r)$, and by weight considerations, one sees that the simple $S_\zeta(n+1,r)$-modules afford precisely the simple $\hat{\mathbf{G}}$-modules $\hat{L}(\lambda)$ with $\lambda \in \hat\pi(r)$. Another important example of an $S_\zeta(n+1,r)$-module is the *quantum symmetric power*

$$S^r E \cong \hat\nabla(r\varepsilon_0),$$

which can be defined as a quotient of $E^{\otimes r}$ (see Section 2.1.(15) and the introduction to Section 2.1 in [Don98])[2]. More generally, for $\alpha = a_0\varepsilon_0 + \cdots + a_n\varepsilon_n \in \hat{X}^+$ with $r = a_0 + \cdots + a_n$, the tensor product of symmetric powers

$$S^\alpha E := S^{a_0} E \otimes \cdots \otimes S^{a_n} E$$

is in a natural way an $S_\zeta(n+1,r)$-module. The reason for our interest in these modules is the following result from Section 2.1.(8) in [Don98].

Proposition 1.20. *Let $\alpha = a_0\varepsilon_0 + \cdots + a_n\varepsilon_n \in \hat{X}$ with $a_0,\dots,a_n \geq 0$ and set $r = a_0 + \cdots + a_n$. Then $S^\alpha E$ is an injective $S_\zeta(n+1,r)$-module. For any finite-dimensional $S_\zeta(n+1,r)$-module M, we have*

$$\mathrm{Hom}_{S_\zeta(n+1,r)}(M, S^\alpha E) \cong M_\alpha,$$

the α-weight space of M.

Now for $\lambda \in \hat\pi(r)$, let us denote by $I_r(\lambda)$ the injective hull of the simple $S_\zeta(n+1,r)$-module $\hat{L}(\lambda)$. As pointed out in Section 2.1.(13) in [Don98], the algebra $S_\zeta(n+1,r)$ is quasi-hereditary, so the injective indecomposable $S_\zeta(n+1,r)$-module $I_r(\lambda)$ has a good filtration with multiplicities

$$\big[I_r(\lambda) : \hat\nabla(\mu)\big]_\nabla = \big[\hat\nabla(\mu) : \hat{L}(\lambda)\big]$$

for $\mu \in \hat\pi(r)$; see Proposition A2.2 in [Don98].

The following result is a quantum analogue of Proposition 1.15.

Proposition 1.21. *Let $a,b \in \mathbb{Z}_{\geq 0}$ and write $a = a_0 + \ell a_1$ and $b = b_0 + \ell b_1$ with $0 \leq a_0, b_0 < \ell$. Furthermore, let*

$$c := \begin{cases} a_0 + b_0 - (\ell - 1) & \text{if } a_0 + b_0 \geq \ell - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $I_{a+b}\big((a+b)\cdot\varepsilon_0 - c\cdot(\varepsilon_0 - \varepsilon_1)\big)$ is the unique indecomposable direct summand of $S^a E \otimes S^b E$ with a non-zero $(a+b)\cdot\varepsilon_0$-weight space.

Proof. Let us first show that $I_{a+b}\big((a+b)\cdot\varepsilon_0 - c\cdot(\varepsilon_0 - \varepsilon_1)\big)$ has a non-zero $(a+b)\cdot\varepsilon_0$-weight space. By the above discussion, we have

$$\big[I_{a+b}\big((a+b)\cdot\varepsilon_0 - c\cdot(\varepsilon_0-\varepsilon_1)\big) : \hat\nabla\big((a+b)\cdot\varepsilon_0\big)\big]_\nabla = \big[\hat\nabla\big((a+b)\cdot\varepsilon_0\big) : \hat{L}\big((a+b)\cdot\varepsilon_0 - c\cdot(\varepsilon_0-\varepsilon_1)\big)\big]$$

and it suffices to prove that the composition multiplicity on the right hand side is non-zero. Using the Schur algebra analogue of truncation to a Levi subgroup from Section 4.2.(5) in [Don98], it further suffices to do this in the case $n = 1$, i.e. for the ζ-Schur algebra $S_\zeta(2, a+b)$. It is straightforward to see that the composition multiplicity in question is non-zero if $c = 0$, so now assume that $a_0 + b_0 \geq \ell$.

[2] The existence of symmetric powers of representations is not obvious in the quantum case because the braiding on the category of $\hat{\mathbf{G}}$-modules is not the standard one.

Then we can write $a+b=(a_1+b_1+1)\cdot\ell+(a_0+b_0-\ell)$, with $0\leq a_0+b_0-\ell<\ell$, and as explained in Section 3.4 in [Don98], the $S_\zeta(2,a+b)$-module $\hat{\nabla}\big((a+b)\cdot\varepsilon_0\big)$ has a composition factor of highest weight

$$(a+b)\cdot\varepsilon_0-(a_0+b_0-\ell+1)\cdot(\varepsilon_0-\varepsilon_1)=(a+b)\cdot\varepsilon_0-c\cdot(\varepsilon_0-\varepsilon_1),$$

as required.

It remains to show that $I_{a+b}\big((a+b)\cdot\varepsilon_0-c\cdot(\varepsilon_0-\varepsilon_1)\big)$ is a direct summand of $S^aE\otimes S^bE$. Recall from Proposition 1.20 that $S^aE\otimes S^bE$ is an injective $S_\zeta(n+1,a+b)$-module and that

$$\mathrm{Hom}_{S_\zeta(n+1,a+b)}(M,S^aE\otimes S^bE)\cong M_{a\varepsilon_0+b\varepsilon_1}$$

for every finite-dimensional $S_\zeta(n+1,a+b)$-module M. Therefore, it suffices to prove that the $a\varepsilon_0+b\varepsilon_1$-weight space of the simple module $\hat{L}\big((a+b)\cdot\varepsilon_0-c\cdot(\varepsilon_0-\varepsilon_1)\big)$ is non-zero. (Compare with the proof of Proposition 1.15.) As before, we can use Section 4.2.(5) in [Don98] to reduce to the case $n=1$, and the claim follows exactly as in Proposition 1.15, using the version of Steinberg's tensor product theorem from Section 3.2.(5) in [Don98]. $\square$

The preceding result allows us to determine the generic direct summands $G_\nabla(t_{a\varpi_1},t_{b\varpi_1})$ of tensor products of induced modules $\nabla(t_{a\varpi_1}\cdot 0)\otimes\nabla(t_{b\varpi_1}\cdot 0)$ for all $a,b\in\mathbb{Z}_{\geq 0}$. Suppose that $\ell\geq h=n+1$.

Theorem 1.22. *For $a,b\in\mathbb{Z}_{\geq 0}$, we have $G_\nabla(t_{a\varpi_1},t_{b\varpi_1})\cong\nabla\big(t_{(a+b)\cdot\varpi_1}\cdot 0\big)=\nabla\big(\ell\cdot(a+b)\cdot\varpi_1\big)$.*

Proof. As $G_\nabla(t_{a\varpi_1},t_{b\varpi_1})$ is the unique regular indecomposable direct summand of the tensor product $\nabla(t_{a\varpi_1}\cdot 0)\otimes\nabla(t_{b\varpi_1}\cdot 0)$ and as the costandard module $\nabla(t_{(a+b)\cdot\varpi_1}\cdot 0)$ is regular by Lemma II.4.3, it suffices to prove that $\nabla(t_{(a+b)\cdot\varpi_1}\cdot 0)$ is a direct summand of $\nabla(t_{a\varpi_1}\cdot 0)\otimes\nabla(t_{b\varpi_1}\cdot 0)$. Recall that we have

$$\nabla(r\varpi_1)\cong\mathrm{res}^{\hat{\mathbf{G}}}_{\mathbf{G}}\hat{\nabla}(r\varepsilon_0)\cong\mathrm{res}^{\hat{\mathbf{G}}}_{\mathbf{G}}S^rE,$$

for all $r\in\mathbb{Z}_{\geq 0}$; hence it further suffices to prove that $S^{\ell\cdot(a+b)}E$ is a direct summand of $S^{\ell a}E\otimes S^{\ell b}E$.

Note that the $\hat{\mathbf{G}}$-module $S^{\ell\cdot(a+b)}E\cong\hat{\nabla}\big(\ell\cdot(a+b)\cdot\varepsilon_0\big)$ is injective as an $S_\zeta\big(n+1,\ell\cdot(a+b)\big)$-module by Proposition 1.20 and that it has simple socle $\hat{L}\big(\ell\cdot(a+b)\cdot\varepsilon_0\big)$. Therefore, we have

$$S^{\ell\cdot(a+b)}E\cong I_{\ell\cdot(a+b)}\big(\ell\cdot(a+b)\cdot\varepsilon_0\big),$$

and Proposition 1.21 implies that $S^{\ell\cdot(a+b)}E$ is a direct summand of $S^{\ell a}E\otimes S^{\ell b}E$, as required. $\square$

Finally, let us return to the case $n=1$ and $\mathbf{G}=U_\zeta\big(\mathfrak{sl}_2(\mathbb{C})\big)$.

Corollary 1.23. *For $x,y\in W^+_{\mathrm{ext}}$, let $a,b\in\mathbb{Z}_{\geq 0}$ and $\omega,\omega'\in\Omega$ such that $x\cdot 0=t_a\omega\cdot 0$ and $y\cdot 0=t_b\omega'\cdot 0$. Then*

$$G_\nabla(x,y)\cong T^{\omega\omega'}\nabla\big(\ell\cdot(a+b)\big).$$

Proof. By Lemma II.5.5 and Theorem 1.22, we have

$$G_\nabla(x,y)\cong T^{\omega\omega'}G_\nabla(t_a,t_b)\cong T^{\omega\omega'}\nabla\big(\ell\cdot(a+b)\big),$$

as required. $\square$

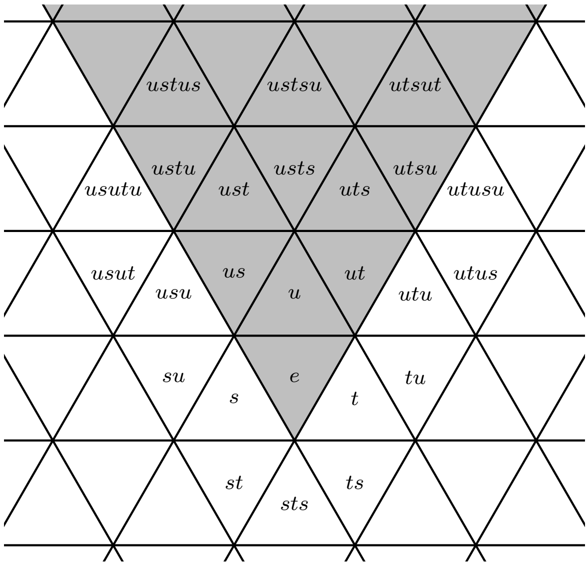

Figure 2.1: Alcoves for $\mathbf{G}$ of type A_2. For some elements $x \in W_{\mathrm{aff}}$, we have labeled $x \cdot C_{\mathrm{fund}}$ by x. The gray-colored region is the set of dominant alcoves.

2 Type A_2

In this Section, we consider the case where $\mathbf{G}$ is of type A_2 and $\ell \geq 3$. Our discussion of regular indecomposable direct summands (and generic direct summands) of tensor products of $\mathbf{G}$-modules strongly relies on the two articles [BDM15] (by C. Bowman, S. Doty and S. Martin) and [CW15] (by X. Chan and J. Wang), whose contents we will briefly discuss here. The main result of [BDM15] is a description of the set of indecomposable $\mathbf{G}$-modules that arise as direct summands of tensor products of simple $\mathbf{G}$-modules with ℓ-restricted highest weights. Strictly speaking, the article only covers the modular case, but none of its methods are specific to that case, and the results that we will use hold in the quantum case as well. The analogous problem (of finding the indecomposable direct summands of tensor products) for costandard modules with ℓ-restricted highest weights was considered in [CW15]. Again, the authors discuss only the modular case, but the results are valid in the quantum case as well (as is pointed out at the end of the introduction of that article). We can combine the results of [BDM15] with the techniques developed in Section II.6 to give a complete description of the generic direct summands of tensor products of simple $\mathbf{G}$-modules with arbitrary ℓ-regular (not necessarily ℓ-restricted) highest weights. For costandard modules, there is (at present) no method for reducing the study of generic direct summands to the case of ℓ-restricted highest weights, so we do not go any further than to point out which of the indecomposable $\mathbf{G}$-modules from [CW15] are the generic direct summands. Note however that for some specific highest weights, the generic direct summands of tensor products of costandard modules can be determined using Corollary 1.17 and Theorem 1.22.

Before we go into any more detail, let us fix some notation. According to the conventions from Section I.1, we have $\Pi = \{\alpha_1, \alpha_2\}$ and $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_{\mathrm{h}}\}$ with $\alpha_{\mathrm{h}} = \alpha_1 + \alpha_2$. The weight lattice $X \cong \mathbb{Z}^2$ is spanned by the fundamental dominant weights ϖ_1 and ϖ_2, and the affine Weyl group W_{aff} is generated by the simple reflections $S = \{s, t, u\}$, where $s = s_{\alpha_1}$, $t = s_{\alpha_2}$ and $u = s_{\alpha_{\mathrm{h}},1}$. Recall that W_{aff} is in bijection with the set of alcoves (or ℓ-alcoves) in $X_{\mathbb{R}}$ via $x \mapsto x(A_{\mathrm{fund}})$ (or $x \mapsto x \cdot C_{\mathrm{fund}}$). In Figure 2.1, we display some alcoves for $\mathbf{G}$, and we label some of them by the corresponding elements of W_{aff}. The only ℓ-alcoves containing ℓ-restricted weights are C_{fund} and $u \cdot C_{\mathrm{fund}}$.

Simple modules

Let us start by recalling the results of C. Bowman, S. Doty and S. Martin in more detail. According to Section 8.5 in [BDM15], the tilting module $T(ust \cdot \nu)$, for $\nu \in C_{\mathrm{fund}} \cap X$, has a unique contravariantly self-dual submodule $M(\nu)$ (denoted there by $M(u \cdot \nu)$) with

$$\operatorname{ch} M(\nu) = \operatorname{ch} L(us \cdot \nu) + \operatorname{ch} L(ut \cdot \nu) + 2 \cdot \operatorname{ch} L(u \cdot \nu) + \operatorname{ch} L(\nu),$$

and the latter has simple head and socle isomorphic to $L(u \cdot \nu)$. Furthermore, the quotient of $\mathrm{rad}_{\mathbf{G}} M(\nu)$ by $\mathrm{soc}_{\mathbf{G}} M(\nu)$ is completely reducible; more precisely, we have

$$\mathrm{rad}_{\mathbf{G}} M(\nu)/\mathrm{soc}_{\mathbf{G}} M(\nu) \cong L(us \cdot \nu) \oplus L(ut \cdot \nu) \oplus L(\nu).$$

Observe that the definition of $M(\nu)$ implies that $T_\nu^\lambda M(\nu) \cong M(\lambda)$ for all $\lambda \in C_{\mathrm{fund}} \cap X$. Following the conventions of [BDM15], we can depict the structure of $M(\nu)$ in an 'Alperin diagram', where we replace a simple module $L(x \cdot \nu)$ by the label $x \in W_{\mathrm{aff}}^+$.

$$M(\nu) = \begin{tikzcd}[column sep=small] & u \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\ us \arrow[dr, no head] & e \arrow[d, no head] & ut \arrow[dl, no head] \\ & u & \end{tikzcd}$$

With this notation in place, a simplified version of the main result of [BDM15] can be stated as follows:

Theorem 2.1. *Let $\lambda, \mu \in X_1$ and let M be an indecomposable direct summand of $L(\lambda) \otimes L(\mu)$. Then either M is a tilting module or $M = L(u \cdot \nu)$ or $M \cong M(\nu)$, for some $\nu \in C_{\mathrm{fund}} \cap X$. The third case $M \cong M(\nu)$ can occur only if $\lambda, \mu \in u \cdot C_{\mathrm{fund}}$.*

Remark 2.2. It is pointed out at the end of Section 3 (below Theorem B) in [BDM15] that an indecomposable direct summand M of $L(\lambda) \otimes L(\mu)$ as in the preceding theorem has simple socle with ℓ-restricted highest weight as a $\mathbf{G}_1\mathbf{T}$-module, unless possibly when

$$M \cong T(ustus \cdot \nu) \qquad \text{or} \qquad M \cong T(utsut \cdot \nu)$$

for some $\nu \in \overline{C}_{\mathrm{fund}} \cap X$. As the $\mathbf{G}_1\mathbf{T}$-socle of a $\mathbf{G}$-module coincides with its $\mathbf{G}_1$-socle (see Remark 1 in Section II.9.6 of [Jan03]), this implies that M is indecomposable as a $\mathbf{G}_1$-module (with the same exceptions).

Lemma 2.3. *We have $G(u,u) \cong M(0)$ and*

$$\big(L(u \cdot 0) \otimes L(u \cdot 0)\big)_{\mathrm{reg}} \cong M(0) \oplus L(0)$$

Proof. Recall that $G(u,u)$ belongs to the linkage class of 0 and that $\mathrm{gfd}\big(G(u,u)\big) = \ell(u) + \ell(u) = 2$; see Proposition II.5.7. By Theorem 2.1, all indecomposable direct summands of $L(u \cdot 0) \otimes L(u \cdot 0)$ that are not of the form $M(\nu)$, for some $\nu \in C_{\mathrm{fund}} \cap X$, have good filtration dimension either zero (because they are tilting modules) or one (by Corollary II.2.7, because $\ell(u) = 1$), and it follows that

$$G(u,u) \cong M(0).$$

In particular, $M(\nu) \cong T_0^\nu M(0)$ is regular for all $\nu \in C_{\mathrm{fund}} \cap X$.

By Section 8.6 in [BDM15], there exist $\lambda, \mu, \nu \in C_{\mathrm{fund}} \cap X$ such that

$$\mathrm{pr}_\nu\big(L(u \cdot \lambda) \otimes L(u \cdot \mu)\big) \cong M(\nu) \oplus L(\nu).$$

Since $M(\nu)$ and $L(\nu)$ are both regular, Theorem II.4.14 yields

$$T_0^\nu\big(M(0) \oplus L(0)\big) \cong M(\nu) \oplus L(\nu) \cong \mathrm{pr}_\nu\big(L(u \cdot \lambda) \otimes L(u \cdot \mu)\big)_{\mathrm{reg}} \cong T_0^\nu\big(L(u \cdot 0) \otimes L(u \cdot 0)\big)_{\mathrm{reg}}^{\oplus c_{\lambda,\mu}^\nu},$$

and we conclude that $c_{\lambda,\mu}^\nu = 1$ and $\big(L(u \cdot 0) \otimes L(u \cdot 0)\big)_{\mathrm{reg}} \cong M(0) \oplus L(0)$. □

Also note that we have

$$G(e,e) \cong L(0) \cong \big(L(0) \otimes L(0)\big)_{\mathrm{reg}} \qquad \text{and} \qquad G(u,e) \cong L(u \cdot 0) \cong \big(L(u \cdot 0) \otimes L(0)\big)_{\mathrm{reg}}.$$

As C_{fund} and $u \cdot C_{\mathrm{fund}}$ are the only ℓ-alcoves containing ℓ-restricted weights, this gives a complete description of the regular parts (and the generic direct summands) of tensor products of simple **G**-modules with ℓ-regular ℓ-restricted highest weights. Furthermore, all regular indecomposable direct summands of such tensor products are strongly regular (because simple **G**-modules with ℓ-regular highest weights and generic direct summands of tensor products of simple **G**-modules are strongly regular, by Remarks II.4.17 and II.5.12) and indecomposable (with simple socle) as $\mathbf{G}_1$-modules by Remark 2.2. (These are important observations in view of Corollaries II.6.10 and II.6.19.)

Now as in Section II.6, let us fix $x, y \in W_{\mathrm{ext}}^+$ and write

$$x \cdot 0 = x_0 \cdot 0 + \ell\lambda \qquad \text{and} \qquad y \cdot 0 = y_0 \cdot 0 + \ell\mu$$

with $\lambda, \mu \in X^+$ and $x_0, y_0 \in W_{\mathrm{ext}}^+$ such that $x_0 \cdot 0, y_0 \cdot 0 \in X_1$. As C_{fund} and $u \cdot C_{\mathrm{fund}}$ are the only ℓ-alcoves containing ℓ-restricted weights, we have $x_0, y_0 \in \Omega \cup u\Omega$, and we write $x_0 = u^\varepsilon\omega$ and $y_0 = u^{\varepsilon'}\omega'$ with $\varepsilon, \varepsilon' \in \{0,1\}$ and $\omega, \omega' \in \Omega$. Note that by Lemma II.4.15, we have

$$\big(L(x_0 \cdot 0) \otimes L(y_0 \cdot 0)\big)_{\mathrm{reg}} \cong \big(T^\omega L(u^\varepsilon \cdot 0) \otimes T^{\omega'} L(u^{\varepsilon'} \cdot 0)\big)_{\mathrm{reg}} \cong T^{\omega\omega'}\big(L(u^\varepsilon \cdot 0) \otimes L(u^{\varepsilon'} \cdot 0)\big)_{\mathrm{reg}}.$$

In the quantum case, we get the following complete desccription of regular parts and generic direct summands of tensor products of simple **G**-modules, from Corollaries II.6.8, II.6.10 and II.6.11.

Theorem 2.4. *Suppose that we are in the quantum case and write*

$$L_{\mathbb{C}}(\lambda) \otimes L_{\mathbb{C}}(\mu) \cong \bigoplus_{\nu \in X^+} L_{\mathbb{C}}(\nu)^{\oplus d_{\lambda,\mu}^\nu}.$$

(1) If $\varepsilon = \varepsilon' = 0$ then

$$\begin{aligned}\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} &\cong \bigoplus_{\nu \in X^+} \big(L(\omega\omega' \cdot 0) \otimes L_{\mathbb{C}}(\nu)^{[1]}\big)^{\oplus d_{\lambda,\mu}^\nu} \\ &\cong \bigoplus_{\nu \in X^+} L(\omega\omega' \cdot 0 + \ell\nu)^{\oplus d_{\lambda,\mu}^\nu}\end{aligned}$$

is a Krull-Schmidt decomposition and $G(x,y) \cong L(\omega\omega' \cdot 0) \otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]} \cong L(\omega\omega' \cdot 0 + \ell\lambda + \ell\mu)$.

(2) If $\varepsilon + \varepsilon' = 1$ then

$$\begin{aligned}\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} &\cong \bigoplus_{\nu \in X^+} \big(L(u\omega\omega' \cdot 0) \otimes L_{\mathbb{C}}(\nu)^{[1]}\big)^{\oplus d_{\lambda,\mu}^\nu} \\ &\cong \bigoplus_{\nu \in X^+} L(u\omega\omega' \cdot 0 + \ell\nu)^{\oplus d_{\lambda,\mu}^\nu}\end{aligned}$$

is a Krull-Schmidt decomposition and $G(x,y) \cong L(u\omega\omega' \cdot 0) \otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]} \cong L(u\omega\omega' \cdot 0 + \ell\lambda + \ell\mu)$.

(3) If $\varepsilon = \varepsilon' = 1$ then

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in X^+} \big(M(\omega\omega' \cdot 0) \otimes L_{\mathbb{C}}(\nu)^{[1]}\big)^{\oplus d^{\nu}_{\lambda,\mu}} \oplus \bigoplus_{\nu \in X^+} L\big(\omega\omega' \cdot 0 + \ell\nu\big)^{\oplus d^{\nu}_{\lambda,\mu}}$$

is a Krull-Schmidt decomposition and $G(x,y) \cong M(\omega\omega' \cdot 0) \otimes L_{\mathbb{C}}(\lambda+\mu)^{[1]}$.

Recall that in the modular case, we write $M(\lambda,\mu)$ for the unique indecomposable direct summand of $L(\lambda) \otimes L(\mu)$ with a non-zero $\lambda+\mu$-weight space. The modular analogue of the preceding theorem follows from Corollaries II.6.16, II.6.19 and II.6.20.

Theorem 2.5. *Suppose that we are in the modular case and fix a Krull-Schmidt decomposition*

$$L(\lambda) \otimes L(\mu) \cong M_1 \oplus \cdots \oplus M_r.$$

(1) If $\varepsilon = \varepsilon' = 0$ then

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{i=1}^{r} L(\omega\omega' \cdot 0) \otimes M_i^{[1]}$$

is a Krull-Schmidt decomposition and $G(x,y) \cong L(\omega\omega' \cdot 0) \otimes M(\lambda,\mu)^{[1]}$.

(2) If $\varepsilon + \varepsilon' = 1$ then

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{i=1}^{r} L(u\omega\omega' \cdot 0) \otimes M_i^{[1]}$$

is a Krull-Schmidt decomposition and $G(x,y) \cong L(u\omega\omega' \cdot 0) \otimes M(\lambda,\mu)^{[1]}$.

(3) If $\varepsilon = \varepsilon' = 1$ then

$$\big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{i=1}^{r} M(\omega\omega' \cdot 0) \otimes M_i^{[1]} \oplus \bigoplus_{i=1}^{r} L(\omega\omega' \cdot 0) \otimes M_i^{[1]}$$

is a Krull-Schmidt decomposition and $G(x,y) \cong M(\omega\omega' \cdot 0) \otimes M(\lambda,\mu)^{[1]}$.

For the rest of this section, suppose that we are in the modular case. Let us write $\lambda = \sum_{i \geq 0} \ell^i \lambda_i$ and $\mu = \sum_{i \geq 0} \ell^i \mu_i$, with $\lambda_i, \mu_i \in X_1$ for all $i \geq 0$. Our next goal is to show that $M(\lambda,\mu)$ is a tensor product of Frobenius twists of the $\mathbf{G}$-modules $M(\lambda_i,\mu_i)$, for $i \geq 0$, as in Lemma II.6.17. We start by determining the module $M(\lambda',\mu')$, for $\lambda',\mu' \in X_1$, more explicitly.

Lemma 2.6. *Let $\lambda',\mu' \in X_1$. We have*

$$M(\lambda',\mu') \cong L(\lambda'+\mu')$$

whenever $\lambda'+\mu' \in u \cdot C_{\mathrm{fund}}$ and either $\lambda' \in u \cdot C_{\mathrm{fund}}$ or $\mu' \in u \cdot C_{\mathrm{fund}}$, and

$$M(\lambda',\mu') \cong T(\lambda'+\mu')$$

in all other cases.

Proof. First suppose that neither of λ' and μ' belongs to $u \cdot C_{\mathrm{fund}}$. As $\lambda', \mu' \in X_1 \subseteq \overline{C}_{\mathrm{fund}} \cup u \cdot \overline{C}_{\mathrm{fund}}$, this implies that $\lambda', \mu' \in C_{\mathrm{fund}}$ or that at least one of λ' and μ' is ℓ-singular. If $\lambda', \mu' \in C_{\mathrm{fund}}$ then

$$L(\lambda') \otimes L(\mu') \cong T(\lambda') \otimes T(\mu')$$

is a tilting module and it follows that $M(\lambda', \mu') \cong T(\lambda' + \mu')$. If either of the weights λ' or μ' is ℓ-singular then one of the simple modules $L(\lambda')$ and $L(\mu')$ is singular by Lemma II.4.3, and $M(\lambda', \mu')$ is singular because singular modules form a thick tensor ideal. On the other hand, $L(u \cdot \nu)$ is regular for all $\nu \in C_{\mathrm{fund}} \cap X$ (again by Lemma II.4.3), and Theorem 2.1 implies that $M(\lambda', \mu')$ is a tilting module, so $M(\lambda', \mu') \cong T(\lambda' + \mu')$.

By symmetry, we may now suppose that $\lambda' \in u \cdot C_{\mathrm{fund}}$ and that μ' is ℓ-regular. If $\lambda' + \mu' \in u \cdot C_{\mathrm{fund}}$ then μ' is the unique dominant weight in the W_{fin}-orbit of $u \cdot (\lambda' + \mu') - u \cdot \lambda' = s_{\alpha_{\mathrm{h}}}(\mu')$, and it follows that

$$L(\lambda' + \mu') \cong T_{u \cdot \lambda'}^{u \cdot (\lambda' + \mu')} L(\lambda') = \mathrm{pr}_{u \cdot (\lambda' + \mu')}\big(L(\lambda') \otimes L(\mu')\big),$$

whence $M(\lambda', \mu') \cong L(\lambda' + \mu')$. If $\mu' \in C_{\mathrm{fund}}$ and $\lambda' + \mu' \notin u \cdot C_{\mathrm{fund}}$ then $M(\lambda', \mu')$ is singular because the regular part

$$\begin{aligned} \big(L(\lambda') \otimes L(\mu')\big)_{\mathrm{reg}} &\cong \big(T_0^{u \cdot \lambda'} L(u \cdot 0) \otimes T_0^{\mu'} L(0)\big)_{\mathrm{reg}} \\ &\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T_0^{\nu}\big(L(u \cdot 0) \otimes L(0)\big)_{\mathrm{reg}}^{\oplus c_{u \cdot \lambda', \mu'}^{\nu}} \\ &\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_0^{\nu} L(u \cdot 0)\big)^{\oplus c_{u \cdot \lambda', \mu'}^{\nu}} \\ &\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} L(u \cdot \nu)^{\oplus c_{u \cdot \lambda', \mu'}^{\nu}} \end{aligned}$$

is a direct sum of simple $\mathbf{G}$-modules with highest weights in $u \cdot C_{\mathrm{fund}}$; see Theorem II.4.14. As before, Theorem 2.1 implies that $M(\lambda', \mu')$ is a tilting module and therefore $M(\lambda', \mu') \cong T(\lambda' + \mu')$.

Finally, suppose that $\mu' \in u \cdot C_{\mathrm{fund}}$. By Lemma 2.3 and Theorem II.4.14, we have

$$\begin{aligned} \big(L(\lambda') \otimes L(\mu')\big)_{\mathrm{reg}} &\cong \big(T_0^{u \cdot \lambda'} L(u \cdot 0) \otimes T_0^{u \cdot \mu'} L(u \cdot 0)\big)_{\mathrm{reg}} \\ &\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T_0^{\nu}\big(L(u \cdot 0) \otimes L(u \cdot 0)\big)_{\mathrm{reg}}^{\oplus c_{u \cdot \lambda', u \cdot \mu'}^{\nu}} \\ &\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(T_0^{\nu} M(0) \oplus T_0^{\nu} L(0)\big)^{\oplus c_{u \cdot \lambda', u \cdot \mu'}^{\nu}} \\ &\cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} \big(M(\nu) \oplus L(\nu)\big)^{\oplus c_{u \cdot \lambda', u \cdot \mu'}^{\nu}}, \end{aligned}$$

and all singular indecomposable direct summands of $L(\lambda') \otimes L(\mu')$ are tilting modules by Theorem 2.1. On the other hand, we have $(\lambda' + \mu' + \rho, \alpha_{\mathrm{h}}^{\vee}) \geq 2\ell$ because $(\lambda' + \rho, \alpha_{\mathrm{h}}^{\vee}) \geq \ell + 1$ and $(\mu' + \rho, \alpha_{\mathrm{h}}^{\vee}) \geq \ell + 1$, hence

$$\lambda' + \mu' \notin C_{\mathrm{fund}} \cup u \cdot C_{\mathrm{fund}} \cup us \cdot C_{\mathrm{fund}} \cup ut \cdot C_{\mathrm{fund}}$$

and $L(\lambda' + \mu')$ is not a composition factor of $M(\nu)$ or $L(\nu)$, for any $\nu \in C_{\mathrm{fund}} \cap X$. We conclude that $M(\lambda', \mu')$ is a tilting module, so $M(\lambda', \mu') \cong T(\lambda' + \mu')$. □

Corollary 2.7. *Let $\lambda', \mu' \in X_1$. Then $M(\lambda', \mu')$ is indecomposable as a $\mathbf{G}_1$-module.*

Proof. By Lemma 2.6, we have either

$$M(\lambda',\mu') \cong T(\lambda'+\mu') \qquad \text{or} \qquad M(\lambda',\mu') \cong L(\lambda'+\mu').$$

In the second case, we further have $\lambda'+\mu' \in u \cdot C_{\mathrm{fund}} \cap X \subseteq X_1$ and it follows that $L(\lambda'+\mu')$ is simple as a $\mathbf{G}_1$-module. As $\lambda',\mu' \in X_1$, we have

$$\lambda'+\mu' \in \big\{ \gamma \in X^+ \bigm| (\gamma,\alpha^\vee) \leq 2\ell-2 \text{ for all } \alpha \in \Pi \big\}.$$

This set of weights is disjoint from $ustus \cdot \overline{C}_{\mathrm{fund}}$ and $utsut \cdot \overline{C}_{\mathrm{fund}}$ (see Figure 2.1), so Remark 2.2 implies that $T(\lambda'+\mu')$ is indecomposable as a $\mathbf{G}_1$-module. □

Recall that we write $\lambda = \sum_{i\geq 0} \ell^i \lambda_i$ and $\mu = \sum_{i \geq 0} \ell^i \mu_i$, with $\lambda_i, \mu_i \in X_1$ for all $i \geq 0$.

Corollary 2.8. *We have $M(\lambda,\mu) \cong \bigotimes_{i\geq 0} M(\lambda_i,\mu_i)^{[i]}$.*

Proof. The $\mathbf{G}$-modules $M(\lambda_i,\mu_i)$, for $i \geq 0$, are indecomposable as $\mathbf{G}_1$-modules by Corollary 2.7 and the claim follows from Lemma II.6.17. □

Finally, let us point out that not all of the indecomposable direct summands of $L(\lambda)\otimes L(\mu)$ can be obtained as tensor products of Frobenius twists of indecomposable direct summands of $L(\lambda_i)\otimes L(\mu_i)$, for $i \geq 0$. Indeed, for $\lambda = (\ell-1)\cdot\rho$ and $\mu = (2\ell-1)\cdot\varpi_1 + 2\varpi_2$, we have

$$L(\lambda_0)\otimes L(\mu_0) = L\big((\ell-1)\cdot\rho\big) \otimes L\big((\ell-1)\cdot\varpi_1+2\varpi_2\big) \cong T\big((\ell-1)\cdot\rho\big)\otimes T\big((\ell-1)\cdot\varpi_1+2\varpi_2\big)$$

by the linkage principle, and it is straightforward to see (by weight considerations) that the tilting module $T\big(\ell\varpi_1+(\ell-1)\cdot\rho\big)$ is a direct summand of $L(\lambda_0)\otimes L(\mu_0)$. Observe that, again by the linkage principle and by Steinberg's tensor product theorem, the tilting module $T\big(\ell\varpi_1+(\ell-1)\cdot\rho\big)$ has a tensor product decomposition

$$T\big(\ell\varpi_1+(\ell-1)\cdot\rho\big) \cong L\big(\ell\varpi_1+(\ell-1)\cdot\rho\big) \cong L\big((\ell-1)\cdot\rho\big)\otimes L(\varpi_1)^{[1]}.$$

Furthermore, we have $\lambda_1 = 0$ and $\mu_1 = \varpi_1$; so $L(\varpi_1)$ is the unique indecomposable direct summand of $L(\lambda_1)\otimes L(\mu_1)$. Now the tensor product

$$T\big(\ell\varpi_1+(\ell-1)\cdot\rho\big)\otimes L(\varpi_1)^{[1]} \cong L\big((\ell-1)\cdot\rho\big)\otimes\big(L(\varpi_1)\otimes L(\varpi_1)\big)^{[1]}$$

is decomposable because $L(\varpi_1)\otimes L(\varpi_1) \cong L(2\varpi_1)\oplus L(\varpi_2)$ is decomposable (since $\ell \geq h = 3$), and the simple module

$$L\big(2\ell\varpi_1+(\ell-1)\cdot\rho\big) \cong L\big((\ell-1)\rho\big)\otimes L(2\varpi_1)^{[1]}$$

is an indecomposable direct summand of $L(\lambda)\otimes L(\mu)$. Now suppose for a contradiction that

$$L\big(2\ell\varpi_1+(\ell-1)\cdot\rho\big) \cong M\otimes N^{[1]},$$

for indecomposable direct summands M and N of $L(\lambda_0)\otimes L(\mu_0)$ and $L(\lambda_1)\otimes L(\mu_1)$, respectively. Then M and N are simple and, as observed above, we must have $N \cong L(\varpi_1)$. By weight considerations, it follows that

$$M \cong L\big(\ell\varpi_1+(\ell-1)\cdot\rho\big) \cong T\big(\ell\varpi_1+(\ell-1)\cdot\rho\big)$$

and we arrive at the contradiction that $M\otimes N^{[1]}$ is decomposable.

Costandard modules

Let us once again start by recalling the main results of X. Chan and J. Wang in more detail. We return to our strategy of discussing the modular case and the quantum case at the same time.

According to Theorem 2.1(3) in [CW15], there is, for every $\nu \in C_{\mathrm{fund}} \cap X$, a unique indecomposable $\mathbf{G}$-module $M_\nabla(\nu)$ (denoted there by $Q(u \cdot \nu)$) that admits a short exact sequence

$$0 \to \nabla(u \cdot \nu) \to M_\nabla(\nu) \to \nabla(us \cdot \nu) \oplus \nabla(ut \cdot \nu) \to 0.$$

See also Lemma 3.1 in [CW15] and the computation thereafter. Furthermore, the $\mathbf{G}$-module $M_\nabla(\nu)$ is isomorphic to the injective hull $I_\pi(u \cdot \nu)$ of the simple $\mathbf{G}$-module $L(u \cdot \nu)$ in the truncated category $\mathrm{Rep}(\mathbf{G}, \pi)$ corresponding to the set of weights $\pi = \{\nu, u \cdot \nu, us \cdot \nu, ut \cdot \nu\}$. Now (a shortened version of) Theorem 2.1 in [CW15] is as follows:

Theorem 2.9. *Let $\lambda, \mu \in C_{\mathrm{fund}} \cap X$.*

(1) The tensor product $\nabla(\lambda) \otimes \nabla(\mu)$ is a direct sum of indecomposable tilting modules.

(2) The tensor product $\nabla(u \cdot \lambda) \otimes \nabla(\mu)$ is a direct sum of induced modules $\nabla(u \cdot \nu)$, with $\nu \in C_{\mathrm{fund}} \cap X$, and of negligible tilting modules.

(3) The tensor product $\nabla(u \cdot \lambda) \otimes \nabla(u \cdot \mu)$ is a direct sum of $\mathbf{G}$-modules of the form $M_\nabla(\nu)$, with $\nu \in C_{\mathrm{fund}} \cap X$, and of negligible tilting modules.

Using the preceding theorem, it is straightforward to work out the generic direct summands of tensor products of induced modules with ℓ-restricted ℓ-regular highest weights. First note that

$$\nabla(0) \cong \nabla(0) \otimes \nabla(0) \cong \big(\nabla(0) \otimes \nabla(0)\big)_{\mathrm{reg}} \cong G_\nabla(e, e)$$

and similarly

$$\nabla(u \cdot 0) \cong \nabla(u \cdot 0) \otimes \nabla(0) \cong \big(\nabla(u \cdot 0) \otimes \nabla(0)\big)_{\mathrm{reg}} \cong G_\nabla(u, e),$$

even without using the theorem. The generic direct summand $G_\nabla(u, u)$ of $\nabla(u \cdot 0) \otimes \nabla(u \cdot 0)$ is regular and belongs to $\mathrm{Rep}_0(\mathbf{G})$, so part (3) of Theorem 2.9 implies that

$$G_\nabla(u, u) \cong M_\nabla(0).$$

Remark 2.10. In Corollary 3.15 in [CW15], it is shown that, for $\lambda, \mu, \nu \in C_{\mathrm{fund}} \cap X$, the multiplicity of $M_\nabla(\nu)$ in a Krull-Schmidt decomposition of $\nabla(u \cdot \lambda) \otimes \nabla(u \cdot \mu)$ is given by the structure constant $c_{\lambda,\mu}^{\nu}$ of the Verlinde algebra. The idea that these structure constants should govern the multiplicities of regular indecomposable direct summands in tensor products (as in Theorem II.4.14) arose when the author was studying this result, but the proof from [CW15] does not carry over to our more general setting. A similar argument to X. Chan and J. Wang's proof can, however, be used to show that $c_{\lambda,\mu}^{\nu}$ is the multiplicity of $T_0^\nu G_\nabla(x, y)$ in a Krull-Schmidt decomposition of $\nabla(x \cdot \lambda) \otimes \nabla(y \cdot \mu)$, for $x, y \in W_{\mathrm{aff}}^+$. (In our setting, this follows from Remarks II.5.2 and II.5.4.)

IV. The second alcove

The most basic example of generic direct summands is given by the observation that

$$G(e,x) \cong L(x \cdot 0) \qquad \text{and} \qquad G_{\nabla}(e,x) \cong \nabla(x \cdot 0)$$

for all $x \in W_{\mathrm{ext}}^{+}$ because $L(0) \cong \nabla(0)$ is the trivial $\mathbf{G}$-module. In this chapter, we study the family of generic direct summands $G(s_{\alpha_{\mathrm{h}},1}, x)$, for $x \in W_{\mathrm{ext}}^{+}$, under the assumption that $\mathbf{G}$ is of type A_n. Note that this can be considered as the smallest non-trivial example of such a family of generic direct summands because $s_{\alpha_{\mathrm{h}},1}$ is the unique element of length one in W_{aff}^{+} (see Lemma I.2.12). We loosely refer to $s_{\alpha_{\mathrm{h}},1} \cdot C_{\mathrm{fund}}$ as the *second alcove* (C_{fund} being the first).

Let us briefly outline our strategy. We first observe that

$$C_{\min}\big(L(s_{\alpha_{\mathrm{h}},1} \cdot 0)\big) = \big(\quad 0 \to T(0) \to T(s_{\alpha_{\mathrm{h}},1} \cdot 0) \to T(0) \to 0 \quad \big).$$

This minimal complex gives rise to natural transfomations

$$e \colon \mathrm{id}_{\mathrm{Rep}(\mathbf{G})} \cong \big(T(0) \otimes -\big) \Longrightarrow \big(T(s_{\alpha_{\mathrm{h}},1} \cdot 0) \otimes -\big)$$

and

$$p \colon \big(T(s_{\alpha_{\mathrm{h}},1} \cdot 0) \otimes -\big) \Longrightarrow \big(T(0) \otimes -\big) \cong \mathrm{id}_{\mathrm{Rep}(\mathbf{G})}$$

such that the components e_M and p_M of e and p at any $\mathbf{G}$-module M satisfy $\mathrm{im}(e_M) \subseteq \ker(p_M)$ and

$$\ker(p_M)/\mathrm{im}(p_M) \cong L(s_{\alpha_{\mathrm{h}},1} \cdot 0) \otimes M.$$

Therefore, we could try to understand tensor products of the form $L(s_{\alpha_{\mathrm{h}},1} \cdot 0) \otimes L(x \cdot 0)$, for $x \in W_{\mathrm{ext}}^{+}$, via a detailed study of the functor $\big(T(s_{\alpha_{\mathrm{h}},1} \cdot 0) \otimes -\big)$ and the natural transformations e and p. We will simplify this task in two ways:

Firstly, we denote by $\omega \in \Omega$ the image of the translation by the first fundamental dominant weight under the epimorphism $W_{\mathrm{ext}} \to \Omega$ and consider the tilting module $T(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0)$ instead of $T(s_{\alpha_{\mathrm{h}},1} \cdot 0)$. By the translation principle, we have

$$C_{\min}\big(L(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0)\big) = \big(\quad 0 \to T(\omega \cdot 0) \to T(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0) \to T(\omega \cdot 0) \to 0 \quad \big),$$

so we replace the natural transformations e and p by a pair of natural transformations

$$\big(T(\omega \cdot 0) \otimes -\big) \Longrightarrow \big(T(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0) \otimes -\big) \Longrightarrow \big(T(\omega \cdot 0) \otimes -\big).$$

The advantage is that the tilting module $T(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0)$ is much easier to compute with than $T(s_{\alpha_{\mathrm{h}},1} \cdot 0)$ because the weight $s_{\alpha_{\mathrm{h}},1}\omega \cdot 0$ lies 'just above' the hyperplane $H_{\alpha_{\mathrm{h}},1}^{\ell}$ separating C_{fund} and $s_{\alpha_{\mathrm{h}},1} \cdot C_{\mathrm{fund}}$. Furthermore, we do not lose any information about generic direct summands when replacing the simple $\mathbf{G}$-module $L(s_{\alpha_{\mathrm{h}},1} \cdot 0)$ by $L(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0)$ because $G(s_{\alpha_{\mathrm{h}},1}\omega, x) \cong T^{\omega} G(s_{\alpha_{\mathrm{h}},1}, x)$ for all $x \in W_{\mathrm{ext}}^{+}$, by Lemma II.5.10.

Secondly, the $\mathbf{G}$-module $G(s_{\alpha_{\mathrm{h}},1}\omega, x)$ belongs to the linkage class of $\omega \cdot 0$ for all $x \in W_{\mathrm{aff}}^{+}$, so we can project to this linkage class and consider the pair of natural transformations

$$T_0^{\omega\cdot 0} = \mathrm{pr}_{\omega\cdot 0}\big(T(\omega \cdot 0) \otimes -\big) \Longrightarrow \mathrm{pr}_{\omega\cdot 0}\big(T(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0) \otimes -\big) \Longrightarrow \mathrm{pr}_{\omega\cdot 0}\big(T(\omega \cdot 0) \otimes -\big) = T_0^{\omega\cdot 0}$$

of functors from $\mathrm{Rep}_0(\mathbf{G})$ to $\mathrm{Rep}_{\omega\cdot 0}(\mathbf{G})$. It turns out that the functor $\Psi \coloneqq \mathrm{pr}_{\omega\cdot 0}\big(T(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0) \otimes -\big)$ decomposes as a direct sum of functors Ψ_s, for $s \in S$, which behave in many ways like the *wall crosing functors* $\Theta_s = T_{\mu_s}^{\omega\cdot 0} T_0^{\mu_s}$, for $\mu_s \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_s) = \{e, s\}$. By projecting onto the direct summands Ψ_s, we obtain natural transformations

$$T_0^{\omega\cdot 0} \Longrightarrow \Psi_s \Longrightarrow T_0^{\omega\cdot 0},$$

for $s \in S$, and these will be our main objects of study in Sections 4 and 5. Before that, we need to establish some additional results about the affine Weyl group and the associated alcove geometry (see Section 1) and about *quasi-translation functors* of the form

$$T_{\lambda}^{\mu,\delta} \coloneqq \mathrm{pr}_{\mu}\big(\nabla(\delta) \otimes -\big) \colon \mathrm{Rep}_{\lambda}(\mathbf{G}) \longrightarrow \mathrm{Rep}_{\mu}(\mathbf{G}),$$

for $\lambda, \mu \in \overline{C}_{\mathrm{fund}} \cap X$ and $\delta \in X^{+}$, not necessarily W_{fin}-conjugate to $\mu - \lambda$ (see Section 2). The details of the strategy which was explained above will be discussed in Section 3, and in Section 6, we use the results from Sections 4 and 5 to study the $\mathbf{G}$-modules $\mathrm{pr}_{\omega\cdot 0}\big(L(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0) \otimes L(x \cdot 0)\big)$ and $G(s_{\alpha_{\mathrm{h}},1}\omega, x)$ for $x \in W_{\mathrm{aff}}^{+}$. While the description that we can give for $\mathrm{pr}_{\omega\cdot 0}\big(L(s_{\alpha_{\mathrm{h}},1}\omega \cdot 0) \otimes L(x \cdot 0)\big)$ is fairly explicit, the structure of $G(s_{\alpha_{\mathrm{h}},1}\omega, x)$ remains somewhat elusive. Nevertheless, we achieve a classification of the elements $x \in W_{\mathrm{aff}}^{+}$ such that $G(s_{\alpha_{\mathrm{h}},1}\omega, x)$ is simple.

1 More alcove geometry

A number of proofs in the following sections rely on intricate properties of the alcove geometry associated with the affine Weyl group W_{aff} (see Section I.2). The aim of this section is to establish these properties. We start by introducing some new tools, namely the notion of a *minimal gallery* connecting two alcoves and the corresponding *distance function*, and by proving some of their elementary properties. Recall that two alcoves $A, A' \subseteq X_{\mathbb{R}}$ are called *adjacent* if they are separated by a unique reflection hyperplane H. In that case, we have $A' = s_H(A)$ by Remark I.2.7 and the fact that W_{aff} acts transitively on the set of alcoves; see Theorem I.2.5.

Definition 1.1. Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves. A *gallery* from A to A' of *length* d is a sequence of alcoves

$$A = A_0, A_1, \ldots, A_d = A'$$

such that A_{i-1} and A_i are adjacent for $i = 1, \ldots, d$. The gallery is called *minimal* if there exists no gallery of smaller length from A to A'. The *distance* $d(A, A')$ between A and A' is the length of a minimal gallery from A to A'.

A priori, it is not clear that the distance between alcoves A and A' is well-defined (because there might not exist any gallery from A to A'). The following remark shows that every pair of alcoves is connected by a gallery. Once well-definedness is established, it is straightforward to see that the distance defines a W_{aff}-invariant metric on the set of alcoves.

Remark 1.2. For alcoves $A, A' \subseteq X_{\mathbb{R}}$, we can choose $x, y \in W_{\mathrm{aff}}$ with $x(A_{\mathrm{fund}}) = A$ and $y(A_{\mathrm{fund}}) = A'$ and write $x^{-1}y = s_1 \cdots s_d$ with $s_1, \ldots, s_d \in S$. Let $x_0 = e$ and $x_i = s_1 \cdots s_i$ for $i = 1, \ldots, d$. We claim that

$$A = x(A_{\mathrm{fund}}), xx_1(A_{\mathrm{fund}}), \ldots, xx_d(A_{\mathrm{fund}}) = A'$$

is a gallery from A to A'. Indeed, for $i = 1, \ldots, d$, the alcoves A_{fund} and $s_i(A_{\mathrm{fund}})$ are adjacent by Remark I.2.7, and it follows that $xx_{i-1}(A_{\mathrm{fund}})$ is adjacent to $xx_{i-1}s_i(A_{\mathrm{fund}}) = xx_i(A_{\mathrm{fund}})$.

The following lemma and corollary will be extremely useful when working with galleries and the distance function.

Lemma 1.3. *Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves with $A \neq A'$ and let H be a hyperplane separating A and A'. Then $d\big(A, s_H(A')\big) < d(A, A')$.*

Proof. Let $A = A_0, A_1, \ldots, A_d = A'$ be a minimal gallery from A to A'. As H separates A and A', there exists $i \in \{1, \ldots, d\}$ such that H separates A_{i-1} and A_i. Then $A_{i-1} = s_H(A_i)$ because A_{i-1} is adjacent to A_i, and there is a gallery

$$A = A_0, A_1, \ldots, A_{i-1} = s_H(A_i), s_H(A_{i+1}), \ldots, s_H(A_d) = s_H(A')$$

of length $d-1$ from A to $s_H(A')$. Hence $d\big(A, s_H(A')\big) < d = d(A, A')$. □

Corollary 1.4. *Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves and let H be a reflection hyperplane. Then H separates A and A' if and only if $d(A, s_H(A')) < d(A, A')$.*

Proof. If H separates A and A' then $d(A, s_H(A')) < d(A, A')$ by Lemma 1.3. If H does not separate A and A' then H separates A and $s_H(A')$. Again by Lemma 1.3, we get

$$d(A, A') = d(A, s_H s_H(A')) < d(A, s_H(A'))$$

and the claim follows. □

Definition 1.5. The *number of times* a gallery $A_0, \ldots, A_d$ *crosses* a reflection hyperplane H is the cardinality of the set $\{i \mid 1 \leq i \leq d \text{ and } A_i = s_H(A_{i-1})\}$. We say that a gallery *crosses* H if it crosses H at least once.

Remark 1.6. Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves and let $A = A_0, \ldots, A_d = A'$ be a gallery from A to A'. It is straightforward to see that a hyperplane H separates A and A' if and only if $A = A_0, \ldots, A_d = A'$ crosses H an odd number of times.

Lemma 1.7. *A minimal gallery crosses any given reflection hyperplane at most once.*

Proof. Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves and let $A = A_0, \ldots, A_d = A'$ be a gallery which crosses a given reflection hyperplane H at least twice. Let $1 \leq i < j \leq d$ such that $A_i = s_H(A_{i-1})$ and $A_j = s_H(A_{j-1})$. Then $A_{i-1} = s_H(A_i)$, and there is a gallery

$$A = A_0, \ldots, A_{i-1} = s_H(A_i), s_H(A_{i+1}), \ldots, s_H(A_{j-1}) = A_j, \ldots, A_d = A'$$

of length $d-2$ from A to A'. Hence $A = A_0, \ldots, A_d = A'$ is not minimal. □

Corollary 1.8. *Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves and let $A = A_0, \ldots, A_d = A'$ be a minimal gallery. A hyperplane H separates A and A' if and only if $A = A_0, \ldots, A_d = A'$ crosses H.*

Proof. It is clear that the gallery $A = A_0, \ldots, A_d = A'$ crosses every hyperplane that separates the alcoves A and A'. Conversely, if $A = A_0, \ldots, A_d = A'$ crosses H then $A = A_0, \ldots, A_d = A'$ crosses H exactly once by Lemma 1.7, and it follows that H separates A and A'. □

Recall that for $\beta \in \Phi^+$ and an alcove $A \subseteq X_{\mathbb{R}}$, we write $n_\beta(A) = \max\{m \in \mathbb{Z} \mid A \subseteq H^+_{\beta,m}\}$.

Corollary 1.9. *Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves. Then $d(A, A')$ equals the number of reflection hyperplanes that separate A and A'. In particular, we have*

$$d(A, A') = \sum_{\beta \in \Phi^+} |n_\beta(A) - n_\beta(A')|.$$

Proof. By Corollary 1.8, the set of hyperplanes that are crossed by a minimal gallery from A to A' is precisely the set of hyperplanes that separate A and A'. As a minimal gallery crosses any given reflection hyperplane at most once by Lemma 1.7, we conclude that the length $d(A, A')$ of a minimal gallery from A to A' equals the number of reflection hyperplanes that separate A and A'. The second claim follows from the observation that, for all $\beta \in \Phi^+$ and $m \in \mathbb{Z}$, the hyperplane $H_{\beta,m}$ separates A and A' if and only if $n_\beta(A) < m \leq n_\beta(A')$ or $n_\beta(A') < m \leq n_\beta(A)$. □

In Section 4, we will need to verify that a certain set $\mathcal{A}$ of alcoves that satisfies a specific symmetry property around an alcove A is of the form $\{A, s(A)\}$, for some reflection $s = s_H$ in a wall H of A. In Lemma 1.13 below, we show that this follows once we know that $s(A)$ is the unique alcove in $\mathcal{A}$ that is adjacent to A. We first define the symmetry property we want to consider.

Definition 1.10. A non-empty set $\mathcal{A}$ of alcoves is *centered* at an alcove $A \subseteq X_{\mathbb{R}}$ if, for every alcove $A' \in \mathcal{A}$ and every reflection hyperplane H separating A and A', we have $s_H(A') \in \mathcal{A}$.

Lemma 1.11. *Let $\mathcal{A}$ be a set of alcoves centered at an alcove $A \subseteq X_{\mathbb{R}}$ and let $A' \in \mathcal{A}$. For any minimal gallery $A = A_0, \ldots, A_d = A'$, we have $A_i \in \mathcal{A}$ for $i = 0, \ldots, d$. In particular $A \in \mathcal{A}$.*

Proof. If H is the hyperplane separating $A' = A_d$ and A_{d-1} then H also separates A_d and A by Corollary 1.8, so $A_{d-1} = s_H(A_d) \in \mathcal{A}$ as $\mathcal{A}$ is centered at A. The claim follows by induction on d. □

Corollary 1.12. *Let $\mathcal{A}$ be a set of alcoves centered at an alcove $A \subseteq X_{\mathbb{R}}$ and let $A' \in \mathcal{A}$. For any wall H of A that separates A and A', we have $s_H(A) \in \mathcal{A}$.*

Proof. Note that A and $s_H(A)$ are adjacent and that $d\big(s_H(A), A'\big) < d(A, A')$ by Lemma 1.3. This implies that a minimal gallery $s_H(A) = A_0, \ldots, A_d = A'$ from $s_H(A)$ to A' can be completed to a minimal gallery $A, s_H(A) = A_0, \ldots, A_d = A'$ from A to A'. Now Lemma 1.11 yields $s_H(A) \in \mathcal{A}$. □

Lemma 1.13. *Let $\mathcal{A}$ be a set of alcoves centered at an alcove $A \subseteq X_{\mathbb{R}}$, and suppose that there exists a unique wall H of A with $s_H(A) \in \mathcal{A}$. Then $\mathcal{A} = \{A, s_H(A)\}$.*

Proof. First note that $A \in \mathcal{A}$ by Lemma 1.11. For an alcove $A' \in \mathcal{A}$ with $A' \neq A$ and a wall H' of A that separates A and A', we have $s_{H'}(A) \in \mathcal{A}$ by Corollary 1.12. By the assumption on $\mathcal{A}$, it follows that $H' = H$. Now H does not separate the alcoves A and $s_H(A')$, and as before, we see that no wall H'' of A with $H'' \neq H$ separates A and $s_H(A')$ either. As no wall of A separates A and $s_H(A')$, we conclude that $A = s_H(A')$ and $A' = s_H(A)$. □

Recall from Theorem I.2.5 that the closure $\overline{A}$ of every alcove $A \subseteq X_{\mathbb{R}}$ is a fundamental domain for the action of W_{aff} on $X_{\mathbb{R}}$, so every W_{aff}-orbit in $X_{\mathbb{R}}$ intersects with $\overline{A}$ in a unique point.

Definition 1.14. For $x \in X_\mathbb{R}$ and $A \subseteq X_\mathbb{R}$ an alcove, denote by $(x)_A$ the unique element of $\overline{A}$ lying in the W_{aff}-orbit of x.

For a subset $S \subseteq X_\mathbb{R}$, we denote by $\mathrm{conv}(S)$ the convex hull of S. The situation in which we want to apply Lemma 1.13 in Section 4 is the following.

Lemma 1.15. *Let $x, y, z \in X_\mathbb{R}$ and consider the set*

$$\mathcal{A}(x,y,z) \coloneqq \big\{ A \subseteq X_\mathbb{R} \;\big|\; A \text{ is an alcove and } (y)_A - x \in \mathrm{conv}\big(W_{\mathrm{fin}}(z)\big) \big\}.$$

If $\mathcal{A}(x,y,z)$ is non-empty and A is an alcove with $x \in \overline{A}$ then $\mathcal{A}(x,y,z)$ is centered at A.

Proof. Let $A' \in \mathcal{A}(x,y,z)$ with $A' \neq A$ and let $H = H_{\beta,m}$ be a hyperplane separating A and A'. Suppose that $A \subseteq H^-$ and $A' \subseteq H^+$ and set $y' \coloneqq (y)_{A'}$, so $(y', \beta^\vee) \geq m \geq (x, \beta^\vee)$. We have

$$(y)_{s_H(A')} - x = s_{\beta,m}(y') - x = y' - x - \big((y', \beta^\vee) - m\big) \cdot \beta,$$

and if $(y', \beta^\vee) = m$ then it follows that

$$(y)_{s_H(A')} - x = y' - x = (y)_{A'} - x \in \mathrm{conv}\big(W_{\mathrm{fin}}(z)\big),$$

so $s_H(A') \in \mathcal{A}(x,y,z)$. If $(y', \beta^\vee) > m \geq (x, \beta^\vee)$ then

$$\begin{aligned} (y)_{s_H(A')} - x &= y' - x - \big((y', \beta^\vee) - m\big) \cdot \beta \\ &= (y' - x) - \frac{(y', \beta^\vee) - m}{(y' - x, \beta^\vee)} \cdot (y' - x, \beta^\vee) \cdot \beta \\ &= (y' - x) + \frac{(y', \beta^\vee) - m}{(y' - x, \beta^\vee)} \cdot \big(s_\beta(y' - x) - (y' - x)\big) \end{aligned}$$

is an element of $\mathrm{conv}\big\{y' - x, s_\beta(y' - x)\big\}$. Now $y' - x \in \mathrm{conv}\big(W_{\mathrm{fin}}(z)\big)$ because $A' \in \mathcal{A}(x,y,z)$ by assumption, and $s_\beta(y' - x) \in \mathrm{conv}\big(W_{\mathrm{fin}}(z)\big)$ because $\mathrm{conv}\big(W_{\mathrm{fin}}(z)\big)$ is W_{fin}-invariant. We conclude that

$$(y)_{s_H(A')} - x \in \mathrm{conv}\big\{y' - x, s_\beta(y' - x)\big\} \subseteq \mathrm{conv}\big(W_{\mathrm{fin}}(z)\big),$$

so $s_H(A') \in \mathcal{A}(x,y,z)$ as required. The case $A \subseteq H^+$ and $A' \subseteq H^-$ is analogous. □

Recall that the linkage order $\uparrow$ on the set of alcoves is the reflexive and transitive closure of the relation that is given by $A \uparrow A'$ if there exists a reflection $s \in W_{\mathrm{aff}}$ with $A \subseteq H_s^-$ and $A' \in H_s^+$ such that $A' = s(A)$. For our further study of the linkage order, the following function on the set of alcoves will be of central importance.

Definition 1.16. For an alcove $A \subseteq X_\mathbb{R}$, let $d(A) \coloneqq \sum_{\beta \in \Phi^+} n_\beta(A)$.

Note that for $\gamma \in X$ and alcoves $A, A' \subseteq X_\mathbb{R}$, we have $A \uparrow A'$ if and only if $A + \gamma \uparrow A' + \gamma$, and

$$d(A' + \gamma) - d(A + \gamma) = d(A') - d(A).$$

Furthermore, if $A \subseteq X_\mathbb{R}$ is a dominant alcove then $n_\beta(A) \geq 0$ for all $\beta \in \Phi^+$ and therefore

$$d(A) = \sum_{\beta \in \Phi^+} n_\beta(A) = \sum_{\beta \in \Phi^+} |n_\beta(A)| = d(A_{\mathrm{fund}}, A)$$

by Corollary 1.9 and Example I.2.3. The connection between the function d and the linkage order comes from the following lemma, which is proven in Section II.6.6 in [Jan03].

Lemma 1.17. *Let $A \subseteq X_{\mathbb{R}}$ be an alcove and let $s \in W_{\mathrm{aff}}$ be a reflection. Then $A \uparrow s(A)$ if and only if $d(A) < d\big(s(A)\big)$.*

The two following results are immediate consequences of Lemma 1.17 and the definition of the linkage order.

Corollary 1.18. *Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves with $A \uparrow A'$. Then $d(A) \leq d(A')$, with equality if and only if $A = A'$.*

Corollary 1.19. *Let $A, A' \subseteq X_{\mathbb{R}}$ be alcoves with $A \uparrow A'$ and $d(A') = d(A) + 1$. Then there is a reflection $s \in W_{\mathrm{aff}}$ with $A' = s(A)$.*

We are now ready to show that the linkage order on the set of dominant alcoves is equivalent to the Bruhat order on W_{aff}^+, with the help of some results from [BB05] and [Wan87]. This was already stated in Theorem I.2.14, but the proof was postponed to this section. The article [Wan87] has been published only in Chinese, but a translation of the main result into English is available as an appendix to [GHS18].

Theorem 1.20. *For $x, y \in W_{\mathrm{aff}}^+$, we have $x \leq y$ if and only if $x(A_{\mathrm{fund}}) \uparrow y(A_{\mathrm{fund}})$.*

Proof. First suppose that $x \leq y$. By Theorem 2.5.5 in [BB05], there exist elements $x_0, \ldots, x_r \in W_{\mathrm{aff}}^+$ with

$$x = x_0 < x_1 < \cdots < x_r = y$$

and such that

$$\ell(x) + i = \ell(x_i) = d\big(A_{\mathrm{fund}}, x_i(A_{\mathrm{fund}})\big) = d\big(x_i(A_{\mathrm{fund}})\big)$$

for $i = 1, \ldots, r$, where the second equality follows from Corollary 1.9 and the third equality holds because $x_i(A_{\mathrm{fund}})$ is a dominant alcove. As $x_{i-1} < x_i$ and $\ell(x_i) - \ell(x_{i-1}) = 1$, for $i = 1, \ldots, r$, there exists a reflection $s_i \in W_{\mathrm{aff}}$ with $x_i = x_{i-1}s_i$. Furthermore, as

$$d\big(x_{i-1}s_i(A_{\mathrm{fund}})\big) = d\big(x_i(A_{\mathrm{fund}})\big) > d\big(x_{i-1}(A_{\mathrm{fund}})\big),$$

we have $x_{i-1}(A_{\mathrm{fund}}) \uparrow x_{i-1}s_i(A_{\mathrm{fund}}) = x_i(A_{\mathrm{fund}})$ by Lemma 1.17 (applied to the reflection $x_{i-1}s_ix_{i-1}^{-1}$), for $i = 1, \ldots, r$, and we conclude that $x(A_{\mathrm{fund}}) \uparrow y(A_{\mathrm{fund}})$.

Now suppose that $x(A_{\mathrm{fund}}) \uparrow y(A_{\mathrm{fund}})$. By Theorem A.1.1 in [GHS18], there exists a sequence of dominant alcoves

$$x(A_{\mathrm{fund}}) = A_0 \uparrow A_1 \uparrow \cdots \uparrow A_r = y(A_{\mathrm{fund}})$$

such that $d(A_i) - d(A_{i-1}) = 1$ for $i = 0, \ldots, r$. As W_{aff} acts (simply) transitively on the set of alcoves (see Theorem I.2.5), there exist $x_0, \ldots, x_r \in W_{\mathrm{aff}}^+$ with $x_i(A_{\mathrm{fund}}) = A_i$ for $i = 0, \ldots, r$. Furthermore, by Corollary 1.19, there exist reflections $s_1, \ldots, s_r \in W_{\mathrm{aff}}$ with $A_i = s_i(A_{i-1})$ for $i = 1, \ldots, r$, and it follows that $x_i = s_ix_{i-1}$. Now as $x_{i-1}, x_i \in W_{\mathrm{aff}}^+$, we have

$$\ell(x_i) = d\big(x_i(A_{\mathrm{fund}})\big) = d(A_i) = d(A_{i-1}) + 1 = d\big(x_{i-1}(A_{\mathrm{fund}})\big) + 1 = \ell(x_{i-1}) + 1 > \ell(x_{i-1})$$

for $i = 1, \ldots, r$, and we conclude that $x = x_0 < x_1 < \cdots < x_r = y$. $\square$

The next result describes the stabilizer of a point $x \in X_{\mathbb{R}}$ in terms of the walls of an alcove $A \subseteq X_{\mathbb{R}}$ with $x \in \overline{A}$; see Section 6.3 in [Jan03].

Lemma 1.21. *Let $x \in X_{\mathbb{R}}$ and let $A \subseteq X_{\mathbb{R}}$ be an alcove with $x \in \overline{A}$. Then $\mathrm{Stab}_{W_{\mathrm{aff}}}(x)$ is generated by the reflections in the walls H of A with $x \in H$, that is*

$$\mathrm{Stab}_{W_{\mathrm{aff}}}(x) = \langle s_H \mid H \textit{ is a wall of } A \textit{ with } x \in H \rangle.$$

In particular, we have $\mathrm{Stab}_{W_{\mathrm{aff}}}(x) = \{e\}$ if and only if $x \in A$.

For applications in Chapter V, we prove some further results about stabilizers in W_{aff}. Recall that we write $X_{\mathbb{R}}^{+} = \{x \in X_{\mathbb{R}} \mid (x, \alpha^{\vee}) > 0 \text{ for all } \alpha \in \Phi^{+}\}$ for the dominant Weyl chamber.

Lemma 1.22. *For $x \in \overline{A}_{\mathrm{fund}}$ and $w \in W_{\mathrm{aff}}$, we have $w(x) \in X_{\mathbb{R}}^{+}$ if and only if $w\mathrm{Stab}_{W_{\mathrm{aff}}}(x) \subseteq W_{\mathrm{aff}}^{+}$.*

Proof. Suppose that $w(x) \in X_{\mathbb{R}}^{+}$ and let $w' \in \mathrm{Stab}_{W_{\mathrm{aff}}}(x)$. Then $w(x) = ww'(x) \in ww'(\overline{A}_{\mathrm{fund}})$, so

$$0 < \big(w(x), \alpha^{\vee}\big) \leq n_{\alpha}\big(ww'(A_{\mathrm{fund}})\big) + 1$$

for all $\alpha \in \Phi^{+}$, and it follows that $n_{\alpha}\big(ww'(A_{\mathrm{fund}})\big) \geq 0$ and $ww' \in W_{\mathrm{aff}}^{+}$.

Now suppose that $w\mathrm{Stab}_{W_{\mathrm{aff}}}(x) \subseteq W_{\mathrm{aff}}^{+}$. For any simple root $\alpha \in \Pi$, we have

$$w(w^{-1} s_{\alpha} w) = s_{\alpha} w \notin W_{\mathrm{aff}}^{+}$$

because $w \in W_{\mathrm{aff}}^{+}$, and it follows that $w^{-1} s_{\alpha} w \notin \mathrm{Stab}_{W_{\mathrm{aff}}}(x)$ and $s_{\alpha} \notin \mathrm{Stab}_{W_{\mathrm{aff}}}\big(w(x)\big)$. This implies that $\big(w(x), \alpha^{\vee}\big) \neq 0$, and as

$$0 \leq n_{\alpha}\big(w(A_{\mathrm{fund}})\big) \leq \big(w(x), \alpha^{\vee}\big),$$

we conclude that $w(x) \in X_{\mathbb{R}}^{+}$, as required. □

We can also characterize the *upper closure* of an alcove in terms of stabilizers.

Definition 1.23. The *upper closure* of an alcove $A \subseteq X_{\mathbb{R}}$ is the set

$$\widehat{A} \coloneqq \big\{ x \in X_{\mathbb{R}} \bigm| n_{\alpha}(A) < (x, \alpha^{\vee}) \leq n_{\alpha}(A) + 1 \text{ for all } \alpha \in \Phi^{+} \big\}.$$

It is straightforward to see that, for every point $x \in X_{\mathbb{R}}$, there is a unique alcove $A \subseteq X_{\mathbb{R}}$ such that $x \in \widehat{A}$.

Lemma 1.24. *Let $x \in \overline{A}_{\mathrm{fund}}$ and $w \in W_{\mathrm{aff}}$. Then the following are equivalent:*

(1) *$w(x)$ belongs to the upper closure of $w(A_{\mathrm{fund}})$;*

(2) *$w(A_{\mathrm{fund}})$ is minimal (in the linkage order) among the alcoves whose closure contains $w(x)$;*

(3) *$w(A_{\mathrm{fund}}) \uparrow ws(A_{\mathrm{fund}})$ for all $s \in S \cap \mathrm{Stab}_{W_{\mathrm{aff}}}(x)$.*

Proof. Suppose first that $w(x)$ belongs to the upper closure of $w(A_{\mathrm{fund}})$, and let $A \subseteq X_{\mathbb{R}}$ be an alcove with $w(x) \in \overline{A}$. If $A \neq w(A_{\mathrm{fund}})$ then there exists a reflection hyperplane $H = H_{\beta,m}$ separating $w(A_{\mathrm{fund}})$ and A, that is

$$n_{\beta}\big(w(A_{\mathrm{fund}})\big) + 1 \leq m \leq n_{\beta}(A) \qquad \text{or} \qquad n_{\beta}(A) + 1 \leq m \leq n_{\beta}\big(w(A_{\mathrm{fund}})\big).$$

As $w(x)$ belongs to the upper closure of $w(A_{\mathrm{fund}})$ and to the closure of A, we have

$$n_{\beta}\big(w(A_{\mathrm{fund}})\big) < \big(w(x), \beta^{\vee}\big) \leq n_{\beta}(A) + 1$$

and
$$n_\beta(A) \leq \big(w(x), \beta^\vee\big) \leq n_\beta\big(w(A_{\mathrm{fund}})\big) + 1,$$
and we conclude that
$$m = n_\beta\big(w(A_{\mathrm{fund}})\big) + 1 = n_\beta(A) = \big(w(x), \beta^\vee\big).$$
This implies that $s_{\beta,m}(A) \uparrow A$ and that $s_{\beta,m} \in \mathrm{Stab}_{W_{\mathrm{aff}}}\big(w(x)\big)$, so $w(x) \in s_{\beta,m}(\overline{A})$. By induction on the distance between $w(A_{\mathrm{fund}})$ and A (see Corollary 1.4), we conclude that
$$w(A_{\mathrm{fund}}) \uparrow s_{\beta,m}(A) \uparrow A,$$
so (1) implies (2). It is straightforward to see that (2) implies (3).

We proceed to prove that (3) implies (1), so now suppose that $w(A_{\mathrm{fund}}) \uparrow ws(A_{\mathrm{fund}})$ for all simple reflections $s \in S \cap \mathrm{Stab}_{W_{\mathrm{aff}}}(x)$. Let $A \subseteq X_{\mathbb{R}}^+$ be the unique alcove whose upper closure contains $w(x)$, and suppose for a contradiction that $A \neq w(A_{\mathrm{fund}})$. Then there exists a simple reflection $s \in S$ such that the wall $w(H_s) = H_{wsw^{-1}}$ of $w(A_{\mathrm{fund}})$ separates $w(A_{\mathrm{fund}})$ and A. Since $w(x)$ belongs to the closures of both of the alcoves $w(A_{\mathrm{fund}})$ and A, we also have
$$w(x) \in \overline{A} \cap w(\overline{A}_{\mathrm{fund}}) \subseteq w(H_s)$$
and it follows that $s \in \mathrm{Stab}_{W_{\mathrm{aff}}}(x)$. Now let us write $wsw^{-1} = s_{\beta,m}$, for some $\beta \in \Phi^+$ and $m \in \mathbb{Z}$, and note that we have $m = \big(w(x), \beta^\vee\big)$ because $wsw^{-1} \in \mathrm{Stab}_{W_{\mathrm{aff}}}\big(w(x)\big)$. As x belongs to the upper closure of A, we further have
$$n_\beta(A) < \big(w(x), \beta^\vee\big) \leq n_\beta(A) + 1$$
and it follows that $n_\beta(A) + 1 = m$ and $A \subseteq H_{\beta,m}^-$. Finally, as $H_{\beta,m} = H_{wsw^{-1}}$ separates the alcoves $w(A_{\mathrm{fund}})$ and A, we conclude that $w(A_{\mathrm{fund}}) \subseteq H_{\beta,m}^+$ and $ws(A_{\mathrm{fund}}) \uparrow w(A_{\mathrm{fund}})$, contradicting the assumption. □

Next, for an alcove $A \subseteq X_{\mathbb{R}}$ and reflections $s, t \in W_{\mathrm{aff}}$, we want to investigate the linkage relation between $s(A)$ and $ts(A)$.

Lemma 1.25. *Let $A \subseteq X_{\mathbb{R}}$ be an alcove and let H be a wall of A with corresponding reflection $s = s_H$. For any reflection $t \in W_{\mathrm{aff}}$ with $t \neq s$ and $A \uparrow t(A)$, we have $s(A) \uparrow ts(A)$.*

Proof. Note that the alcoves A and $s(A)$ are adjacent and that $H = H_s$ is the unique reflection hyperplane separating them. Now the assumption $A \uparrow t(A)$ implies that $A \subseteq H_t^-$. As $t \neq s$, the hyperplane H_t does not separate A and $s(A)$, whence $s(A) \subseteq H_t^-$ and $s(A) \uparrow ts(A)$. □

Corollary 1.26. *Let $A \subseteq X_{\mathbb{R}}$ be an alcove and let H be a wall of A with corresponding reflection $s = s_H$. For any reflection $t \in W_{\mathrm{aff}}$ with $t \neq s$ and $d(A) < d\big(t(A)\big)$, we have $d\big(s(A)\big) < d\big(ts(A)\big)$.*

Proof. This is immediate from Lemmas 1.17 and 1.25. □

As W_{aff} is in bijection with the set of of alcoves (via $x \mapsto x(A_{\mathrm{fund}})$, see Theorem I.2.5), we can also consider d as a function on W_{aff}.

Definition 1.27. For $x \in W_{\mathrm{aff}}$, let $d(x) \coloneqq d\big(x(A_{\mathrm{fund}})\big)$.

Note that for $x \in W_{\mathrm{aff}}^+$, we have $d(x) = d\big(A_{\mathrm{fund}}, x(A_{\mathrm{fund}})\big) = \ell(x)$. The following result is only a reformulation of Corollary 1.26 in terms of elements of W_{aff}.

Corollary 1.28. *Let $x \in W_{\mathrm{aff}}$ and $s \in S$. For any reflection $t \in W_{\mathrm{aff}}$ with $t \neq s$ and $d(x) < d(xt)$, we have $d(xs) < d(xts)$.*

Proof. Apply Corollary 1.26 to the alcove $x(A_{\mathrm{fund}})$ and the reflections xsx^{-1} and xtx^{-1}. □

Lemma 1.29. *For $x \in W_{\mathrm{aff}}$ and $s \in S$, we have $d(xs) \in \{d(x)+1, d(x)-1\}$.*

Proof. As the alcoves A_{fund} and $s(A_{\mathrm{fund}})$ are adjacent (by Remark I.2.7), so are $x(A_{\mathrm{fund}})$ and $xs(A_{\mathrm{fund}})$. Now Corollary 1.9 implies that

$$1 = d\big(x(A_{\mathrm{fund}}), xs(A_{\mathrm{fund}})\big) = \sum_{\beta \in \Phi^+} \big| n_\beta\big(x(A_{\mathrm{fund}})\big) - n_\beta\big(xs(A_{\mathrm{fund}})\big) \big| ,$$

from which the claim is immediate. □

Lemma 1.30. *Let $x \in W_{\mathrm{aff}}^+$ with $x \neq e$ and let $s \in S$ with $xs < x$. Then*

$$xs \in W_{\mathrm{aff}}^+, \qquad xs(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}}) \qquad \text{and} \qquad d(xs) = d(x) - 1.$$

Proof. We have $xs \in W_{\mathrm{aff}}^+$ by Corollary I.2.13 and therefore

$$d(xs) = \ell(xs) = \ell(x) - 1 = d(x) - 1 < d(x).$$

Now Lemma 1.17, applied to $xs(A_{\mathrm{fund}})$ and the reflection xsx^{-1}, implies that $xs(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}})$. □

Let $x \in W_{\mathrm{aff}}$ and $s \in S$ with $x(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}})$. For applications in Section 5, it will be important to consider sequences of elements $x_0, \ldots, x_d \in W_{\mathrm{aff}}$ with

$$x_0(A_{\mathrm{fund}}) \uparrow x_1(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_d(A_{\mathrm{fund}}) = x(A_{\mathrm{fund}})$$

and such that $d(x_i) = d(x_0) + i$ and $x_i(A_{\mathrm{fund}}) \uparrow x_i s(A_{\mathrm{fund}})$ for $i = 0, \ldots, d$. In the following, we denote by $s_0 \coloneqq s_{\alpha_{\mathrm{h}},1}$ the unique simple reflection with $A_{\mathrm{fund}} \uparrow s_0(A_{\mathrm{fund}})$.

Proposition 1.31. *Let $x \in W_{\mathrm{aff}}$ such that $xs_0 \in W_{\mathrm{aff}}^+$ and $x(A_{\mathrm{fund}}) \uparrow xs_0(A_{\mathrm{fund}})$. Then there exist $y_0, \ldots, y_r \in W_{\mathrm{aff}}$ with $y_0 \in W_{\mathrm{aff}}^+$ and*

$$y_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow y_r(A_{\mathrm{fund}}) = x(A_{\mathrm{fund}})$$

such that $d(y_i) = d(y_0) + i$, $y_i(A_{\mathrm{fund}}) \uparrow y_i s_0(A_{\mathrm{fund}})$ and $y_i s_0 \in W_{\mathrm{aff}}^+$ for $i = 0, \ldots, r$.

Proof. We prove the claim by induction on $d = d(xs_0)$. Note that since $xs_0 \in W_{\mathrm{aff}}^+$, we have

$$d = d(xs_0) = d\big(A_{\mathrm{fund}}, xs_0(A_{\mathrm{fund}})\big) = \ell(xs_0).$$

If $d = 0$ then $xs_0 = e$ and $x = s_0$, contradicting the assumption that $x(A_{\mathrm{fund}}) \uparrow xs_0(A_{\mathrm{fund}})$. If $d = 1$ then $xs_0(A_{\mathrm{fund}})$ is adjacent to A_{fund}, so $xs_0 = s$ for some $s \in S$. As $s_\alpha \notin W_{\mathrm{aff}}^+$ for all $\alpha \in \Pi$, we conclude that $xs_0 = s_0$ and $x = e$, and the claim follows with $r = 0$.

Now suppose that $d \geq 2$ and that the claim is true for all $y \in W_{\mathrm{aff}}$ with $d(ys_0) < d$ that satisfy the hypotheses of the proposition. If $x \in W_{\mathrm{aff}}^+$ then the claim follows with $r = 0$, so let us further assume that $x \notin W_{\mathrm{aff}}^+$. We have $d(xs_0) = d(x) + 1$ by Lemmas 1.17 and 1.29. Furthermore, as $d(xs_0) = d > 0$, we have $xs_0 \neq e$ and there exists $s \in S$ with $xs_0 s < xs_0$, whence

$$xs_0 s \in W_{\mathrm{aff}}^+, \qquad xs_0 s(A_{\mathrm{fund}}) \uparrow xs_0(A_{\mathrm{fund}}) \qquad \text{and} \qquad d(xs_0 s) = d(xs_0) - 1$$

by Lemma 1.30. Note that we have $s \neq s_0$ because $xs_0s_0 = x \notin W_{\mathrm{aff}}^+$. As $d(xs_0s) < d(xs_0) = d(xs_0ss)$, Corollary 1.28 yields

$$d(xs_0ss_0) < d(xs_0sss_0) = d(x) = d(xs_0) - 1 = d(xs_0s),$$

and using Lemma 1.29, we conclude that $d(xs_0ss_0) = d(xs_0s) - 1 = d(x) - 1$. Now Lemma 1.17 yields

$$xs_0ss_0(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}}) \qquad \text{and} \qquad xs_0ss_0(A_{\mathrm{fund}}) \uparrow xs_0s(A_{\mathrm{fund}}) = (xs_0ss_0)s_0(A_{\mathrm{fund}}),$$

and as $(xs_0ss_0)s_0 = xs_0s \in W_{\mathrm{aff}}^+$, the element xs_0ss_0 satisfies the hypothesis of the proposition. Furthermore, we have $d\big((xs_0ss_0)s_0\big) = d(xs_0s) < d(xs_0) = d$, and by the induction hypothesis, there exist $y_0, \dots y_r \in W_{\mathrm{aff}}$ with $y_0 \in W_{\mathrm{aff}}^+$ and

$$y_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow y_r(A_{\mathrm{fund}}) = xs_0ss_0(A_{\mathrm{fund}})$$

such that $d(y_i) = d(y_0) + i$, $y_i(A_{\mathrm{fund}}) \uparrow y_is_0(A_{\mathrm{fund}})$ and $y_is_0 \in W_{\mathrm{aff}}^+$ for $i = 0, \dots, r$. Then the chain

$$y_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow y_r(A_{\mathrm{fund}}) = xs_0ss_0(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}})$$

has the required properties. □

For the following result, we suppose that Φ is of type A_n. By Example I.2.6, this ensures that, for all $s, t \in S$, we have either $st = ts$ or $sts = tst$. Recall that we write $s_0 = s_{\alpha_{\mathrm{h}},1}$.

Proposition 1.32. *Suppose that Φ is of type A_n. Let $x \in W_{\mathrm{aff}}^+$ such that $x(A_{\mathrm{fund}}) \uparrow xs_0(A_{\mathrm{fund}})$ and set $d = d(x)$. Then there exist $x_0, \dots, x_d \in W_{\mathrm{aff}}$ with*

$$A_{\mathrm{fund}} = x_0(A_{\mathrm{fund}}) \uparrow x_1(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_d(A_{\mathrm{fund}}) = x(A_{\mathrm{fund}})$$

and such that $x_i(A_{\mathrm{fund}}) \uparrow x_is_0(A_{\mathrm{fund}})$ and $d(x_i) = i$ for $i = 0, \dots, d$.

Proof. We prove the claim by induction on $d = d(x) = d\big(A_{\mathrm{fund}}, x(A_{\mathrm{fund}})\big)$. If $d = 0$ then $x = e$ and the claim follows with $x_0 = e$. Now suppose that $d > 0$ and that the claim is true for all $y \in W_{\mathrm{aff}}^+$ such that $d(y) < d$ and $y(A_{\mathrm{fund}}) \uparrow ys_0(A_{\mathrm{fund}})$. As $d(x) = d > 0$, we have $x \neq e$ and there exists a simple reflection $s \in S$ with $xs < x$. Then Lemma 1.30 implies that

$$xs \in W_{\mathrm{aff}}^+, \qquad xs(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}}) \qquad \text{and} \qquad d(xs) = d(x) - 1.$$

Also note that $d(xs_0) = d(x) + 1 = d + 1$ by Lemmas 1.17 and 1.29, in particular $s \neq s_0$.

If $xs(A_{\mathrm{fund}}) \uparrow xss_0(A_{\mathrm{fund}})$ then, by induction, there exist $x_0, \dots, x_{d-1} \in W_{\mathrm{aff}}$ with

$$A_{\mathrm{fund}} = x_0(A_{\mathrm{fund}}) \uparrow x_1(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_{d-1}(A_{\mathrm{fund}}) = xs(A_{\mathrm{fund}})$$

and such that $x_i(A_{\mathrm{fund}}) \uparrow x_is_0(A_{\mathrm{fund}})$ and $d(x_i) = i$ for $i = 0, \dots, d-1$. In this case, the claim follows with $x_d = x$. Now suppose that $xss_0(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}})$ and therefore $d(xss_0) = d(xs) - 1 = d(x) - 2$ by Lemmas 1.17 and 1.29. If s and s_0 commute then

$$d(xss_0) = d(xs_0s) \geq d(xs_0) - 1 = d(x) > d(xs)$$

by Lemma 1.29 and therefore $xs(A_{\mathrm{fund}}) \uparrow xss_0(A_{\mathrm{fund}})$ by Lemma 1.17, a contradiction. Hence s does not commute with s_0, and it follows that $ss_0s = s_0ss_0$ (see Example I.2.6). Applying Lemma 1.29 three times, we obtain

$$d(x) - 1 = d(xss_0) + 1 \geq d(xss_0s) = d(xs_0ss_0) \geq d(xs_0s) - 1 \geq d(xs_0) - 2 = d(x) - 1,$$

and therefore

$$d(x) - 1 = d(xss_0) + 1 = d(xss_0s) = d(xs_0s) - 1.$$

Now Lemma 1.17 yields

$$xss_0(A_{\mathrm{fund}}) \uparrow xss_0s(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}}) \qquad \text{and} \qquad xss_0s(A_{\mathrm{fund}}) \uparrow xs_0s(A_{\mathrm{fund}}) = (xss_0s)s_0(A_{\mathrm{fund}}).$$

Recall that $xs \in W_{\mathrm{aff}}^+$ and that $xss_0(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}}) = (xss_0)s_0(A_{\mathrm{fund}})$ by assumption. By Proposition 1.31 (applied to xss_0), there exist $y_0, \ldots, y_r \in W_{\mathrm{aff}}$ with $y_0 \in W_{\mathrm{aff}}^+$ and

$$y_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow y_r(A_{\mathrm{fund}}) = xss_0(A_{\mathrm{fund}})$$

such that $y_i(A_{\mathrm{fund}}) \uparrow y_is_0(A_{\mathrm{fund}})$ and $d(y_i) = d(y_0) + i$ for $i = 0, \ldots, r$. Then $d(y_0) \leq d(xss_0) < d(x)$, and by the induction hypothesis, there exist $x_0, \ldots, x_{r'} \in W_{\mathrm{aff}}$ with

$$A_{\mathrm{fund}} = x_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_{r'}(A_{\mathrm{fund}}) = y_0(A_{\mathrm{fund}})$$

and such that $x_i(A_{\mathrm{fund}}) \uparrow x_is_0(A_{\mathrm{fund}})$ and $d(x_i) = i$ for $i = 0, \ldots, r'$. Now the chain

$$\begin{aligned} A_{\mathrm{fund}} &= x_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_{r'}(A_{\mathrm{fund}}) \\ &= y_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow y_r(A_{\mathrm{fund}}) = xss_0(A_{\mathrm{fund}}) \uparrow xss_0s(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}}) \end{aligned}$$

has the required properties. □

Chains of alcoves as in the preceding proposition will also be of interest if we replace $s_0 = s_{\alpha_{\mathrm{h}},1}$ by an arbitrary simple reflection $s \in S$ (and the fundamental alcove A_{fund} by $A_{\mathrm{fund}} + \gamma$ for some $\gamma \in X$). Before stating a corollary of the proposition that establishes the existence of such chains, we need to discuss the notion of *extremal points* of the fundamental alcove.

Remark 1.33. Recall from Remark I.2.7 that $\{H_{\alpha,0} \mid \alpha \in \Pi\} \cup \{H_{\alpha_{\mathrm{h}},1}\}$ is the set of walls of A_{fund}. A point that lies in the intersection of all but one wall of A_{fund} is called an *extremal point* of A_{fund}. It is straightforward to see that 0 is the unique extremal point of A_{fund} that does not lie on $H_{\alpha_{\mathrm{h}},1}$. Furthermore, for each simple root $\alpha \in \Pi$, the unique extremal point of A_{fund} that does not lie on the wall $H_{\alpha,0}$ is given by $\frac{1}{c_\alpha} \cdot \varpi_\alpha$, where $c_\alpha := (\varpi_\alpha, \alpha_{\mathrm{h}}^\vee)$.

As the action of $\Omega = \mathrm{Stab}_{W_{\mathrm{ext}}}(A_{\mathrm{fund}})$ on $X_{\mathbb{R}}$ permutes the walls of A_{fund}, it also permutes the set of extremal points of A_{fund}. Furthermore, the only affine linear transformation of $X_{\mathbb{R}}$ that fixes all of the extremal points is the identity; hence the action of Ω on the set of extremal points of A_{fund} is faithful. This implies that the action of Ω on $X_{\mathbb{R}}$ faithfully permutes the walls of A_{fund} and that the action of Ω on W_{aff} by conjugation faithfully permutes the simple reflections.

Example 1.34. Suppose that Φ is of type A_n. Then $\alpha_{\mathrm{h}}^\vee = \sum_{\alpha \in \Pi} \alpha^\vee$ and by Remark 1.33, the set of extremal points of A_{fund} is given by $\{\varpi_\alpha \mid \alpha \in \Pi\} \cup \{0\}$. Note that all extremal points belong to X (which may not be true for other types of root systems).

Conjugation by $\omega \in \Omega$ is an automorphism of W_{aff} which permutes the simple reflections, hence it induces a graph automorphism of the Coxeter diagram of W_{aff}. As the action of

$$\Omega \cong X/\mathbb{Z}\Phi \cong \mathbb{Z}/(n+1)\mathbb{Z}$$

by conjugation on the set S of simple reflections is faithful (again by Remark 1.33) and as the Coxeter diagram of W_{aff} (for Φ of type A_n) is a cycle of length $n+1$ (see Example I.2.6), the action of Ω on S by conjugation is transitive.

Corollary 1.35. *Suppose that Φ is of type A_n. Let $x \in W_{\mathrm{aff}}$ and $s \in S$ such that $x(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}})$. Then there exist $\gamma \in X$ and $x_0, \ldots, x_d \in W_{\mathrm{aff}}$ with*

$$A_{\mathrm{fund}} + \gamma = x_0(A_{\mathrm{fund}}) \uparrow x_1(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_d(A_{\mathrm{fund}}) = x(A_{\mathrm{fund}})$$

and such that $x_i(A_{\mathrm{fund}}) \uparrow x_i s(A_{\mathrm{fund}})$ and $d(x_i) = d(t_\gamma) + i$ for $i = 0, \ldots, d$.

Proof. The action of Ω on S by conjugation is transitive by Example 1.34, so let $\omega \in \Omega$ with $s = \omega s_0 \omega^{-1}$ and write $x' = \omega^{-1} x \omega$ and $\omega = t_\mu y$ with $\mu \in X$ and $y \in W_{\mathrm{fin}}$. Observe that we have

$$xs(A_{\mathrm{fund}}) = x\omega s_0 \omega^{-1}(A_{\mathrm{fund}}) = x\omega s_0(A_{\mathrm{fund}}) = \omega x' s_0(A_{\mathrm{fund}}) = yx' s_0(A_{\mathrm{fund}}) + \mu$$

and

$$x(A_{\mathrm{fund}}) = x\omega(A_{\mathrm{fund}}) = \omega x'(A_{\mathrm{fund}}) = yx'(A_{\mathrm{fund}}) + \mu,$$

so

$$yx'(A_{\mathrm{fund}}) + \mu = x(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}}) = yx' s_0(A_{\mathrm{fund}}) + \mu,$$

and it follows that $yx'(A_{\mathrm{fund}}) \uparrow yx' s_0(A_{\mathrm{fund}})$. Let us fix $\nu \in \mathbb{Z}\Phi$ with $t_\nu y x' \in W_{\mathrm{aff}}^+$, and note that

$$t_\nu y x'(A_{\mathrm{fund}}) = yx'(A_{\mathrm{fund}}) + \nu \uparrow yx' s_0(A_{\mathrm{fund}}) + \nu = t_\nu y x' s_0(A_{\mathrm{fund}}).$$

By Proposition 1.32, there exist $y_0, \ldots, y_d \in W_{\mathrm{aff}}$ with

$$A_{\mathrm{fund}} = y_0(A_{\mathrm{fund}}) \uparrow y_1(A_{\mathrm{fund}}) \uparrow \cdots \uparrow y_d(A_{\mathrm{fund}}) = t_\nu y x'(A_{\mathrm{fund}})$$

and such that $d(y_i) = i$ and $y_i(A_{\mathrm{fund}}) \uparrow y_i s_0(A_{\mathrm{fund}})$ for $i = 0, \ldots, d$. We define $\gamma \coloneqq \mu - \nu$ and

$$x_i \coloneqq t_\gamma y_i \omega^{-1} = t_\mu t_{-\nu} y_i \omega^{-1} = \omega(y^{-1} t_{-\nu} y_i)\omega^{-1} \in W_{\mathrm{aff}}$$

for $i = 0, \ldots, d$, so that $x_i(A_{\mathrm{fund}}) = y_i(A_{\mathrm{fund}}) + \gamma$. Thus

$$A_{\mathrm{fund}} + \gamma = x_0(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_d(A_{\mathrm{fund}}) = t_\nu y x'(A_{\mathrm{fund}}) + \gamma = yx'(A_{\mathrm{fund}}) + \mu = x(A_{\mathrm{fund}})$$

and

$$d(x_i) - d(t_\gamma) = d\big(y_i(A_{\mathrm{fund}}) + \gamma\big) - d(A_{\mathrm{fund}} + \gamma) = d(y_i) = i$$

for $i = 0, \ldots, d$. Furthermore, we have

$$x_i(A_{\mathrm{fund}}) = y_i(A_{\mathrm{fund}}) + \gamma \uparrow y_i s_0(A_{\mathrm{fund}}) + \gamma = t_\gamma y_i \omega^{-1} \cdot \omega s_0 \omega^{-1}(A_{\mathrm{fund}}) = x_i s(A_{\mathrm{fund}})$$

for $i = 1, \ldots, d$, so the elements $x_0, \ldots, x_d \in W_{\mathrm{aff}}$ have the required properties. ∎

We conclude this section with another result that is only valid for Φ of type A_n; it will be needed in Section 6.

Lemma 1.36. *Suppose that Φ is of type A_n. For every alcove $A \subseteq X_{\mathbb{R}}$, there is a wall H of A such that $A \uparrow s_H(A)$. If there is a unique wall H of A with $A \uparrow s_H(A)$ then $A = A_{\mathrm{fund}} + \gamma$ for some $\gamma \in X$.*

Proof. Suppose that there are (at least) n walls $H_1,\dots,H_n$ of A such that $s_{H_i}(A)\uparrow A$ for $i=1,\dots,n$ and let $w\in W_{\mathrm{aff}}$ such that $w(A)=A_{\mathrm{fund}}$. Then $w(H_1),\dots,w(H_n)$ are walls of A_{fund} and there is an extremal point δ of A_{fund} such that $\bigcap_i w(H_i)=\{\delta\}$. By Example 1.34, we have $\delta\in X$ (recall that Φ is of type A_n), so $\gamma\coloneqq w^{-1}(\delta)\in X$ and $\{\gamma\}=\bigcap_i H_i$. Thus $A-\gamma$ is an alcove whose closure contains 0, and the hyperplanes $H_i-\gamma$, for $i=1,\dots,n$, are precisely the walls of $A-\gamma$ containing 0. As A_{fund} is the unique dominant alcove whose closure contains 0, we have $A-\gamma=w'(A_{\mathrm{fund}})$ for some $w'\in W_{\mathrm{fin}}$.

If $w'=e$ then $A=A_{\mathrm{fund}}+\gamma$, as required. Suppose for a contradiction that $w'\neq e$, and choose a simple root $\alpha\in\Pi$ such that $w'(\alpha)\in-\Phi^+$. As $H_{\alpha,0}$ is a wall of A_{fund} (see Remark I.2.7), the hyperplane $H\coloneqq H_{-w'(\alpha),0}=w'(H_{\alpha,0})$ is a wall of the alcove $A-\gamma=w'(A_{\mathrm{fund}})$, and as $0\in H$, we conclude that $H=H_i-\gamma$ for some $i\in\{1,\dots,n\}$. Furthermore, we have

$$s_H(A-\gamma)=s_{H_i-\gamma}(A-\gamma)=s_{H_i}(A)-\gamma\uparrow A-\gamma,$$

because $s_{H_i}(A)\uparrow A$. However, for $x\in A-\gamma=w'(A_{\mathrm{fund}})$, there exists $y\in A_{\mathrm{fund}}$ such that $x=w'(y)$ and we have

$$\big(x,-w'(\alpha)^\vee\big)=\big(w'(y),-w'(\alpha)^\vee\big)=-(y,\alpha^\vee)<0,$$

that is $A-\gamma\subseteq H^-$, a contradiction. Hence $w'=e$ and $A-\gamma=A_{\mathrm{fund}}$. The first claim follows because the hyperplane $H'\coloneqq H_{\alpha_{\mathrm{h}},1}+\gamma$ is a wall of $A=A_{\mathrm{fund}}+\gamma$ with $A\uparrow s_{H'}(A)$. □

2 Quasi-translation functors

In the following sections, we will be concerned with functors of the form

$$T_\lambda^{\mu,\delta}\coloneqq \mathrm{pr}_\mu(\nabla(\delta)\otimes-)\colon \mathrm{Rep}_\lambda(\mathbf{G})\longrightarrow\mathrm{Rep}_\mu(\mathbf{G}),$$

for $\lambda,\mu\in\overline{C}_{\mathrm{fund}}\cap X$ and $\delta\in X^+$ not necessarily W_{fin}-conjugate to $\mu-\lambda$. We call such functors *quasi-translation functors*, because of their similarity with the translation functors T_λ^μ from Section I.6. The purpose of this section is to discuss some properties of quasi-translation functors. We rely on the following result from Lemma II.7.5 in [Jan03], which we will use to describe the action of quasi-translation functors on the level of characters.

Lemma 2.1. *Let M be a $\mathbf{G}$-module in $\mathrm{Rep}_\lambda(\mathbf{G})$, for some $\lambda\in\overline{C}_{\mathrm{fund}}\cap X$, and write*

$$\mathrm{ch}(M)=\sum_{w\in W_{\mathrm{aff}}}a_w\cdot\chi(w\cdot\lambda),$$

with $a_w\in\mathbb{Z}$ for all $w\in W_{\mathrm{aff}}$ and $a_w=0$ for all but finitely many w. For any $\mathbf{G}$-module V and any weight $\mu\in\overline{C}_{\mathrm{fund}}\cap X$, we have

$$\mathrm{ch}\big(\mathrm{pr}_\mu(V\otimes M)\big)=\sum_{w\in W_{\mathrm{aff}}}a_w\cdot\sum_\nu \dim V_\nu\cdot\chi\big(w\cdot(\lambda+\nu)\big),$$

where we sum over all $\nu\in X$ with $\lambda+\nu\in W_{\mathrm{aff}}\cdot\mu$.

Now let us consider a quasi-translation functor

$$T_\lambda^{\mu,\delta}=\mathrm{pr}_\mu(\nabla(\delta)\otimes-)\colon\mathrm{Rep}_\lambda(G)\longrightarrow\mathrm{Rep}_\mu(G),$$

for $\lambda,\mu\in\overline{C}_{\mathrm{fund}}\cap X$ and $\delta\in X^+$. By the character formula in Lemma 2.1, we have

$$\mathrm{ch}\,T_\lambda^{\mu,\delta}\nabla(x\cdot\lambda)=\mathrm{ch}\,\mathrm{pr}_\mu\big(\nabla(\delta)\otimes\nabla(x\cdot\lambda)\big)=\sum_\nu\dim\nabla(\delta)_\nu\cdot\chi\big(x\cdot(\lambda+\nu)\big)$$

for all $x \in W_{\mathrm{aff}}$ with $x \cdot \lambda \in X^+$, where we sum over all $\nu \in X$ such that $\lambda + \nu \in W_{\mathrm{aff}} \cdot \mu$. The summand corresponding to $\nu \in X$ contributes a non-zero term to this sum only if ν is a weight of the costandard module $\nabla(\delta)$; hence, setting

$$\Lambda(\lambda,\mu,\delta) \coloneqq \big\{ \nu \in X \bigm| \lambda+\nu \in W_{\mathrm{aff}} \cdot \mu \text{ and } \nu \text{ is a weight if } \nabla(\delta) \big\},$$

we have

$$\mathrm{ch}\, T_\lambda^{\mu,\delta} \nabla(x \cdot \lambda) = \sum_{\nu \in \Lambda(\lambda,\mu,\delta)} \dim \nabla(\delta)_\nu \cdot \chi\big(x \cdot (\lambda+\nu)\big). \tag{2.1}$$

Now for $\gamma \in X$ and an ℓ-alcove $C \subseteq X_{\mathbb{R}}$, let us write $(\gamma)_C$ for the unique W_{aff}-conjugate of γ in the closure of C (cf. Definition 1.14), with respect to the ℓ-dilated dot action. Furthermore, let us define

$$\mathcal{C}(\lambda,\mu,\delta) \coloneqq \big\{ C \subseteq X_{\mathbb{R}} \bigm| C \text{ is an } \ell\text{-alcove and } (\mu)_C - \lambda \text{ is a weight of } \nabla(\delta) \big\}.$$

Lemma 2.2. *We have*

$$\Lambda(\lambda,\mu,\delta) = \big\{ (\mu)_C - \lambda \bigm| C \in \mathcal{C}(\lambda,\mu,\delta) \big\}$$

and

$$\mathcal{C}(\lambda,\mu,\delta) = \big\{ C \subseteq X_{\mathbb{R}} \bigm| C \textit{ is an } \ell\textit{-alcove with } (\mu)_C - \lambda \in \Lambda(\lambda,\mu,\delta) \big\}.$$

In particular, $\Lambda(\lambda,\mu,\delta)$ is non-empty if and only if $\mathcal{C}(\lambda,\mu,\delta)$ is non-empty.

Proof. For $\nu \in \Lambda(\lambda,\mu,\delta)$ and $C \subseteq X_{\mathbb{R}}$ an ℓ-alcove with $\lambda+\nu \in \overline{C}$, we have $(\mu)_C = \lambda+\nu$ and it follows that $C \in \mathcal{C}(\lambda,\mu,\delta)$. Conversely, for $C \in \mathcal{C}(\lambda,\mu,\delta)$ and $\nu \coloneqq (\mu)_C - \lambda$, we have $\nu+\lambda \in W_{\mathrm{aff}} \cdot \mu$ and ν is a weight of $\nabla(\delta)$, so $\nu \in \Lambda(\lambda,\mu,\delta)$. □

Note that $\Lambda(\lambda,\mu,\delta)$ is empty unless δ lies in the same $\mathbb{Z}\Phi$-coset as $x \cdot \mu - \lambda$ for some (and hence all) $x \in W_{\mathrm{aff}}$. Using the well-known fact that the set of weights of $\nabla(\delta)$ equals $\mathrm{conv}\big(W_{\mathrm{fin}}(\delta)\big) \cap (\delta + \mathbb{Z}\Phi)$, it follows that

$$\mathcal{C}(\lambda,\mu,\delta) = \big\{ C \subseteq X_{\mathbb{R}} \bigm| C \text{ is an } \ell\text{-alcove and } (\mu)_C - \lambda \in \mathrm{conv}\big(W_{\mathrm{fin}}(\delta)\big) \big\},$$

whenever $\mathcal{C}(\lambda,\mu,\delta)$ is non-empty.

Lemma 2.3. *If $\mathcal{C}(\lambda,\mu,\delta)$ is non-empty then $\mathcal{C}(\lambda,\mu,\delta)$ is centered at C_{fund} and $C_{\mathrm{fund}} \in \mathcal{C}(\lambda,\mu,\delta)$.*

Proof. Suppose that $\mathcal{C}(\lambda,\mu,\delta)$ is non-empty, so that

$$\mathcal{C}(\lambda,\mu,\delta) = \big\{ C \subseteq X_{\mathbb{R}} \bigm| C \text{ is an } \ell\text{-alcove and } (\mu)_C - \lambda \in \mathrm{conv}\big(W_{\mathrm{fin}}(\delta)\big) \big\}$$

by the above discussion. As explained in Section I.6, we have a correspondence between alcoves and ℓ-alcoves in $X_{\mathbb{R}}$, which sends an alcove $A \subseteq X_{\mathbb{R}}$ to the ℓ-alcove $\ell \cdot A - \rho$. Let us define

$$x \coloneqq (\lambda+\rho)/\ell, \qquad y \coloneqq (\mu+\rho)/\ell \qquad \text{and} \qquad z = \delta/\ell$$

and consider the set of alcoves

$$\mathcal{A}(x,y,z) = \big\{ A \subseteq X_{\mathbb{R}} \bigm| A \text{ is an alcove and } (y)_A - x \in \mathrm{conv}\big(W_{\mathrm{fin}}(z)\big) \big\},$$

which is either empty or centered at A_{fund} by Lemma 1.15, because $\lambda \in \overline{C}_{\mathrm{fund}}$ and thus

$$x = (\lambda+\rho)/\ell \in (\overline{C}_{\mathrm{fund}} + \rho)/\ell = \overline{A}_{\mathrm{fund}}.$$

Now it is straightforward to see that

$$\begin{aligned}\mathcal{C}(\lambda,\mu,\delta) &= \big\{ C \subseteq X_{\mathbb{R}} \bigm| C \text{ is an } \ell\text{-alcove and } (\mu)_C - \lambda \in \operatorname{conv}\big(W_{\mathrm{fin}}(\delta)\big) \big\} \\ &= \big\{ \ell \cdot A - \rho \bigm| A \subseteq X_{\mathbb{R}} \text{ is an alcove and } (y)_A - x \in \operatorname{conv}\big(W_{\mathrm{fin}}(z)\big) \big\} \\ &= \big\{ \ell \cdot A - \rho \bigm| A \in \mathcal{A}(x,y,z) \big\}\end{aligned}$$

is centered at $\ell \cdot A_{\mathrm{fund}} - \rho = C_{\mathrm{fund}}$, as claimed. The second claim follows from Lemma 1.11. $\square$

Corollary 2.4. *If $T_\lambda^{\mu,\delta}$ is non-zero then $\mu - \lambda$ is a weight of $\nabla(\delta)$.*

Proof. If $T_\lambda^{\mu,\delta}$ is non-zero then $\Lambda(\lambda,\mu,\delta)$ is non-empty by equation (2.1), so $\mathcal{C}(\lambda,\mu,\delta)$ is non-empty by Lemma 2.2 and $C_{\mathrm{fund}} \in \mathcal{C}(\lambda,\mu,\delta)$ by Lemma 2.3. By the definition of $\mathcal{C}(\lambda,\mu,\delta)$, this means that $\mu - \lambda = (\mu)_{C_{\mathrm{fund}}} - \lambda$ is a weight of $\nabla(\delta)$, as claimed. $\square$

An immediate application of the preceding corollary is the following result, which will be very useful later on.

Proposition 2.5. *Let M be a $\mathbf{G}$-module in $\mathrm{Rep}_\lambda(\mathbf{G})$ and let V be a minuscule $\mathbf{G}$-module. Then*

$$V \otimes M \cong \bigoplus_{\nu} T_\lambda^\nu M,$$

where we sum over all $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ such that $\nu - \lambda$ is a weight of V.

Proof. Recall that a $\mathbf{G}$-module is called minuscule if all of its weights belong to the same W_{fin}-orbit and that the minuscule $\mathbf{G}$-module V is of the form $V \cong L(\varpi) = \nabla(\varpi)$ for a minuscule weight $\varpi \in X^+$, that is, a dominant weight ϖ with $(\varpi, \alpha_{\mathrm{h}}^\vee) = 1$. By the linkage principle, we have

$$V \otimes M = \bigoplus_{\nu \in \overline{C}_{\mathrm{fund}} \cap X} \mathrm{pr}_\nu\big(V \otimes M\big) \cong \bigoplus_{\nu \in \overline{C}_{\mathrm{fund}} \cap X} T_\lambda^{\nu,\varpi} M,$$

where $T_\lambda^{\nu,\varpi} M = 0$ if $\nu - \lambda$ is not a weight of V, by Corollary 2.4. If $\nu - \lambda$ is a weight of V then $\nu - \lambda$ is W_{fin}-conjugate to the highest weight ϖ of V because V is minuscule, so

$$T_\lambda^{\nu,\varpi} M \cong \mathrm{pr}_\nu\big(\nabla(\varpi) \otimes M\big) \cong T_\lambda^\nu M$$

and the claim follows. $\square$

3 The setup

From now on until the end of this chapter, we suppose that $\mathbf{G}$ is of type A_n and that $\ell \geq n+1$. We fix a numbering of the simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, in accordance with the Dynkin diagram in Figure I.1.1 and denote by $\varpi_i = \varpi_{\alpha_i}$ and $s_i = s_{\alpha_i}$ the fundamental dominant weight and the simple reflection corresponding to α_i, for $i = 1, \dots, n$. Furthermore, we write $s_0 = s_{\alpha_{\mathrm{h}},1}$ and adopt the convention that $\varpi_0 = 0$ and $\varpi_{n+1} = 0$. The positive roots in Φ are given by

$$\Phi^+ = \{\beta_{i,j} \mid 1 \leq i \leq j \leq n\},$$

where

$$\beta_{i,j} = \alpha_i + \cdots + \alpha_j = \varpi_{i-1} + \varpi_i + \varpi_j - \varpi_{j+1},$$

and we have $\alpha_{\mathrm{h}} = \beta_{1,n} = \varpi_1 + \varpi_n$.

Recall from the introduction to the chapter (see page 95) that we want to study the generic direct summand $G(s_0\omega, x)$ of $L(s_0\omega \cdot 0) \otimes L(x \cdot 0)$, for $x \in W_{\mathrm{aff}}^{+}$ and a certain fixed element $\omega \in \Omega$ (to be specified below), by realizing $L(s_0\omega \cdot 0)$ as a subquotient of the tilting module $T(s_0\omega \cdot 0)$ and giving a detailed description of the functor

$$\mathrm{pr}_{\omega \cdot 0}\big(T(s_0\omega \cdot 0) \otimes -\big) \colon \mathrm{Rep}_0(\mathbf{G}) \longrightarrow \mathrm{Rep}_{\omega \cdot 0}(\mathbf{G}).$$

The starting point for our strategy is the following elementary lemma (which does not yet require the hypothesis that $\mathbf{G}$ is of type A_n):

Lemma 3.1. *For $\lambda \in C_{\mathrm{fund}} \cap X$, the minimal tilting complex of $L(s_0 \cdot \lambda)$ is of the form*

$$C_{\min}\big(L(s_0 \cdot \lambda)\big) = \big(\quad 0 \to T(\lambda) \to T(s_0 \cdot \lambda) \to T(\lambda) \to 0 \quad \big),$$

with the tilting module $T(s_0 \cdot \lambda)$ in homological degree 0.

Proof. This follows from Proposition II.2.6 because $\ell(s_0) = 1$ and because e and s_0 are the only elements of W_{aff}^{+} of length at most 1 (by Lemma I.2.12). □

Remark 3.2. Let us explain another way of computing the minimal complex from Lemma 3.1. For a weight $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s_0\}$, we have $L(\mu) \cong T(\mu)$ by the linkage principle, and it follows that $T_\mu^\lambda L(\mu)$ is a tilting module. Now $T_\mu^\lambda L(\mu)$ is indecomposable with simple head and socle

$$\mathrm{soc}_{\mathbf{G}} T_\mu^\lambda L(\mu) \cong \mathrm{head}_{\mathbf{G}} T_\mu^\lambda L(\mu) \cong L(\lambda)$$

and with

$$\mathrm{rad}_{\mathbf{G}} T_\mu^\lambda L(\mu) / \mathrm{soc}_{\mathbf{G}} T_\mu^\lambda L(\mu) \cong L(s_0 \cdot \lambda)$$

by Proposition I.6.10. By weight considerations, we conclude that $T_\mu^\lambda L(\mu) \cong T(s_0 \cdot \lambda)$. Furthermore, the monomorphism $T(\lambda) \cong L(\lambda) \to T(s_0 \cdot \lambda)$ and the epimorphism $T(s_0 \cdot \lambda) \to L(\lambda) \cong T(\lambda)$ give rise to a complex

$$C = \big(\quad 0 \to T(\lambda) \to T(s_0 \cdot \lambda) \to T(\lambda) \to 0 \quad \big),$$

where $T(s_0 \cdot \lambda)$ is in homological degree zero, with $H^0(C) \cong L(s_0 \cdot \lambda)$ and $H^i(C) = 0$ for $i \neq 0$. It is straightforward to see that C is minimal, and we conclude that $C = C_{\min}\big(L(s_0 \cdot \lambda)\big)$.

As mentioned before, we will apply Lemma 3.1 in the case where $\lambda = \omega \cdot 0$ for an element $\omega \in \Omega$, which we define in the following lemma:

Lemma 3.3. *We have $\omega \coloneqq t_{\varpi_1} s_1 s_2 \cdots s_n \in \Omega$.*

Proof. Note that we have $s_n s_{n-1} \cdots s_1(\alpha_1) = -\alpha_{\mathrm{h}}$ and $s_n s_{n-1} \cdots s_1(\alpha_i) = \alpha_{i-1}$ for $1 < i \leq n$. For $1 \leq i, j \leq n$, we further have

$$\big(\omega(\varpi_i), \alpha_j^\vee\big) = \big(s_1 s_2 \cdots s_n(\varpi_i) + \varpi_1, \alpha_j^\vee\big) = \big(\varpi_i, s_n s_{n-1} \cdots s_1(\alpha_j^\vee)\big) + \delta_{1,j},$$

and for $j = 1$, it follows that $\big(\omega(\varpi_i), \alpha_1^\vee\big) = -\big(\varpi_i, \alpha_{\mathrm{h}}^\vee\big) + 1 = 0$. For $j > 1$, we obtain

$$\big(\omega(\varpi_i), \alpha_j^\vee\big) = (\varpi_i, \alpha_{j-1}^\vee) = \delta_{i,j-1},$$

and we conclude that $\omega(\varpi_n) = 0$ and $\omega(\varpi_i) = \varpi_{i+1}$ for $1 \leq i < n$. Furthermore, we have $\omega(0) = \varpi_1$, so ω permutes the set $\{0, \varpi_1, \ldots, \varpi_n\}$ of extremal points of A_{fund} (see Example 1.34), and it follows that $\omega(A_{\mathrm{fund}}) = A_{\mathrm{fund}}$ and $\omega \in \mathrm{Stab}_{W_{\mathrm{ext}}}(A_{\mathrm{fund}}) = \Omega$, as claimed. □

For the remainder of this chapter, we fix

$$\omega := t_{\varpi_1} s_1 s_2 \cdots s_n \in \Omega \qquad \text{and} \qquad \lambda := \omega \cdot 0 \in C_{\mathrm{fund}} \cap X.$$

Lemma 3.4. *We have $\lambda = \omega \cdot 0 = (\ell - n - 1) \cdot \varpi_1$.*

Proof. It is straightforward to see, by induction on i, that

$$s_{n-i} s_{n-i+1} \cdots s_n \cdot 0 = -\alpha_n - 2\alpha_{n-1} - \cdots - (i+1) \cdot \alpha_{n-i} = (i+1) \cdot \varpi_{n-i-1} - (i+2) \cdot \varpi_{n-i}$$

for $i = 0, \ldots, n-1$, with the convention that $\varpi_0 = 0$. In particular, we have $s_1 s_2 \cdots s_n \cdot 0 = -(n+1) \cdot \varpi_1$ and $\omega \cdot 0 = t_{\varpi_1} s_1 s_2 \cdots s_n \cdot 0 = (\ell - n - 1) \cdot \varpi_1$, as claimed. □

One advantage of working with $T(s_0 \cdot \lambda) = T(s_0 \omega \cdot 0)$, rather than $T(s_0 \cdot 0)$, is that we have a tensor product decomposition of the former tilting module.

Proposition 3.5. *We have $s_0 \cdot \lambda = (\ell - n) \cdot \varpi_1 + \varpi_n$ and*

$$T(s_0 \cdot \lambda) \cong \nabla(\varpi_n) \otimes \nabla\big((\ell - n) \cdot \varpi_1\big).$$

Proof. By Lemma 3.4, we have $\lambda = (\ell - n - 1) \cdot \varpi_1$, and it is straightforward to compute that

$$s_0 \cdot \lambda = \lambda - (\lambda + \rho, \alpha_{\mathrm{h}}^\vee) \cdot \alpha_{\mathrm{h}} + \ell \alpha_{\mathrm{h}} = \lambda + \alpha_{\mathrm{h}} = (\ell - n) \cdot \varpi_1 + \varpi_n.$$

Now the $\mathbf{G}$-module $\nabla(\varpi_n)$ is minuscule with set of weights $\{\varpi_n, -\varpi_1\} \cup \{\varpi_{i-1} - \varpi_i \mid 1 < i \leq n\}$. Furthermore, we have $\mu := (\ell - n) \cdot \varpi_1 \in \overline{C}_{\mathrm{fund}}$ and using Proposition 2.5, it follows that

$$\nabla(\varpi_n) \otimes \nabla(\mu) \cong \bigoplus_{\nu} T_\mu^\nu \nabla(\mu),$$

where we sum over all $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ such that $\nu - \mu$ is a weight of $\nabla(\varpi_n)$.

For $1 < i \leq n$, the weight $\delta_i := \mu + \varpi_{i-1} - \varpi_i$ is non-dominant and we have and $\delta_i \in \overline{F}_\mu \subseteq \overline{C}_{\mathrm{fund}}$, so Proposition I.6.8 implies that $T_\mu^{\delta_i} \nabla(\mu) = 0$. As $\mu + \varpi_n \notin \overline{C}_{\mathrm{fund}}$ and $\mu - \varpi_1 = \lambda \in C_{\mathrm{fund}}$, we conclude that

$$\nabla(\varpi_n) \otimes \nabla(\mu) \cong T_\mu^\lambda \nabla(\mu).$$

Now $\nabla(\mu) \cong T(\mu)$ by the linkage principle (because $\mu \in \overline{C}_{\mathrm{fund}}$) and it follows that $T_\mu^\lambda \nabla(\mu) \cong T_\mu^\lambda T(\mu)$ is a tilting module. Furthermore, as s_0 is the only simple reflection that stabilizes μ, Lemma 1.21 implies that $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s_0\}$. By Proposition I.6.9, the $\mathbf{G}$-module $T_\mu^\lambda \nabla(\mu)$ is indecomposable and has a good filtration with subquotients $\nabla(\lambda)$ and $\nabla(s_0 \cdot \lambda)$. We conclude that

$$\nabla(\varpi_n) \otimes \nabla(\mu) \cong T_\mu^\lambda \nabla(\mu) \cong T(s_0 \cdot \lambda),$$

as claimed. □

Now let us consider the functor

$$\Psi := \mathrm{pr}_\lambda\big(T(s_0 \cdot \lambda) \otimes -\big) \colon \mathrm{Rep}_0(\mathbf{G}) \longrightarrow \mathrm{Rep}_\lambda(\mathbf{G}).$$

By Lemma 3.1, we can choose a monomorphism

$$e \colon T(\lambda) \longrightarrow T(s_0 \cdot \lambda)$$

and an epimorphism

$$p\colon T(s_0 \cdot 0) \longrightarrow T(\lambda)$$

such that $\mathrm{im}(e) \subseteq \ker(p)$ and $\ker(p)/\mathrm{im}(e) \cong L(s_0 \cdot \lambda)$, and as $\lambda \in C_{\mathrm{fund}}$, these homomorphisms give rise to natural transformations

$$\vartheta = \mathrm{pr}_\lambda(e \otimes -)\colon \quad T_0^\lambda = \mathrm{pr}_\lambda\big(T(\lambda) \otimes -\big) \implies \mathrm{pr}_\lambda\big(T(s_0 \cdot \lambda) \otimes -\big) = \Psi$$

and

$$\pi = \mathrm{pr}_\lambda(p \otimes -)\colon \quad \Psi = \mathrm{pr}_\lambda\big(T(s_0 \cdot \lambda) \otimes -\big) \implies \mathrm{pr}_\lambda\big(T(\lambda) \otimes -\big) = T_0^\lambda.$$

For every $\mathbf{G}$-module M in $\mathrm{Rep}_0(\mathbf{G})$, the components of ϑ and π at M satisfy $\mathrm{im}(\vartheta_M) \subseteq \ker(\pi_M)$ and

$$\ker(\pi_M)/\mathrm{im}(\vartheta_M) \cong \mathrm{pr}_\lambda\Big(\big(\ker(p)/\mathrm{im}(e)\big) \otimes M\Big) \cong \mathrm{pr}_\lambda\big(L(s_0 \cdot \lambda) \otimes M\big),$$

by the construction of ϑ and π.

Next, let us set $\mu \coloneqq (\ell - n) \cdot \varpi_1$ and fix an isomorphism

$$f\colon T(s_0 \cdot \lambda) \longrightarrow \nabla(\varpi_n) \otimes \nabla(\mu),$$

as given by Proposition 3.5. Then f and the associativity of tensor products give rise to a natural isomorphism

$$\Psi = \mathrm{pr}_\lambda\big(T(s_0 \cdot 0) \otimes -\big) \implies \mathrm{pr}_\lambda\big(\nabla(\varpi_n) \otimes -\big) \circ \big(\nabla(\mu) \otimes -\big),$$

where we view $\big(\nabla(\mu) \otimes -\big)$ as a functor from $\mathrm{Rep}_0(\mathbf{G})$ to $\mathrm{Rep}(\mathbf{G})$ and $\mathrm{pr}_\lambda\big(\nabla(\varpi_n) \otimes -\big)$ as a functor from $\mathrm{Rep}(\mathbf{G})$ to $\mathrm{Rep}_\lambda(\mathbf{G})$. By the linkage principle, we have an isomorphism of functors

$$\big(\nabla(\mu) \otimes -\big) \cong \bigoplus_{\nu \in \overline{C}_{\mathrm{fund}} \cap X} \mathrm{pr}_\nu\big(\nabla(\mu) \otimes -\big),$$

and by composing with the functor $\mathrm{pr}_\lambda\big(\nabla(\varpi_n) \otimes -\big)$, we obtain

$$\begin{aligned} \Psi &\cong \mathrm{pr}_\lambda\big(\nabla(\varpi_n) \otimes -\big) \circ \big(\nabla(\mu) \otimes -\big) \\ &\cong \bigoplus_{\nu \in \overline{C}_{\mathrm{fund}} \cap X} \mathrm{pr}_\lambda\big(\nabla(\varpi_n) \otimes -\big) \circ \mathrm{pr}_\nu\big(\nabla(\mu) \otimes -\big) \\ &= \bigoplus_{\nu \in \overline{C}_{\mathrm{fund}} \cap X} T_\nu^{\lambda,\varpi_n} \circ T_0^{\nu,\mu}. \end{aligned}$$

Recall from Corollary 2.4 that T_ν^{λ,ϖ_n} is zero unless $\lambda - \nu$ is a weight of $\nabla(\varpi_n)$ and from the proof of Proposition 3.5 that $\nabla(\varpi_n)$ is minuscule with set of weights $\{\varpi_n, -\varpi_1\} \cup \{\varpi_{i-1} - \varpi_i \mid 1 < i \leq n\}$. Let us define

$$\begin{aligned} \mu_0 &\coloneqq \lambda + \varpi_1 = (\ell - n) \cdot \varpi_1 = \mu, \\ \mu_1 &\coloneqq \lambda - (\varpi_1 - \varpi_2) = (\ell - n - 2) \cdot \varpi_1 + \varpi_2 = \mu - \alpha_1, \\ \mu_i &\coloneqq \lambda - (\varpi_i - \varpi_{i+1}) = (\ell - n - 1) \cdot \varpi_1 - \varpi_i + \varpi_{i+1} = \mu - \beta_{1,i} \qquad \text{for } 2 \leq i \leq n, \end{aligned}$$

and note that $\mu_i \in \overline{C}_{\mathrm{fund}} \cap X$ for $0 \leq i \leq n$. By construction, $\{\mu_0, \mu_1, \ldots, \mu_n\}$ is precisely the set of weights $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ such that $\lambda - \nu$ is a weight of $\nabla(\varpi_n)$, and it follows that

$$\Psi \cong \bigoplus_{i=0}^{n} T_{\mu_i}^{\lambda,\varpi_n} \circ T_0^{\mu_i,\mu} = \bigoplus_{i=0}^{n} T_{\mu_i}^{\lambda} \circ T_0^{\mu_i,\mu},$$

where $T_{\mu_i}^{\lambda,\varpi_n} = T_{\mu_i}^{\lambda}$ because $\lambda - \mu_i$ is W_{fin}-conjugate to ϖ_n for $i = 0, \ldots, n$. We write

$$\mathrm{pr}_i \colon \Psi \Longrightarrow T_{\mu_i}^{\lambda} \circ T_0^{\mu_i,\mu} =: \Theta_i$$

for the natural transformation that projects onto a direct summand in the above direct sum decomposition of Ψ and consider the natural transformations

$$\vartheta_i := \mathrm{pr}_i \circ \vartheta \colon \; T_0^{\lambda} \Longrightarrow \Theta_i,$$

for $i = 0, \ldots, n$. In the following sections, we will show that

(1) the functor $\Theta_i = T_{\mu_i}^{\lambda} \circ T_0^{\mu_i,\mu}$ behaves essentially like the wall crossing functor $T_{\mu_i}^{\lambda} \circ T_0^{\mu_i}$ corresponding to the simple reflection s_i (with some adjustments in the case $i = 1$);

(2) the component $(\vartheta_i)_{L(x\cdot 0)}$ of ϑ_i at a simple $\mathbf{G}$-module $L(x \cdot 0)$, for $x \in W_{\mathrm{aff}}^+$, is non-zero if and only if $x(A_{\mathrm{fund}}) \uparrow x s_i(A_{\mathrm{fund}})$.

This will enable us to give a description of the generic direct summands $G(s_0\omega, x)$ for all $x \in W_{\mathrm{aff}}^+$.

4 Properties of Θ_i

We keep the notation and assumptions from Section 3. Recall that we consider the functors

$$\Theta_i = T_{\mu_i}^{\lambda} \circ T_0^{\mu_i,\mu}$$

for $i = 0, \ldots, n$, where $\lambda = \omega \cdot 0 = (\ell - n - 1) \cdot \varpi_1$, $\mu = (\ell - n) \cdot \varpi_1$ and

$$\begin{aligned}
\mu_0 &= \lambda + \varpi_1 = (\ell - n) \cdot \varpi_1 = \mu, \\
\mu_1 &= \lambda - (\varpi_1 - \varpi_2) = (\ell - n - 2) \cdot \varpi_1 + \varpi_2, \\
\mu_i &= \lambda - (\varpi_i - \varpi_{i+1}) = (\ell - n - 1) \cdot \varpi_1 - \varpi_i + \varpi_{i+1} \qquad \text{for } 2 \leq i \leq n.
\end{aligned}$$

Note that for $i \neq 1$, we have $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_i) = \{e, s_i\}$ because s_i is the unique simple reflection that stabilizes μ_i (see Lemma 1.21). For $i = 1$, we still have $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_1) = \{e, s_1\}$ in case $\ell = n + 1$, but for $\ell > n + 1$, the weight μ_1 is ℓ-regular (so that $T_{\mu_1}^{\lambda}$ is an equivalence by Proposition I.6.7). In either case, the translation functors $T_{\mu_i}^{\lambda}$ are well-understood by the results from Section I.6 (see in particular Propositions I.6.9 and I.6.10), so we will focus on understanding the quasi-translation functors $T_0^{\mu_i,\mu}$. For $i = 0$, we have $\mu_0 = \mu$ and therefore $T_0^{\mu_0,\mu} = T_0^{\mu_0} = T_0^{\mu}$. Our first aim will be to show that for $i \geq 2$, the quasi-translation functor $T_0^{\mu_i,\mu}$ acts like the translation functor $T_0^{\mu_i}$ on the level of characters. (We do not know if there exists a natural isomorphism between these functors.)

Recall from equation (2.1) that we have

$$\operatorname{ch} T_0^{\mu_i,\mu} \nabla(x \cdot 0) = \sum_{\nu \in \Lambda(0,\mu_i,\mu)} \dim \nabla(\mu)_\nu \cdot \chi(x \cdot \nu) \tag{4.1}$$

for all $x \in W_{\mathrm{aff}}^+$, where

$$\Lambda(0, \mu_i, \mu) = \big\{ \nu \in X \bigm| \nu \in W_{\mathrm{aff}} \cdot \mu_i \text{ and } \nu \text{ is a weight of } \nabla(\mu) \big\}.$$

Furthermore, we have $\Lambda(0, \mu_i, \mu) = \big\{ (\mu_i)_C \bigm| C \in \mathcal{C}(0, \mu_i, \mu) \big\}$, where

$$\mathcal{C}(0, \mu_i, \mu) = \big\{ C \subseteq X_{\mathbb{R}} \bigm| C \text{ is an } \ell\text{-alcove and } (\mu_i)_C \text{ is a weight of } \nabla(\mu) \big\},$$

and the set $\mathcal{C}(0, \mu_i, \mu)$ is centered at C_{fund} (or empty); see Lemmas 2.2 and 2.3. In order to describe the action of the functors $T_0^{\mu_i,\mu}$ on the level of characters, we first compute the sets of ℓ-alcoves $\mathcal{C}(0, \mu_i, \mu)$. This will be achieved in Corollary 4.4 below, but we start with some weight considerations.

Lemma 4.1. *For $i = 0, \ldots, n$, we have* $\dim \nabla(\mu)_{\mu_i} = 1$.

Proof. For $i = 0$, the claim is obvious because $\mu_0 = \mu$, and for $i = 1$, it follows by truncation to to the Levi subgroup corresponding to the subset $\{\alpha_1\} \subseteq \Pi$, since $\mu_1 = \mu - \alpha_1$. For $i > 1$, it is straightforward to see that $\mu_{i-1} = s_i(\mu_i)$ and therefore

$$\dim \nabla(\mu)_{\mu_n} = \dim \nabla(\mu)_{\mu_{n-1}} = \cdots = \dim \nabla(\mu_1)_{\mu_1} = 1,$$

as claimed. □

Lemma 4.2. *Let $i, j \in \{1, \ldots, n\}$. Then $\mu - \beta_{1,i} + \alpha_j$ is a weight of $\nabla(\mu)$ if and only if $i = j$.*

Proof. If $j > i$ then $\mu - \beta_{1,i} + \alpha_j \not\leq \mu$ and if $j < i$ then

$$s_{\beta_{j+1,i}}(\mu - \beta_{1,i} + \alpha_j) = \mu - \beta_{1,j-1} + \beta_{j+1,i} \not\leq \mu,$$

with the convention that $\beta_{1,0} = 0$. In both cases, it follows that $\mu - \beta_{1,i} + \alpha_j$ is not a weight of $\nabla(\mu)$. If $i = j$ then we have

$$\mu - \beta_{1,i} + \alpha_i = \mu - \beta_{1,i-1} = \mu_{i-1}$$

for $i > 1$ and $\mu - \beta_{1,1} + \alpha_1 = \mu$, so the claim follows from Lemma 4.1. □

Proposition 4.3. *For $i = 0, \ldots, n$ and $s \in S$, we have $s \cdot C_{\mathrm{fund}} \in \mathcal{C}(0, \mu_i, \mu)$ if and only if $s = s_i$.*

Proof. First note that we have $(\mu_i)_{s \cdot C_{\mathrm{fund}}} = s \cdot \mu_i$ for all $s \in S$ because $\mu_i \in \overline{C}_{\mathrm{fund}}$. Therefore, our claim is equivalent to proving that $s \cdot \mu_i$ is a weight of $\nabla(\mu)$ if and only if $s = s_i$. For $i \neq 1$, we have $s_i \cdot \mu_i = \mu_i$ (as observed before) and $\dim \nabla(\mu)_{\mu_i} = 1$ by Lemma 4.1. For $i = 1$, we have

$$s_1 \cdot \mu_1 = s_1(\mu_1) - \alpha_1 = s_1(\mu_1 + \alpha_1) = s_1(\mu)$$

and $\dim \nabla(\mu)_{s_1(\mu)} = \dim \nabla(\mu)_\mu = 1$. It remains to show that $s_j \cdot \mu_i$ is not a weight of $\nabla(\mu)$ for $j \neq i$.

First suppose that $j = 0$. We have

$$s_0 \cdot \mu_i = \mu_i + \alpha_{\mathrm{h}} = \mu + \beta_{i+1,n} > \mu$$

for $1 \leq i < n$ and

$$s_0 \cdot \mu_n = \mu_n + 2\alpha_{\mathrm{h}} = \mu + \alpha_{\mathrm{h}} > \mu,$$

and it follows that $s_0 \cdot \mu_i$ is not a weight of $\nabla(\mu)$ for $i \neq 0$.

Now suppose that $j \geq 1$ and note that $s_j \cdot x = s_j(x) - \alpha_j = s_j(x + \alpha_j)$ for all $x \in X_{\mathbb{R}}$. For $i = 0$, it follows that $s_j \cdot \mu_0 = s_j(\mu + \alpha_j)$ is not a weight of $\nabla(\mu)$ because $\mu + \alpha_j > \mu$. For $1 \leq i \leq n$, we find that

$$s_j \cdot \mu_i = s_j(\mu_i + \alpha_j) = s_j(\mu - \beta_{1,i} + \alpha_j)$$

is a weight of $\nabla(\mu)$ if and only if $i = j$ by Lemma 4.2. □

Corollary 4.4. *For $i = 0, \ldots, n$, we have $\mathcal{C}(0, \mu_i, \mu) = \{C_{\mathrm{fund}}, s_i \cdot C_{\mathrm{fund}}\}$.*

Proof. Recall from Proposition 4.3 that s_i is the unique simple reflection with $s_i \cdot C_{\mathrm{fund}} \in \mathcal{C}(0, \mu_i, \mu)$ and from Lemma 2.3 that the set $\mathcal{C}(0, \mu_i, \mu)$ is centered at C_{fund}. As the walls of C_{fund} are precisely the reflection hyperplanes corresponding to the simple reflections (see Remark I.2.7), the claim follows from Lemma 1.13. □

Now we are ready to prove that the quasi-translation functors $T_0^{\mu_i,\mu}$ act like translation functors on the level of chatacters, first for $i \neq 1$ and then for $i = 1$.

Proposition 4.5. *Let M be a $\mathbf{G}$-module in $\mathrm{Rep}_0(\mathbf{G})$ and let $i \in \{0, \ldots, n\}$. Furthermore, suppose that either $i \neq 1$ or $\ell = n+1$. Then we have*

$$\mathrm{ch}\,\big(T_0^{\mu_i,\mu}M\big) = \mathrm{ch}\,\big(T_0^{\mu_i}M\big).$$

Proof. If $\ell = n+1$ then $\mu_i = -\varpi_i + \varpi_{i+1}$ is W_{fin}-conjugate to $\mu = \varpi_1$ for $i = 0, \ldots, n$, so $T_0^{\mu_i,\mu} = T_0^{\mu_i}$ and the claim is immediate. Now suppose that $i \neq 1$ and $\ell \geq n+1$. As the characters of the induced modules form a basis of $\mathbb{Z}[X]^{W_{\mathrm{fin}}}$, it suffices to prove the claim in the case where $M = \nabla(x \cdot 0)$ for some $x \in W_{\mathrm{aff}}^+$. As $\mathrm{Stab}_{W_{\mathrm{aff}}}(0) = \{e\}$, Proposition I.6.5 yields

$$\mathrm{ch}\,\big(T_0^{\mu_i}\nabla(x \cdot 0)\big) = \chi(x \cdot \mu_i).$$

Furthermore, we have $\mathcal{C}(0, \mu_i, \mu) = \{C_{\mathrm{fund}}, s_i \cdot C_{\mathrm{fund}}\}$ by Corollary 4.4 and

$$\Lambda(0, \mu_i, \mu) = \big\{(\mu_i)_{C_{\mathrm{fund}}}, (\mu_i)_{s_i \cdot C_{\mathrm{fund}}}\big\} = \{\mu_i, s_i \cdot \mu_i\} = \{\mu_i\},$$

by Lemma 2.2, and using equation (4.1), we conclude that

$$\mathrm{ch}\,\big(T_0^{\mu_i,\mu}\nabla(x \cdot 0)\big) = \sum_{\nu \in \Lambda(0,\mu_i,\mu)} \dim \nabla(\mu)_\nu \cdot \chi\big(x \cdot \nu\big) = \dim \nabla(\mu)_{\mu_i} \cdot \chi(x \cdot \mu_i).$$

Now the claim follows because $\dim \nabla(\mu)_{\mu_i} = 1$ by Lemma 4.1. $\square$

Proposition 4.6. *Suppose that $\ell > n+1$ and let M be a $\mathbf{G}$-module in $\mathrm{Rep}_0(\mathbf{G})$. Then, for any weight $\delta \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta) = \{e, s_1\}$, we have*

$$\mathrm{ch}\,\big(T_0^{\mu_1,\mu}M\big) = \mathrm{ch}\,\big(T_\delta^{\mu_1}T_0^{\delta}M\big)$$

Proof. As in the proof of Proposition 4.5, it suffices to prove the claim in the case where $M = \nabla(x \cdot 0)$ for some $x \in W_{\mathrm{aff}}^+$. By Proposition I.6.5, we have

$$\mathrm{ch}\,\big(T_0^{\delta}\nabla(x \cdot 0)\big) = \chi(x \cdot \delta) \qquad \text{and} \qquad \mathrm{ch}\,\big(T_\delta^{\mu_1}T_0^{\delta}\nabla(x \cdot 0)\big) = \chi(x \cdot \mu_1) + \chi(xs_1 \cdot \mu_1)$$

because $\mathrm{Stab}_{W_{\mathrm{aff}}}(0) = \{e\} = \mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_1)$ and $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta) = \{e, s_1\}$. Furthermore, we have

$$\Lambda(0, \mu_1, \mu) = \big\{(\mu_1)_{C_{\mathrm{fund}}}, (\mu_1)_{s_1 \cdot C_{\mathrm{fund}}}\big\} = \{\mu_1, s_1 \cdot \mu_1\}$$

by Lemma 2.2 and Corollary 4.4, and equation (4.1) yields

$$\begin{aligned} \mathrm{ch}\,\big(T_0^{\mu_1,\mu}\nabla(x \cdot 0)\big) &= \sum_{\nu \in \Lambda(0,\mu_1,\mu)} \dim \nabla(\mu)_\nu \cdot \chi\big(x \cdot \nu\big) \\ &= \dim \nabla(\mu)_{\mu_1} \cdot \chi(x \cdot \mu_1) + \dim \nabla(\mu)_{s_1 \cdot \mu_1} \cdot \chi(xs_1 \cdot \mu_1). \end{aligned}$$

The claim follows because $\dim \nabla(\mu)_{\mu_1} = 1$ (by Lemma 4.1) and $\dim \nabla(\mu)_{s_1 \cdot \mu_1} = 1$ (since $s_1 \cdot \mu_1 = s_1(\mu)$, as computed in the proof of Proposition 4.3). $\square$

For $i \neq 1$, we can now explicitly determine $T_0^{\mu_i,\mu}\nabla(x \cdot 0)$ and $T_0^{\mu_i,\mu}L(x \cdot 0)$ for $x \in W_{\mathrm{aff}}^+$.

Corollary 4.7. *Let $x \in W_{\mathrm{aff}}^+$ and $i \in \{0, \ldots, n\}$. Furthermore, suppose that either $i \neq 1$ or $\ell = n+1$. Then we have*

$$T_0^{\mu_i,\mu} \nabla(x \cdot 0) \cong \begin{cases} \nabla(x \cdot \mu_i) & \text{if } x \cdot \mu_i \in X^+, \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_0^{\mu_i,\mu} L(x \cdot 0) \cong \begin{cases} L(x \cdot \mu_i) & \text{if } x \cdot C_{\mathrm{fund}} \uparrow_\ell xs_i \cdot C_{\mathrm{fund}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that $T_0^{\mu_i,\mu} \nabla(x \cdot 0) = \mathrm{pr}_{\mu_i}\big(\nabla(\mu) \otimes \nabla(x \cdot 0)\big)$ has a good filtration and that the multiplicity of an induced module as a subquotient in a good filtration is determined by the character of $T_0^{\mu_i,\mu} \nabla(x \cdot 0)$. Analogously, the multiplicity of a simple $\mathbf{G}$-module in a composition series of $T_0^{\mu_i,\mu} L(x \cdot 0)$ is determined by the character of $T_0^{\mu_i,\mu} L(x \cdot 0)$. As

$$\mathrm{ch}\big(T_0^{\mu_i,\mu} \nabla(x \cdot 0)\big) = \mathrm{ch}\big(T_0^{\mu_i} \nabla(x \cdot 0)\big) \qquad \text{and} \qquad \mathrm{ch}\big(T_0^{\mu_i,\mu} L(x \cdot 0)\big) = \mathrm{ch}\big(T_0^{\mu_i} L(x \cdot 0)\big)$$

by Proposition 4.5, the claim follows from Proposition I.6.8 and the observation that $x \cdot \mu_i$ belongs to the upper closure of $x \cdot C_{\mathrm{fund}}$ if and only if $x \cdot C_{\mathrm{fund}} \uparrow_\ell xs_i \cdot C_{\mathrm{fund}}$ (see Lemma 1.24). ☐

Let us conclude this Section with some observations about the functors $\Theta_i = T_{\mu_i}^{\lambda} \circ T_0^{\mu_i,\mu}$, which will be important later on.

Corollary 4.8. *For $x \in W_{\mathrm{aff}}^+$ and $i \in \{0, \ldots, n\}$ with $xs_i(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}})$, we have $\Theta_i L(x \cdot 0) = 0$.*

Proof. If $i \neq 1$ or $\ell = n+1$ then $T_0^{\mu_i,\mu} L(x \cdot 0) = 0$ by Corollary 4.7 and it follows that $\Theta_i L(x \cdot 0) = 0$. Now suppose that $i = 1$ and $\ell > n+1$, and let $\delta \in \overline{C}_{\mathrm{fund}} \cap X$ such that $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta) = \{e, s_1\}$. By Proposition 4.6, we have

$$\mathrm{ch}\big(T_0^{\mu_1,\mu} L(x \cdot 0)\big) = \mathrm{ch}\big(T_\delta^{\mu_1} T_0^{\delta} L(x \cdot 0)\big).$$

As $xs_1(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}})$, the weight $x \cdot \delta$ does not belong to the upper closure of the alcove $x \cdot C_{\mathrm{fund}}$ by Lemma 1.24, so Proposition I.6.8 implies that $T_0^{\delta} L(x \cdot 0) = 0$ and the claim follows. ☐

Corollary 4.9. *For $x \in W_{\mathrm{aff}}^+$ and $i \in \{0, \ldots, n\}$ such that $x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$, there is a non-split short exact sequence*

$$0 \longrightarrow \nabla(x \cdot \lambda) \longrightarrow \Theta_i \nabla(x \cdot 0) \longrightarrow \nabla(xs_i \cdot \lambda) \longrightarrow 0.$$

Proof. If $i \neq 1$ or $\ell = n+1$ then $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_i) = \{e, s_i\}$ and the assumption that $x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$ implies that $x \cdot \mu_i \in X^+$. Now Corollary 4.7 yields $T_0^{\mu_i,\mu} \nabla(x \cdot 0) \cong \nabla(x \cdot \mu_i)$, so $\Theta_i \nabla(x \cdot 0) \cong T_{\mu_i}^{\lambda} \nabla(x \cdot \mu_i)$ and the claim follows from Proposition I.6.9.

Now suppose that $i = 1$ and $\ell > n+1$, and note that $T_0^{\mu_1,\mu} \nabla(x \cdot 0) = \mathrm{pr}_{\mu_1}\big(\nabla(\mu) \otimes \nabla(x \cdot 0)\big)$ has a good filtration. Furthermore, for $\delta \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta) = \{e, s_1\}$, we have

$$\mathrm{ch}\big(T_0^{\mu_1,\mu} \nabla(x \cdot 0)\big) = \mathrm{ch}\big(T_\delta^{\mu_1} T_0^{\delta} \nabla(x \cdot 0)\big)$$

by Proposition 4.6. Using Propositions I.6.8 and I.6.9, we see that $T_0^{\delta} \nabla(x \cdot 0) \cong \nabla(x \cdot \delta)$ and that there is a short exact sequence

$$0 \longrightarrow \nabla(x \cdot \mu_1) \longrightarrow T_\delta^{\mu_1} T_0^{\delta} \nabla(x \cdot 0) \longrightarrow \nabla(xs_1 \cdot \mu_1) \longrightarrow 0.$$

In particular, we have

$$\mathrm{ch}\big(T_0^{\mu_1,\mu} \nabla(x \cdot 0)\big) = \mathrm{ch}\big(T_\delta^{\mu_1} T_0^{\delta} \nabla(x \cdot 0)\big) = \mathrm{ch}\, \nabla(x \cdot \mu_1) + \mathrm{ch}\, \nabla(xs_1 \cdot \mu_1)$$

and it follows that $T_0^{\mu_1,\mu}\nabla(x\cdot 0)$ has a good filtration with subquotients $\nabla(x\cdot\mu_1)$ and $\nabla(xs_1\cdot\mu_1)$, both appearing with multiplicity one. As $x\cdot\mu_1 < xs_1\cdot\mu_1$, the good filtration can be chosen with $\nabla(x\cdot\mu_1)$ as a submodule and $\nabla(xs_1\cdot\mu_1)$ as a quotient of $T_0^{\mu_1,\mu}\nabla(x\cdot 0)$ (see the remarks after Proposition I.5.1), so there is a short exact sequence

$$0 \longrightarrow \nabla(x\cdot\mu_1) \longrightarrow T_0^{\mu_1,\mu}\nabla(x\cdot 0) \longrightarrow \nabla(xs_1\cdot\mu_1) \longrightarrow 0.$$

Note that μ is ℓ-singular, so $\nabla(\mu)$ is a singular $\mathbf{G}$-module by Lemma II.4.2. As singular $\mathbf{G}$-modules form a thick tensor ideal, it follows that $T_0^{\mu_1,\mu}\nabla(x\cdot 0)$ is singular. In particular, as $\nabla(x\cdot\mu_1)$ is regular by Lemma II.4.3 (and taking duals), the above short exact sequence is non-split. Now the claim follows by applying the translation functor $T_{\mu_1}^{\lambda}$ to this short exact sequence. □

Proposition 4.10. *Let $x \in W_{\mathrm{aff}}^+$ and $i \in \{0,\dots,n\}$ such that $x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$. Then there exists a weight $\delta \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta) = \{e, s_i\}$ such that $\Theta_i L(x\cdot 0) \cong T_\delta^\lambda L(x\cdot\delta)$.*

Proof. If $i \neq 1$ or $\ell = n+1$ then $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_i) = \{e, s_i\}$ and by Corollary 4.7, we have

$$T_0^{\mu_i,\mu} L(x\cdot 0) \cong L(x\cdot\mu_i).$$

This implies that $\Theta_i L(x\cdot 0) \cong T_{\mu_i}^{\lambda} L(x\cdot\mu_i)$ and the claim follows with $\delta = \mu_i$. Now suppose that $i = 1$ and $\ell > n+1$, and choose any weight $\delta \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta) = \{e, s_1\}$. By Proposition I.6.9, there is a non-split short exact sequence

$$0 \longrightarrow \nabla(x\cdot\lambda) \longrightarrow T_\delta^\lambda\nabla(x\cdot\delta) \longrightarrow \nabla(xs_1\cdot\lambda) \longrightarrow 0$$

and we have $\mathrm{soc}_{\mathbf{G}} T_\delta^\lambda \nabla(x\cdot\delta) \cong L(x\cdot\lambda)$. Furthermore, the $\mathbf{G}$-module $T_\delta^\lambda\nabla(x\cdot\delta)$ is the unique non-split extension of $\nabla(xs_1\cdot\lambda)$ by $\nabla(x\cdot\lambda)$, as remarked after Proposition I.6.10. Now by Corollary 4.9, there is a non-split short exact sequence

$$0 \longrightarrow \nabla(x\cdot\lambda) \longrightarrow \Theta_1\nabla(x\cdot 0) \longrightarrow \nabla(xs_1\cdot\lambda) \longrightarrow 0$$

and it follows that $\Theta_1\nabla(x\cdot 0) \cong T_\delta^\lambda\nabla(x\cdot\delta)$. As Θ_1 is exact, there is an embedding

$$\Theta_1 L(x\cdot 0) \longrightarrow \Theta_1\nabla(x\cdot 0) \cong T_\delta^\lambda\nabla(x\cdot\delta),$$

and in particular, we have either $\mathrm{soc}_{\mathbf{G}}\Theta_1 L(x\cdot 0) \cong L(x\cdot\lambda)$ or $\Theta_1 L(x\cdot 0) = 0$. We claim that the above embedding factors through an embedding of $\Theta_1 L(x\cdot 0)$ into $T_\delta^\lambda L(x\cdot\delta)$.

First observe that we have

$$\mathrm{ch}\,\Theta_1 L(x\cdot 0) = \mathrm{ch}\left(T_{\mu_1}^\lambda T_\delta^{\mu_1} T_0^\delta L(x\cdot 0)\right) = \mathrm{ch}\, T_\delta^\lambda L(x\cdot\delta)$$

by Propositions 4.6 and I.6.8 and that $T_\delta^\lambda L(x\cdot\delta)$ is non-zero by Proposition I.6.10. Furthermore, we have $\nabla(\mu) \cong L(\mu)$ because $\mu \in \overline{C}_{\mathrm{fund}}$, so $\Theta_1 L(x\cdot 0) = T_{\mu_1}^\lambda T_0^{\mu_1,\mu} L(x\cdot 0)$ is contravariantly self-dual and it follows that

$$\mathrm{head}_{\mathbf{G}}\Theta_1 L(x\cdot 0) \cong \mathrm{soc}_{\mathbf{G}}\Theta_1 L(x\cdot 0) \cong L(x\cdot\lambda).$$

By Propositions I.6.10 and I.6.11, we have $[T_\delta^\lambda L(x\cdot\delta) : L(x\cdot\lambda)] = 2$ and

$$[\nabla(xs_1\cdot\lambda) : L(x\cdot\lambda)] = [\nabla(x\cdot\lambda) : L(x\cdot\lambda)] = 1,$$

so $[T_\delta^\lambda\nabla(x\cdot\delta) : L(x\cdot\lambda)] = 2$ by our first short exact sequence. This implies that $L(x\cdot\lambda)$ is not a composition factor of the quotient Q of $T_\delta^\lambda\nabla(x\cdot\delta)$ by the naturally embedded submodule $T_\delta^\lambda L(x\cdot\delta)$. As $\mathrm{head}_{\mathbf{G}}\Theta_1 L(x\cdot 0) \cong L(x\cdot\lambda)$, there is no non-zero homomorphism from $\Theta_1 L(x\cdot 0)$ to Q, and we conclude that the embedding $\Theta_1 L(x\cdot 0) \to T_\delta^\lambda\nabla(x\cdot\delta)$ factors through $T_\delta^\lambda L(x\cdot\delta)$, as claimed. Finally, as the characters of $\Theta_1 L(x\cdot 0)$ and $T_\delta^\lambda L(x\cdot\delta)$ coincide, it follows that $\Theta_1 L(x\cdot 0) \cong T_\delta^\lambda L(x\cdot\delta)$. □

5 Properties of ϑ_i

We keep the notation and assumptions from Section 3. Recall that we consider the functor

$$\Psi = \mathrm{pr}_\lambda\big(T(s_0 \cdot \lambda) \otimes -\big),$$

which decomposes as a direct sum

$$\Psi \cong \Theta_0 \oplus \Theta_1 \oplus \cdots \oplus \Theta_n$$

of the functors Θ_i studied in Section 4. Furthermore, the fixed homomorphisms

$$e \colon T(\lambda) \longrightarrow T(s_0 \cdot \lambda) \qquad \text{and} \qquad p \colon T(s_0 \cdot \lambda) \longrightarrow T(\lambda)$$

give rise to natural transformations

$$\vartheta = \mathrm{pr}_\lambda(e \otimes -) \colon T_0^\lambda \Longrightarrow \Psi \qquad \text{and} \qquad \pi = \mathrm{pr}_\lambda(p \otimes -) \colon \Psi \Longrightarrow T_0^\lambda$$

such that, for every $\mathbf{G}$-module M in $\mathrm{Rep}_0(\mathbf{G})$, we have $\ker(\pi_M)/\mathrm{im}(\vartheta_M) \cong \mathrm{pr}_\lambda\big(L(s_0 \cdot \lambda) \otimes M\big)$.

For $x \in W_{\mathrm{aff}}^+$, we have

$$\Psi L(x \cdot 0) \cong \Theta_0 L(x \cdot 0) \oplus \cdots \oplus \Theta_n L(x \cdot 0),$$

and each of the $\mathbf{G}$-modules $\Theta_i L(x \cdot 0)$ is either zero or admits a description as the 'translation from an s_i-wall' $T_{\delta_i}^\lambda L(x \cdot \delta_i)$ of a simple $\mathbf{G}$-module $L(x \cdot \delta_i)$, for $\delta_i \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta_i) = \{e, s_i\}$ (see Corollary 4.8 and Proposition 4.10). The component at $L(x \cdot 0)$ of the natural transformation ϑ gives an embedding

$$\vartheta_{L(x \cdot 0)} \colon L(x \cdot \lambda) \cong T_0^\lambda L(x \cdot \lambda) \longrightarrow \Psi L(x \cdot 0),$$

which induces homomorphisms $L(x \cdot \lambda) \to \Theta_i L(x \cdot 0)$ for $i = 0, \ldots, n$ by composition with the projections onto the direct summands. In order to describe the subquotient

$$\ker\big(\pi_{L(x \cdot 0)}\big) \big/ \mathrm{im}\big(\vartheta_{L(x \cdot 0)}\big) \cong \mathrm{pr}_\lambda\big(L(s_0 \cdot \lambda) \otimes L(x \cdot 0)\big),$$

we need to understand precisely which of the homomorphisms $L(x \cdot \lambda) \to \Theta_i L(x \cdot 0)$ are non-zero. This requires a detailed analysis of the natural transformations

$$\vartheta_i = \mathrm{pr}_i \circ \vartheta \colon \quad T_0^\lambda \Longrightarrow \Theta_i$$

(where $\mathrm{pr}_i \colon \Psi \Longrightarrow \Theta_i$ denotes the canonical projection), which will be carried out in this section. We start by giving a sufficient condition for the non-vanishing of the component $(\vartheta_i)_{L(x \cdot 0)}$ of ϑ_i at a simple $\mathbf{G}$-module $L(x \cdot 0)$.

Lemma 5.1. *Let $x \in W_{\mathrm{aff}}^+$ and $i \in \{0, \ldots, n\}$ such that $x(A_{\mathrm{fund}}) \uparrow x s_i(A_{\mathrm{fund}})$. If $x(A_{\mathrm{fund}}) = A_{\mathrm{fund}} + \gamma$ for some $\gamma \in X$ then $(\vartheta_i)_{L(x \cdot 0)} \neq 0$.*

Proof. As observed before, the alcove A_{fund} has a unique wall $H = H_{\alpha_{\mathrm{h}},1}$ with $A_{\mathrm{fund}} \uparrow s_H(A_{\mathrm{fund}})$, hence $x(A_{\mathrm{fund}}) = A_{\mathrm{fund}} + \gamma$ has a unique wall $H' = H + \gamma$ with $x(A_{\mathrm{fund}}) \uparrow s_{H'} x(A_{\mathrm{fund}})$. This implies that s_i is the unique simple reflection with $x(A_{\mathrm{fund}}) \uparrow x s_i(A_{\mathrm{fund}})$, so $x s_j(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}})$ for $j \neq i$ and $\Theta_j L(x \cdot 0) = 0$ by Corollary 4.8. In particular, we have $(\vartheta_j)_{L(x \cdot 0)} = 0$ for $j \neq i$, and the claim follows because

$$\vartheta_{L(x \cdot 0)} \colon L(x \cdot \lambda) \cong T_0^\lambda L(x \cdot 0) \longrightarrow \Psi L(x \cdot 0)$$

is injective (hence non-zero) and $\vartheta = \vartheta_0 \oplus \cdots \oplus \vartheta_n$. $\square$

Our next goal is to show that the non-vanishing of $(\vartheta_i)_{L(x\cdot 0)}$ is 'invariant under translation by the root lattice'. More precisely, we want to establish that, for $x \in W_{\mathrm{aff}}^+$ and $\gamma \in \mathbb{Z}\Phi$ such that $t_\gamma x \in W_{\mathrm{aff}}^+$, we have $(\vartheta_i)_{L(x\cdot 0)} \neq 0$ if and only if $(\vartheta_i)_{L(t_\gamma x\cdot 0)} \neq 0$. To that end, we will consider analogues of Θ_i and ϑ_i for $\mathbf{G}_1\mathbf{T}$-modules (rather than $\mathbf{G}$-modules) and use the fact that tensoring with the one-dimensional $\mathbf{G}_1\mathbf{T}$-module $\ell\gamma$ is an equivalence with $\ell\gamma \otimes \widehat{L}_1(x\cdot\lambda) \cong \widehat{L}_1(t_\gamma x\cdot\lambda)$ (see Section I.8). Our goal will then be achieved by a comparison of Θ_i and ϑ_i with their $\mathbf{G}_1\mathbf{T}$-versions. In order to carry out this strategy, we will need some more notation for functors and natural transformations, which we introduce in the following remark.

Remark 5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let F_1 and F_2 be functors from $\mathcal{C}$ to $\mathcal{D}$. As before, for a natural transformation $\psi\colon F_1 \to F_2$ and M an object of $\mathcal{C}$, we write $\psi_M\colon F_1(M) \to F_2(M)$ for the component of ψ at M. For a category $\mathcal{E}$ and a functor $F\colon \mathcal{D} \to \mathcal{E}$, we have a natural transformation

$$F\,\psi\colon F\circ F_1 \to F\circ F_2$$

with component $(F\,\psi)_M = F(\psi_M)\colon F\circ F_1(M) \to F\circ F_2(M)$ at an object M of $\mathcal{C}$. Analogously, for a category $\mathcal{B}$ and a functor $F'\colon \mathcal{B} \to \mathcal{C}$, there is a natural transformation

$$\psi\,F'\colon F_1\circ F' \to F_2\circ F'$$

with component $(\psi\,F')_N = \psi_{F'(N)}\colon F_1\circ F'(N) \to F_2\circ F'(N)$ at an object N of $\mathcal{B}$. For another functor $F_3\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\varphi\colon F_2 \to F_3$, we have

$$F\,(\varphi\circ\psi) = F\,\varphi \circ F\,\psi \qquad \text{and} \qquad (\varphi\circ\psi)\,F' = \varphi\,F' \circ \psi\,F',$$

and it is straightforward to see that $F\,\psi\,F' \coloneqq (F\,\psi)\,F' = F\,(\psi\,F')$.

Recall that from Section 3 that the functor $\Theta_i\colon \mathrm{Rep}_0(\mathbf{G}) \to \mathrm{Rep}_\lambda(\mathbf{G})$ is defined by

$$\Theta_i = T_{\mu_i}^{\lambda} \circ T_0^{\mu_i,\mu} = \mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes -\big) \circ \mathrm{pr}_{\mu_i}\big(\nabla(\mu)\otimes -\big),$$

for $i \in \{0,\ldots,n\}$. Furthermore, the natural transformation $\vartheta_i\colon T_0^\lambda \to \Theta_i$ is the composition of

$$\mathrm{pr}_\lambda\big((f\circ e)\otimes -\big)\colon\ T_0^\lambda = \mathrm{pr}_\lambda\big(T(\lambda)\otimes -\big) \implies \mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes\nabla(\mu)\otimes -\big)$$

with

$$\mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes -\big)\ \mathrm{pr}_{\mu_i}\ \big(\nabla(\mu)\otimes -\big)\colon\ \mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes\nabla(\mu)\otimes -\big) \implies \mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes -\big)\circ \mathrm{pr}_{\mu_i}\circ\big(\nabla(\mu)\otimes -\big),$$

where $e\colon T(\lambda) \to T(s_0\cdot\lambda)$ and $f\colon T(s_0\cdot\lambda) \to \nabla(\varpi_1)\otimes\nabla(\mu)$ are the fixed homomorphisms from Section 3. Here, by abuse of notation, we consider pr_{μ_i} as a natural transformation from the identity functor on $\mathrm{Rep}(\mathbf{G})$ to the projection functor pr_{μ_i}, whose component at a $\mathbf{G}$-module M is the natural projection $M \to \mathrm{pr}_{\mu_i}M$. Thus $\mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes -\big)\ \mathrm{pr}_{\mu_i}\ \big(\nabla(\mu)\otimes -\big)$ is indeed a natural transformation from the functor

$$\mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes\nabla(\mu)\otimes -\big) \cong \mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes -\big)\circ \mathrm{id}_{\mathrm{Rep}(\mathbf{G})}\circ\big(\nabla(\mu)\otimes -\big)$$

to the functor $\mathrm{pr}_\lambda\big(\nabla(\varpi_1)\otimes -\big)\circ\mathrm{pr}_{\mu_i}\circ\big(\nabla(\mu)\otimes -\big)$; see Remark 5.2.

Now let us simplify notation by writing $r = \mathrm{res}_{\mathbf{G}_1\mathbf{T}}^{\mathbf{G}}$ and define

$$\widehat{\Theta}_i \coloneqq \mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes -\big)\circ\mathrm{pr}_{\mu_i}\big(r\nabla(\mu)\otimes -\big)\colon\ \mathrm{Rep}_0(\mathbf{G}_1\mathbf{T}) \longrightarrow \mathrm{Rep}_\lambda(\mathbf{G}_1\mathbf{T}),$$

for $i = 0, \ldots, n$. Furthermore, we define natural transformations $\widehat{\vartheta}_i \colon T_0^\lambda \Longrightarrow \widehat{\Theta}_i$ by

$$\widehat{\vartheta}_i \coloneqq \Big(\mathrm{pr}_\lambda\big(r\nabla(\varpi_1) \otimes -\big)\ \mathrm{pr}_{\mu_i}\ \big(r\nabla(\mu) \otimes -\big)\Big) \circ \mathrm{pr}_\lambda\big(r(f \circ e) \otimes -\big);$$

see the last paragraph of Section I.8 for a discussion of translation functors for $\mathbf{G}_1\mathbf{T}$-modules. By construction, we have equalities of functors $r \circ \Theta_i = \widehat{\Theta}_i \circ r$ and of natural transformations $r\,\vartheta_i = \widehat{\vartheta}_i\, r$ (from $r \circ T_0^\lambda = T_0^\lambda \circ r$ to $r \circ \Theta_i = \widehat{\Theta}_i \circ r$). This allows us to relate the non-vanishing of the components of ϑ_i and $\widehat{\vartheta}_i$ at simple $\mathbf{G}$-modules and simple $\mathbf{G}_1\mathbf{T}$-modules, respectively.

Lemma 5.3. *For $x \in W_{\mathrm{aff}}^+$ and $i \in \{0, \ldots, n\}$, we have $(\vartheta_i)_{L(x \cdot 0)} \neq 0$ if and only if $(\widehat{\vartheta}_i)_{\widehat{L}_1(x \cdot 0)} \neq 0$.*

Proof. Let us write $x \cdot 0 = \lambda_0 + \ell\lambda_1$ with $\lambda_0 \in X_1$ and set $L = L(\lambda_1)$ or $L = L_{\mathbb{C}}(\lambda_1)$, in the modular case or in the quantum case, respectively. As explained in Section I.8, we have

$$\widehat{L}_1(x \cdot 0) \cong \widehat{L}_1(\lambda_0) \otimes \ell\lambda_1 \qquad \text{and} \qquad L(x \cdot 0) \cong L(\lambda_0) \otimes L^{[1]},$$

where $\widehat{L}_1(\lambda_0) \cong rL(\lambda_0)$. Furthermore, the restriction to $\mathbf{G}_1\mathbf{T}$ of the Frobenius twist $L^{[1]}$ decomposes as a direct sum of one-dimensional $\mathbf{G}_1\mathbf{T}$-modules $\ell\nu$ for the different weights ν of L, each occurring $\dim L_\nu$ times; hence there exists an embedding of $\mathbf{G}_1\mathbf{T}$-modules $\ell\lambda_1 \to rL^{[1]}$. It is straightforward to see that the latter induces an embedding of $\mathbf{G}_1\mathbf{T}$-modules

$$\iota \colon \widehat{L}_1(x \cdot 0) \longrightarrow rL(x \cdot 0).$$

By the above discussion, we have a commutative diagram

$$\begin{tikzcd}
\widehat{L}_1(x \cdot \lambda) \arrow[r, "(\widehat{\vartheta}_i)_{\widehat{L}_1(x \cdot 0)}"] \arrow[d, "T_0^\lambda \iota"'] & \widehat{\Theta}_i \widehat{L}_1(x \cdot 0) \arrow[d, "\widehat{\Theta}_i \iota"] \\
rL(x \cdot \lambda) \arrow[r, "(\widehat{\vartheta}_i)_{rL(x \cdot 0)}"'] & \widehat{\Theta}_i rL(x \cdot 0)
\end{tikzcd}$$

where $\widehat{\Theta}_i rL(x \cdot 0) = r\Theta_i L(x \cdot 0)$ and $(\widehat{\vartheta}_i)_{rL(x \cdot 0)} = r\big((\vartheta_i)_{L(x \cdot 0)}\big)$. If $(\vartheta_i)_{L(x \cdot 0)}$ is non-zero then $(\vartheta_i)_{L(x \cdot 0)}$ is injective because $L(x \cdot \lambda)$ is simple, hence $r\big((\vartheta_i)_{L(x \cdot 0)}\big) \circ T_0^\lambda \iota$ is injective. It follows that $(\widehat{\vartheta}_i)_{\widehat{L}_1(x \cdot 0)}$ is injective and therefore non-zero. Conversely, if $(\widehat{\vartheta}_i)_{\widehat{L}_1(x \cdot 0)}$ is non-zero then $\widehat{\Theta}_i \iota \circ (\widehat{\vartheta}_i)_{\widehat{L}_1(x \cdot 0)}$ is non-zero because $\widehat{\Theta}_i \iota$ is injective by exactness of $\widehat{\Theta}_i$; hence $(\vartheta_i)_{L(x \cdot 0)}$ is non-zero, as claimed. $\square$

Now recall that for $\gamma \in \mathbb{Z}\Phi$, tensoring with the one-dimensional simple $\mathbf{G}_1\mathbf{T}$-module $\ell\gamma = \widehat{L}_1(\ell\gamma)$ gives rise to an auto-equivalence of $\mathrm{Rep}(\mathbf{G}_1\mathbf{T})$ with $\mathrm{pr}_\nu \circ (\ell\gamma \otimes -) = (\ell\gamma \otimes -) \circ \mathrm{pr}_\nu$ for all $\nu \in \overline{C}_{\mathrm{fund}} \cap X$. Any fixed choice of isomorphisms of $\mathbf{G}_1\mathbf{T}$-modules

$$r\nabla(\varpi_1) \otimes \ell\gamma \cong \ell\gamma \otimes r\nabla(\varpi_1) \qquad \text{and} \qquad r\nabla(\mu) \otimes \ell\gamma \cong \ell\gamma \otimes r\nabla(\mu) \tag{5.1}$$

gives rise to a commutative diagram of functors and natural transformations

$$\begin{tikzcd}
\mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes r\nabla(\mu)\otimes \ell\gamma\otimes -\big) \arrow[r] \arrow[d] & \mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes -\big)\circ \mathrm{pr}_{\mu_i}\circ \big(r\nabla(\mu)\otimes \ell\gamma\otimes -\big) \arrow[d] \\
\mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes \ell\gamma\otimes r\nabla(\mu)\otimes -\big) \arrow[r] \arrow[d] & \mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes -\big)\circ \mathrm{pr}_{\mu_i}\circ \big(\ell\gamma\otimes r\nabla(\mu)\otimes -\big) \arrow[d] \\
\mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes \ell\gamma\otimes r\nabla(\mu)\otimes -\big) \arrow[r] \arrow[d] & \mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes \ell\gamma\otimes -\big)\circ \mathrm{pr}_{\mu_i}\circ \big(r\nabla(\mu)\otimes -\big) \arrow[d] \\
\mathrm{pr}_\lambda\big(\ell\gamma\otimes r\nabla(\varpi_1)\otimes r\nabla(\mu)\otimes -\big) \arrow[r] \arrow[d] & \mathrm{pr}_\lambda\big(\ell\gamma\otimes r\nabla(\varpi_1)\otimes -\big)\circ \mathrm{pr}_{\mu_i}\circ \big(r\nabla(\mu)\otimes -\big) \arrow[d] \\
\ell\gamma\otimes \mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes r\nabla(\mu)\otimes -\big) \arrow[r] & \ell\gamma\otimes \mathrm{pr}_\lambda\big(r\nabla(\varpi_1)\otimes -\big)\circ \mathrm{pr}_{\mu_i}\circ \big(r\nabla(\mu)\otimes -\big)
\end{tikzcd}$$

for $i = 0, \dots, n$, where the vertical arrows are natural isomorphisms, induced by the isomorphisms from (5.1) and the equalities

$$\mathrm{pr}_{\mu_i}\circ(\ell\gamma\otimes -) = (\ell\gamma\otimes -)\circ \mathrm{pr}_{\mu_i} \qquad \text{and} \qquad \mathrm{pr}_\lambda\circ(\ell\gamma\otimes -) = (\ell\gamma\otimes -)\circ \mathrm{pr}_\lambda,$$

and where the horizontal arrows are obtained, like ϑ_i, by considering pr_{μ_i} as a natural transformation from the identity functor on $\mathrm{Rep}(\mathbf{G}_1\mathbf{T})$ to the projection functor pr_{μ_i}. Furthermore, the isomorphisms from (5.1) give rise to an isomorphism

$$r\nabla(\varpi_1)\otimes r\nabla(\mu)\otimes \ell\gamma \cong r\nabla(\varpi_1)\otimes \ell\gamma\otimes r\nabla(\mu) \cong \ell\gamma\otimes r\nabla(\varpi_1)\otimes r\nabla(\mu),$$

and we claim that the latter induces an isomorphism $rT(\lambda)\otimes\ell\gamma \cong \ell\gamma\otimes rT(\lambda)$, such that the following diagram commutes:

$$\begin{tikzcd}
rT(\lambda)\otimes\ell\gamma \arrow[r, "r(f\circ e)\otimes\ell\gamma"] \arrow[d] & r\nabla(\varpi_1)\otimes r\nabla(\mu)\otimes\ell\gamma \arrow[d] \\
\ell\gamma\otimes rT(\lambda) \arrow[r, "\ell\gamma\otimes r(f\circ e)"] & \ell\gamma\otimes r\nabla(\varpi_1)\otimes r\nabla(\mu)
\end{tikzcd} \tag{5.2}$$

Indeed, $T(\lambda)\cong L(\lambda)$ is isomorphic to the unique simple submodule of $\nabla(\varpi_1)\otimes\nabla(\mu)\cong T(s_0\cdot\lambda)$ by Remark 3.2, and arguing as in the proof of Lemma III.1.9, we see that $rT(\lambda)\cong \widehat{L}_1(\lambda)$ is in fact isomorphic to the unique simple $\mathbf{G}_1\mathbf{T}$-submodule of $r\nabla(\varpi_1)\otimes r\nabla(\mu)$. Hence the isomorphism

$$r\nabla(\varpi_1)\otimes r\nabla(\mu)\otimes\ell\gamma \cong \ell\gamma\otimes r\nabla(\varpi_1)\otimes r\nabla(\mu)$$

identifies the image of $r(f\circ e)\otimes\ell\gamma$ with the image of $\ell\gamma\otimes r(f\circ e)$; therefore it induces an isomorphism

$$rT(\lambda)\otimes\ell\gamma \cong \ell\gamma\otimes T(\lambda)$$

that makes the diagram (5.2) commute, as claimed.

Remark 5.4. In the modular case, we could of course choose the isomorphisms in (5.1) to be the canonical ones (arising from the standard braiding). Then the canonical isomorphism

$$rT(\lambda) \otimes \ell\gamma \cong \ell\gamma \otimes T(\lambda)$$

would make the diagram (5.2) commute. In the quantum case, we could give a similar argument if we were given a braiding on the monoidal category $\mathrm{Rep}(\mathbf{G}_1\mathbf{T})$, or just a natural isomorphism between the functors $(\ell\gamma \otimes -)$ and $(- \otimes \ell\gamma)$, respecting the associativity of tensor products. The existence of a braiding on $\mathrm{Rep}(\mathbf{G}_1\mathbf{T})$ seems to be widely accepted, but we were not able to find an explicit proof in the literature. This is the reason why we have chosen the direct approach above.

Now the commutative diagram (5.2) affords a commutative diagram of functors and natural transformations

$$\begin{tikzcd}
\mathrm{pr}_\lambda\big(rT(\lambda) \otimes \ell\gamma \otimes -\big) \arrow[r] \arrow[d] & \mathrm{pr}_\lambda\big(r\nabla(\varpi_1) \otimes r\nabla(\mu) \otimes \ell\gamma \otimes -\big) \arrow[d] \\
\mathrm{pr}_\lambda\big(\ell\gamma \otimes rT(\lambda) \otimes -\big) \arrow[r] \arrow[d] & \mathrm{pr}_\lambda\big(\ell\gamma \otimes r\nabla(\varpi_1) \otimes r\nabla(\mu) \otimes -\big) \arrow[d] \\
\ell\gamma \otimes \mathrm{pr}_\lambda\big(rT(\lambda) \otimes -\big) \arrow[r] & \ell\gamma \otimes \mathrm{pr}_\lambda\big(r\nabla(\varpi_1) \otimes r\nabla(\mu) \otimes -\big)
\end{tikzcd}$$

where the vertical arrows are natural isomorphisms and the horizontal arrows are induced by $r(f \circ e)$. By combining the two commutative diagrams of functors and natural transformations above, we see that, for $i = 0, \ldots, n$, there are natural isomorphisms

$$\varphi_\gamma \colon\ T_0^\lambda \circ (\ell\gamma \otimes -) = \mathrm{pr}_\lambda\big(rT(\lambda) \otimes \ell\gamma \otimes -\big) \implies \ell\gamma \otimes \mathrm{pr}_\lambda\big(rT(\lambda) \otimes -\big) = (\ell\gamma \otimes -) \circ T_0^\lambda$$

and

$$\begin{aligned}\psi_\gamma \colon\ \widehat{\Theta}_i \circ (\ell\gamma \otimes -) &= \mathrm{pr}_\lambda\big(r\nabla(\varpi_1) \otimes -\big) \circ \mathrm{pr}_{\mu_i} \circ \big(r\nabla(\mu) \otimes \ell\gamma \otimes -\big) \\ &\implies \ell\gamma \otimes \mathrm{pr}_\lambda\big(r\nabla(\varpi_1) \otimes -\big) \circ \mathrm{pr}_{\mu_i} \circ \big(r\nabla(\mu) \otimes -\big) = (\ell\gamma \otimes -) \circ \widehat{\Theta}_i\end{aligned}$$

such that the following diagram commutes:

$$\begin{tikzcd}
T_0^\lambda \circ (\ell\gamma \otimes -) \arrow[r, "\widehat{\vartheta}_i\,(\ell\gamma \otimes -)"] \arrow[d, "\varphi_\gamma"'] & \widehat{\Theta}_i \circ (\ell\gamma \otimes -) \arrow[d, "\psi_\gamma"] \\
(\ell\gamma \otimes -) \circ T_0^\lambda \arrow[r, "(\ell\gamma \otimes -)\,\widehat{\vartheta}_i"] & (\ell\gamma \otimes -) \circ \widehat{\Theta}_i
\end{tikzcd} \tag{5.3}$$

Using this observation, we can now prove that 'translation by the root lattice' does not affect the non-vanishing of the components of $\widehat{\vartheta}_i$ or ϑ_i at a simple $\mathbf{G}_1\mathbf{T}$-module or a simple $\mathbf{G}$-module, respectively.

Lemma 5.5. *For $x \in W_{\mathrm{aff}}$, $\gamma \in \mathbb{Z}\Phi$ and $i \in \{0, \ldots, n\}$, we have*

$$(\widehat{\vartheta}_i)_{\widehat{L}_1(x \cdot 0)} \neq 0 \qquad \textit{if and only if} \qquad (\widehat{\vartheta}_i)_{\widehat{L}_1(t_\gamma x \cdot 0)} \neq 0.$$

Proof. We have $\widehat{L}_1(t_\gamma x \cdot 0) = \widehat{L}_1(x \cdot 0 + \ell\gamma) \cong \ell\gamma \otimes \widehat{L}_1(x \cdot 0)$ and

$$\begin{aligned}(\widehat{\vartheta}_i)_{\ell\gamma\otimes\widehat{L}_1(x\cdot 0)} \circ (\varphi_\gamma)_{\widehat{L}_1(x\cdot 0)} &= \big(\widehat{\vartheta}_i\,(\ell\gamma\otimes -)\big)_{\widehat{L}_1(x\cdot 0)} \circ (\varphi_\gamma)_{\widehat{L}_1(x\cdot 0)} \\ &= (\psi_\gamma)_{\widehat{L}_1(x\cdot 0)} \circ \big((\ell\gamma\otimes -)\,\widehat{\vartheta}_i\big)_{\widehat{L}_1(x\cdot 0)} \\ &= (\psi_\gamma)_{\widehat{L}_1(x\cdot 0)} \circ \big(\ell\gamma \otimes (\widehat{\vartheta}_i)_{\widehat{L}_1(x\cdot 0)}\big)\end{aligned}$$

by the commutative diagram (5.3). Now the claim follows because $(\ell\gamma \otimes -)$ is an equivalence and φ_γ and ψ_γ are natural isomorphisms. □

Corollary 5.6. *Let $x \in W_{\mathrm{aff}}^+$ and $\gamma \in \mathbb{Z}\Phi$ such that $t_\gamma x \in W_{\mathrm{aff}}^+$. For $i \in \{0, \ldots, n\}$, we have*

$$(\vartheta_i)_{L(x\cdot 0)} \neq 0 \qquad \text{if and only if} \qquad (\vartheta_i)_{L(t_\gamma x\cdot 0)} \neq 0.$$

Proof. This is immediate from Lemmas 5.3 and 5.5. □

We continue to examine the non-vanishing of $(\vartheta_i)_{L(x\cdot 0)}$ for $x \in W_{\mathrm{aff}}^+$. Our next goal is to show that, for any pair of elements $x, y \in W_{\mathrm{aff}}^+$ with $d(y) = d(x) + 1$ and such that

$$x(A_{\mathrm{fund}}) \uparrow y(A_{\mathrm{fund}}) \uparrow ys_i(A_{\mathrm{fund}}) \qquad \text{and} \qquad x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}}),$$

the non-vanishing $(\vartheta_i)_{L(x\cdot 0)} \neq 0$ implies that $(\vartheta_i)_{L(y\cdot 0)} \neq 0$. This will be achieved by a comparison of the components $(\vartheta_i)_{L(x\cdot 0)}$ and $(\vartheta_i)_{\nabla(x\cdot 0)}$ and by an application of the snake lemma to a non-split extension of $L(y \cdot \lambda)$ by $\nabla(x \cdot \lambda)$. We will need the following elementary lemma:

Lemma 5.7. *Let M and N be $\mathbf{G}$-modules and let $\varphi\colon M \to N$ be a homomorphism. If the restriction of φ to $\mathrm{soc}_{\mathbf{G}} M$ is injective then φ is injective.*

Proof. If the restriction of φ to $\mathrm{soc}_{\mathbf{G}} M$ is injective then

$$\mathrm{soc}_{\mathbf{G}} \ker(\varphi) \subseteq \mathrm{soc}_{\mathbf{G}}(M) \cap \ker(\varphi) = 0$$

and therefore $\ker(\varphi) = 0$, as claimed. □

Lemma 5.8. *For $x \in W_{\mathrm{aff}}^+$, we have $(\vartheta_i)_{L(x\cdot 0)} \neq 0$ if and only if $(\vartheta_i)_{\nabla(x\cdot 0)}$ is injective.*

Proof. Note that $(\vartheta_i)_{L(x\cdot 0)} \neq 0$ if and only if $(\vartheta_i)_{L(x\cdot 0)}$ is injective because $T_0^\lambda L(x \cdot 0) \cong L(x \cdot \lambda)$ is simple. As ϑ_i is a natural transformation, the canonical embedding $\iota\colon L(x \cdot 0) \to \nabla(x \cdot 0)$ gives rise to a commutative diagram

$$\begin{tikzcd} L(x\cdot\lambda) \arrow[r, "T_0^\lambda \iota"] \arrow[d, "(\vartheta_i)_{L(x\cdot 0)}"'] & \nabla(x\cdot\lambda) \arrow[d, "(\vartheta_i)_{\nabla(x\cdot 0)}"] \\ \Theta_i L(x\cdot 0) \arrow[r, "\Theta_i \iota"'] & \Theta_i \nabla(x\cdot 0) \end{tikzcd}$$

where $T_0^\lambda \iota$ and $\Theta_i\iota$ are injective by exactness of T_0^λ and Θ_i. This implies that $(\vartheta_i)_{L(x\cdot 0)}$ is injective whenever $(\vartheta_i)_{\nabla(x\cdot 0)}$ is injective. Conversely, if $(\vartheta_i)_{L(x\cdot 0)}$ is injective then $(\vartheta_i)_{\nabla(x\cdot 0)}$ is injective by Lemma 5.7 because $L(x \cdot \lambda) = \mathrm{soc}_{\mathbf{G}} \nabla(x \cdot \lambda)$. □

In order to apply Lemma 5.7 to a non-split extension of a simple $\mathbf{G}$-module $L(y \cdot \lambda)$ by an induced module $\nabla(x \cdot \lambda)$, for $x, y \in W_{\mathrm{aff}}^+$, we make the following observation:

Lemma 5.9. *Let $x, y \in W_{\mathrm{aff}}^+$ and $\nu \in C_{\mathrm{fund}} \cap X$, and suppose that there is a non-split extension M of $L(y \cdot \nu)$ by $\nabla(x \cdot \nu)$. Then $\mathrm{soc}_{\mathbf{G}} M \cong L(x \cdot \nu)$.*

Proof. The existence of M implies that $x \neq y$ because $\mathrm{Ext}^1_{\mathbf{G}}\big(L(x \cdot \nu), \nabla(x \cdot \nu)\big) = 0$ by Remark I.4.1. By assumption, there is a non-split short exact sequence

$$0 \longrightarrow \nabla(x \cdot \nu) \xrightarrow{a} M \xrightarrow{b} L(y \cdot \nu) \longrightarrow 0,$$

and the latter gives rise to an exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{G}}\big(L(z \cdot \nu), \nabla(x \cdot \nu)\big) \to \mathrm{Hom}_{\mathbf{G}}\big(L(z \cdot \nu), M\big) \to \mathrm{Hom}_{\mathbf{G}}\big(L(z \cdot \nu), L(y \cdot \nu)\big)$$

for all $z \in W_{\mathrm{aff}}^+$. Therefore, it suffices to show that $\mathrm{Hom}_{\mathbf{G}}\big(L(y \cdot \nu), M\big) = 0$.

Indeed, if there is a non-zero homomorphism $h \colon L(y \cdot \nu) \to M$ then $\mathrm{im}(h) \cong L(y \cdot \nu)$ intersects trivially with $\ker(b) = \mathrm{im}(a) \cong \nabla(x \cdot \nu)$ because $x \neq y$. This implies that $b \circ h$ is a non-zero endomorphism of $L(y \cdot \nu)$. Using Schur's lemma, it follows that $b \circ h$ is an automorphism of $L(y \cdot \nu)$, contradicting the assumption that the above short exact sequence is non-split. □

Proposition 5.10. *Let $x, y \in W_{\mathrm{aff}}^+$ and $i \in \{0, \ldots, n\}$ with $x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$. Furthermore, suppose that $y \neq xs_i$ and that there exists a non-split extension M of $L(y \cdot 0)$ by $\nabla(x \cdot 0)$. If $(\vartheta_i)_{\nabla(x \cdot 0)}$ is injective then so is $(\vartheta_i)_{L(y \cdot 0)}$.*

Proof. By applying the snake lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \nabla(x \cdot \lambda) \arrow[r] \arrow[d, "(\vartheta_i)_{\nabla(x \cdot 0)}"] & T_0^\lambda M \arrow[r] \arrow[d, "(\vartheta_i)_M"] & L(y \cdot \lambda) \arrow[r] \arrow[d, "(\vartheta_i)_{L(y \cdot 0)}"] & 0 \\
0 \arrow[r] & \Theta_i \nabla(x \cdot 0) \arrow[r] & \Theta_i M \arrow[r] & \Theta_i L(y \cdot 0) \arrow[r] & 0
\end{tikzcd},$$

we obtain an exact sequence

$$0 \to \ker\big((\vartheta_i)_{\nabla(x \cdot 0)}\big) \to \ker\big((\vartheta_i)_M\big) \to \ker\big((\vartheta_i)_{L(y \cdot 0)}\big) \to \mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big).$$

By Lemma 5.9, we have $\mathrm{soc}_{\mathbf{G}}\big(T_0^\lambda M\big) \cong T_0^\lambda \mathrm{soc}_{\mathbf{G}} M \cong L(x \cdot \lambda)$, and it follows that $\mathrm{soc}_{\mathbf{G}}(T_0^\lambda M)$ is contained in the image of $\nabla(x \cdot \lambda)$ in $T_0^\lambda M$. Now suppose that $(\vartheta_i)_{\nabla(x \cdot 0)}$ is injective. Then the restriction of $(\vartheta_i)_M$ to $\mathrm{soc}_{\mathbf{G}}\big(T_0^\lambda M\big)$ is injective, and by Lemma 5.7, we have $\ker\big((\vartheta_i)_M\big) = 0$. Thus, the above exact sequence reduces to

$$0 \to \ker\big((\vartheta_i)_{L(y \cdot 0)}\big) \to \mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big).$$

We claim that $\mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big) \cong \nabla(xs_i \cdot \lambda)$. Indeed, part (3) of Lemma I.7.4, applied to the short exact sequence

$$0 \longrightarrow \nabla(x \cdot \lambda) \longrightarrow \Theta_i \nabla(x \cdot 0) \longrightarrow \mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big) \longrightarrow 0,$$

implies that $\mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big)$ has a good filtration. Furthermore, we have

$$\mathrm{ch}\left(\mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big)\right) = \mathrm{ch}\left(\Theta_i \nabla(x \cdot 0)\right) - \mathrm{ch}\left(\nabla(x \cdot \lambda)\right) = \mathrm{ch}\left(\nabla(xs_i \cdot \lambda)\right)$$

by Corollary 4.9, and it follows that $\mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big) \cong \nabla(xs_i \cdot \lambda)$, as claimed. Now $\ker\big((\vartheta_i)_{L(y \cdot 0)}\big)$ is a submodule of $L(y \cdot \lambda)$ and embeds into $\mathrm{cok}\big((\vartheta_i)_{\nabla(x \cdot 0)}\big) \cong \nabla(xs_i \cdot \lambda)$. As $y \neq xs_i$ by assumption, we conclude that $\ker\big((\vartheta_i)_{L(y \cdot 0)}\big) = 0$, as required. □

The existence of a non-split extension of $L(y \cdot \lambda)$ by $\nabla(x \cdot \lambda)$, in the setting described above Lemma 5.7, is guaranteed by a result of S. Ryom-Hansen; see Theorem 2.4 in [RH03a].

Theorem 5.11. *Let $\nu \in C_{\mathrm{fund}} \cap X$ and $x, y \in W_{\mathrm{aff}}^+$ such that $x(A_{\mathrm{fund}}) \uparrow y(A_{\mathrm{fund}})$. Then*

$$\mathrm{Ext}_{\mathbf{G}}^{d(y)-d(x)}\big(L(y \cdot \nu), \nabla(x \cdot \nu)\big) \cong \Bbbk.$$

Corollary 5.12. *Let $i \in \{0, \ldots, n\}$ and $x, y \in W_{\mathrm{aff}}^+$ with $d(y) = d(x) + 1$ and such that*

$$x(A_{\mathrm{fund}}) \uparrow y(A_{\mathrm{fund}}) \uparrow ys_i(A_{\mathrm{fund}}) \qquad \text{and} \qquad x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}}).$$

If $(\vartheta_i)_{L(x \cdot 0)} \neq 0$ then $(\vartheta_i)_{L(y \cdot 0)} \neq 0$.

Proof. If $(\vartheta_i)_{L(x\cdot 0)} \neq 0$ then $(\vartheta_i)_{\nabla(x \cdot 0)}$ is injective by Lemma 5.8. By Theorem 5.11, we have

$$\mathrm{Ext}_{\mathbf{G}}^{1}\big(L(y \cdot 0), \nabla(x \cdot 0)\big) \cong \Bbbk,$$

so there exists a non-split extension of $L(y \cdot 0)$ by $\nabla(x \cdot 0)$. Furthermore, we have $y \neq xs_i$ as

$$x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}}) \qquad \text{and} \qquad y(A_{\mathrm{fund}}) \uparrow ys_i(A_{\mathrm{fund}}),$$

and the claim follows from Proposition 5.10. □

Recall from Corollary 4.8 that for all $x \in W_{\mathrm{aff}}^+$ and $i \in \{0, \ldots, n\}$ such that $xs_i(A_{\mathrm{fund}}) \uparrow A_{\mathrm{fund}}$, we have $\Theta_i L(x \cdot 0) = 0$ and therefore $(\vartheta_i)_{L(x \cdot 0)} = 0$. We can now prove the strongest possible converse of this statement.

Theorem 5.13. *For $i \in \{0, \ldots, n\}$ and $x \in W_{\mathrm{aff}}^+$, the following are equivalent:*

(1) $(\vartheta_i)_{L(x \cdot 0)} \neq 0$;

(2) $\Theta_i L(x \cdot 0) \neq 0$;

(3) $x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$.

Proof. First observe that (1) implies (2) because $(\vartheta_i)_{L(x \cdot 0)}$ is a homomorphism from $T_0^\lambda L(x \cdot 0)$ to $\Theta_i L(x \cdot 0)$ and that (2) implies (3) by Corollary 4.8. Now suppose that $x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$. By Corollary 1.35, there exist $\delta \in X$ and $x_0, \ldots, x_r \in W_{\mathrm{aff}}$ with

$$A_{\mathrm{fund}} + \delta = x_0(A_{\mathrm{fund}}) \uparrow x_1(A_{\mathrm{fund}}) \uparrow \cdots \uparrow x_r(A_{\mathrm{fund}}) = x(A_{\mathrm{fund}})$$

and such that $x_j(A_{\mathrm{fund}}) \uparrow x_j s_i(A_{\mathrm{fund}})$ and $d(x_j) = d(t_\delta) + j$ for $j = 0, \ldots, r$. We can choose $\gamma \in \mathbb{Z}\Phi$ such that $y_j := t_\gamma x_j \in W_{\mathrm{aff}}^+$ for $j = 0, \ldots, r$ (we could take $\gamma = 2a\rho$ for $a \in \mathbb{Z}_{\geq 0}$ sufficiently large), and it is straightforward to see that

$$A_{\mathrm{fund}} + \gamma + \delta = y_0(A_{\mathrm{fund}}) \uparrow y_1(A_{\mathrm{fund}}) \uparrow \cdots \uparrow y_r(A_{\mathrm{fund}}) = t_\gamma x(A_{\mathrm{fund}}),$$

where for $j = 0, \ldots, r$, we have

$$y_j(A_{\mathrm{fund}}) = x_j(A_{\mathrm{fund}}) + \gamma \uparrow x_j s_i(A_{\mathrm{fund}}) + \gamma = y_j s_i(A_{\mathrm{fund}})$$

because $x_j(A_{\mathrm{fund}}) \uparrow x_j s_i(A_{\mathrm{fund}})$, and

$$d(y_j) - d(t_{\gamma+\delta}) = d(t_\gamma x_j) - d(t_\gamma t_\delta) = d(x_j) - d(t_\delta) = j.$$

As $y_0(A_{\mathrm{fund}}) = A_{\mathrm{fund}} + \gamma + \delta$, we have $(\vartheta_i)_{L(y_0 \cdot 0)} \neq 0$ by Lemma 5.1, and using Corollary 5.12 and induction on j, it follows that $(\vartheta_i)_{L(y_j \cdot 0)} \neq 0$ for $j = 0, \ldots, r$. In particular, we have $(\vartheta_i)_{L(t_\gamma x \cdot 0)} \neq 0$ and therefore $(\vartheta_i)_{L(x \cdot 0)} \neq 0$ by Corollary 5.6. Hence (3) implies (1), as required. □

6 Conclusions

In this section, we explain how the results from Sections 3, 4 and 5 can be used to describe the structure of the **G**-modules $\mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big)$ and to study the generic direct summands $G(s_0\omega,x)$ of the tensor products $L(s_0\omega\cdot 0)\otimes L(x\cdot 0)$, for $x\in W_{\mathrm{aff}}^+$. Let us start by recalling some of the notation and the key results from Sections 3, 4 and 5.

We set $\omega=t_{\varpi_1}s_1\cdots s_n\in\Omega$ and $\lambda=\omega\cdot 0=(\ell-n-1)\cdot\varpi_1\in C_{\mathrm{fund}}\cap X$ (see Lemmas 3.3 and 3.4) and consider the functor

$$\Psi=\mathrm{pr}_\lambda\big(T(s_0\cdot\lambda)\otimes -\big)\colon \mathrm{Rep}_0(\mathbf{G})\longrightarrow\mathrm{Rep}_\lambda(\mathbf{G}).$$

By Lemma 3.1, the minimal tilting complex of $L(s_0\cdot\lambda)$ is given by

$$C_{\min}\big(L(s_0\cdot\lambda)\big)=\big(\quad 0\to T(\lambda)\to T(s_0\cdot\lambda)\to T(\lambda)\to 0\quad\big),$$

and as explained after Proposition 3.5, this complex gives rise to natural transformations

$$\vartheta\colon\ T_0^\lambda\Longrightarrow\Psi\qquad\text{and}\qquad\pi\colon\ \Psi\Longrightarrow T_0^\lambda$$

such that, for every **G**-module M in $\mathrm{Rep}_0(\mathbf{G})$, we have $\mathrm{im}(\vartheta_M)\subseteq\ker(\pi_M)$ and

$$\mathrm{im}(\pi_M)/\ker(\vartheta_M)\cong L(s_0\cdot\lambda)\otimes M.$$

Furthermore, the functor Ψ decomposes as a direct sum

$$\Psi\cong\Theta_0\oplus\Theta_1\oplus\cdots\Theta_n,$$

and we write $\mathrm{pr}_i\colon\Psi\Longrightarrow\Theta_i$ for the natural transformations that project onto the direct summands. In many ways, the functor Θ_i, for $i=0,\dots,n$, behaves like an 's_i-wall crossing functor'. For instance, for $x\in W_{\mathrm{aff}}^+$, we have $\Theta_iL(x\cdot 0)\neq 0$ if and only if $x(A_{\mathrm{fund}})\uparrow xs_i(A_{\mathrm{fund}})$, and in that case, there exists a weight $\delta_i\in\overline{C}_{\mathrm{fund}}\cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta_i)=\{e,s_i\}$ (i.e. in the s_i-wall of C_{fund}) such that

$$\Theta_iL(x\cdot 0)\cong T_{\delta_i}^\lambda L(x\cdot\delta_i);$$

see Proposition 4.10. Now Proposition I.6.10 gives a (partial) description of the structure of $\Theta_iL(x\cdot 0)$: For $x\in W_{\mathrm{aff}}^+$ with $x(A_{\mathrm{fund}})\uparrow xs_i(A_{\mathrm{fund}})$, the **G**-module $\Theta_iL(x\cdot 0)$ is indecomposable, with simple socle and head

$$\mathrm{soc}_{\mathbf{G}}\Theta_iL(x\cdot 0)\cong\mathrm{head}_{\mathbf{G}}\Theta_iL(x\cdot 0)\cong L(x\cdot\lambda),$$

and for $y\in W_{\mathrm{aff}}^+$ with $y\neq x$ and $[\Theta_iL(x\cdot 0):L(y\cdot\lambda)]\neq 0$, we have

$$ys_i(A_{\mathrm{fund}})\uparrow y(A_{\mathrm{fund}})\uparrow xs_i(A_{\mathrm{fund}}).$$

In particular, $\Psi L(x\cdot 0)$ has $L(x\cdot\lambda)$-isotypical socle (and head), for all $x\in W_{\mathrm{aff}}^+$, and the number of simple direct summands $L(x\cdot 0)$ of the socle (or the head) coincides with the number of simple reflections $s\in S$ such that $x(A_{\mathrm{fund}})\uparrow xs(A_{\mathrm{fund}})$. Finally, by Theorem 5.13, the component $(\vartheta_i)_{L(x\cdot 0)}$ of the natural transformation

$$\vartheta_i=\mathrm{pr}_i\circ\vartheta\colon\ T_0^\lambda\Longrightarrow\Theta_i$$

at $L(x\cdot 0)$ is non-zero if and only if $x(A_{\mathrm{fund}})\uparrow xs_i(A_{\mathrm{fund}})$; hence the embedding

$$\vartheta_{L(x\cdot 0)}\colon L(x\cdot\lambda)\cong T_0^\lambda L(x\cdot 0)\longrightarrow\Psi L(x\cdot 0)\cong\Theta_0L(x\cdot 0)\oplus\cdots\oplus\Theta_nL(x\cdot 0)$$

induces an embedding

$$(\vartheta_i)_{L(x\cdot 0)} \colon L(x\cdot\lambda) \cong T_0^\lambda L(x\cdot 0) \longrightarrow \Theta_i L(x\cdot 0)$$

whenever $\Theta_i L(x\cdot 0) \neq 0$. In this way, we can consider the image of $\vartheta_{L(x\cdot 0)}$ as being embedded 'diagonally' into the direct sum of the non-zero $\Theta_i L(x\cdot 0)$, all of which have simple socle isomorphic to $L(x\cdot\lambda)$. Analogously, one can show that the epimorphism

$$\pi_{L(x\cdot 0)} \colon \Psi L(x\cdot 0) \longrightarrow T_0^\lambda L(x\cdot 0) \cong L(x\cdot\lambda)$$

induces an epimorphism

$$(\pi_i)_{L(x\cdot 0)} \colon \Theta_i L(x\cdot 0) \longrightarrow T_0^\lambda L(x\cdot 0) \cong L(x\cdot\lambda)$$

whenever $\Theta_i L(x\cdot 0) \neq 0$. Loosely speaking, this means that we can obtain the $\mathbf{G}$-module

$$\mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big) \cong \ker\big(\pi_{L(x\cdot 0)}\big)/\mathrm{im}\big(\vartheta_{L(x\cdot 0)}\big)$$

from the direct sum

$$\Psi L(x\cdot 0) \cong \Theta_0 L(x\cdot 0) \oplus \cdots \oplus \Theta_n L(x\cdot 0)$$

by gluing together the various non-zero $\Theta_i L(x\cdot 0)$ along their socles and along their heads, which are all isomorphic to $L(x\cdot\lambda)$. Let us illustrate this by an example for $\mathbf{G}$ of type A_2. We warn the reader that the following is not a rigorous mathematical discussion, but it may still be helpful in understanding and unraveling the results that we discussed above.

Example 6.1. Suppose that $\mathbf{G}$ is of type A_2. By the above discussion, we have

$$\Theta_0 L(s_0\cdot 0) = 0 \qquad\text{and}\qquad \Theta_i L(s_0\cdot 0) \cong T_{\delta_i}^\lambda L(s_0\cdot\delta_i) \quad \text{for } i = 1,2,$$

for certain weights $\delta_i \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta_i) = \{e, s_i\}$. As in Remark 3.2, one sees that

$$\Theta_i L(s_0\cdot 0) \cong T_{\delta_i}^\lambda L(s_0\cdot\delta_i) \cong T_{\delta_i}^\lambda T(s_0\cdot\delta_i) \cong T(s_0 s_i\cdot\lambda)$$

for $i = 1,2$. The submodule structure of these tilting modules has been determined in Theorem B of [BDM15]; it can be described by the Alperin diagrams below (where as before, we replace a simple $\mathbf{G}$-module $L(x\cdot\lambda)$ by the label $x \in W_{\mathrm{aff}}^+$).

$$\Theta_1 L(x\cdot 0) = \begin{array}{ccc} & s_0 & \\ \diagup & & \diagdown \\ s_0 s_1 & & e \\ \diagdown & & \diagup \\ & s_0 & \end{array} \qquad\text{and}\qquad \Theta_2 L(x\cdot 0) = \begin{array}{ccc} & s_0 & \\ \diagup & & \diagdown \\ s_0 s_2 & & e \\ \diagdown & & \diagup \\ & s_0 & \end{array} \tag{6.1}$$

By Theorem II.4.14 and Lemma III.2.3, we have $\big(L(s_0\cdot\lambda)\otimes L(s_0\cdot 0)\big)_{\mathrm{reg}} \cong M(\lambda) \oplus L(\lambda)$, where the Alperin diagram of $M(\lambda)$ is as follows:

$$M(\lambda) = \begin{array}{ccc} & s_0 & \\ \diagup & | & \diagdown \\ s_0 s_1 & e & s_0 s_2 \\ \diagdown & | & \diagup \\ & s_0 & \end{array}$$

One sees that the diagram for $M(\lambda)$ is obtained from the diagrams for $\Theta_1 L(x\cdot 0)$ and $\Theta_2 L(x\cdot 0)$ by identifying with each other the two bottom nodes and the two top nodes of the diagrams in (6.1) and by discarding one of the nodes labeled by e in the middle layer. The discarded node corresponds to the simple direct summand $L(\lambda)$ of $\big(L(s_0\cdot\lambda)\otimes L(s_0\cdot 0)\big)_{\mathrm{reg}}$, which appears because both of the diagrams in (6.1) have a node labeled by e in the middle layer.

In order to give an example of how our description of the **G**-modules $\mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big)$, for $x\in W_{\mathrm{aff}}^+$, can be used to study the generic direct summands $G(s_0\omega,x)$, we give a classification of the elements $x\in W_{\mathrm{aff}}^+$ such that $G(s_0\omega,x)$ is simple. The key tool for this classification is the following proposition.

Proposition 6.2. *Let $x,y\in W_{\mathrm{aff}}^+$ with $y\neq x$ and suppose that there exists a non-zero homomorphism*

$$L(y\cdot\lambda)\longrightarrow \mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big).$$

For all $i\in\{0,\ldots,n\}$ with $x(A_{\mathrm{fund}})\uparrow xs_i(A_{\mathrm{fund}})$, we have

$$ys_i(A_{\mathrm{fund}})\uparrow y(A_{\mathrm{fund}})\uparrow xs_i(A_{\mathrm{fund}}).$$

Proof. Let $i\in\{0,\ldots,n\}$ such that $x(A_{\mathrm{fund}})\uparrow xs_i(A_{\mathrm{fund}})$, and recall from Proposition 4.10 that there exists a weight $\delta\in\overline{C}_{\mathrm{fund}}\cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\delta)=\{e,s_i\}$ such that $\Theta_i L(x\cdot 0)\cong T_\delta^\lambda L(x\cdot\delta)$. By Proposition I.6.10, every element $z\in W_{\mathrm{aff}}^+$ with $z\neq x$ and

$$0\neq[T_\delta^\lambda L(x\cdot\delta):L(z\cdot\lambda)]=[\Theta_i L(x\cdot 0):L(z\cdot\lambda)]$$

satisfies $zs_i\cdot\lambda\uparrow_\ell z\cdot\lambda\uparrow_\ell xs_i\cdot\lambda$. Therefore, it suffices to prove that $L(y\cdot\lambda)$ appears as a composition factor of $\Theta_i L(x\cdot 0)$.

As explained in Section 3, we have

$$\mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big)\cong\ker\big(\pi_{L(x\cdot 0)}\big)/\mathrm{im}\big(\vartheta_{L(x\cdot 0)}\big),$$

where $\vartheta\colon T_0^\lambda\Longrightarrow\Psi$ and $\pi\colon\Psi\Longrightarrow T_0^\lambda$ are natural transformations, and

$$\Psi\cong\Theta_0\oplus\Theta_1\oplus\cdots\oplus\Theta_n,$$

with natural transformations $\mathrm{pr}_i\colon\Psi\Longrightarrow\Theta_i$ projecting onto the direct summands. By Theorem 5.13, the component $(\vartheta_i)_{L(x\cdot 0)}$ of the natural transformation $\vartheta_i=\mathrm{pr}_i\circ\vartheta$ at $L(x\cdot 0)$ is non-zero, hence injective because $T_0^\lambda L(x\cdot 0)\cong L(x\cdot\lambda)$ is simple. Therefore, by applying the snake lemma to the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & T_0^\lambda L(x\cdot 0) \arrow[r] \arrow[d, "\vartheta_{L(x\cdot 0)}"] & T_0^\lambda L(x\cdot 0) \arrow[r] \arrow[d, "(\vartheta_i)_{L(x\cdot 0)}"] & 0 \\
0 \arrow[r] & \ker\big((\mathrm{pr}_i)_{L(x\cdot 0)}\big) \arrow[r] & \Psi L(x\cdot 0) \arrow[r, "(\mathrm{pr}_i)_{L(x\cdot 0)}"'] & \Theta_i L(x\cdot 0) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, we obtain a short exact sequence

$$0\longrightarrow\ker\big((\mathrm{pr}_i)_{L(x\cdot 0)}\big)\longrightarrow\mathrm{cok}\big(\vartheta_{L(x\cdot 0)}\big)\longrightarrow\mathrm{cok}\big((\vartheta_i)_{L(x\cdot 0)}\big)\longrightarrow 0.$$

We claim that the socle of $\ker\big((\mathrm{pr}_i)_{L(x\cdot 0)}\big)$ is $L(x\cdot\lambda)$-isotypical. Indeed, we have

$$\ker\big((\mathrm{pr}_i)_{L(x\cdot 0)}\big)\cong\bigoplus_{j\neq i}\Theta_j L(x\cdot 0)$$

by definition of pr_i. For $j \in \{0,\ldots,n\}$, we further have $\Theta_j L(x\cdot 0)=0$ unless $x(A_{\mathrm{fund}}) \uparrow xs_j(A_{\mathrm{fund}})$ (see Corollary 4.8). If $x(A_{\mathrm{fund}}) \uparrow xs_j(A_{\mathrm{fund}})$ then, by Proposition 4.10, there exists a weight $\nu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\nu)=\{e,s_j\}$ such that $\Theta_j L(x\cdot 0) \cong T_\nu^\lambda L(x\cdot\nu)$, and by Proposition I.6.10, we have

$$\mathrm{soc}_{\mathbf{G}} \Theta_j L(x\cdot 0) \cong \mathrm{soc}_{\mathbf{G}} T_\nu^\lambda L(x\cdot\nu) \cong L(x\cdot\lambda).$$

We conclude that the socle of $\ker\big((\mathrm{pr}_i)_{L(x\cdot 0)}\big)$ is $L(x\cdot\lambda)$-isotypical, as claimed.

Now observe that

$$\mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big) \cong \ker\big(\pi_{L(x\cdot 0)}\big)/\mathrm{im}\big(\vartheta_{L(x\cdot 0)}\big),$$

naturally embeds into $\mathrm{cok}\big(\vartheta_{L(x\cdot 0)}\big) = \Psi L(x\cdot 0)/\mathrm{im}\big(\vartheta_{L(x\cdot 0)}\big)$. As $y \neq x$, there is no non-zero homomorphism from $L(y\cdot\lambda)$ to $\ker\big((\mathrm{pr}_i)_{L(x\cdot 0)}\big)$, and using the short exact sequence

$$0 \longrightarrow \ker\big((\mathrm{pr}_i)_{L(x\cdot 0)}\big) \longrightarrow \mathrm{cok}\big(\vartheta_{L(x\cdot 0)}\big) \longrightarrow \mathrm{cok}\big((\vartheta_i)_{L(x\cdot 0)}\big) \longrightarrow 0,$$

it follows that the non-zero homomorphism

$$L(y\cdot\lambda) \longrightarrow \mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big) \longrightarrow \mathrm{cok}\big(\vartheta_{L(x\cdot 0)}\big)$$

affords a non-zero homomorphism $L(y\cdot\lambda) \to \mathrm{cok}\big((\vartheta_i)_{L(x\cdot 0)}\big)$. In particular, $L(y\cdot\lambda)$ is a composition factor of $\Theta_i L(x\cdot 0)$, as required. $\square$

The first part of the aforementioned classification result is given by the following theorem:

Theorem 6.3. *For $x \in W_{\mathrm{aff}}^+$, the generic direct summand $G(s_0,x)$ of $L(s_0\cdot 0)\otimes L(x\cdot 0)$ is non-simple unless $x(A_{\mathrm{fund}}) = A_{\mathrm{fund}}+\gamma$ for some $\gamma \in X^+$.*

Proof. Suppose that $G(s_0,x)$ is simple. As $G(s_0,x)$ has good filtration dimension $\ell(x)+\ell(s_0)=\ell(x)+1$ and belongs to the linkage class of 0, Corollary II.2.7 implies that $G(s_0,x)\cong L(y\cdot 0)$ for some $y \in W_{\mathrm{aff}}^+$ with $\ell(y)=\ell(x)+1=d(x)+1$. By Lemma II.5.10, we have

$$G(s_0\omega,x) \cong T^\omega G(s_0,x) \cong T^\omega L(y\cdot 0) \cong L(y\omega\cdot 0) = L(y\cdot\lambda),$$

and it follows that there is a split embedding

$$L(y\cdot\lambda) \cong G(s_0\omega,x) \longrightarrow \mathrm{pr}_\lambda\big(L(s_0\cdot\lambda)\otimes L(x\cdot 0)\big).$$

By Lemma 1.36, there exists $i \in \{0,\ldots,n\}$ such that $x(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$, and by Proposition 6.2, it follows that $y(A_{\mathrm{fund}}) \uparrow xs_i(A_{\mathrm{fund}})$. Furthermore, we have $d(xs_i)=d(x)+1$ by Lemmas 1.17 and 1.29, so $d(xs_i)=\ell(y)=d(y)$ and $xs_i=y$ by Corollary 1.18. In particular, i is uniquely determined by x and y, and we conclude that $xs_j(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}})$ for all $j \in \{0,\ldots,n\}$ with $j \neq i$. As $s_0,\ldots,s_n$ are precisely the reflections in the walls of A_{fund} (see Remark I.2.7), we conclude that there is a unique wall H of $x(A_{\mathrm{fund}})$ with $x(A_{\mathrm{fund}}) \uparrow s_H x(A_{\mathrm{fund}})$. Now Lemma 1.36 shows that $x(A_{\mathrm{fund}}) = A_{\mathrm{fund}}+\gamma$ for some $\gamma \in X$, and as $x \in W_{\mathrm{aff}}^+$, we further have $\gamma \in X^+$. $\square$

In the following lemma, we show that the converse of Theorem 6.3 is true when $n \geq 2$. For $\mathbf{G}$ of type A_1, the converse of Theorem 6.3 is also true in the quantum case (by Lemma III.1.7), but not in the modular case: By Lemmas II.5.10 and III.1.11, we have

$$G(s_0,t_{\ell-1}) \cong T^\omega G(t_1,t_{\ell-1}) \cong T^\omega J(\ell,0,\ldots)^{[1]} \cong T^\omega T(\ell)^{[1]},$$

and $T(\ell)^{[1]}$ is non-simple by Lemma III.1.2.

Lemma 6.4. *Suppose that $n \geq 2$ and let $x \in W_{\mathrm{ext}}^{+}$ such that $x(A_{\mathrm{fund}}) = A_{\mathrm{fund}} + \gamma$ for some $\gamma \in X^{+}$. Then $G(s_0, x)$ is simple.*

Proof. As $x(A_{\mathrm{fund}}) = A_{\mathrm{fund}} + \gamma$, we have $\omega' \coloneqq t_{-\gamma} x \in \mathrm{Stab}_{W_{\mathrm{ext}}}(A_{\mathrm{fund}}) = \Omega$, and Lemma II.5.10 yields

$$G(s_0, x) = G(s_0, t_\gamma \omega') \cong T^{\omega'} G(s_0, t_\gamma).$$

Since $s_0 \cdot 0$ is ℓ-restricted (recall that $n \geq 2$), we have

$$L(s_0 \cdot 0) \otimes L(t_\gamma \cdot 0) = L(s_0 \cdot 0) \otimes L(\ell\gamma) \cong L(s_0 \cdot 0 + \ell\gamma) = L(t_\gamma s_0 \cdot 0)$$

by the Steinberg-Lusztig tensor product theorem, and it follows that $G(s_0, t_\gamma) \cong L(t_\gamma s_0 \cdot 0)$ and

$$G(s_0, x) \cong T^{\omega'} G(s_0, t_\gamma) \cong T^{\omega'} L(t_\gamma s_0 \cdot 0) \cong L(t_\gamma s_0 \omega' \cdot 0)$$

are simple, as required. $\square$

V. Further results in type A_n

The main objective of this chapter is to prove the following theorem, which was our original motivation for developing the theory of generic direct summands.

Complete reducibility theorem. *Suppose that we are in the modular case, that* $\mathbf{G}$ *is of type* A_n *and that* $\ell \geq n+1$. *Let* $\lambda, \mu \in X^+$ *be* ℓ*-restricted and* ℓ*-regular, and further suppose that* $L(\lambda) \otimes L(\mu)$ *is completely reducible. Then either* $\lambda \in C_{\mathrm{fund}}$ *or* $\mu \in C_{\mathrm{fund}}$.

Let us briefly explain our strategy for proving the theorem. For $x, y \in W_{\mathrm{aff}}$ such that $\lambda \in x \cdot C_{\mathrm{fund}}$ and $\mu \in y \cdot C_{\mathrm{fund}}$, the theory developed in Chapter II tells us that there exists a weight $\nu \in C_{\mathrm{fund}} \cap X$ such that $T_0^\nu G(x,y)$ is a direct summand of $L(\lambda) \otimes L(\mu)$. In particular, if $L(\lambda) \otimes L(\mu)$ is completely reducible then $G(x,y)$ is simple. Therefore, the theorem would follow if we could prove that, for any two elements $x, y \in W_{\mathrm{aff}}^+$ such that the ℓ-alcoves $x \cdot C_{\mathrm{fund}}$ and $y \cdot C_{\mathrm{fund}}$ contain ℓ-restricted weights, the generic direct summand $G(x,y)$ of $L(x \cdot 0) \otimes L(y \cdot 0)$ is non-simple, unless $x = e$ or $y = e$.

Unfortunately, the last statement still seems to be intractable, with the tools that are available at present (though we are not aware of any counterexamples in type A_n). We can overcome this problem by combining our initial approach with the classical technique of truncation to Levi subgroups, which allows us to give an inductive argument. For $\mathbf{G}$ of type A_n, there are two Levi subgroups of type A_{n-1}, and if $L(\lambda) \otimes L(\mu)$ is completely reducible then so is the truncation of $L(\lambda) \otimes L(\mu)$ to either of these Levi subgroups. Supposing that the statement of the theorem is true for groups of type A_{n-1}, it follows that, for each of the two Levi subgroups, a suitable truncation of one of the weights λ and μ belongs to the fundamental ℓ-alcove (with respect to the Levi subgroup). This allows us to impose certain conditions on λ and μ, and we can thus drastically reduce the number of pairs of elements $x, y \in W_{\mathrm{aff}}^+$ that we need to consider.

Essentially, the restrictions on λ and μ that we obtain leave open two cases. In the first case, we have $x = s_{\alpha_{\mathrm{h},1}}$ and the element $y \in W_{\mathrm{aff}}^+$ is arbitrary. This situation was studied in detail in Chapter IV. In the second case, the conditions on λ and μ allow us to explicitly determine a (reasonably small) subset $\mathfrak{X} \subseteq W_{\mathrm{aff}}^+$ that contains x and y, and we will show in Section 5 of this chapter that $G(x,y)$ is non-simple, for the pairs of elements $x, y \in \mathfrak{X}$ that the conditions allow.

The proof of the non-simplicity of $G(x,y)$, for $x, y \in \mathfrak{X}$ as above, relies on a detailed study of the composition multiplicities and the Loewy structure of the Weyl modules with highest weights in the ℓ-alcoves $z \cdot C_{\mathrm{fund}}$ for $z \in \mathfrak{X}$, and these results take up the first four sections of this chapter. We first explain in Section 1 how the composition multiplicities of certain Weyl modules can be computed via the Jantzen sum formula and a so-called *recursion formula* from [Gru22]. In Section 2, we establish some preliminary combinatorial results about the set $\mathfrak{X}$, and in Section 3, we use these results and the recursion formula to compute composition multiplicities in certain Weyl modules with highest weights in $z \cdot C_{\mathrm{fund}}$, for $z \in \mathfrak{X}$. We then determine the socle filtrations and the radical filtrations for some (but not all) of these Weyl modules in Section 4. Let us remark that the results of the computations in Sections 3 and 4 may be of independent interest, beyond the applications that we give here. In Section 5, we can finally prove the non-simplicity of $G(x,y)$, for the pairs of elements $x,y \in \mathfrak{X}$ that we consider, by combining results from the previous sections with extensive computations of certain maximal vectors in tensor products, using the distribution algebra $\mathrm{Dist}(\mathbf{G})$ of $\mathbf{G}$.[1] The proof of the complete reducibility theorem is given in Section 6.

1 The Jantzen sum formula

As explained in the introduction, our proof of the complete reducibility theorem relies on a detailed study of the submodule structure of certain Weyl modules. The main tool in our investigation of these Weyl modules will be the *Jantzen filtration* and the *Jantzen sum formula*, which we will compute via the *recursion formula* from [Gru22] (see equation (1.2) below).

For $\lambda \in X^+$, the *Jantzen filtration* of $\Delta(\lambda)$, as defined in Section II.8.19 in [Jan03], is an exhaustive descending filtration

$$\Delta(\lambda) \supseteq \Delta(\lambda)^1 \supseteq \Delta(\lambda)^2 \supseteq \cdots$$

such that $\Delta(\lambda)/\Delta(\lambda)^1 \cong L(\lambda)$. Partial information about the layers $\Delta(\lambda)^i/\Delta(\lambda)^{i+1}$ of this filtration can be obtained from the *Jantzen sum formula*

$$\sum_{i>0} \operatorname{ch} \Delta(\lambda)^i = \sum_{\alpha \in \Phi^+} \sum_{0<m\ell<(\lambda+\rho,\alpha^\vee)} \nu_\ell(m\ell) \cdot \chi(s_{\alpha,m} \cdot \lambda),$$

where ν_ℓ denotes the ℓ-adic evaluation in the modular case and where ν_ℓ is the constant function with value 1 in the quantum case (see Proposition II.8.19 in [Jan03] and Theorem 6.3 in [AK08]). Note that the weight $s_{\alpha,m} \cdot \lambda$ on the right hand side of the Jantzen sum formula may be non-dominant. Therefore, computing the sum formula in concrete examples often involves finding the dominant W_{fin}-conjugates of many non-dominant weights. (Recall that $\chi(w \cdot \mu) = \det(w) \cdot \chi(\mu)$ for $\mu \in X$ and $w \in W_{\mathrm{fin}}$.) When the weight λ is ℓ-regular, we can avoid these computations by considering the Jantzen sum formula as an element of the *anti-spherical module* over the integral group ring $\mathbb{Z}[W_{\mathrm{aff}}]$ of the affine Weyl group, as we explain below.

First, let us denote by $[\mathrm{Rep}(\mathbf{G})]$ the Grothendieck group of $\mathrm{Rep}(\mathbf{G})$, i.e. the quotient of the free $\mathbb{Z}$-module with basis the isomorphism classes of $\mathbf{G}$-modules by the submodule generated by the elements of the form $[A]-[B]+[C]$, for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathrm{Rep}(\mathbf{G})$. For any $\mathbf{G}$-module M, we denote by $[M]$ the image in $[\mathrm{Rep}(\mathbf{G})]$ of the isomorphism class of M. We can define a $\mathbb{Z}$-module homomorphism $[\mathrm{Rep}(\mathbf{G})] \to \mathbb{Z}[X]^{W_{\mathrm{fin}}}$ with $[M] \mapsto \operatorname{ch} M$ for every $\mathbf{G}$-module M, and as the characters $\operatorname{ch} \Delta(\mu) = \chi(\mu)$, for $\mu \in X^+$, form a basis of $\mathbb{Z}[X]^{W_{\mathrm{fin}}}$, it follows that the classes $[\Delta(\mu)]$, for $\mu \in X^+$, form a basis of $[\mathrm{Rep}(\mathbf{G})]$. Similarly, for $\nu \in \overline{C}_{\mathrm{fund}} \cap X$, the Grothendieck group $[\mathrm{Rep}_\nu(\mathbf{G})]$ of the linkage class $\mathrm{Rep}_\nu(\mathbf{G})$ has a basis given by the classes of the Weyl modules in $\mathrm{Rep}_\nu(\mathbf{G})$.

Next consider the *anti-spherical* $\mathbb{Z}[W_{\mathrm{aff}}]$*-module*

$$M_{\mathrm{asph}} \coloneqq \mathrm{sign} \otimes_{\mathbb{Z}[W_{\mathrm{fin}}]} \mathbb{Z}[W_{\mathrm{aff}}],$$

where sign denotes the sign representation of W_{fin}, and for all $x \in W_{\mathrm{aff}}$, let $N_x \coloneqq 1 \otimes x$ be the image of x in M_{asph}. For $w \in W_{\mathrm{fin}}$ and $x, y \in W_{\mathrm{aff}}$, we have

$$N_{wx} = \mathrm{sign}(w) \cdot N_x \qquad \text{and} \qquad N_{xy} = N_x \cdot y,$$

[1] Our usage of $\mathrm{Dist}(\mathbf{G})$ for this part of the proof is the main reason why we only get the complete reducibility theorem in the modular case. We were not able to find an approach that bypasses this computational argument.

and as W_{aff}^+ is a set of W_{fin}-coset representatives in W_{aff}, the elements N_x with $x \in W_{\mathrm{aff}}^+$ form a $\mathbb{Z}$-basis of M_{asph}. The action of the simple reflections in W_{aff} on this basis is given by

$$(1.1) \qquad N_x \cdot s = \begin{cases} N_{xs} & \text{if } xs \in W_{\mathrm{aff}}^+, \\ -N_x & \text{if } xs \notin W_{\mathrm{aff}}^+ \end{cases}$$

for $x \in W_{\mathrm{aff}}^+$ and $s \in S$ because $xs \notin W_{\mathrm{aff}}^+$ if and only if $xsx^{-1} \in W_{\mathrm{fin}}$.

Now suppose that $\ell \geq h$ and fix a weight $\lambda \in C_{\mathrm{fund}} \cap X$. By the above discussion, there is a canonical $\mathbb{Z}$-module isomorphism

$$\psi_\lambda \colon M_{\mathrm{asph}} \longrightarrow [\mathrm{Rep}_\lambda(\mathbf{G})]$$

with $N_x \mapsto [\Delta(x \cdot \lambda)]$ for all $x \in W_{\mathrm{aff}}^+$.

Remark 1.1. For every simple reflection $s \in S$, let us fix $\mu_s \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu_s) = \{e, s\}$ and consider the s-wall crossing functor $\Theta_s \coloneqq T_{\mu_s}^{\lambda} \circ T_{\lambda}^{\mu_s}$. We can endow $[\mathrm{Rep}_\lambda(\mathbf{G})]$ with a right $\mathbb{Z}[W_{\mathrm{aff}}]$-module structure via

$$[M] \cdot (s+1) \coloneqq [\Theta_s M],$$

for every $\mathbf{G}$-module M in $\mathrm{Rep}_\lambda(\mathbf{G})$ and every simple reflection $s \in S$. Using equation (1.1) and Propositions I.6.8 and I.6.9, it is straightforward to see that $\psi_\lambda \colon M_{\mathrm{asph}} \to [\mathrm{Rep}_\lambda(\mathbf{G})]$ is an isomorphism of $\mathbb{Z}[W_{\mathrm{aff}}]$-modules.

For $x \in W_{\mathrm{aff}}^+$, we define elements $\mathrm{JSF}_x^\lambda \in [\mathrm{Rep}_\lambda(\mathbf{G})]$ and $\mathrm{JSF}_x \in M_{\mathrm{asph}}$ by

$$\mathrm{JSF}_x^\lambda \coloneqq \sum_{i>0} [\Delta(x \cdot \lambda)^i] \qquad \text{and} \qquad \mathrm{JSF}_x \coloneqq \psi_\lambda^{-1}(\mathrm{JSF}_x^\lambda).$$

Using the Jantzen sum formula and the translation principle, it is straightforward to see that JSF_x does not depend on the choice of $\lambda \in C_{\mathrm{fund}} \cap X$. The elements $\mathrm{JSF}_x \in M_{\mathrm{asph}}$ can now be computed inductively via the following simple *recursion formula*; see Theorem 4.1 in [Gru22].

Recursion formula. *Let $x \in W_{\mathrm{aff}}^+$ and $s \in S$ such that $x < xs$ and $xs \in W_{\mathrm{aff}}^+$. Write $xsx^{-1} = s_{\beta,m}$ for some $m > 0$ and $\beta \in \Phi^+$. Then*

$$(1.2) \qquad \mathrm{JSF}_{xs} = \nu_\ell(m\ell) \cdot N_x + \mathrm{JSF}_x \cdot s.$$

2 Alcove combinatorics

From now on and for the rest of this chapter, suppose that $\mathbf{G}$ is of type A_n for some $n \geq 3$ and that $\ell \geq h = n+1$. As in Chapter IV, we fix a numbering $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of the simple roots, in accordance with the Dynkin diagram in Figure I.1.1, and denote by $\varpi_i = \varpi_{\alpha_i}$ and $s_i = s_{\alpha_i}$ the fundamental dominant weight and the simple reflection corresponding to α_i, for $i = 1, \ldots, n$. Furthermore, we adopt the convention that $\varpi_0 = 0$ and $\varpi_{n+1} = 0$, and we write $s_0 = s_{\alpha_{\mathrm{h}},1}$ for the affine simple reflection in W_{aff}. The positive roots in Φ are given by $\Phi^+ = \{\beta_{i,j} \mid 1 \leq i \leq j \leq n\}$, where

$$\beta_{i,j} = \alpha_i + \cdots + \alpha_j = -\varpi_{i-1} + \varpi_i + \varpi_j - \varpi_{j+1},$$

and we have $\alpha_{\mathrm{h}} = \beta_{1,n} = \varpi_1 + \varpi_n$.

For $0 \leq i < n$ and $0 \leq j \leq n$, we define

$$x(i,j) \coloneqq s_0 s_1 \cdots s_i s_n s_{n-1} \cdots s_{n-j+1},$$

with the convention that $x(i,0) = s_0 s_1 \cdots s_i$. Furthermore, we set

$$\mathfrak{X} \coloneqq \{x(i,j) \mid 0 \leq i < n,\, 0 \leq j \leq n\} \cup \{e\}.$$

In this section, we carry out some preliminary computations which will enable us to compute the Jantzen sum formula JSF_x for all $x \in \mathfrak{X}$. More precisely, we describe the action by right multiplication of the simple reflections $s_1, \ldots, s_n$ on $\mathfrak{X}$ and compute the integers $n_\alpha(x \cdot C_{\mathrm{fund}})$ for $x \in \mathfrak{X}$ and $\alpha \in \Phi^+$. We start by showing that $x(i,j) \in W_{\mathrm{aff}}^+$ for $0 \leq i < n$ and $0 \leq j \leq n$ by explicitly computing the weights $x(i,j) \cdot 0$.

Lemma 2.1. *For $0 \leq i < n$ and $0 \leq j \leq n$, we have*

$$x(i,j) \cdot 0 = \begin{cases} (\ell - n - 1 + j) \cdot \varpi_1 + \varpi_{i+1} + \varpi_{n-j} + (\ell - n - 1 + i) \cdot \varpi_n & \text{if } i + j < n, \\ (\ell - n - 1 + j) \cdot \varpi_1 + \varpi_{n-j+1} + \varpi_{i+1} + (\ell - n + i) \cdot \varpi_n & \text{if } i + j \geq n. \end{cases}$$

In particular, we have $x(i,j) \in W_{\mathrm{aff}}^+$.

Proof. If $j > 0$ then

$$s_{n-j+1} \cdot 0 = -\alpha_{n-j+1} = \varpi_{n-j} - 2\varpi_{n-j+1} + \varpi_{n-j+2},$$

and it is straightforward to see by induction on k that

$$s_{n-j+k} \cdots s_{n-j+2} s_{n-j+1} \cdot 0 = \varpi_{n-j} - (k+1) \cdot \varpi_{n-j+k} + k \cdot \varpi_{n-j+k+1}$$

for $k = 1, \ldots, j$, that is $s_n \cdots s_{n-j+1} \cdot 0 = \varpi_{n-j} - (j+1) \cdot \varpi_n$ (even when $j = 0$). First suppose that $i + j < n$, so that $i < n - j$. Then $(s_n \cdots s_{n-j+1} \cdot 0, \alpha_i^\vee) = 0$ and

$$s_i s_n \cdots s_{n-j+1} \cdot 0 = s_n \cdots s_{n-j+1} \cdot 0 - \alpha_i = \varpi_{i-1} - 2\varpi_i + \varpi_{i+1} + \varpi_{n-j} - (j+1) \cdot \varpi_n.$$

Again, an easy induction argument yields

$$s_{i-k} \cdots s_{i-1} s_i s_n \cdots s_{n-j+1} \cdot 0 = (k+1) \cdot \varpi_{i-k-1} - (k+2) \cdot \varpi_{i-k} + \varpi_{i+1} + \varpi_{n-j} - (j+1) \cdot \varpi_n$$

for $k = 0, \ldots, i-1$, and we conclude that

$$s_1 \cdots s_i s_n \cdots s_{n-j+1} \cdot 0 = -(i+1) \cdot \varpi_1 + \varpi_{i+1} + \varpi_{n-j} - (j+1) \cdot \varpi_n.$$

It follows that $(s_1 \cdots s_i s_n \cdots s_{n-j+1} \cdot 0 + \rho, \alpha_{\mathrm{h}}^\vee) = n - i - j$ and therefore

$$\begin{aligned} s_0 s_1 \cdots s_i s_n \cdots s_{n-j+1} \cdot 0 &= s_1 \cdots s_i s_n \cdots s_{n-j+1} \cdot 0 + (\ell - n + i + j) \cdot \alpha_{\mathrm{h}} \\ &= (\ell - n - 1 + j) \cdot \varpi_1 + \varpi_{i+1} + \varpi_{n-j} + (\ell - n - 1 + i) \cdot \varpi_n, \end{aligned}$$

as claimed. Now suppose that $i + j \geq n$. If $i > n - j$ then as before $(s_n \cdots s_{n-j+1} \cdot 0, \alpha_i^\vee) = 0$, so

$$s_i s_n \cdots s_{n-j+1} \cdot 0 = s_n \cdots s_{n-j+1} \cdot 0 - \alpha_i = \varpi_{n-j} + \varpi_{i-1} - 2\varpi_i + \varpi_{i+1} - (j+1) \cdot \varpi_n$$

and induction yields

$$s_{i-k}\cdots s_{i-1}s_is_n\cdots s_{n-j+1}\bullet 0=\varpi_{n-j}+(k+1)\cdot\varpi_{i-k-1}-(k+2)\cdot\varpi_{i-k}+\varpi_{i+1}-(j+1)\cdot\varpi_n$$

for $k=0,\dots,i+j-n-1$. In particular, we have

$$\begin{aligned}&s_{n-j+1}\cdots s_{i-1}s_is_n\cdots s_{n-j+1}\bullet 0\\ &\qquad=\varpi_{n-j}+(i+j-n)\cdot\varpi_{n-j}-(i+j-n+1)\cdot\varpi_{n-j+1}+\varpi_{i+1}-(j+1)\cdot\varpi_n\\ &\qquad=(i+j-n+1)\cdot\varpi_{n-j}-(i+j-n+1)\cdot\varpi_{n-j+1}+\varpi_{i+1}-(j+1)\cdot\varpi_n\end{aligned}$$

and

$$\begin{aligned}s_{n-j}s_{n-j+1}\cdots s_{i-1}s_is_n\cdots s_{n-j+1}\bullet 0&=s_{n-j+1}\cdots s_{i-1}s_is_n\cdots s_{n-j+1}\bullet 0-(i+j-n+2)\cdot\alpha_{n-j}\\ &=(i+j-n+2)\cdot\varpi_{n-j-1}-(i+j-n+3)\cdot\varpi_{n-j}+\varpi_{n-j+1}+\varpi_{i+1}-(j+1)\cdot\varpi_n.\end{aligned}$$

Observe that the last equality is also satisfied when $i=n-j$. As before, induction yields

$$\begin{aligned}&s_{n-j-k}\cdots s_{n-j-1}s_{n-j}s_{n-j+1}\cdots s_{i-1}s_is_n\cdots s_{n-j+1}\bullet 0\\ &\quad=(i+j-n+k+2)\cdot\varpi_{n-j-k-1}-(i+j-n+k+3)\cdot\varpi_{n-j-k}+\varpi_{n-j+1}+\varpi_{i+1}-(j+1)\cdot\varpi_n\end{aligned}$$

for $k=0,\dots,n-j-1$ and therefore

$$s_1\cdots s_is_n\cdots s_{n-j+1}=-(i+2)\cdot\varpi_1+\varpi_{n-j+1}+\varpi_{i+1}-(j+1)\cdot\varpi_n.$$

We conclude that $(s_1\cdots s_is_n\cdots s_{n-j+1}\bullet 0+\rho,\alpha_{\mathrm{h}}^\vee)=n-i-j-1$ and

$$\begin{aligned}s_0s_1\cdots s_is_n\cdots s_{n-j+1}\bullet 0&=s_1\cdots s_is_n\cdots s_{n-j+1}\bullet 0+(\ell-n+i+j+1)\cdot\alpha_{\mathrm{h}}\\ &=(\ell-n-1+j)\cdot\varpi_1+\varpi_{n-j+1}+\varpi_{i+1}+(\ell-n+i)\cdot\varpi_n,\end{aligned}$$

as claimed. □

Corollary 2.2. *Let $0\le i<n$, $0\le j\le n$ and $1\le u\le v\le n$. If $i+j<n$ then*

$$n_{\beta_{u,v}}\big(x(i,j)\bullet C_{\mathrm{fund}}\big)=\begin{cases}1 & \text{if } u=1 \text{ and } v\ge n-j \text{ or } v=n \text{ and } u\le i+1,\\ 0 & \text{otherwise,}\end{cases}$$

and if $i+j\ge n$ then

$$n_{\beta_{u,v}}\big(x(i,j)\bullet C_{\mathrm{fund}}\big)=\begin{cases}2 & \text{if } u=1 \text{ and } v=n,\\ 1 & \text{if } u=1 \text{ and } n>v\ge n-j+1 \text{ or } v=n \text{ and } 1<u\le i+1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The integers $n_{\beta_{u,v}}\big(x(i,j)\bullet C_{\mathrm{fund}}\big)$ are uniquely determined by the inequalities

$$n_{\beta_{u,v}}\big(x(i,j)\bullet C_{\mathrm{fund}}\big)\cdot\ell<\big(x(i,j)\bullet 0+\rho,\beta_{u,v}^\vee\big)<\big(n_{\beta_{u,v}}\big(x(i,j)\bullet C_{\mathrm{fund}}\big)+1\big)\cdot\ell,$$

and the claim is easily verified using Lemma 2.1. □

Observe that all elements of $\mathfrak{X}$ belong to the coset $s_0 W_{\mathrm{fin}}$ in W_{aff}. In order to apply the recursion formula (1.2) to recursively compute the Jantzen sum formula JSF_x for the Weyl modules $\Delta(x \cdot \lambda)$ with $x \in \mathfrak{X}$ and $\lambda \in C_{\mathrm{fund}} \cap X$, we need to describe the action of the simple reflections $s \in S \cap W_{\mathrm{fin}}$ on the basis elements $N_x \in M_{\mathrm{asph}}$ with $x \in \mathfrak{X}$. By equation (1.1), it suffices to describe the action of these reflections on the elements of $\mathfrak{X}$ themselves, under right multiplication.

Lemma 2.3. *Let $0 \leq i < n$, $0 \leq j \leq n$ and $1 \leq k \leq n$. Then*

$$x(i,j)s_k \in W_{\mathrm{aff}}^+ \quad \Leftrightarrow \quad \begin{cases} k \in \{i, i+1, n-j, n-j+1\} & \text{if } i+j<n, \\ k \in \{i+1, i+2, n-j, n-j+1\} & \text{if } i+j \geq n. \end{cases}$$

More precisely, we have $x(i,j)s_{n-j} = x(i,j+1)$ (if $j<n$) and $x(i,j)s_{n-j+1} = x(i,j-1)$ (if $j>0$). If $i+j<n$ then $x(i,j)s_i = x(i-1,j)$ (if $i>0$) and

$$x(i,j)s_{i+1} = \begin{cases} x(i+1,j) & \text{if } i+j<n-1, \\ x(i,j+1) & \text{if } i+j=n-1, \end{cases}$$

and if $i+j \geq n$ then $x(i,j)s_{i+2} = x(i+1,j)$ (if $i<n-1$) and

$$x(i,j)s_{i+1} = \begin{cases} x(i-1,j) & \text{if } i+j>n, \\ x(i,j-1) & \text{if } i+j=n. \end{cases}$$

Proof. Recall from Example I.2.6 that we have $s_a s_b s_a = s_b s_a s_b$ for all $a,b \in \{0,\dots,n\}$ such that either $|a-b|=1$ or $\{a,b\}=\{0,n\}$ and that s_a commutes with s_b if $|a-b| \neq 1$ and $\{a,b\} \neq \{0,n\}$. First suppose that $i+j<n$. If $i+1<k<n-j$ then $x(i,j)s_k = s_k x(i,j) \notin W_{\mathrm{aff}}^+$ because $x(i,j) \in W_{\mathrm{aff}}^+$ by Lemma 2.1. If $k>n-j+1$ then

$$\begin{aligned} x(i,j)s_k &= s_0 \cdots s_i s_n \cdots s_{k+1} s_k s_{k-1} s_k s_{k-2} \cdots s_{n-j+1} \\ &= s_0 \cdots s_i s_n \cdots s_{k+1} s_{k-1} s_k s_{k-1} s_{k-2} \cdots s_{n-j+1} = s_{k-1} x(i,j), \end{aligned}$$

and if $k<i$ then

$$\begin{aligned} x(i,j)s_k &= s_0 \cdots s_{k-1} s_k s_{k+1} s_k \cdots s_i s_n \cdots s_{n-j+1} \\ &= s_0 \cdots s_{k-1} s_{k+1} s_k s_{k+1} \cdots s_i s_n \cdots s_{n-j+1} = s_{k+1} x(i,j). \end{aligned}$$

In both cases, it follows that $x(i,j)s_k \notin W_{\mathrm{aff}}^+$. It is straightforward to see from the definition of $x(i,j)$ that we have $x(i,j)s_{n-j} = x(i,j+1)$ (if $j<n$) and $x(i,j)s_{n-j+1} = x(i,j-1)$ (if $j>0$). If $i>0$ then s_i commutes with $s_{n-j+1},\dots,s_n$ and we have $x(i,j)s_i = x(i-1,j)$. Finally, if $i+j<n-1$ then s_{i+1} commutes with $s_{n-j+1},\dots,s_n$ and $x(i,j)s_{i+1} = x(i+1,j)$, and if $i+j=n-1$ then $i+1=n-j$ and $x(i,j)s_{i+1} = x(i,j+1)$, as observed before.

Now suppose that $i+j \geq n$. If $k<n-j \leq i$ then

$$\begin{aligned} x(i,j)s_k &= s_0 \cdots s_{k-1} s_k s_{k+1} s_k s_{k+2} \cdots s_i s_n \cdots s_{n-j+1} \\ &= s_0 \cdots s_{k-1} s_{k+1} s_k s_{k+1} s_{k+2} \cdots s_i s_n \cdots s_{n-j+1} = s_{k+1} x(i,j), \end{aligned}$$

and if $k>i+2 \geq n-j+2$ then

$$\begin{aligned} x(i,j)s_k &= s_0 \cdots s_i s_n \cdots s_{k+1} s_k s_{k-1} s_k s_{k-2} \cdots s_{n-j+1} \\ &= s_0 \cdots s_i s_n \cdots s_{k+1} s_{k-1} s_k s_{k-1} s_{k-2} \cdots s_{n-j+1} = s_{k-1} x(i,j). \end{aligned}$$

In both cases, we conclude that $x(i,j)s_k \notin W_{\text{aff}}^+$. If $n-j+1 < k < i+1$ then

$$\begin{aligned}
x(i,j)s_k &= s_0 \cdots s_i s_n \cdots s_{k+1} s_k s_{k-1} s_k s_{k-2} \cdots s_{n-j+1} \\
&= s_0 \cdots s_i s_n \cdots s_{k+1} s_{k-1} s_k s_{k-1} s_{k-2} \cdots s_{n-j+1} \\
&= s_0 \cdots s_{k-2} s_{k-1} s_k s_{k-1} s_{k+1} \cdots s_i s_n \cdots s_{k+1} s_k s_{k-1} s_{k-2} \cdots s_{n-j+1} \\
&= s_0 \cdots s_{k-2} s_k s_{k-1} s_k s_{k+1} s_i s_n \cdots s_{n-j+1} \\
&= s_k x(i,j) \notin W_{\text{aff}}^+.
\end{aligned}$$

As before, we have $x(i,j)s_{n-j} = x(i,j+1)$ (if $j < n$) and $x(i,j)s_{n-j+1} = x(i,j-1)$ (if $j > 0$). As $i+j \geq n$, we have $i+2 > n-j+1$, and for $i < n-1$, it follows that

$$\begin{aligned}
x(i,j)s_{i+2} &= s_0 \cdots s_i s_n \cdots s_{i+3} s_{i+2} s_{i+1} s_{i+2} s_i \cdots s_{n-j+1} \\
&= s_0 \cdots s_i s_n \cdots s_{i+3} s_{i+1} s_{i+2} s_{i+1} s_i \cdots s_{n-j+1} \\
&= s_0 \cdots s_i s_{i+1} s_n \cdots s_{i+3} s_{i+2} s_{i+1} s_i \cdots s_{n-j+1} \\
&= x(i+1,j).
\end{aligned}$$

Finally, if $i+j > n$ then

$$\begin{aligned}
x(i,j)s_{i+1} &= s_0 \cdots s_i s_n \cdots s_{i+2} s_{i+1} s_i s_{i+1} s_{i-1} \cdots s_{n-j+1} \\
&= s_0 \cdots s_i s_n \cdots s_{i+2} s_i s_{i+1} s_i s_{i-1} \cdots s_{n-j+1} \\
&= s_0 \cdots s_{i-1} s_n \cdots s_{i+2} s_{i+1} s_i s_{i-1} \cdots s_{n-j+1} \\
&= x(i-1,j),
\end{aligned}$$

and if $i+j=n$ then $i+1 = n-j+1$ and $x(i,j)s_{i+1} = x(i,j-1)$. □

Corollary 2.4. *Let $0 \leq i < n$, $0 \leq j \leq n$ and $1 \leq k \leq n$. If $i+j < n$ then*

$$N_{x(i,j)} \cdot s_k = \begin{cases} N_{x(i,j)s_k} & \text{if } k \in \{i, i+1, n-j, n-j+1\}, \\ -N_{x(i,j)} & \text{otherwise,} \end{cases}$$

and if $i+j \geq n$ then

$$N_{x(i,j)} \cdot s_k = \begin{cases} N_{x(i,j)s_k} & \text{if } k \in \{i+1, i+2, n-j, n-j+1\}, \\ -N_{x(i,j)} & \text{otherwise.} \end{cases}$$

Proof. This is straightforward to see from equation (1.1) and Lemma 2.3. □

Remark 2.5. Let $0 \leq i < n$ and $0 \leq j \leq n$. Using Corollary 2.2, it is straightforward to see that

$$\ell\big(x(i,j)\big) = d\big(x(i,j)\big) = \sum_{\beta \in \Phi^+} n_\beta\big(x(i,j) \cdot C_{\text{fund}}\big) = i+j+1.$$

If $j < n$ then $x(i,j+1) = x(i,j)s_{n-j}$ and it follows that $x(i,j) < x(i,j+1)$. Analogously, if $i < n-1$ then $x(i+1,j) = x(i,j)s$, where

$$s = \begin{cases} s_{i+1} & \text{if } i+j < n-1, \\ s_{i+1}s_{i+2}s_{i+1} & \text{if } i+j = n-1, \\ s_{i+2} & \text{if } i+j \geq n \end{cases}$$

by Lemma 2.3, and we conclude that $x(i,j) < x(i+1,j)$. Furthermore, we can use Lemma 2.1 to see that $x(i,j) \cdot 0 < x(i,j+1) \cdot 0$ if $j < n$ and $x(i,j) \cdot 0 < x(i+1,j) \cdot 0$ if $i < n-1$.

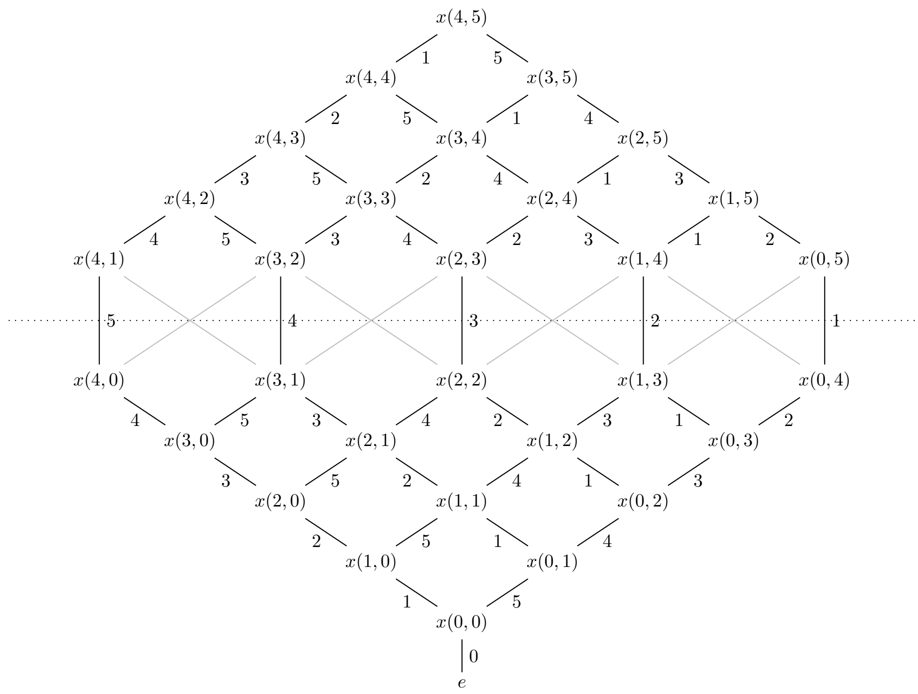

Figure 2.1: The Bruhat graph of $\mathfrak{X}$ for $n=5$.

In Figure 2.1, we give the Bruhat graph of the set $\mathfrak{X} \subseteq W_{\mathrm{aff}}^{+}$ for $n=5$, i.e. the graph with vertices labeled by $\mathfrak{X}$, where two elements $x, y \in \mathfrak{X}$ are joined by an edge if $|\ell(x)-\ell(y)|=1$ and $y=xs$ for a reflection $s \in W_{\mathrm{aff}}$. An edge labeled by i stands for right multiplication by the simple reflection s_i. The dotted line indicates where braid relations are visible in the graph, and the gray lines correspond to multiplication by non-simple reflections.

3 Composition series of Weyl modules

In this section, we use the recursive version of the Jantzen sum formula from Section 1 and the alcove combinatorics established in Section 2 to compute JSF_x in the anti-spherical module, for all $x \in \mathfrak{X}$. Throughout this section, we fix $\lambda \in C_{\mathrm{fund}} \cap X$ and write $\Delta_x = \Delta(x \cdot \lambda)$, $\nabla_x = \nabla(x \cdot \lambda)$ and $L_x = L(x \cdot \lambda)$, for $x \in W_{\mathrm{ext}}^{+}$. Recall that we assume that $\ell \geq h = n+1$.

For $0 \leq i < n$ and $0 \leq j \leq n$ with $i+j \leq n$, the Jantzen sum formula will allow us to compute all composition multiplicities in the Weyl module $\Delta_{x(i,j)}$. When $i+j>n$, there remains some ambiguity about the multiplicity of the simple module L_e in $\Delta_{x(i,j)}$, but we can still determine the multiplicities of the simple $\mathbf{G}$-modules L_y for $y \in \mathfrak{X} \backslash \{e\}$. In order to better understand the 'patterns' of composition factors in $\Delta_{x(i,j)}$ that we will establish below, we encourage the reader to visualize them using the Bruhat graph from Figure 2.1.

The following observation will be very useful for reducing the amount of computations that is necessary to compute the composition multiplicities in the Weyl modules $\Delta_{x(i,j)}$ (which is considerable nevertheless).

Remark 3.1. For $x \in W_{\mathrm{ext}}$ with $x = t_\gamma w$ for some $\gamma \in X$ and $w \in W_{\mathrm{fin}}$, we define

$$x_- \coloneqq t_{-\gamma} w \qquad \text{and} \qquad x^* \coloneqq w_0 \cdot x_- \cdot w_0^{-1} = t_{-w_0(\gamma)} w_0 w w_0^{-1},$$

where w_0 denotes the longest element of W_{fin}. Note that the maps $x \mapsto x_-$ and $x \mapsto x^*$ are automorphisms of W_{ext} and preserve the subgroup W_{aff}. For $x = t_\gamma w \in W_{\mathrm{ext}}^+$, the dual of the simple $\mathbf{G}$-module L_x is simple of highest weight

$$-w_0\big(x \cdot \lambda\big) = -w_0\big(w(\lambda+\rho)+\ell\gamma-\rho\big) = w_0ww_0^{-1}\big(-w_0(\lambda)+\rho\big) - \ell\cdot w_0(\gamma) - \rho = x^* \cdot \big(-w_0(\lambda)\big),$$

where $-w_0(\lambda) \in C_{\mathrm{fund}} \cap X$, and the dual of the Weyl module Δ_x is the induced module of highest weight $x^* \cdot \big(-w_0(\lambda)\big)$. By the translation principle, it follows that

$$[\Delta_x : L_y] = [\Delta_x^* : L_y^*] = [\nabla_{x^*} : L_{y^*}] = [\Delta_{x^*} : L_{y^*}]$$

for all $x, y \in W_{\mathrm{ext}}^+$.

Let us now describe the action of the automorphism $x \mapsto x^*$ on the set $\mathfrak{X}$. As $\mathbf{G}$ is of type A_n, we have $w_0(\alpha_{\mathrm{h}}) = -\alpha_{\mathrm{h}}$ and $w_0(\alpha_i) = -\alpha_{n+1-i}$ for $i = 1,\dots,n$, so $s_0^* = s_0$ and $s_i^* = s_{n+1-i}$ for $i = 1,\dots,n$. We can use Lemma 2.3 and induction on $i+j$ to see that

$$x(i,j)^* = \begin{cases} x(j,i) & \text{if } i+j < n, \\ x(j-1,i+1) & \text{if } i+j \geq n \end{cases}$$

for $0 \leq i < n$ and $0 \leq j \leq n$.

We first compute the composition multiplicities in the Weyl modules $\Delta_{x(i,j)}$ with $i+j<n$. The Jantzen sum formula for these Weyl modules is given in the following result.

Proposition 3.2. *For $0 \leq i < n$ and $0 \leq j < n$ with $i+j<n$, we have*

$$\mathrm{JSF}_{x(i,j)} = (-1)^{i+j} \cdot N_e + \sum_{k=1}^{i} (-1)^{k-1} \cdot N_{x(i-k,j)} + \sum_{k=1}^{j} (-1)^{k-1} \cdot N_{x(i,j-k)}.$$

Proof. We prove the claim by induction on $i+j$. If $i+j=0$ then $i=j=0$ and

$$\mathrm{JSF}_{x(0,0)} = \mathrm{JSF}_{s_0} = N_e + \mathrm{JSF}_e \cdot s_0 = N_e$$

by the recursion formula (1.2), where $\mathrm{JSF}_e = 0$ because $\Delta_e = L_e$.

Now suppose that $i+j>0$ and that the proposition holds for all i' and j' with $i'+j' < i+j$. Then either $i>0$ or $j>0$, and by Remark 3.1, we may assume that $i>0$. By the induction hypothesis, we have

$$\mathrm{JSF}_{x(i-1,j)} = (-1)^{i+j-1} \cdot N_e + \sum_{k=1}^{i-1} (-1)^{k-1} \cdot N_{x(i-1-k,j)} + \sum_{k=1}^{j} (-1)^{k-1} \cdot N_{x(i-1,j-k)}.$$

Now $x(i-1,j) < x(i,j) = x(i-1,j)s_i$ by Remark 2.5, and using Lemma 2.3, Corollary 2.4 and the recursion formula (1.2), we obtain

$$\begin{aligned}
\mathrm{JSF}_{x(i,j)} &= N_{x(i-1,j)} + \mathrm{JSF}_x(i-1,j) \cdot s_i \\
&= N_{x(i-1,j)} + (-1)^{i+j-1} \cdot N_e \cdot s_i + \sum_{k=1}^{i-1} (-1)^{k-1} \cdot N_{x(i-1-k,j)} \cdot s_i + \sum_{k=1}^{j} (-1)^{k-1} \cdot N_{x(i-1,j-k)} \cdot s_i \\
&= N_{x(i-1,j)} + (-1)^{i+j} \cdot N_e + \sum_{k=1}^{i-1} (-1)^{k} \cdot N_{x(i-1-k,j)} + \sum_{k=1}^{j} (-1)^{k-1} \cdot N_{x(i,j-k)}
\end{aligned}$$

$$= (-1)^{i+j} \cdot N_e + \sum_{k=1}^{i} (-1)^{k-1} \cdot N_{x(i-k,j)} + \sum_{k=1}^{j} (-1)^{k-1} \cdot N_{x(i,j-k)},$$

as claimed. □

Recall from Section 1 that we have an isomorphism $\psi_\lambda \colon M_{\mathrm{asph}} \to [\mathrm{Rep}_\lambda(\mathbf{G})]$ such that

$$\psi_\lambda(N_x) = [\Delta_x] \qquad \text{and} \qquad \psi_\lambda(\mathrm{JSF}_x) = \mathrm{JSF}_x^\lambda = \sum_{i>0} [\Delta_x^i]$$

for all $x \in W_{\mathrm{aff}}^+$.

Lemma 3.3. *For $0 < i < n$, we have $\mathrm{JSF}_{x(i,0)}^\lambda = [L_{x(i-1,0)}]$ and $\mathrm{JSF}_{x(0,0)}^\lambda = [L_e]$. In particular,*

$$[\Delta_{x(i,0)}] = [L_{x(i,0)}] + [L_{x(i-1,0)}] \qquad \textit{and} \qquad [\Delta_{x(0,0)}] = [L_{x(0,0)}] + [L_e].$$

Proof. We prove the claim by induction on i. By Proposition 3.2, we have

$$\mathrm{JSF}_{x(0,0)}^\lambda = \psi_\lambda(\mathrm{JSF}_{x(0,0)}) = \psi_\lambda(N_e) = [\Delta_e] = [L_e],$$

hence $\Delta_{x(0,0)}^1 \cong L_e$ and $[\Delta_{x(0,0)}] = [L_{x(0,0)}] + [L_e]$, as claimed. For $i > 0$, Proposition 3.2 and induction on i yield

$$\begin{aligned} \mathrm{JSF}_{x(i,0)}^\lambda &= (-1)^i \cdot [\Delta_e] + \sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,0)}] \\ &= (-1)^i \cdot [L_e] + (-1)^{i-1} \cdot \big([L_{x(0,0)}] + [L_e]\big) + \sum_{k=1}^{i-1} (-1)^{k-1} \cdot \big([L_{x(i-k,0)}] + [L_{x(i-k-1,0)}]\big) \\ &= [L_{x(i-1,0)}], \end{aligned}$$

so $\Delta_{x(i,0)}^1 \cong L_{x(i-1,0)}$ and $[\Delta_{x(i,0)}] = [L_{x(i,0)}] + [L_{x(i-1,0)}]$, as claimed. □

Remark 3.4. From the character formulas in Lemma 3.3, it follows that the Weyl modules $\Delta_{x(0,0)}$ and $\Delta_{x(i,0)}$ are uniserial of composition length 2, with $\mathrm{soc}_{\mathbf{G}} \Delta_{x(0,0)} \cong L_e$ and $\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,0)} \cong L_{x(i-1,0)}$, for $0 < i < n$. By taking duals (see Remark 3.1), we see that the Weyl module $\Delta_{x(0,j)}$ is uniserial of composition length 2, with $\mathrm{soc}_{\mathbf{G}} \Delta_{x(0,j)} \cong L_{x(0,j-1)}$, for $0 < j < n$. We can depict the structure of these Weyl modules in the following diagrams:

$$\Delta_{x(0,0)} = \frac{L_{x(0,0)}}{L_e} \qquad\qquad \Delta_{x(i,0)} = \frac{L_{x(i,0)}}{L_{x(i-1,0)}} \qquad\qquad \Delta_{x(0,j)} = \frac{L_{x(0,j)}}{L_{x(0,j-1)}}$$

Lemma 3.5. *Let $0 < i < n$ and $0 < j < n$ such that $i + j < n$. Then*

$$[\Delta_{x(i,j)}] = [L_{x(i,j)}] + [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] + \delta_{i,j} \cdot [L_e].$$

Proof. We prove the claim by induction on $i + j$. Recall from Proposition 3.2 that

$$\mathrm{JSF}_{x(i,j)}^\lambda = \psi_\lambda(\mathrm{JSF}_{x(i,j)}) = (-1)^{i+j} \cdot [\Delta_e] + \sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] + \sum_{k=1}^{j} (-1)^{k-1} \cdot [\Delta_{x(i,j-k)}],$$

so

$$\mathrm{JSF}_{x(1,1)}^\lambda = [\Delta_{x(1,0)}] + [\Delta_{x(0,1)}] + [\Delta_e] = [L_{x(1,0)}] + [L_{x(0,1)}] + 2 \cdot [L_{x(0,0)}] + [L_e]$$

by Lemma 3.3. This implies that each of the simple **G**-modules $L_{x(1,0)}$, $L_{x(0,1)}$ and L_e appears with multiplicity one as a composition factor of $\Delta_{x(1,1)}$. By Proposition I.6.11 and Lemma 3.3, we further have

$$[\Delta_{x(1,1)} : L_{x(0,0)}] = [\Delta_{x(0,1)} : L_{x(0,0)}] = 1$$

because $x(0,0) \cdot 0 < x(1,0) \cdot 0 = x(0,0)s_1 \cdot 0$ and $x(0,1)s_1 = x(1,1)$. Now suppose that $i + j > 2$, and that the lemma holds for all $i', j' > 0$ with $i' + j' < i + j$. Possibly after taking duals, we may further assume that $i \geq j$ (see Remark 3.1), and Lemma 3.3 and the induction hypothesis yield

$$\begin{aligned}
\sum_{k=1}^{j} (-1)^{k-1} \cdot [\Delta_{x(i,j-k)}] &= (-1)^{j-1} \cdot [\Delta_{x(i,0)}] + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot [\Delta_{x(i,j-k)}] \\
&= (-1)^{j-1} \cdot \big([L_{x(i,0)}] + [L_{x(i-1,0)}]\big) \\
&\qquad + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot \big([L_{x(i,j-k)}] + [L_{x(i-1,j-k)}] + [L_{x(i,j-k-1)}] + [L_{x(i-1,j-k-1)}]\big) \\
&= (-1)^{j-1} \cdot \big([L_{x(i,0)}] + [L_{x(i-1,0)}]\big) + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot \big([L_{x(i,j-k)}] + [L_{x(i-1,j-k)}]\big) \\
&\qquad\qquad + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot \big([L_{x(i,j-k-1)}] + [L_{x(i-1,j-k-1)}]\big) \\
&= \sum_{k=1}^{j} (-1)^{k-1} \cdot \big([L_{x(i,j-k)}] + [L_{x(i-1,j-k)}]\big) \\
&\qquad\qquad + \sum_{k=2}^{j} (-1)^{k-2} \cdot \big([L_{x(i,j-k)}] + [L_{x(i-1,j-k)}]\big) \\
&= [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}],
\end{aligned}$$

hence

$$\sum_{k=1}^{j} (-1)^{k-1} \cdot [\Delta_{x(i,j-k)}] = [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}]. \tag{3.1}$$

Analogously, we compute that

$$\sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] = \begin{cases} [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] & \text{if } i = j, \\ [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] + (-1)^{i-j-1} \cdot [L_e] & \text{if } i > j, \end{cases} \tag{3.2}$$

where the summand corresponding to $k = i - j$ on the left hand side contributes $(-1)^{i-j-1} \cdot [L_e]$ on the right hand side in the case $i > j$ by the induction hypothesis. We conclude that

$$\begin{aligned}
\mathrm{JSF}^{\lambda}_{x(i,j)} &= (-1)^{i+j} \cdot [\Delta_e] + \sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] + \sum_{k=1}^{j} (-1)^{k-1} \cdot [\Delta_{x(i,j-k)}] \\
&= (-1)^{i+j} \cdot [L_e] + [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] + (1 - \delta_{i,j}) \cdot (-1)^{i-j-1} \cdot [L_e] \\
&\qquad\qquad + [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] \\
&= [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + 2 \cdot [L_{x(i-1,j-1)}] + \delta_{i,j} \cdot [L_e],
\end{aligned}$$

whence $L_{x(i-1,j)}$ and $L_{x(i,j-1)}$ appear with multiplicity one as composition factors of $\Delta_{x(i,j)}$, and L_e appears as a composition factor of $\Delta(i,j)$ (with multiplicity one) if and only if $i=j$. Furthermore, Proposition I.6.11 and the induction hypothesis (or Lemma 3.3, if $j=1$) yield

$$[\Delta_{x(i,j)} : L_{x(i-1,j-1)}] = [\Delta_{x(i,j-1)} : L_{x(i-1,j-1)}] = 1$$

because $x(i-1,j-1)\cdot 0 < x(i-1,j)\cdot 0 = x(i-1,j-1)s_{n-j+1}\cdot 0$ and $x(i,j)s_{n-j+1} = x(i,j-1)$; see Remark 2.5. This completes the proof. $\square$

For later use, we note the following consequence of the proof of Lemma 3.5.

Lemma 3.6. *Let $0<i<n$ and $0<j<n$ such that $i+j\leq n$. Then*

$$\sum_{k=1}^{j}(-1)^{k-1}\cdot[\Delta_{x(i,j-k)}] = \begin{cases} [L_{x(i,j-1)}]+[L_{x(i-1,j-1)}] & \text{if } i\geq j, \\ [L_{x(i,j-1)}]+[L_{x(i-1,j-1)}]+(-1)^{j-i-1}\cdot[L_e] & \text{if } i<j \end{cases}$$

and

$$\sum_{k=1}^{i}(-1)^{k-1}\cdot[\Delta_{x(i-k,j)}] = \begin{cases} [L_{x(i-1,j)}]+[L_{x(i-1,j-1)}] & \text{if } i\leq j, \\ [L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+(-1)^{i-j-1}\cdot[L_e] & \text{if } i>j. \end{cases}$$

Proof. This was shown in the proof of Lemma 3.5 under the assumptions that $i+j<n$ and $i\geq j$, see equations (3.1) and (3.2). It is straightforward to verify that the condition $i+j<n$ can be relaxed to $i+j\leq n$, and for $i\leq j$, the formulas are obtained by taking duals (see Remark 3.1). $\square$

In Lemmas 3.3 and 3.5, the composition multiplicities of the Weyl modules $\Delta_{x(i,j)}$ were computed for all $0\leq i<n$ and $0\leq j<n$ such that $i+j<n$. Now we turn to the Weyl modules $\Delta_{x(i,j)}$ with $i+j\geq n$. As before, we first determine the Jantzen sum formula for these Weyl modules. For ease of notation, we define

$$y(i,j) \coloneqq \begin{cases} x(i,j) & \text{if } i+j\geq n \\ x(i+1,j-1) & \text{if } i+j<n \end{cases}$$

for $0\leq i<n$ and $0<j\leq n$. Observe that by Lemma 2.3 and Corollary 2.4, we have

$$N_{y(i-1,j)}\cdot s_{i+1} = N_{y(i,j)}$$

for $1\leq i\leq n-1$. If $i+j\geq n$ then the elements $y(i,j-k)\in\mathfrak{X}$, for $0\leq k<j$, lie on a line pointing southwest from $x(i,j)=y(i,j)$ in the Bruhat graph from Figure 2.1, whereas the elements $x(i-k,j)$, for $0\leq k\leq i$, lie on a line pointing southeast.

Proposition 3.7. *Let $0\leq i<n$ and $1\leq j\leq n$ such that $i+j\geq n$. Then*

$$\mathrm{JSF}_{x(i,j)} = (-1)^{i+j}\cdot N_e + N_{x(n-j,n-i-1)} + \sum_{k=1}^{i}(-1)^{k-1}\cdot N_{x(i-k,j)} + \sum_{k=1}^{j-1}(-1)^{k-1}\cdot N_{y(i,j-k)}.$$

Proof. We prove the claim by induction on $i+j$. First suppose that $i+j=n$, and recall from Proposition 3.2 that

$$\mathrm{JSF}_{x(i,j-1)} = (-1)^{i+j-1}\cdot N_e + \sum_{k=1}^{i}(-1)^{k-1}\cdot N_{x(i-k,j-1)} + \sum_{k=1}^{j-1}(-1)^{k-1}\cdot N_{x(i,j-1-k)}.$$

By Remark 2.5, we have $x(i,j-1) < x(i,j) = x(i,j-1)s_{n-j+1}$, and as $i = n-j$, the recursion formula (1.2) and Corollary 2.4 yield

$$\begin{aligned}
\mathrm{JSF}_{x(i,j)} &= N_{x(i,j-1)} + \mathrm{JSF}_{x(i,j-1)} \cdot s_{n-j+1} \\
&= N_{x(i,j-1)} + (-1)^{i+j-1} \cdot N_e \cdot s_{n-j+1} + \sum_{k=1}^{i} (-1)^{k-1} \cdot N_{x(i-k,j-1)} \cdot s_{n-j+1} \\
&\qquad\qquad + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot N_{x(i,j-1-k)} \cdot s_{n-j+1} \\
&= N_{x(i,j-1)} + (-1)^{i+j} \cdot N_e + \sum_{k=1}^{i} (-1)^{k-1} \cdot N_{x(i-k,j)} + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot N_{x(i+1,j-1-k)} \\
&= N_{x(n-j,n-i-1)} + (-1)^{i+j} \cdot N_e + \sum_{k=1}^{i} (-1)^{k-1} \cdot N_{x(i-k,j)} + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot N_{y(i,j-k)},
\end{aligned}$$

as required. Now suppose that $i+j > n$. Then $i > 0$ and by induction, we may assume that

$$\mathrm{JSF}_{x(i-1,j)} = (-1)^{i+j-1} \cdot N_e + N_{x(n-j,n-i)} + \sum_{k=1}^{i-1} (-1)^{k-1} \cdot N_{x(i-1-k,j)} + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot N_{y(i-1,j-k)}.$$

Now $x(i-1,j) < x(i,j) = x(i-1,j)s_{i+1}$ by Remark 2.5, and again using the recursion formula (1.2), Lemma 2.3 and Corollary 2.4, it follows that

$$\begin{aligned}
\mathrm{JSF}_{x(i,j)} &= N_{x(i-1,j)} + \mathrm{JSF}_{x(i-1,j)} \cdot s_{i+1} \\
&= N_{x(i-1,j)} + (-1)^{i+j-1} \cdot N_e \cdot s_{i+1} + N_{x(n-j,n-i)} \cdot s_{i+1} \\
&\qquad + \sum_{k=1}^{i-1} (-1)^{k-1} \cdot N_{x(i-1-k,j)} \cdot s_{i+1} + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot N_{y(i-1,j-k)} \cdot s_{i+1} \\
&= N_{x(i-1,j)} + (-1)^{i+j} \cdot N_e + N_{x(n-j,n-i-1)} \\
&\qquad + \sum_{k=1}^{i-1} (-1)^{k} \cdot N_{x(i-1-k,j)} + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot N_{y(i,j-k)} \\
&= (-1)^{i+j} \cdot N_e + N_{x(n-j,n-i-1)} + \sum_{k=1}^{i} (-1)^{k-1} \cdot N_{x(i-k,j)} + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot N_{y(i,j-k)},
\end{aligned}$$

as claimed. □

In the next two lemmas, we compute the composition multiplicities in the Weyl modules $\Delta_{x(i,j)}$, for $0 \leq i < n$ and $1 \leq j \leq n$ such that $i+j = n$.

Lemma 3.8. *Let $1 \leq i < n-1$ and $2 \leq j < n$ such that $i+j = n$. Then*

$$\begin{aligned}
[\Delta_{x(i,j)}] = [L_{x(i,j)}] &+ [L_{x(i+1,j-2)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j)}] \\
&+ [L_{x(i-1,j-1)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-2)}] + (\delta_{i,j} + \delta_{i,j-2}) \cdot [L_e].
\end{aligned}$$

Proof. Recall from Proposition 3.7 that

$$\mathrm{JSF}^{\lambda}_{x(i,j)} = (-1)^{i+j} \cdot [\Delta_e] + [\Delta_{x(n-j,n-i-1)}] + \sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot [\Delta_{y(i,j-k)}].$$

By Lemma 3.6, we have

$$\sum_{k=1}^{i}(-1)^{k-1}\cdot[\Delta_{x(i-k,j)}]=\begin{cases}[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}] & \text{if } i\leq j,\\ [L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+(-1)^{i-j-1}\cdot[L_e] & \text{if } i>j\end{cases}$$

and

$$\begin{aligned}\sum_{k=1}^{j-1}(-1)^{k-1}\cdot[\Delta_{y(i,j-k)}]&=\sum_{k=1}^{j-1}(-1)^{k-1}\cdot[\Delta_{x(i+1,j-1-k)}]\\ &=\begin{cases}[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}] & \text{if } i+1\geq j-1,\\ [L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]+(-1)^{j-i-3}\cdot[L_e] & \text{if } i+1<j-1.\end{cases}\end{aligned}$$

For $i>j$, we obtain (using Lemma 3.5 in the last step)

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)}&=(-1)^{i+j}\cdot[L_e]+[\Delta_{x(n-j,n-i-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]\\ &\qquad+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+(-1)^{i-j-1}\cdot[L_e]\\ &=[\Delta_{x(i,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]\\ &=[L_{x(i,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-2)}]+2\cdot\big([L_{x(i-1,j-1)}]+[L_{x(i,j-2)}]\big),\end{aligned}$$

and analogously, for $j>i+2$, we have

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)}&=(-1)^{i+j}\cdot[L_e]+[\Delta_{x(n-j,n-i-1)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]\\ &\qquad+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]+(-1)^{j-i-3}\cdot[L_e]\\ &=[\Delta_{x(i,j-1)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]\\ &=[L_{x(i,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-2)}]+2\cdot\big([L_{x(i-1,j-1)}]+[L_{x(i,j-2)}]\big).\end{aligned}$$

If $i\in\{j,j-2\}$ then

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)}&=(-1)^{i+j}\cdot[L_e]+[\Delta_{x(n-j,n-i-1)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]\\ &=[L_e]+[\Delta_{x(i,j-1)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]\\ &=[L_e]+[L_{x(i,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-2)}]\\ &\qquad+2\cdot\big([L_{x(i-1,j-1)}]+[L_{x(i,j-2)}]\big),\end{aligned}$$

and if $i=j-1$ then

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)}&=(-1)^{i+j}\cdot[L_e]+[\Delta_{x(n-j,n-i-1)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]\\ &=-[L_e]+[\Delta_{x(i,j-1)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i,j-2)}]\\ &=[L_{x(i,j-1)}]+[L_{x(i+1,j-2)}]+[L_{x(i-1,j)}]+[L_{x(i-1,j-2)}]+2\cdot\big([L_{x(i-1,j-1)}]+[L_{x(i,j-2)}]\big)\end{aligned}$$

because $[\Delta_{x(i,j-1)}:L_e]=1$ by Lemma 3.5. In all cases, it remains to show that

$$[\Delta_{x(i,j)}:L_{x(i-1,j-1)}]=1=[\Delta_{x(i,j)}:L_{x(i,j-2)}].$$

Note that we have $x(i,j)=x(i,j-1)s_{n-j+1}$ and $x(i-1,j-1)s_{n-j+1}=x(i-1,j)>x(i-1,j-1)$ by Remark 2.5, so Proposition I.6.11 and Lemma 3.5 yield

$$[\Delta_{x(i,j)}:L_{x(i-1,j-1)}]=[\Delta_{x(i,j-1)}:L_{x(i-1,j-1)}]=1.$$

Analogously, we have $x(i,j-2)s_{n-j+1} = x(i,j-2)s_{i+1} = x(i+1,j-2) > x(i,j-2)$ by Remark 2.5, and as before, it follows that

$$[\Delta_{x(i,j)} : L_{x(i,j-2)}] = [\Delta_{x(i,j-1)} : L_{x(i,j-2)}] = 1,$$

as required. □

Lemma 3.9. *We have*

$$[\Delta_{x(n-1,1)}] = [L_{x(n-1,1)}] + [L_{x(n-1,0)}] + [L_{x(n-2,1)}] + [L_{x(n-2,0)}]$$

and

$$[\Delta_{x(0,n)}] = [L_{x(0,n)}] + [L_{x(0,n-1)}] + [L_{x(1,n-2)}] + [L_{x(0,n-2)}].$$

Proof. We prove the character formula for $\Delta_{x(n-1,1)}$; the formula for $\Delta_{x(0,n)}$ follows by taking duals (see Remark 3.1). By Proposition 3.7, we have

$$\mathrm{JSF}^{\lambda}_{x(n-1,1)} = (-1)^n \cdot [L_e] + [\Delta_{x(n-1,0)}] + \sum_{k=1}^{n-1} (-1)^{k-1} \cdot [\Delta_{x(n-1-k,1)}],$$

where $[\Delta_{x(n-1,0)}] = [L_{x(n-1,0)}] + [L_{x(n-2,0)}]$ by Lemma 3.3 and

$$\sum_{k=1}^{n-1} (-1)^{k-1} \cdot [\Delta_{x(n-1-k,1)}] = [L_{x(n-2,1)}] + [L_{x(n-2,0)}] + (-1)^{n-3} \cdot [L_e]$$

by Lemma 3.6. We conclude that

$$\mathrm{JSF}^{\lambda}_{x(n-1,1)} = [L_{x(n-1,0)}] + [L_{x(n-2,1)}] + 2 \cdot [L_{x(n-2,0)}].$$

Now $x(n-1,1) = x(n-1,0)s_n$ and $x(n-2,0)s_n = x(n-2,1) > x(n-2,0)$ by Remark 2.5, so

$$[\Delta_{x(n-1,1)} : L_{x(n-2,0)}] = [\Delta_{x(n-1,0)} : L_{x(n-2,0)}] = 1$$

by Proposition I.6.11 and Lemma 3.3, and the claim follows. □

In Lemmas 3.8 and 3.9, the composition multiplicities in the Weyl modules $\Delta_{x(i,j)}$ were computed for all $0 \leq i < n$ and $1 \leq j \leq n$ such that $i + j = n$. Now we turn to the Weyl modules $\Delta_{x(i,j)}$ where $i + j > n$. Here, our methods will not be sufficient to determine the composition multiplicities of all simple **G**-modules. Indeed, when computing the Jantzen sum formula, there remains some ambiguity about the multiplicity $[\Delta_{x(i,j)} : L_e]$ of the simple **G**-module with highest weight in C_{fund}, but all the remaining multiplicities can be determined. For the sake of notational simplicity, we define

$$c_{i,j} \coloneqq [\Delta_{x(i,j)} : L_e]$$

for $0 \leq i < n$ and $0 \leq j \leq n$, so $c_{i,j} = \delta_{i,j}$ when $i + j < n$ and $c_{i,j} = \delta_{i,j} + \delta_{i,j-2}$ when $i + j = n$ (see Lemmas 3.5 and 3.8). Furthermore, we write $c'_{i,j} \coloneqq [\Delta_{y(i,j)} : L_e]$ (recall the notation introduced before Proposition 3.7) and denote by $d_{i,j}$ the multiplicity of $[L_e]$ in $\mathrm{JSF}^{\lambda}_{x(i,j)}$.

Lemma 3.10. *Let $2 \leq i < n-1$ and $3 \leq j < n$ such that $i + j = n + 1$. Then*

$$\begin{aligned}[\Delta_{x(i,j)}] = [L_{x(i,j)}] &+ [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-2)}] \\ &+ [L_{x(i-2,j-2)}] + [L_{x(i-1,j-3)}] + [L_{x(i-2,j-3)}] + c_{i,j} \cdot [L_e],\end{aligned}$$

where $c_{i,j} = 1$ if $j \in \{i, i+2\}$, $c_{i,j} \in \{1, 2\}$ if $j = i + 1$ and $c_{i,j} = 0$ otherwise.

Proof. Recall from Proposition 3.7 that

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)} &= (-1)^{i+j}\cdot[\Delta_e] + [\Delta_{x(n-j,n-i-1)}] + \sum_{k=1}^{i}(-1)^{k-1}\cdot[\Delta_{x(i-k,j)}] + \sum_{k=1}^{j-1}(-1)^{k-1}\cdot[\Delta_{y(i,j-k)}] \\ &= (-1)^{n+1}\cdot[L_e] + [\Delta_{x(i-1,j-2)}] + [\Delta_{x(i-1,j)}] + [\Delta_{x(i,j-1)}] \\ &\qquad - \sum_{k=1}^{i-1}(-1)^{k-1}\cdot[\Delta_{x(i-1-k,j)}] - \sum_{k=1}^{j-2}(-1)^{k-1}\cdot[\Delta_{y(i,j-1-k)}].\end{aligned}$$

By Lemma 3.6, we have

$$\sum_{k=1}^{i-1}(-1)^{k-1}\cdot[\Delta_{x(i-1-k,j)}] = \begin{cases}[L_{x(i-2,j)}] + [L_{x(i-2,j-1)}] & \text{if } i-1\le j,\\ [L_{x(i-2,j)}] + [L_{x(i-2,j-1)}] + (-1)^{i-j-2}\cdot[L_e] & \text{if } i-1>j\end{cases}$$

and

$$\begin{aligned}\sum_{k=1}^{j-2}(-1)^{k-1}\cdot[\Delta_{y(i,j-1-k)}] &= \sum_{k=1}^{j-2}(-1)^{k-1}\cdot[\Delta_{x(i+1,j-2-k)}] \\ &= \begin{cases}[L_{x(i+1,j-3)}] + [L_{x(i,j-3)}] & \text{if } i+1\ge j-2,\\ [L_{x(i+1,j-3)}] + [L_{x(i,j-3)}] + (-1)^{j-i-4}\cdot[L_e] & \text{if } i+1<j-2.\end{cases}\end{aligned}$$

Furthermore, we know from Lemma 3.8 that

$$\begin{aligned}[\Delta_{x(i-1,j)}] &= [L_{x(i-1,j)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-1)}] + [L_{x(i-2,j)}] \\ &\qquad + [L_{x(i-2,j-1)}] + [L_{x(i-1,j-2)}] + [L_{x(i-2,j-2)}] + (\delta_{i-1,j} + \delta_{i,j-1})\cdot[L_e]\end{aligned}$$

and

$$\begin{aligned}[\Delta_{x(i,j-1)}] &= [L_{x(i,j-1)}] + [L_{x(i+1,j-3)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-1)}] \\ &\qquad + [L_{x(i-1,j-2)}] + [L_{x(i,j-3)}] + [L_{x(i-1,j-3)}] + (\delta_{i,j-1} + \delta_{i,j-3})\cdot[L_e]\end{aligned}$$

and from Lemma 3.5 that

$$[\Delta_{x(i-1,j-2)}] = [L_{x(i-1,j-2)}] + [L_{x(i-2,j-2)}] + [L_{x(i-1,j-3)}] + [L_{x(i-2,j-3)}] + \delta_{i-1,j-2}\cdot[L_e].$$

We conclude that

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)} - d_{i,j}\cdot[L_e] &= \big((-1)^{n+1} - d_{i,j}\big)\cdot[L_e] + [\Delta_{x(i-1,j-2)}] + [\Delta_{x(i-1,j)}] + [\Delta_{x(i,j-1)}] \\ &\qquad - \sum_{k=1}^{i-1}(-1)^{k-1}\cdot[\Delta_{x(i-1-k,j)}] - \sum_{k=1}^{j-2}(-1)^{k-1}\cdot[\Delta_{y(i,j-1-k)}] \\ &= \big([L_{x(i-1,j-2)}] + [L_{x(i-2,j-2)}] + [L_{x(i-1,j-3)}] + [L_{x(i-2,j-3)}]\big) \\ &\qquad + \big([L_{x(i-1,j)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-1)}] + [L_{x(i-2,j)}] \\ &\qquad\qquad + [L_{x(i-2,j-1)}] + [L_{x(i-1,j-2)}] + [L_{x(i-2,j-2)}]\big) \\ &\qquad + \big([L_{x(i,j-1)}] + [L_{x(i+1,j-3)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-1)}] \\ &\qquad\qquad + [L_{x(i-1,j-2)}] + [L_{x(i,j-3)}] + [L_{x(i-1,j-3)}]\big) \\ &\qquad - \big([L_{x(i-2,j)}] + [L_{x(i-2,j-1)}]\big) - \big([L_{x(i+1,j-3)}] + [L_{x(i,j-3)}]\big) \\ &= [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + 2\cdot\big([L_{x(i-1,j-1)}] + [L_{x(i,j-2)}]\big) \\ &\qquad + 3\cdot[L_{x(i-1,j-2)}] + 2\cdot\big([L_{x(i-2,j-2)}] + [L_{x(i-1,j-3)}]\big) + [L_{x(i-2,j-3)}].\end{aligned}$$

Furthermore, we have

$$\begin{aligned} d_{i,j} &= (-1)^{n+1} + c_{i-1,j-2} + c_{i-1,j} + c_{i,j-1} - \sum_{k=1}^{i-1} (-1)^{k-1} \cdot c_{i-1-k,j} - \sum_{k=1}^{j-2} (-1)^{k-1} \cdot c'_{i,j-1-k} \\ &= (-1)^{n+1} + \delta_{i-1,j-2} + \delta_{i-1,j} + \delta_{i-1,j-2} + \delta_{i,j-1} + \delta_{i,j-3} \\ &\qquad\qquad - \sum_{k=1}^{i-1} (-1)^{k-1} \cdot \delta_{i-1-k,j} - \sum_{k=1}^{j-2} (-1)^{k-1} \cdot \delta_{i+1,j-2-k} \\ &= (-1)^{n+1} + 3 \cdot \delta_{i,j-1} + \sum_{k=0}^{i-1} (-1)^{k} \cdot \delta_{i-1-k,j} + \sum_{k=0}^{j-2} (-1)^{k} \cdot \delta_{i+1,j-2-k}, \end{aligned}$$

and as $i + j = n + 1$, it follows that

$$d_{i,j} = \begin{cases} 2 & \text{if } i = j - 1, \\ 1 & \text{if } i = j - 2 \text{ or } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In all cases, it is straightforward to see that the value of $c_{i,j}$ is as claimed above, and it remains to show that the simple **G**-modules $L_{x(i-1,j-1)}$, $L_{x(i,j-2)}$, $L_{x(i-1,j-2)}$, $L_{x(i-2,j-2)}$ and $L_{x(i-1,j-3)}$ appear with multiplicity one as composition factors of $\Delta_{x(i,j)}$.

Indeed, we have

$$x(i,j) = x(i,j-1)s_{n-j+1} = x(i-1,j)s_{i+1}$$

by Lemma 2.3, and as $x(i-1,j-1)s_{n-j+1} = x(i-1,j) > x(i-1,j-1)$, Proposition I.6.11 and Lemma 3.8 yield

$$[\Delta_{x(i,j)} : L_{x(i-1,j-1)}] = [\Delta_{x(i,j-1)} : L_{x(i-1,j-1)}] = 1.$$

Analogously, we have $x(i,j-2)s_{i+1} = x(i,j-2)s_{n-j+2} = x(i,j-1) > x(i,j-2)$ and therefore

$$[\Delta_{x(i,j)} : L_{x(i,j-2)}] = [\Delta_{x(i-1,j)} : L_{x(i,j-2)}] = 1$$

and $x(i-1,j-2)s_{i+1} = x(i-1,j-2)s_{n-j+2} = x(i-1,j-1) > x(i-1,j-2)$, so

$$[\Delta_{x(i,j)} : L_{x(i-1,j-2)}] = [\Delta_{x(i-1,j)} : L_{x(i-1,j-2)}] = 1.$$

Finally, we have $x(i-2,j-2)s_{i+1} = x(i-2,j-2)s_{n-j+2} = x(i-2,j-1) > x(i-2,j-2)$, so

$$[\Delta_{x(i,j)} : L_{x(i-2,j-2)}] = [\Delta_{x(i-1,j)} : L_{x(i-2,j-2)}] = 1,$$

and $x(i-1,j-3)s_{n-j+1} = x(i-1,j-3)s_i = x(i,j-3) > x(i-1,j-3)$, so

$$[\Delta_{x(i,j)} : L_{x(i-1,j-3)}] = [\Delta_{x(i,j-1)} : L_{x(i-1,j-3)}] = 1,$$

as required. □

Lemma 3.11. *We have*

$$\begin{aligned} [\Delta_{x(n-1,2)}] = [L_{x(n-1,2)}] &+ [L_{x(n-1,1)}] + [L_{x(n-2,2)}] + [L_{x(n-1,0)}] + [L_{x(n-2,1)}] \\ &+ [L_{x(n-2,0)}] + [L_{x(n-3,0)}] + \delta_{n,3} \cdot [L_e] \end{aligned}$$

and

$$\begin{aligned} [\Delta_{x(1,n)}] = [L_{x(1,n)}] &+ [L_{x(0,n)}] + [L_{x(1,n-1)}] + [L_{x(0,n-1)}] + [L_{x(1,n-2)}] \\ &+ [L_{x(0,n-2)}] + [L_{x(0,n-3)}] + \delta_{n,3} \cdot [L_e]. \end{aligned}$$

Proof. We prove the first character formula, the second one follows from the first by taking duals (see Remark 3.1). By Proposition 3.7, we have

$$\mathrm{JSF}^{\lambda}_{x(n-1,2)} = (-1)^{n+1} \cdot [\Delta_e] + [\Delta_{x(n-2,0)}] + [\Delta_{x(n-1,1)}] + [\Delta_{x(n-2,2)}] - \sum_{k=1}^{n-2} (-1)^{k-1} \cdot [\Delta_{x(n-2-k,2)}].$$

If $n > 4$ then Lemma 3.6 gives

$$\sum_{k=1}^{n-2} (-1)^{k-1} \cdot [\Delta_{x(n-2-k,2)}] = [L_{x(n-3,2)}] + [L_{x(n-3,1)}] + (-1)^{n-5} \cdot [L_e],$$

and using Lemmas 3.3, 3.8 and 3.9, it follows that

$$\begin{aligned}
\mathrm{JSF}^{\lambda}_{x(n-1,2)} = (-1)^{n+1} \cdot [L_e] &+ \big([L_{x(n-2,0)}] + [L_{x(n-3,0)}]\big) \\
&+ \big([L_{x(n-1,1)}] + [L_{x(n-2,1)}] + [L_{x(n-1,0)}] + [L_{x(n-2,0)}]\big) \\
&+ \big([L_{x(n-2,2)}] + [L_{x(n-1,0)}] + [L_{x(n-2,1)}] + [L_{x(n-3,2)}] \\
&\qquad + [L_{x(n-2,0)}] + [L_{x(n-3,1)}] + [L_{x(n-3,0)}]\big) \\
&- \big([L_{x(n-3,2)}] + [L_{x(n-3,1)}] + (-1)^{n+5} \cdot [L_e]\big) \\
= [L_{x(n-1,1)}] + [L_{x(n-2,2)}] &+ 2 \cdot \big([L_{x(n-1,0)}] + [L_{x(n-2,1)}]\big) \\
&+ 3 \cdot [L_{x(n-2,0)}] + 2 \cdot [L_{x(n-3,0)}].
\end{aligned}$$

As before, we can use Lemma 2.3, Remark 2.5 and Proposition I.6.11 to see that each of the simple $\mathbf{G}$-modules $L_{x(n-1,0)}$, $L_{x(n-2,1)}$, $L_{x(n-2,0)}$ and $L_{x(n-3,0)}$ appears with multiplicity one as a composition factor of $\Delta_{x(i,j)}$, as claimed. If $n = 3$ then $[\Delta_{x(2,2)} : L_e] = 1$ because

$$\sum_{k=1}^{n-2} (-1)^{k-1} \cdot [\Delta_{x(n-2-k,2)}] = [\Delta_{x(0,2)}] = [L_{x(0,2)}] + [L_{x(0,1)}],$$

and the rest of the proof is as in the case $n > 4$. If $n = 4$ then

$$\sum_{k=1}^{n-2} (-1)^{k-1} \cdot [\Delta_{x(n-2-k,2)}] = [\Delta_{x(1,2)}] - [\Delta_{x(0,2)}] = [L_{x(1,2)}] + [L_{x(1,1)}]$$

and $[\Delta_{x(2,2)} : L_e] = 1$ by Lemma 3.8, hence $[\Delta_{x(3,2)} : L_e] = 0$. Again, the rest of the proof is as in the case $n > 4$. ☐

It remains to consider the Weyl modules $\Delta_{x(i,j)}$ for $0 \leq i < n$ and $0 \leq j \leq n$ such that $i+j > n+1$. We first assume that $i < n-1$ and $j < n$, and we start with the case $i + j = n + 2$.

Lemma 3.12. *Let $3 \leq i < n-1$ and $4 \leq j < n$ such that $i + j = n + 2$. Then*

$$\begin{aligned}
[\Delta_{x(i,j)}] = [L_{x(i,j)}] + [L_{x(i,j-1)}] &+ [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] \\
&+ [L_{x(i-2,j-3)}] + [L_{x(i-3,j-3)}] + [L_{x(i-2,j-4)}] + [L_{x(i-3,j-4)}] \\
&+ c_{i,j} \cdot [L_e],
\end{aligned}$$

where $c_{i,j} \in \{1, 2\}$ if $i \leq j \leq i+2$ and $c_{i,j} = 0$ otherwise.

Proof. By Proposition 3.7, we have

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)} = (-1)^{i+j} \cdot [\Delta_e] + [\Delta_{x(n-j,n-i-1)}] \\ + [\Delta_{x(i-1,j)}] - [\Delta_{x(i-2,j)}] + \sum_{k=3}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] \\ + [\Delta_{x(i,j-1)}] - [\Delta_{x(i,j-2)}] + \sum_{k=3}^{j-1} (-1)^{k-1} \cdot [\Delta_{y(i,j-k)}],\end{aligned}$$

where $[\Delta_{x(n-j,n-i-1)}] = [\Delta_{x(i-2,j-3)}]$, and by Lemma 3.6, we have

$$\sum_{k=3}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] = \begin{cases} [L_{x(i-3,j)}] + [L_{x(i-3,j-1)}] & \text{if } i-2 \leq j \\ [L_{x(i-3,j)}] + [L_{x(i-3,j-1)}] + (-1)^{i-j-3} \cdot [L_e] & \text{if } i-2 > j \end{cases}$$

and

$$\sum_{k=3}^{j-1} (-1)^{k-1} \cdot [\Delta_{y(i,j-k)}] = \begin{cases} [L_{x(i+1,j-4)}] + [L_{x(i,j-4)}] & \text{if } i+1 \geq j-3 \\ [L_{x(i+1,j-4)}] + [L_{x(i,j-4)}] + (-1)^{j-i-5} \cdot [L_e] & \text{if } i+1 < j-3. \end{cases}$$

Using Lemmas 3.5, 3.8 and 3.10, we obtain

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)} - d_{i,j} \cdot [L_e] &= \big([L_{x(i-2,j-3)}] + [L_{x(i-3,j-3)}] + [L_{x(i-2,j-4)}] + [L_{x(i-3,j-4)}]\big) \\ &\quad + \big([L_{x(i-1,j)}] + [L_{x(i-2,j)}] + [L_{x(i-1,j-1)}] + [L_{x(i-2,j-1)}] + [L_{x(i-1,j-2)}] \\ &\qquad + [L_{x(i-2,j-2)}] + [L_{x(i-3,j-2)}] + [L_{x(i-2,j-3)}] + [L_{x(i-3,j-3)}]\big) \\ &\quad - \big([L_{x(i-2,j)}] + [L_{x(i-2,j-1)}] + [L_{x(i-1,j-2)}] + [L_{x(i-3,j)}] \\ &\qquad + [L_{x(i-3,j-1)}] + [L_{x(i-2,j-2)}] + [L_{x(i-3,j-2)}]\big) \\ &\quad + \big([L_{x(i-3,j)}] + [L_{x(i-3,j-1)}]\big) \\ &\quad + \big([L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-2)}] + [L_{x(i,j-3)}] \\ &\qquad + [L_{x(i-1,j-3)}] + [L_{x(i-2,j-3)}] + [L_{x(i-1,j-4)}] + [L_{x(i-2,j-4)}]\big) \\ &\quad - \big([L_{x(i,j-2)}] + [L_{x(i,j-3)}] + [L_{x(i+1,j-4)}] + [L_{x(i-1,j-2)}] \\ &\qquad + [L_{x(i,j-4)}] + [L_{x(i-1,j-3)}] + [L_{x(i-1,j-4)}]\big) \\ &\quad + \big([L_{x(i+1,j-4)}] + [L_{x(i,j-4)}]\big) \\ &= [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + [L_{x(i-3,j-4)}] \\ &\quad + 2 \cdot \big([L_{x(i-1,j-1)}] + [L_{x(i-2,j-4)}] + [L_{x(i-3,j-3)}]\big) + 3 \cdot [L_{x(i-2,j-3)}].\end{aligned}$$

As before, Proposition I.6.11 yields

$$[\Delta_{x(i,j)} : L_{x(i-2,j-3)}] = [\Delta_{x(i-1,j)} : L_{x(i-2,j-3)}] = 1,$$

and analogously, we see that the simple **G**-modules $L_{x(i-1,j-1)}$, $L_{x(i-2,j-4)}$, and $L_{x(i-3,j-3)}$ appear with multiplicity 1 as composition factors of $\Delta_{x(i,j)}$. It remains to determine the multiplicity of L_e as a composition factor of $\Delta_{x(i,j)}$. We have

$$\begin{aligned} d_{i,j} &= (-1)^{i+j} + c_{n-j,n-i-1} + c_{i-1,j} + c_{i,j-1} - c_{i-2,j} - c_{i,j-2} \\ &\qquad + \sum_{k=3}^{i} (-1)^{k-1} \cdot c_{i-k,j} + \sum_{k=3}^{j-1} (-1)^{k-1} \cdot c'_{i,j-k} \\ &= (-1)^{i+j} + \delta_{n-j,n-i-1} + c_{i-1,j} + c_{i,j-1} - \delta_{i-2,j} - \delta_{i-2,j-2} - \delta_{i,j-2} - \delta_{i,j-4} \\ &\qquad + \sum_{k=3}^{i} (-1)^{k-1} \cdot \delta_{i-k,j} + \sum_{k=3}^{j-1} (-1)^{k-1} \cdot \delta_{i+1,j-1-k}, \end{aligned}$$

where $c_{i-1,j} = \delta_{i-1,j} + \delta_{i+1,j}$ unless $i = j$ and $c_{i,j-1} = \delta_{i+1,j} + \delta_{i,j-3}$ unless $i = j-2$. For $i \notin \{j-2, j\}$, it follows that

$$d_{i,j} = (-1)^{i+j} + 3 \cdot \delta_{i+1,j} - \sum_{k=0}^{i} (-1)^k \cdot \delta_{i-k,j} - \sum_{k=0}^{j-1} (-1)^k \cdot \delta_{i+1,j-1-k},$$

and therefore $d_{i,j} = c_{i,j} = 0$ for $i < j-2$ and for $i > j$. For $i = j-1$, we obtain $d_{i,j} = 2$ and therefore $c_{i,j} \in \{1,2\}$. Finally, if $i = j$ then $d_{i,j} = c_{i-1,j}$ and if $i = j-2$ then $d_{i,j} = c_{i,j-1}$, and in both cases, it follows that $c_{i,j} \in \{1,2\}$. □

Lemma 3.13. *Let $3 \leq i < n-1$ and $4 \leq j < n$ such that $i + j \geq n+2$. Then*

$$\begin{aligned} [\Delta_{x(i,j)}] &= [L_{x(i,j)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] + c_{i,j} \cdot [L_e] \\ &\qquad + [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] + [L_{x(n-j,n-i-2)}] + [L_{x(n-j-1,n-i-2)}], \end{aligned}$$

where $c_{i,j} = 0$ if $n > 2j$ or $n > 2i+2$.

Proof. We prove the claim by induction on $r \coloneqq i + j - (n+2)$. The case $r = 0$ is Lemma 3.12, so now assume that $r \geq 1$. By Proposition 3.7, we have

$$\mathrm{JSF}^{\lambda}_{x(i,j)} = (-1)^{i+j} \cdot [\Delta_e] + [\Delta_{x(n-j,n-i-1)}] + \sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot [\Delta_{y(i,j-k)}].$$

Note that we can write

$$\begin{aligned} \sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] &= \sum_{k=1}^{i+j-(n+2)} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] \\ &\qquad + (-1)^{i+j-n-2} \cdot \big([\Delta_{x(n+1-j,j)}] - [\Delta_{x(n-j,j)}] \big) \\ &\qquad + (-1)^{i+j-n} \cdot \sum_{k=1}^{n-j} (-1)^{k-1} \cdot [\Delta_{x(n-j-k,j)}] \end{aligned}$$

and that the characters of all the Weyl modules in this sum are known, either by induction or by our previous results. By Lemma 3.6, we have

$$\sum_{k=1}^{n-j} (-1)^{k-1} \cdot [\Delta_{x(n-j-k,j)}] = \begin{cases} [L_{x(n-j-1,j)}] + [L_{x(n-j-1,j-1)}] & \text{if } n \leq 2j, \\ [L_{x(n-j-1,j)}] + [L_{x(n-j-1,j-1)}] + (-1)^{n-1} \cdot [L_e] & \text{if } n > 2j, \end{cases} \tag{3.3}$$

and by Lemmas 3.8 and 3.10, we have

$$
\begin{aligned}
\Delta_{x(n-j+1,j)} - \Delta_{x(n-j,j)} &= [L_{x(n-j+1,j)}] + [L_{x(n-j,j)}] + [L_{x(n-j+1,j-1)}] + [L_{x(n-j,j-1)}] + [L_{x(n-j+1,j-2)}] \\
&\qquad + [L_{x(n-j,j-2)}] + [L_{x(n-j-1,j-2)}] + [L_{x(n-j,j-3)}] + [L_{x(n-j-1,j-3)}] \\
&\qquad - [L_{x(n-j,j)}] - [L_{x(n-j+1,j-2)}] - [L_{x(n-j-1,j)}] - [L_{x(n-j,j-1)}] \\
&\qquad - [L_{x(n-j-1,j-1)}] - [L_{x(n-j,j-2)}] - [L_{x(n-j-1,j-2)}] \\
&\qquad + (c_{n-j+1,j} - c_{n-j,j}) \cdot [L_e] \\
&= [L_{x(n-j+1,j)}] + [L_{x(n-j+1,j-1)}] + [L_{x(n-j,j-3)}] + [L_{x(n-j-1,j-3)}] \\
&\qquad - [L_{x(n-j-1,j)}] - [L_{x(n-j-1,j-1)}] + (c_{n-j+1,j} - c_{n-j,j}) \cdot [L_e].
\end{aligned}
$$

Furthermore, induction yields

$$
\begin{aligned}
&\sum_{k=1}^{i+j-(n+2)} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] \\
&\quad = \sum_{k=1}^{i+j-(n+2)} (-1)^{k-1} \cdot \Big([L_{x(i-k,j)}] + [L_{x(i-k-1,j)}] + [L_{x(i-k,j-1)}] + [L_{x(i-k-1,j-1)}] \\
&\qquad\qquad + [L_{x(n-j,n-i+k-1)}] + [L_{x(n-j-1,n-i+k-1)}] \\
&\qquad\qquad + [L_{x(n-j,n-i+k-2)}] + [L_{x(n-j-1,n-i+k-2)}] + c_{i-k,j} \cdot [L_e] \Big) \\
&\quad = [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] + [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] \\
&\qquad + (-1)^{i+j-n-3} \cdot \Big([L_{x(n+1-j,j)}] + [L_{x(n+1-j,j-1)}] \\
&\qquad\qquad + [L_{x(n-j,j-3)}] + [L_{x(n-j-1,j-3)}] \Big) \\
&\qquad + \Bigg(\sum_{k=1}^{i+j-(n+2)} (-1)^{k-1} \cdot c_{i-k,j} \Bigg) \cdot [L_e],
\end{aligned}
$$

and we conclude that

$$
\begin{aligned}
\sum_{k=1}^{i} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] &= \sum_{k=1}^{i+j-(n+2)} (-1)^{k-1} \cdot [\Delta_{x(i-k,j)}] \\
&\qquad + (-1)^{i+j-n-2} \cdot \big([\Delta_{x(n+1-j,j)}] - [\Delta_{x(n-j,j)}] \big) \\
&\qquad + (-1)^{i+j-n} \cdot \sum_{k=1}^{n-j} (-1)^{k-1} \cdot [\Delta_{x(n-j-k,j)}] \\
&= [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] + [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] \\
&\qquad + \Bigg(\sum_{k=1}^{i} (-1)^{k-1} \cdot c_{i-k,j} \Bigg) \cdot [L_e].
\end{aligned}
$$

Analogously (or by taking duals), we see that

$$
\begin{aligned}
\sum_{k=1}^{j-1} (-1)^{k-1} \cdot [\Delta_{y(i,j-k)}] &= [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] + [L_{x(n-j,n-i-1)}] + [L_{x(n-j,n-i-2)}] \\
&\qquad + \Bigg(\sum_{k=1}^{j-1} (-1)^{k-1} \cdot c'_{i,j-k} \Bigg) \cdot [L_e]
\end{aligned}
$$

(recall that $c'_{i,j-k} = [\Delta_{y(i,j-k)} : L_e]$), and as

$$[\Delta_{x(n-j,n-i-1)}] = [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] + [L_{x(n-j,n-i-2)}] + [L_{x(n-j-1,n-i-2)}] + \delta_{j,i+1} \cdot [L_e]$$

by Lemma 3.5, it follows that

$$\begin{aligned}
\mathrm{JSF}^{\lambda}_{x(i,j)} - d_{i,j} \cdot [L_e] = {} & [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] + [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] \\
& + [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] + [L_{x(n-j,n-i-1)}] + [L_{x(n-j,n-i-2)}] \\
& + [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] \\
& + [L_{x(n-j,n-i-2)}] + [L_{x(n-j-1,n-i-2)}] \\
= {} & [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + [L_{x(n-j-1,n-i-2)}] + 3 \cdot [L_{x(n-j,n-i-1)}] \\
& + 2 \cdot \big([L_{x(i-1,j-1)}] + [L_{x(n-j-1,n-i-1)}] + [L_{x(n-j,n-i-2)}] \big).
\end{aligned}$$

As before, we can use Proposition I.6.11 to see that the simple $\mathbf{G}$-modules $L_{x(i-1,j-1)}$, $L_{x(n-j-1,n-i-1)}$, $L_{x(n-j,n-i-2)}$ and $L_{x(n-j,n-i-1)}$ appear with multiplicity one in a composition series of $\Delta_{x(i,j)}$, and it remains to prove that $c_{i,j} = 0$ if $n > 2j$ or $n > 2i+2$.

First assume that $n > 2j$, and observe that

$$d_{i,j} = (-1)^{i+j} + \delta_{j,i+1} + \sum_{k=1}^{i} (-1)^{k-1} \cdot c_{i-k,j} + \sum_{k=1}^{j-1} (-1)^{k-1} \cdot c'_{i,j-k}.$$

By induction, we have $c_{i-k,j} = 0$ and $c'_{i,j-k} = c_{i,j-k} = 0$ for all $k > 0$ with $i + j - k \geq n + 2$. Furthermore, Lemmas 3.8 and 3.10 imply that $c_{n-j+1,j} = c_{n-j,j} = 0$, and by equation (3.3) (see also Lemma 3.6), we have

$$\sum_{k=1}^{n-j} (-1)^{k-1} \cdot c_{n-j-k,j} = (-1)^{n-1}.$$

As $i + j \geq n + 2$, the assumption that $n > 2j$ implies that $i \geq j + 3$ and $n \leq 2i - 5$, and it is straightforward to see that $c'_{i,n-i+1} = c_{i,n-i+1} = 0$ and $c'_{i,n-i} = c_{i,n-i} = 0$. Finally, by Lemma 3.5, we have $c'_{i,j-k} = c_{i+1,j-1-k} = 0$ for $i + j - n < k \leq j - 1$ because $i + 1 > j - 1 - k$, and we conclude that

$$d_{i,j} = (-1)^{i+j} + (-1)^{i+j-n} \cdot (-1)^{n-1} = 0,$$

whence $c_{i,j} = 0$. Similarly (or by taking duals), we see that $c_{i,j} = 0$ when $n > 2i + 2$, as claimed. □

Finally, we turn to the Weyl modules $\Delta_{x(i,n)}$ and $\Delta_{x(n-1,j)}$ for $i \geq 2$ and $j \geq 3$.

Lemma 3.14. *For $2 \leq i < n-1$ and $3 \leq j < n$, we have*

$$\begin{aligned}
[\Delta_{x(n-1,j)}] = {} & [L_{x(n-1,j)}] + [L_{x(n-1,j-1)}] + [L_{x(n-2,j)}] + [L_{x(n-2,j-1)}] \\
& + [L_{x(n-j,0)}] + [L_{x(n-j-1,0)}] + c_{n-1,j} \cdot [L_e],
\end{aligned}$$

with $c_{n-1,j} = 0$ if $n > 2j$, and

$$\begin{aligned}
[\Delta_{x(i,n)}] = {} & [L_{x(i,n)}] + [L_{x(i,n-1)}] + [L_{x(i-1,n)}] + [L_{x(i-1,n-1)}] \\
& + [L_{x(0,n-i-1)}] + [L_{x(0,n-i-2)}] + c_{i,n} \cdot [L_e],
\end{aligned}$$

with $c_{i,n} = 0$ if $n > 2i + 2$. Furthermore, we have

$$\begin{aligned}
[\Delta_{x(n-1,n)}] = {} & [L_{x(n-1,n)}] + [L_{x(n-1,n-1)}] + [L_{x(n-2,n)}] + [L_{x(n-2,n-1)}] \\
& + [L_{x(0,0)}] + c_{n-1,n} \cdot [L_e].
\end{aligned}$$

Proof. We prove the first character formula by induction on j. By Proposition 3.7, we have

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(n-1,j)} = (-1)^{n-1+j}\cdot[\Delta_e] + [\Delta_{x(n-j,0)}]&\\ +\sum_{k=1}^{n-1}(-1)^{k-1}\cdot[\Delta_{x(n-1-k,j)}] + \sum_{k=1}^{j-1}(-1)^{k-1}\cdot[\Delta_{y(n-1,j-k)}],&\end{aligned}$$

and as in the proof of Lemma 3.13, we see that

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^{k-1}\cdot[\Delta_{x(n-1-k,j)}] = [L_{x(n-2,j)}]+[L_{x(n-2,j-1)}]+[L_{x(n-j,0)}]+[L_{x(n-j-1,0)}]&\\ +\left(\sum_{k=1}^{n-1}(-1)^{k-1}\cdot c_{n-1-k,j}\right)\cdot[L_e].&\end{aligned}$$

Furthermore, we have

$$\sum_{k=1}^{j-1}(-1)^{k-1}\cdot[\Delta_{y(n-1,j-k)}] = \sum_{k=1}^{j-3}(-1)^{k-1}\cdot[\Delta_{y(n-1,j-k)}] + (-1)^{j-3}\cdot\big([\Delta_{y(n-1,2)}]-[\Delta_{y(n-1,1)}]\big),$$

where

$$\begin{aligned}[\Delta_{y(n-1,2)}]-[\Delta_{y(n-1,1)}] &= [\Delta_{x(n-1,2)}]-[\Delta_{x(n-1,1)}]\\ &= [L_{x(n-1,2)}]+[L_{x(n-1,1)}]+[L_{x(n-2,2)}]+[L_{x(n-1,0)}]\\ &\qquad+[L_{x(n-2,1)}]+[L_{x(n-2,0)}]+[L_{x(n-3,0)}]\\ &\qquad-[L_{x(n-1,1)}]-[L_{x(n-1,0)}]-[L_{x(n-2,1)}]-[L_{x(n-2,0)}]\\ &= [L_{x(n-1,2)}]+[L_{x(n-2,2)}]+[L_{x(n-3,0)}]\end{aligned}$$

by Lemmas 3.9 and 3.11 (recall that $n > j \geq 3$), and

$$[\Delta_{x(n-j,0)}] = [L_{x(n-j,0)}]+[L_{x(n-j-1,0)}]$$

by Lemma 3.3. For $j = 3$, it follows that

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(n-1,3)} - d_{n-1,3}\cdot[L_e] &= [L_{x(n-2,3)}]+[L_{x(n-2,2)}]+[L_{x(n-3,0)}]+[L_{x(n-4,0)}]\\ &\qquad+[L_{x(n-1,2)}]+[L_{x(n-2,2)}]+[L_{x(n-3,0)}]\\ &\qquad+[L_{x(n-3,0)}]+[L_{x(n-4,0)}]\\ &= [L_{x(n-2,3)}]+[L_{x(n-1,2)}]\\ &\qquad+2\cdot\big([L_{x(n-2,2)}]+[L_{x(n-4,0)}]\big)+3\cdot[L_{x(n-3,0)}],\end{aligned}$$

and as before, we can use Proposition I.6.11 to see that the simple $\mathbf{G}$-modules $[L_{x(n-2,2)}]$, $[L_{x(n-4,0)}]$ and $[L_{x(n-3,0)}]$ appear with multiplicity one in a composition series of $\Delta_{x(n-1,3)}$. Furthermore, we can argue as in the proof of Lemma 3.13 to see that $d_{n-1,3} = c_{n-1,3} = 0$ if $n > 6$.

Now suppose that $j > 3$. Then induction yields

$$\sum_{k=1}^{j-3}(-1)^{k-1}\cdot[\Delta_{y(n-1,j-k)}] = \sum_{k=1}^{j-3}(-1)^{k-1}\cdot[\Delta_{x(n-1,j-k)}]$$

$$
\begin{aligned}
&= \sum_{k=1}^{j-3}(-1)^{k-1}\big([L_{x(n-1,j-k)}]+[L_{x(n-1,j-k-1)}]+[L_{x(n-2,j-k)}]+[L_{x(n-2,j-k-1)}]\\
&\qquad\qquad\qquad\qquad + [L_{x(n-j+k,0)}]+[L_{x(n-j+k-1,0)}]+c_{n-1,j-k}\cdot[L_e]\big)\\
&= [L_{x(n-1,j-1)}]+[L_{x(n-2,j-1)}]+[L_{x(n-j,0)}]\\
&\qquad\qquad + (-1)^{j-4}\cdot\big([L_{x(n-1,2)}]+[L_{x(n-2,2)}]+[L_{x(n-3,0)}]\big)\\
&\qquad\qquad + \Big(\sum_{k=1}^{j-3}(-1)^{k-1}\cdot c_{n-1,j-k}\Big)\cdot[L_e],
\end{aligned}
$$

and we conclude that

$$
\begin{aligned}
\sum_{k=1}^{j-1}(-1)^{k-1}\cdot[\Delta_{y(n-1,j-k)}] = [L_{x(n-1,j-1)}]+[L_{x(n-2,j-1)}]+[L_{x(n-j,0)}]&\\
+\Big(\sum_{k=1}^{j-1}(-1)^{k-1}\cdot c_{n-1,j-k}\Big)\cdot[L_e]&.
\end{aligned}
$$

As in the case $j=3$, we obtain

$$
\begin{aligned}
\mathrm{JSF}^{\lambda}_{x(n-1,j)} - d_{n-1,j}\cdot[L_e] &= [L_{x(n-2,j)}]+[L_{x(n-2,j-1)}]+[L_{x(n-j,0)}]+[L_{x(n-j-1,0)}]\\
&\qquad + [L_{x(n-1,j-1)}]+[L_{x(n-2,j-1)}]+[L_{x(n-j,0)}]\\
&\qquad + [L_{x(n-j,0)}]+[L_{x(n-j-1,0)}]\\
&= [L_{x(n-2,j)}]+[L_{x(n-1,j-1)}]\\
&\qquad + 2\cdot\big([L_{x(n-2,j-1)}]+[L_{x(n-j-1,0)}]\big)+3\cdot[L_{x(n-j,0)}],
\end{aligned}
$$

and we can use Proposition I.6.11 to see that each of the simple **G**-modules $L_{x(n-2,j-1)}$, $L_{x(n-j-1,0)}$ and $L_{x(n-j,0)}$ appears with multiplicity one in a composition series of $\Delta_{x(n-1,j)}$. Furthermore, we can argue as in the proof of Lemma 3.13 to see that $d_{n-1,j}=c_{n-1,j}=0$ if $n>2j$.

The second character formula follows from the first by taking duals (see Remark 3.1), so it remains to compute the composition multiplicities in $\Delta_{x(n-1,n)}$. Again by Proposition 3.7, we have

$$
\mathrm{JSF}^{\lambda}_{x(n-1,n)} = (-1)\cdot[\Delta_e]+[\Delta_{x(0,0)}]+\sum_{k=1}^{n-1}(-1)^{k-1}\cdot[\Delta_{x(n-k-1,n)}]+\sum_{k=1}^{n-1}(-1)^{k-1}\cdot[\Delta_{y(n-1,n-k)}],
$$

and as before, we see that

$$
\sum_{k=1}^{n-1}(-1)^{k-1}\cdot[\Delta_{y(n-1,n-k)}] = [L_{x(n-1,n-1)}]+[L_{x(n-2,n-1)}]+[L_{x(0,0)}]+\Big(\sum_{k=1}^{n-1}(-1)^{k-1}\cdot c_{n-1,n-k}\Big)\cdot[L_e]
$$

and (by dualizing)

$$
\sum_{k=1}^{n-1}(-1)^{k-1}\cdot[\Delta_{x(n-k-1,n)}] = [L_{x(n-2,n)}]+[L_{x(n-2,n-1)}]+[L_{x(0,0)}]+\Big(\sum_{k=1}^{n-1}(-1)^{k-1}\cdot c_{n-k-1,n}\Big)\cdot[L_e].
$$

As $[\Delta_{x(0,0)}]=[L_{x(0,0)}]+[L_e]$ by Lemma 3.3, we conclude that

$$
\begin{aligned}
\mathrm{JSF}^{\lambda}_{x(n-1,n)} - d_{n-1,n}\cdot[L_e] &= [L_{x(0,0)}]+[L_{x(n-1,n-1)}]+[L_{x(n-2,n-1)}]+[L_{x(0,0)}]\\
&\qquad + [L_{x(n-2,n)}]+[L_{x(n-2,n-1)}]+[L_{x(0,0)}]
\end{aligned}
$$

$$= [L_{x(n-1,n-1)}] + [L_{x(n-2,n)}] + 2 \cdot [L_{x(n-2,n-1)}] + 3 \cdot [L_{x(0,0)}],$$

and as before, we can use Proposition I.6.11 to see that the simple $\mathbf{G}$-modules $L_{x(n-2,n-1)}$ and $L_{x(0,0)}$ appear with multiplicity one in a composition series of $\Delta_{x(n-1,n)}$. $\square$

4 Loewy structure of Weyl modules

The *socle filtration* of a $\mathbf{G}$-module M is defined inductively by $\mathrm{soc}^0_{\mathbf{G}} M = 0$ and

$$\mathrm{soc}^k_{\mathbf{G}} M / \mathrm{soc}^{k-1}_{\mathbf{G}} M = \mathrm{soc}_{\mathbf{G}} \big(M / \mathrm{soc}^{k-1}_{\mathbf{G}} M \big)$$

for $k > 0$. Analogously, the *radical filtration* is defined by $\mathrm{rad}^0_{\mathbf{G}} M = M$ and

$$\mathrm{rad}^k_{\mathbf{G}} M = \mathrm{rad}_{\mathbf{G}} \big(\mathrm{rad}^{k-1}_{\mathbf{G}} M \big)$$

for $k > 0$. By construction, the successive quotients (called *layers*) of both the socle and the radical filtration are completely reducible, and if $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ is a filtration of M by $\mathbf{G}$-submodules such that the successive quotients M_i/M_{i-1} are completely reducible then

$$M_i \subseteq \mathrm{soc}^i_{\mathbf{G}} M \qquad \text{and} \qquad \mathrm{rad}^i_{\mathbf{G}} M \subseteq M_{r-i}$$

for $i = 0, \ldots, r$. Consequently, we have

$$m \coloneqq \min \big\{ k \geq 0 \bigm| \mathrm{soc}^k_{\mathbf{G}} M = M \big\} = \min \big\{ k \geq 0 \bigm| \mathrm{rad}^k_{\mathbf{G}} M = 0 \big\}$$

and $\mathrm{rad}^{m-i}_{\mathbf{G}} M \subseteq \mathrm{soc}^i_{\mathbf{G}} M$ for $i = 0, \ldots, r$. The integer m is called the *Loewy length* of M, and M is called *rigid* if $\mathrm{rad}^{m-i}_{\mathbf{G}} M = \mathrm{soc}^i_{\mathbf{G}} M$ for $i = 0, \ldots, m$. The data of socle and radical filtration is loosely referred to as the *Loewy structure* of M.

In this section, we use the results about composition series and Jantzen filtrations from the previous section to examine the Loewy structure of the Weyl modules Δ_x for $x \in \mathfrak{X}$. We keep the notation and assumptions from Sections 2 and 3. In the cases where the multiplicities of all composition factors of the Weyl module Δ_x can be computed, it turns out that Δ_x is rigid, with socle filtration and radical filtration equal to the Jantzen filtration. When $x = x(i,j)$ for $0 \leq i < n$ and $0 \leq j \leq n$ with $i + j > n$ then there is some ambiguity about the multiplicity of the simple $\mathbf{G}$-module L_e in a composition series of $\Delta_{x(i,j)}$ (see Lemma 3.13), and it is unclear if the layers of the Jantzen filtration are completely reducible. Nevertheless, we can compute the socle of $\Delta_{x(i,j)}$ using translation arguments.

Generalities

We start with some general observations about the Jantzen filtration and complete reducibility. First, we give a sufficient condition for the complete reducibility of a contravariantly self-dual $\mathbf{G}$-module. (Recall that a $\mathbf{G}$-module M is called contravariantly self-dual if $M^\tau \cong M$; see Section I.4.)

Definition 4.1. A $\mathbf{G}$-module M is called *multiplicity free* if $[M : L(\lambda)] \leq 1$ for all $\lambda \in X^+$.

A special case of the following lemma was already given as Lemma 4.12 in [Gru21]; the proof is essentially the same.

Lemma 4.2. *Let M be a contravariantly self-dual $\mathbf{G}$-module. If M is multiplicity free then M is completely reducible.*

Proof. Suppose that M is not completely reducible, so $\mathrm{rad}_{\mathbf{G}}M \neq 0$ and $\mathrm{soc}_{\mathbf{G}}(\mathrm{rad}_{\mathbf{G}}M) \neq 0$. As M is contravariantly self-dual, we have $\mathrm{soc}_{\mathbf{G}}M \cong M/\mathrm{rad}_{\mathbf{G}}M$ and therefore

$$0 \neq \mathrm{soc}_{\mathbf{G}}(\mathrm{rad}_{\mathbf{G}}M) \subseteq \mathrm{soc}_{\mathbf{G}}M \cong M/\mathrm{rad}_{\mathbf{G}}M.$$

It follows that $\mathrm{rad}_{\mathbf{G}}M$ and $M/\mathrm{rad}_{\mathbf{G}}M$ have a composition factor in common; hence M is not multiplicity free. □

Recall that we fix a weight $\lambda \in C_{\mathrm{fund}} \cap X$ and write $\Delta_x = \Delta(x \cdot \lambda)$ and $L_x = L(x \cdot \lambda)$ for $x \in W_{\mathrm{ext}}^+$. For $k > 0$, we further denote by $\Delta_{x,k} \coloneqq \Delta_x^k/\Delta_x^{k+1}$ the k-th layer of the Jantzen filtration of Δ_x. By Remark 1 in Section II.8.19 in [Jan03], the $\mathbf{G}$-modules $\Delta_{x,k}$ are contravariantly self-dual for $x \in W_{\mathrm{ext}}^+$ and $k > 0$. Therefore, the following corollary is an immediate consequence of Lemma 4.2.

Corollary 4.3. *Let $x \in W_{\mathrm{ext}}^+$ and $k > 0$ such that $\Delta_{x,k}$ is multiplicity-free. Then $\Delta_{x,k}$ is completely reducible.*

Whenever we are able to determine all composition multiplicities of the Weyl module Δ_x, for $x \in \mathfrak{X}$ as in the previous section, this module is in fact multiplicity free, and it follows that all layers $\Delta_{x,k}$ of the Jantzen filtration of Δ_x are completely reducible. Furthermore, we can use the following lemma to determine these layers precisely.

Lemma 4.4. *Let $x \in W_{\mathrm{aff}}^+$ such that Δ_x is multiplicity free and write*

$$\mathrm{JSF}_x^\lambda = \sum_{y \in W_{\mathrm{aff}}^+} a_y \cdot [L_y].$$

Then the layers of the Jantzen filtration of Δ_x are given by

$$\Delta_{x,k} \cong \bigoplus_{\substack{y \in W_{\mathrm{aff}}^+ \\ a_y = k}} L_y$$

for $k > 0$.

Proof. Recall from Section 1 that we have

$$\mathrm{JSF}_x^\lambda = \sum_{k>0} [\Delta_x^k] = \sum_{k>0} k \cdot [\Delta_{x,k}].$$

As Δ_x is multiplicity free, so are the layers $\Delta_{x,k}$ for $k > 0$, and furthermore, no two layers have a composition factor in common. By linear independence of the classes $[L_y]$, for $y \in W_{\mathrm{aff}}^+$, in the Grothendieck group $[\mathrm{Rep}(\mathbf{G})]$, we conclude that

$$[\Delta_{x,k}] = \sum_{\substack{y \in W_{\mathrm{aff}}^+ \\ a_y = k}} [L_y].$$

Now the claim follows from Corollary 4.3. □

The preceding Lemma shows that the layers of the Jantzen filtration are completely reducible and can be uniquely determined from the Jantzen sum formula for every multiplicity free Weyl module. In order to show that the Jantzen filtration coincides with the socle filtration and the radical filtration for the Weyl modules Δ_x, for certain (but not all) $x \in \mathfrak{X}$, we use translation arguments, as will be explained in the following.

Recall from Proposition I.6.8 that, for $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ and $x \in W_{\mathrm{aff}}^+$, we have $T_\lambda^\mu \Delta_x \neq 0$ if and only if $x \cdot \mu \in X^+$ and $T_\lambda^\mu L_x \neq 0$ if and only if $x \cdot \mu$ belongs to the upper closure of $x \cdot C_{\mathrm{fund}}$ (because the ℓ-alcove $x \cdot C_{\mathrm{fund}}$ is the unique ℓ-facet containing $x \cdot \lambda$). We now reformulate the conditions for the non-vanishing of $T_\lambda^\mu \Delta_x$ and $T_\lambda^\mu L_x$ purely in terms of x and $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$.

Lemma 4.5. *For $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ and $x \in W_{\mathrm{aff}}^+$, we have $T_\lambda^\mu \Delta_x \neq 0$ if and only if $x\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) \subseteq W_{\mathrm{aff}}^+$. In that case, we further have $T_\lambda^\mu \Delta_x \cong \Delta(x \cdot \mu)$.*

Proof. By Proposition I.6.8 (see also the discussion above), it suffices to prove that $x \cdot \mu \in X^+$ if and only if $x\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) \subseteq W_{\mathrm{aff}}^+$. An analogous statement was proven in Lemma IV.1.22, for the standard action of W_{aff} on $X_{\mathbb{R}}$ rather than the ℓ-dilated dot action, and we can essentially copy the proof of that lemma.

First suppose that $x \cdot \mu \in X^+$ and let $w \in \mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$. Then $x \cdot \mu = xw \cdot \mu \in xw \cdot \overline{C}_{\mathrm{fund}}$, so

$$0 < (x \cdot \mu + \rho, \alpha^\vee) \leq \big(n_\alpha(xw \cdot C_{\mathrm{fund}}) + 1\big) \cdot \ell$$

for all $\alpha \in \Phi^+$, and it follows that $n_\alpha(xw \cdot C_{\mathrm{fund}}) \geq 0$ and $xw \in W_{\mathrm{aff}}^+$.

Now suppose that $x\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) \subseteq W_{\mathrm{aff}}^+$ and let $\alpha \in \Pi$. Then $x \cdot (x^{-1}s_\alpha x) = s_\alpha x \notin W_{\mathrm{aff}}^+$ as $x \in W_{\mathrm{aff}}^+$, so $x^{-1}s_\alpha x \notin \mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$ and $s_\alpha \notin \mathrm{Stab}_{W_{\mathrm{aff}}}(x \cdot \mu)$. This implies that $(x \cdot \mu, \alpha^\vee) \neq -1$, and as

$$0 \leq n_\alpha(x \cdot C_{\mathrm{fund}}) \cdot \ell \leq (x \cdot \mu + \rho, \alpha^\vee) = (x \cdot \mu, \alpha^\vee) + 1,$$

we conclude that $(x \cdot \mu, \alpha^\vee) \geq 0$ and $x \cdot \mu \in X^+$, as required. □

Remark 4.6. Let $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ and $x \in W_{\mathrm{aff}}$ such that $x \cdot \mu \in X^+$. Then $x\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) \subseteq W_{\mathrm{aff}}^+$ (see the proof of Lemma 4.5), and by Proposition I.6.6, the $\mathbf{G}$-module $T_\mu^\lambda \Delta(x \cdot \mu)$ has a Weyl filtration with subquotients the Weyl modules Δ_y with $y \in x\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$, each occurring precisely once. Writing

$$x\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{y_1, \ldots, y_r\}$$

with $i < j$ whenever $y_j \cdot C_{\mathrm{fund}} \uparrow_\ell y_i \cdot C_{\mathrm{fund}}$, we can choose a Weyl filtration

$$0 = M_0 \subseteq \cdots \subseteq M_r = T_\mu^\lambda \Delta(x \cdot \mu)$$

with $M_i/M_{i-1} \cong \Delta_{y_i}$ for $i = 0, \ldots, r$, as explained in Section I.5. In particular there exists an embedding $\Delta_{y_1} \to T_\mu^\lambda \Delta(x \cdot \mu)$.

Lemma 4.7. *For $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ and $x \in W_{\mathrm{aff}}^+$, we have $T_\lambda^\mu L_x \neq 0$ if and only if $xs \in W_{\mathrm{aff}}^+$ and $x < xs$ for all $s \in S \cap \mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$. In that case, we further have $T_\lambda^\mu L_x \cong L(x \cdot \mu)$.*

Proof. By Proposition I.6.8 (see also the discussion above Lemma 4.5), it suffices to prove that the weight $x \cdot \mu$ belongs to the upper closure of $x \cdot C_{\mathrm{fund}}$ if and only if $xs \in W_{\mathrm{aff}}^+$ and $x < xs$ for all simple reflections $s \in S \cap \mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$. Recall from Lemma IV.1.24 that the weight $x \cdot \mu$ belongs to the upper closure of $x \cdot C_{\mathrm{fund}}$ if and only if $x(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}})$ for all $s \in S \cap \mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$. As $x \in W_{\mathrm{aff}}^+$, we have $xs(A_{\mathrm{fund}}) \uparrow x(A_{\mathrm{fund}})$ for all $s \in S$ with $xs \notin W_{\mathrm{aff}}^+$, and for $xs \in W_{\mathrm{aff}}^+$, we have $x < xs$ if and only if $x(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}})$ by Theorem IV.1.20. Combining these observations, we see that $x(A_{\mathrm{fund}}) \uparrow xs(A_{\mathrm{fund}})$ if and only if $xs \in W_{\mathrm{aff}}^+$ and $x < xs$, for all $s \in S$, and the claim follows. □

Now we are ready to employ translation functors in order to study the Loewy structure of Weyl modules.

Lemma 4.8. *Let $x, y \in W_{\mathrm{aff}}^+$ and suppose that L_x is isomorphic to a submodule of Δ_y. For any $s \in S$ such that $ys < y$, we have $xs \in W_{\mathrm{aff}}^+$ and $x < xs$.*

Proof. Let $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s\}$. By Corollary I.2.13, we have $ys \in W_{\mathrm{aff}}^+$ and therefore $y\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{y, ys\} \subseteq W_{\mathrm{aff}}^+$, and it follows that $y \cdot \mu \in X^+$ (see the proof of Lemma 4.5). As $ys < y$, we have $ys \cdot C_{\mathrm{fund}} \uparrow_\ell y \cdot C_{\mathrm{fund}}$ by Theorem IV.1.20, and using Remark 4.6, we see that there is an embedding $\Delta_y \to T_\mu^\lambda \Delta(y \cdot \mu)$. By assumption, we have $\mathrm{Hom}_{\mathbf{G}}(L_x, \Delta_y) \neq 0$ and therefore

$$0 \neq \mathrm{Hom}_{\mathbf{G}}\big(L_x, T_\mu^\lambda \Delta(y \cdot \mu)\big) \cong \mathrm{Hom}_G\big(T_\lambda^\mu L_x, \Delta(y \cdot \mu)\big).$$

It follows that $T_\lambda^\mu L_x \neq 0$, and Lemma 4.7 implies that $xs \in W_{\mathrm{aff}}^+$ and $x < xs$. $\square$

Lemma 4.9. *Let $x \in W_{\mathrm{aff}}^+$ and let $0 = M_0 \subseteq \cdots \subseteq M_r = \Delta_x$ be a filtration such that the successive quotients M_i/M_{i-1} are completely reducible. Further, let $s \in S$ such that $xs \in W_{\mathrm{aff}}^+$, and let $j \in \mathbb{Z}_{>0}$ be maximal with the property that M_j/M_{j-1} has a composition factor L_y with $ys \in W_{\mathrm{aff}}^+$ and $y < ys$. Then one of the simple $\mathbf{G}$-modules L_x and L_{xs} is a composition factor of M_j/M_{j-1}, and if $w \in W_{\mathrm{aff}}^+$ with $ws \in W_{\mathrm{aff}}^+$ and $w < ws$ such that L_w is a composition factor of M_j/M_{j-1} then $w \in \{x, xs\}$.*

Proof. Fix $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ with $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{e, s\}$. We first observe that j as in the statement of the lemma exists: The assumption that $\{x, xs\} \subseteq W_{\mathrm{aff}}^+$ implies that

$$T_\lambda^\mu \Delta_x \cong \Delta(x \cdot \mu) \neq 0$$

by Lemma 4.5, and it follows that $T_\lambda^\mu(M_i/M_{i-1}) \neq 0$ for some $i \in \{1, \ldots, r\}$. Then M_i/M_{i-1} has a composition factor L_y with $T_\lambda^\mu L_y \neq 0$, and using Lemma 4.7, we conclude that $ys \in W_{\mathrm{aff}}^+$ and $y < ys$.

The submodules $T_\lambda^\mu M_i$ of $T_\lambda^\mu \Delta_x$ afford a filtration whose successive quotients are completely reducible, and again by Lemma 4.7, j is maximal with the property that

$$T_\lambda^\mu M_j / T_\lambda^\mu M_{j-1} \cong T_\lambda^\mu(M_j/M_{j-1}) \neq 0.$$

Hence $T_\lambda^\mu(M_j/M_{j-1})$ is a non-zero and completely reducible quotient of $T_\lambda^\mu \Delta_x \cong \Delta(x \cdot \mu)$, and we conclude that $T_\lambda^\mu(M_j/M_{j-1}) \cong L(x \cdot \mu)$. Now let $w \in W_{\mathrm{aff}}^+$ with $ws \in W_{\mathrm{aff}}^+$ and $w < ws$, and suppose that L_w is a composition factor of M_j/M_{j-1}. By Lemma 4.7, we have $T_\lambda^\mu L_w \cong L(w \cdot \mu)$, and it follows that $L(w \cdot \mu)$ is a composition factor of $T_\lambda^\mu(M_j/M_{j-1}) \cong L(x \cdot \mu)$. This implies that $x \cdot \mu = w \cdot \mu$ and therefore $w \in x\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{x, xs\}$, as claimed. $\square$

Now we are ready to determine the Loewy structures (or in some cases, only the socles) of the Weyl modules Δ_x for $x \in \mathfrak{X}$. We do this over the course of the four following subsections, distinguishing the four cases $i + j < n$, $i + j = n$, $i + j = n + 1$ and $i + j > n + 1$.

The case $i + j < n$

Let $0 \leq i < n$ and $0 \leq j < n$ such that $i + j < n$. By Remark 3.4, the Weyl modules $\Delta_{x(i,0)}$, $\Delta_{x(0,j)}$ and $\Delta_{x(0,0)}$ are uniserial, and their Loewy structure can be depicted in the following diagrams:

$$\Delta_{x(i,0)} = \frac{L_{x(i,0)}}{L_{x(i-1,0)}} \qquad \Delta_{x(0,j)} = \frac{L_{x(0,j)}}{L_{x(0,j-1)}} \qquad \Delta_{x(0,0)} = \frac{L_{x(0,0)}}{L_e} \tag{4.1}$$

Now suppose that $i > 0$ and $j > 0$. By Lemma 3.5 and its proof, we have

$$[\Delta_{x(i,j)}] = [L_{x(i,j)}] + [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] + \delta_{i,j} \cdot [L_e]$$

and

$$\mathrm{JSF}^{\lambda}_{x(i,j)} = [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + 2 \cdot [L_{x(i-1,j-1)}] + \delta_{i,j} \cdot [L_e],$$

and if $i \neq j$ then Lemma 4.4 implies that

$$\Delta_{x(i,j),1} \cong L_{x(i-1,j)} \oplus L_{x(i,j-1)} \qquad \text{and} \qquad \Delta_{x(i,j),2} \cong L_{x(i-1,j-1)},$$

so $\Delta_{x(i,j)}$ has Loewy length at most 3. Furthermore, we have

$$x(i,j)s_i = x(i-1,j) < x(i,j) \qquad \text{and} \qquad x(i,j-1)s_i = x(i-1,j-1) < x(i,j-1),$$

whence $L_{x(i,j-1)}$ is not contained in $\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)}$ by Lemma 4.8. Analogously, we see that $L_{x(i-1,j)}$ is not contained in $\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)}$, and it follows that $\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)} = \Delta^2_{x(i,j)} \cong L_{x(i-1,j-1)}$. As

$$\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)} = \Delta^1_{x(i,j)} \supsetneq \Delta^2_{x(i,j)} = \mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)},$$

the submodule $\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)}$ of $\Delta_{x(i,j)}$ is not completely reducible, and we conclude that $\Delta_{x(i,j)}$ has Loewy length 3. Now the fact that

$$0 \neq \mathrm{rad}^2_{\mathbf{G}}\Delta_{x(i,j)} \subseteq \mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)}$$

forces that $\mathrm{rad}^2_{\mathbf{G}}\Delta_{x(i,j)} = \mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)}$, and as $\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)}$ is the unique maximal submodule of $\Delta_{x(i,j)}$ and

$$\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)} \subseteq \mathrm{soc}^2_{\mathbf{G}}\Delta_{x(i,j)} \subsetneq \Delta_{x(i,j)},$$

we further have $\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)} = \mathrm{soc}^2_{\mathbf{G}}\Delta_{x(i,j)}$. We conclude that $\Delta_{x(i,j)}$ is rigid.

If $i = j$ then Lemma 4.4 yields

$$\Delta_{x(i,i),1} \cong L_{x(i-1,i)} \oplus L_{x(i,i-1)} \oplus L_e \qquad \text{and} \qquad \Delta_{x(i,i),2} \cong L_{x(i-1,i-1)},$$

and as before, the simple $\mathbf{G}$-modules $L_{x(i-1,i)}$ and $L_{x(i,i-1)}$ do not belong to $\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,i)}$. Furthermore, we have $x(i,i)s_i = x(i-1,i) < x(i,i)$ and $s_i \notin W^+_{\mathrm{aff}}$, whence L_e does not belong to $\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,i)}$ by Lemma 4.8, and we conclude that $\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,i)} \cong L_{x(i-1,i-1)}$. Arguing as in the case $i \neq j$, we see that the Weyl module $\Delta_{x(i,i)}$ is rigid of Loewy length 3 and that the socle filtration and the radical filtration both coincide with the Jantzen filtration.

We depict the Loewy structure of the Weyl module $\Delta_{x(i,j)}$ (for $i,j > 0$) in the following diagrams, for $i \neq j$ on the left and for $i = j$ on the right.

$$\Delta_{x(i,j)} = \begin{array}{c} L_{x(i,j)} \\ \hline L_{x(i-1,j)} \oplus L_{x(i,j-1)} \\ \hline L_{x(i-1,j-1)} \end{array} \qquad\qquad \Delta_{x(i,i)} = \begin{array}{c} L_{x(i,i)} \\ \hline L_{x(i-1,i)} \oplus L_e \oplus L_{x(i,i-1)} \\ \hline L_{x(i-1,i-1)} \end{array} \tag{4.2}$$

The case $i+j=n$

Let $0 \leq i < n$ and $0 \leq j \leq n$ such that $i+j=n$. If $j=1$ then

$$[\Delta_{x(n-1,1)}] = [L_{x(n-1,1)}] + [L_{x(n-1,0)}] + [L_{x(n-2,1)}] + [L_{x(n-2,0)}]$$

and

$$\mathrm{JSF}^{\lambda}_{x(n-1,1)} = [L_{x(n-1,0)}] + [L_{x(n-2,1)}] + 2 \cdot [L_{x(n-2,0)}],$$

by Lemma 3.9 and its proof, so Lemma 4.4 yields

$$\Delta_{x(n-1,1),1} \cong L_{x(n-1,0)} \oplus L_{x(n-2,1)} \qquad \text{and} \qquad \Delta_{x(n-1,1),2} \cong L_{x(n-2,0)}.$$

As in the previous subsection, we can use Lemma 4.8 to see that the simple $\mathbf{G}$-module $L_{x(n-2,1)}$ does not belong to $\mathrm{soc}_{\mathbf{G}} \Delta_{x(n-1,1)}$. Furthermore, as

$$x(n-1,1)s_n = x(n-1,0) < x(n-1,1) \qquad \text{and} \qquad x(n-2,0)s_n = x(n-2,1) > x(n-2,0),$$

Lemma 4.9 implies that the simple $\mathbf{G}$-modules $L_{x(n-1,0)}$ and $L_{x(n-2,0)}$ cannot belong to the same socle layer of $\Delta_{x(n-1,1)}$. We conclude that $\mathrm{soc}_{\mathbf{G}} \Delta_{x(n-1,1)} \cong L_{x(n-2,0)}$, and arguing as in the previous subsection, it follows that $\Delta_{x(n-1,1)}$ is rigid of Loewy length 3, with socle filtration and radical filtration both equal to the Jantzen filtration. Analogously, if $i=0$ then

$$\Delta_{x(0,n),1} \cong L_{x(0,n-1)} \oplus L_{x(1,n-2)} \qquad \text{and} \qquad \Delta_{x(0,n),2} \cong L_{x(0,n-2)},$$

and $\Delta_{x(0,n)}$ is rigid of Loewy length 3, with socle filtration and radical filtration equal to the Jantzen filtration. Below, we depict the Loewy structure of the Weyl modules $\Delta_{x(n-1,1)}$ and $\Delta_{x(0,n)}$.

$$(4.3) \qquad \Delta_{x(n-1,1)} = \begin{array}{c} L_{x(n-1,1)} \\ \hline L_{x(n-1,0)} \oplus L_{x(n-2,1)} \\ \hline L_{x(n-2,0)} \end{array} \qquad\qquad \Delta_{x(0,n)} = \begin{array}{c} L_{x(0,n)} \\ \hline L_{x(0,n-1)} \oplus L_{x(1,n-2)} \\ \hline L_{x(0,n-2)} \end{array}$$

Now suppose that $i>0$ and $j>1$. Then

$$\begin{aligned}[\Delta_{x(i,j)}] = [L_{x(i,j)}] &+ [L_{x(i+1,j-2)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j)}] \\ &+ [L_{x(i-1,j-1)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-2)}] + (\delta_{i,j} + \delta_{i,j-2}) \cdot [L_e]\end{aligned}$$

and

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)} = [L_{x(i+1,j-2)}] &+ [L_{x(i,j-1)}] + [L_{x(i-1,j)}] + [L_{x(i-1,j-2)}] \\ &+ 2 \cdot [L_{x(i-1,j-1)}] + 2 \cdot [L_{x(i,j-2)}] + (\delta_{i,j} + \delta_{i,j-2}) \cdot [L_e],\end{aligned}$$

by Lemma 3.8 and its proof. If $j \notin \{i, i+2\}$ then Lemma 4.4 yields

$$\begin{aligned}\Delta_{x(i,j),1} &\cong L_{x(i+1,j-2)} \oplus L_{x(i,j-1)} \oplus L_{x(i-1,j)} \oplus L_{x(i-1,j-2)}, \\ \Delta_{x(i,j),2} &\cong L_{x(i-1,j-1)} \oplus L_{x(i,j-2)};\end{aligned}$$

in particular, $\Delta_{x(i,j)}$ has Loewy length at most 3 and $\Delta^2_{x(i,j)} \subseteq \mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)}$. As before, we can use Lemma 4.8 to see that none of the simple $\mathbf{G}$-modules $L_{x(i+1,j-2)}$, $L_{x(i-1,j)}$ and $L_{x(i-1,j-2)}$ belongs to the socle of $\Delta_{x(i,j)}$; hence

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} \subsetneq \Delta^1_{x(i,j)} = \mathrm{rad}_{\mathbf{G}} \Delta_{x(i,j)},$$

and it follows that $\Delta_{x(i,j)}$ has Loewy length 3. This implies that

$$\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)} \subseteq \mathrm{soc}^2_{\mathbf{G}}\Delta_{x(i,j)} \subsetneq \Delta_{x(i,j)},$$

and as $\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)}$ is the unique maximal submodule of $\Delta_{x(i,j)}$, we obtain $\mathrm{rad}_{\mathbf{G}}\Delta_{x(i,j)} = \mathrm{soc}^2_{\mathbf{G}}\Delta_{x(i,j)}$. Finally, we have

$$\begin{aligned} x(i,j)s_{i+1} &= x(i,j-1) < x(i,j), \\ x(i,j-2)s_{i+1} &= x(i+1,j-2) > x(i,j-2), \\ x(i-1,j-1)s_{i+1} &= x(i-1,j) > x(i-1,j-1) \end{aligned}$$

by Remark 2.5, and Lemma 4.9 implies that $L_{x(i,j-1)}$ cannot belong to the same radical layer or socle layer as either of the simple $\mathbf{G}$-modules $L_{x(i-1,j-1)}$ or $L_{x(i,j-2)}$. We conclude that

$$\mathrm{rad}^2_{\mathbf{G}}\Delta_{x(i,j)} = \mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)} = \Delta^2_{x(i,j)};$$

hence $\Delta_{x(i,j)}$ is rigid, and the Loewy structure is as follows:

$$\Delta_{x(i,j)} = \begin{array}{c} L_{x(i,j)} \\ \hline L_{x(i+1,j-2)} \oplus L_{x(i,j-1)} \oplus L_{x(i-1,j-2)} \oplus L_{x(i-1,j)} \\ \hline L_{x(i,j-2)} \oplus L_{x(i-1,j-1)} \end{array} \tag{4.4}$$

If $j \in \{i, i+2\}$ then

$$\begin{aligned} \Delta_{x(i,j),1} &\cong L_{x(i+1,j-2)} \oplus L_{x(i,j-1)} \oplus L_{x(i-1,j)} \oplus L_{x(i-1,j-2)} \oplus L_e, \\ \Delta_{x(i,j),2} &\cong L_{x(i-1,j-1)} \oplus L_{x(i,j-2)}, \end{aligned}$$

and arguing as above, we see that $\Delta_{x(i,j)}$ is rigid of Loewy length 3. Both the socle filtration and the radical filtration coincide with the Jantzen filtration, and the Loewy structure is displayed below, for the cases $j = i$ and $j = i+2$, respectively.

$$\begin{aligned} \Delta_{x(i,i)} &= \begin{array}{c} L_{x(i,i)} \\ \hline L_{x(i+1,i-2)} \oplus L_{x(i,i-1)} \oplus L_e \oplus L_{x(i-1,i-2)} \oplus L_{x(i-1,i)} \\ \hline L_{x(i,i-2)} \oplus L_{x(i-1,i-1)} \end{array} \\ \Delta_{x(i,i+2)} &= \begin{array}{c} L_{x(i,i+2)} \\ \hline L_{x(i+1,i)} \oplus L_{x(i,i+1)} \oplus L_e \oplus L_{x(i-1,i)} \oplus L_{x(i-1,i+2)} \\ \hline L_{x(i,i)} \oplus L_{x(i-1,i+1)} \end{array} \end{aligned} \tag{4.5}$$

The case $i + j = n + 1$

Let $0 \leq i < n$ and $0 \leq j \leq n$ such that $i + j = n + 1$ and suppose first that $i \geq 2$ and $j \geq 3$. Then, by Lemma 3.10 and its proof, we have

$$\begin{aligned} [\Delta_{x(i,j)}] = [L_{x(i,j)}] + [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j-1)}] + [L_{x(i,j-2)}] + [L_{x(i-1,j-2)}] \\ + [L_{x(i-2,j-2)}] + [L_{x(i-1,j-3)}] + [L_{x(i-2,j-3)}] + c_{i,j} \cdot [L_e] \end{aligned}$$

and

$$\begin{aligned} \mathrm{JSF}^{\lambda}_{x(i,j)} = [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + 2 \cdot [L_{x(i-1,j-1)}] + 2 \cdot [L_{x(i,j-2)}] + 3 \cdot [L_{x(i-1,j-2)}] \\ + 2 \cdot [L_{x(i-2,j-2)}] + 2 \cdot [L_{x(i-1,j-3)}] + [L_{x(i-2,j-3)}] + d_{i,j} \cdot [L_e], \end{aligned}$$

where $c_{i,j} = d_{i,j} = 0$ unless $i \leq j \leq i+2$. For $i \leq j \leq i+2$, we have $c_{i,j}, d_{i,j} \in \{1,2\}$ and $c_{i,j} = d_{i,j} = 1$ for $j \in \{i, i+2\}$. If $j < i$ or $j > i+2$ then Lemma 4.4 yields

$$\begin{aligned}
&\Delta_{x(i,j),1} \cong L_{x(i-1,j)} \oplus L_{x(i,j-1)} \oplus L_{x(i-2,j-3)},\\
&\Delta_{x(i,j),2} \cong L_{x(i-1,j-1)} \oplus L_{x(i,j-2)} \oplus L_{x(i-2,j-2)} \oplus L_{x(i-1,j-3)},\\
&\Delta_{x(i,j),3} \cong L_{x(i-1,j-2)},
\end{aligned}$$

so $\Delta_{x(i,j)}$ has Loewy length at most 4. As

$$x(i,j)s_i = x(i,j-1) < x(i,j) \qquad \text{and} \qquad x(i,j)s_{i+1} = x(i-1,j) < x(i,j),$$

any element $y \in W_{\mathrm{aff}}^+$ such that L_y is isomorphic to a submodule of $\Delta_{x(i,j)}$ satisfies $y < ys_i \in W_{\mathrm{aff}}^+$ and $y < ys_{i+1} \in W_{\mathrm{aff}}^+$ by Lemma 4.8, and using Lemma 2.3 and Remark 2.5, it follows that

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} = \Delta_{x(i,j)}^3 \cong L_{x(i-1,j-2)}.$$

Furthermore, we have

$$x(i,j-1)s_i = x(i,j) > x(i,j-1) \qquad \text{and} \qquad x(i-1,j-1)s_i = x(i-1,j) > x(i-1,j-1),$$

and both of the simple $\mathbf{G}$-modules $L_{x(i,j-1)}$ and $L_{x(i-1,j-1)}$ are composition factors of

$$\mathrm{rad}_{\mathbf{G}} \Delta_{x(i,j)} / \mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} = \Delta_{x(i,j)}^1 / \Delta_{x(i,j)}^3.$$

Now Lemma 4.9 implies that $\mathrm{rad}_{\mathbf{G}} \Delta_{x(i,j)} / \mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)}$ is not completely reducible, and we conclude that $\Delta_{x(i,j)}$ has Loewy length 4. As before, we have $\mathrm{soc}_{\mathbf{G}}^3 \Delta_{x(i,j)} = \mathrm{rad}_{\mathbf{G}} \Delta_{x(i,j)}$ because $\mathrm{rad}_{\mathbf{G}} \Delta_{x(i,j)}$ is the unique maximal submodule of $\Delta_{x(i,j)}$, and

$$0 \neq \mathrm{rad}_{\mathbf{G}}^3 \Delta_{x(i,j)} \subseteq \mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} \cong L_{x(i-1,j-2)},$$

whence $\mathrm{rad}_{\mathbf{G}}^3 \Delta_{x(i,j)} = \mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)}$. Applying Lemma 4.9 again (with the reflections s_i and s_{i+1}, respectively), we see that neither of the simple $\mathbf{G}$-modules $L_{x(i-1,j-3)}$ and $L_{x(i-1,j-1)}$ can belong to the same radical layer as $L_{x(i,j-1)}$ and that neither of the simple $\mathbf{G}$-modules $L_{x(i-2,j-2)}$ and $L_{x(i,j-2)}$ can belong to the same radical layer as $L_{x(i-1,j)}$; hence $\mathrm{rad}_G^2 \Delta(i,j) = \Delta_{x(i,j)}^2$. Analogously, neither of the simple $\mathbf{G}$-modules $L_{x(i-1,j)}$ and $L_{x(i,j-1)}$ can belong to the second socle layer of $\Delta_{x(i,j)}$. In order to show that $\Delta_{x(i,j)}$ is rigid, it remains to see that $L_{x(i-2,j-3)}$ belongs to the third socle layer. Suppose for a contradiction that $L_{x(i-2,j-3)}$ belongs to the second socle layer of $\Delta_{x(i,j)}$. Then there exists a non-split extension of $L_{x(i-2,j-3)}$ by $L_{x(i-1,j-2)} \cong \mathrm{soc}_{\mathbf{G}} \Delta(i,j)$. By Remark I.4.1, we have

$$\mathrm{Ext}_{\mathbf{G}}^i\big(\Delta_{x(i-1,j-2)}, L_{x(i-2,j-3)}\big) \cong \mathrm{Ext}_{\mathbf{G}}^i\big(L_{x(i-2,j-3)}, \nabla_{x(i-1,j-2)}\big) = 0$$

for all $i \geq 0$, and as all simple $\mathbf{G}$-modules are contravariantly self-dual, it follows that

$$\begin{aligned}
0 \neq \mathrm{Ext}_{\mathbf{G}}^1\big(L_{x(i-2,j-3)}, L_{x(i-1,j-2)}\big) &\cong \mathrm{Ext}_{\mathbf{G}}^1\big(L_{x(i-1,j-2)}, L_{x(i-2,j-3)}\big)\\
&\cong \mathrm{Hom}_{\mathbf{G}}\big(\mathrm{rad}_{\mathbf{G}} \Delta_{x(i-1,j-2)}, L_{x(i-2,j-3)}\big).
\end{aligned}$$

This contradicts the fact that the simple $\mathbf{G}$-module $L_{x(i-2,j-3)}$ belongs to the third radical layer of the Weyl module $\Delta_{x(i-1,j-2)}$; see the diagrams in (4.2). We conclude that $L_{x(i-2,j-3)}$ belongs to the third socle layer of $\Delta_{x(i,j)}$, so

$$\mathrm{soc}_{\mathbf{G}}^2 \Delta_{x(i,j)} = \mathrm{rad}_{\mathbf{G}}^2 \Delta_{x(i,j)} = \Delta_{x(i,j)}^2$$

and $\Delta_{x(i,j)}$ is rigid, with socle filtration and radical filtration equal to the Jantzen filtration. The Loewy structure can be depicted as follows:

$$\Delta_{x(i,j)} = \frac{\dfrac{L_{x(i,j)}}{L_{x(i,j-1)} \oplus L_{x(i-2,j-3)} \oplus L_{x(i-1,j)}}}{\dfrac{L_{x(i-1,j-3)} \oplus L_{x(i,j-2)} \oplus L_{x(i-1,j-1)} \oplus L_{x(i-2,j-2)}}{L_{x(i-1,j-2)}}} \tag{4.6}$$

If $j \in \{i, i+2\}$ then

$$\begin{aligned}
\Delta_{x(i,j),1} &\cong L_{x(i-1,j)} \oplus L_{x(i,j-1)} \oplus L_{x(i-2,j-3)} \oplus L_e, \\
\Delta_{x(i,j),2} &\cong L_{x(i-1,j-1)} \oplus L_{x(i,j-2)} \oplus L_{x(i-2,j-2)} \oplus L_{x(i-1,j-3)}, \\
\Delta_{x(i,j),3} &\cong L_{x(i-1,j-2)},
\end{aligned}$$

and arguing as before, we see that $\Delta_{x(i,j)}$ is rigid of Loewy length 4, with socle filtration and radical filtration both equal to the Jantzen filtration. Below, we depict the Loewy structure in the cases $j = i$ and $j = i + 2$, respectively.

$$\begin{gathered}
\Delta_{x(i,i)} = \frac{\dfrac{L_{x(i,i)}}{L_{x(i,i-1)} \oplus L_{x(i-2,i-3)} \oplus L_{x(i-1,i)} \oplus L_e}}{\dfrac{L_{x(i-1,i-3)} \oplus L_{x(i,i-2)} \oplus L_{x(i-1,i-1)} \oplus L_{x(i-2,i-2)}}{L_{x(i-1,i-2)}}} \\
\Delta_{x(i,i+2)} = \frac{\dfrac{L_{x(i,i+2)}}{L_{x(i,i+1)} \oplus L_{x(i-2,i-1)} \oplus L_{x(i-1,i+2)} \oplus L_e}}{\dfrac{L_{x(i-1,i-1)} \oplus L_{x(i,i)} \oplus L_{x(i-1,i+1)} \oplus L_{x(i-2,i)}}{L_{x(i-1,i)}}}
\end{gathered} \tag{4.7}$$

In the case $j = i+1$, we cannot determine the the Loewy structure of $\Delta_{x(i,j)}$ because we do not know the multiplicity of L_e in a composition series of $\Delta_{x(i,i+1)}$. Nevertheless, we can use Lemma 4.8, as before, to show that

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,i+1)} \cong L_{x(i-1,i-1)}. \tag{4.8}$$

Now suppose that $j = 2$, so

$$\begin{aligned}
[\Delta_{x(n-1,2)}] = [L_{x(n-1,2)}] &+ [L_{x(n-1,1)}] + [L_{x(n-2,2)}] + [L_{x(n-1,0)}] + [L_{x(n-2,1)}] \\
&+ [L_{x(n-2,0)}] + [L_{x(n-3,0)}] + \delta_{n,3} \cdot [L_e]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{JSF}^{\lambda}_{x(n-1,2)} = [L_{x(n-1,1)}] &+ [L_{x(n-2,2)}] + 2 \cdot [L_{x(n-1,0)}] + 2 \cdot [L_{x(n-2,1)}] \\
&+ 2 \cdot [L_{x(n-3,0)}] + 3 \cdot [L_{x(n-2,0)}] + \delta_{n,3} \cdot [L_e]
\end{aligned}$$

by Lemma 3.11 and its proof. If $n \neq 3$ then Lemma 4.4 yields

$$\begin{aligned}
\Delta_{x(n-1,2),1} &\cong L_{x(n-1,1)} \oplus L_{x(n-2,2)}, \\
\Delta_{x(n-1,2),2} &\cong L_{x(n-1,0)} \oplus L_{x(n-2,1)} \oplus L_{x(n-3,0)}, \\
\Delta_{x(n-1,2),3} &\cong L_{x(n-2,0)},
\end{aligned}$$

and using the same arguments as before, we see that $\Delta_{x(n-1,2)}$ is rigid of Loewy length 4 and that the socle filtration and the radical filtration both coincide with the Jantzen filtration. Analogously, we see that the layers of the Jantzen filtration of $\Delta_{x(1,n)}$ (for $n \neq 3$) are given by

$$\begin{aligned}\Delta_{x(1,n),1} &\cong L_{x(0,n)} \oplus L_{x(1,n-1)},\\ \Delta_{x(1,n),2} &\cong L_{x(0,n-1)} \oplus L_{x(1,n-2)} \oplus L_{x(0,n-3)},\\ \Delta_{x(1,n),3} &\cong L_{x(0,n-2)}\end{aligned}$$

and that $\Delta_{x(1,n)}$ is rigid of Loewy length 4, with socle filtration and radical filtration equal to the Jantzen filtration. The Loewy structure of the Weyl modules $\Delta_{x(n-1,2)}$ and $\Delta_{x(1,n)}$ is given below.

$$\begin{aligned}\Delta_{x(n-1,2)} &= \begin{array}{c} L_{x(n-1,2)} \\ \hline L_{x(n-1,1)} \oplus L_{x(n-2,2)} \\ \hline L_{x(n-1,0)} \oplus L_{x(n-3,0)} \oplus L_{x(n-2,1)} \\ \hline L_{x(n-2,0)} \end{array}\\ \Delta_{x(1,n)} &= \begin{array}{c} L_{x(1,n)} \\ \hline L_{x(1,n-1)} \oplus L_{x(0,n)} \\ \hline L_{x(0,n-1)} \oplus L_{x(0,n-3)} \oplus L_{x(1,n-2)} \\ \hline L_{x(0,n-2)} \end{array}\end{aligned} \tag{4.9}$$

If $n = 3$ then the Loewy structure of the Weyl modules $\Delta_{x(2,2)}$ and $\Delta_{x(1,3)}$ is as depicted above, but with an additional composition factor L_e in the second radical layer.

The case $i + j > n + 1$

Let $0 \leq i < n$ and $0 \leq j \leq n$ such that $i + j > n + 1$. First suppose that $i < n - 1$ and $j < n$, so that

$$\begin{aligned}[\Delta_{x(i,j)}] = [L_{x(i,j)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}]\\ + [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] + [L_{x(n-j,n-i-2)}] + [L_{x(n-j-1,n-i-2)}]\\ + c_{i,j} \cdot [L_e]\end{aligned}$$

and

$$\begin{aligned}\mathrm{JSF}^{\lambda}_{x(i,j)} = [L_{x(i-1,j)}] + [L_{x(i,j-1)}] + [L_{x(n-j-1,n-i-2)}]\\ + 2 \cdot [L_{x(i-1,j-1)}] + 2 \cdot [L_{x(n-j-1,n-i-1)}] + 2 \cdot [L_{x(n-j,n-i-2)}]\\ + 3 \cdot [L_{x(n-j,n-i-1)}] + d_{i,j} \cdot [L_e]\end{aligned}$$

by Lemma 3.13 and its proof. If $n > 2j$ or $n > 2i + 2$ then $c_{i,j} = d_{i,j} = 0$ and Lemma 4.4 yields

$$\begin{aligned}\Delta_{x(i,j),1} &\cong L_{x(i-1,j)} \oplus L_{x(i,j-1)} \oplus L_{x(n-j-1,n-i-2)},\\ \Delta_{x(i,j),2} &\cong L_{x(i-1,j-1)} \oplus L_{x(n-j-1,n-i-1)} \oplus L_{x(n-j,n-i-2)},\\ \Delta_{x(i,j),3} &\cong L_{x(n-j,n-i-1)}.\end{aligned}$$

As before, we can use Lemma 4.8 to see that none of the simple $\mathbf{G}$-modules

$$L_{x(i-1,j)}, \qquad L_{x(i,j-1)}, \qquad L_{x(n-j-1,n-i-2)}, \qquad L_{x(n-j-1,n-i-1)} \qquad \text{and} \qquad L_{x(n-j,n-i-2)}$$

belong to the socle of $\Delta_{x(i,j)}$. However, Lemma 4.8 does not rule out the possibility that $L_{x(i-1,j-1)}$ belongs to the socle of $\Delta_{x(i,j)}$. We will give a more subtle argument to show that

$$\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)} \cong L_{x(n-j,n-i-1)},$$

even when $n \leq 2j$ and $n \leq 2i+2$, and also when $i = n-1$ or $j = n$. Our strategy is to use translation arguments as in the proofs of Lemmas 4.8 and 4.9, but contrary to the proofs of these lemmas, we will use translation functors T_λ^μ and T_μ^λ for weights $\mu \in \overline{C}_{\mathrm{fund}} \cap X$ such that $\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$ is generated by two simple reflections (rather than just one).

Proposition 4.10. *Let $0 \leq i < n$ and $0 \leq j \leq n$ such that $i + j > n+1$. Then*

$$\mathrm{soc}_{\mathbf{G}}\Delta_{x(i,j)} \cong L_{x(n-j,n-i-1)}.$$

Proof. Recall from Lemmas 3.13 and 3.14 that

$$\begin{aligned}[\Delta_{x(i,j)}] = [L_{x(i,j)}] + [L_{x(i,j-1)}] + [L_{x(i-1,j)}] + [L_{x(i-1,j-1)}] \\ + [L_{x(n-j,n-i-1)}] + [L_{x(n-j-1,n-i-1)}] + [L_{x(n-j,n-i-2)}] + [L_{x(n-j-1,n-i-2)}] \\ + c_{i,j} \cdot [L_e],\end{aligned}$$

with the convention that $[L_{x(a,b)}] = 0$ if $a < 0$ or $b < 0$. Using Lemma 4.8, it is straightforward to see that $L_{x(i-1,j-1)}$ and $L_{x(n-j,n-i-1)}$ are the only simple $\mathbf{G}$-modules that could appear in the socle of $\Delta_{x(i,j)}$. Therefore, it suffices to prove that $L_{x(i-1,j-1)}$ does not appear in the socle of $\Delta_{x(i,j)}$.

Consider the weight $\mu := -\varpi_{i+1} - \varpi_{n-j+1} \in \overline{C}_{\mathrm{fund}} \cap X$ and observe that

$$\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \langle s_{i+1}, s_{n-j+1} \rangle = \{e, s_{i+1}, s_{n-j+1}, s_{i+1}s_{n-j+1}\}$$

by Lemma IV.1.21 and Example I.2.6. Using Lemma 2.3, it is straightforward to see that

$$x(i,j)\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu) = \{x(i,j), x(i-1,j), x(i,j-1), x(i-1,j-1)\} \subseteq W_{\mathrm{aff}}^+,$$

and Lemma 4.5 implies that $0 \neq T_\lambda^\mu \Delta_{x(i,j)} \cong \Delta\big(x(i,j) \cdot \mu\big)$. Now T_λ^μ is exact and takes simple $\mathbf{G}$-modules in $\mathrm{Rep}_\lambda(\mathbf{G})$ to simple $\mathbf{G}$-modules in $\mathrm{Rep}_\mu(\mathbf{G})$ or to zero (see Lemma 4.7), so we obtain a composition series of $\Delta\big(x(i,j)\cdot\mu\big)$ by applying T_λ^μ to all composition factors of $\Delta_{x(i,j)}$ (and forgetting about those simple $\mathbf{G}$-modules which are mapped to zero). For $y \in W_{\mathrm{aff}}^+$, we have $T_\lambda^\mu L_y \neq 0$ if and only if $y < ys_{i+1} \in W_{\mathrm{aff}}^+$ and $y < ys_{n-j+1} \in W_{\mathrm{aff}}^+$ (again by Lemma 4.7), and using Lemma 2.3 and Remark 2.5, it is straightforward to see that $L_{x(i-1,j-1)}$ and $L_{x(n-j,n-i-1)}$ are the only composition factors of $\Delta_{x(i,j)}$ that are not mapped to zero by T_λ^μ. We conclude that $\Delta\big(x(i,j)\cdot\mu\big)$ is uniserial of composition length 2, with

$$\mathrm{head}_{\mathbf{G}}\Delta\big(x(i,j)\cdot\mu\big) \cong T_\lambda^\mu L_{x(i-1,j-1)} \cong L\big(x(i-1,j-1)\cdot\mu\big) = L\big(x(i,j)\cdot\mu\big)$$

and

$$\mathrm{soc}_{\mathbf{G}}\Delta\big(x(i,j)\cdot\mu\big) \cong T_\lambda^\mu L_{x(n-j,n-i-1)} \cong L\big(x(n-j,n-i-1)\cdot\mu\big).$$

Now suppose for a contradiction that $L_{x(i-1,j-1)}$ appears in the socle of $\Delta_{x(i,j)}$. Then the simple $\mathbf{G}$-module $T_\lambda^\mu L_{x(i-1,j-1)} \cong L\big(x(i,j)\cdot\mu\big)$ appears in the socle of $T_\lambda^\mu\Delta_{x(i,j)} \cong \Delta\big(x(i,j)\cdot\mu\big)$; hence

$$x(i,j)\cdot\mu = x(n-j,n-i-1)\cdot\mu,$$

and we arrive at the contradiction $x(n-j, n-i-1) \in x(i,j)\mathrm{Stab}_{W_{\mathrm{aff}}}(\mu)$. Therefore, $L_{x(i-1,j-1)}$ does not appear in the socle of $\Delta_{x(i,j)}$, and we conclude that

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} \cong L_{x(n-j,n-i-1)},$$

as required. □

Now let us return to the Loewy structure of the Weyl module $\Delta_{x(i,j)}$ for $i < n-1$ and $j < n$ such that $i + j > n+1$ and either $n > 2j$ or $n > 2i+2$. Recall that by Lemma 4.4, we have

$$\begin{aligned}
\Delta_{x(i,j),1} &\cong L_{x(i-1,j)} \oplus L_{x(i,j-1)} \oplus L_{x(n-j-1,n-i-2)}, \\
\Delta_{x(i,j),2} &\cong L_{x(i-1,j-1)} \oplus L_{x(n-j-1,n-i-1)} \oplus L_{x(n-j,n-i-2)}, \\
\Delta_{x(i,j),3} &\cong L_{x(n-j,n-i-1)},
\end{aligned}$$

where $\Delta^3_{x(i,j)} = \mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)}$ by Proposition 4.10. Arguing as in the previous subsection, we see that the Weyl module $\Delta_{x(i,j)}$ is rigid of Loewy length 4, with socle filtration and radical filtration equal to the Jantzen filtration. The Loewy structure of $\Delta_{x(i,j)}$ is depicted in the following diagram.

$$\Delta_{x(i,j)} = \quad \begin{array}{c}
L_{x(i,j)} \\ \hline
L_{x(i-1,j)} \oplus L_{x(n-j-1,n-i-2)} \oplus L_{x(i,j-1)} \\ \hline
L_{x(n-j-1,n-i-1)} \oplus L_{x(i-1,j-1)} \oplus L_{x(n-j,n-i-2)} \\ \hline
L_{x(n-j,n-i-1)}
\end{array}$$

Now suppose that $2 \leq i < n-1$ and $3 \leq j < n$ with $n > 2i+2$ and $n > 2j$. As before, one can show that the Weyl modules $\Delta_{x(i,n)}$ and $\Delta_{x(n-1,j)}$ are rigid of Loewy length 4, and that their socle filtrations and radical filtrations coincide with the respective Jantzen filtrations. The Loewy structure of these Weyl modules is depicted below.

$$\Delta_{x(i,n)} = \quad \begin{array}{c}
L_{x(i,n)} \\ \hline
L_{x(i-1,n)} \oplus L_{x(i,n-1)} \\ \hline
L_{x(i-1,n-1)} \oplus L_{x(0,n-i-2)} \\ \hline
L_{x(0,n-i-1)}
\end{array} \qquad\qquad \Delta_{x(n-1,j)} = \quad \begin{array}{c}
L_{x(n-1,j)} \\ \hline
L_{x(n-2,j)} \oplus L_{x(n-1,j-1)} \\ \hline
L_{x(n-j-1,0)} \oplus L_{x(n-2,j-1)} \\ \hline
L_{x(n-j,0)}
\end{array}$$

We conclude this section by recalling the information we have obtained about the socles of the Weyl modules Δ_x for $x \in \mathfrak{X}$.

Remark 4.11. Let $0 \leq i < n$ and $0 \leq j \leq n$. For $j < n$, we have

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(0,0)} \cong L_e, \qquad \mathrm{soc}_{\mathbf{G}} \Delta_{x(i,0)} \cong L_{x(i-1,0)}, \qquad \mathrm{soc}_{\mathbf{G}} \Delta_{x(0,j)} \cong L_{x(0,j-1)}$$

by equation (4.1), and if $i > 0$, $j > 0$ and $i + j < n$ then

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} \cong L_{x(i-1,j-1)}$$

by equation (4.2). Furthermore, we have

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(n-1,1)} \cong L_{x(n-2,0)} \qquad \text{and} \qquad \mathrm{soc}_{\mathbf{G}} \Delta_{x(0,n)} \cong L_{x(0,n-2)}$$

by equation (4.3), and if $i > 0$, $j > 1$ and $i + j = n$ then

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} \cong L_{x(i-1,j-1)} \oplus L_{x(i,j-2)}$$

by equations (4.4) and (4.5). Finally, if $i + j = n + 1$ then

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} \cong L_{x(i-1,j-2)}$$

by equations (4.6), (4.7), (4.8) and (4.9), and if $i + j > n + 1$ then

$$\mathrm{soc}_{\mathbf{G}} \Delta_{x(i,j)} \cong L_{x(n-j,n-i-1)}$$

by Proposition 4.10. Observe that for $0 \leq a < n$ and $0 \leq b \leq n$ such that $L_{x(a,b)}$ is isomorphic to a submodule of $\Delta_{x(i,j)}$, we have $a + b \leq n - 2$, and Corollary II.2.7 and Remark 2.5 yield

$$\mathrm{gfd}\big(L_{x(a,b)}\big) = \ell\big(x(a,b)\big) = a + b + 1 \leq n - 1.$$

Hence the good filtration dimension of any simple $\mathbf{G}$-module in the socle of $\Delta_{x(i,j)}$ is at most $n - 1$.

5 Non-simplicity of generic direct summands

In this section, we apply the results from the previous sections to study tensor products of simple $\mathbf{G}$-modules. As before, we assume that $\mathbf{G}$ is of type A_n and that $\ell \geq h = n + 1$, and we adopt the notation and conventions from Section 2. In particular, we consider the set

$$\mathfrak{X} := \{x(i,j) \mid 0 \leq i < n,\, 0 \leq j \leq n\} \cup \{e\} \subseteq W_{\mathrm{aff}}^{+},$$

where for $0 \leq i < n$ and $0 \leq j \leq n$, we have

$$x(i,j) = s_0 s_1 \cdots s_i s_n s_{n-1} \cdots s_{n-j+1}.$$

Our aim is to show that, in the modular case, the generic direct summand $G\big(x(i,0), x(0,j)\big)$ of the tensor product $L\big(x(i,0) \cdot 0\big) \otimes L\big(x(0,j) \cdot 0\big)$ is non-simple, for all $0 \leq i < n - 1$ and $0 \leq j < n - 1$.

As in Chapter IV, we will fix an element $\omega \in \Omega = \mathrm{Stab}_{W_{\mathrm{ext}}}(A_{\mathrm{fund}})$ and make use of the fact that

$$G\big(x(i,0)\omega^{-1}, x(0,j)\omega\big) \cong G\big(x(i,0), x(0,j)\big);$$

see Lemma II.5.10. Then, in order to prove the non-simplicity of $G\big(x(i,0), x(0,j)\big)$, we will distinguish two cases:

When $i + j \geq n - 2$, we can use an embedding

$$L\big(x(i,0)\omega^{-1} \cdot 0\big) \otimes L\big(x(0,j)\omega \cdot 0\big) \longrightarrow \Delta\big(x(i+1,0)\omega^{-1} \cdot 0\big) \otimes \Delta\big(x(0,j+1)\omega \cdot 0\big)$$

and the information about the socles of the Weyl modules $\Delta(x \cdot 0)$ for $x \in \mathfrak{X}$, established in the previous section, to see that $L\big(x(i,0)\omega^{-1} \cdot 0\big) \otimes L\big(x(0,j)\omega \cdot 0\big)$ has no simple submodule belonging to the linkage class of 0 and having good filtration dimension $\ell\big(x(i,0)\big) + \ell\big(x(0,j)\big) = i + j + 2$.

When $i + j < n - 2$, we will use the distribution algebra of $\mathbf{G}$ to show that certain maximal vectors in the tensor product $L\big(x(i,0)\omega^{-1} \cdot 0\big) \otimes L\big(x(0,j)\omega \cdot 0\big)$ generate non-simple submodules, and the non-simplicity of $G\big(x(i,0), x(0,j)\big)$ will follow by weight considerations. This is the only part of the proof (of the aim formulated above and of the complete reducibility theorem on page 131) where it is necessary to assume that we are in the modular case. We believe that the complete reducibility theorem should still hold in the quantum case, but we were not able to find a proof that bypasses the direct computation of maximal vectors, which is feasible only in the modular case.

Before we start dealing with the two cases outlined above, let us fix some more notation. As in Section IV.3, we define

$$\omega := t_{\varpi_1} s_1 s_2 \cdots s_n \in \Omega,$$

so that $\omega \cdot 0 = (\ell - n - 1) \cdot \varpi_1$ (see Lemmas IV.3.3 and IV.3.4). Furthermore, for $x \in W_{\mathrm{ext}}^+$, we write

$$\Delta_x = \Delta(x \cdot 0) \qquad \text{and} \qquad L_x = L(x \cdot 0).$$

The highest weights of the simple $\mathbf{G}$-modules $L_{x(i,0)\omega^{-1}}$ and $L_{x(0,j)\omega}$, whose tensor product we want to study below, are made explicit in the following lemma:

Lemma 5.1. *Let $0 \leq i, j < n$. Then*

$$x(i,0)\omega^{-1} \cdot 0 = \varpi_{i+1} + (\ell - n + i) \cdot \varpi_n \qquad \textit{and} \qquad x(0,j)\omega \cdot 0 = (\ell - n + j) \cdot \varpi_1 + \varpi_{n-j}.$$

Proof. Recall from Lemma IV.3.4 that we have $\omega \cdot 0 = (\ell - n - 1) \cdot \varpi_1$. It is straightforward to see by induction on k that

$$s_{n-j+k} s_{n-j+k-1} \cdots s_{n-j+1} \omega \cdot 0 = (\ell - n - 1) \cdot \varpi_1 + \varpi_{n-j} - (k+1) \cdot \varpi_{n-j+k} + k \cdot \varpi_{n-j+k+1}$$

for $k = 1, \ldots, j$, and in particular

$$s_n s_{n-1} \cdots s_{n-j+1} \omega \cdot 0 = (\ell - n - 1) \cdot \varpi_1 + \varpi_{n-j} - (j+1) \cdot \varpi_n.$$

It follows that $\big(s_n s_{n-1} \cdots s_{n-j+1} \omega \cdot 0 + \rho, \alpha_{\mathrm{h}}^\vee \big) = \ell - (j+1)$ and

$$x(0,j)\omega \cdot 0 = s_0 s_n \cdots s_{n-j+1} \omega \cdot 0 = s_n \cdots s_{n-j+1} \omega \cdot 0 + (j+1) \cdot \alpha_{\mathrm{h}} = (\ell - n + j) \cdot \varpi_1 + \varpi_{n-j},$$

as claimed. Next observe that we have $x(i,0) = x(0,i)^*$ and

$$\omega^{-1} = s_n s_{n-1} \cdots s_1 t_{-\varpi_1} = t_{\varpi_n} s_n s_{n-1} \cdots s_1 = \omega^*;$$

see Remark 3.1. Using the preceding case and again Remark 3.1, we conclude that

$$\begin{aligned} x(i,0)\omega^{-1} \cdot 0 = x(0,i)^* \omega^* \cdot 0 &= -w_0\big(x(0,i)\omega \cdot 0 \big) \\ &= -w_0\big((\ell - n + i) \cdot \varpi_1 + \varpi_{n-i} \big) = \varpi_{i+1} + (\ell - n + i) \cdot \varpi_n, \end{aligned}$$

as required. □

The case $i + j \geq n - 2$

As explained in the introduction to this section, we want to prove that $G\big(x(i,0), x(0,j)\big)$ is non-simple, in the case $i + j \geq n - 2$, by first showing that $G\big(x(i,0), x(0,j)\big)$ can be embedded into the tensor product $\Delta_{x(i+1,0)\omega^{-1}} \otimes \Delta_{x(0,j+1)\omega}$ and then arguing that the latter tensor product does not have any simple submodule that belongs to $\mathrm{Rep}_0(\mathbf{G})$ and has good filtration dimension

$$\ell\big(x(i,0)\big) + \ell\big(x(0,j)\big) = \mathrm{gfd}\big(G\big(x(i,0), x(0,j)\big)\big).$$

For the second step, we will need the following lemma, which gives us some control over the highest weights of the Weyl modules appearing in a Weyl filtration of $\Delta_{x(i,0)\omega^{-1}} \otimes \Delta_{x(0,j)\omega}$.

Lemma 5.2. *Let $0 \leq i, j \leq n-1$ and let $\delta \in X^+$ such that $\Delta(\delta)$ appears as a subquotient in a Weyl filtration of $\Delta_{x(i,0)\omega^{-1}} \otimes \Delta_{x(0,j)\omega}$. Then $\delta \in x \cdot \overline{C}_{\mathrm{fund}}$ for some $x \in \mathfrak{X}$.*

Proof. Recall from Lemma 5.1 that we have

$$x(i,0)\omega^{-1} \cdot 0 = \varpi_{i+1} + (\ell - n + i) \cdot \varpi_n \qquad \text{and} \qquad x(0,j)\omega \cdot 0 = (\ell - n + j) \cdot \varpi_1 + \varpi_{n-j}.$$

It is straightforward to see by weight considerations that the Weyl modules $\Delta_{x(i,0)\omega^{-1}}$ and $\Delta_{x(0,j)\omega}$ appear as subquotients in Weyl filtrations of the tensor products

$$\Delta\big((\ell - n + i) \cdot \varpi_n\big) \otimes \Delta(\varpi_{i+1}) \qquad \text{and} \qquad \Delta\big((\ell - n + j) \cdot \varpi_1\big) \otimes \Delta(\varpi_{n-j}),$$

respectively, so $\Delta(\delta)$ appears as a subquotient in a Weyl filtration of

$$\Delta\big((\ell - n + i) \cdot \varpi_n\big) \otimes \Delta\big((\ell - n + j) \cdot \varpi_1\big) \otimes \Delta(\varpi_{i+1}) \otimes \Delta(\varpi_{n-j}).$$

We claim that the highest weights of the Weyl modules appearing in a Weyl filtration of the tensor product $\Delta\big((\ell - n + i) \cdot \varpi_n\big) \otimes \Delta\big((\ell - n + j) \cdot \varpi_1\big)$ are all of the form

$$\delta_k \coloneqq (\ell - n + j - k) \cdot \varpi_1 + (\ell - n + i - k) \cdot \varpi_n,$$

for $0 \leq k \leq \min\{\ell - n + i, \ell - n + j\}$. Indeed, as the characters of the Weyl modules form a basis of the character lattice $\mathbb{Z}[X]^{W_{\mathrm{fin}}}$, it suffices to show that the character of the tensor product above can be written as a linear combination of the characters of the Weyl modules with highest weights δ_k. This can easily be verified, using the well-known *Pieri rule*; see Proposition 15.25 in [FH91]. It follows that $\Delta(\delta)$ appears in a Weyl filtration of a tensor product of the form $\Delta(\delta_k) \otimes \Delta(\varpi_{i+1}) \otimes \Delta(\varpi_{n-j})$, and by Propositions I.6.6 and IV.2.5, we can write $\delta = \delta_k + \nu_1 + \nu_2$, where ν_1 and ν_2 are W_{fin}-conjugate to the minuscule weights ϖ_{i+1} and ϖ_{n-j}, respectively. In particular, the weight $\nu \coloneqq \nu_1 + \nu_2 = \delta - \delta_k$ satisfies $(\nu, \alpha^\vee) \leq 2$ for all $\alpha \in \Phi^+$. We conclude that

$$\begin{aligned}
(\delta + \rho, \alpha_{\mathrm{h}}^\vee) = (\delta_k, \alpha_{\mathrm{h}}^\vee) + (\nu, \alpha_{\mathrm{h}}^\vee) + (\rho, \alpha_{\mathrm{h}}^\vee) &\leq 2 \cdot (\ell - n - k) + i + j + 2 + n < 3\ell, \\
(\delta + \rho, \beta_{2,n-1}^\vee) = (\delta_k, \beta_{2,n-1}^\vee) + (\nu, \beta_{2,n-1}^\vee) + (\rho, \beta_{2,n-1}^\vee) &\leq 2 + n - 2 < \ell, \\
(\delta + \rho, \beta_{2,n}^\vee) = (\delta_k, \beta_{2,n}^\vee) + (\nu, \beta_{2,n}^\vee) + (\rho, \beta_{2,n}^\vee) &\leq \ell - n + i - k + 2 + n - 1 < 2\ell, \\
(\delta + \rho, \beta_{1,n-1}^\vee) = (\delta_k, \beta_{1,n-1}^\vee) + (\nu, \beta_{1,n-1}^\vee) + (\rho, \beta_{1,n-1}^\vee) &\leq \ell - n + j - k + 2 + n - 1 < 2\ell,
\end{aligned}$$

and the claim follows from Corollary 2.2. $\square$

Using the results about socles of Weyl modules from Section 4, we are now ready to prove the non-simplicity of $G\big(x(i,0), x(0,j)\big)$ for $i + j \geq n - 2$.

Proposition 5.3. *Let $0 \leq i < n-1$ and $0 \leq j < n-1$ such that $i + j \geq n - 2$. Then the generic direct summand $G\big(x(i,0), x(0,j)\big)$ of $L_{x(i,0)} \otimes L_{x(0,j)}$ is non-simple.*

Proof. First observe that we have $G\big(x(i,0), x(0,j)\big) \cong G\big(x(i,0)\omega^{-1}, x(0,j)\omega\big)$ by Lemma II.5.10. Furthermore, by Remark 4.11, we have

$$L_{x(i,0)\omega^{-1}} \cong \mathrm{soc}_{\mathbf{G}} \Delta_{x(i+1,0)\omega^{-1}} \qquad \text{and} \qquad L_{x(0,j)\omega} \cong \mathrm{soc}_{\mathbf{G}} \Delta_{x(0,j+1)\omega};$$

hence there exists an embedding

$$G\big(x(i,0), x(0,j)\big) \longrightarrow L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega} \longrightarrow \Delta_{x(i+1,0)\omega^{-1}} \otimes \Delta_{x(0,j+1)\omega}.$$

Now suppose for a contradiction that $G\big(x(i,0),x(0,j)\big)$ is simple, so that $G\big(x(i,0),x(0,j)\big) \cong L_y$ for some $y \in W_{\mathrm{aff}}^+$ with

$$\mathrm{gfd}(L_y) = \ell\big(x(i,0)\big) + \ell\big(x(0,j)\big) = i+j+2 \geq n;$$

see Proposition II.5.7. The tensor product $\Delta_{x(i+1,0)\omega^{-1}} \otimes \Delta_{x(0,j+1)\omega}$ has a Weyl filtration, and as

$$\mathrm{Hom}_{\mathbf{G}}(L_y, \Delta_{x(i+1,0)\omega^{-1}} \otimes \Delta_{x(0,j+1)\omega}) \neq 0,$$

there exists an element $x \in W_{\mathrm{aff}}^+$ such that $\mathrm{Hom}_{\mathbf{G}}(L_y,\Delta_x) \neq 0$ and the multiplicity of Δ_x in a Weyl filtration of $\Delta_{x(i+1,0)\omega^{-1}} \otimes \Delta_{x(0,j+1)\omega}$ is non-zero. By Lemma 5.2, we have either $x=e$ or $x=x(a,b)$ for some $0 \leq a < n$ and $0 \leq b \leq n$, and by Remark 4.11, every simple $\mathbf{G}$-module in the socle of Δ_x has good filtration dimension at most $n-1$. This contradicts the observation that $\mathrm{gfd}(L_y) \geq n$, and we conclude that $G\big(x(i,0),x(0,j)\big)$ is non-simple. $\square$

The case $i+j<n-2$

For the rest of this chapter, we make the following assumption:

$$\boxed{\text{Suppose that we are in the modular case.}}$$

When $i+j<n-2$, our proof of the non-simplicity of the generic direct summand $G\big(x(i,0),x(0,j)\big)$ involves an explicit computation of certain maximal vectors which generate non-simple submodules in the tensor product $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$.[2] By a detailed study of these maximal vectors and by weight considerations, we will be able show that $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$ does not have any simple submodule that belongs to $\mathrm{Rep}_0(\mathbf{G})$ and has good filtration dimension

$$\ell\big(x(i,0)\big) + \ell\big(x(0,j)\big) = \mathrm{gfd}\big(G\big(x(i,0),x(0,j)\big)\big).$$

We start with two basic lemmas about the highest weights of composition factors of $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$.

Lemma 5.4. *Let $i,j \geq 1$ with $i+j<n-2$ and let $0 \leq a < n$ and $0 \leq b \leq n$ such that*

$$x(a,b) \cdot 0 \leq x(i,0)\omega^{-1} \cdot 0 + x(0,j)\omega \cdot 0.$$

Then either $a+b \leq i+j$ or (a,b) is one of the pairs $(i+1,j)$ or $(i,j+1)$.

Proof. Let us set $\gamma \coloneqq x(i,0)\omega^{-1} \cdot 0 + x(0,j)\omega \cdot 0$ and observe that by Lemma 5.1, we have

$$\gamma = (\ell-n+j) \cdot \varpi_1 + \varpi_{i+1} + \varpi_{n-j} + (\ell-n+i) \cdot \varpi_n$$

and therefore

$$\big(x(a,b) \cdot 0 \,, \alpha_{\mathrm{h}}^\vee\big) \leq \big(\gamma \,, \alpha_{\mathrm{h}}^\vee\big) = 2 \cdot (\ell-n) + i + j + 2.$$

If $a+b \geq n$ then $x(a,b) \cdot 0 = (\ell-n-1+b) \cdot \varpi_1 + \varpi_{n-b+1} + \varpi_{a+1} + (\ell-n+a) \cdot \varpi_n$ by Lemma 2.1, and it follows that

$$\big(x(a,b) \cdot 0, \alpha_{\mathrm{h}}^\vee\big) = 2 \cdot (\ell-n) + a + b + 1 > 2 \cdot (\ell-n) + n > 2 \cdot (\ell-n) + i + j + 2,$$

a contradiction. Hence $a+b<n$ and

$$x(a,b) \cdot 0 = (\ell-n-1+b) \cdot \varpi_1 + \varpi_{a+1} + \varpi_{n-b} + (\ell-n-1+a) \cdot \varpi_n,$$

[2] This computation will be carried out using the distribution algebra of $\mathbf{G}$, which is the reason our argument does not work in the quantum case.

again by Lemma 2.1. Thus $2\cdot(\ell-n)+i+j+2 \geq \big(x(a,b)\cdot 0\,,\alpha_{\mathrm{h}}^\vee\big) = 2\cdot(\ell-n)+a+b$ and we conclude that $a+b \leq i+j+2$. Now let us write

$$\gamma - x(a,b)\cdot 0 = \sum_{k=1}^{n} c_k \cdot \alpha_k,$$

with $c_1,\ldots,c_n \in \mathbb{Z}_{\geq 0}$, so that

$$i+j+2-(a+b) = \big(\gamma - x(a,b)\cdot 0\,,\alpha_{\mathrm{h}}^\vee\big) = c_1 + c_n.$$

If $a+b = i+j+2$ then $c_1 = c_n = 0$ and therefore

$$\begin{aligned}\ell-n+i = (\gamma\,,\alpha_n^\vee) &= \big(x(a,b)\cdot 0\,,\alpha_n^\vee\big) + \sum_{k=2}^{n-1} c_k\cdot(\alpha_k\,,\alpha_n^\vee)\\ &= \big(x(a,b)\cdot 0\,,\alpha_n^\vee\big) - c_{n-1} \leq \big(x(a,b)\cdot 0\,,\alpha_n^\vee\big) = \ell-n-1+a,\end{aligned}$$

that is $a \geq i+1$, and analogously $b \geq j+1$. Now this forces that $a = i+1$ and $b = j+1$, so

$$\gamma - x(a,b)\cdot 0 = \varpi_{i+1} - \varpi_{i+2} - \varpi_{n-j-1} + \varpi_{n-j} = -\beta_{i+2,n-j-1},$$

contradicting the assumption that $x(a,b)\cdot 0 \leq \gamma$.

Next suppose that $a+b = i+j+1$, so that $c_1 + c_n = 1$. If $c_1 = 1$ and $c_n = 0$ then, as before, we obtain $a \geq i+1$ and

$$\begin{aligned}\ell-n+j = (\gamma\,,\alpha_1^\vee) &= \big(x(a,b)\cdot 0\,,\alpha_1^\vee\big) + \sum_{k=1}^{n-1} c_k\cdot(\alpha_k\,,\alpha_1^\vee)\\ &= \big(x(a,b)\cdot 0\,,\alpha_1^\vee\big) + 2 - c_2 \leq \big(x(a,b)\cdot 0\,,\alpha_1^\vee\big) + 2 = \ell-n+1+b,\end{aligned}$$

that is $b \geq j-1$. If $a = i+2$ and $b = j-1$ then

$$\gamma - x(a,b)\cdot 0 = 2\varpi_1 + \varpi_{i+1} - \varpi_{i+3} + \varpi_{n-j} - \varpi_{n-j+1} - \varpi_n = \beta_{1,i+1} + \beta_{1,i+2} - \beta_{n-j+1,n},$$

contradicting the assumption that $x(a,b)\cdot 0 \leq \gamma$, so we conclude that $a = i+1$ and $b = j$. Analogously, the case $c_1 = 0$ and $c_n = 1$ leads to $a = i$ and $b = j+1$, and the claim follows. ☐

Lemma 5.5. *Let $0 < i,j \leq n$ with $i+j < n-2$, and let $x \in W_{\mathrm{aff}}^+$ such that L_x is a composition factor of the tensor product $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$. If $x \neq e$ then $x = x(a,b)$, for $0 \leq a < n$ and $0 \leq b < n$ such that either $a+b \leq i+j$ or (a,b) is one of the pairs $(i+1,j)$ or $(i,j+1)$.*

Proof. As $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$ is isomorphic to a quotient of $\Delta_{x(i,0)\omega^{-1}} \otimes \Delta_{x(0,j)\omega}$, Lemma 5.2 implies that L_x is a composition factor of a Weyl module Δ_y with $y \in \mathfrak{X}$, and using the results about composition series of Weyl modules from Section 3, it follows that $x \in \mathfrak{X}$. Furthermore, we have

$$x\cdot 0 \leq x(i,0)\omega^{-1}\cdot 0 + x(0,j)\omega\cdot 0$$

and the claim follows from Lemma 5.4. ☐

Our key tool for establishing the non-simplicity of $G\big(x(i,0),x(0,j)\big)$ is the following proposition, which will be proven by an explicit computation of certain maximal vectors in $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$, using the distribution algebra $\mathrm{Dist}(\mathbf{G})$ of $\mathbf{G}$.

Proposition 5.6. *Let $i,j \geq 1$ such that $i+j < n-2$, and let $a,b \geq 1$ such that (a,b) is one of the pairs $(i+1,j)$ or $(i,j+1)$. Then*

$$\dim \mathrm{Hom}_{\mathbf{G}}\big(\Delta_{x(a,b)}, L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}\big) = 1,$$

and the restriction to $\mathrm{rad}_{\mathbf{G}}\Delta_{x(a,b)}$ of any non-zero homomorphism from $\Delta_{x(a,b)}$ to $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$ is non-zero.

For the sake of readability, we postpone the proof of Proposition 5.6 to the end of this section (see Proposition 5.17) and directly jump to the main result.

Proposition 5.7. *Let $i,j \geq 1$ such that $i+j < n-2$. Then $G\big(x(i,0),x(0,j)\big)$ is non-simple.*

Proof. Suppose for a contradiction that $G\big(x(i,0),x(0,j)\big)$ is simple, so $G\big(x(i,0),x(0,j)\big) \cong L_y$ for some $y \in W_{\mathrm{aff}}^{+}$ with

$$\ell(y) = \mathrm{gfd}(L_y) = \mathrm{gfd}\Big(G\big(x(i,0),x(0,j)\big)\Big) = \ell\big(x(i,0)\big) + \ell\big(x(0,j)\big) = i+j+2,$$

where the first equality follows from Corollary II.2.7. By Lemma II.5.10, we have

$$G\big(x(i,0),x(0,j)\big) \cong G\big(x(i,0)\omega^{-1},x(0,j)\omega\big),$$

and it follows that there is an embedding

$$L_y \cong G\big(x(i,0),x(0,j)\big) \longrightarrow L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}.$$

In particular, L_y is a composition factor of $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$, and by Lemma 5.5, we have $y = x(a,b)$ for some $a,b \geq 0$, where either $a+b \leq i+j$ or (a,b) is one of the pairs $(i+1,j)$ or $(i,j+1)$. As

$$i+j+2 = \ell(y) = \ell\big(x(a,b)\big) = a+b+1,$$

we conclude that (a,b) is one of the pairs $(i+1,j)$ or $(i,j+1)$.

Now let us write $M \coloneqq L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$, and observe that the short exact sequence

$$0 \longrightarrow \mathrm{rad}_{\mathbf{G}}\Delta_y \longrightarrow \Delta_y \longrightarrow L_y \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{G}}\big(L_y,M\big) \longrightarrow \mathrm{Hom}_{\mathbf{G}}\big(\Delta_y,M\big) \longrightarrow \mathrm{Hom}_{\mathbf{G}}\big(\mathrm{rad}_{\mathbf{G}}\Delta_y,M\big).$$

By Proposition 5.6, the middle term in this exact sequence is one-dimensional, and the map from the middle term to the rightmost term is non-zero. This implies that

$$\mathrm{Hom}_{\mathbf{G}}\big(L_y, L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}\big) = 0,$$

contradicting the observation that there is an embedding of L_y into $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$. $\square$

The remainder of this section is devoted to proving Proposition 5.6. As was mentioned before, we will do this by an explicit computation, using the distribution algebra $\mathrm{Dist}(\mathbf{G})$ of $\mathbf{G}$. Recall from Section I.4 that a choice of root homomorphisms $x_\beta \colon \mathbb{Z} \to \mathbf{U}_{\beta,\mathbb{Z}}$ for the group scheme $\mathbf{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ gives rise to a Chevalley basis $\{X_\beta, H_\alpha \mid \beta \in \Phi, \alpha \in \Pi\}$ of the complex simple Lie algebra $\mathfrak{g}$ with root system Φ and that this Chevalley basis affords divided powers $X_{\beta,r}$ and $H_{\alpha,m}$ in $\mathrm{Dist}(\mathbf{G})$, for $\beta \in \Phi$, $\alpha \in \Pi$ and $r, m \geq 0$. For a $\mathbf{G}$-module M and $v \in M_\lambda$ for some $\lambda \in X$, we have

$$X_{\beta,r} \cdot M_\lambda \subseteq M_{\lambda+r\beta} \qquad \text{and} \qquad H_{\alpha,m} \cdot v = \binom{(\lambda, \alpha^\vee)}{m} \cdot v$$

by Section II.1.19 in [Jan03]. Furthermore, $\mathrm{Dist}(\mathbf{G})$ admits a PBW-type basis, consisting of products of the form

$$\prod_{\beta \in -\Phi^+} X_{\beta,r_\beta} \cdot \prod_{\alpha \in \Pi} H_{\alpha,m_\alpha} \cdot \prod_{\beta \in \Phi^+} X_{\beta,r_\beta},$$

with $r_\beta, m_\alpha \in \mathbb{Z}_{\geq 0}$ for $\beta \in \Phi$ and $\alpha \in \Pi$, for any fixed ordering of the roots in the product. Since we assume $\mathbf{G}$ to be of type A_n, the canonical choice of root homomorphisms gives rise to the standard Chevalley basis of $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, which satisfies the commutator relations

$$[X_{\beta_{i,j}}, X_{\beta_{a,b}}] = \begin{cases} X_{\beta_{i,b}} & \text{if } a = j+1, \\ -X_{\beta_{a,j}} & \text{if } i = b+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$[X_{\beta_{i,j}}, X_{-\beta_{a,b}}] = \begin{cases} X_{\beta_{i,a-1}} & \text{if } j = b \text{ and } i < a, \\ X_{-\beta_{a,i-1}} & \text{if } j = b \text{ and } a < i, \\ -X_{\beta_{b+1,j}} & \text{if } i = a \text{ and } b < j, \\ -X_{-\beta_{j+1,b}} & \text{if } i = a \text{ and } j < b, \\ H_{\beta_{i,j}} & \text{if } i = a \text{ and } j = b, \\ 0 & \text{otherwise,} \end{cases}$$

$$[X_{-\beta_{i,j}}, X_{-\beta_{a,b}}] = \begin{cases} -X_{-\beta_{i,b}} & \text{if } a = j+1, \\ X_{-\beta_{a,j}} & \text{if } i = b+1, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq j \leq n$ and $1 \leq a \leq b \leq n$, where $H_{\beta_{i,j}} := H_{\alpha_i} + \cdots + H_{\alpha_j}$. For the remainder of the section, we fix this Chevalley basis and the corresponding divided powers in $\mathrm{Dist}(\mathbf{G})$.

Next, let us make some observations about *maximal vectors*.

Definition 5.8. A *maximal vector* of weight $\lambda \in X^+$ in a $\mathbf{G}$-module M is a non-zero vector $v^+ \in M_\lambda$ such that $X_{\beta,r} \cdot v^+ = 0$ for all $\beta \in \Phi^+$ and $r > 0$.

For $\lambda \in X^+$, it is straightforward to see that any non-zero vector in $L(\lambda)_\lambda$ is a maximal vector in $L(\lambda)$ and that any non-zero vector in $\Delta(\lambda)_\lambda$ is a maximal vector in $\Delta(\lambda)$. Furthermore, as $\Delta(\lambda)$ has a unique maximal submodule $\mathrm{rad}_{\mathbf{G}}\Delta(\lambda)$ and as the latter does not contain the weight space $\Delta(\lambda)_\lambda$, any maximal vector in $\Delta(\lambda)_\lambda$ generates $\Delta(\lambda)$ over $\mathrm{Dist}(\mathbf{G})$. Now let M be an arbitrary $\mathbf{G}$-module and suppose that there is a maximal vector $w^+ \in M_\lambda$ of weight λ. Using the PBW-type basis of $\mathrm{Dist}(\mathbf{G})$, one sees that λ is maximal among the weights of the submodule $M' := \mathrm{Dist}(\mathbf{G}) \cdot w^+$ of M generated by w^+ and that M' has simple head $\mathrm{head}_{\mathbf{G}} M' \cong L(\lambda)$. In particular, by Lemma I.4.2, there exists a homomorphism $\varphi\colon \Delta(\lambda) \to M$ such that, for some maximal vector $v^+ \in \Delta(\lambda)_\lambda$, we have $\varphi(v^+) = w^+$. Furthermore, φ is unique with this property because v^+ generates $\Delta(\lambda)$.

With the above notation and conventions in place, we can now start proving some preliminary results which will be needed for the proof of Proposition 5.6. We first compute bases for some specific weight spaces of certain Weyl modules in terms of their maximal vectors.

Lemma 5.9. *Let $a,b \geq 1$ and let $v^+ \in \Delta(a\varpi_1 + b\varpi_n)_{a\varpi_1+b\varpi_n}$ be a maximal vector. Then the set*

$$B \coloneqq \{X_{-\beta_{1,i}}X_{-\beta_{i+1,n}} \cdot v^+ \mid 1 \leq i < n\} \cup \{X_{-\beta_{1,n}} \cdot v^+\}$$

is a basis of the weight space $\Delta(a\varpi_1 + b\varpi_n)_{a\varpi_1+b\varpi_n-\beta_{1,n}}$.

Proof. By Weyl's character formula, we have $\dim \Delta(a\varpi_1+b\varpi_n)_{a\varpi_1+b\varpi_n-\beta_{1,n}} = n$, so it suffices to show that the weight space $\Delta(a\varpi_1+b\varpi_n)_{a\varpi_1+b\varpi_n-\beta_{1,n}}$ is spanned by B. Consider the total order on Φ^+ that is defined by $\beta_{i,j} \prec \beta_{i',j'}$ if and only if $i < i'$ or $i = i'$ and $j < j'$. As the Weyl module $\Delta(a\varpi_1 + b\varpi_n)$ is generated by the maximal vector v^+ over $\mathrm{Dist}(\mathbf{G})$, we can use the PBW-type basis of $\mathrm{Dist}(\mathbf{G})$ to see that the weight space $\Delta(a\varpi_1+b\varpi_n)_{a\varpi_1+b\varpi_n}$ is spanned by vectors of the form $w \coloneqq X_{-\gamma_1}^{(r_1)} \cdots X_{-\gamma_m}^{(r_m)} \cdot v^+$, with $\gamma_1 \prec \cdots \prec \gamma_m$ and $\sum_k r_k\gamma_k = \beta_{1,n}$. This implies that $r_1 = \cdots = r_m = 1$, and by the definition of $\prec$, there exist integers $0 = a_0 < a_1 < \cdots < a_m = n$ such that $\gamma_k = \beta_{a_{k-1}+1,a_k}$ for $k = 1,\ldots,m$. If $m \leq 2$ then w belongs to B, so now suppose that $m \geq 3$. For $1 < i \leq j < n$, we have $X_{-\beta_{i,j}} \cdot v^+ = 0$ by weight considerations and therefore

$$X_{-\beta_{i,j}}X_{-\beta_{j+1,n}} \cdot v^+ = [X_{-\beta_{i,j}}, X_{-\beta_{j+1,n}}] \cdot v^+ + X_{-\beta_{j+1,n}}X_{-\beta_{i,j}} \cdot v^+ = -X_{-\beta_{i,n}} \cdot v^+.$$

By induction on m, we see that $X_{-\gamma_2} \cdots X_{-\gamma_m} \cdot v^+$ is a scalar multiple of $X_{-\beta_{a_1+1,n}}$, so w is a scalar multiple of $X_{-\beta_{1,a_1}}X_{-\beta_{a_1+1,n}} \cdot v^+ \in B$. We conclude that B spans $\Delta(a\varpi_1 + b\varpi_n)_{a\varpi_1+b\varpi_n-\beta_{1,n}}$ and the claim follows. □

Corollary 5.10. *Let $a,b \geq 1$ and $1 < k < n$, and let $v^+ \in \Delta(a\varpi_k + b\varpi_n)_{a\varpi_k+b\varpi_n}$ be a maximal vector. Then the set*

$$B \coloneqq \{X_{-\beta_{1,i}}X_{-\beta_{i+1,n}} \cdot v^+ \mid k \leq i < n\} \cup \{X_{-\beta_{1,n}} \cdot v^+\}$$

is a basis of the weight space $\Delta(a\varpi_k + b\varpi_n)_{a\varpi_k+b\varpi_n-\beta_{1,n}}$.

Proof. First observe that we have $s_{\beta_{1,k-1}}(a\varpi_k + b\varpi_n - \beta_{1,n}) = a\varpi_k + b\varpi_n - \beta_{k,n}$. By truncation to the Levi subgroup corresponding to $\{\alpha_k,\ldots,\alpha_n\} \subseteq \Pi$, it is straightforward to see that

$$\dim \Delta(a\varpi_k + b\varpi_n)_{a\varpi_k+b\varpi_n-\beta_{1,n}} = \dim \Delta(a\varpi_k + b\varpi_n)_{a\varpi_k+b\varpi_n-\beta_{k,n}} = n - k + 1,$$

hence it suffices to prove that B spans the weight space $\Delta(a\varpi_k + b\varpi_n)_{a\varpi_k+b\varpi_n-\beta_{1,n}}$. As in the proof of Lemma 5.9, we see that $\Delta(a\varpi_k + b\varpi_n)_{a\varpi_k+b\varpi_n-\beta_{1,n}}$ is spanned by vectors of the form

$$w \coloneqq X_{-\gamma_1} \cdots X_{-\gamma_m} \cdot v^+,$$

where for certain integers $0 = a_0 < a_1 < \cdots < a_m = n$, we have $\gamma_i = \beta_{a_{i-1}+1,a_i}$ for $i = 1,\ldots,m$. We will show that w is a scalar multiple of an element of B.

If $m = 1$ then $w \in B$, so now suppose that $m \geq 2$. By weight considerations, we have $X_{-\beta_{i,j}} \cdot v^+ = 0$ for $k < i \leq j < n$, and it follows that

$$X_{-\beta_{i,j}}X_{-\beta_{j+1,n}} \cdot v^+ = [X_{-\beta_{i,j}}, X_{-\beta_{j+1,n}}] \cdot v^+ = -X_{-\beta_{i,n}} \cdot v^+.$$

Therefore, if $r \coloneqq \min\{i \mid a_i \geq k\} < m$ then $X_{-\gamma_{r+1}} \cdots X_{-\gamma_m} \cdot v^+$ is a scalar multiple of $X_{-\beta_{a_r+1,n}} \cdot v^+$. Furthermore, for $1 \leq i \leq j < k$, we have $X_{-\beta_{i,j}}X_{-\beta_{a_r+1,n}} \cdot v^+ = 0$ by weight considerations and therefore

$$X_{-\beta_{i,j}}X_{-\beta_{j+1,a_r}}X_{-\beta_{a_r+1,n}} \cdot v^+ = [X_{-\beta_{i,j}}X_{-\beta_{j+1,a_r}}]X_{-\beta_{a_r+1,n}} \cdot v^+ = -X_{-\beta_{i,a_r}}X_{-\beta_{a_r+1,n}} \cdot v^+.$$

It follows that $X_{-\gamma_1}\cdots X_{-\gamma_r}X_{-\beta_{a_r+1,n}}\cdot v^+$ is a scalar multiple of the vector $X_{-\beta_{1,a_r}}X_{-\beta_{a_r+1,n}}\cdot v^+\in B$, hence so is w. If $r=m$ then $a_{m-1}<k$, so

$$X_{-\beta_{i,j}}X_{-\beta_{j+1,n}}\cdot v^+=[X_{-\beta_{i,j}}X_{-\beta_{j+1,n}}]\cdot v^+=-X_{-\beta_{i,n}}\cdot v^+$$

for all $1\leq i\leq j\leq a_{m-1}$, and we conclude that w is a scalar multiple of the vector $X_{-\beta_{1,n}}\cdot v^+\in B$. $\square$

Next, we use the distribution algebra of $\mathbf{G}$ to explicitly construct a maximal vector in a tensor product of two Weyl modules.

Lemma 5.11. *Let $a,b>0$ and let $v^+\in\Delta(a\varpi_1)_{a\varpi_1}$ and $w^+\in\Delta(b\varpi_n)_{b\varpi_n}$ be maximal vectors. Then*

$$x:=a\cdot v^+\otimes X_{-\beta_{1,n}}w^+-b\cdot X_{-\beta_{1,n}}v^+\otimes w^++\sum_{k=1}^{n-1}X_{-\beta_{1,k}}v^+\otimes X_{-\beta_{k+1,n}}w^+$$

is a maximal vector of weight $a\varpi_1+b\varpi_n-\alpha_{\mathrm{h}}$ in $\Delta(a\varpi_1)\otimes\Delta(b\varpi_n)$.

Proof. Using the PBW-type basis of $\mathrm{Dist}(\mathbf{G})$ (as in the proof of Lemma 5.9), it is straightforward to see that the vectors $X_{-\beta_{1,1}}\cdot v^+$ and $X_{-\beta_{2,n}}\cdot w^+$ are non-zero, and as

$$\big(\Delta(a\varpi_1)\otimes\Delta(b\varpi_n)\big)_{a\varpi_1+b\varpi_n-\alpha_{\mathrm{h}}}=\bigoplus_{\gamma\in\mathbb{Z}\Phi}\Delta(a\varpi_1)_{a\varpi_1-\gamma}\otimes\Delta(b\varpi_n)_{b\varpi_n-\alpha_{\mathrm{h}}+\gamma},$$

it follows that $x\neq 0$. It remains to show that $X_{\beta,r}\cdot x=0$ for all $\beta\in\Phi^+$ and $r\geq 1$. By weight considerations, we have $X_{\beta,r}\cdot x=0$ for all $r>1$, and as $X_{\beta_{i,j}}=[X_{\alpha_i},X_{\beta_{i+1,j}}]$ for $i<j$, it suffices to verify that $X_{\alpha_i}\cdot x=0$ for $i=1,\ldots,n$. We have

$$\begin{aligned}X_{\alpha_i}\cdot x&=a\cdot X_{\alpha_i}v^+\otimes X_{-\beta_{1,n}}w^+-b\cdot X_{\alpha_i}X_{-\beta_{1,n}}v^+\otimes w^++\sum_{k=1}^{n-1}X_{\alpha_i}X_{-\beta_{1,k}}v^+\otimes X_{-\beta_{k+1,n}}w^+\\&\qquad+a\cdot v^+\otimes X_{\alpha_i}X_{-\beta_{1,n}}w^+-b\cdot X_{-\beta_{1,n}}v^+\otimes X_{\alpha_i}w^++\sum_{k=1}^{n-1}X_{-\beta_{1,k}}v^+\otimes X_{\alpha_i}X_{-\beta_{k+1,n}}w^+\\&=-b\cdot X_{\alpha_i}X_{-\beta_{1,n}}v^+\otimes w^++\sum_{k=1}^{n-1}X_{\alpha_i}X_{-\beta_{1,k}}v^+\otimes X_{-\beta_{k+1,n}}w^+\\&\qquad+a\cdot v^+\otimes X_{\alpha_i}X_{-\beta_{1,n}}w^++\sum_{k=1}^{n-1}X_{-\beta_{1,k}}v^+\otimes X_{\alpha_i}X_{-\beta_{k+1,n}}w^+.\end{aligned}$$

If $1<i<n$ then $[X_{\alpha_i},X_{-\beta_{1,n}}]=0$, so $X_{\alpha_i}X_{-\beta_{1,n}}v^+=0$ and $X_{\alpha_i}X_{-\beta_{1,n}}w^+=0$, and

$$[X_{\alpha_i},X_{-\beta_{1,k}}]=\begin{cases}X_{-\beta_{1,i-1}}&\text{if }k=i,\\0&\text{otherwise}\end{cases}\qquad\text{and}\qquad[X_{\alpha_i},X_{-\beta_{k+1,n}}]=\begin{cases}-X_{-\beta_{i+1,n}}&\text{if }k=i-1,\\0&\text{otherwise}\end{cases}$$

for $1\leq k\leq n-1$, so that

$$X_{\alpha_i}X_{-\beta_{1,k}}v^+=\begin{cases}X_{-\beta_{1,i-1}}v^+&\text{if }k=i,\\0&\text{otherwise}\end{cases}$$

and

$$X_{\alpha_i}X_{-\beta_{k+1,n}}w^+=\begin{cases}-X_{-\beta_{i+1,n}}w^+&\text{if }k=i-1,\\0&\text{otherwise.}\end{cases}$$

We conclude that

$$X_{\alpha_i} \cdot x = X_{-\beta_{1,i-1}} v^+ \otimes X_{-\beta_{i+1,n}} w^+ - X_{-\beta_{1,i-1}} v^+ \otimes X_{-\beta_{i+1,n}} w^+ = 0,$$

as required. For $i = 1$ and $1 \leq k \leq n$, we have

$$[X_{\alpha_1}, X_{-\beta_{1,k}}] = \begin{cases} H_{\alpha_1} & \text{if } k = 1, \\ -X_{-\beta_{2,k}} & \text{otherwise} \end{cases}$$

and therefore

$$X_{\alpha_1} X_{-\beta_{1,k}} v^+ = \begin{cases} H_{\alpha_1} v^+ = a \cdot v^+ & \text{if } k = 1, \\ -X_{-\beta_{2,k}} v^+ = 0 & \text{otherwise,} \end{cases}$$

where the second equality in the first case uses the fact that $(a\varpi_1, \alpha_1^\vee) = a$ and the second equality in the second case follows from the observation that $a\varpi_1 - \beta_{2,k} = s_{\beta_{2,k}}(a\varpi_1 + \beta_{2,k})$ and $a\varpi_1 + \beta_{2,k} > a\varpi_1$. Furthermore, we have $X_{\alpha_1} X_{-\beta_{1,n}} w^+ = -X_{-\beta_{2,n}} w^+$ and $X_{\alpha_1} X_{-\beta_{k+1,n}} w^+ = 0$ for $1 \leq k < n$ because X_{α_1} commutes with $X_{-\beta_{k+1,n}}$, and we conclude that

$$X_{\alpha_1} \cdot x = a \cdot v^+ \otimes X_{-\beta_{2,n}} w^+ - a \cdot v^+ \otimes X_{-\beta_{2,n}} w^+ = 0.$$

Analogously, we obtain

$$X_{\alpha_n} \cdot x = -b \cdot X_{-\beta_{1,n-1}} v^+ \otimes w^+ + b \cdot X_{-\beta_{1,n-1}} v^+ \otimes w^+ = 0,$$

and the claim follows. □

Corollary 5.12. *Let $a, b > 0$ such that $a + b + n \equiv 1 \mod \ell$, and let $x^+ \in \Delta(a\varpi_1 + b\varpi_n)_{a\varpi_1 + b\varpi_n}$ be a maximal vector. Then*

$$x := -b \cdot X_{-\beta_{1,n}} \cdot x^+ + \sum_{i=1}^{n-1} X_{-\beta_{1,i}} X_{-\beta_{i+1,n}} \cdot x^+$$

is a maximal vector of weight $a\varpi_1 + b\varpi_n - \alpha_\mathrm{h}$ in $\Delta(a\varpi_1 + b\varpi_n)$.

Proof. Let $v^+ \in \Delta(a\varpi_1)_{a\varpi_1}$ and $w^+ \in \Delta(b\varpi_n)_{b\varpi_n}$ be maximal vectors. As $a\varpi_1 + b\varpi_n$ is maximal among the highest weights of Weyl modules appearing in a Weyl filtration of $\Delta(a\varpi_1) \otimes \Delta(b\varpi_n)$, there is an embedding of $\mathbf{G}$-modules $\varphi\colon \Delta(a\varpi_1 + b\varpi_n) \to \Delta(a\varpi_1) \otimes \Delta(b\varpi_n)$, and as the $a\varpi_1 + b\varpi_n$-weight space of $\Delta(a\varpi_1) \otimes \Delta(b\varpi_n)$ is one-dimensional, we may assume that $\varphi(x^+) = v^+ \otimes w^+$, possibly after replacing x^+ by a scalar multiple. We will show that $\varphi(x)$ coincides with the maximal vector constructed in Lemma 5.11. Note that we have

$$X_{-\beta_{1,n}} \cdot (v^+ \otimes w^+) = v^+ \otimes X_{-\beta_{1,n}} w^+ + X_{-\beta_{1,n}} v^+ \otimes w^+$$

and

$$\begin{aligned} X_{-\beta_{1,i}} X_{-\beta_{i+1,n}} \cdot (v^+ \otimes w^+) &= X_{-\beta_{1,i}} v^+ \otimes X_{-\beta_{i+1,n}} w^+ + v^+ \otimes X_{-\beta_{1,i}} X_{-\beta_{i+1,n}} w^+ \\ &= X_{-\beta_{1,i}} v^+ \otimes X_{-\beta_{i+1,n}} w^+ - v^+ \otimes X_{-\beta_{1,n}} w^+ \end{aligned}$$

for $1 \leq i < n$ because $X_{-\beta_{i+1,n}} v^+ = 0$, $X_{-\beta_{1,i}} w^+ = 0$ and $[X_{-\beta_{1,i}}, X_{-\beta_{i+1,n}}] = -X_{-\beta_{1,n}}$. It follows that

$$\varphi(x) = -b \cdot X_{-\beta_{1,n}} \cdot (v^+ \otimes w^+) + \sum_{i=1}^{n-1} X_{-\beta_{1,i}} X_{-\beta_{i+1,n}} \cdot (v^+ \otimes w^+)$$

$$= -b \cdot v^+ \otimes X_{-\beta_{1,n}} w^+ - b \cdot X_{-\beta_{1,n}} v^+ \otimes w^+ + \sum_{i=1}^{n-1} \big(X_{-\beta_{1,i}} v^+ \otimes X_{-\beta_{i+1,n}} w^+ - v^+ \otimes X_{-\beta_{1,n}} w^+ \big)$$

$$= (-b - n + 1) \cdot v^+ \otimes X_{-\beta_{1,n}} w^+ - b \cdot X_{-\beta_{1,n}} v^+ \otimes w^+ + \sum_{i=1}^{n-1} X_{-\beta_{1,i}} v^+ \otimes X_{-\beta_{i+1,n}} w^+$$

$$= a \cdot v^+ \otimes X_{-\beta_{1,n}} w^+ - b \cdot X_{-\beta_{1,n}} v^+ \otimes w^+ + \sum_{i=1}^{n-1} \big(X_{-\beta_{1,i}} v^+ \otimes X_{-\beta_{i+1,n}} w^+ \big)$$

because $\Bbbk$ has characteristic ℓ and $a + b + n \equiv 1 \mod \ell$. Now $\varphi(x)$ is a maximal vector in the tensor product $\Delta(a\varpi_1) \otimes \Delta(b\varpi_n)$ by Lemma 5.11, and we conclude that x is a maximal vector. □

Now we combine our results about maximal vectors and about bases of weight spaces in Weyl modules in order to find a basis of a weight space in the simple $\mathbf{G}$-module $L_{x(i,0)\omega^{-1}}$.

Lemma 5.13. *Let $1 < i < n$, set $\lambda = x(i,0)\omega^{-1} \bullet 0$ and let $w^+ \in L(\lambda)_\lambda$ be a maximal vector. Then*

$$(\ell - n + i) \cdot X_{-\beta_{1,n}} \cdot w^+ - \sum_{k=i+1}^{n-1} X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot w^+ = 0$$

and the set

$$\{ X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot w^+ \mid i + 1 \leq k < n \}$$

is a basis of the weight space $L(\lambda)_{\lambda - \alpha_{\mathrm{h}}}$.

Proof. Let $\hat{w}^+ \in \Delta(\lambda)_\lambda$ be a maximal vector and fix an epimorphism

$$\varphi \colon \Delta(\lambda) \longrightarrow L(\lambda)$$

such that $\varphi(\hat{w}^+) = w^+$. By Lemma 3.3, the Weyl module $\Delta(\lambda)$ is uniserial, with two composition factors $L(\lambda)$ and $L(\lambda')$, where

$$\lambda = x(i,0)\omega^{-1} \bullet 0 = \varpi_{i+1} + (\ell - n + i) \cdot \varpi_n \qquad \text{and} \qquad \lambda' = x(i-1,0)\omega^{-1} \bullet 0 = \lambda - \beta_{i+1,n};$$

see Lemma 5.1. By Corollary 5.12 and truncation to the Levi subgroup of type A_{n-i} corresponding to the set of simple roots $\{\alpha_{i+1}, \ldots, \alpha_n\} \subseteq \Pi$, we see that a maximal vector that generates a simple submodule $M \cong L(\lambda')$ in $\Delta(\lambda)$ is given by

$$x := -(\ell - n + i) \cdot X_{-\beta_{i+1,n}} \cdot \hat{w}^+ + \sum_{k=i+1}^{n-1} X_{-\beta_{i+1,k}} X_{-\beta_{k+1,n}} \cdot \hat{w}^+.$$

Since $X_{-\beta_{1,i}} \cdot \hat{w}^+ = 0$ (by weight considerations) and $[X_{-\beta_{1,i}}, X_{-\beta_{i+1,k}}] = -X_{-\beta_{1,k}}$ for $i + 1 \leq k \leq n$, it is straightforward to see that

$$x' := X_{-\beta_{1,i}} \cdot x = (\ell - n + i) \cdot X_{-\beta_{1,n}} \cdot \hat{w}^+ - \sum_{k=i+1}^{n-1} X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot \hat{w}^+,$$

and as M equals the kernel of φ, we conclude that

$$0 = \varphi(x') = (\ell - n + i) \cdot X_{-\beta_{1,n}} \cdot w^+ - \sum_{k=i+1}^{n-1} X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot w^+.$$

Combining this equality with Corollary 5.10 (and the fact that φ is surjective), we see that the set

$$\{X_{-\beta_{1,k}}X_{-\beta_{k+1,n}}\cdot w^+ \mid i+1\leq k<n\}$$

spans the weight space $L(\lambda)_{\lambda-\alpha_\mathrm{h}}$. Furthermore, we have

$$\dim L(\lambda)_{\lambda-\alpha_\mathrm{h}} = \dim \Delta(\lambda)_{\lambda-\alpha_\mathrm{h}} - \dim L(\lambda')_{\lambda-\alpha_\mathrm{h}},$$

where $\dim \Delta(\lambda)_{\lambda-\alpha_\mathrm{h}} = n-i$ by Corollary 5.10 and $\dim L(\lambda')_{\lambda-\alpha_\mathrm{h}}=1$ because $\lambda-\alpha_\mathrm{h}=s_{\beta_{1,i}}(\lambda')$. We conclude that $\dim L(\lambda)_{\lambda-\alpha_\mathrm{h}}=n-i-1$ and the second claim follows. □

In the following proposition, we construct a maximal vector in a tensor product involving the Weyl module $\Delta(\varpi_1)$. Observe that the weight ϖ_1 is minuscule and that $\{-\varpi_{i-1}+\varpi_i \mid 1\leq i\leq n+1\}$ is the set of weights of $\Delta(\varpi_1)\cong L(\varpi_1)$. Let us fix a maximal vector $v_1\in\Delta(\varpi_1)_{\varpi_1}$ and define

$$v_i \coloneqq X_{-\beta_{1,i-1}}\cdot v_1$$

for $i=2,\dots,n+1$. Using the PBW-type basis of $\mathrm{Dist}(\mathbf{G})$ (as in the proof of Lemma 5.9), it is straightforward to see that $v_1,\dots,v_{n+1}$ is a basis of $\Delta(\varpi_1)$, with v_i of weight $-\varpi_{i-1}+\varpi_i$, and that

$$X_{\alpha_r}\cdot v_i = \begin{cases} v_{i-1} & \text{if } i=r+1,\\ 0 & \text{otherwise}\end{cases}$$

for $1\leq i\leq n+1$ and $1\leq r\leq n$. We call $v_1,\dots,v_{n+1}$ a *standard basis* of $\Delta(\varpi_1)$ (for any fixed choice of v_1).

Proposition 5.14. *Suppose that $\ell>n+1$ and consider the weight $\delta\coloneqq\varpi_k+a\varpi_n$ for $1\leq a<\ell-n$ and $1<k<n$. Denote by $v_1,\dots,v_{n+1}$ a standard basis of $\Delta(\varpi_1)$ and let $v^+\in\Delta(\delta)_\delta$ be a maximal vector. Then*

$$\begin{aligned} y \coloneqq \sum_{i=1}^{k}\Big(-a\cdot v_i\otimes X_{-\beta_{i,n}}v^+ + \sum_{j=1}^{n-k} v_i\otimes X_{-\beta_{i,k+j-1}}X_{-\beta_{k+j,n}}v^+\Big)\\ -\sum_{i=k+1}^{n}(a+1+n-k)\cdot v_i\otimes X_{-\beta_{i,n}}v^+ + a\cdot(a+1+n-k)\cdot v_{n+1}\otimes v^+\end{aligned}$$

is a maximal vector of weight $\delta+\varpi_1-\alpha_\mathrm{h}$ in $\Delta(\varpi_1)\otimes\Delta(\delta)$.

Proof. Observe that y is non-zero because $a<\ell-n<\ell$ and $a+1+n-k<\ell+1-k<\ell$. As in the proof of Lemma 5.11, it suffices to show that $X_{\alpha_r}\cdot y=0$ for $r=1,\dots,n$. First note that

$$\begin{aligned} X_{\alpha_r}\cdot y = \sum_{i=1}^{k}\Big(-a\cdot X_{\alpha_r}v_i\otimes X_{-\beta_{i,n}}v^+ + \sum_{j=1}^{n-k} X_{\alpha_r}v_i\otimes X_{-\beta_{i,k+j-1}}X_{-\beta_{k+j,n}}v^+\Big)\\ -\sum_{i=k+1}^{n}(a+1+n-k)\cdot X_{\alpha_r}v_i\otimes X_{-\beta_{i,n}}v^+ + a\cdot(a+1+n-k)\cdot X_{\alpha_r}v_{n+1}\otimes v^+\\ +\sum_{i=1}^{k}\Big(-a\cdot v_i\otimes X_{\alpha_r}X_{-\beta_{i,n}}v^+ + \sum_{j=1}^{n-k} v_i\otimes X_{\alpha_r}X_{-\beta_{i,k+j-1}}X_{-\beta_{k+j,n}}v^+\Big)\\ -\sum_{i=k+1}^{n}(a+1+n-k)\cdot v_i\otimes X_{\alpha_r}X_{-\beta_{i,n}}v^+ + a\cdot(a+1+n-k)\cdot v_{n+1}\otimes X_{\alpha_r}v^+.\end{aligned}$$

In the above equation, we have

$$X_{\alpha_r} v_i = \begin{cases} v_{i-1} & \text{if } i = r+1, \\ 0 & \text{otherwise,} \end{cases}$$

as observed before, and

$$X_{\alpha_r} X_{-\beta_{i,j}} v^+ = [X_{\alpha_r}, X_{-\beta_{i,j}}] \cdot v^+ + X_{-\beta_{i,j}} X_{\alpha_r} v^+ = [X_{\alpha_r}, X_{-\beta_{i,j}}] \cdot v^+,$$

where

$$[X_{\alpha_r}, X_{-\beta_{i,j}}] = \begin{cases} -X_{-\beta_{r+1,j}} & \text{if } r = i < j, \\ X_{-\beta_{i,r-1}} & \text{if } r = j > i, \\ H_{\alpha_r} & \text{if } r = i = j, \\ 0 & \text{otherwise} \end{cases}$$

for $i \leq j$. As v^+ is a maximal vector, it follows that

$$X_{\alpha_r} X_{-\beta_{i,n}} v^+ = [X_{\alpha_r}, X_{-\beta_{i,n}}] \cdot v^+ = \begin{cases} -X_{-\beta_{r+1,n}} \cdot v^+ & \text{if } r = i < n, \\ X_{-\beta_{i,n-1}} \cdot v^+ & \text{if } r = n > i, \\ H_{\alpha_n} \cdot v^+ = a \cdot v^+ & \text{if } r = i = n, \\ 0 & \text{otherwise.} \end{cases}$$

If $r < k$ then we further have

$$X_{\alpha_r} X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} v^+ = \begin{cases} -X_{-\beta_{r+1,k+j-1}} X_{-\beta_{k+j,n}} v^+ & \text{if } r = i, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq n-k$, and we conclude that

$$\begin{aligned} X_{\alpha_r} \cdot y = -a \cdot v_r \otimes X_{-\beta_{r+1,n}} v^+ + \sum_{j=1}^{n-k} v_r \otimes X_{-\beta_{r+1,k+j-1}} X_{-\beta_{k+j,n}} v^+ \\ + a \cdot v_r \otimes X_{-\beta_{r+1,n}} v^+ - \sum_{j=1}^{n-k} v_r \otimes X_{-\beta_{r+1,k+j-1}} X_{-\beta_{k+j,n}} v^+ \\ = 0. \end{aligned}$$

If $r = k$ then

$$X_{\alpha_k} X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} v^+ = \begin{cases} -X_{-\beta_{k+1,k+j-1}} X_{-\beta_{k+j,n}} v^+ & \text{if } i = k \text{ and } j > 1, \\ H_{\alpha_k} X_{-\beta_{k+1,n}} v^+ = 2 \cdot X_{-\beta_{k+1,n}} v^+ & \text{if } i = k \text{ and } j = 1, \\ X_{-\beta_{i,k-1}} X_{-\beta_{k+1,n}} v^+ = 0 & \text{if } i < k \text{ and } j = 1, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq n-k$, where the second equality in the second case follows from the equality $(\delta - \beta_{k+1,n}, \alpha_k^\vee) = 2$ and the second equality in the third case follows from the fact that $X_{-\beta_{i,k-1}}$ commutes with $X_{-\beta_{k+1,n}}$ combined with weight considerations. Furthermore, we have

$$[X_{-\beta_{k+1,k+j-1}} X_{-\beta_{k+j,n}}] = -X_{-\beta_{k+1,n}}$$

for $j > 1$ and $X_{-\beta_{k+1,k+j-1}}v^+ = 0$ by weight considerations, hence

$$-X_{-\beta_{k+1,k+j-1}}X_{-\beta_{k+j,n}}v^+ = X_{-\beta_{k+1,n}}v^+.$$

We conclude that

$$\begin{aligned}
X_{\alpha_k} \cdot y &= -(a+1+n-k) \cdot v_k \otimes X_{-\beta_{k+1,n}}v^+ \\
&\qquad + a \cdot v_k \otimes X_{-\beta_{k+1,n}}v^+ + 2 \cdot v_k \otimes X_{-\beta_{k+1,n}}v^+ + \sum_{j=2}^{n-k} -v_k \otimes X_{-\beta_{k+1,k+j-1}}X_{-\beta_{k+j,n}}v^+ \\
&= -(a+1+n-k) \cdot v_k \otimes X_{-\beta_{k+1,n}}v^+ \\
&\qquad + a \cdot v_k \otimes X_{-\beta_{k+1,n}}v^+ + 2 \cdot v_k \otimes X_{-\beta_{k+1,n}}v^+ + \sum_{j=2}^{n-k} v_k \otimes X_{-\beta_{k+1,n}}v^+ \\
&= 0.
\end{aligned}$$

If $k < r < n$ then

$$X_{\alpha_r} \cdot X_{-\beta_{i,k+j-1}}X_{-\beta_{k+j,n}}v^+ = \begin{cases} X_{-\beta_{i,r-1}}X_{-\beta_{r+1,n}}v^+ & \text{if } j = r-k+1, \\ -X_{-\beta_{i,r-1}}X_{-\beta_{r+1,n}}v^+ & \text{if } j = r-k, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq n-k$ and therefore

$$\begin{aligned}
X_{\alpha_r} \cdot y &= -(a+1+n-k) \cdot v_r \otimes X_{-\beta_{r+1,n}}v^+ \\
&\qquad + \sum_{i=1}^{k} \big(-v_i \otimes X_{-\beta_{i,r-1}}X_{-\beta_{r+1,n}}v^+ + v_i \otimes X_{-\beta_{i,r-1}}X_{-\beta_{r+1,n}}v^+ \big) \\
&\qquad + (a+1+n-k) \cdot v_r \otimes X_{-\beta_{r+1,n}}v^+ \\
&= 0.
\end{aligned}$$

Finally, for $r = n$ we have

$$X_{\alpha_n}X_{-\beta_{i,k+j-1}}X_{-\beta_{k+j,n}}v^+ = \begin{cases} X_{-\beta_{i,n-1}}H_{\alpha_n}v^+ = a \cdot X_{-\beta_{i,n-1}}v^+ & \text{if } j = n-k, \\ X_{-\beta_{i,k+j-1}}X_{-\beta_{k+j,n-1}}v^+ = 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq n-k$, where the second equality in the second case follows from weight considerations. For $k < i \leq n-1$, we have $X_{-\beta_{i,n-1}}v^+ = 0$, again by weight considerations, and it follows that

$$\begin{aligned}
X_{\alpha_n} \cdot y &= a \cdot (a+1+n-k) \cdot v_n \otimes v^+ \\
&\qquad + \sum_{i=1}^{k} \big(-a \cdot v_i \otimes X_{-\beta_{i,n-1}}v^+ + a \cdot v_i \otimes X_{-\beta_{i,n-1}}v^+ \big) \\
&\qquad - \sum_{i=k+1}^{n-1} (a+1+n-k) \cdot v_i \otimes X_{-\beta_{i,n-1}}v^+ - a \cdot (a+1+n-k) \cdot v_n \otimes v^+ \\
&= 0,
\end{aligned}$$

as required. □

Corollary 5.15. *Suppose that $\ell > n+1$, consider the weight $\delta \coloneqq \varpi_1 + \varpi_k + (\ell-n-1)\cdot\varpi_n$ for some $1<k<n$ and let $x^+ \in \Delta(\delta)_\delta$ be a maximal vector. Then*

$$y \coloneqq \sum_{i=2}^{k} \Big(-(\ell-n-1)\cdot X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} \cdot x^+ + \sum_{j=1}^{n-k} X_{-\beta_{1,i-1}} X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} \cdot x^+ \Big) \\ - \sum_{i=k+1}^{n} (\ell-k)\cdot X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} \cdot x^+ + (\ell-n-1)(\ell-k)\cdot X_{-\beta_{1,n}} \cdot x^+$$

is a maximal vector of weight $\delta-\alpha_{\mathrm{h}}$ in $\Delta(\delta)$.

Proof. Set $\delta' = \varpi_k + (\ell-n-1)\cdot\varpi_n$, let $v^+ \in \Delta(\delta')_{\delta'}$ be a maximal vector and let $v_1,\ldots,v_{n+1}$ be a standard basis of $\Delta(\varpi_1)$, as defined above Proposition 5.14. As in the proof of Corollary 5.12, we can choose an embedding of $\mathbf{G}$-modules $\varphi\colon \Delta(\delta) \to \Delta(\varpi_1)\otimes\Delta(\delta')$ with $\varphi(x^+) = v_1 \otimes v^+$. We will show that $\varphi(y)$ coincides with the maximal vector in $\Delta(\varpi_1)\otimes\Delta(\delta')$ constructed in Proposition 5.14. Note that we have

$$X_{-\beta_{1,n}} \cdot (v_1\otimes v^+) = v_1 \otimes X_{-\beta_{1,n}} v^+ + v_{n+1}\otimes v^+$$

and

$$X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} \cdot (v_1\otimes v^+) = v_1 \otimes X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} v^+ + v_i \otimes X_{-\beta_{i,n}} v^+$$

for $1<i\leq n$ as $X_{-\beta_{i,n}} v_1 = 0$. For $1<i\leq k$, we further have $X_{-\beta_{1,i-1}} v^+ = 0$ by weight considerations and therefore

$$X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} v^+ = [X_{-\beta_{1,i-1}}, X_{-\beta_{i,n}}]\cdot v^+ = -X_{-\beta_{1,n}} v^+$$

and

$$X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} \cdot (v_1\otimes v^+) = -v_1 \otimes X_{-\beta_{1,n}} v^+ + v_i \otimes X_{-\beta_{i,n}} v^+.$$

Analogously, we see that

$$X_{-\beta_{1,i-1}} X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} \cdot (v_1\otimes v^+) = -v_1 \otimes X_{-\beta_{1,k+j-1}} X_{-\beta_{k+j,n}} v^+ + v_i \otimes X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} v^+$$

for $1<i\leq k$ and $1\leq j\leq n-k$. It follows that

$$\begin{aligned}
\varphi(y) = {} & \sum_{i=2}^{k} \Big(-(\ell-n-1)\cdot X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} \cdot (v_1\otimes v^+) + \sum_{j=1}^{n-k} X_{-\beta_{1,i-1}} X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} \cdot (v_1\otimes v^+) \Big) \\
& - \sum_{i=k+1}^{n} (\ell-k)\cdot X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} \cdot (v_1\otimes v^+) \\
& + (\ell-n-1)(\ell-k)\cdot X_{-\beta_{1,n}} \cdot (v_1\otimes v^+) \\
= {} & \sum_{i=2}^{k} \Big((\ell-n-1)\cdot v_1 \otimes X_{-\beta_{1,n}} v^+ - (\ell-n-1)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ \\
& \qquad + \sum_{j=1}^{n-k} \big(-v_1 \otimes X_{-\beta_{1,k+j-1}} X_{-\beta_{k+j,n}} v^+ + v_i \otimes X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} v^+ \big) \Big) \\
& - \sum_{i=k+1}^{n} \big((\ell-k)\cdot v_1 \otimes X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} v^+ + (\ell-k)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ \big) \\
& + (\ell-n-1)(\ell-k)\cdot \big(v_1 \otimes X_{-\beta_{1,n}} v^+ + v_{n+1}\otimes v^+ \big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=2}^{k} \Big(-(\ell-n-1)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ + \sum_{j=1}^{n-k} v_i \otimes X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} v^+ \Big) \\
&\qquad + \sum_{i=2}^{k} \Big((\ell-n-1)\cdot v_1 \otimes X_{-\beta_{1,n}} v^+ + \sum_{j=1}^{n-k} -v_1 \otimes X_{-\beta_{1,k+j-1}} X_{-\beta_{k+j,n}} v^+ \Big) \\
&\qquad - \sum_{i=k+1}^{n} (\ell-k)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ + (\ell-n-1)(\ell-k)\cdot v_{n+1} \otimes v^+ \\
&\qquad - \sum_{i=k+1}^{n} (\ell-k)\cdot v_1 \otimes X_{-\beta_{1,i-1}} X_{-\beta_{i,n}} v^+ + (\ell-n-1)(\ell-k)\cdot v_1 \otimes X_{-\beta_{1,n}} v^+ \\
&= \sum_{i=2}^{k} \Big(-(\ell-n-1)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ + \sum_{j=1}^{n-k} v_i \otimes X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} v^+ \Big) \\
&\qquad + (k-1)\cdot \Big((\ell-n-1)\cdot v_1 \otimes X_{-\beta_{1,n}} v^+ + \sum_{j=1}^{n-k} -v_1 \otimes X_{-\beta_{1,k+j-1}} X_{-\beta_{k+j,n}} v^+ \Big) \\
&\qquad - \sum_{i=k+1}^{n} (\ell-k)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ + (\ell-n-1)(\ell-k)\cdot v_{n+1} \otimes v^+ \\
&\qquad + (\ell-k)\cdot \Big((\ell-n-1)\cdot v_1 \otimes X_{-\beta_{1,n}} v^+ + \sum_{j=1}^{n-k} -v_1 \otimes X_{-\beta_{1,k+j-1}} X_{-\beta_{k+j,n}} v^+ \Big) \\
&= \sum_{i=1}^{k} \Big(-(\ell-n-1)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ + \sum_{j=1}^{n-k} v_i \otimes X_{-\beta_{i,k+j-1}} X_{-\beta_{k+j,n}} v^+ \Big) \\
&\qquad - \sum_{i=k+1}^{n} (\ell-k)\cdot v_i \otimes X_{-\beta_{i,n}} v^+ + (\ell-n-1)(\ell-k)\cdot v_{n+1} \otimes v^+,
\end{aligned}
$$

where the last equality follows from the fact that $(k-1)+(\ell-k)=\ell-1=-1$ in $\Bbbk$, because $\Bbbk$ has characteristic ℓ. Now $\varphi(y)$ is a maximal vector by Proposition 5.14, and we conclude that y is a maximal vector. □

The final result that we need in order to prove Proposition 5.6 is the following lemma, which compares the actions of different elements of $\mathrm{Dist}(\mathbf{G})$ on maximal vectors in certain simple $\mathbf{G}$-modules.

Lemma 5.16. *For integers $c>0$ and $1<k<n$, set $\mu=\varpi_k+c\varpi_n$ and let $v^+\in L(\mu)_\mu$ be a maximal vector. Then, for all $k+1\leq a\leq n-1$ and $a+1\leq b\leq n-1$, we have*

$$
\begin{aligned}
X_{-\beta_{k+1,n}} X_{-\beta_{1,k}} \cdot v^+ &= X_{-\beta_{1,n}} \cdot v^+ + X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot v^+, \\
X_{-\beta_{k+1,a}} X_{-\beta_{a+1,n}} X_{-\beta_{1,k}} \cdot v^+ &= X_{-\beta_{1,a}} X_{-\beta_{a+1,n}} \cdot v^+ - X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot v^+, \\
X_{-\beta_{k+1,a}} X_{-\beta_{a+1,b}} X_{-\beta_{b+1,n}} X_{-\beta_{1,k}} \cdot v^+ &= -X_{-\beta_{1,a}} X_{-\beta_{a+1,n}} \cdot v^+ + X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot v^+.
\end{aligned}
$$

Proof. We have $[X_{-\beta_{k+1,n}}, X_{-\beta_{1,k}}] = X_{-\beta_{1,n}}$, from which the first equality is immediate. For the second equality, note that $X_{-\beta_{1,k}}$ commutes with $X_{-\beta_{a+1,n}}$ and that $[X_{-\beta_{k+1,a}}, X_{-\beta_{1,k}}] = X_{-\beta_{1,a}}$, so

$$X_{-\beta_{k+1,a}} X_{-\beta_{a+1,n}} X_{-\beta_{1,k}} \cdot v^+ = X_{-\beta_{1,a}} X_{-\beta_{a+1,n}} \cdot v^+ + X_{-\beta_{1,k}} X_{-\beta_{k+1,a}} X_{-\beta_{a+1,n}} \cdot v^+.$$

Furthermore, we have $[X_{-\beta_{k+1,a}}, X_{-\beta_{a+1,n}}] = -X_{-\beta_{k+1,n}}$ and $X_{-\beta_{k+1,a}} \cdot v^+ = 0$ by weight considerations. This implies that $X_{-\beta_{k+1,a}} X_{-\beta_{a+1,n}} \cdot v^+ = -X_{-\beta_{k+1,n}} \cdot v^+$, and we conclude that

$$X_{-\beta_{k+1,a}} X_{-\beta_{a+1,n}} X_{-\beta_{1,k}} \cdot v^+ = X_{-\beta_{1,a}} X_{-\beta_{a+1,n}} \cdot v^+ - X_{-\beta_{1,k}} X_{-\beta_{k+1,n}} \cdot v^+,$$

as claimed. The proof of the third equality is analogous. □

Equipped with the results about maximal vectors and bases of weight spaces which we established above, we can now give the proof of Proposition 5.6. Let us recall the statement of that proposition:

Proposition 5.17. *Let $i, j \geq 1$ such that $i + j < n - 2$, and let $a, b \geq 1$ such that (a, b) is one of the pairs $(i + 1, j)$ or $(i, j + 1)$. Then*

$$\dim \mathrm{Hom}_{\mathbf{G}}\big(\Delta_{x(a,b)}, L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}\big) = 1,$$

and the restriction to $\mathrm{rad}_{\mathbf{G}} \Delta_{x(a,b)}$ *of any non-zero homomorphism from* $\Delta_{x(a,b)}$ *to* $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$ *is non-zero.*

Proof. We prove the claim for $(a, b) = (i + 1, j)$; the case where $(a, b) = (i, j + 1)$ is analogous. By Lemmas 5.1 and 2.1, the highest weights of the simple $\mathbf{G}$-modules $L_{x(i,0)\omega^{-1}}$ and $L_{x(0,j)\omega}$ are given by

$$x(i, 0)\omega^{-1} \cdot 0 = \varpi_{i+1} + (\ell - n + i) \cdot \varpi_n \qquad \text{and} \qquad x(0, j)\omega \cdot 0 = (\ell - n + j) \cdot \varpi_1 + \varpi_{n-j},$$

respectively, and the highest weight of $\Delta_{x(i+1,j)}$ is

$$x(i + 1, j) \cdot 0 = (\ell - n - 1 + j) \cdot \varpi_1 + \varpi_{i+2} + \varpi_{n-j} + (\ell - n + i) \cdot \varpi_n.$$

Observe that we have $x(i + 1, j) \cdot 0 = x(i, 0)\omega^{-1} \cdot 0 + x(0, j)\omega \cdot 0 - \beta_{1,i+1}$. By truncation to the Levi subgroup corresponding to $\{\alpha_1, \ldots, \alpha_{i+1}\} \subseteq \Pi$, it is straightforward to see that $L_{x(i+1,j)}$ appears with multiplicity one as a composition factor of the tensor product $L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$, and it follows that

$$\dim \mathrm{Hom}_{\mathbf{G}}\big(\Delta_{x(i+1,j)}, L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}\big) \leq 1.$$

Let us write $\lambda \coloneqq x(i, 0)\omega^{-1} \cdot 0$ and $\mu \coloneqq x(0, j)\omega \cdot 0$ for the highest weights of the simple $\mathbf{G}$-modules $L_{x(i,0)\omega^{-1}}$ and $L_{x(0,j)\omega}$, and fix maximal vectors $v^+ \in L(\lambda)_\lambda$ and $w^+ \in L(\mu)_\mu$. Using Lemma 5.11 and truncation to the Levi subgroup corresponding to $\{\alpha_1, \ldots, \alpha_{i+1}\} \subseteq \Pi$, we see that the vector

$$x^+ \coloneqq (\ell - n + j) \cdot X_{-\beta_{1,i+1}} v^+ \otimes w^+ - v^+ \otimes X_{-\beta_{1,i+1}} w^+ + \sum_{k=1}^{i} X_{-\beta_{k+1,i+1}} v^+ \otimes X_{-\beta_{1,k}} w^+$$

is a maximal vector of weight $\lambda + \mu - \beta_{1,i+1} = x(i + 1, j) \cdot 0$ in $L(\lambda) \otimes L(\mu)$. In particular, we have

$$\dim \mathrm{Hom}_{\mathbf{G}}\big(\Delta_{x(i+1,j)}, L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}\big) = 1,$$

as claimed. Now let $\hat{x}^+$ be a maximal vector of weight $x(i + 1, j) \cdot 0$ in $\Delta_{x(i+1,j)}$, and let

$$\varphi \colon \Delta_{x(i+1,j)} \longrightarrow L_{x(i,0)\omega^{-1}} \otimes L_{x(0,j)\omega}$$

be the unique homomorphism with $\varphi(\hat{x}^+) = x^+$. By Corollary 5.15 and truncation to the Levi subgroup of type A_{n-i-1} corresponding to $\{\alpha_{i+2}, \ldots, \alpha_n\} \subseteq \Pi$, the vector

$$\begin{aligned} \hat{y}^+ \coloneqq \sum_{k=i+3}^{n-j} \Big(&- (\ell - n + i) \cdot X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n}} \cdot \hat{x}^+ + \sum_{r=1}^{j} X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n-j+r-1}} X_{-\beta_{n-j+r,n}} \cdot \hat{x}^+ \Big) \\ &- \sum_{k=n-j+1}^{n} (\ell - n + i + j + 1) \cdot X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n}} \cdot \hat{x}^+ \\ &+ (\ell - n + i)(\ell - n + i + j + 1) \cdot X_{-\beta_{i+2,n}} \cdot \hat{x}^+ \end{aligned}$$

is a maximal vector of weight $x(i+1,j)\cdot 0-\beta_{i+2,n}=\lambda+\mu-\alpha_{\mathrm{h}}=x(i,j)\cdot 0$ in $\Delta_{x(i+1,j)}$. In particular, $\hat{y}^+$ generates a proper submodule of $\Delta_{x(i+1,j)}$ and we have $\hat{y}^+\in\mathrm{rad}_{\mathbf{G}}\Delta_{x(i+1,j)}$. In order to complete the proof of the proposition, it suffices to verify that $y^+\coloneqq\varphi(\hat{y}^+)\neq 0$.

Observe that

$$\big(L(\lambda)\otimes L(\mu)\big)_{\lambda+\mu-\alpha_{\mathrm{h}}}=\bigoplus_{\gamma\in\mathbb{Z}\Phi}L(\lambda)_{\lambda-\alpha_{\mathrm{h}}+\gamma}\otimes L(\mu)_{\mu-\gamma},$$

and denote by p_0 the linear projection onto the tensor product of weight spaces $L(\lambda)_{\lambda-\alpha_{\mathrm{h}}}\otimes L(\mu)_{\mu}$. We consider the vector $y_0\coloneqq p_0(y^+)$. Let us write $x^+=x+x'$, with

$$x=(\ell-n+j)\cdot X_{-\beta_{1,i+1}}v^+\otimes w^+\qquad\text{and}\qquad x'=x^+-x\in\bigoplus_{0<\gamma\in\mathbb{Z}\Phi}L(\lambda)_{\lambda-\beta_{1,i+1}+\gamma}\otimes L(\mu)_{\mu-\gamma}.$$

As $\varphi(\hat{x}^+)=x^+$ and $\varphi(\hat{y}^+)=y^+$, we have

$$\begin{aligned}y^+=\sum_{k=i+3}^{n-j}\Big(-(\ell-n+i)\cdot X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n}}\cdot x^++\sum_{r=1}^{j}X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n-j+r-1}}X_{-\beta_{n-j+r,n}}\cdot x^+\Big)\\ -\sum_{k=n-j+1}^{n}(\ell-n+i+j+1)\cdot X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n}}\cdot x^+\\ +(\ell-n+i)(\ell-n+i+j+1)\cdot X_{-\beta_{i+2,n}}\cdot x^+,\end{aligned}$$

by the definition of $\hat{y}^+$. Using the observation that

$$X_\gamma\cdot\big(L(\lambda)_\delta\otimes L(\mu)_\nu\big)\subseteq\big(L(\lambda)_{\delta+\gamma}\otimes L(\mu)_\nu\big)\oplus\big(L(\lambda)_\delta\otimes L(\mu)_{\nu+\gamma}\big)\tag{5.1}$$

for all $\gamma\in\Phi$ and $\delta,\nu\in X$, it follows that x' does not contribute to y_0, that is, that $y_0=p_0(y^+)=p_0(z)$ with

$$\begin{aligned}z\coloneqq\sum_{k=i+3}^{n-j}\Big(-(\ell-n+i)\cdot X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n}}\cdot x+\sum_{r=1}^{j}X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n-j+r-1}}X_{-\beta_{n-j+r,n}}\cdot x\Big)\\ -\sum_{k=n-j+1}^{n}(\ell-n+i+j+1)\cdot X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n}}\cdot x\\ +(\ell-n+i)(\ell-n+i+j+1)\cdot X_{-\beta_{i+2,n}}\cdot x.\end{aligned}$$

Next observe that $\ell-n+j$ is invertible in $\Bbbk$, so after replacing x^+ by a scalar multiple, we may assume that $x=X_{-\beta_{1,i+1}}v^+\otimes w^+$. Then, again by (5.1), we have

$$\begin{aligned}y_0=\sum_{k=i+3}^{n-j}\Big(-(\ell-n+i)\cdot X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n}}X_{-\beta_{1,i+1}}v^+\otimes w^+\\ +\sum_{r=1}^{j}X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n-j+r-1}}X_{-\beta_{n-j+r,n}}X_{-\beta_{1,i+1}}v^+\otimes w^+\Big)\\ -\sum_{k=n-j+1}^{n}(\ell-n+i+j+1)\cdot X_{-\beta_{i+2,k-1}}X_{-\beta_{k,n}}X_{-\beta_{1,i+1}}v^+\otimes w^+\\ +(\ell-n+i)(\ell-n+i+j+1)\cdot X_{-\beta_{i+2,n}}X_{-\beta_{1,i+1}}v^+\otimes w^+,\end{aligned}$$

and we can write $y_0 = v_0 \otimes w^+$, where $v_0 \in L(\lambda)_{\lambda-\alpha_{\mathrm{h}}}$ is defined by

$$\begin{aligned} v_0 \coloneqq \sum_{k=i+3}^{n-j} \Big(&-(\ell-n+i)\cdot X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n}} X_{-\beta_{1,i+1}} v^+ \\ &+ \sum_{r=1}^{j} X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n-j+r-1}} X_{-\beta_{n-j+r,n}} X_{-\beta_{1,i+1}} v^+ \Big) \\ &- \sum_{k=n-j+1}^{n} (\ell-n+i+j+1)\cdot X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n}} X_{-\beta_{1,i+1}} v^+ \\ &+ (\ell-n+i)(\ell-n+i+j+1)\cdot X_{-\beta_{i+2,n}} X_{-\beta_{1,i+1}} v^+. \end{aligned}$$

By Lemma 5.16, we have

$$\begin{aligned} X_{-\beta_{i+2,n}} X_{-\beta_{1,i+1}} v^+ &= X_{-\beta_{1,n}} v^+ + X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+, \\ X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n}} X_{-\beta_{1,i+1}} v^+ &= X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ - X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+, \\ X_{-\beta_{i+2,k-1}} X_{-\beta_{k,n-j+r-1}} X_{-\beta_{n-j+r,n}} X_{-\beta_{1,i+1}} v^+ &= -X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ + X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \end{aligned}$$

for all $i+3 \leq k \leq n$ and $1 \leq r \leq j$ (and $k \leq n-j$ in the third equation). Using these equations, we can rewrite the vector v_0 as

$$\begin{aligned} v_0 = \sum_{k=i+3}^{n-j} \Big(&-(\ell-n+i)\cdot \big(X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ - X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \big) \\ &+ \sum_{r=1}^{j} \big(-X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ + X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \big) \Big) \\ &- \sum_{k=n-j+1}^{n} (\ell-n+i+j+1)\cdot \big(X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ - X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \big) \\ &+ (\ell-n+i)(\ell-n+i+j+1)\cdot \big(X_{-\beta_{1,n}} v^+ + X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \big) \\ = \sum_{k=i+3}^{n-j} &-(\ell-n+i+j)\cdot \big(X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ - X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \big) \\ &- \sum_{k=n-j+1}^{n} (\ell-n+i+j+1)\cdot \big(X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ - X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \big) \\ &+ (\ell-n+i)(\ell-n+i+j+1)\cdot \big(X_{-\beta_{1,n}} v^+ + X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+ \big) \\ = -(\ell-n+i+j)\cdot &\sum_{k=i+3}^{n-j} X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ - (\ell-n+i+j+1)\cdot \sum_{k=n-j+1}^{n} X_{-\beta_{1,k-1}} X_{-\beta_{k,n}} v^+ \\ &+ (\ell-n+i)(\ell-n+i+j+1)\cdot X_{-\beta_{1,n}} v^+ \\ &+ c\cdot X_{-\beta_{1,i+1}} X_{-\beta_{i+2,n}} v^+, \end{aligned}$$

where the scalar $c \in \Bbbk$ is given by

$$\begin{aligned} c &= (n-i-j-2)\cdot(\ell-n+i+j) + j\cdot(\ell-n+i+j+1) + (\ell-n+i)\cdot(\ell-n+i+j+1) \\ &= (n-i-j-2)\cdot(-n+i+j) + j\cdot(-n+i+j+1) + (-n+i)\cdot(-n+i+j+1) \end{aligned}$$

$$
\begin{aligned}
&= (n-i-j-2)\cdot(-n+i+j)+(-n+i+j)\cdot(-n+i+j+1)\\
&= -(-n+i+j)\\
&= -(\ell-n+i+j)
\end{aligned}
$$

because $\Bbbk$ is of characteristic ℓ. As c coincides with the coefficient of the vectors $X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+$, for $i+3\leq k\leq n-j$, in the preceding formula for v_0, we obtain

$$
\begin{aligned}
v_0 &= -(\ell-n+i+j)\cdot\sum_{k=i+2}^{n-j}X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+-(\ell-n+i+j+1)\cdot\sum_{k=n-j+1}^{n}X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+\\
&\qquad\qquad +(\ell-n+i)(\ell-n+i+j+1)\cdot X_{-\beta_{1,n}}v^+\\
&= (\ell-n+i+j+1)\cdot\Big((\ell-n+i)\cdot X_{-\beta_{1,n}}v^+-\sum_{k=i+2}^{n}X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+\Big)\\
&\qquad\qquad +\sum_{k=i+2}^{n-j}X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+.
\end{aligned}
$$

By Lemma 5.13, we have

$$(\ell-n+i)\cdot X_{-\beta_{1,n}}v^+-\sum_{k=i+2}^{n}X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+=0,$$

and it follows that

$$v_0=\sum_{k=i+2}^{n-j}X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+.$$

The vectors $X_{-\beta_{1,k-1}}X_{-\beta_{k,n}}v^+$, for $i+2\leq k\leq n$, form a basis of the weight space $L(\lambda)_{\lambda-\alpha_{\mathrm{h}}}$ (again by Lemma 5.13), and we conclude that $v_0\neq 0$. This implies that $p_0(y^+)=y_0=v_0\otimes w^+\neq 0$ and therefore $\varphi(\hat{y}^+)=y^+\neq 0$, as required. □

6 The complete reducibility theorem

Before we can prove the complete reducibility theorem from the introduction to this chapter, we need two more lemmas about weights.

Lemma 6.1. *Let $n\geq 2$, let $\lambda\in X^+$ be ℓ-regular and suppose that*

$$(\lambda+\rho,\beta_{1,n-1}^\vee)<\ell\qquad\textit{and}\qquad(\lambda+\rho,\beta_{2,n}^\vee)<\ell.$$

Then either $\lambda\in C_{\mathrm{fund}}$ or $\lambda\in s_0\cdot C_{\mathrm{fund}}$.

Proof. Let C be the ℓ-alcove with $\lambda\in C$. For $1\leq i\leq j\leq n-1$, we have

$$n_{\beta_{i,j}}(C)\cdot\ell<(\lambda+\rho,\beta_{i,j}^\vee)\leq(\lambda+\rho,\beta_{1,n-1}^\vee)<\ell$$

and it follows that $n_{\beta_{i,j}}(C)=0$. Analogously, we can use the inequality $(\lambda+\rho,\beta_{2,n}^\vee)<\ell$ to see that $n_{\beta_{i,j}}(C)=0$ for $2\leq i\leq j\leq n$. Furthermore, we have

$$n_{\beta_{1,n}}(C)\cdot\ell<(\lambda+\rho,\beta_{1,n}^\vee)\leq(\lambda+\rho,\beta_{1,n-1}^\vee)+(\lambda+\rho,\beta_{2,n}^\vee)<2\ell$$

and it follows that $n_{\beta_{1,n}}(C) \in \{0,1\}$. Now the claim follows because C is uniquely determined by the integers $n_\beta(C)$, for $\beta \in \Phi^+$, and because $n_\beta(C_{\mathrm{fund}}) = 0$ and

$$n_\beta(s_0 \cdot C_{\mathrm{fund}}) = n_\beta\big(x(0,0) \cdot C_{\mathrm{fund}}\big) = \begin{cases} 1 & \text{if } \beta = \beta_{1,n}, \\ 0 & \text{otherwise} \end{cases}$$

for all $\beta \in \Phi^+$, by Corollary 2.2. $\square$

Lemma 6.2. *Let $n \geq 2$, let $\lambda \in X^+$ be ℓ-regular and ℓ-restricted and suppose that $(\lambda + \rho, \beta_{1,n-1}) < \ell$. Then either $\lambda \in C_{\mathrm{fund}}$ or $\lambda \in x(i,0) \cdot C_{\mathrm{fund}}$ for some $0 \leq i < n-1$. Analogously, if $\mu \in X^+$ is ℓ-regular and ℓ-restricted with $(\mu + \rho, \beta_{2,n}) < \ell$ then $\mu \in C_{\mathrm{fund}}$ or $\mu \in x(0,j) \cdot C_{\mathrm{fund}}$ for some $0 \leq j < n-1$.*

Proof. Let C be the ℓ-alcove with $\lambda \in C$ and observe that, for $1 \leq i \leq j \leq n$, we have

$$n_{\beta_{i,j}}(C) \cdot \ell < (\lambda + \rho, \beta_{i,j}^\vee) \leq (\lambda + \rho, \beta_{1,n-1}^\vee) + (\lambda + \rho, \alpha_n^\vee) < 2\ell$$

because λ is ℓ-restricted, and therefore $n_{\beta_{i,j}}(C) \in \{0,1\}$. Furthermore, if $j < n$ then $n_{\beta_{i,j}}(C) = 0$, as in the proof of Lemma 6.1. If $n_{\beta_{2,n}}(C) = 0$ then $(\lambda + \rho, \beta_{2,n}^\vee) < \ell$ and the claim follows from Lemma 6.1. Now suppose that $n_{\beta_{2,n}}(C) = 1$ and let $i \in \{1, \ldots, n-1\}$ be maximal with the property that $n_{\beta_{i+1,n}}(C) = 1$. As λ is ℓ-restricted, we have $n_{\beta_{n,n}}(C) = n_{\alpha_n}(C) = 0$ and therefore $i < n-1$. It is straightforward to see that $n_{\beta_{k,n}}(C) \leq n_{\beta_{j,n}}(C)$ for $1 \leq j \leq k \leq n$, and it follows that

$$n_{\beta_{j,k}}(C) = \begin{cases} 1 & \text{if } j \leq i+1 \text{ and } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 2.2, we have $n_\beta(C) = n_\beta\big(x(i,0) \cdot C_{\mathrm{fund}}\big)$ for all $\beta \in \Phi^+$, and as C is uniquely determined by the integers $n_\beta(C)$, we conclude that $C = x(i,0) \cdot C_{\mathrm{fund}}$. The proof of the second claim is analogous. $\square$

Now we are ready to prove the complete reducibility theorem.

Theorem 6.3. *Suppose that we are in the modular case, that $\mathbf{G}$ is of type A_n for some $n \geq 1$ and that $\ell \geq n+1$. Let $\lambda, \mu \in X^+$ be ℓ-restricted and ℓ-regular. If the tensor product $L(\lambda) \otimes L(\mu)$ is completely reducible then either $\lambda \in C_{\mathrm{fund}}$ or $\mu \in C_{\mathrm{fund}}$.*

Proof. We prove the claim by induction on n. For $n = 1$, there is nothing to show because all ℓ-restricted ℓ-regular weights belong to C_{fund} (see Section III.1).

Now suppose that $n \geq 2$, and let $x, y \in W_{\mathrm{aff}}^+$ and $\lambda', \mu' \in C_{\mathrm{fund}}$ such that $\lambda = x \cdot \lambda'$ and $\mu = y \cdot \mu'$. By Proposition I.6.8 and Theorem II.4.14, we have

$$\big(L(\lambda) \otimes L(\mu)\big)_{\mathrm{reg}} \cong \big(T_0^{\lambda'} L(x \cdot 0) \otimes T_0^{\mu'} L(y \cdot 0)\big)_{\mathrm{reg}} \cong \bigoplus_{\nu \in C_{\mathrm{fund}} \cap X} T_0^\nu \big(L(x \cdot 0) \otimes L(y \cdot 0)\big)_{\mathrm{reg}}^{\oplus c_{\lambda',\mu'}^\nu},$$

and by Lemma I.9.3, there exists a weight $\nu \in C_{\mathrm{fund}} \cap X$ such that $c_{\lambda',\mu'}^\nu \neq 0$. As the generic direct summand $G(x,y)$ of $L(x \cdot 0) \otimes L(y \cdot 0)$ is regular (see Proposition II.5.7), we conclude that $T_0^\nu G(x,y)$ is a direct summand of $L(\lambda) \otimes L(\mu)$. In particular, if $L(\lambda) \otimes L(\mu)$ is completely reducible then $G(x,y)$ is simple (because T_0^ν is an equivalence).

For $n = 2$, the only ℓ-alcoves containing ℓ-restricted weights are C_{fund} and $s_0 \cdot C_{\mathrm{fund}}$ (as observed in Section III.2), and we have $G(s_0, s_0) \cong M(0)$ by Lemma III.2.3, where $M(0)$ denotes the non-simple $\mathbf{G}$-module defined on page 89. Hence the statement of the theorem is true for $\mathbf{G}$ of type A_2.

Now let $n \geq 3$ and suppose that the statement of the theorem is true for groups of type A_{n-1}. Consider the set of simple roots $I = \{\alpha_1, \ldots, \alpha_{n-1}\} \subseteq \Pi$ and let $\mathbf{L}_I$ be the derived subgroup of the corresponding Levi subgroup (see Remark I.4.3). Then $\mathbf{L}_I \cong \mathrm{SL}_n(\Bbbk)$ is of type A_{n-1}, and $\mathbf{L}_I$ has weight lattice $X_I = \bigoplus_{\alpha \in I} \mathbb{Z}\varpi_\alpha$, root system $\Phi_I = \mathbb{Z}I \cap \Phi$ and positive roots $\Phi_I^+ = \Phi_I \cap \Phi^+$ with respect to the base I. Let $\lambda, \mu \in X^+$ be ℓ-regular and ℓ-restricted, and suppose that $L(\lambda) \otimes L(\mu)$ is completely reducible. It is straightforward to see that the weights

$$\lambda_I = \sum_{\alpha \in I} (\lambda, \alpha^\vee) \cdot \varpi_\alpha \qquad \text{and} \qquad \mu_I = \sum_{\alpha \in I} (\mu, \alpha^\vee) \cdot \varpi_\alpha$$

are ℓ-regular and ℓ-restricted, and again by Remark I.4.3, the tensor product $L_I(\lambda_I) \otimes L_I(\mu_I)$ of simple $\mathbf{L}_I$-modules is completely reducible. By the induction hypothesis, this implies that at least one of the weights λ_I and μ_I belongs to the fundamental alcove with respect to $\mathbf{L}_I$, and as $\beta_{1,n-1} \in \Phi_I^+$, it follows that

$$(\lambda + \rho, \beta_{1,n-1}^\vee) < \ell \qquad \text{or} \qquad (\mu + \rho, \beta_{1,n-1}^\vee) < \ell.$$

Analogously, by considering the derived subgroup of the Levi subgroup of $\mathbf{G}$ corresponding to the set of simple roots $\{\alpha_2, \ldots, \alpha_n\} \subseteq \Pi$, we see that either

$$(\lambda + \rho, \beta_{2,n}^\vee) < \ell \qquad \text{or} \qquad (\mu + \rho, \beta_{2,n}^\vee) < \ell.$$

Possibly after interchanging λ and μ, we may assume that $(\lambda + \rho, \beta_{1,n-1}^\vee) < \ell$, and we consider the two possibilities $(\lambda + \rho, \beta_{2,n}^\vee) < \ell$ and $(\mu + \rho, \beta_{2,n}^\vee) < \ell$ in turn.

First suppose that $(\lambda + \rho, \beta_{2,n}^\vee) < \ell$. Then we have either $\lambda \in C_{\mathrm{fund}}$, as required, or $\lambda \in s_0 \cdot C_{\mathrm{fund}}$ by Lemma 6.1. Suppose that $\lambda \in s_0 \cdot C_{\mathrm{fund}}$, and let $y \in W_{\mathrm{aff}}^+$ such that $\mu \in y \cdot C_{\mathrm{fund}}$. As observed above, the complete reducibility of $L(\lambda) \otimes L(\mu)$ forces that $G(s_0, y)$ is simple. Now Theorem IV.6.3 implies that $y(A_{\mathrm{fund}}) = A_{\mathrm{fund}} + \gamma$ for some $\gamma \in X^+$, and as $\mu \in y \cdot C_{\mathrm{fund}} = C_{\mathrm{fund}} + \ell\gamma$ is ℓ-restricted, it follows that $\gamma = 0$ and $\mu \in C_{\mathrm{fund}}$.

Now suppose that $(\mu + \rho, \beta_{2,n}^\vee) < \ell$. If neither of λ and μ belongs to C_{fund} then, by Lemma 6.2, we have $\lambda \in x(i,0) \cdot C_{\mathrm{fund}}$ and $\mu \in x(0,j) \cdot C_{\mathrm{fund}}$ for some $0 \leq i < n-1$ and $0 \leq j < n-1$. By the previous case, we may further assume that $i > 0$ and $j > 0$ (because $x(0,0) = s_0$). Then the generic direct summand $G\big(x(i,0), x(0,j)\big)$ of $L\big(x(i,0) \cdot 0\big) \otimes L\big(x(0,j) \cdot 0\big)$ is non-simple by Propositions 5.3 and 5.7, contradicting the complete reducibility of $L(\lambda) \otimes L(\mu)$. $\square$

Appendices

A Additive categories and ideals

In this section, we discuss the notion of ideals (of morphisms) in categories and prove some properties of quotient categories. We start with the most basic definitions.

Definition. A *preadditive category* is a category in which all Hom-sets are abelian groups and composition of morphisms is bilinear. An *additive category* is a preadditive category that admits finite direct sums.

Note that the definition of an additive category includes the empty direct sum, which is both an initial and a final object, called the *zero object*.

Definition. An *additive functor* between additive categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F\colon \mathcal{A} \to \mathcal{B}$ such that the maps

$$F(A, A')\colon \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}\big(F(A), F(A')\big)$$

are group homomorphisms for all pairs of objects A and A' of $\mathcal{A}$.

Additive categories can be thought of as a generalization of rings, where multiplication is spread out over multiple objects. Accordingly, we can define a notion of ideals and quotient categories.

Definition. An *ideal (of morphisms)* $\mathcal{I}$ in $\mathcal{A}$ is a collection of subgroups $\mathcal{I}(A, B) \subseteq \mathrm{Hom}_{\mathcal{A}}(A, B)$, for every pair of objects A and B of $\mathcal{A}$, that is stable under composition. More specifically, this means that $b \circ f \circ a \in \mathcal{I}(A', B')$ for all $a \in \mathrm{Hom}_{\mathcal{A}}(A', A)$, $b \in \mathrm{Hom}_{\mathcal{A}}(B, B')$ and $f \in \mathcal{I}(A, B)$.

Definition. Let $\mathcal{I}$ be an ideal of morphisms in $\mathcal{A}$. The *quotient category* $\mathcal{A}/\mathcal{I}$ has the same objects as $\mathcal{A}$ and Hom-sets

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A, B) \coloneqq \mathrm{Hom}_{\mathcal{A}}(A, B)/\mathcal{I}(A, B).$$

Composition of morphisms in $\mathcal{A}/\mathcal{I}$ is induced by the composition law in $\mathcal{A}$.

Note that the composition of morphisms in the quotient category is well-defined because $\mathcal{I}$ is stable under composition. It is straightforward to check that $\mathcal{A}/\mathcal{I}$ is an additive category and that the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$, sending an object to itself and a morphism to its residue class, is additive.

Definition. A *Krull-Schmidt category* is an additive category where every object is isomorphic to a finite direct sum of objects having local endomorphism rings.

Note that an object whose endomorphism ring is local is a fortiori *indecomposable*, i.e. it does not admit a non-trivial direct sum decomposition. In a Krull-Schmidt category, the converse of this statement is also true, that is, the endomorphism rings of all indecomposable objects are local. The name Krull-Schmidt category is justified by the fact that a version of the Krull-Schmidt theorem holds in such categories; see Theorem 4.2 in [Kra15].

Theorem A.1. *Let $\mathcal{A}$ be a Krull-Schmidt category and let A be an object of $\mathcal{A}$ with two decompositions*

$$A_1 \oplus \cdots \oplus A_r \cong A \cong B_1 \oplus \cdots \oplus B_s$$

as a direct sum of objects with local endomorphism rings. Then $r = s$ and there exists a permutation τ such that $B_i \cong A_{\tau(i)}$ for $1 \leq i \leq r$.

A direct sum decomposition as in the preceding theorem is called a *Krull-Schmidt decomposition.* Let us cite another result about Krull-Schmidt categories from Corollary 4.4 in [Kra15].

Lemma A.2. *A Krull-Schmidt category $\mathcal{A}$ has split idempotents, that is, for every idempotent $e = e^2$ in the endomorphism ring of an object A of $\mathcal{A}$, there exist an object B of A and morphisms $f\colon B \to A$ and $g\colon A \to B$ such that $f \circ g = e$ and $g \circ f = \mathrm{id}_B$.*

Next we show that Krull-Schmidt categories are well-behaved with respect to taking quotients by ideals of morphisms.

Lemma A.3. *Let $\mathcal{A}$ be a Krull-Schmidt category and let $\mathcal{I}$ be an ideal of morphisms in $\mathcal{A}$. Then $\mathcal{A}/\mathcal{I}$ is a Krull-Schmidt category.*

Proof. For any object A of $\mathcal{A}$, the endomorphism ring $\mathrm{End}_{\mathcal{A}/\mathcal{I}}(A)$ is a quotient of $\mathrm{End}_{\mathcal{A}}(A)$ by a two-sided ideal, so $\mathrm{End}_{\mathcal{A}/\mathcal{I}}(A)$ is local or zero whenever $\mathrm{End}_{\mathcal{A}}(A)$ is local. Now the endomorphism ring of an object in an additive category is zero if and only if the object is isomorphic to the zero object. Hence a decomposition of an object of $\mathcal{A}$ as a finite direct sum of objects with local endomorphism rings gives rise to such a decomposition in $\mathcal{A}/\mathcal{I}$, by omitting the objects whose endomorphism rings in the quotient are zero. $\square$

Corollary A.4. *Let $\mathcal{A}$ be a Krull-Schmidt category and let $\mathcal{I}$ be an ideal of morphisms in $\mathcal{A}$. Furthermore, let A be an object of $\mathcal{A}$ and fix a Krull-Schmidt decomposition*

$$A \cong A_1 \oplus \cdots \oplus A_r \oplus B_1 \oplus \cdots \oplus B_s$$

of A in $\mathcal{A}$ such that $A_i \ncong 0$ in $\mathcal{A}/\mathcal{I}$ for $1 \leq i \leq r$ and $B_i \cong 0$ in $\mathcal{A}/\mathcal{I}$ for $1 \leq i \leq s$. Then

$$A \cong A_1 \oplus \cdots \oplus A_r$$

is a Krull-Schmidt decomposition of A in $\mathcal{A}/\mathcal{I}$.

Proof. This follows from the proof of Lemma A.3. $\square$

Example A.5. Let $\mathcal{A}$ be an additive category and let $\mathcal{J}$ be a non-empty set of objects of $\mathcal{A}$ such that $A \oplus B \in \mathcal{J}$ for all $A, B \in \mathcal{J}$. Then $\mathcal{J}$ defines an ideal of morphisms via

$$\mathcal{J}(A,B) \coloneqq \big\{ f \in \mathrm{Hom}_{\mathcal{A}}(A,B) \bigm| f \text{ factors through an object in } \mathcal{J} \big\}.$$

Indeed, stability under composition is obvious and the sum of two morphisms $f, g \in \mathcal{J}(A,B)$ that factor through objects C and C' in $\mathcal{J}$, respectively, factors though the direct sum $C \oplus C'$.

Now suppose that $\mathcal{A}$ is a Krull-Schmidt category and that $\mathcal{J}$ is closed under retracts, i.e. that $A \oplus B \in \mathcal{J}$ implies that $A \in \mathcal{J}$ and $B \in \mathcal{J}$. Then an object of $\mathcal{A}$ is isomorphic to the zero object in the quotient category $\mathcal{A}/\mathcal{J}$ if and only if it belongs to $\mathcal{J}$. Indeed, an object A of $\mathcal{A}$ is isomorphic to the zero object in $\mathcal{A}/\mathcal{J}$ if and only if $\mathrm{id}_A \in \mathcal{J}(A,A)$, and if id_A factors through an object in $\mathcal{J}$ then $A \in \mathcal{J}$ because $\mathcal{A}$ has split idempotents and $\mathcal{J}$ is closed under retracts.

B Homological algebra

This section serves as a reminder on some important constructions in homological algebra. Namely, we will recall *homotopy categories* and *derived categories* and discuss their triangulated structure. For a more detailed overview of these topics with further references, see [Kra07]. Throughout the section, we fix an additive category $\mathcal{A}$ (which we will later assume to be abelian).

A *complex* $A = (A_\bullet, d^A_\bullet)$ over $\mathcal{A}$ is a sequence of objects $(A_i)_{i\in\mathbb{Z}}$ of $\mathcal{A}$ with morphisms

$$d^A_i \in \mathrm{Hom}_{\mathcal{A}}(A_i, A_{i+1})$$

such that $d^A_{i+1} \circ d^A_i = 0$ for all $i \in \mathbb{Z}$. We call A_i the *term in homological degree* i and d^A_i the *i-th differential* of A. The complex A is called *bounded* if all but finitely many terms are zero. For two complexes A and B, a *chain map* (or *homomorphism of complexes*) $f = (f_\bullet)\colon A \to B$ is a sequence of morphisms $f_i \in \mathrm{Hom}_{\mathcal{A}}(A_i, B_i)$ such that $d^B_i \circ f_i = f_{i+1} \circ d^A_i$ for all $i \in \mathbb{Z}$. The category $C(\mathcal{A})$ of complexes over $\mathcal{A}$ has objects the complexes over $\mathcal{A}$ and morphisms the chain maps. We write $C^b(\mathcal{A})$ for the full subcategory of bounded complexes.

Remark. What we call a *complex* here is often referred to as a *cochain complex* in the literature. In some situations, it is useful to distinguish between *cochain complexes* and *chain complexes*, where the latter have differentials going in the 'opposite direction', i.e. from degree i to degree $i-1$. It should also be noted that many authors write cochain complexes with indices as superscripts (like A^i or d^i_A) and chain complexes with indices as subscripts. As all complexes that will be considered here are cochain complexes, we can ignore this distinction.

Note that the additive structure on $\mathcal{A}$ induces an additive structure on $C(\mathcal{A})$. Next we discuss some important constructions related to complexes. For $j \in \mathbb{Z}$, the *j-th homological shift* $A[j]$ of a complex $A = (A_\bullet, d^A_\bullet)$ is the complex with terms $A[j]_i := A_{i+j}$ and differentials $d^{A[j]}_i := (-1)^j \cdot d_{i+j}$. Given a chain map $f\colon A \to B$, the *cone* of f is the complex $C = \mathrm{cone}(f)$ with terms $C_i = A_{i+1} \oplus B_i$ and differentials

$$d^C_i = \begin{pmatrix} -d^A_{i+1} & 0 \\ f_{i+1} & d^B_i \end{pmatrix},$$

acting as though on column vectors. Note that the inclusions $B_i \to C_i$ and the projections $C_i \to A_{i+1}$ define canonical chain maps

$$B \longrightarrow \mathrm{cone}(f) \longrightarrow A[1].$$

We say that two chain maps $f, g\colon A \to B$ are *homotopic* if there exists a *homotopy* $h = (h_\bullet)$ from f to g, i.e. a sequence of morphisms $h_i \in \mathrm{Hom}_{\mathcal{A}}(A_i, B_{i-1})$ such that $f_i - g_i = d^B_{i-1} \circ h_i + h_{i+1} \circ d^A_i$. A chain map is called *nullhomotopic* if it is homotopic to the zero chain map. The nullhomotopic chain maps form an ideal in the category $C(\mathcal{A})$, that is, the nullhomotopic chain maps $f\colon A \to B$ form a subgroup of $\mathrm{Hom}_{C(\mathcal{A})}(A, B)$ and if $f\colon A \to B$ is nullhomotopic then so are $x \circ f$ and $f \circ y$ for any chain maps $x\colon B \to B'$ and $y\colon A' \to A$. Therefore, we can define the *homotopy category* $K(\mathcal{A})$ of $\mathcal{A}$ as the quotient of $C(\mathcal{A})$ by the ideal of nullhomotopic chain maps: The objects of $K(\mathcal{A})$ are just the complexes over $\mathcal{A}$, and the morphisms from A to B are defined as the quotient of $\mathrm{Hom}_{C(\mathcal{A})}(A, B)$ by the subgroup of nullhomotopic chain maps. Two complexes that are isomorphic in $K(\mathcal{A})$ are called *homotopy equivalent*, and a chain map that becomes an isomorphism in $K(\mathcal{A})$ is called a *homotopy equivalence*. Note that $K(\mathcal{A})$ inherits the additive structure from $C(\mathcal{A})$. We write $K^b(\mathcal{A})$ for the *bounded homotopy category* of $\mathcal{A}$, i.e. the full subcategory of $K(\mathcal{A})$ whose objects are the bounded complexes. The categories $K^b(\mathcal{A})$ and $K(\mathcal{A})$ admit the structure of *triangulated categories*, that is, they are additive categories with a *shift functor* (denoted on objects by $A \mapsto A[1]$) and a class of *distinguished triangles* $A \to B \to C \to A[1]$ satisfying certain axioms (which will not be recalled here). In the present case, the shift functor is the homological shift of complexes and the distinguished triangles are those that are isomorphic to a triangle of the form

$$A \xrightarrow{f} B \longrightarrow \operatorname{cone}(f) \longrightarrow A[1],$$

where the chain maps $B \to \operatorname{cone}(f) \to A[1]$ are those that were discussed above. One important consequence of this definition is that for any distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1],$$

the triangle

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is also distinguished. We refer to this property as *triangle rotation*.

Now suppose that $\mathcal{A}$ is an abelian category. For a complex A over $\mathcal{A}$, the condition that $d_i^A \circ d_{i-1}^A = 0$ means that the image of d_{i-1}^A is contained in the kernel of d_i^A, and we define the *i-th cohomology* of A as

$$H^i(A) := \ker(d_i^A)/\operatorname{im}(d_{i-1}^A).$$

A complex is called *exact in degree i* if its i-th cohomology is zero and *exact* if it is exact in all degrees. We also say *exact sequence* for a bounded exact complex and *short exact sequence* for an exact sequence with at most three non-zero terms. A chain map $f\colon A \to B$ induces homomorphisms

$$H^i(f)\colon H^i(A) \to H^i(B)$$

for all $i \in \mathbb{Z}$, and we call f a *quasi-isomorphism* if all $H^i(f)$ are isomorphisms. The *derived category* $D(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $K(\mathcal{A})$ at the class of quasi-isomorphisms. Its objects are the complexes over $\mathcal{A}$ and its morphisms can be constructed as certain equivalence classes of 'roofs' of chain maps $A \leftarrow M \to B$ where $A \leftarrow M$ is a quasi-isomorphism. As before, we write $D^b(\mathcal{A})$ for the *bounded derived category* of $\mathcal{A}$, i.e. the full subcategory whose objects are the bounded complexes. The categories $D(\mathcal{A})$ and $D^b(\mathcal{A})$ inherit from $K(\mathcal{A})$ the structure of triangulated categories, where as before, the shift functor is the homological shift and the distinguished triangles are those that arise from cones of chain maps. Thus the natural functor $K(\mathcal{A}) \to D(\mathcal{A})$ that sends a complex to itself and a homotopy class of chain maps to its equivalence class in $D(\mathcal{A})$ is a triangulated functor, i.e. it commutes with the shift functors and takes distinguished triangles to distinguished triangles. Every object of $\mathcal{A}$ can be viewed as a complex with a single non-zero term in degree zero, so there also is a natural functor $\mathcal{A} \to D^b(\mathcal{A})$ which turns out to be fully faithful, i.e. it induces isomorphisms between the respective homomorphism groups. If A is a complex over $\mathcal{A}$ with $H^i(A) = 0$ for all $i \neq 0$ then A is isomorphic to the one-term complex with $H^0(A)$ in degree zero, as an object of $D^b(\mathcal{A})$. Moreover, if $f\colon A \to B$ is a monomorphism in $\mathcal{A}$ then the cone of f is isomorphic to the cokernel of f (viewed as a complex) in $D^b(\mathcal{A})$. In particular, any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ gives rise to a distinguished triangle $A \to B \to C \to A[1]$ in $D^b(\mathcal{A})$. Another important property of derived categories (or triangulated categories in general) is that Hom-functors are *cohomological*: For any complex D, applying the functors

$$\operatorname{Hom}_{D(\mathcal{A})}(D, -) \qquad \text{and} \qquad \operatorname{Hom}_{D(\mathcal{A})}(-, D)$$

to a distinguished triangle $A \to B \to C \to A[1]$ gives rise to exact sequences

$$\begin{aligned}\cdots \to \mathrm{Hom}_{D(\mathcal{A})}(D, A[i]) \to \mathrm{Hom}_{D(\mathcal{A})}(D, B[i]) \to \mathrm{Hom}_{D(\mathcal{A})}(D, C[i])\\ \to \mathrm{Hom}_{D(\mathcal{A})}(D, A[i+1]) \to \mathrm{Hom}_{D(\mathcal{A})}(D, B[i+1]) \to \mathrm{Hom}_{D(\mathcal{A})}(D, C[i+1]) \to \cdots\end{aligned}$$

and

$$\begin{aligned}\cdots \to \mathrm{Hom}_{D(\mathcal{A})}(C[i], D) \to \mathrm{Hom}_{D(\mathcal{A})}(B[i], D) \to \mathrm{Hom}_{D(\mathcal{A})}(A[i], D)\\ \to \mathrm{Hom}_{D(\mathcal{A})}(C[i-1], D) \to \mathrm{Hom}_{D(\mathcal{A})}(B[i-1], D) \to \mathrm{Hom}_{D(\mathcal{A})}(A[i-1], D) \to \cdots,\end{aligned}$$

respectively.[3] Yet another example of a cohomological functor is the degree zero cohomology $H^0(-)$. As $H^0(D[i]) = H^i(D)$ for all $i \in \mathbb{Z}$ and for any complex D over $\mathcal{A}$, this implies that a distinguished triangle $A \to B \to C \to A[1]$ affords an exact sequence

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots.$$

For objects A and B of $\mathcal{A}$, we define

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) := \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])$$

for $i \in \mathbb{Z}$, viewing A and B as one-term complexes as before. We note that

$$\mathrm{Ext}^0_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, B)$$

and $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0$ for $i < 0$. As a special case of the exact sequences of Hom-groups in $D(\mathcal{A})$ discussed above, we see that any short exact sequence $0 \to A \to B \to C \to 0$ gives rise to exact sequences

$$0 \to \mathrm{Hom}_{\mathcal{A}}(D, A) \to \mathrm{Hom}_{\mathcal{A}}(D, B) \to \mathrm{Hom}_{\mathcal{A}}(D, C) \to \mathrm{Ext}^1_{\mathcal{A}}(D, A) \to \cdots$$

and

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C, D) \to \mathrm{Hom}_{\mathcal{A}}(B, D) \to \mathrm{Hom}_{\mathcal{A}}(A, D) \to \mathrm{Ext}^1_{\mathcal{A}}(C, D) \to \cdots,$$

for any object D of $\mathcal{A}$.

If the category $\mathcal{A}$ is additive monoidal (i.e. if $\mathcal{A}$ has a bi-additive tensor product bifunctor $\otimes$, subject to some axioms which we do not recall here) then we can define a tensor product $M \otimes N$ of bounded complexes $M = (M_\bullet, d^M_\bullet)$ and $N = (N_\bullet, d^N_\bullet)$ as follows: The terms of $M \otimes N$ are defined by

$$(M \otimes N)_k = \bigoplus_{i+j=k} M_i \otimes N_j,$$

and the k-th differential $d^{M\otimes N}_k$ can be written as a matrix with entries

$$\left(d^{M\otimes N}_k\right)_{(i,j),(i',j')} \colon M_i \otimes M_j \longrightarrow M_{i'} \otimes M_{j'}$$

for $i + j = k$ and $i' + j' = k + 1$, where

$$\left(d^{M\otimes N}_k\right)_{(i,j),(i',j')} = \begin{cases} d^M_i \otimes \mathrm{id}_{N_j} & \text{if } (i', j') = (i+1, j), \\ (-1)^i \cdot \mathrm{id}_{M_i} \otimes d^N_j & \text{if } (i', j') = (i, j+1), \\ 0 & \text{otherwise.} \end{cases}$$

[3]To be more precise, one should say that $\mathrm{Hom}_{D(\mathcal{A})}(-, D)$ is a cohomological functor from $D(\mathcal{A})$ to the opposite category of the category of abelian groups.

Thus the category of bounded complexes $C^b(\mathcal{A})$ inherits a monoidal structure from $\mathcal{A}$, which descends to $K^b(\mathcal{A})$ and $D^b(\mathcal{A})$. If $\mathcal{A}$ is abelian and the tensor product bifunctor is exact in both components then the cohomology of the tensor product complex $M \otimes N$ can be computed by the Künneth formula:

$$H^k(M \otimes N) \cong \bigoplus_{i+j=k} H^i(M) \otimes H^j(N)$$

This is a special case of the main theorem of [Big07]. We give a sketch of a proof for the reader's convenience. First note that the claim is certainly true if all differentials of N are trivial, and in that case $H^j(N) \cong N_j$ for all j. In general, we have short exact sequences

$$0 \longrightarrow \ker(d_j^N) \longrightarrow N_j \longrightarrow \operatorname{im}(d_j^N) \longrightarrow 0$$

for all j and we consider the complexes $K = (K_\bullet, d_\bullet^K)$ and $I = (I_\bullet, d_\bullet^I)$ with trivial differentials and terms $K_j = \ker(d_j^N)$ and $I_j = \operatorname{im}(d_j^N)$. Then there is a short exact sequence of complexes

$$0 \longrightarrow M \otimes K \longrightarrow M \otimes N \longrightarrow M \otimes I \longrightarrow 0,$$

and the latter gives rise to an exact sequence

$$\cdots \to H^{k-1}(M \otimes N) \to H^{k-1}(M \otimes I) \to H^k(M \otimes K) \to H^k(M \otimes N) \to H^k(M \otimes I) \to \cdots$$

via the snake lemma. By the initial observation, we have

$$H^{k-1}(M \otimes I) \cong \bigoplus_{i+j=k} H^i(M) \otimes \operatorname{im}(d_{j-1}^N) \qquad \text{and} \qquad H^k(M \otimes K) \cong \bigoplus_{i+j=k} H^i(M) \otimes \ker(d_j^N)$$

for all k, and one can check that the homomorphism from $H^{k-1}(M \otimes I)$ to $H^k(M \otimes K)$ restricts to the tensor product of the identity on $H^i(M)$ with the embedding of $\operatorname{im}(d_{j-1}^N)$ into $\ker(d_j^N)$, on the different components of the direct sum. Now the claim follows since $H^j(N) = \ker(d_j^N)/\operatorname{im}(d_{j-1}^N)$.

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List of notations

$[M : \Delta(\lambda)]_\Delta$: Weyl filtration multiplicity, 23
$[M : L(\lambda)]$: composition multiplicity, 20
$[M : N]_\oplus$: Krull-Schmidt multiplicity, 20
$[M : \nabla(\lambda)]_\nabla$: good filtration multiplicity, 22
$[M]$: the class of M in $[\mathrm{Rep}(\mathbf{G})]$, 132
$[\mathrm{Rep}(\mathbf{G})]$: Grothendieck group of $\mathrm{Rep}(\mathbf{G})$, 132

A_{fund}: the fundamental alcove, 12
$\widehat{A}$: upper closure of an alcove, 101
α_{h}: the highest short root in Φ, 9
$\tilde{\alpha}_{\mathrm{h}}$: the highest root in Φ, 9
$\uparrow$: the linkage order on alcoves, 14
$\uparrow_\ell$: the linkage order on ℓ-alcoves, 25

$\mathbf{B} = \mathbf{B}_\Bbbk$ in the modular case, 15
$\mathbf{B} = U_\zeta^-(\mathfrak{g})$ in the quantum case, 19

$C(\mathcal{A})$: the category of complexes over $\mathcal{A}$, 191
$C^b(\mathcal{A})$: bounded complexes over $\mathcal{A}$, 191
C_{fund}: the fundamental ℓ-alcove, 25
$\chi(\lambda) = \mathrm{ch}\,\nabla(\lambda)$, 20
$\mathrm{ch}\,M$: the character of M, 19
$c_{\lambda,\mu}^\nu = [T(\lambda) \otimes T(\mu) : T(\nu)]_\oplus$, 37
$C_{\min}(M)$: minimal tilting complex of M, 43

$D(\mathcal{A})$: the derived category of $\mathcal{A}$, 192
$d(A) = \sum_{\beta\in\Phi^+} n_\beta(A)$, 99
$d(A, A')$: the distance between A and A', 96
$d(x) = d\big(x(A_{\mathrm{fund}})\big)$, 102
$D^b(\mathcal{A})$: bounded derived category of $\mathcal{A}$, 192
$\Delta(\lambda) = \nabla(-w_0\lambda)^*$, 20
$\Delta(\lambda)^k$: term in the Jantzen filtration, 132
$\Delta_{x,k}$: layer of the Jantzen filtration of Δ_x, 156
$\mathrm{Dist}(\mathbf{G})$: the distribution algebra of $\mathbf{G}$, 17

$\widehat{F}$: upper closure of an ℓ-facet, 28
F_λ: the ℓ-facet containing λ, 28

Fr: the Frobenius morphism, 31

$\mathbf{G} = \mathbf{G}_\Bbbk$ in the modular case, 15
$\mathbf{G} = U_\zeta(\mathfrak{g})$ in the quantum case, 19
$G(x, y)$: a generic direct summand, 63
$G_\Delta(x, y)$: a generic direct summand, 61
$G_\nabla(x, y)$: a generic direct summand, 61
$\mathbf{G}_1$: the first Frobenius kernel or the small quantum group, 34
$\mathrm{gfd}(M)$: good filtration dimension of M, 29
$\mathbf{G}_r$: a Frobenius kernel of $\mathbf{G}$, 31

$H_{\beta,m}$: hyperplane of fixed points of $s_{\beta,m}$, 11
$H_{\beta,m}^\ell = \{\ell \cdot x - \rho \mid x \in H_{\beta,m}\}$, 25
$H_{\beta,m}^+$: positive half space w.r.t. $H_{\beta,m}$, 12
$H_{\beta,m}^-$: negative half space w.r.t. $H_{\beta,m}$, 12
$\mathrm{head}_{\mathbf{G}} M = M/\mathrm{rad}_{\mathbf{G}} M$: the head of M, 19
$H^i(A)$: i-th cohomology of a complex A, 192

$J(\mathbf{u}) = \bigotimes_{i\geq 0} T(u_i)^{[i]}$, for $\mathbf{G}$ of type A_1, 76
$\mathrm{JSF}_x = \psi_\lambda^{-1}(\mathrm{JSF}_x^\lambda) \in M_{\mathrm{asph}}$, 133
$\mathrm{JSF}_x^\lambda = \sum_{i>0} [\Delta(x \cdot \lambda)^i] \in [\mathrm{Rep}(\mathbf{G})]$, 133

$K(\mathcal{A})$: the homotopy category of $\mathcal{A}$, 191
$K^b(\mathcal{A})$: bounded homotopy category of $\mathcal{A}$, 191

$\ell = \mathrm{char}(\Bbbk)$ in the modular case, 15
$\ell = \mathrm{ord}(\zeta)$ in the quantum case, 16
$L(\lambda) = \mathrm{soc}_{\mathbf{G}} \nabla(\lambda)$, 20
$\mathcal{L}(w)$: set of hyperplanes separating A_{fund} and $w(A_{\mathrm{fund}})$, 13
$\ell(w)$: the length of $w \in W_{\mathrm{ext}}$, 13
$L_r(\lambda)$: a simple $\mathbf{G}_r$-module, 32
$\widehat{L}_r(\lambda)$: a simple $\mathbf{G}_r\mathbf{T}$-module, 33

$M(\lambda, \mu)$: direct summand of $L(\lambda) \otimes L(\mu)$, 70
$M(\nu)$, for $\mathbf{G}$ of type A_2, 89

M_{asph}: the anti-spherical $\mathbb{Z}[W_{\mathrm{aff}}]$-module, 132
$M^{[r]}$: the r-th Frobenius twist of M, 31
M_{reg}: the regular part of M, 55
M_{sing}: the singular part of M, 55
M^*: the dual of M, 19
M^τ: the contravariant dual of M, 19

$\mathcal{N}$: negligible tilting modules, 37
$\nabla(\lambda) = \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_\lambda)$, 20
$n_\beta(A)$, for an alcove A, 12
$n_\beta(C)$, for an ℓ-alcove C, 25
$N_x = 1 \otimes x \in M_{\mathrm{asph}}$, 132

$\Omega = \mathrm{Stab}_{W_{\mathrm{ext}}}(A_{\mathrm{fund}})$, 13
ω_x: image of $x \in W_{\mathrm{ext}}$ under $W_{\mathrm{ext}} \to \Omega$, 13

Φ: a simple root system, 9
Φ^+: a positive system in Φ, 9
Π: a base of Φ, 9
pr_μ: $\mathrm{Rep}(\mathbf{G}) \to \mathrm{Rep}_\mu(\mathbf{G})$, 27

$\mathrm{rad}_{\mathbf{G}} M$: the radical of M, 19
$\mathrm{rad}_{\mathbf{G}}^k M$: term in the radical filtration, 155
$\mathrm{Rep}(\mathbf{G})$, in the modular case, 17
$\mathrm{Rep}(\mathbf{G})$, in the quantum case, 18
$\mathrm{Rep}_\lambda(\mathbf{G})$: the linkage class of λ, 26
$\mathrm{Rep}_{\Omega\cdot 0}(\mathbf{G})$: the extended principal block, 27
$\underline{\mathrm{Rep}}(\mathbf{G}) = \mathrm{Rep}(\mathbf{G})/\langle\mathcal{N}\rangle$, 51
$\mathrm{Rep}(\mathbf{G}, \pi)$: a truncated subcategory, 78
$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, 9

S: simple reflections in W_{aff}, 12
$s_{\beta,m} = t_{m\beta}s_\beta$: an affine reflection, 11

$\mathrm{soc}_{\mathbf{G}} M$: the socle of M, 19
$\mathrm{soc}_{\mathbf{G}}^k M$: term in the socle filtration, 155

$\mathbf{T} = \mathbf{T}_\Bbbk$ in the modular case, 15
$\mathbf{T} = U_\zeta^0(\mathfrak{g})$ in the quantum case, 19
$T(\lambda)$: an indecomposable tilting module, 23
$\mathrm{Tilt}(\mathbf{G})$: tilting modules in $\mathrm{Rep}(\mathbf{G})$, 23
T_λ^μ: a translation functor, 27
$T_\lambda^{\mu,\delta}$: a quasi-translation functor, 107
$T^\omega = \bigoplus_{\lambda\in\Omega\cdot 0} T_\lambda^{\omega\cdot\lambda}$, 56

$U_q(\mathfrak{g})$: quantum group over $\mathbb{Q}(q)$, 16
$U_q^{\mathbb{Z}}(\mathfrak{g})$: integral form of $U_q(\mathfrak{g})$, 16
$U_\zeta(\mathfrak{g}) = U'_\zeta(\mathfrak{g})/\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 \mid \alpha \in \Pi\rangle$, 17
$U'_\zeta(\mathfrak{g}) = U_q^{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \Bbbk$, 17

$w_0 \in W_{\mathrm{fin}}$: the longest element, 10
$W_{\mathrm{aff}} = \mathbb{Z}\Phi \rtimes W_{\mathrm{fin}}$: the affine Weyl group, 11
$W_{\mathrm{aff}}^+ = \{w \in W_{\mathrm{aff}} \mid w(A_{\mathrm{fund}}) \text{ is dominant}\}$, 13
$W_{\mathrm{ext}} = X \rtimes W_{\mathrm{fin}}$: extended aff. Weyl group, 11
$W_{\mathrm{ext}}^+ = \{w \in W_{\mathrm{ext}} \mid w(A_{\mathrm{fund}}) \text{ is dominant}\}$, 13
$\mathrm{wfd}(M)$: Weyl filtration dimension of M, 29
W_{fin}: the (finite) Weyl group of Φ, 9

X: the weight lattice, 9
X^+: the dominant weights, 9
$\mathfrak{X} = \{x(i,j) \mid 0 \le i < n,\ 0 \le j \le n\} \cup \{e\}$, 134
$x(i,j) = s_0 s_1 \cdots s_i s_n s_{n-1} \cdots s_{n-j+1}$, 134
$X_r = \{\lambda \in X^+ \mid (\lambda, \alpha^\vee) < \ell^r \text{ for all } \alpha \in \Pi\}$, 32
$X_{\mathbb{R}}^+$: the dominant Weyl chamber, 12

ζ: a primitive ℓ-th root of unity, 16

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