

Classical soft graviton theorem rewritten

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ABSTRACT: Classical soft graviton theorem gives the gravitational wave-form at future null infinity at late retarded time u for a general classical scattering. The large u expansion has three known universal terms: the constant term, the term proportional to $1/u$ and the term proportional to $\ln u/u^2$, whose coefficients are determined solely in terms of the momenta of incoming and the outgoing hard particles, including the momenta carried by outgoing gravitational and electromagnetic radiation produced during scattering. For the constant term, also known as the memory effect, the dependence on the momenta carried away by the final state radiation / massless particles is known as non-linear memory or null memory. It was shown earlier that for the coefficient of the $1/u$ term the dependence on the momenta of the final state massless particles / radiation cancels and the result can be written solely in terms of the momenta of the incoming particles / radiation and the final state massive particles. In this note we show that the same result holds for the coefficient of the $\ln u/u^2$ term. Our result implies that for scattering of massless particles the coefficients of the $1/u$ and $\ln u/u^2$ terms are determined solely by the incoming momenta, even if the particles coalesce to form a black hole and massless radiation. We use our result to compute the low frequency flux of gravitational radiation from the collision of massless particles at large impact parameter.

KEYWORDS: Scattering Amplitudes, Classical Theories of Gravity, Effective Field Theories

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1 Introduction and summary

Let us consider a general classical scattering process in which a set of m objects carrying four momenta p'_1, \dots, p'_m in the asymptotic past come together, interact and then disperse as a set of n objects carrying four momenta p_1, \dots, p_n . We shall choose the origin of our space-time coordinate system close to the region where the particles interact and consider a gravitational wave detector far away from the scattering region, whose space-time coordinates will be denoted by (t, \vec{x}) . Our object of interest will be the gravitational wave-form at the detector:

$$h_{\mu\nu}(t, \vec{x}) = \frac{1}{2}(g_{\mu\nu} - \eta_{\mu\nu}), \tag{1.1}$$

but we shall find it convenient to state the result in terms of a slightly different quantity that carries the same information:

$$e_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_\rho{}^\rho \quad \Leftrightarrow \quad h_{\mu\nu} \equiv e_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}e_\rho{}^\rho. \tag{1.2}$$

All indices are raised and lowered by the flat metric $\eta_{\rho\sigma}$. We also define

$$R \equiv |\vec{x}|, \quad \hat{n} = \frac{\vec{x}}{R}, \quad n = (1, \hat{n}), \tag{1.3}$$

and the retarded time at the detector:

$$u = t - t_0, \quad t_0 = \frac{R}{c} + \text{correction}, \tag{1.4}$$

where t_0 is taken to be the time around which the peak of the gravitational wave-form reaches the detector. The ‘correction’ proportional to $\ln R$ represents the effect of the time delay due to gravitational drag on the gravitational waves due to the objects involved in the scattering. We denote by G and c respectively the Newton’s gravitational constant and

the speed of light in flat space-time. We also use mostly + signature metric and compute the inner products with flat metric unless mentioned otherwise. Since we are displaying explicit factors of c , the zeroth component of the momentum will be given by the energy divided by c . In this convention, results based on soft theorem determine the form of $e_{\mu\nu}$ at the detector, up to gauge transformation, at late and early retarded time [1–4]:

$$\begin{aligned} e_{\mu\nu} &= A_{\mu\nu} + \frac{1}{u} B_{\mu\nu} + u^{-2} \ln |u| F_{\mu\nu} + \mathcal{O}(u^{-2}) + \mathcal{O}(R^{-2}), \quad \text{for large positive } u, \\ &= \frac{1}{u} C_{\mu\nu} + u^{-2} \ln |u| G_{\mu\nu} + \mathcal{O}(u^{-2}) + \mathcal{O}(R^{-2}), \quad \text{for large negative } u, \end{aligned} \quad (1.5)$$

where,

$$A^{\mu\nu} = \frac{2G}{Rc^3} \left[-\sum_{i=1}^n p_i^\mu p_i^\nu \frac{1}{n \cdot p_i} + \sum_{i=1}^m p_i'^\mu p_i'^\nu \frac{1}{n \cdot p_i'} \right], \quad (1.6)$$

$$\begin{aligned} B^{\mu\nu} &= -\frac{4G^2}{Rc^7} \left[\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{p_i \cdot p_j}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \left\{ \frac{3}{2} p_i^2 p_j^2 - (p_i \cdot p_j)^2 \right\} \frac{p_i^\mu}{n \cdot p_i} (n \cdot p_j p_i^\nu - n \cdot p_i p_j^\nu) \right. \\ &\quad \left. - \left\{ \sum_{j=1}^n p_j \cdot n \sum_{i=1}^n \frac{1}{p_i \cdot n} p_i^\mu p_i^\nu - \sum_{j=1}^m p_j' \cdot n \sum_{i=1}^m \frac{1}{p_i' \cdot n} p_i'^\mu p_i'^\nu \right\} \right], \end{aligned} \quad (1.7)$$

$$C^{\mu\nu} = \frac{4G^2}{Rc^7} \left[\sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \frac{p_i' \cdot p_j'}{\{(p_i' \cdot p_j')^2 - p_i'^2 p_j'^2\}^{3/2}} \left\{ \frac{3}{2} p_i'^2 p_j'^2 - (p_i' \cdot p_j')^2 \right\} \frac{p_i'^\mu}{n \cdot p_i'} (n \cdot p_j' p_i'^\nu - n \cdot p_i' p_j'^\nu) \right],$$

$$\begin{aligned} F^{\mu\nu} &= \frac{2G^3}{Rc^{11}} \left[4 \sum_{j=1}^n p_j \cdot n \sum_{\ell=1}^n p_\ell \cdot n \sum_{i=1}^n \frac{p_i^\mu p_i^\nu}{p_i \cdot n} - 4 \sum_{j=1}^m p_j' \cdot n \sum_{\ell=1}^m p_\ell' \cdot n \sum_{i=1}^m \frac{p_i'^\mu p_i'^\nu}{p_i' \cdot n} \right. \\ &\quad + 4 \sum_{\ell=1}^n p_\ell \cdot n \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{p_i \cdot n} \\ &\quad \times \frac{p_i \cdot p_j}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \{2(p_i \cdot p_j)^2 - 3p_i^2 p_j^2\} \{n \cdot p_j p_i^\mu p_i^\nu - n \cdot p_i p_i^\mu p_j^\nu\} \\ &\quad + 2 \sum_{\ell=1}^m p_\ell' \cdot n \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \frac{1}{p_i' \cdot n} \\ &\quad \times \frac{p_i' \cdot p_j'}{\{(p_i' \cdot p_j')^2 - p_i'^2 p_j'^2\}^{3/2}} \{2(p_i' \cdot p_j')^2 - 3p_i'^2 p_j'^2\} \{n \cdot p_j' p_i'^\mu p_i'^\nu - n \cdot p_i' p_i'^\mu p_j'^\nu\} \\ &\quad + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \sum_{\ell=1}^n \frac{1}{p_i \cdot n} \frac{p_i \cdot p_j}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \{2(p_i \cdot p_j)^2 - 3p_i^2 p_j^2\} \\ &\quad \times \frac{p_i \cdot p_\ell}{\{(p_i \cdot p_\ell)^2 - p_i^2 p_\ell^2\}^{3/2}} \{2(p_i \cdot p_\ell)^2 - 3p_i^2 p_\ell^2\} \{n \cdot p_j p_i^\mu - n \cdot p_i p_j^\mu\} \{n \cdot p_\ell p_i^\nu - n \cdot p_i p_\ell^\nu\} \Big], \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} G^{\mu\nu} &= -\frac{2G^3}{Rc^{11}} \left[2 \sum_{\ell=1}^m p_\ell' \cdot n \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \frac{1}{p_i' \cdot n} \right. \\ &\quad \times \frac{p_i' \cdot p_j'}{\{(p_i' \cdot p_j')^2 - p_i'^2 p_j'^2\}^{3/2}} \{2(p_i' \cdot p_j')^2 - 3p_i'^2 p_j'^2\} \{n \cdot p_j' p_i'^\mu p_i'^\nu - n \cdot p_i' p_i'^\mu p_j'^\nu\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \sum_{\substack{\ell=1 \\ \ell \neq i}}^m \frac{1}{p'_i \cdot n} \frac{p'_i \cdot p'_j}{\{(p'_i \cdot p'_j)^2 - p_i'^2 p_j'^2\}^{3/2}} \{2(p'_i \cdot p'_j)^2 - 3p_i'^2 p_j'^2\} \frac{p'_i \cdot p'_\ell}{\{(p'_i \cdot p'_\ell)^2 - p_i'^2 p_\ell'^2\}^{3/2}} \\
& \times \{2(p'_i \cdot p'_\ell)^2 - 3p_i'^2 p_\ell'^2\} \{n \cdot p'_j p_i'^\mu - n \cdot p'_i p_j'^\mu\} \{n \cdot p'_\ell p_i'^\nu - n \cdot p'_i p_\ell'^\nu\} \Big]. \quad (1.9)
\end{aligned}$$

Note that the coefficients $A^{\mu\nu}$, $B^{\mu\nu}$, $C^{\mu\nu}$, $F^{\mu\nu}$ and $G^{\mu\nu}$ are given only by the momenta of the incoming and the outgoing objects and do not depend on the details of the scattering process. One can also check that each of these coefficients is gauge invariant, i.e. vanishes upon contraction with the four vector n .

In the above formulæ, $A_{\mu\nu}$ represents a permanent change in metric due to the passage of the gravitational wave, and is known as the memory effect [5–14]. Its connection to the leading soft graviton theorem has been discussed in [15]. The coefficients $B_{\mu\nu}$ and $C_{\mu\nu}$, representing long range tail of the gravitational wave-form, are related to the logarithmic correction to the subleading soft graviton theorem [3]. The coefficients $F_{\mu\nu}$ and $G_{\mu\nu}$ are related to the leading logarithmic correction to the subsubleading soft graviton theorem [3, 4].

If a significant fraction of energy is carried away by radiation, then the sum over the final state momenta in the expressions for $A_{\mu\nu}$, $B_{\mu\nu}$ and $F_{\mu\nu}$ should include integration over outgoing flux of radiation, regarded as a flux of massless particles. $C_{\mu\nu}$ and $G_{\mu\nu}$ are given in terms of incoming momenta only and are not sensitive to the momenta of outgoing particles or radiation.

The contribution to $A_{\mu\nu}$ due to the final state gravitational waves is some time referred to as non-linear memory [9] or null memory [14]. This makes the computation of $A_{\mu\nu}$ somewhat hard since we need to first find the angular distribution of the flux of energy carried away by the gravitational waves. *A priori*, computation of $B_{\mu\nu}$ and $F_{\mu\nu}$ suffers from the same difficulty. However it was found in [2] that due to some miraculous cancellations, the contribution to $B_{\mu\nu}$ due to final state massless particles, including gravitational waves, can be expressed in terms of the momenta of massive objects only. If we denote by \tilde{p}_i the momenta carried by the final state massive objects only and by \tilde{n} the number of such objects, then the modified formula takes the form:

$$\begin{aligned}
B^{\mu\nu} = & - \frac{4G^2}{Rc^7} \left[\sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \left\{ \frac{3}{2} \tilde{p}_i^2 \tilde{p}_j^2 - (\tilde{p}_i \cdot \tilde{p}_j)^2 \right\} \frac{\tilde{p}_i^\mu}{n \cdot \tilde{p}_i} (n \cdot \tilde{p}_j \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\nu) \right. \\
& \left. - \left\{ \sum_{j=1}^{\tilde{n}} \tilde{p}_j \cdot n \sum_{i=1}^{\tilde{n}} \frac{1}{\tilde{p}_i \cdot n} \tilde{p}_i^\mu \tilde{p}_i^\nu - \sum_{j=1}^m p'_j \cdot n \sum_{i=1}^m \frac{1}{p'_i \cdot n} p_i'^\mu p_i'^\nu \right\} + \tilde{P}^\mu \tilde{P}^\nu - P'^\mu P'^\nu \right], \quad (1.10)
\end{aligned}$$

where

$$P' = \sum_{i=1}^m p'_i, \quad \tilde{P} = \sum_{i=1}^{\tilde{n}} \tilde{p}_i. \quad (1.11)$$

We emphasize that (1.10) is not an independent formulæ but follows from (1.7) after setting $p_i^2 = 0$ for the massless final state particles.

In this paper we shall show that a similar rewriting is possible for the quantity $F^{\mu\nu}$ as well. In particular, the expression for $F^{\mu\nu}$ can be manipulated into the form:

$$\begin{aligned}
F^{\mu\nu} = & \frac{2G^3}{Rc^{11}} \left[4 \sum_{j=1}^{\tilde{n}} \tilde{p}_j \cdot n \sum_{\ell=1}^{\tilde{n}} \tilde{p}_\ell \cdot n \sum_{i=1}^{\tilde{n}} \frac{\tilde{p}_i^\mu \tilde{p}_i^\nu}{\tilde{p}_i \cdot n} - 4 \sum_{j=1}^m p'_j \cdot n \sum_{\ell=1}^m p'_\ell \cdot n \sum_{i=1}^m \frac{p'_i{}^\mu p'_i{}^\nu}{p'_i \cdot n} \right. \\
& + 4 \sum_{\ell=1}^{\tilde{n}} \tilde{p}_\ell \cdot n \sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \frac{1}{\tilde{p}_i \cdot n} \\
& \quad \times \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{2(\tilde{p}_i \cdot \tilde{p}_j)^2 - 3\tilde{p}_i^2 \tilde{p}_j^2\} \{n \cdot \tilde{p}_j \tilde{p}_i^\mu \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\mu \tilde{p}_j^\nu\} \\
& + 2 \sum_{\ell=1}^m p'_\ell \cdot n \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \frac{1}{p'_i \cdot n} \\
& \quad \times \frac{p'_i \cdot p'_j}{\{(p'_i \cdot p'_j)^2 - p_i'^2 p_j'^2\}^{3/2}} \{2(p'_i \cdot p'_j)^2 - 3p_i'^2 p_j'^2\} \{n \cdot p'_j p_i'^\mu p_i'^\nu - n \cdot p'_i p_j'^\mu p_j'^\nu\} \\
& + \sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \sum_{\substack{\ell=1 \\ \ell \neq i}}^{\tilde{n}} \frac{1}{\tilde{p}_i \cdot n} \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{2(\tilde{p}_i \cdot \tilde{p}_j)^2 - 3\tilde{p}_i^2 \tilde{p}_j^2\} \frac{\tilde{p}_i \cdot \tilde{p}_\ell}{\{(\tilde{p}_i \cdot \tilde{p}_\ell)^2 - \tilde{p}_i^2 \tilde{p}_\ell^2\}^{3/2}} \\
& \quad \times \{2(\tilde{p}_i \cdot \tilde{p}_\ell)^2 - 3\tilde{p}_i^2 \tilde{p}_\ell^2\} \{n \cdot \tilde{p}_j \tilde{p}_i^\mu - n \cdot \tilde{p}_i \tilde{p}_j^\mu\} \{n \cdot \tilde{p}_\ell \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_\ell^\nu\} \\
& \left. + 4n \cdot P' P'^\mu P'^\nu - 4n \cdot \tilde{P} \tilde{P}^\mu \tilde{P}^\nu \right]. \tag{1.12}
\end{aligned}$$

This corresponds to restricting the sum over final state particles in (1.8) to over massive particles only, and adding the compensating term given in the last line of (1.12).

It follows from (1.10) and (1.12) that if the final state contains only massless particles, then $B_{\mu\nu}$ and $F_{\mu\nu}$ will depend on only the momenta of the objects in the initial state. In fact it can also be seen with little effort that if the final state contains one massive object and arbitrary number of massless objects, then $B_{\mu\nu}$ and $F_{\mu\nu}$ still depend only on the momenta of the objects in the initial state. In particular the terms proportional to $\tilde{P}^\mu \tilde{P}^\nu$ in these expressions exactly cancel the terms involving \tilde{p}_i .

Finally we would like to mention that if some of the initial and /or final state particles are charged then we also need to take into account the effect of long range electromagnetic interaction among these particles. These effects have been studied in [2–4]. Following the same procedure that will be described in section 2 and section 3, it is easy to show that even in the presence of charged particles in the initial and the final states, the late time gravitational wave-form at future null infinity continues to be independent of the individual momenta of the final state massless particles. This has been demonstrated in appendix A.

In section 4 and 5 we apply our results to analyze gravitational radiation emitted during scattering of massless particles and compare the results with those in [16–18].

2 Review of the analysis of the subleading term

In this section we shall briefly review the analysis of the subleading term leading to (1.10). For this we divide the set of final momenta $\{p_i\}$ into the massive particle momenta \tilde{p}_i and

the massless particle momenta $\{\hat{p}_i\}$. We now use the fact that if either p_i or p_j represents a massless particle, then we have:

$$\frac{p_i \cdot p_j}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \left\{ \frac{3}{2} p_i^2 p_j^2 - (p_i \cdot p_j)^2 \right\} = 1. \quad (2.1)$$

Therefore the first term in the expression for $B^{\mu\nu}$ given in (1.7) may be written as:

$$\begin{aligned} & \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{p_i \cdot p_j}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \left\{ \frac{3}{2} p_i^2 p_j^2 - (p_i \cdot p_j)^2 \right\} \frac{p_i^\mu}{n \cdot p_i} (n \cdot p_j p_i^\nu - n \cdot p_i p_j^\nu) \\ &= \sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \left\{ \frac{3}{2} \tilde{p}_i^2 \tilde{p}_j^2 - (\tilde{p}_i \cdot \tilde{p}_j)^2 \right\} \frac{\tilde{p}_i^\mu}{n \cdot \tilde{p}_i} (n \cdot \tilde{p}_j \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\nu) \\ &+ \sum_i \sum_j \frac{\tilde{p}_i^\mu}{n \cdot \tilde{p}_i} (n \cdot \hat{p}_j \tilde{p}_i^\nu - n \cdot \tilde{p}_i \hat{p}_j^\nu) + \sum_i \sum_j \frac{\hat{p}_i^\mu}{n \cdot \hat{p}_i} (n \cdot \tilde{p}_j \hat{p}_i^\nu - n \cdot \hat{p}_i \tilde{p}_j^\nu) \\ &+ \sum_i \sum_j \frac{\hat{p}_i^\mu}{n \cdot \hat{p}_i} (n \cdot \hat{p}_j \hat{p}_i^\nu - n \cdot \hat{p}_i \hat{p}_j^\nu). \end{aligned} \quad (2.2)$$

We can now use the result of momentum conservation:

$$\sum_j \tilde{p}_j^\nu + \sum_j \hat{p}_j^\nu = P^{\nu}, \quad P' \equiv \sum_i p_i', \quad (2.3)$$

to express (2.2) as,

$$\begin{aligned} & \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{p_i \cdot p_j}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \left\{ \frac{3}{2} p_i^2 p_j^2 - (p_i \cdot p_j)^2 \right\} \frac{p_i^\mu}{n \cdot p_i} (n \cdot p_j p_i^\nu - n \cdot p_i p_j^\nu) \\ &= \sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \left\{ \frac{3}{2} \tilde{p}_i^2 \tilde{p}_j^2 - (\tilde{p}_i \cdot \tilde{p}_j)^2 \right\} \frac{\tilde{p}_i^\mu}{n \cdot \tilde{p}_i} (n \cdot \tilde{p}_j \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\nu) \\ &+ \sum_i \frac{\tilde{p}_i^\mu \tilde{p}_i^\nu}{n \cdot \tilde{p}_i} n \cdot (P' - \tilde{P}) + \sum_i \frac{\hat{p}_i^\mu \hat{p}_i^\nu}{n \cdot \hat{p}_i} n \cdot P' + \tilde{P}^\mu \tilde{P}^\nu - P'^\mu P'^\nu. \end{aligned} \quad (2.4)$$

We also have

$$-\sum_{j=1}^n p_j \cdot n \sum_{i=1}^n \frac{1}{p_i \cdot n} p_i^\mu p_i^\nu = -n \cdot P' \sum_{i=1}^n \frac{1}{\tilde{p}_i \cdot n} \tilde{p}_i^\mu \tilde{p}_i^\nu - n \cdot P' \sum_i \frac{1}{\hat{p}_i \cdot n} \hat{p}_i^\mu \hat{p}_i^\nu. \quad (2.5)$$

Substituting (2.4) and (2.5) into (1.7), we get,

$$\begin{aligned} B^{\mu\nu} &= -\frac{4G^2}{Rc^7} \left[\sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \left\{ \frac{3}{2} \tilde{p}_i^2 \tilde{p}_j^2 - (\tilde{p}_i \cdot \tilde{p}_j)^2 \right\} \frac{\tilde{p}_i^\mu}{n \cdot \tilde{p}_i} (n \cdot \tilde{p}_j \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\nu) \right. \\ &\quad \left. - \left\{ \sum_{j=1}^{\tilde{n}} \tilde{p}_j \cdot n \sum_{i=1}^{\tilde{n}} \frac{1}{\tilde{p}_i \cdot n} \tilde{p}_i^\mu \tilde{p}_i^\nu - \sum_{j=1}^m p_j' \cdot n \sum_{i=1}^m \frac{1}{p_i' \cdot n} p_i'^\mu p_i'^\nu \right\} + \tilde{P}^\mu \tilde{P}^\nu - P'^\mu P'^\nu \right]. \end{aligned} \quad (2.6)$$

This reproduces (1.10).

3 Analysis of the subsubleading term

We shall now rewrite the expression (1.8) for $F^{\mu\nu}$ by dividing the sum over final state momenta into the contribution from massless and massive particle momenta, denoted by \hat{p}_i and \tilde{p}_i respectively. $F_{(n)}^{\mu\nu}$ will denote contribution from terms where in each term, we have n factors of \hat{p}_i . Therefore we have:

$$\begin{aligned}
 F_{(0)}^{\mu\nu} = & \frac{2G^3}{Rc^{11}} \left[4 \sum_{j=1}^{\tilde{n}} \tilde{p}_j \cdot n \sum_{\ell=1}^{\tilde{n}} \tilde{p}_\ell \cdot n \sum_{i=1}^{\tilde{n}} \frac{\tilde{p}_i^\mu \tilde{p}_i^\nu}{\tilde{p}_i \cdot n} - 4 \sum_{j=1}^m p'_j \cdot n \sum_{\ell=1}^m p'_\ell \cdot n \sum_{i=1}^m \frac{p'_i{}^\mu p'_i{}^\nu}{p'_i \cdot n} \right. \\
 & + 4 \sum_{\ell=1}^{\tilde{n}} \tilde{p}_\ell \cdot n \sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \frac{1}{\tilde{p}_i \cdot n} \\
 & \quad \times \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{2(\tilde{p}_i \cdot \tilde{p}_j)^2 - 3\tilde{p}_i^2 \tilde{p}_j^2\} \{n \cdot \tilde{p}_j \tilde{p}_i^\mu \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\mu \tilde{p}_j^\nu\} \\
 & + 2 \sum_{\ell=1}^m p'_\ell \cdot n \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \frac{1}{p'_i \cdot n} \\
 & \quad \times \frac{p'_i \cdot p'_j}{\{(p'_i \cdot p'_j)^2 - p_i'^2 p_j'^2\}^{3/2}} \{2(p'_i \cdot p'_j)^2 - 3p_i'^2 p_j'^2\} \{n \cdot p'_j p'_i{}^\mu p'_i{}^\nu - n \cdot p'_i p'_j{}^\mu p'_j{}^\nu\} \\
 & + \sum_{i=1}^{\tilde{n}} \sum_{\substack{j=1 \\ j \neq i}}^{\tilde{n}} \sum_{\substack{\ell=1 \\ \ell \neq i}}^{\tilde{n}} \frac{1}{\tilde{p}_i \cdot n} \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{2(\tilde{p}_i \cdot \tilde{p}_j)^2 - 3\tilde{p}_i^2 \tilde{p}_j^2\} \frac{\tilde{p}_i \cdot \tilde{p}_\ell}{\{(\tilde{p}_i \cdot \tilde{p}_\ell)^2 - \tilde{p}_i^2 \tilde{p}_\ell^2\}^{3/2}} \\
 & \quad \times \{2(\tilde{p}_i \cdot \tilde{p}_\ell)^2 - 3\tilde{p}_i^2 \tilde{p}_\ell^2\} \{n \cdot \tilde{p}_j \tilde{p}_i^\mu - n \cdot \tilde{p}_i \tilde{p}_j^\mu\} \{n \cdot \tilde{p}_\ell \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_\ell^\nu\} \left. \right]. \quad (3.1)
 \end{aligned}$$

In writing down the expressions for $F_{(n)}^{\mu\nu}$ for $n \geq 1$, we shall try to simplify the sum over final state momenta using the relations:

$$\sum_j \tilde{p}_j^\mu = \tilde{P}^\mu, \quad \sum_j \hat{p}_j^\mu = (P' - \tilde{P})^\mu. \quad (3.2)$$

Another simplification follows from the observation that the expression $\{n \cdot p_j p_i^\mu p_i^\nu - n \cdot p_i p_i^\mu p_j^\nu\}$ vanishes for $j = i$. Therefore unless the factor multiplying it diverges for $j = i$, we can include in the sum over j the term $j = i$ even if the original sum excludes this. This trick can often be used to make the sum over i and j into independent sums as long as either i or j represents a massless particle, since in this case we can first use (2.1) to replace the apparently divergent factor at $j = i$ by a finite term, and then include the contribution from the $j = i$ term in the sum. This gives:

$$\begin{aligned}
 F_{(1)}^{\mu\nu} = & \frac{2G^3}{Rc^{11}} \left[8n \cdot (P' - \tilde{P}) n \cdot \tilde{P} \sum_{i=1}^{\tilde{n}} \frac{\tilde{p}_i^\mu \tilde{p}_i^\nu}{\tilde{p}_i \cdot n} + 4n \cdot \tilde{P} n \cdot \tilde{P} \sum_i \frac{\hat{p}_i^\mu \hat{p}_i^\nu}{\hat{p}_i \cdot n} \right. \\
 & + 4n \cdot (P' - \tilde{P}) \sum_i \sum_{\substack{j=1 \\ j \neq i}} \frac{1}{\tilde{p}_i \cdot n} \\
 & \quad \times \frac{\tilde{p}_i \cdot \tilde{p}_j}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{2(\tilde{p}_i \cdot \tilde{p}_j)^2 - 3\tilde{p}_i^2 \tilde{p}_j^2\} \{n \cdot \tilde{p}_j \tilde{p}_i^\mu \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\mu \tilde{p}_j^\nu\}
 \end{aligned}$$

$$\begin{aligned}
 & -8n.\tilde{P} \sum_i \frac{1}{\tilde{p}_i.n} \{n.(P' - \tilde{P}) \tilde{p}_i^\mu \tilde{p}_i^\nu - n.\tilde{p}_i \tilde{p}_i^{(\mu} (P' - \tilde{P})^{\nu)}\} \\
 & -8n.\tilde{P} \sum_i \frac{1}{\hat{p}_i.n} \{n.\tilde{P} \hat{p}_i^\mu \hat{p}_i^\nu - n.\hat{p}_i \hat{p}_i^{(\mu} \tilde{P}^{\nu)}\} \quad non \quad (3.3) \\
 & +4 \sum_i \frac{1}{\hat{p}_i.n} \{n.\tilde{P} \hat{p}_i^\mu - n.\hat{p}_i \tilde{P}^\mu\} \{n.\tilde{P} \hat{p}_i^\nu - n.\hat{p}_i \tilde{P}^\nu\} \\
 & -4 \sum_i \sum_{j \neq i} \frac{1}{\tilde{p}_i.n} \frac{\tilde{p}_i.\tilde{p}_j}{\{(\tilde{p}_i.\tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{2(\tilde{p}_i.\tilde{p}_j)^2 - 3\tilde{p}_i^2 \tilde{p}_j^2\} \\
 & \quad \times \left\{ n.\tilde{p}_j n.(P' - \tilde{P}) \tilde{p}_i^\mu \tilde{p}_i^\nu - n.\tilde{p}_i n.(P' - \tilde{P}) \tilde{p}_j^\mu \tilde{p}_j^\nu - n.\tilde{p}_j n.\tilde{p}_i \tilde{p}_i^{(\mu} (P' - \tilde{P})^{\nu)} \right. \\
 & \quad \left. + (n.\tilde{p}_i)^2 \tilde{p}_j^{(\mu} (P' - \tilde{P})^{\nu)} \right\} \Big]. \quad (3.4)
 \end{aligned}$$

This can be simplified to:

$$\begin{aligned}
 F_{(1)}^{\mu\nu} &= \frac{2G^3}{Rc^{11}} \left[4n.\tilde{P} (P' - \tilde{P})^\mu \tilde{P}^\nu + 4n.\tilde{P} (P' - \tilde{P})^\nu \tilde{P}^\mu + 4n.(P' - \tilde{P}) \tilde{P}^\mu \tilde{P}^\nu \right. \\
 & \quad - 4 \sum_i \sum_{j \neq i} \frac{\tilde{p}_i.\tilde{p}_j}{\{(\tilde{p}_i.\tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{2(\tilde{p}_i.\tilde{p}_j)^2 - 3\tilde{p}_i^2 \tilde{p}_j^2\} \\
 & \quad \left. \times \left\{ -n.\tilde{p}_j \tilde{p}_i^{(\mu} (P' - \tilde{P})^{\nu)} + n.\tilde{p}_i \tilde{p}_j^{(\mu} (P' - \tilde{P})^{\nu)} \right\} \right]. \quad (3.5)
 \end{aligned}$$

We now note that the last line is anti-symmetric under the exchange of i and j while the second line is symmetric under this exchange. Therefore this term vanishes after summing over i and j , and we get:

$$F_{(1)}^{\mu\nu} = \frac{2G^3}{Rc^{11}} \left[4n.\tilde{P} (P' - \tilde{P})^\mu \tilde{P}^\nu + 4n.\tilde{P} (P' - \tilde{P})^\nu \tilde{P}^\mu + 4n.(P' - \tilde{P}) \tilde{P}^\mu \tilde{P}^\nu \right]. \quad (3.6)$$

We also have,

$$\begin{aligned}
 F_{(2)}^{\mu\nu} &= \frac{2G^3}{Rc^{11}} \left[4n.(P' - \tilde{P}) n.(P' - \tilde{P}) \sum_{i=1}^{\tilde{n}} \frac{\tilde{p}_i^\mu \tilde{p}_i^\nu}{\tilde{p}_i.n} + 8n.(P' - \tilde{P}) n.\tilde{P} \sum_i \frac{\hat{p}_i^\mu \hat{p}_i^\nu}{\hat{p}_i.n} \right. \\
 & \quad - 8n.(P' - \tilde{P}) \sum_i \frac{1}{\hat{p}_i.n} \{n.\tilde{P} \hat{p}_i^\mu \hat{p}_i^\nu - n.\hat{p}_i \hat{p}_i^\mu \tilde{P}^\nu\} \\
 & \quad - 8n.(P' - \tilde{P}) \sum_i \frac{1}{\tilde{p}_i.n} \{n.(P' - \tilde{P}) \tilde{p}_i^\mu \tilde{p}_i^\nu - n.\tilde{p}_i \tilde{p}_i^\mu (P' - \tilde{P})^\nu\} \\
 & \quad - 8n.\tilde{P} \sum_i \frac{1}{\hat{p}_i.n} \{n.(P' - \tilde{P}) \hat{p}_i^\mu \hat{p}_i^\nu - n.\hat{p}_i \hat{p}_i^\mu (P' - \tilde{P})^\nu\} \\
 & \quad + 4 \sum_i \frac{1}{\tilde{p}_i.n} \left\{ n.(P' - \tilde{P}) n.(P' - \tilde{P}) \tilde{p}_i^\mu \tilde{p}_i^\nu - n.\tilde{p}_i n.(P' - \tilde{P}) (P' - \tilde{P})^\mu \tilde{p}_i^\nu \right. \\
 & \quad \left. - n.(P' - \tilde{P}) n.\tilde{p}_i \tilde{p}_i^\mu (P' - \tilde{P})^\nu + (n.\tilde{p}_i)^2 (P' - \tilde{P})^\mu (P' - \tilde{P})^\nu \right\} \\
 & \quad \left. + 8 \sum_i \frac{1}{\hat{p}_i.n} \left\{ n.(P' - \tilde{P}) \hat{p}_i^{(\mu} - n.\hat{p}_i (P' - \tilde{P})^{(\mu)} \right\} \left\{ n.\tilde{P} \hat{p}_i^{\nu)} - n.\hat{p}_i \tilde{P}^{\nu)} \right\} \right]. \quad (3.7)
 \end{aligned}$$

This can be simplified to:

$$F_{(2)}^{\mu\nu} = \frac{2G^3}{Rc^{11}} \left[8n.(P' - \tilde{P})(P' - \tilde{P})^{(\mu} \tilde{P}^{\nu)} + 4n.\tilde{P}(P' - \tilde{P})^\mu (P' - \tilde{P})^\nu \right]. \quad (3.8)$$

Finally we have,

$$\begin{aligned} F_{(3)}^{\mu\nu} = & \frac{2G^3}{Rc^{11}} \left[4n.(P' - \tilde{P})n.(P' - \tilde{P}) \sum_i \frac{\hat{p}_i^\mu \hat{p}_i^\nu}{\hat{p}_i \cdot n} \right. \\ & - 8n.(P' - \tilde{P}) \sum_i \frac{1}{\hat{p}_i \cdot n} \{n.(P' - \tilde{P}) \hat{p}_i^\mu \hat{p}_i^\nu - n.\hat{p}_i \hat{p}_i^\mu (P' - \tilde{P})^\nu\} \\ & \left. + 4 \sum_i \frac{1}{\hat{p}_i \cdot n} \{n.(P' - \tilde{P}) \hat{p}_i^\mu - n.\hat{p}_i (P' - \tilde{P})^\mu\} \{n.(P' - \tilde{P}) \hat{p}_i^\nu - n.\hat{p}_i (P' - \tilde{P})^\nu\} \right]. \end{aligned} \quad (3.9)$$

This can be simplified to:

$$F_{(3)}^{\mu\nu} = \frac{2G^3}{Rc^{11}} \left[4n.(P' - \tilde{P})(P' - \tilde{P})^\mu (P' - \tilde{P})^\nu \right]. \quad (3.10)$$

Using (3.6), (3.8) and (3.10), we get,

$$F_{(1)}^{\mu\nu} + F_{(2)}^{\mu\nu} + F_{(3)}^{\mu\nu} = \frac{8G^3}{Rc^{11}} \left[n.P' P'^\mu P'^\nu - n.\tilde{P} \tilde{P}^\mu \tilde{P}^\nu \right]. \quad (3.11)$$

Adding this to (3.1) we get (1.12).

4 Example involving scattering of massless particles

In this section we shall compare our results to that of [16, 17] on the emission of soft gravitational radiation during the scattering of a pair of massless particles. For this comparison we shall set $c = 1$ since the results of [17] were given in that convention.¹ Let $\tilde{e}_{\mu\nu}(\omega, \vec{x})$ be the time Fourier transform of $e_{\mu\nu}(t, \vec{x})$:

$$\tilde{e}_{\mu\nu}(\omega, \vec{x}) = \int du e^{i\omega u} e_{\mu\nu}(t, \vec{x}), \quad u = t - t_0. \quad (4.1)$$

Then $\tilde{e}_{\mu\nu}(\omega, \vec{x})$ has a small ω expansion of the form [3]:

$$\tilde{e}_{\mu\nu}(\omega, \vec{x}) = i A_{\mu\nu} \omega^{-1} - (B_{\mu\nu} - C_{\mu\nu}) \ln \omega + \frac{i}{2} (F_{\mu\nu} - G_{\mu\nu}) \omega (\ln \omega)^2 + \dots, \quad (4.2)$$

where $A_{\mu\nu}$, $B_{\mu\nu}$, $C_{\mu\nu}$, $F_{\mu\nu}$ and $G_{\mu\nu}$ are the same coefficients that appeared in the large $|u|$ expansion of $e_{\mu\nu}$. The $i\epsilon$ prescription inside $\ln \omega$ captures separate information on $B_{\mu\nu}$ and $C_{\mu\nu}$ and also on $F_{\mu\nu}$ and $G_{\mu\nu}$ [3], but at present we shall proceed ignoring the $i\epsilon$ prescription.

¹A similar result was found in [18] where part of the contribution associated to the Coulomb phase of the soft radiation (e.g. the terms in the second line of (1.7)) was not included.

When all the incoming and outgoing particles are massless, we get from (1.8), (1.9), (1.10) and (1.12),

$$B^{\mu\nu} = -C^{\mu\nu} = -\frac{4G^2}{R} \left[n.P' \sum_i \frac{p_i^\mu p_i^{\nu'}}{n.p_i'} - P'^\mu P'^\nu \right], \quad (4.3)$$

and

$$F^{\mu\nu} = -G^{\mu\nu} = -\frac{16G^3}{R} n.P' \left[n.P' \sum_i \frac{p_i^\mu p_i^{\nu'}}{n.p_i'} - P'^\mu P'^\nu \right]. \quad (4.4)$$

We also have, from (1.6),

$$A^{\mu\nu} = \frac{2G}{R} \left[-\sum_{i=1}^n p_i^\mu p_i^\nu \frac{1}{n.p_i} + \sum_{i=1}^m p_i^{\prime\mu} p_i^{\prime\nu} \frac{1}{n.p_i'} \right]. \quad (4.5)$$

We now apply these results to the specific case of scattering of two massless particles into two massless particles and soft gravitational radiation. Following [17], we label the momenta of the incoming and outgoing hard particles as:

$$\begin{aligned} p_1' &= E(1, 0, 0, 1), & p_2' &= E(1, 0, 0, -1), \\ p_1 &= E(1, \sin \Theta_s \cos \phi, \sin \Theta_s \sin \phi, \cos \Theta_s), \\ p_2 &= E(1, -\sin \Theta_s \cos \phi, -\sin \Theta_s \sin \phi, -\cos \Theta_s). \end{aligned} \quad (4.6)$$

On the other hand, the direction of emission of the soft gravitational radiation, encoded in the four vector $n = (1, \hat{n})$, takes the form:

$$n = (1, \sin \theta, 0, \cos \theta). \quad (4.7)$$

Our choice of frame is rotated by an angle ϕ about the z -axis relative to the frame used in [17], so that the direction of propagation of the soft gravitational wave, and not the momenta of the outgoing hard particles, lies in the x - z plane. We now define

$$\hat{e}^+ = (0, \cos \theta, i, -\sin \theta), \quad \hat{e}^- = (0, \cos \theta, -i, -\sin \theta), \quad (4.8)$$

so that,

$$\epsilon_{\mu\nu}^+ = \frac{1}{2} \hat{e}_\mu^+ \hat{e}_\nu^+, \quad \epsilon_{\mu\nu}^- = \frac{1}{2} \hat{e}_\mu^- \hat{e}_\nu^-, \quad (4.9)$$

denote left and right circular polarizations of soft gravitational waves traveling along n .

Using (4.2) we now get,

$$\epsilon_{\mu\nu}^\pm \tilde{e}^{\mu\nu}(\omega, \vec{x}) = \epsilon_{\mu\nu}^\pm \left[i A^{\mu\nu} \omega^{-1} - (B^{\mu\nu} - C^{\mu\nu}) \ln \omega + \frac{i}{2} (F^{\mu\nu} - G^{\mu\nu}) \omega (\ln \omega)^2 + \dots \right]. \quad (4.10)$$

Since [17] gives the result for small θ and Θ_s , we shall also make this approximation. However, in section 5 we have given the results for finite θ and Θ_s . Now, for small θ, Θ_s ,

$$\begin{aligned} \hat{e}^\pm.p_1' &= -E \sin \theta \simeq -E \theta, \\ \hat{e}^\pm.p_2' &= E \sin \theta \simeq E \theta, \end{aligned}$$

$$\begin{aligned}
 \hat{e}^\pm \cdot p_1 &= E (\sin \Theta_s \cos \theta \cos \phi - \cos \Theta_s \sin \theta \pm i \sin \Theta_s \sin \phi) \\
 &\simeq E(\Theta_s \cos \phi - \theta \pm i \Theta_s \sin \phi), \\
 \hat{e}^\pm \cdot p_2 &= -E (\sin \Theta_s \cos \theta \cos \phi - \cos \Theta_s \sin \theta \pm i \sin \Theta_s \sin \phi) \\
 &\simeq -E(\Theta_s \cos \phi - \theta \pm i \Theta_s \sin \phi), \\
 n \cdot p'_1 &= -E (1 - \cos \theta) \simeq -E \theta^2/2, \\
 n \cdot p'_2 &= -E (1 + \cos \theta) \simeq -2E \\
 n \cdot p_1 &= -E (1 - \sin \Theta_s \sin \theta \cos \phi - \cos \Theta_s \cos \theta) \simeq -E (\Theta_s^2 + \theta^2 - 2 \Theta_s \theta \cos \phi)/2 \\
 n \cdot p_2 &= -E (1 + \sin \Theta_s \sin \theta \cos \phi + \cos \Theta_s \cos \theta) \simeq -2E. \tag{4.11}
 \end{aligned}$$

It follows from these equations that $\epsilon_{\mu\nu}^\pm p_i^\mu p_i^\nu$ and $\epsilon_{\mu\nu}^\pm p_i^{\prime\mu} p_i^{\prime\nu}$ are quadratic in the small parameters Θ_s and θ . Therefore only terms with $n \cdot p_1$ or $n \cdot p'_1$ in the denominator will survive in this limit. This gives:

$$\begin{aligned}
 \epsilon_{\mu\nu}^\pm A^{\mu\nu} &= \frac{G}{R} \left[-\frac{\hat{e}^\pm \cdot p_1 \hat{e}^\pm \cdot p_1}{n \cdot p_1} + \frac{\hat{e}^\pm \cdot p'_1 \hat{e}^\pm \cdot p'_1}{n \cdot p'_1} \right] \\
 &= \frac{2GE}{R} \left[\frac{(\Theta_s \cos \phi - \theta \pm i \Theta_s \sin \phi)^2}{(\Theta_s^2 + \theta^2 - 2 \Theta_s \theta \cos \phi)} - 1 \right], \tag{4.12}
 \end{aligned}$$

$$\epsilon_{\mu\nu}^\pm (B^{\mu\nu} - C^{\mu\nu}) = -\frac{4G^2}{R} n \cdot P' \frac{\hat{e}^\pm \cdot p'_1 \hat{e}^\pm \cdot p'_1}{n \cdot p'_1} = -\frac{16G^2 E^2}{R}, \tag{4.13}$$

and

$$\epsilon_{\mu\nu}^\pm (F^{\mu\nu} - G^{\mu\nu}) = -\frac{16G^3}{R} (n \cdot P')^2 \frac{\hat{e}^\pm \cdot p'_1 \hat{e}^\pm \cdot p'_1}{n \cdot p'_1} = \frac{128G^3 E^3}{R}. \tag{4.14}$$

Following [17], we introduce the variable ψ via:

$$\sin \psi = \frac{\Theta_s \sin \phi}{(\Theta_s^2 + \theta^2 - 2 \Theta_s \theta \cos \phi)^{1/2}}, \quad \cos \psi = \frac{\Theta_s \cos \phi - \theta}{(\Theta_s^2 + \theta^2 - 2 \Theta_s \theta \cos \phi)^{1/2}}, \tag{4.15}$$

so that (4.12) may be expressed as:

$$\epsilon_{\mu\nu}^\pm A^{\mu\nu} = \frac{2GE}{R} \left[e^{\pm 2i\psi} - 1 \right]. \tag{4.16}$$

Substituting (4.13), (4.14) and (4.16) into (4.10), we get,

$$\epsilon_{\mu\nu}^\pm \tilde{e}^{\mu\nu} = i \frac{2GE}{R} \omega^{-1} \left[e^{\pm 2i\psi} - 1 - i 8GE \omega \ln \omega + 32G^2 E^2 \omega^2 (\ln \omega)^2 \right]. \tag{4.17}$$

Up to an overall normalization this agrees with the small ω expansion of eq. (6.20) of [17] after identifying the variable R of [17], describing the Schwarzschild radius of the system, with $4GE$. We shall now verify that the overall normalization also agrees.

To check the overall normalization, we compute the energy flux associated with (4.17) at the leading order in ω . This can be done using the formula for the angular distribution of the energy flux with a given polarization. In the $8\pi G = 1$ unit the flux is given by (see e.g. [19]):

$$\frac{dE_\pm}{d\omega d\Omega} = \frac{\omega^2}{\pi} R^2 |\epsilon_{\mu\nu}^\pm \tilde{e}^{\mu\nu}|^2 = \frac{E^2}{16\pi^3} |e^{\pm 2i\psi} - 1|^2. \tag{4.18}$$

On the other hand, the same flux computed in [17] at the leading order in ω is given by (see eq. (6.13)):

$$\frac{4G E^2}{8\pi^2} \left| e^{\pm 2i\psi} - 1 \right|^2 = \frac{E^2}{16\pi^3} \left| e^{\pm 2i\psi} - 1 \right|^2. \quad (4.19)$$

Comparing (4.18) and (4.19) we see that the overall normalizations also match.

Even though we have derived the various formulæ in the limit of small Θ_s and θ for comparison with the results of [17], it follows from our general result that even for general values of Θ_s and θ , our expressions for $B_{\mu\nu} = -C_{\mu\nu}$ and $F_{\mu\nu} = -G_{\mu\nu}$ remain the same as those given in (4.13) and (4.14). What is perhaps more striking is that even if the incoming states have a small enough impact parameter so that they form a black hole, possibly accompanied by hard radiation, the expressions for $B_{\mu\nu}$ and $F_{\mu\nu}$ do not change. This follows from the discussion in the last paragraph of section 1 since we have only one massive object in the final state.

Before concluding this section, we would like to discuss another aspect of the results given in [17]. [17] used the wave-form (4.17) to compute the total flux of soft radiation to subsubleading order. However on dimensional grounds, the $\ln \omega$ in (4.17) should be regarded as $\ln(\omega a)$ where a is some scale of the order of the size of the scattering region, giving non universal contribution of order $\omega \ln a$. Therefore there could be terms of order ω inside the square bracket that have not been computed. This would give an additional contribution at the subleading order, spoiling the subsubleading results of [17]. It is easy to see that an imaginary term of order ω inside the square bracket in (4.17) will not contribute to the energy flux at the subleading order after summing over polarizations of the gravitational radiation. On the other hand a real term proportional to ω will violate the reality condition $\tilde{e}_{\mu\nu}(\omega)^* = \tilde{e}_{\mu\nu}(-\omega)$ that is required for the reality of the gravitational field. However there could be a contribution proportional to $\omega \{H(\omega) - H(-\omega)\}$ inside the square bracket, with H denoting the Heaviside function, that satisfies the reality condition. If present, such a term would give subleading contribution to the energy flux, spoiling the result of [17].

We shall now show that such a term is absent, but for this we need to carefully keep track of the $i\epsilon$ prescription in the argument of the logarithms in (4.2). It follows from the analysis of [3], that at the subleading order, the time Fourier transform of the wave-form, including the $i\epsilon$ prescription, is given by:

$$-\frac{1}{2} \{ B_{\mu\nu} \ln(\omega + i\epsilon) - C_{\mu\nu} \ln(\omega - i\epsilon) + B_{\mu\nu} \ln(-\omega - i\epsilon) - C_{\mu\nu} \ln(-\omega + i\epsilon) \}. \quad (4.20)$$

Using (4.3) this can be written as:

$$\frac{1}{2} C_{\mu\nu} \{ \ln(\omega + i\epsilon) + \ln(\omega - i\epsilon) + \ln(-\omega - i\epsilon) + \ln(-\omega + i\epsilon) \} = 2 C_{\mu\nu} \ln |\omega|. \quad (4.21)$$

Therefore at the subleading order we do not have terms proportional to $H(\omega) - H(-\omega)$. Note that this is a consequence of the relation $B_{\mu\nu} = -C_{\mu\nu}$ and seems to be present when all the incoming and outgoing particles are massless. From this it follows that (4.17) can be used to compute the total flux of energy carried by the gravitational radiation to order $\omega^2 (\ln \omega)^2$, reproducing the result of [17].

5 Energy flux from massless particle scattering

In section 4 we compared our results for radiation during scattering of massless particles at small angle with those of [17]. In this section we shall compute the energy flux of low frequency gravitational radiation produced during such a scattering without making the small angle approximation.

Using (4.3)–(4.5) and (4.11) without making the small θ, Θ_s approximation, we get

$$\begin{aligned}\epsilon_{\mu\nu}^{\pm} A^{\mu\nu} &= \frac{2GE}{R} \left[e^{\pm 2i\psi} - 1 \right], \\ \sin \psi &= \frac{\sin \Theta_s \sin \phi}{\{1 - (\sin \Theta_s \sin \theta \cos \phi + \cos \Theta_s \cos \theta)^2\}^{1/2}}, \\ \cos \psi &= \frac{\sin \Theta_s \cos \theta \cos \phi - \cos \Theta_s \sin \theta}{\{1 - (\sin \Theta_s \sin \theta \cos \phi + \cos \Theta_s \cos \theta)^2\}^{1/2}}, \\ \epsilon_{\mu\nu}^{\pm} (B^{\mu\nu} - C^{\mu\nu}) &= -\frac{16G^2E^2}{R}, \quad \epsilon_{\mu\nu}^{\pm} (F^{\mu\nu} - G^{\mu\nu}) = \frac{128G^3E^3}{R}.\end{aligned}\quad (5.1)$$

Using (4.2) and the first equality in (4.18), and summing over polarizations, we now get the total differential flux in the $8\pi G = 1$ unit:

$$\begin{aligned}\frac{dE}{d\omega d\Omega} &\equiv \sum_{\pm} \frac{dE_{\pm}}{d\omega d\Omega} \\ &= \sum_{\pm} \frac{\omega^2}{\pi} R^2 \left| \epsilon_{\mu\nu}^{\pm} \left\{ i A^{\mu\nu} \omega^{-1} - \ln \omega (B^{\mu\nu} - C^{\mu\nu}) + \frac{i}{2} \omega (\ln \omega)^2 (F^{\mu\nu} - G^{\mu\nu}) \right\} \right|^2.\end{aligned}\quad (5.2)$$

Using (5.1), (5.1), this can be written as

$$P(\theta, \phi) + Q(\theta, \phi) \omega^2 (\ln \omega)^2 + \mathcal{O}(\omega^2 \ln \omega), \quad (5.3)$$

where

$$P(\theta, \phi) = \frac{E^2}{2\pi^3} \sin^2 \psi, \quad Q(\theta, \phi) = \frac{E^4}{8\pi^5} (1 - 2 \sin^2 \psi). \quad (5.4)$$

Therefore for computing the total flux, we need to evaluate the integral:

$$\begin{aligned}I &\equiv \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin^2 \psi \\ &= \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \frac{\sin^2 \Theta_s \sin^2 \phi}{\{1 - (\sin \Theta_s \sin \theta \cos \phi + \cos \Theta_s \cos \theta)^2\}}.\end{aligned}\quad (5.5)$$

Now, writing

$$\begin{aligned}\frac{\sin^2 \Theta_s \sin^2 \phi}{\{1 - (\sin \Theta_s \sin \theta \cos \phi + \cos \Theta_s \cos \theta)^2\}} &= \frac{1}{2} \frac{\sin^2 \Theta_s \sin^2 \phi}{\{1 - (\sin \Theta_s \sin \theta \cos \phi + \cos \Theta_s \cos \theta)\}} \\ &\quad + (\theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi),\end{aligned}\quad (5.6)$$

and noting that both terms produce the same integral, we can express I as:

$$I = \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \frac{\sin^2 \Theta_s \sin^2 \phi}{\{1 - (\sin \Theta_s \sin \theta \cos \phi + \cos \Theta_s \cos \theta)\}}. \quad (5.7)$$

We can carry out the ϕ integral by defining $z = e^{i\phi}$, replacing $\sin \phi$ by $(z - z^{-1})/(2i)$, $\cos \phi$ by $(z + z^{-1})/2$, and regarding (5.7) as a contour integral over z along the unit circle. The resulting integrand has double pole at $z = 0$ and single poles at $z = z_{\pm}$ where

$$z_{\pm} = \frac{(1 \pm \cos \theta_s)(1 \mp \cos \theta)}{\sin \theta \sin \theta_s}. \quad (5.8)$$

It is easy to see that for $\theta > \Theta_s$, $z_+ > 1$, $z_- < 1$ and for $\theta < \Theta_s$, $z_+ < 1$, $z_- > 1$. We can now perform the contour integral by picking up the residues at the poles inside the unit circle. The result is:

$$I = 2\pi \int_0^{\pi} \sin \theta d\theta \left[H(\Theta_s - \theta) \frac{1 + \cos \Theta_s}{1 + \cos \theta} + H(\theta - \Theta_s) \frac{1 - \cos \Theta_s}{1 - \cos \theta} \right], \quad (5.9)$$

where H is the step function. This arises due to the fact that as θ varies from being below Θ_s to above Θ_s , the poles of the integrand move across the integration contour. After carrying out the θ integration we get:

$$I = 2\pi [2 \ln 2 - (1 + \cos \Theta_s) \ln(1 + \cos \Theta_s) - (1 - \cos \Theta_s) \ln(1 - \cos \Theta_s)]. \quad (5.10)$$

We can now use (5.4) to calculate the energy flux integrated over all angles, up to order $\omega^2(\ln \omega)^2$:

$$\begin{aligned} \int \frac{dE}{d\omega d\Omega} d\Omega &= \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi [P(\theta, \phi) + Q(\theta, \phi) \omega^2 (\ln \omega)^2] \\ &= \frac{E^2}{\pi^2} [2 \ln 2 - (1 + \cos \Theta_s) \ln(1 + \cos \Theta_s) - (1 - \cos \Theta_s) \ln(1 - \cos \Theta_s)] \\ &\quad + \frac{E^4}{2\pi^4} \omega^2 (\ln \omega)^2 [1 - 2 \ln 2 \\ &\quad + (1 + \cos \Theta_s) \ln(1 + \cos \Theta_s) + (1 - \cos \Theta_s) \ln(1 - \cos \Theta_s)]. \end{aligned} \quad (5.11)$$

The small Θ_s expansion of this function takes the form:

$$\begin{aligned} \int \frac{dE}{d\omega d\Omega} d\Omega &= \frac{E^2}{\pi^2} \frac{\Theta_s^2}{2} \left\{ 1 + 2 \ln 2 + \ln \Theta_s^{-2} + \mathcal{O}(\Theta_s^3) \right\} \\ &\quad + \frac{E^4}{2\pi^4} \omega^2 (\ln \omega)^2 \left[1 - \frac{\Theta_s^2}{2} \left\{ 1 + 2 \ln 2 + \ln \Theta_s^{-2} \right\} + \mathcal{O}(\Theta_s^3) \right] + \mathcal{O}(\omega^2 \ln \omega). \end{aligned} \quad (5.12)$$

Even though we have evaluated (5.11) for general Θ_s , during this derivation we have ignored the possible modification of $A_{\mu\nu}$ due to radiation emitted during the scattering. Now for scattering at large impact parameter b , we have $\Theta_s \sim E/b$ [20]. If we assume that the spectrum of gravitational radiation falls off rapidly for $\omega > b^{-1} \sim \Theta_s/E$, then integrating (5.12) in the range $0 \leq \omega \leq \Theta_s/E$ we see that the total radiated energy during the scattering is of order $E\Theta_s^3$ times possible factors of $\ln \Theta_s^{-1}$. Since the correction to $\epsilon_{\mu\nu}^{\pm} A^{\mu\nu}$ from a final state particle is proportional to the energy carried by the particle, we see that $\epsilon_{\mu\nu}^{\pm} A^{\mu\nu}$ can receive correction of order $E\Theta_s^3$, and this in turn can affect the coefficients appearing in (5.11) by terms of order Θ_s^3 . However expansion up to order Θ_s^2 , given in (5.12) can be trusted.

It was shown in [16] however that in the near forward direction, the actual cut-off on ω extends beyond $1/b$ and as a result the net energy of emitted radiation is of order Θ_s^2 with possible logarithmic corrections. Therefore one might worry that this will give corrections to $\epsilon_{\mu\nu}^{\pm} A^{\mu\nu}$ of order Θ_s^2 and affect the order Θ_s^2 coefficient of the $\omega^2(\ln \omega)^2$ term. However one can see as follows that this is not the case. If the energy is emitted strictly in the forward direction, then it effectively amounts to a redistribution of the energy among the various final state particles in the forward direction and does not affect $A_{\mu\nu}$. For example a final state particle of momentum λp and another particle of momentum $(1 - \lambda)p$ give the same contribution to $A_{\mu\nu}$ as a single particle of momentum p . Therefore the relevant correction to $A_{\mu\nu}$ can be estimated from the difference between the emission in the forward direction and in a direction making an angle of order $\delta\Theta_s \sim \Theta_s$ with the forward direction. We see from (5.1) that when we change Θ_s by $\delta\Theta_s$, the change in $\epsilon_{\mu\nu}^{\pm} A^{\mu\nu}$ is of order $E\delta\Theta_s$ even when $\delta\Theta_s \sim \Theta_s$ but θ is finite, but is of order E when θ and $\delta\Theta_s$ are both of order Θ_s . Using $E \sim \Theta_s^2$ and $\delta\Theta_s \sim \Theta_s$, one finds that for finite θ , the correction to $\epsilon_{\mu\nu}^{\pm} A^{\mu\nu}$ is still of order Θ_s^3 (possibly multiplied by powers of $\ln \Theta_s^{-1}$) while for $\theta \sim \Theta_s$ the correction to $\epsilon_{\mu\nu}^{\pm} A^{\mu\nu}$ is of order Θ_s^2 . Taking into account the extra factor of Θ_s^2 that we get when θ integration is restricted to be of order Θ_s , we see that the correction to (5.11) due to the modification of $A_{\mu\nu}$ by the final state radiation is of order Θ_s^3 times possible logarithmic corrections.

The positivity of the coefficient of the $\omega^2(\ln \omega)^2$ term for small Θ_s shows that the flux has a local minimum at $\omega = 0$ and therefore has a maximum elsewhere, presumably around $\omega \sim b^{-1}$. This confirms the prediction of [17]. The actual coefficient of this term differs from that of [17] by a factor of 2 in the small Θ_s limit. On the other hand the coefficient of the $E^2\Theta_s^2/(2\pi^2)$ term differs from that of [17] by the additive constant $2\ln 2$. Both of these can be attributed to the fact that [17] computed the differential flux in the small θ approximation, whereas (5.12) receives contribution also from the finite θ region.

6 Discussion

In this paper we have observed that the late time radiation during a scattering process is independent of the final state massless particle momenta. A similar phenomenon has now also been observed for the spin dependent part of the final state radiation [21]. These results are based on the analysis of the general formula for the radiation tail, but we do not know of any deeper explanation. The usual explanation of soft theorems based on the analysis of asymptotic symmetries does not work for logarithmic corrections. Some attempts to modify the asymptotic symmetry analysis to include the logarithmic corrections has been made in [22–25] for electrodynamics. Possible generalization of these results to gravity could give some insight into the phenomena described here.

We also observed that for a binary black-hole merger process the gravitational tail memories at order u^{-1} , $u^{-2} \ln u$ vanish when we treat the inspiral binary as a single initial bound state object which decays to one massive black-hole and lots of massless radiation. A similar vanishing phenomena for spin dependent gravitational tail memory at order u^{-2} has also been observed in [21]. Vanishing of the late time gravitational waveform at future null infinity is consistent with the prediction from Price’s law for nearly spherical gravitational

collapse [26–28].² Late time behaviour of scalar perturbation near future time-like and null hyper-surfaces has been widely studied in a spherically symmetric background to determine Price’s law for scalar fluctuation [26, 29–35]. According to our current understanding a non-linear version of Price’s law for gravitational fluctuation has not been studied rigorously so far. In that case our observation on the vanishing of gravitational tail memories at the orders described above can be thought of as new predictions for a non-linear version of Price’s law for gravitational fluctuation.

We have restricted our analysis to four dimensions, but one could ask if similar results may hold in higher dimensions. It is known that due to phase space suppression in the Fourier integral, the leading soft term in the amplitude proportional to $1/\omega$ does not produce a late time memory effect in the final state radiation. Furthermore, the infrared effects are suppressed in higher dimensions and do not generate logarithmic corrections at low orders. However it is possible that they would generate $\omega^n \ln \omega$ type of terms in the Fourier transform of the final state radiation with high power of n . As these are non-analytic despite having a finite $\omega \rightarrow 0$ limit, these could give rise to late time tail in the retarded time with high power $1/u$. It will be interesting to explore whether the coefficient of these tail terms enjoy the same cancelation properties as described in this paper.

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A Effect of electromagnetic interaction

If the incoming and the outgoing particles carry electric charge then the coefficients $B_{\mu\nu}$, $C_{\mu\nu}$, $F_{\mu\nu}$ and $G_{\mu\nu}$ receive additional corrections [2–4]. In this appendix we shall show that even in the presence of these corrections, $B_{\mu\nu}$ and $F_{\mu\nu}$ continue to be independent of the momenta (and charges) carried by individual massless particles in the final state.

The extra contribution to $B_{\mu\nu}$ due to electromagnetic interaction is given by [3]:

$$\Delta B^{\mu\nu} = -\frac{2G}{Rc^5} \left[\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \frac{p_i^\mu}{n \cdot p_i} (n \cdot p_j p_i^\nu - n \cdot p_i p_j^\nu) \times \frac{1}{4\pi} q_i q_j p_i^2 p_j^2 \right], \tag{A.1}$$

²We wish to thank Luc Blanchet and Thibault Damour for pointing out possible connection between our results and Price’s law.

where q_i is the charge carried by the i -th final state particles, in units where the electrostatic force between a pair of charges separated by distance r is given by $q_i q_j / (4\pi r^2)$. (A.1) clearly vanishes if either i or j represents a massless particle.

The correction to $F^{\mu\nu}$ is given by [4]:

$$\begin{aligned}
 \Delta F^{\mu\nu} = & \frac{2G}{R c^9} \left[-4G \sum_{\ell=1}^n p_\ell \cdot n \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{p_i \cdot n} \right. \\
 & \times \frac{1}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \{n \cdot p_j p_i^\mu p_i^\nu - n \cdot p_i p_i^\mu p_j^\nu\} \times \frac{q_i q_j}{4\pi} p_i^2 p_j^2 \\
 & -2G \sum_{\ell=1}^m p'_\ell \cdot n \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \frac{1}{p'_i \cdot n} \\
 & \times \frac{1}{\{(p'_i \cdot p'_j)^2 - p_i'^2 p_j'^2\}^{3/2}} \{n \cdot p'_j p_i'^\mu p_i'^\nu - n \cdot p'_i p_i'^\mu p_j'^\nu\} \times \frac{q'_i q'_j}{4\pi} p_i'^2 p_j'^2 \\
 & -2G \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \sum_{\substack{\ell=1 \\ \ell \neq i}}^n \frac{1}{p_i \cdot n} \frac{p_i \cdot p_j}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \{2(p_i \cdot p_j)^2 - 3p_i^2 p_j^2\} \\
 & \times \frac{1}{\{(p_i \cdot p_\ell)^2 - p_i^2 p_\ell^2\}^{3/2}} \times \frac{q_i q_\ell}{4\pi} p_i^2 p_\ell^2 \{n \cdot p_j p_i^\mu - n \cdot p_i p_j^\mu\} \{n \cdot p_\ell p_i^\nu - n \cdot p_i p_\ell^\nu\} \\
 & + c^2 \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \sum_{\substack{\ell=1 \\ \ell \neq i}}^n \frac{1}{p_i \cdot n} \frac{1}{\{(p_i \cdot p_j)^2 - p_i^2 p_j^2\}^{3/2}} \frac{1}{\{(p_i \cdot p_\ell)^2 - p_i^2 p_\ell^2\}^{3/2}} \\
 & \left. \times \frac{q_i q_j}{4\pi} p_i^2 p_j^2 \times \frac{q_i q_\ell}{4\pi} p_i^2 p_\ell^2 \{n \cdot p_j p_i^\mu - n \cdot p_i p_j^\mu\} \{n \cdot p_\ell p_i^\nu - n \cdot p_i p_\ell^\nu\} \right]. \quad (\text{A.2})
 \end{aligned}$$

It is understood that the expression needs to be symmetrized under the exchange of μ and ν . As before we divide the final state particles into massive particles carrying momenta \tilde{p}_i and charges \tilde{q}_i and massless particles with momenta \hat{p}_i and charges \hat{q}_i . Examining this expression we see that the only contributions from massless final state particles can come when p_ℓ represents a massless particle momentum in the first term inside the square bracket and when p_j represents a massless particle momentum in the third term inside the square bracket. The first contribution may be expressed as:

$$-4G \sum_{\ell} \hat{p}_\ell \cdot n \sum_i \sum_{\substack{j \\ j \neq i}} \frac{1}{\tilde{p}_i \cdot n} \frac{1}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{n \cdot \tilde{p}_j \tilde{p}_i^\mu \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_i^\mu \tilde{p}_j^\nu\} \times \frac{\tilde{q}_i \tilde{q}_j}{4\pi} \tilde{p}_i^2 \tilde{p}_j^2. \quad (\text{A.3})$$

On the other hand when the sum over j runs over massless particles in the third term within the square bracket, the contribution takes the form:

$$\begin{aligned}
 & 4G \sum_j \sum_i \sum_{\substack{\ell \\ \ell \neq i}} \frac{1}{\tilde{p}_i \cdot n} \frac{1}{\{(\tilde{p}_i \cdot \tilde{p}_\ell)^2 - \tilde{p}_i^2 \tilde{p}_\ell^2\}^{3/2}} \\
 & \times \frac{\tilde{q}_i \tilde{q}_\ell}{4\pi} \tilde{p}_i^2 \tilde{p}_\ell^2 \{n \cdot \hat{p}_j \tilde{p}_i^\mu - n \cdot \tilde{p}_i \hat{p}_j^\mu\} \{n \cdot \tilde{p}_\ell \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_\ell^\nu\}. \quad (\text{A.4})
 \end{aligned}$$

In the above expression, relabelling ℓ as j and j as ℓ and simplifying we get:

$$\begin{aligned}
& 4G \sum_{\ell} \sum_i \sum_{\substack{j \\ j \neq i}} \frac{1}{\tilde{p}_i \cdot n} \frac{1}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \\
& \quad \times \frac{\tilde{q}_i \tilde{q}_j}{4\pi} \tilde{p}_i^2 \tilde{p}_j^2 \{n \cdot \hat{p}_\ell \tilde{p}_i^\mu - n \cdot \tilde{p}_i \hat{p}_\ell^\mu\} \{n \cdot \tilde{p}_j \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\nu\} \\
& = 4G \sum_{\ell} \hat{p}_\ell \cdot n \sum_i \sum_{\substack{j \\ j \neq i}} \frac{1}{\tilde{p}_i \cdot n} \frac{1}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \{n \cdot \tilde{p}_j \tilde{p}_i^\mu \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_i^\mu \tilde{p}_j^\nu\} \times \frac{\tilde{q}_i \tilde{q}_j}{4\pi} \tilde{p}_i^2 \tilde{p}_j^2 \\
& \quad - 4G \sum_{\ell} \sum_i \sum_{\substack{j \\ j \neq i}} \frac{1}{\{(\tilde{p}_i \cdot \tilde{p}_j)^2 - \tilde{p}_i^2 \tilde{p}_j^2\}^{3/2}} \hat{p}_\ell^\mu \{n \cdot \tilde{p}_j \tilde{p}_i^\nu - n \cdot \tilde{p}_i \tilde{p}_j^\nu\} \times \frac{\tilde{q}_i \tilde{q}_j}{4\pi} \tilde{p}_i^2 \tilde{p}_j^2. \quad (\text{A.5})
\end{aligned}$$

We now see that the first term on the right hand side of (A.5) cancels (A.3) and the second term vanishes since the summand is anti-symmetric under the exchange of i and j . Hence we do not get any contribution involving final state massless particles due to long-range electromagnetic interaction.

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