A FORMAL MODEL OF EXTENDED TONAL HARMONY

Martin Rohrmeier
Digital and Cognitive Musicology Lab, EPFL
martin.rohrmeier@epfl.ch

Fabian C. Moss
Digital and Cognitive Musicology Lab, EPFL
fabian.moss@epfl.ch

ABSTRACT

Extended tonality is a central system that characterizes the music from the 19th up to the 21st century, including styles like popular music, film music or Jazz. Developing from classical major-minor tonality, the harmonic language of extended tonality forms its own set of rules and regularities, which are a result of the freer combinatoriality of chords within phrases, non-standard chord forms, the emancipation of dissonance, and the loosening of the concept of key. These phenomena posit a challenge for formal, mathematical theory building. The theoretical model proposed in this paper proceeds from Neo-Riemannian and Tonfeld theory, a systematic but informal music-theoretical framework for extended tonality. Our model brings together three fundamental components: the underlying algebraic structure of the Tonnetz, the three basic analytical categories from Tonfeld theory (octatonic and hexatonic collections as well as stacks of fifths), and harmonic syntax in terms of formal language theory. The proposed model is specified to a level of detail that lends itself for implementation and empirical investigation.

1. INTRODUCTION

Harmony is a central latent structure governing Western music since centuries until today [1]. While a considerable amount of research has focused on theoretical, mathematical, and computational exploration of harmony in common-practice major-minor tonality, comparably less attention has been devoted to the challenges that come with the paradigm shift of extended tonality, as it is in place since the 19th century up until the present day in various styles like Jazz, popular or film music.

Extended tonality exhibits harmonic sequences that defy the logic of common-practice (major-minor) tonality. In this paper, we address the problem of accounting for such phenomena with a grammar approach that bridges formal language theory and (mathematical) music theory.

1.1 The Challenge of Extended Tonality

Briefly construed, harmonic progressions in pieces in diatonic common-practice tonality involve chords that are mostly stacks of thirds derived from the seven diatonic degrees of major or minor scales. Larger harmonic structures emerge through modulations between different keys that are usually close to one another on the line of fifths [1–4], thus forming a system that governs the global hierarchy of pieces [5–9].

In contrast, pieces employing extended tonality may rely on a variety of different scales (e.g., pentatonic, hexatonic, and octatonic scales; see Section 2), freely use harmonies that are not necessarily construed by stacking diatonic thirds, and modulate to or immediately combine chords from relatively distant keys [10–13]. For example, late-Romantic pieces often distinguish themselves from earlier diatonic ones by featuring frequent enharmonic exchanges of pitches, resulting in uncommon chord combinations [14–16], and by the frequent usage of symmetrical scales that impede a listener's orientation towards a unique tonic, possibly resulting in multiple parallel tonal centers [17, 18].

However, extended tonality is not restricted to late 19th-century pieces, but reaches into many more recent styles, in particular in Jazz with its highly chromatic harmonies [19, 20], and film music, such as scores by Korngold or Williams [21–24]. It also plays a role in Rock and Pop [25–27], and minimalist music, such as by Glass, Frahm or Richter. Extended tonality thus describes not a historical time span but rather captures characteristic features of a harmonic language that extends common-practice major-minor tonality with a variety of phenomena reaching from the late 18th century until the present day [28].

1.2 Related Work

1.2.1 The Tonnetz and Neo-Riemannian Theory

A major analytical approach to extended tonality is neo-Riemannian theory (NRT), which models harmonic progressions between pairs of triads or keys through parsimonious voice-leading transformations [16, 29, 30]. For instance, the *relative* transformation R converts C major into A minor, the *parallel* transformation P converts C major into C minor, and the *leading-tone exchange* transforms C major into E minor. All transformations are involutions, i.e. they are self-inverse. Repeated application of (combinations of) NRT transformations leads to patterns on the *Tonnetz* (Section 2.1) that visualize a particular analysis.

Figure 1 shows an excerpt of the Tonnetz. Note that the alternation of P and R transformations creates a pattern

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on the Tonnetz that covers all pitches from an octatonic scale (shown in blue), the alternation of P and L transformations creates a path on the Tonnetz that contains all members of a hexatonic scale (shown in orange), and combinations of R and L transformations generate a sequence of triads that modulates through all diatonic scales (not shown; the diatonic is encompassed by a horizontal line of six fifths or two adjacent horizontal lines connecting 6 triangles). The green rectangle delineates a stack of fifths (see below). Due to its focus on triadic transformations, and in particular those that form hexatonic or octatonic cycles, NRT analyses are commonly restricted such that the assumed algebraic spaces imposes some inflexibility with regard to the chord form. Further, stacks of fifths (see below) are commonly not addressed in NRT. Also, though some analyses work with reductions and abstractions from the score, there is no formalized theory of harmonic hierarchy in NRT. In further work, mathematical spaces have been extended or generalized [31–33], used in computational models of harmony or tonality [34–38], or explored empirically [39–42].

1.2.2 Tonfeld Theory

Another recent theory addressing the challenges of extended tonality is Tonfeld theory (TFT) [43–49]. Unlike NRT, it does not fundamentally rely on triads or keys. Instead, it departs directly from three so-called *Tonfelder* (‘tone fields’) that correspond to hexatonic, octatonic, and fifths-related (e.g., pentatonic, diatonic) tone collections (see Section 2.2 for details) that are assumed to govern segments of pieces at the granularity of the pitch level. It mostly focuses on late 19th-century compositions but some analyses for 18th and 20th century pieces exist as well [28,50,51]. Moreover, TFT subscribes to a fundamentally hierarchical conception of compositions by analyzing a piece’s tonal coherence through the presence and interactions of Tonfelder on several structural levels of abstraction/reduction. This allows in principle for the expression of nested structures and non-local dependencies.

1.2.3 Harmonic Syntax

Syntactic formalisms derive musical (e.g. harmonic or melodic) sequences through generative models that result in trees or similar hierarchical dependency structures [6, 52–57]. They frequently adopt frameworks from formal language theory and adapt them to the particular needs for the case of music. In recent years, several formal and computational approaches have been developed for Western classical music [7, 58, 59], Blues [60, 61], Jazz [20, 62–65]. We build on such previous approaches and expand their scope to extended tonality.

2. THE MODEL

2.1 The Tonnetz

One foundation of the present model is the *Tonnetz*. It goes back to Leonard Euler’s definition of intervals in just intonation [66], leading to an abstract pitch class space [67]. Accordingly, every just interval can be expressed by a frequency ratio:

$$f_1/f_2 = 2^x \cdot 3^y \cdot 5^z, \quad x, y, z \in \mathbb{Z}. \tag{1}$$

Since the factor 2 defines the octave, the two other integer factors y and z span a coordinate system of pitch classes (modulo the octave) that defines an infinite plane of fifths and major thirds. Taking into account that a fifth is composed of a major and a minor third, the plane corresponds to a triangular graph such that there are three main axes of major, minor thirds and fifths, in which each triangle defines a major or minor triad (see Figure 1).

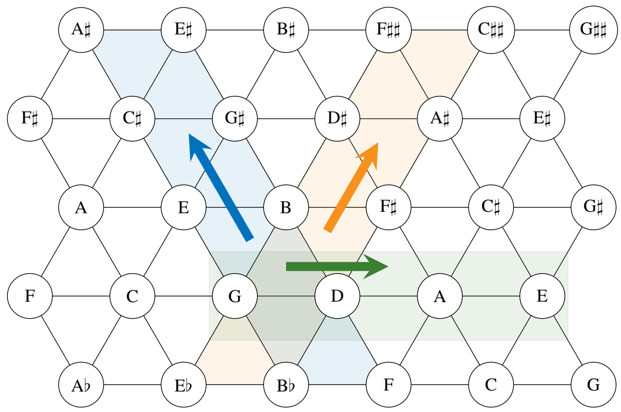

Figure 1: The Tonnetz and the construction of the three types of *Tonfeld* structures. The pitch class set shaded in blue defines one octatonic, the set shaded in orange defines one hexatonic, the set in green defines a stack of fifths.

The Tonnetz defines an *infinite* space of spelled pitch classes none of which are identical, i.e. different nodes in the graph with the same label are indeed distinct. If one identifies nodes with the same labels, the infinite line of fifths wraps itself around the so-called *Spiral Array* [37]. If, further, enharmonic equivalence is assumed (e.g. $D\sharp \equiv E\flat$), the space becomes a torus [39, 68] and the resulting pitch-class space is isomorphic to $\mathbb{Z}_{12}$.

2.2 The Tonfelder

Tonfeld theory comprises three fundamental concepts that are expressed in terms of pitch collections: octatonic, hexatonic, and stacks of fifths. This section introduces these building blocks that the theory operates on. All concepts are formulated in the toroidal $\mathbb{Z}_{12}$ pitch space, yet they could be easily generalized to the spelled pitch space and the infinite Tonnetz. At first, a stack of intervals SI is defined by a starting pitch p, and an interval i with k iterations. If the iterations reach the starting pitch, SI defines a cyclic group (simply written as $SI_{p,i}$).[1]

$$SI_{p,i,k} := \{p + im \mid m \leq k; k, m \in \mathbb{N}_0\} \tag{2}$$

Interval cycles are musically meaningful units to ground a tonal theory [69, 70]. Here, the three Tonfelder are constructed from the three directions in the Tonnetz. The *octatonic* is defined by shifting a fifth along the axis of minor

[1] An analogous definition in the infinite Tonnetz space would not result in a cyclic group.

thirds, the *hexatonic* by shifting a fifth along the axis of major thirds, and the *stack of fifths* by collecting consecutive fifths along the axis of fifths (see Figure 1).

$$Oct_p := \{i, i+7 \mid i \in SI_{p,3}\} \equiv SI_{p,3} \uplus SI_{p+7,3} \quad (3)$$
$$Hex_p := \{i, i+7 \mid i \in SI_{p,4}\} \equiv SI_{p,4} \uplus SI_{p+7,4} \quad (4)$$
$$Fif_{p,k} := SI_{p,7,k} \quad (5)$$

Notably, all three Tonfelder are based on the foundational tonal interval of the fifth. This construction results in the three different octatonic (halftone–wholetone) scales and the four hexatonic (minor-third–halftone) scales ($Oct_i = Oct_{i+3}$; $Hex_i = Hex_{i+4}$):

$$Oct_0 = \{C, D\flat, E\flat, E\natural, G\flat, G\natural, A, B\flat\} \quad (6)$$
$$Oct_1 = \{D\flat, D, E\natural, F, G, A\flat, B\flat, C\flat\} \quad (7)$$
$$Oct_2 = \{C, D, E\flat, F, G\flat, A\flat, A\natural, B\} \quad (8)$$
$$Hex_0 = \{C, E\flat, E, G, A\flat, B\} \quad (9)$$
$$Hex_1 = \{C\sharp, E, F, G\sharp, A, C\} \quad (10)$$
$$Hex_2 = \{D, F, F\sharp, A, B\flat, C\sharp\} \quad (11)$$
$$Hex_3 = \{E\flat, F\sharp, G, B\flat, B\natural, D\} \quad (12)$$

Since stacks of fifths do not form a mode of limited transposition [71], there are 12 different types of stacks of fifths until the whole chromatic is reached.

$$Fif_{C,2} = \{C, G, D\} \quad (13)$$
$$Fif_{C,3} = \{C, G, D, A\} \quad (14)$$
$$\ldots Fif_{C,5} = \{C, G, D, A, E, B\} \quad (15)$$

The set of all Tonfelder $\mathbb{T}$ is defined as $\mathbb{T} := \{Oct_0, Oct_1, Oct_2\} \cup \{Hex_0, \ldots, Hex_3\} \cup \{Fif_{p,k} \mid p \in 0, \ldots, 11, k \in 2, \ldots, 10\}$. One can apply *filters* to a Tonfeld t to arrive at basic musical units, such as triads or tetrads. For instance, all triads in a Tonfeld are filtered out by

$$f : t \mapsto \{\{a, b, c\} \mid iv(\{a, b, c\}) = (0, 0, 1, 1, 1, 0)\} \quad (16)$$

where iv is the interval vector of a given pitch-class set, which counts all possible interval classes [72]. One can easily define filters for other chord types or, in fact, arbitrary pitch-class sets [72, 73].

In terms of common chord types, the octatonic scale yields four major, minor, dominant seventh, minor seventh and half-diminished chords each, all related by minor thirds, and two fully diminished tetrads a fifth/a semitone apart. The hexatonic scale in turn yields three major, minor, major seventh, minor-major-seventh chords a major third apart, and two augmented triads a fifth/a semitone apart. Stacks of fifths yield chords that are often classified as sus-chords or quartal voicings in Jazz harmony terminology, [2] and also cover complex add 6, 9th or 11th chords as they appear in Jazz [74]. Notably, also complex chords such as the "Tristan chord", the "Petrouchka chord", Scriabin's "mystical chord" as well as many upper structure chords in Jazz (e.g. G-F-B-D-E-$G\sharp$-B) [75], are captured within the octatonic set. Non-diatonic minor chords with major sevenths as they occur in Jazz are captured by the hexatonic set. The set of all chords derived by suitable filters (for the particular surface to be modeled) from a Tonfeld $t \in \mathbb{T}$ is denoted by C_t. This definition encompasses occurrences of non-standard chord forms, such as the ones above.

Tonfelder further bear generalizing expressive power with respect to central tonal relations. The set of chords with a dominant function may, for instance, involve V, V^7, vii^0, $\flat VII$, $\flat II^7$ (tritone substitution), or III^7. Similar, subdomantic/pre-dominantic chords may, e.g., involve, IV, ii, II, iv^6, $\flat VI$. Both of these sets of equivalences are all encompassed by the set of chords from the two octatonics neighboring the reference tonic chord [44, 70]. Therefore, the octatonic Tonfeld can be understood as a generalization over the concept of tonal harmonic function (tonic, dominant, predominant) as well as intra-functional prolongation/substitution (within the same octatonic).

Conversely, Neo-Riemannian theories (as well as TFT) have identified chords from the hexatonic to establish contrastive relationships, such as the hexatonic pole (e.g., C major – $A\flat$ minor), [14, 76]. Such relations between harmonies are the basis of the hierarchical dependencies that are modelled by the grammar outlined in the next subsection.

2.3 The Grammar

The proposed harmonic grammar formalism is based on abstract context-free grammars (ACFGs) [63] and extends previous models of harmonic syntax [7, 20, 58]. It consists of four components: $\mathcal{G} = (K, \Sigma, P, s)$, non-terminal categories K, terminal symbols Σ, production rules P, and a start symbol $s \in K$. The set of all terminals Σ encompasses all (potentially non-standard) chords derived from the Tonfelder: $\Sigma := \{c \mid c \in C_t \ \forall t \in \mathbb{T}\}$.

There are three kinds of non-terminal category symbols: $K = \{s\} \cup \{\Theta_t \mid t \in \mathbb{T}\} \cup \{c_t \mid c \in C_t, t \in \mathbb{T}\}$. s denotes the abstract start symbol. Except for the start symbol, non-terminal category symbols have a feature t, which indicates the assigned Tonfeld of the category. Abstract Tonfeld categories Θ_t ($\in \{\Theta\} \times \mathbb{T}$) define an unspecific Tonfeld $t \in \mathbb{T}$ that has not yet been instantiated in terms of a concrete chord category. Chord categories c_t ($\in \Sigma \times \mathbb{T}$) are defined in terms of any chord symbol c derived from its assigned Tonfeld t.

In ACFGs, the production rules P are defined as functions mapping the left-hand side to the right-hand side. Here, P involves three kinds of rules: general rules (*start*, *instantiation*, *Tonfeld cast*, *termination*), rules characterizing hierarchical functional relations (*prolongation*, (*substitution*), *preparation*, *plagal dependency*, *contrast*), and rules with set operations for manipulating stacks of fifths (*fifth shift*, *fifth expansion & contraction*, *fifth split*). Notably, it is not necessary to formally assume substitution

[2] The stack of fifths chords (e.g. $C - G - D$) are technically not suspension chords since they do commonly not imply a resolution of the dissonant fourth into a third as in standard common-practice. [3]

because substitutable equivalences are already expressed at the level of the octatonic Tonfeld. The following paragraphs define each rule type:

2.3.1 General Rules

Start. A piece is modeled as a sequence of different Tonfeld categories Θ.

$$s \longrightarrow \Theta_{t_1}^{(1)} \ \dots \ \Theta_{t_n}^{(n)} \tag{17}$$

Note that in contrast to previous (diatonic) syntax theories [7], there is no requirement of an overarching single tonic node of a derivation tree (although this 'downward compatability' can be achieved by a single overarching octatonic chord category). This comes from the different logic of Tonfeld structures [28,44]. Thus, the top level may consist in a sequence of different Tonfelder. For abridged derivations or analyses of partial sequences, the trees can also directly be headed by a single Tonfeld or chord category, omitting the start symbol (see the examples below).

Tonfeld instantiation. An abstract Tonfeld symbol Θ_t of the Tonfeld t may be instantiated with one or more member chords from its set.

$$\Theta_t \longrightarrow Y_t^{(1)} \ \dots \ Y_t^{(n)} \tag{18}$$

Tonfeld casting. Generalizing modulations, a chord category can change its underlying Tonfeld and recursively yield different generations. Importantly, this operation can only be performed over chord categories, since the abstract Tonfeld categories are ambiguous with regard to their chord instantiation, and therefore their cast to a different Tonfeld is not well-defined. An abstract Tonfeld category can only be cast into another through *instantiation* in terms of a pivot chord category. Therefore, Tonfeld casting necessarily involves the set intersection of two Tonfelder (which in turn enforces the chord type category for X).

$$X_m \longrightarrow X_k, \quad X \in C_m \cap C_k \tag{19}$$

Terminal rules. The grammar needs to ensure that the sequence generation terminates. The production can terminate when there are no more abstract categories in the sequence. Since chord categories are already absolute chords matching surface chord forms, the only final step is to cast the chord category into a terminal chord without a Tonfeld feature: $X_t \longrightarrow X$. For stacks of fifths, the resulting chord forms may be non-standard, e.g. non-triadic (see Figure 6).

2.3.2 Rules for functional relations

Prolongation. Going beyond previous models [7,20], prolongation need not only combine identical categories but may combine elements of the same Tonfeld. Following the generalization by Steedman, prolongation can be understood as an instance of syntactic coordination [62, 77]. Prolongation can be established with two different types, abstract Tonfeld categories and chord categories, and may combine two or more categories.

$$\Theta_t \ \rightarrow \ \Theta_t \ \dots \ \Theta_t \tag{20}$$

$$X_t \ \rightarrow \ Y_t^{(1)} \ \dots \ Y_t^{(n)}, \ X_t = \Theta_t \vee \exists i : Y^{(i)} = X \tag{21}$$

(Substitution.) The octatonic moreover generalizes over possible substitutions of certain sets of chords. It is useful to formulate this as a rule even though, in most cases, the direct derivation of dominant or subdominant substitutions may be achieved directly though the preparation- and plagal-dependency rules (and therefore, the rule may not be necessary in computational implementations).

$$X_{Oct_a} \longrightarrow Y_{Oct_a}, X \neq Y \tag{22}$$

Preparation. The octatonic generalizes over the class of chords that may prepare other chords [20]. Since there are only three different octatonics (3) there are only two possible preparations (motions between them): preparation and plagal dependency (see below). A preparation is derived as the preceding left child of the prepared chord. The types of X and Y can be an abstract octatonic Tonfeld Θ_{Oct_a} or a chord category c_t.

$$X_{Oct_a} \longrightarrow Y_{Oct_{a+1}} \, X_{Oct_a} \tag{23}$$

Plagal dependency. The octatonic also generalizes over the set of subdominants. A plagal dependency/relaxation into a chord is modeled as its left child. Although the plagal relaxation has a similar form as the preparation rule, its semantics is different. This implies that the semantic of the applied dependency type (preparation or plagal) cannot be inferred from its shape in the tree (i.e. a left child is not necessarily a dominant – pace GTTM [6]). Similarly to the preparation rule, the types of X and Y can either be both abstract octatonic Tonfelder or chord categories.

$$X_{Oct_a} \longrightarrow Y_{Oct_{a-1}} \, X_{Oct_a} \tag{24}$$

Contrast. The hexatonic and the stack of fifths allow for the instantiation of the relation of a contrast to a given element. For the hexatonic, contrast is established *within* the same hexatonic. The most frequent form of the hexatonic contrast is the *hexatonic pole* [78]. For the stack of fifths, contrast is established between two quasi-complementary stacks of fifths (*fifth flipover*).

$$X_{Hex_i} \ \longrightarrow \ Y_{Hex_i} \, X_{Hex_i} \quad , X \neq Y \tag{25}$$

$$X_{Hex_i} \ \longrightarrow \ X_{Hex_i} \, Y_{Hex_i} \quad , X \neq Y \tag{26}$$

$$\Theta_{Fif_{a,b}} \ \longrightarrow \ \Theta_{Fif_{c,d}} \, \Theta_{Fif_{a,b}}, \quad \text{if } inv(Fif_{a,b}, Fif_{c,d}) \tag{27}$$

$$\Theta_{Fif_{a,b}} \ \longrightarrow \ \Theta_{Fif_{a,b}} \, \Theta_{Fif_{c,d}}, \quad \text{if } inv(Fif_{a,b}, Fif_{c,d}) \tag{28}$$

The Boolean *inverse* relation inv between two stacks of fifths $Fif_{p,k}$ and $Fif_{q,l}$ is fulfilled if they are sufficiently distinct for some distance function d and threshold δ (29). Since the stacks are sets, a suitable candidate is the Jaccard distance (30).

$$inv(Fif_{p,k}, Fif_{q,l}) \ := \ d(Fif_{p,k}, Fif_{q,l}) \leq \delta \tag{29}$$

$$d(A, B) \ := \ \frac{|A \cap B|}{|A \cup B|} \tag{30}$$

2.3.3 Rules for Stacks of Fifths

Fifth shift. A stack of fifths can be shifted one step in either direction of the circle of fifths. The rule can instantiate an instance before or after the parent category.

$$\Theta_{Fif_{p,k}} \longrightarrow \Theta_{Fif_{p,k}} \Theta_{Fif_{p\pm7,k}} \tag{31}$$

$$\Theta_{Fif_{p,k}} \longrightarrow \Theta_{Fif_{p\pm7,k}} \Theta_{Fif_{p,k}} \tag{32}$$

Fifth expansion and contraction. A stack of fifths can be extended or reduced by a number of fifths. This results in a stack of fifths with a different number m of fifths, respecting the defining criterion of a stack of fifths of $2 < m < 11$. Fifth expansion and contraction may propagate to the left or the right within the sequence.

$$\Theta_{Fif_{p,k}} \longrightarrow \Theta_{Fif_{p,k}} \Theta_{Fif_{p,m}} \quad , m \neq k \tag{33}$$

$$\Theta_{Fif_{p,k}} \longrightarrow \Theta_{Fif_{p,m}} \Theta_{Fif_{p,k}} \quad , m \neq k \tag{34}$$

Fifth split. A stack of fifths can be split into two different, potentially overlapping, stacks of fifths.

$$\Theta_{Fif_{p,k}} \longrightarrow \Theta_{Fif_{q,l}} \Theta_{Fif_{r,m}} , Fif_{q,l} \cup Fif_{r,m} = Fif_{p,k} \tag{35}$$

3. EXAMPLES

Figure 2 illustrates how the use of the octatonic generalizes over remote variants of authentic preparation progressions or cadences, without requiring modulation, borrowing, tritone substitutions, or chromatic operations.

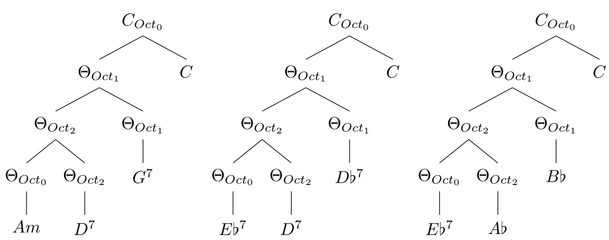

Figure 2: Generalizing over 3 preparatory progressions.

One example that well illustrates the octatonic set is given by two phrases from the second movement of Antonin Dvořák's ninth Symphony (Figure 3). It illustrates (a) that the chords used in sequence defy a purely diatonic (e.g. D♭ major) analysis, thus requiring a different analytical framework, and (b) that the two excerpts sound very similar even though they use different and remote chords at the surface. Our analysis illuminates that the chords stem from two octatonic Tonfelder which are identical for both examples. The very similar impression of both excerpts is modeled by the identical deep structure of the tree derivations. Further octatonic examples include Schubert's *Ganymed* D544 and Scriabin's *Prelude* op. 74/2.

One paradigmatic example for hexatonic Tonfelder is John Coltrane's piece "Giant Steps" (Figure 5). Similarly to the previous example, the chord sequence here also defies diatonic derivations because of the overarching major-third relations of the harmonic centers B major, G major and $E\flat$ major. Notably, the piece abandons the sense of an overarching key, osciallating between the harmonic centers establishing an overarching *abstract* hexatonic Tonfeld instead. The tree analysis demonstrates that the chord sequence is simple to derive once hexatonic relations are assumed at the top level. The local $ii-V-I$ progressions are

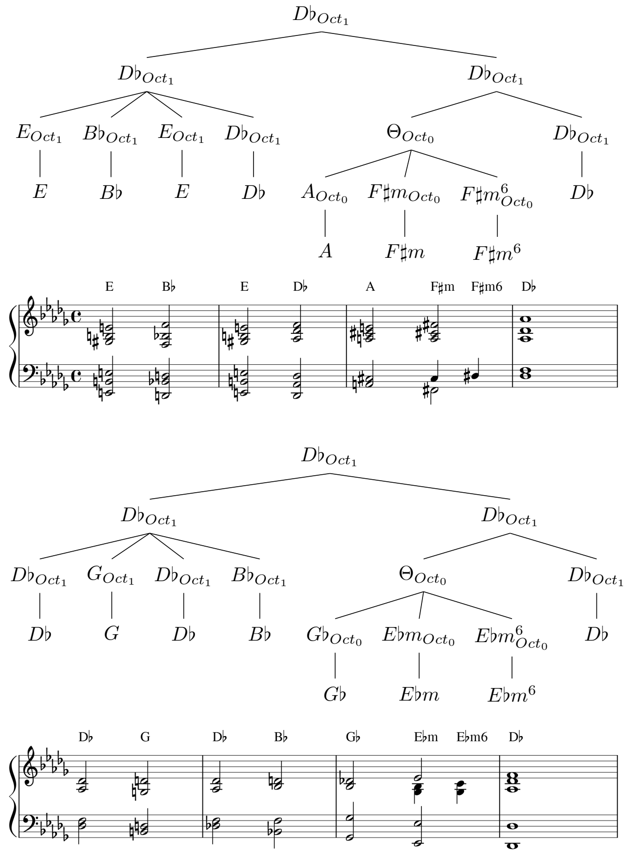

Figure 3: Dvořák, Symphony IX, op. 95–II, mm. 22-25, mm. 120-123

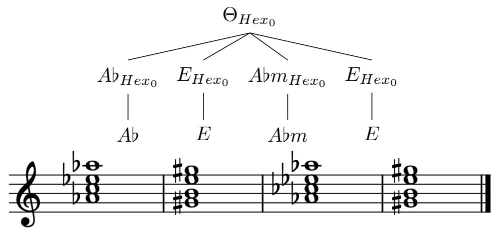

Figure 4: Chord progression for Aragorn and Arwen's love scene from *Lord of the Rings* (Howard Shore).

well-modeled with octatonic preparatory relations. Thus, this example also illustrates a case of hierarchical intertwining of two different types of Tonfelder.

Another example for a hexatonic progression is taken from Howard Shore's score to the film *Lord of the Rings* for the scene in which the characters Aragorn and Arwen, a couple that embodies contrast (human/mortal vs. elf/immortal), engage in intimate conversation. Underlying this scene is a loop of the chord progression $A\flat - E - A\flat m - E$. Similar to the octatonic example above, these chords can not be subsumed under a single key and an analysis where each chord change entails a key modulation seems implausible. Rather, these chords are all taken from the hexatonic Tonfeld Hex_0, as shown by the analysis in Figure 4. Not all triads possible in this Tonfeld do occur but the sequence expresses all pitch classes from Hex_0, except G. Further hexatonic examples can be found

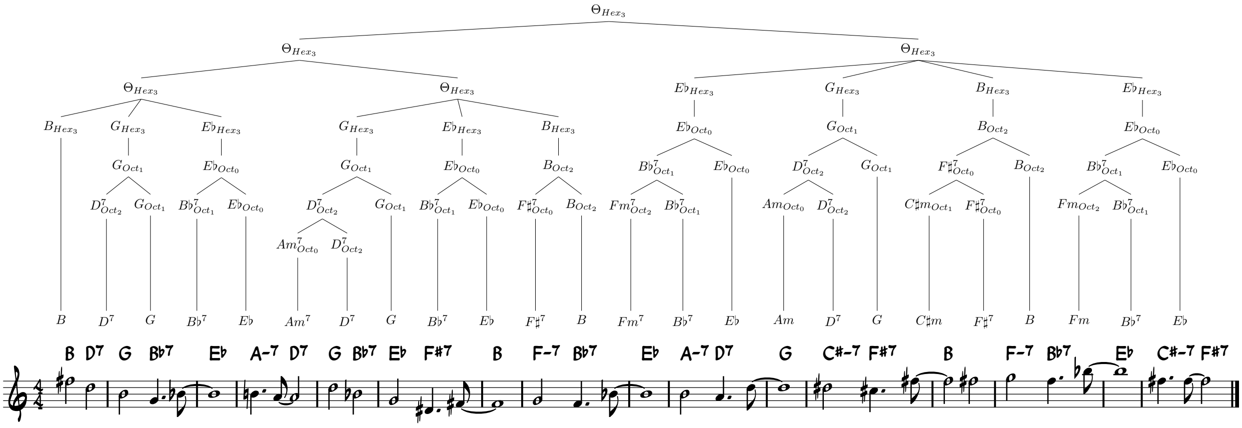

Figure 5: John Coltrane's "Giant Steps". The three harmonic centers B, G, $E\flat$ span a complete hexatonic Hex_3.

in the prelude of Wagner's *Parsifal* and Bruckner's *Ecce Sacerdos Magnus* WAB 13. As shown in [25], octatonic and hexatonic structures occur frequently in Popular music, e.g. *Shake the Disease* by Depeche Mode, *Easy Meat* by Frank Zappa, *Creep* by Radiohead, or *Lay, Lady, Lay* by Bob Dylan.

"Maiden Voyage" by Herbie Hancock provides a good illustration for the use of stacks of fifths (Figure 6). First, none of the *sus* chords in the leadsheet are proper *suspension* chords because they do not imply a resolution to an omitted harmonic interval. In fact, they are implicit notations for quartal voicings, which are in fact stacks of fifths. Both chord pairs in both sections constitute a *split* of an overarching stack of fifths. The relation between both parts is that the overarching *prolonged* stack of fifths is contrasted by *fifth flipover*. Other examples for this Tonfeld are Bartók's *Boating* from Mikrokosmos V, Tailleferre's *Pastorale*, Ligeti's piano etude no. 8, Kraftwerk's *Trans Europa Express*, Zimmer's film music to *Interstellar*.

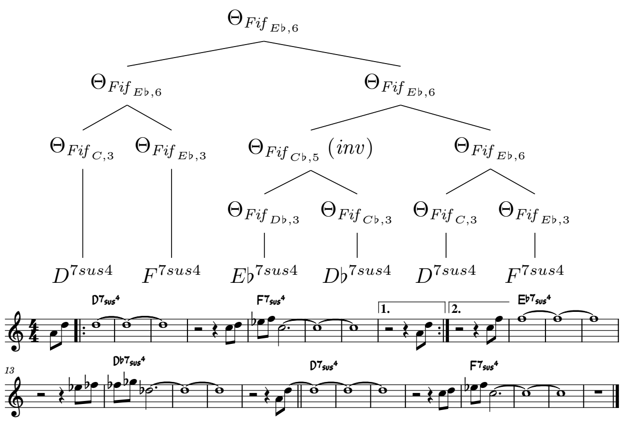

Figure 6: Analyzing Herbie Hancock's "Maiden Voyage" illustrating operations on stacks of fifths.

4. CONCLUSION

We present a model of extended tonality bridging music theoretical accounts and formal grammars. It captures chord forms and chord sequences that are challenging for other (diatonic) approaches. While some details, e.g. cases not yet covered, may be subject to debate, we argue that the core innovation and main benefit lies in providing a well-formalized hierarchical model of extended tonal harmony. As an aside, our model offers an explanation why some previous computational models find octatonic and hexatonic structures as efficient structures in their latent space [40, 42].

Our model is based on three Tonfelder as construed from the Tonnetz and the foundational interval of the perfect fifth, both of which we consider fundamental for extended tonality. We argue that the collections that can be constructed from the Tonnetz have a special status in establishing (extended) tonality compared with the manifold other scales. For instance, the whole-tone scale cannot be directly represented on the Tonnetz, and we argue that it can thus not form a deep structure, despite appearing on the surface. Our model constitutes an extension of formal grammars for diatonic music, meaning that it can also generate purely diatonic sequences, in analogy to extended tonal compositions containing also purely tonal sections.

The aim of the theory is to not only capture musical surface events but to model the kinds of underlying dependencies with theoretically meaningful concepts and assumptions, i.e. *strong generativity* [79]. Similarly to previous syntactic theories of music, the latent analytic derivations link *structure* and *interpretation* in terms of dependencies and chord functions (e.g. preparation, contrast) [20, 80, 81]. Because of the expressive richness of the model multiple concurrent analyses are possible for a given sequence. This makes it possible to express diverging hearings and nuances of a passage that different listeners may experience. The theory will not suffice as a forward generative model for computational composition on its own. It would require additional (inferable) style-specific parameters, since extended harmony works differently in Dvořák, Ravel, the Beatles, Coltrane, or Richter, and may as well benefit from a joint model of rhythm [20, 82]. The theory is sufficiently well-specified such that it is testable, debatable, and lends itself for empirical investigation in a probabilistic formulation in future work.

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