Decentralized and distributed robust control invariance for constrained linear systems

Stefano Riverso, Konstantinos Kouramas and Giancarlo Ferrari-Trecate

Abstract— **In this paper we propose an efficient procedure to compute time-varying Robust Control Invariant (RCI) sets for large-scale systems arising from the interconnection of M Linear Time-Invariant (LTI) constrained subsystems. In particular, the associated state-feedback controllers are nonlinear and decentralized or distributed. Our algorithm is structured in three stages: the computation of a Control Invariant (CI) set for each subsystem, the design of coupling attenuation control terms and the construction of a family or RCI sets for the overall system affected by disturbances. The last stage, which is based on the notion of practical invariance, is the only one requiring centralized computations for analyzing the stability of an M-th order system. Differently from existing approaches based on Linear Matrix Inequalities (LMIs), in case of polytopic constraints our method requires to solve local Linear Programming (LP) problems.**

Key Words: large-scale systems, decentralized control, distributed control, robust control, invariant sets

I. Introduction

In last decades, several methods have been proposed for computing invariant sets, which are particularly relevant to the study of stability properties. For instance, they provide a framework to prove stability and constraint satisfaction in model predictive control [1]. We can categorize approaches to the computation of invariant sets as follows: (i) methods for linear systems [2], [3], [4], [5], [6] or nonlinear systems [7], [8], [9], and (ii) algorithms for Robust Positive Invariant (RPI) sets [4], [9], [6], nominal invariant sets [8], [6], or CI and RCI sets [9], [5], [6].

Most of the algorithms proposed in literature are not scalable, meaning that the computational complexity for obtaining the desired invariant set can increase considerably with the total order of the system. One of the reasons, in the case of polytopic constrained LTI systems, is that existing methods are based on the enumeration of the vertices of full-dimensional polytopes. When the order of the system increases, the number of vertices becomes large and the computation of invariant sets quickly becomes intractable. For these reasons new methods with reduced computational burden are needed for large-scale systems [10], [11], [12], [13], [14], [15]. An approach to tackle this problem is to first build local invariant sets $\mathbb{S}_i$ for each subsystem $i = 1, \ldots, M$ and then analyze the coupling among subsystems in order to guarantee that $\mathbb{S} = \prod_{i=1}^{M} \mathbb{S}_i$ is invariant for the whole system.

S. Riverso and K. Kouramas is with United Technologies Research Center Ireland, Ltd. 4th Floor, Penrose Business Center, Penrose Wharf, Cork, Republic of Ireland. (riverss@utrc.utc.com,kouramk@utrc.utc.com)

G. Ferrari-Trecate is with Automatic Control Laboratory, École Polytechnique Fédérale de Lausanne (EPFL), Switzerland. (giancarlo.ferraritrecate@epfl.ch)

For control invariance, an additional goal is to design decentralized or distributed controllers associated to CI and RCI sets. In this case, the sets themselves are termed decentralized or distributed after the controllers. These approaches require, as a preliminary step, to represent the large-scale system as a family of interacting subsystems, i.e. as a directed graph where nodes are subsystems and edges are coupling channels representing the influence of parent subsystems on child nodes.

the authors propose to compute a CI set as an ellipsoid described by a matrix P that is block-diagonal and constraint satisfaction is guaranteed if the initial state of the system is inside the ellipsoid or, equivalently, if the initial state of each subsystem is inside the corresponding local ellipsoids. To achieve control invariance, the authors design a stabilizing state-feedback decentralized controller through LMIs. The optimization problem is however centralized and, therefore, the resulting algorithm is not scalable. In order to reduce the computational burden, [11] and [13] consider distributed state-feedback *linear* controllers and then exploit the notion of practical invariance [11], [12] to build a family of RCI sets for the whole system. The main advantage of this approach is that invariance is implied by the stability of an M-th order system: this avoids any computation based on the overall system model that usually has $n >> M$ states.

In [14] the authors propose a plug-and-play decentralized MPC controller for large-scale LTI systems. In order to guarantee the stability of the closed-loop system, algorithms for computing decentralized RCI sets are given. The idea is to compute a local robust invariant set for each subsystem considering the coupling terms coming from other subsystems as disturbances. Obviously, this method might be conservative because local controllers do not have access to the state of parent subsystems. RCI sets for large-scale systems have been also studied in [16], but only for specific subsystems with network storage dynamics.

In this paper we propose a new procedure for computing a time-varying family of RCI sets based on the notion of practical invariance introduced in [11]. However, differently from [11] and [13], we allow for *nonlinear* state feedback controllers. The algorithm is complementary to those in [11] and [13], in the sense that it can successfully produce RCI sets when the approaches in [11] and [13] fail (and vice-versa).

Our approach requires to first compute a local CI sets for each subsystem as it was isolated. Then, in order to broaden the applicability of the method, the coupling between subsystems is actively reduced by designing control terms utilizing the state of parent subsystems. This task is optional and, if executed, it leads to *distributed* controllers requiring a communication network with the same topology of the coupling graph. In the last step, similarly to [11] and [13], we build a family of RCI sets for the whole system leveraging the notion of practical invariance. In terms of computations, our procedure requires the solutions of local LP problems only, plus the stability analysis of an M-th order system. This is in contrast with the approaches in [11] and [13] based on loosely-coupled LMIs [13] or centralized optimization problems [11].

The paper is organized as follows. In Section II, we introduce large-scale systems and the class of distributed controllers associated to the RCI sets considered in the sequel. Then, in Section III, we summarize the results in [6] for the LP-based computation of CI sets. This method will be used in Section IV as a part of a more complex procedure for computing the family of RCI sets. The design procedure of local controllers and corresponding RCI sets is detailed in Section V. A simulation example is given in Section VI and some concluding remarks are provided in Section VII.

Notation. We use $a : b$ for the set of integers $\{a, a+1, \dots, b\}$. The symbol $\mathbb{R}^n_+$ stands for the vectors in $\mathbb{R}^n$ with nonnegative elements. The column vector with s components $v_1, \dots, v_s$ is $\mathbf{v} = (v_1, \dots, v_s)$. The symbol $\oplus$ denotes the Minkowski sum, i.e. $A = B \oplus C$ if and only if $A = \{a : a = b + c,\ \forall b \in B,\ \forall c \in C\}$. Moreover, $\bigoplus_{i=1}^{s} G_i = G_1 \oplus \dots \oplus G_s$. The symbol $\mathbf{1}$ denotes a matrix or a column vector with all elements equal to 1. Given a vector $x \in \mathbb{R}^n$ and a set $\mathbb{S} \subseteq \mathbb{R}^n$, $\mathrm{dist}(x, \mathbb{S}) = \inf_{s \in \mathbb{S}} ||x - s||$. A polyhedron $\mathbb{X}$ is the intersection of finitely many half spaces and a polytope is a bounded polyhedron. A $\mathcal{C}$-set is a set that is compact, convex and contains the origin.

Definition 1: Consider the discrete-time system $x(t+1) = f(x(t), u(t), w(t))$, with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ and disturbance $w(t) \in \mathbb{W} \subset \mathbb{R}^n$. The set $\mathbb{X} \subseteq \mathbb{R}^n$ is RCI if $\forall x(t) \in \mathbb{X}$ there exists $u(t) \in \mathbb{U}$ such that $x(t+1) \in \mathbb{X}$, $\forall w(t) \in \mathbb{W}$. If the system is autonomous, $\mathbb{X}$ is RPI if $x(t) \in \mathbb{X} \Rightarrow x(t+1) \in \mathbb{X}$, $\forall w(t) \in \mathbb{W}$. The RPI set $\bar{\mathbb{X}}$ is maximal if it includes every other RPI set. A positively invariant (resp. CI) set is an RPI (resp. RCI) set when $\mathbb{W} = \{0\}$. For an RCI set $\mathbb{X}$, a control law $\kappa(x)$ such that $\mathbb{X}$ is RPI for $x(t+1) = f(x(t), \kappa(x(t)), w(t))$ is termed *associated* to $\mathbb{X}$. The set $\mathbb{X} \subseteq \mathbb{R}^n$ is a λ-contractive RCI set for system $x(t+1) = f(x(t), u(t))$, with $\lambda \in [0, 1)$, if $\forall x(t) \in \mathbb{X}$ there exist $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ such that $x(t+1) \in \lambda\mathbb{X}$.

II. Distributed controllers for RCI sets

We consider a discrete-time LTI system

$$\mathbf{x}^+ = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{D}\mathbf{w} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^r$ are the state, the input and the disturbance, respectively, at time t and $\mathbf{x}^+$ stands for $\mathbf{x}$ at time $t+1$. The state is partitioned into M state vectors $x_{[i]} \in \mathbb{R}^{n_i}$, $i \in \mathcal{M} = 1 : M$ such that $\mathbf{x} = (x_{[1]}, \dots, x_{[M]})$, and $n = \sum_{i \in \mathcal{M}} n_i$. Similarly, the input and the disturbance are partitioned into M vectors $u_{[i]} \in \mathbb{R}^{m_i}$, $w_{[i]} \in \mathbb{R}^{r_i}$, $i \in \mathcal{M}$ such that $\mathbf{u} = (u_{[1]}, \dots, u_{[M]})$, $m = \sum_{i \in \mathcal{M}} m_i$, $\mathbf{w} = (w_{[1]}, \dots, w_{[M]})$ and $r = \sum_{i \in \mathcal{M}} r_i$. We assume the dynamics of the i-th subsystem is given by

$$\Sigma_{[i]}: \quad {x^+}_{[i]} = A_{ii} x_{[i]} + B_i u_{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij} x_{[j]} + D_i w_{[i]} \tag{2}$$

where $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $i, j \in \mathcal{M}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $D_i \in \mathbb{R}^{n_i \times r_i}$ and $\mathcal{N}_i$ is the set of parents of subsystem i defined as $\mathcal{N}_i = \{j \in \mathcal{M} : A_{ij} \neq 0, i \neq j\}$. We note that, since $x_{[i]}$ depends on the local input $u_{[i]}$ only, subsystems $\Sigma_{[i]}$ are input-decoupled and then $\mathbf{B} = \mathrm{diag}(B_1, \dots, B_M)$. Similarly, subsystems $\Sigma_{[i]}$ are disturbance-decoupled, hence $\mathbf{D} = \mathrm{diag}(D_1, \dots, D_M)$. We also define $\mathbf{A_D} = \mathrm{diag}(A_{11}, \dots, A_{MM})$ and $\mathbf{A_C} = \mathbf{A} - \mathbf{A_D}$, i.e. $\mathbf{A_D}$ collects the state transition matrices of every subsystem and $\mathbf{A_C}$ collects coupling terms between subsystems. We assume

$$x_{[i]} \in \mathbb{X}_i, \qquad u_{[i]} \in \mathbb{U}_i, \qquad w_{[i]} \in \mathbb{W}_i \tag{3}$$

where $\mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, $\mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ and $\mathbb{W}_i \subset \mathbb{R}^{r_i}$.

Next, we introduce a distributed regulator for (1). As it will be clear in the sequel, this is the controller associated with the family of RCI sets we will compute. We define for all $i \in \mathcal{M}$ the following state-feedback law

$$\mathcal{C}_{[i]}: \ u_{[i]} = \kappa_i(x_{[i]}) + \sum_{j \in \mathcal{N}_i} \delta_{ij} K_{ij} x_{[j]}. \tag{4}$$

where $\kappa_i(\cdot) : \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$ is a nonlinear function, $K_{ij} \in \mathbb{R}^{m_i \times n_j}$ are gain matrices and $\delta_{ij} \in \{0, 1\}$ are binary variables. Hereafter we assume $\delta_{ij} = 0$ and $K_{ij} = 0$ if $j \notin \mathcal{N}_i$. This implies that $\mathcal{C}_{[i]}$ depends only on the local state ($x_{[i]}$) and the parents' states ($x_{[j]}$, $j \in \mathcal{N}_i$). Binary parameters δ_{ij}, $j \in \mathcal{N}_i$ can be chosen to take advantage of the knowledge of parents' states ($\delta_{ij} = 1$) or not ($\delta_{ij} = 0$). In the first (resp. second) case, the control scheme is distributed (resp. decentralized).

Our main goal is to solve the following problem.

Problem 1: Design RCI local controllers $\mathcal{C}_{[i]}$, $i \in \mathcal{M}$ such that, for a nontrivial set of initial states

(a) the origin of the closed-loop system is nominally stable, i.e. when $\mathbb{W} = \{0\}$ it holds

$$||x_{[i]}(t)|| \to 0 \text{ as } t \to \infty \tag{5}$$

(b) state and input constraints are fulfilled at all time instants, i.e.

$$x_{[i]}(t) \in \mathbb{X}_i,\ u_{[i]}(t) \in \mathbb{U}_i,\ \forall t \geq 0 \text{ and } \forall w_{[i]}(t) \in \mathbb{W}_i. \tag{6}$$

Using the collective variables, from (2) and (4) one obtains the collective closed-loop dynamics

$$\mathbf{x}^+ = \mathbf{A_D}\mathbf{x} + \mathbf{B}\kappa(\mathbf{x}) + (\mathbf{A_C} + \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{D}\mathbf{w} \tag{7}$$

where $\mathbf{K}$ is composed of matrices $\delta_{ij} K_{ij}$ and $\kappa(\mathbf{x}) = (\kappa(x_{[1]}), \dots, \kappa(x_{[M]}))$.

We equip system (7) with constraints $\mathbf{x} \in \mathbb{X} = \prod_{i \in \mathcal{M}} \mathbb{X}_i$, $\mathbf{u} \in \mathbb{U} = \prod_{i \in \mathcal{M}} \mathbb{U}_i$ and $\mathbf{w} \in \mathbb{W} = \prod_{i \in \mathcal{M}} \mathbb{W}_i$. In Section IV we solve Problem 1 under the following assumptions

Assumption 1: The matrix pair (A_{ii}, B_i), $i \in \mathcal{M}$ is controllable.

Assumption 2: Constraints $\mathbb{X}_i$, $\mathbb{U}_i$ and $\mathbb{W}_i$ are $\mathcal{C}$-sets.

From Assumption 2 constraints $\mathbb{X}_i$, $\mathbb{U}_i$ and $\mathbb{W}_i$ are polytopes that can be written as

$$\mathbb{X}_i = \{x_{[i]} \in \mathbb{R}^{n_i} : c^T_{x_{i,\tau}} x_{[i]} \leq 1,\ \forall \tau \in 1 : \tau_i^x\} \tag{8a}$$
$$\mathbb{U}_i = \{u_{[i]} \in \mathbb{R}^{m_i} : c^T_{u_{i,\tau}} u_{[i]} \leq 1,\ \forall \tau \in 1 : \tau_i^u\} \tag{8b}$$
$$\mathbb{W}_i = \{w_{[i]} \in \mathbb{R}^{r_i} : c^T_{w_{i,\tau}} w_{[i]} \leq 1,\ \forall \tau \in 1 : \tau_i^w\}, \tag{8c}$$

where $c_{x_{i,\tau}} \in \mathbb{R}^{n_i}$, $c_{u_{i,\tau}} \in \mathbb{R}^{m_i}$ and $c_{w_{i,\tau}} \in \mathbb{R}^{r_i}$.

In absence of coupling between subsystems (i.e. $A_{ij} = 0$, $i \neq j$) and disturbances (i.e. $\mathbb{W}_i = \{0\}$, $\forall i \in \mathcal{M}$) the controller (4) is decentralized and hence the collective closed-loop system (7) is composed by decoupled subsystems. Therefore, from [5], Assumptions 1 and 2 guarantee that there exists a CI set Ω_i for each subsystem $\Sigma_{[i]}$ with $\mathcal{N}_i = \emptyset$. This implies that Problem 1-(a) is solvable in a totally decentralized fashion. Furthermore, since $\mathbb{X}_i$ and $\mathbb{U}_i$ are polytopes, from [5] one has that the computation of set Ω_i requires the solution of an LP problem for subsystem $\Sigma_{[i]}$ (the precise algorithm is summarized in Section III). In Section IV we will derive a partially decentralized method for computing RCI sets (and the associated controllers) in presence of coupling between subsystems and local disturbances. This aim will be achieved by accounting, in a suitable way, of coupling propagation over the interconnection graph.

III. COMPUTATION OF CI SETS FOR DECOUPLED SYSTEMS

In this section, we assume that subsystems $\Sigma_{[i]}$, $\forall i \in \mathcal{M}$ are decoupled and not affected by disturbances, i.e. $A_{ij} = 0$, for $i \neq j$, $j \in \mathcal{M}$ and $\mathbb{W}_i = \{0\}$. In this case, for all $i \in \mathcal{M}$ we can compute a CI set $\mathbb{S}_i \subseteq \mathbb{X}_i$ and the associated control law $u_{s[i]} = \kappa_i(x_{[i]}) \in \mathbb{U}_{s_i} \subseteq \mathbb{U}_i$ using the LP-based procedure proposed in [6]. For the sake of completeness, we summarize it in the following. As in Section VI of [6], we define $\forall i \in \mathcal{M}$ the set of variables ζ_i as

$$\begin{aligned}
\zeta_i = \{&\bar{s}_{[i]}{}^{(s,f)} \in \mathbb{R}^{n_i} && \forall s \in \mathcal{A}_i^1,\ \forall f \in \mathcal{A}_i^2;\\
&\bar{u}_{[i]}{}^{(s,f)} \in \mathbb{R}^{m_i} && \forall s \in \mathcal{A}_i^3,\ \forall f \in \mathcal{A}_i^2;\\
&\tilde{\rho}_i^{(f_1,f_2)} \in \mathbb{R} && \forall f_1 \in \mathcal{A}_i^2,\ \forall f_2 \in \mathcal{A}_i^2;\\
&\tilde{\psi}_i^{(r,s)} \in \mathbb{R} && \forall r \in \mathcal{A}_i^4,\ \forall s \in \mathcal{A}_i^3;\\
&\tilde{\gamma}_i^{(r,s)} \in \mathbb{R} && \forall r \in \mathcal{A}_i^5,\ \forall s \in \mathcal{A}_i^3;\\
&\tilde{\alpha}_i \in \mathbb{R};\\
&\tilde{\beta}_i \in \mathbb{R}\}
\end{aligned}$$

with $\mathcal{A}_i^1 = 1 : k_i$, $\mathcal{A}_i^2 = 1 : q_i$, $\mathcal{A}_i^3 = 0 : k_i - 1$, $\mathcal{A}_i^4 = 1 : l_i$ and $\mathcal{A}_i^5 = 1 : g_i$, where k_i, $q_i \in \mathbb{N}$ are parameters of the procedure that can be chosen by the user as well as the set

$$\bar{\mathbb{S}}_i^0 = \text{convh}(\{\bar{s}_{[i]}{}^{(0,f)} \in \mathbb{R}^{n_i},\ \forall f \in \mathcal{A}_i^2\}), \tag{10}$$

We assume $\bar{s}_{[i]}{}^{(0,1)} = 0$, thus $\bar{\mathbb{S}}_i^0$ contains the origin in its non-empty interior. Let us define the sets

$$\bar{\mathbb{S}}_i^s = \text{convh}(\{\bar{s}_{[i]}{}^{(s,f)} \in \mathbb{R}^{n_i},\ \forall f \in 1 : q_i\}),\ \forall s \in \mathcal{A}_i^1$$

and

$$\bar{\mathbb{U}}_{s_i}^s = \text{convh}(\{\bar{u}_{[i]}{}^{(s,f)} \in \mathbb{R}^{m_i},\ \forall f \in 1 : q_i\}),\ \forall s \in \mathcal{A}_i^3$$

with $\bar{s}_{[i]}{}^{(s,1)} = 0$ and $\bar{u}_{[i]}{}^{(s,1)} = 0$. Moreover let us also consider the following set of affine constraints on the decision variable ζ_i

$$\mathbb{Z}_i = \{\zeta_i : \tag{11a}$$
$$\tilde{\alpha}_i < 1;\ -\tilde{\alpha}_i \leq 0; \tag{11b}$$
$$-\tilde{\beta}_i \leq 0; \tag{11c}$$
$$\bar{s}_{[i]}{}^{(k_i,f_1)} - \sum_{f_2=1}^{q_i} \tilde{\rho}_i^{(f_1,f_2)} \bar{s}_{[i]}{}^{(0,f_2)} = 0; \tag{11d}$$
$$-\tilde{\alpha}_i + \sum_{f_2=1}^{q_i} \tilde{\rho}_i^{(f_1,f_2)} \leq 0; \tag{11e}$$
$$-\tilde{\rho}_i^{(f_1,f_3)} \leq 0; \tag{11f}$$
$$\sum_{s=0}^{k_i-1} \tilde{\psi}_i^{(f_4,s)} \leq \tilde{\beta}_i d_{u_{i,f_4}}; \tag{11g}$$
$$c^T_{u_{i,f_4}} \bar{u}_{[i]}{}^{(f_5,f_1)} - \tilde{\psi}_i^{(f_4,f_5)} \leq 0; \tag{11h}$$
$$\sum_{s=0}^{k_i-1} \tilde{\gamma}_i^{(f_6,s)} \leq \tilde{\beta}_i d_{x_{i,f_6}}; \tag{11i}$$
$$c^T_{x_{i,f_6}} \bar{s}_{[i]}{}^{(f_5,f_1)} - \tilde{\gamma}_i^{(f_6,f_5)} \leq 0; \tag{11j}$$
$$\bar{s}_{[i]}{}^{(f_5+1,f_1)} = A_{ii}\bar{s}_{[i]}{}^{(f_5,f_1)} + B_i\bar{u}_{[i]}{}^{(f_5,f_1)}\}. \tag{11k}$$

$\forall f_1, f_3 \in \mathcal{A}_i^2$, $\forall f_4 \in \mathcal{A}_i^4$, $\forall f_5 \in \mathcal{A}_i^3$, $\forall f_6 \in \mathcal{A}_i^5$. The relation between elements of $\mathbb{Z}_i$ and the control invariant set $\mathbb{S}_i$ is established in the next proposition.

Proposition 1 ([6]): Let Assumption 1 and 2 hold. Let $k_i \geq \mathcal{CI}(A_{ii}, B_i)$[1] and set $\bar{\mathbb{S}}_i^0$ as in (10). Then

- there exist a feasible point $\zeta_i \in \mathbb{Z}_i$;
- the set
$$\mathbb{S}_i = \tilde{\beta}_i^{-1} \bigoplus_{s=0}^{k_i-1} \bar{\mathbb{S}}_i^s \subseteq \mathbb{X}_i \tag{12}$$
is CI and the corresponding set $\mathbb{U}_{s_i}$ is given by
$$\mathbb{U}_{s_i} = \tilde{\beta}_i^{-1} \bigoplus_{s=0}^{k_i-1} \bar{\mathbb{U}}_{s_i}^s \subseteq \mathbb{U}_i;$$
- set $\mathbb{S}_i$ is a λ_i-contractive set with $\lambda_i = \frac{\tilde{\delta}_i + \tilde{\alpha}_i - 1}{\tilde{\delta}_i} \in [0, 1)$ where $\tilde{\delta}_i = \min_\delta\{\delta \mid \bigoplus_{s=0}^{k_i-1} \mathbb{S}_i^s \subseteq \delta\mathbb{S}_i^0,\ \delta \geq 1\}$;
- There exists a control law $\kappa_i(\cdot)$ associated to $\mathbb{S}_i$ such that the origin of $x^+{}_{[i]} = A_{ii}x_{[i]} + B_i\kappa_i(x_{[i]})$ is an equilibrium point and $\kappa_i(\mathbb{S}_i) \subseteq \mathbb{U}_i$. Furthermore there exist a Lyapunov function defined on $\mathbb{S}_i$ certifying the stability of the origin.

The feasibility problem (11) is an LP problem, since the constraints in $\mathbb{Z}_i$ are affine. In Remark 6.3 of [6] the authors propose to find $\zeta_i \in \mathbb{Z}_i$ while minimizing different cost functions. In the sequel, we always assume to minimize $\tilde{\alpha}_i$. The definition of sets $\bar{\mathbb{S}}_i^s$, $\forall s \in 0 : k_i$, ensures that

[1] $\mathcal{CI}(A_{ii}, B_i)$ is the controllability index of the matrix pair (A_{ii}, B_i).

$\bar{\mathbb{S}}_i^s$ contains the origin and hence, since $\bar{\mathbb{S}}_i^0$ contains the origin in its nonempty interior, $\mathbb{S}_i$ contains the origin in its nonempty interior as well. In the following, we show how to compute the control law $\kappa_i(x_{[i]})$ appearing in Proposition 1 by exploiting the implicit representation of set $\mathbb{S}_i$ proposed in Section VI.B of [6]. Recalling that $\mathbb{S}_i$ is the Minkowski sum of k_i polytopes and that, for all $s \in \mathcal{A}_i^3$, polytope $\bar{\mathbb{S}}_i^s$ is described by the convex combination of points ${\bar{s}_{[i]}}^{(s,f)}$, we have

$$\begin{aligned}&{\tilde{s}_{[i]}}^{s} \in \bar{\mathbb{S}}_i^s \quad \text{ if } \forall f \in \mathcal{A}_i^2, \exists \eta_i^{(s,f)} \geq 0\\ &\text{such that } \sum_{f=1}^{q_i} \eta_i^{(s,f)} = 1, {\tilde{s}_{[i]}}^{s} = \sum_{f=1}^{q_i} \eta_i^{(s,f)} {\bar{s}_{[i]}}^{(s,f)}.\end{aligned}$$

Hence we have that $x_{[i]} \in \mathbb{S}_i$ if and only if $\forall f \in \mathcal{A}_i^2, \forall s \in \mathcal{A}_i^3$ there exist $\eta_i^{(s,f)} \in \mathbb{R}$ such that

$$\eta_i^{(s,f)} \geq 0 \tag{13a}$$

$$\sum_{f=1}^{q_i} \eta_i^{(s,f)} = 1 \tag{13b}$$

$$x_{[i]} = \tilde{\beta}_i^{-1} \sum_{s=0}^{k_i-1} \sum_{f=1}^{q_i} \eta_i^{(s,f)} {\bar{s}_{[i]}}^{(s,f)}. \tag{13c}$$

In [6] the authors propose also an implicit representation of controller $\kappa_i(x_{[i]})$ based on the implicit representation (13) of set $\mathbb{S}_i$. The control law $\kappa_i(\cdot)$ is computed by solving the following LP problem

$$\bar{\mathbb{P}}_i(x_{[i]}) : \min_{\mu, \eta_i^{(s,f)}} \mu \tag{14a}$$

$$\eta_i^{(s,f)} \geq 0 \quad \forall f \in \mathcal{A}_i^2, \forall s \in \mathcal{A}_i^3 \tag{14b}$$

$$\sum_{f=1}^{q_i} \eta_i^{(s,f)} = \mu \quad \forall s \in \mathcal{A}_i^3 \tag{14c}$$

$$\mu \geq 0 \tag{14d}$$

$$x_{[i]} = \tilde{\beta}_i^{-1} \sum_{s=0}^{k_i-1} \sum_{f=1}^{q_i} \eta_i^{(s,f)} {\bar{s}_{[i]}}^{(s,f)} \tag{14e}$$

and setting

$$\kappa_i(x_{[i]}) = \tilde{\beta}_i^{-1} \sum_{s=0}^{k_i-1} \kappa_i^s(x_{[i]}),\ \kappa_i^s(x_{[i]}) = \sum_{f=1}^{q_i} \bar{\eta}_i^{(s,f)} {\bar{u}_{[i]}}^{(s,f)} \tag{15}$$

where $\bar{\eta}_i^{(s,f)}$ are the optimizers to (14). According to [17] we can assume without loss of generality that $\bar{\kappa}_i(\cdot)$ is a continuous piecewise affine map.

Next we prove a new result for CI sets defined in Proposition 1: in particular we show that control law $\kappa_i(x_{[i]})$ defined in (15) is homogeneous. This result will be used in the next section to exploit a practical robust control invariance. In [14] a similar result is proposed for RCI sets designed as in [6].

Lemma 1: For $x_{[i]} \in \mathbb{S}_i$ and $\rho \geq 0$ one has

$$\kappa_i^s(\rho x_{[i]}) = \rho \kappa_i^s(x_{[i]}),\ \forall s \in \mathcal{A}_i^3$$

and hence $\kappa_i(\rho x_{[i]}) = \rho \kappa_i(x_{[i]})$.

Proof: Let $\bar{\eta}_i^{(s,f)}$, $f \in \mathcal{A}_i^2$, $s \in \mathcal{A}_i^3$ and $\bar{\mu}$ be the optimizers to $\bar{\mathbb{P}}_i(x_{[i]})$. One can easily verify that $\eta_i^{(s,f)} = \rho\bar{\eta}_i^{(s,f)}$ and $\mu = \rho\bar{\mu}$ fulfill the constraints (14b)-(14e) for $\bar{\mathbb{P}}_i(\rho x_{[i]})$. We show now that these values are also optimal for $\bar{\mathbb{P}}_i(\rho x_{[i]})$. By contradiction, assume that $\tilde{\eta}_i^{(s,f)}$, $\tilde{\mu}$ are the optimizers to $\bar{\mathbb{P}}_i(\rho x_{[i]})$ giving the optimal cost $\tilde{\mu} < \rho\bar{\mu}$. One can easily verify that $\eta_i^{(s,f)} = \rho^{-1}\tilde{\eta}_i^{(s,f)}$ and $\mu = \rho^{-1}\tilde{\mu}$ verify the constraints (14b)-(14e) for $\bar{\mathbb{P}}_i(x_{[i]})$ and yield a cost $\rho^{-1}\tilde{\mu} < \bar{\mu}$. This contradicts the optimality of $\bar{\mu}$ for $\bar{\mathbb{P}}_i(x_{[i]})$. ■

IV. PRACTICAL RCI SETS

In this section, we show how to leverage the main results of [11] and [12] in order to guarantee properties (5) and (6) for the dynamics (7) equipped with constraints $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{W}$. Indeed, in presence of coupling, Problem 1 can be solved using the notion of practical Robust Control Invariant (pRCI) proposed in [11] and [12]. This approach also offers a computationally affordable, yet conservative, procedure. In particular in [11], functions $\kappa_i(x_{[i]})$ are constrained to be *linear*. In this section, we will propose a notion of pRCI using the *nonlinear* controllers $\mathcal{C}_i$ in (4).

Given a collection of sets $\mathbb{S} = \{\mathbb{S}_i,\ i \in \mathcal{M}\}$, $\mathbb{S}_i \subset \mathbb{R}^{n_i}$, with $\mathbb{S}_i$ computed as in Proposition 1, and a set $\Theta \subset \mathbb{R}_+^M$, we define a parameterized family of sets $\mathcal{S}(\mathbb{S}, \Theta) = \{(\theta_1\mathbb{S}_1, \ldots, \theta_M\mathbb{S}_M) : \theta \in \Theta\}$, where $\theta = (\theta_1, \ldots, \theta_M)$. Intuitively, scalars θ_i can be interpreted as zooming factors.

Definition 2: The family of sets $\mathcal{S}(\mathbb{S}, \theta)$ is pRCI for the constrained local dynamics given by (2), equipped with local controller $\mathcal{C}_{[i]}$ in (4) and constraints (3), if for all $i \in \mathcal{M}$ and all $(\theta_1\mathbb{S}_1, \ldots, \theta_M\mathbb{S}_M) \in \mathcal{S}(\mathbb{S}, \Theta)$, one has

$$\theta_i\mathbb{S}_i \subseteq \mathbb{X}_i \tag{16a}$$

$$\mathbb{S}_i^+ \subseteq \theta_i^+\mathbb{S}_i \tag{16b}$$

$$\kappa_i(\theta_i\mathbb{S}_i) \oplus \bigoplus_{j\in\mathcal{N}_i} \delta_{ij}K_{ij}\theta_j\mathbb{S}_j \subseteq \mathbb{U}_i \tag{16c}$$

$$(\theta_1^+\mathbb{S}_1, \ldots, \theta_M^+\mathbb{S}_M) \in \mathcal{S}(\mathbb{S}, \Theta). \tag{16d}$$

where $\mathbb{S}_i^+ = A_{ii}\theta_i\mathbb{S}_i \oplus B_i\kappa_i(\theta_i\mathbb{S}_i) \oplus \bigoplus_{j\in\mathcal{N}_i}(A_{ij} + B_i\delta_{ij}K_{ij})\theta_j\mathbb{S}_j \oplus D_i\mathbb{W}_i$. Without loss of generality, we consider sets $\mathbb{S}_i$ and $\mathbb{U}_{s_i}$ defined as $\mathbb{S}_i = \{G_i s_{[i]} \leq \mathbf{1}\}$.

The main issue we will address in the sequel is the following: given $\mathbb{S}$ is there any set $\Theta \subset \mathbb{R}_+^M$ such that the family $\mathcal{S}(\mathbb{S}, \Theta)$ is pRCI? As in [11] and [12], we propose to first derive the dynamics of the scaling factors θ_i. More precisely, for all $i, j \in \mathcal{M}$ we set μ_{ij} as

$$\begin{cases}\lambda_i & \text{if } i = j\\ \min_{\mu\geq 0}\{\mu : (A_{ij} + \delta_{ij}B_iK_{ij})\mathbb{S}_j \subseteq \mu\mathbb{S}_i\} & \text{if } j \in \mathcal{N}_i\\ 0 & \text{otherwise}\end{cases} \tag{17}$$

$$\alpha_i = \min_{\beta\geq 0}\{\beta : D_i\mathbb{W}_i \subseteq \beta\mathbb{S}_i\}. \tag{18}$$

this allows us to write the collective dynamics of the scaling factors

$$\theta^+ = T\theta + \alpha \tag{19}$$

where the entries of $T \in \mathbb{R}^{M\times M}$ are $T_{ij} = \mu_{ij}$ and $\alpha = (\alpha_1, \ldots, \alpha_M)$. For fulfilling (16), using Lemma 1, let us define

$$\begin{aligned}\Theta_0 = \{\theta \in \mathbb{R}^M : \forall i \in \mathcal{M},\ \theta_i \mathbb{S}_i \subseteq \mathbb{X}_i \\ \theta_i \mathbb{U}_{s_i} \oplus \bigoplus_{j\in\mathcal{N}_i} \delta_{ij} K_{ij} \theta_j \mathbb{S}_j \subseteq \mathbb{U}_i\}\end{aligned} \tag{20}$$

The key assumption for providing a set Θ that makes $\mathcal{S}(\mathbb{S}, \Theta)$ a pRCI family is the following.

Assumption 3: (i) T is Schur.

(ii) The unique equilibrium point $\bar{\theta}$ of system (19) is such that $\bar{\theta} \in \Theta_0$.

(iii) The set Θ is an invariant set for system (19) and constraint set Θ_0, i.e. $\forall \theta \in \Theta \subseteq \Theta_0$, $\theta^+ \in \Theta$.

Lemma 2: Let Assumptions 1-3 hold and sets $\mathbb{S}_i$, $i \in \mathcal{M}$ defined as in Proposition 1. Then

(i) there is a non-trivial convex and compact positively invariant set Θ for system (19) equipped with constraints $\theta \in \Theta_0$;

(ii) $\mathcal{S}(\mathbb{S}, \Theta)$ is pRCI for (1) with constraints (3).

Proof: Point (i) has been proved in [11] and [12]. Differently from [11] and [12], the proof of robust invariance of $\mathcal{S}(\mathbb{S}, \Theta)$ is based on Lemma 1. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_M) \in \Theta$. For all $i \in \mathcal{M}$ we can write

$$\begin{aligned}&\theta_i A_{ii} \mathbb{S}_i \oplus B_i \kappa_i(\theta_i \mathbb{S}_i) \oplus \bigoplus_{j\in\mathcal{N}_i} (A_{ij} + B_i \delta_{ij} K_{ij}) \theta_j \mathbb{S}_j \oplus D_i \mathbb{W}_i \\ &= \theta_i (A_{ii} \mathbb{S}_i \oplus B_i \kappa_i(\mathbb{S}_i)) \oplus \bigoplus_{j\in\mathcal{N}_i} (A_{ij} + B_i \delta_{ij} K_{ij}) \theta_j \mathbb{S}_j \oplus D_i \mathbb{W}_i \\ &\subseteq \lambda_i \theta_i \mathbb{S}_i \oplus \bigoplus_{j\in\mathcal{N}_i} \mu_{ij} \theta_j \mathbb{S}_i \oplus \alpha_i \mathbb{S}_i \\ &= (\lambda_i \theta_i + \sum_{j\in\mathcal{N}_i} \mu_{ij} + \alpha_i) \\ &= \theta_i^+ \mathbb{S}_i.\end{aligned}$$

where in the first step we used Lemma 1 and in the second step we used equations (17) and (18). Since Θ is an invariant set for system (19), $\theta^+ \in \Theta$ and hence $(\theta_1^+ \mathbb{S}_1, \ldots, \theta_M^+ \mathbb{S}_M) \in \mathcal{S}(\mathbb{S}, \Theta)$. Moreover since $\Theta \subseteq \Theta_0$, state and input constraints are fulfilled $\forall \theta \in \Theta$. Therefore $\mathcal{S}(\mathbb{S}, \Theta)$ is pRCI. ■

Lemma 2 guarantees that

$$\begin{aligned}\theta(0) \in \Theta \text{ and } x_{[i]}(0) \in \theta_i(0) \mathbb{S}_i,\ \forall i \in \mathcal{M} \\ \Rightarrow x_{[i]}(t) \in \theta_i(t) \mathbb{S}_i,\ \forall i \in \mathcal{M},\ \forall t \geq 0\end{aligned} \tag{21}$$

Furthermore, as shown in [12], $\text{dist}(x_{[i]}(t), \bar{\theta}_i \mathbb{S}_i) \to 0$ as $t \to \infty$. In the nominal case, i.e. $\mathbb{W} = \{0\}$, one has $\alpha = 0$ in (19) and then $\bar{\theta} = 0$. In summary, property (5) is guaranteed. Also state and input constraints hold since $\mathcal{S}(\mathbb{S}, \Theta)$ is pRCI. Therefore, Problem 1 is solved if we can design local controllers $\mathcal{C}_{[i]}$ fulfilling the assumptions of Lemma 2. A design procedure to achieve this goal is proposed in Section V.

Remark 1: Note that, according to (21), constraints (3) can be satisfied if $x_{[i]}(0) \in \theta_i(0)\mathbb{S}_i, \forall i \in \mathcal{M}$ and $\theta(0) \in \Theta$. Clearly this is a centralized operation. In order to allow each controller to locally verify (21), one can compute offline an inner-box approximation $\bar{\Theta} = \prod_{i=1}^M [0, \bar{\theta}_i]$ contained in Θ, therefore each controller must verify locally $x_{[i]}(0) \in \theta_i(0)\mathbb{S}_i$ and $\theta_i(0) \in [0, \bar{\theta}_i]$.

V. Design procedure for controllers $\mathcal{C}_{[i]}$

In this section, we propose a method to design the distributed controllers $\mathcal{C}_{[i]}$. The key issue is how to compute suitable gains K_{ij} and binary variables δ_{ij} such that Assumption 3 holds. Differently from [11], we propose a design procedure in order to reduce as possible the number of centralized operations. The design procedure is summarized in Algorithm 1 that is composed by three parts.

Algorithm 1

Input: Dynamics (2) and polytopic sets $\mathbb{X}_i$, $\mathbb{U}_i$, $\mathbb{W}_i$, $i \in \mathcal{M}$ verifying Assumption 1 and 2.
Output: A pRCI family of sets $\mathcal{S}(\mathbb{S}, \Theta)$.

(A) *Decentralized steps*. For all $i \in \mathcal{M}$,
- (I) compute a λ_i-contractive set $\mathbb{S}_i$ for $x^+_{[i]} = A_{ii} x_{[i]} + B_i u_{[i]}$ equipped with constraints $\mathbb{X}_i$ and constraints $\mathbb{U}_i$, solving LP problem $\mathbb{Z}_i$ and set $\mu_{ii} = \lambda_i$;
- (II) compute α_i as in (18).

(B) *Distributed steps*. For all $i \in \mathcal{M}$,
- (I) if $\delta_{ij} = 1$, compute the matrix K_{ij}, $\forall j \in \mathcal{N}_i$ solving
$$\min_{K_{ij}} ||G_i (A_{ij} + \delta_{ij} B_i K_{ij}) G_j^\flat||_p \tag{22}$$
where p is a generic norm.
- (II) compute μ_{ij} as in (17).

(C) *Centralized steps*
- (I) if matrix T is not Schur **stop**;
- (II) compute set Θ_0 as in (20) and the equilibrium point $\bar{\theta}$ of system (19). If $\bar{\theta} \notin \Theta_0$ **stop**;
- (III) compute the maximal invariant set Θ_∞ of system (19) equipped with constraint Θ_0;
- (IV) compute an inner box approximation $\bar{\Theta}$ of Θ_∞.

Operations in part (A) can be executed in parallel using computational resources associated with subsystems, i.e. in a decentralized fashion. Steps in part (B) have a distributed nature, meaning that computations are decentralized but they can be performed only after each system has received suitable pieces of information from its parents. Finally, design steps in part (C) require centralized computations involving only the M-th order system (19). Next, we comment each step of Algorithm 1 in details.

A. Part (A)

Step (AI) is the easiest one. The computation of sets $\mathbb{S}_i$ as in step (AI) is the major improvement compared to [11] and [12] and it allows pRCI. The idea of compute set $\mathbb{S}_i$ as a λ_i-contractive set is based on the argument that sets $(1-\lambda_i)S_i$

can be used for compensating coupling terms in the dynamics. Remarkably, using the efficient procedures proposed in Section III, the computation of a set $\mathbb{S}_i$ amounts to solving an LP problem. Step (AII) focuses on the computation of scalars α_i. From (18) and (8c), using procedures proposed in [18], we can compute α_i by means of LP problems.

B. Part (B)

For the computation of matrices K_{ij} and parameters μ_{ij}, each system $\Sigma_{[i]}$ needs to receive the set $\mathbb{S}_j$ from neighbors $j \in \mathcal{N}_i$ such that $\delta_{ij} = 1$. In step (BI), if $\delta_{ij} = 1$, the computation of matrices K_{ij}, $j \in \mathcal{N}_i$ is required. Following the idea proposed in [19], since K_{ij} affects the stability of matrix T on the off-diagonal terms, we propose to reduce the magnitude of coupling by minimizing the magnitude of $\bar{A}_{ij}$ as in (22), where G_i and $G_j^{\flat}$ allows us to take into account the size of sets $\mathbb{S}_i$ and $\mathbb{S}_j$ respectively. So far the parameters δ_{ij} have been considered fixed. However, if in step (BI) one obtains $K_{ij} = 0$ for some $j \in \mathcal{N}_i$, it is impossible to reduce the magnitude of the coupling term $\bar{A}_{ij}$ and, from controller $\mathcal{C}_{[i]}$, the knowledge of $x_{[j]}$ is useless. This suggests to revise the choice of δ_{ij} and set $\delta_{ij} = 0$. In step (BII), since $\mathbb{S}_i$ are polytopes, using procedures proposed in [18] we can compute scalars μ_{ij} by means of LP problems.

C. Part (C)

In step (CI) we check the Schurness of matrix T. If the test fails it is not possible to fulfill Assumption 3-(i) and the only possibility is to restart the algorithm after increasing the number of variables δ_{ij} that are equal to one.

In step (CII), since the sets $\mathbb{S}_i$ and $\mathbb{X}_i$ are polytopes, using results from [18] the computation of the set Θ_0 can be done as follows

$$
\begin{aligned}
\Theta_0 &= \prod_{i=1}^{M}[0,\tilde{\theta}_i] \bigcap \bigcap_{i=1\in\mathcal{M}} \tilde{\Theta}_i \\
\tilde{\theta}_i &= (\max_{\tau\in 1:\bar{\tau}_i} \{\sup_{s_{[i]}} h_{i,\tau}^T s_{[i]} : G_i s_{[i]} \leq \mathbf{1}\})^{-1} \\
\tilde{\Theta}_i &= \{\theta \in \mathbb{R}^M : \sum_{j\in\mathcal{N}_i} \tilde{\theta}_{u_{ij}}\theta_j + \tilde{\theta}_{u_i}\theta_i \leq 1\} \\
\tilde{\theta}_{u_i} &= (\max_{\sigma\in 1:\bar{\sigma}_i} \{\sup_{u_{s[i]}} l_{i,\sigma}^T u_{s[i]} : \mathcal{D}_i u_{s[i]} \leq \mathbf{1}\}) \\
\tilde{\theta}_{u_{ij}} &= (\max_{\sigma\in 1:\bar{\sigma}_i} \{\sup_{s_{[j]}} l_{i,\sigma}^T \delta_{ij} K_{ij} s_{[j]} : G_j s_{[j]} \leq \mathbf{1}\})
\end{aligned} \tag{23}
$$

Moreover, in step (CII) we compute the equilibrium point $\bar{\theta}$ of system (19). If $\bar{\theta} \notin \Theta_0$ we can not guarantee Assumption 3 and therefore the algorithm stops. Note that if $\mathbb{W}_i = \{0\}$, $\forall i \in \mathcal{M}$, the equilibrium point $\bar{\theta}$ is the origin and hence $\bar{\theta} \in \Theta_0$ by construction. According to Assumption 3-(iii), the set Θ of all feasible contractions θ is computed as an RPI set for system (19) and constraints $\theta \in \Theta_0$. In particular, since T is Schur and Θ_0 is a polytopic, using results from [2] we can compute the maximal RPI set Θ_∞. As discussed in Remark 1, a decentralized initialization of state estimators is possible computing an inner-box approximation $\bar{\Theta}$ contained in Θ_∞. This is done in step (CIV) by solving a suitable LP problem.

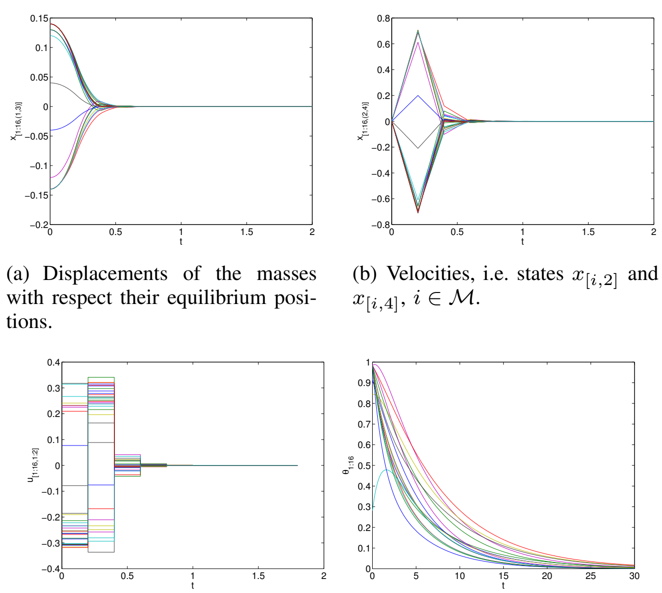
Remark 2: We note that, similarly to Section 5 of [19], the plug-in and unplugging of subsystems do not require to retune all the controllers $\mathcal{C}_{[i]}$, $i \in \mathcal{M}$. In particular Part (A) and Part (C) of Algorithm 1 are executed only for the plugged-in subsystem and at most for the children of the plugged-in or unplugged subsystem. Part (C) is the only step which may require a complete execution. For more details about plugging-in and unplugging operations we refer the interested reader to Section 5 of [19].

Remark 3: In this paper we consider that each subsystem is linearly coupled with his parent subsystems. As shown in [13], the concept of practical invariance for large-scale systems can be extended to subsystems nonlinearly interconnected, i.e. $\sum_{j\in\mathcal{N}_i} A_{ij}x_{[j]}$ is replaced with $\sum_{j\in\mathcal{N}_i} \phi_i(x_{[j]})$, where $\phi_i(\cdot) : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$.

VI. Simulation example

A. Example 1

We consider a system composed by 16 masses coupled in a grid 4×4 through springs and dampers. The description of the system as well as the values of parameters and constraints are the same as in Section 5 of [14]. We synthesized controllers $\mathcal{C}_{[i]}$, $i \in \mathcal{M}$ using Algorithm 1 and setting $\delta_{ij} = 1$ if mass i is coupled with mass j. In Figure 1 we show a simulation where the control action $u_{[i]}(t)$ computed by the controller $\mathcal{C}_{[i]}$, for all $i \in \mathcal{M}$, is kept constant during the sampling interval and applied to the continuous-time system. Convergence is obtained for all masses to their equilibrium position while fulfilling input and state constraints. State and input variables are depicted in Figure 1. Moreover in Figure 1(d) we note that the zooming factors θ_i converge to the origin since the masses are not affected by disturbances.



(a) Displacements of the masses with respect their equilibrium positions.

(b) Velocities, i.e. states $x_{[i,2]}$ and $x_{[i,4]}$, $i \in \mathcal{M}$.

(c) Inputs $u_{[i]}$, $i \in \mathcal{M}$.

(d) Zooming factors trajectories.

Fig. 1. State and input trajectories of the 16 masses and zooming factor trajectories.

B. Example 2

We consider that each continuous-time model of the mass is affected by a disturbance $w_{[i]} \in \mathbb{W}_i = \{w_{[i]} \in \mathbb{R}^4 : ||w_{[i]}||_\infty \leq 0.0064\}$, therefore we can consider errors of 0.8% of the maximum velocity of the mass. In Figure 2 we note that the zooming factors θ_i converge to an equilibrium point $\bar{\theta}$ since the masses are affected by disturbances. In particular the equilibrium point guarantees that asymptotically $x_{[i]} \in \bar{\theta}_i \mathbb{S}_i$, irrespectively of disturbances.

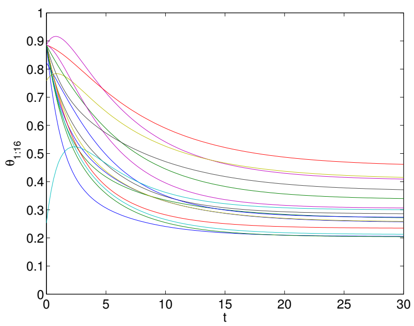


Fig. 2. Zooming factors trajectories for Example 2.

VII. Conclusions

In this paper we have presented an algorithm for the computation of time-varying RCI sets. Our approach requires reduced centralized computations and, as such, is suitable for large-scale systems. The method hinges on the notion of practical RPI sets and provides, as a byproduct, the associated decentralized (or distributed) control laws. Each local controller depends in a linear way on parents' states. Future research will focus on relaxing this assumption by considering the use of fully nonlinear state-feedback laws and on combining the proposed pRCI sets with the distributed state estimator proposed in [19] in order to design an output-feedback robust control invariance family of sets for LTI large-scale systems.

References

[1] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*. Madison, WI, USA: Nob Hill Pub., 2009.

[2] E. G. Gilbert and K. T. Tan, "Linear Systems with State and Control Constraints: The Theory and Application of Maximal Output Admissible Sets," *IEEE Transactions on Automatic Control*, vol. 36, no. 9, pp. 1008–1020, 1991.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Philadelphia, Pennsylvania, USA: SIAM Studies in Applied Mathematics, vol. 15, 1994.

[4] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 406–410, 2005.

[5] S. V. Raković and M. Baric, "Local Control Lyapunov Functions for Constrained Linear Discrete-Time Systems: The Minkowski Algebra Approach," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2686 – 2692, 2009.

[6] ——, "Parameterized Robust Control Invariant Sets for Linear Systems: Theoretical Advances and Computational Remarks," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1599–1614, 2010.

[7] M. Cannon, V. Deshmukh, and B. Kouvaritakis, "Nonlinear model predictive control with polytopic invariant sets," *Automatica*, vol. 39, no. 8, pp. 1487–1494, 2003.

[8] M. Fiacchini, T. Alamo, and E. F. Camacho, "On the computation of local invariant sets for nonlinear systems," in *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, LA, USA, December 12-14, 2007, pp. 3989–3994.

[9] ——, "On the computation of convex robust control invariant sets for nonlinear systems," *Automatica*, vol. 46, no. 8, pp. 1334–1338, 2010.

[10] C. Conte, C. N. Jones, M. Morari, and M. N. Zeilinger, "Distributed synthesis and stability of cooperative distributed model predictive control for linear systems," *Automatica*, vol. 69, pp. 117 – 125, 2016. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109816300413

[11] S. V. Raković, B. Kern, and R. Findeisen, "Practical Set Invariance for Decentralized Discrete Time Systems," in *Proceedings of the 49th IEEE Conference on Decision and Control*, Hilton Atlanta Hotel, Atlanta, GA, USA, December 15-17, 2010, pp. 3283–3288.

[12] ——, "Practical Robust Positive Invariance for Large Scale Discrete Time Systems," in *Proceedings of the 16th IFAC World Congress*, Milano, Italy, August 28 - September 2, 2011, pp. 6425–6430.

[13] B. Kern and R. Findeisen, "Analysis and constrained control of nonlinear interconnected systems exploiting positively invariant family of sets," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, Florence, Italy, December 10-13, 2013, pp. 3806–3811.

[14] S. Riverso, M. Farina, and G. Ferrari-Trecate, "Plug-and-Play Model Predictive Control based on robust control invariant sets," *Automatica*, vol. 50, pp. 2179–2186, 2014.

[15] A. I. Zečević and D. D. Šiljak, *Control of complex systems: Structural constraints and uncertainty*. Berlin, Germany: Springer, Communications and Control Engineering, 2010.

[16] M. Baric and F. Borrelli, "Decentralized Robust Control Invariance for a Network of Storage Devices," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 1018–1024, 2012.

[17] T. Gal, *Postoptimal Analyses, Parametric Programming and Related Topics*, 2nd ed. Berlin, Germany: de Gruyter, 1995.

[18] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, no. 4, pp. 317–363, 1998.

[19] S. Riverso, D. Rubini, and G. Ferrari-Trecate, "Distributed bounded-error state estimation based on practical robust positive invariance," *International Journal of Control*, vol. 88, no. 11, pp. 2277–2290, 2015.