Resource-Aware Stochastic Self-Triggered Model Predictive Control

Yingzhao Lian, Yuning Jiang, Naomi Stricker, Lothar Thiele, Colin N. Jones

***Abstract*— This paper considers the control of uncertain systems operated under limited resource factors, such as battery life or hardware longevity. We consider here resource-aware self-triggered control techniques that schedule system operation non-uniformly in time in order to balance performance against resource consumption.**

When running in an uncertain environment, unknown disturbances may deteriorate system performance by acting adversarially against the planned event triggering schedule. In this work, we propose a resource-aware stochastic predictive control scheme to tackle this challenge, where a novel zero-order hold feedback control scheme is proposed to accommodate a time-inhomogeneous predictive control update.

***Index Terms*— Stochastic optimal control, self-triggered model predictive control**

I. INTRODUCTION

MOST devices in Internet of Things (IoT) networks and wireless sensing systems are operated with some limited resource factors, such as battery life or hardware longevity. In order to maintain desirable performance, a minimal number of triggers are required to best exploit the limited resource. Event-triggered control and self-triggered control are two main control schemes [1] accommodating this issue. In particular, control under an event-triggered scheme is updated *reactively* by determining a trigger condition, for which a sensor has to continuously monitor the trigger condition. Contrarily, a self-triggered scheme updates *proactively* by planning the next trigger in advance, leaving the sensor and controller in idle mode. Due to the limitation of the resource factors, especially battery life, a self-triggered scheme is often preferable and is, therefore, the research object of this work.

The key ingredient of a self-triggered controller is the decision of the triggering time sequence. The triggering time can be chosen as long as possible to minimize resource consumption as in [2], [3]. However, to balance performance and resource consumption more effectively, the response of the resource is explicitly considered in the model predictive control (MPC) problem in [4], [5]. The former work solves a mixed-integer problem and is designed for discrete-time systems, while the latter work solves a non-convex continuous-time optimal control problem and has been later generalized to a distributed control scheme [6].

Running a triggered system within an uncertain environment while maintaining system performance is challenging. Especially for the self-triggered controllers, the lack of sensor measurement between consecutive triggers requires extra consideration of the uncertainty propagation. In [7], a nominal control law is determined based on a nominal system, while the discrepancy between the nominal and measured trajectories serves as the triggering condition. In [8]–[10], the idea of tube-MPC enables the design of robust self-triggered controllers for both discrete-time and continuous-time linear systems. [11] used a min-max optimization to optimize the worst-case performance. Although it is capable of handling general uncertainties in nonlinear systems, the resulting non-convex robust optimization problem is NP-hard [12]. Except for [11], other previous works mainly decouple the effects of uncertainty from the nominal system, and the feedback control laws are updated with a fixed frequency.

In this work, a resource-aware stochastic predictive control scheme is designed for a stochastic linear system where the process noise is explicitly considered in the predictive control problem. In particular, a discrete-time zero-order-hold linear feedback control law is integrated into the closed-loop predictive control problem. The update time instances of this feedback control law distribute non-uniformly on the time axis, which we term *time-inhomogeneous*, and are optimized within the predictive control problem. The contributions of this work are summarized into two aspects:

- A sigma field decomposition strategy is proposed to enable the analysis of a time-inhomogeneous control.
- A discrete-time closed-loop feedback control law for stochastic self-triggered MPC is proposed.

The rest of this paper is organized as follows: Section II reviews deterministic resource-aware self-triggered MPC, after which the stochastic extension is elaborated in Section III. This section further details the sigma field decomposition and the continuous-time dynamics of a discrete-time feedback law. The effectiveness of the proposed method is validated in Section IV and conclusions are given in Section V.

Notation: $\{x_i\}_{i=0}^K$ denotes a finite set of size K whose elements x_i are indexed by i. $\mathbb{Z}_a^b$ is the set of integers $\{a, a+1 \ldots, b\}$. $A \backslash B := \{x \,|\, x \in A,\, x \notin B\}$. $\mathbb{E}\{\cdot\}$ denotes the expectation operator and $\mathbb{P}(\cdot)$ represents the probability.

II. DETERMINISTIC SELF-TRIGGERED MPC

This section recaps the main idea of deterministic resource-aware self-triggered control. We consider a linear time invari-

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ant (LTI) system in continuous time:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{1}$$

with state $x(\cdot) : [0, \infty) \to \mathbb{R}^{n_x}$ and control input $u(\cdot) : [0, \infty) \in \mathbb{R}^{n_u}$. A self-triggered controller determines both the value of the control inputs and the time instances at which the control input is changed. In the framework of direct optimal control [13], a self-triggered controller parameterizes its control inputs over the time horizon $[0, t_N]$ by

$$u(t) = \sum_{k=0}^{N-1} v_k \cdot \zeta_k(t, t_k, t_{k+1}), \tag{2}$$

where the orthogonal functions $\zeta_k \in \mathcal{L}^2[t_0, t_N],\ k \in \mathbb{Z}_0^{N-1}$ model the triggering property with a piece-wise constant function

$$\zeta_k(t, t_k, t_{k+1}) = \begin{cases} 1 & t \in (t_k, t_{k+1}] \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For the sake of compactness, we use notation $v \in \mathbb{R}^{Nn_u} := [v_0^\top, v_1^\top \ldots, v_{N-1}^\top]^\top$ to stack the control coefficients, and define the triggering time interval $\Delta_k := t_{k+1} - t_k$ and use the notation $\Delta = [\Delta_0, ..., \Delta_{N-1}]^\top$.

A self-triggered agent updates its control inputs at triggering time instances $\{t_k\}_{k=0}^{N-1}$. When the control law is fixed within $(t_k, t_{k+1}]$, the resource r is recharged at a constant rate ρ until saturation. More specifically, $\dot{r}(t) = h(\overline{r} - r(t))\rho$ for all $t \in [t_k, t_{k+1})$, where $\overline{r}$ is the saturated value and $h(\cdot)$ is a heaviside function with $h(s) = 1$ if $s > 0$ and 0 elsewhere. When the agent is triggered to update the control input, the resource is discharged by an amount $\eta(\Delta_k)$ to pay the update cost. Hence, the resource at triggering time instants $\{t_k\}_{k=0}^{N-1}$ is

$$r(t) = \begin{cases} r_0 & t = t_0 \\ \lim\limits_{t \to t_k^-} r(t) - \eta(\Delta_k) & t \in \{t_k\}_{k=1}^{N-1} \end{cases} \tag{4}$$

with an initially available resource r_0 at t_0. Here, $t \to t_k^-$ represents the left limits, i.e., $t \to t_k$ and $t < t_k$. Moreover, the resource r is required to be lower bounded by $\underline{r}$. In conclusion, a resource-aware self-triggered agent can update its control input when its resource is sufficiently high to stay above the lower bound $\underline{r}$. Otherwise, it must wait until enough resource is available. Once the controller is triggered at time t_0, the resource-aware self-triggered control solves the following optimization problem to plan the next trigger time t_1 and the control input within $[t_0, t_1]$,

$$\min_{x(\cdot), v, \Delta} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} l(x(\tau), v_k) d\tau + M(x(t_N)) \tag{5a}$$

$$\text{s.t.} \quad x(t_0) = x_0\ ,\ r(t_0) = r_0 \tag{5b}$$

$$\forall t \in [t_0, t_N],\ \frac{dx(t)}{dt} = Ax(t) + Bu(t), \tag{5c}$$

$$\forall t \in [t_0, t_N],\ x(t) \in \mathcal{X}\ ,\ u(t) \in \mathcal{U}\ , \tag{5d}$$

$$\forall k \in \{0, 1, ..., N-1\}$$

$$r(t_{k+1}) = \min\{\rho\Delta_k + r(t_k) - \eta(\Delta_k), \overline{r}\} \tag{5e}$$

$$r(t_{k+1}) \in [\underline{r}, \overline{r}]\ , \tag{5f}$$

$$\Delta_k \in [\underline{\Delta}, \overline{\Delta}] \tag{5g}$$

where $l(\cdot, \cdot)$ and $M(\cdot)$ in (5a) are stage cost and terminal cost, respectively. (5e) is a simplified yet equivalent formulation of the resource dynamics (4) [5] and the resource is bounded by (5f). The constraints of the triggering time interval in (5g) protects the system from being Zeno[1] or frozen. $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ and $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ in (5d) model the state and input constraints. The initial state and resource are given by (5b).

III. Stochastic Self-Triggered MPC

In this section, we consider the LTI system (1) contaminated by a Wiener process noise. This is described by the stochastic differential equation (SDE)

$$dx(t) = (Ax(t) + Bu(t))dt + dW\ , \tag{6}$$

where W denotes a multi-dimensional Wiener process with statistics $\mathbb{E}\{W(t)W(s)\} = Q\min(s, t)$ and $\mathbb{E}\{W(t)\} = 0$.

The open-loop evolution of the system's state distribution (6) is widely studied in filter theory [14] and the state evolution remains Gaussian $\mathcal{N}(\mu(t),\ P(t))$, where

$$\frac{d\mu(t)}{dt} = A\mu(t) + Bu(t)\ , \tag{7a}$$

$$\frac{dP(t)}{dt} = AP(t) + P(t)A^\top + Q\ . \tag{7b}$$

Above all, given the dynamics in (7), it is straightforward to adapt the deterministic formulation in (5) to generate an open-loop resource-aware stochastic MPC. The main focus and contribution of this work are to develop a **closed-loop** scheme with respect to the dynamics (7). In particular, a feedback control law is explicitly considered in the predictive control problem and this feedback control law should satisfy the following requirements:

1) The feedback control law can only change its value when the controller is triggered, otherwise, the control inputs remain constant.
2) The feedback control law is not updated at a fixed frequency, and its update time instances are decision variables of the self-triggered problem.

In the following, the dynamics of the state distribution driven by a discrete time feedback control law are developed by using the technique of sigma field decomposition. This results in a resource-aware stochastic self-triggered MPC and its numerical implementation is discussed at the end of this section. In order to convey the main idea of the proposed scheme, we state the main results intuitively and provide the mathematical details in the Appendix.

A. Stochastic Process Decomposition

Considering an ordered triggering time sequence $\{t_k\}_{k=0}^{N}$, a sigma field $\mathcal{F}_k$ collects all the stochastic events occurring between $[t_0, t_k]$, whereas $\mathcal{F}_0$ includes all the deterministic events. Because the controller can update only when it is

[1]Zeno means that the triggering time Δ can be zero.

triggered, we propose to partition the stochastic events by time intervals. In particular, the collection of stochastic events between two consecutive triggers is defined by $\mathcal{F}_{k,k+1} := \sigma(\mathcal{F}_{k+1} \setminus \mathcal{F}_k)$, where $\sigma(\cdot)$ denotes the minimal sigma field. The following lemma indicates that there is no information loss with the partitioning $\{\mathcal{F}_{k,k+1}\}_{k=0}^{N-1}$.

Lemma 1 *For a given Wiener process W with a stopping time sequence $\{t_k\}_{k=0}^{N-1}$, if $t_j > t_i$ holds almost surely for all $j > i$, the sigma field at time t_N can be decomposed as $\mathcal{F}_N = \sigma(\cup_{i=0}^{N-1} \mathcal{F}_{i,i+1})$, where $\mathcal{F}_{i,i+1}$ is independent from $\mathcal{F}_{j,j+1}$ for all $i \neq j$.*

The proof can be found in Appendix VI-A. Lemma 1 is the key component of the feedback control law analysis, which enables us to decompose the statistics of the state evolution into non-overlapping time intervals. Here, we focus on the decomposition of the covariance matrix $P(t)$ because of its close link with the feedback control law. The projection[2] of the covariance matrix $P(t)$ onto the stochastic events within $(t_k, t_{k+1}]$ is defined by $P_k(t) := \mathbb{E}(P(t) | \mathcal{F}_{k,k+1})$, and Lemma 1 implies that

$$\forall\, t \in [t_0, t_N]\,, \quad P(t) = \sum_{i=0}^{N-1} P_i(t)\,. \tag{8}$$

Based on this decomposition, the open-loop evolution of the conditional dynamics of $P_k(t)$ are given by

$$\frac{dP_k(t)}{dt} = \begin{cases} 0 & t \in [t_0, t_k] \\ AP_k(t) + P_k(t)A^\top + Q & t \in (t_k, t_{k+1}] \\ AP_k(t) + P_k(t)A^\top & t > t_k\,. \end{cases} \tag{9}$$

Note that substituting (9) into (8) yields the dynamics in (7b).

B. Discrete-Time Feedback Covariance Dynamics

To alleviate the perturbation caused by the process noise in (6), a feedback control law is introduced to regulate the state deviation around the expected trajectory $\mu(t) := \mathbb{E}\{x(t)\}$. Based on the standard self-triggered scheme in (2) and (3), the feedback control law is defined by

$$u(t) = \sum_{k=0}^{N-1} (v_k + K(x(t_k) - \mu(t_k)))\zeta(t, t_k, t_{k+1})\,, \tag{10}$$

where v_k is the nominal control input determined by the expected dynamics of $\mu(t)$. Note that this is a **discrete-time** linear control law written in continuous time, and it respects the self-triggered control scheme such that the control input remains constant within time interval $(t_k, t_{k+1}]$ given by

$$u(t) = v_k + K(x(t_k) - \mu(t_k))\,,\ t \in (t_k, t_{k+1}]\,.$$

Meanwhile, due to the fact that the state is accurately measured at time instance t_0, there is no feedback at t_0. Recall (7), the evolution of the state distribution under the control law (10) is characterized by its mean and covariance, where the nominal input v_k governs the mean dynamics by

$$\frac{d\mu(t)}{dt} = A\mu(t) + Bv_k, \ \ t \in (t_k, t_{k+1}], \ \ k \in \mathbb{Z}_0^{N-1}. \tag{11}$$

As the feedback part in (10) reacts to the deviation from the nominal dynamics of $\mu(t)$, the covariance dynamics is therefore governed by the feedback control gain $K \in \mathbb{R}^{n_u \times n_x}$. The following theorem gives the covariance dynamics.

Theorem 1 *Let the feedback control law be defined by* (10). *The dynamics of the covariance is given by*

$$\begin{aligned} \frac{dP(t)}{dt} =& AP(t) + P(t)A^\top \qquad (12a) \\ & + BKP_{k,t}(t) + P_{t,k}(t)(BK)^\top + Q\,, \\ \frac{dP_{t,k}(t)}{dt} =& AP_{t,k}(t) + BKP(t_k) \qquad (12b) \end{aligned}$$

with correlation matrix $\lim\limits_{t \to t_k, t > t_k} P_{t,k}(t) = P(t_k)$ *and*

$$\begin{aligned} & P_{t,k}(t) := \qquad (13) \\ & \quad \mathbb{E}\{\mathbb{E}\{(x(t) - \mathbb{E}\{x(t)\})(x(t_k) - \mathbb{E}\{x(t_k)\})^\top | \mathcal{F}_k\}\} \end{aligned}$$

for all $t \in (t_k, t_{k+1}]$ *and* $k \in \mathbb{Z}_0^{N-1}$.

The proof first derives the dynamics of $P_k(t)$ based on Equation (9) and then, summarizes the dynamics of $P(t)$ by Equation (8). A detailed proof of Theorem 1 can be found in Appendix VI-B. Before proceeding to the predictive control problem, we discuss the physical meaning behind Theorem 1. Equation (13) is the definition of $P_{t,k}(t),\ \forall\ k \in \mathbb{Z}_0^{N-1}$ and the rest of Theorem 1 summarizes its dynamics. In particular, Matrix Q in (12a) models the uncontrolled uncertainty happening during interval $(t_k, t_{k+1}]$ and $P_{t,k}$ models the stabilization effect of the feedback control law, $P_{t,k}$ in (12b) is the correlation between current time instance t and the previous trigger moment t_k, which reflects the fact that the feedback control law within $(t_k, t_{k+1}]$ only uses information up to t_k to generate a constant feedback. The final piece of Theorem 1, $\lim\limits_{t \to t_k, t > t_k} P_{t,k}(t) = P(t_k)$, links the dynamics between $(t_{k-1}, t_k]$ and $(t_k, t_{k+1}]$. In particular, as the feedback control law updates at t_k, $P_{t,k}$ gets reset at t_k and drops the information $P_{t,k-1}(t_k)$ from the last interval.

Remark 1 *We have $P_{t,0}(t_0) = 0$ and $P(t_0) = 0$ in the first time interval $[t_0, t_1]$, . Hence, by Theorem 1, the covariance dynamics in $t \in [t_0, t_1]$ is*

$$\frac{dP(t)}{dt} = AP(t) + P(t)A^\top + Q\,, \quad \frac{dP_{t,0}(t)}{dt} = 0, \tag{14}$$

which is consistent with the fact that there is no effective feedback within the first interval $[t_0, t_1]$.

C. Model Predictive Control Scheme

In this section, we summarize a stochastic MPC controller, that incorporates the dynamics derived in Theorem 1 into the self-triggered MPC scheme. In particular, the controller optimizes the expected performance while ensuring input/state

[2] This is a geometric interpretation of conditional expectation [15]. Given a squared-integrable random variable X in Sigma field $\mathcal{F}$ and a sub-Sigma field $\mathcal{G} \subset \mathcal{F}$, then $\mathbb{E}(X|\mathcal{G}) = \arg\min_{Y \in \mathcal{G}} \mathbb{E}((X - Y)^2)$.

constraint satisfaction up to some user defined probability. For the sake of compactness, the saturated resource dynamics are denoted by $g(r(t_k), \Delta_k) := \min\{\rho\Delta_k + r(t_k) - \eta(\Delta_k), \overline{r}\}$. In general, the nominal inputs $\{v_k\}_{k=0}^{N-1}$, the feedback control K, the triggering time instances $\{t_k\}_{k=1}^{N}$ are determined by following problem.

$$\min_{\substack{K,v,\Delta \\ \mu(\cdot),P(\cdot)}} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left(l(x(\tau), u(\tau))\right) d\tau + \mathbb{E}(M(x(t_N))) \quad \text{(15a)}$$

$$\text{s.t. } \forall t \in (t_k, t_{k+1}],\ \forall k \in \{0, 1, ..., N-1\},$$

$$\begin{cases} \dfrac{d\mu(t)}{dt} = A\mu(t) + Bv_k\ , \\ \dfrac{dP(t)}{dt} = AP(t) + P(t)A^\top + Q \\ \qquad\qquad + BKP_{k,t}(t) + P_{t,k}(t)(BK)^\top, \\ \dfrac{dP_{t,k}(t)}{dt} = AP_{t,k}(t) + BKP(t_k), \end{cases} \quad \text{(15b)}$$

$$\forall t \in [t_0, t_N]\ , \begin{cases} x(t) \sim \mathcal{N}(\mu(t), P(t)), \\ \mathbb{P}(x(t) \in \mathcal{X}) \geq 1 - \epsilon_x, \end{cases} \quad \text{(15c)}$$

$$\forall t \in \{t_k\}_{k=0}^{N-1}, \begin{cases} u(t_k) \sim \mathcal{N}(K\mu(t_k), KP(t_k)K^\top), \\ \mathbb{P}(u(t_k) \in \mathcal{U}) \geq 1 - \epsilon_u, \end{cases} \quad \text{(15d)}$$

$$\forall k \in \{0, 1, ..., N-1\}\ ,$$

$$\begin{cases} r(t_{k+1}) = g(r(t_k), \Delta_k),\ \Delta_k \in [\underline{\Delta}, \overline{\Delta}], \\ P_{t,k}(t_k) = P(t_k),\ r(t_{k+1}) \in [\underline{r}, \overline{r}], \end{cases} \quad \text{(15e)}$$

where ϵ_x and ϵ_u are the threshold that the chance constraints (15c) and (15d) are required to stay above. Notice that due to the feedback with respect to a random event, the actual input value $u(t)$ is uncertain as well. On the practical side, if the feasible $\mathcal{X}$ and $\mathcal{U}$ are polytopic, the chance constraints can be conservatively approximated by an explicit reformulation [16, Chapter 3]. Without loss of generality, we consider $\mathbb{P}(H_{x,i}^\top x(t) \leq h_{x,i})$, $i \in \mathbb{Z}_{i=1}^{n_h}$, where n_h is the number of inequality constraints with respect to x and $H_{x,i} \in \mathbb{R}^{n_x}$ and $h_{x,i} \in \mathbb{R}$. As $x(t)$ follows a Gaussian distribution, any of these constraints can be reformulated as

$$H_{x,i}^\top \mu(t) \leq h_{x,i} - \sqrt{H_{x,i}^\top P(t) H_{x,i}}\, \mathcal{N}^{-1}(1 - \epsilon_x)\ ,$$

where $\mathcal{N}^{-1}(\cdot)$ is the inverse cumulative probability distribution function, i.e., $\mathbb{P}(x \leq \mathcal{N}^{-1}(1-\epsilon_x)) = 1 - \epsilon_x$. Please refer to [16] regarding the computation of the cost (15a).

D. Implementation Discussion

When Problem (15) is solved within a direct optimal control scheme, the integration of the ordinary differential equations can be achieved by numerical integration methods such as the Runge-Kutta method or the collocation method [17]. We recommend to use the collocation method, because the triggering time instances are decision variables. If Runge-Kutta is used, the integration depends on high order terms of $\{\Delta_k\}_{k=0}^{N-1}$, which results in low numerical stability. Instead, a collocation method depends linearly on $\{\Delta_k\}_{k=0}^{N-1}$ and hence is numerically more stable.

IV. Numerical Result

The proposed algorithm is tested on a double integrator with state $x(t) = (x_1(t), x_2(t))$, whose SDE is

$$dx(t) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \right) dt + dW.$$

The controller is designed to track a reference signal oscillating between 1 and -0.4. Only the stage cost is considered with $l(x(t), u(t)) = 10(x_1(t) - \text{ref}(t))^2 + 0.1u(t)^2$ and $\text{ref}(t)$ is the tracking reference. The parameters for the chance constraints in (15c) and (15d) are $\epsilon_x = 0.01$, $\epsilon_u = 0.01$. The recharging rate is 1 with a trigger cost of 0.4. To show the effectiveness of the proposed algorithm, we consider two different cases, $Q = 0.01I$ and $Q = 10^{-4}I$. The former case can be considered as more dangerous than the second case as it has larger process noise. In both cases, we consider a input chance constraint in $[-10, 10]$ with a prediction horizon $N = 10$. The triggering time is bounded within $[0.1, 0.8]$.

In the first case, the covariance of the process noise is set to be $Q = 0.01I$ and the output is bounded by $y \in [-2, 1]$. In this case, the reference overlaps with the output's upper bound, and the standard deviation of the process noise is around 10% the scale of the output.

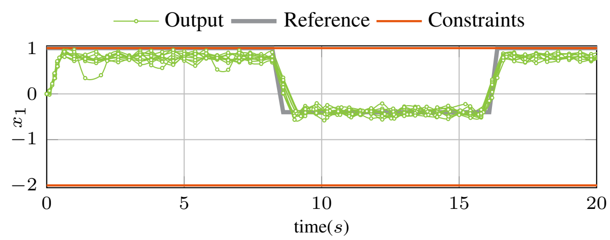

Fig. 1: Output of the stochastic self-triggered MPC ($Q = 10^{-2}I$)

A Monte-Carlo simulation of the output responses is shown in Figure 1, where the controller tries to stay close to the reference, however, as the output is upper bounded by 1, it stays below the upper reference to ensure safety. Regarding another reference signal at -0.4, because it is far away from both constraints, the fluctuations of all the sampled experiments center around the reference -0.4.

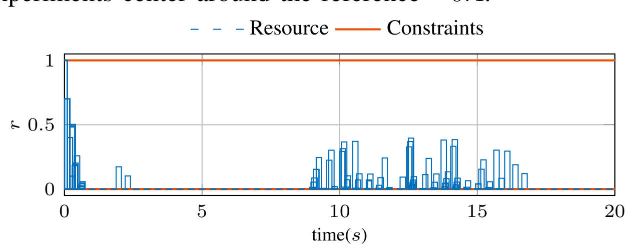

Fig. 2: Resource response of the stochastic self-triggered MPC ($Q = 10^{-2}I$)

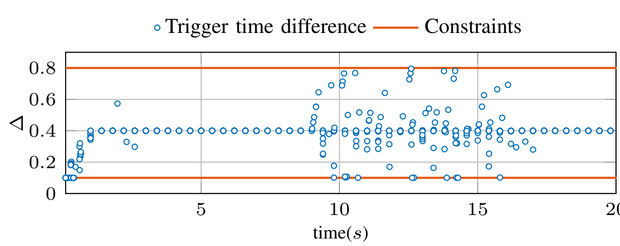

Fig. 3: Triggering time response of the stochastic self-triggered MPC ($Q = 10^{-2}I$)

Figure 2 and Figure 3 show the responses of the resource and the triggering time difference Δ of all Monte-Carlo runs.

When the reference is close to the bound, the controller uses the shortest triggering time confined by the resource dynamics (*i.e.* $0.4s$) and consumes all the resource. When the reference is farther away from the bound, the resource starts to recharge, which is also reflected as the time between triggering times is larger than 0.4 in Figure 3. However, the resource level is lower in comparison with another case because the process noise is large and a more frequent triggers are required to guarantee the controller performance.

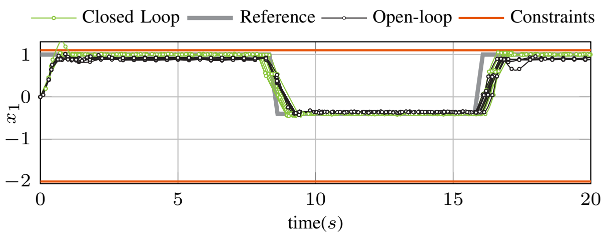

Fig. 4: Output of the stochastic self-triggered MPC ($Q = 10^{-4}I$)

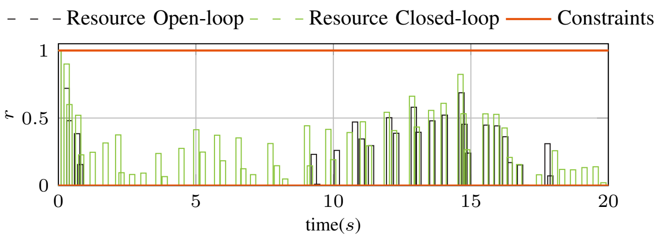

Fig. 5: Resource response of the stochastic self-triggered MPC ($Q = 10^{-4}I$)

In the second case, a smaller process noise $Q = 10^{-4}I$ is considered and the output is bounded by $y \in [-2, 1.1]$. In this case, we compare our proposed scheme against the open-loop scheme to show the necessity of the closed-loop control. Monte-Carlo samples of the output responses are shown in Figure 4, where the output of the proposed scheme tightly tracks the reference. Meanwhile, as a stochastic control scheme, one can see that there is a sampled trajectory that violates the upper bound at around $1s$. The open-loop scheme also performs the task properly, however, we can see that it stays farther away from the reference 1. To make a cleaner and more informative plot, the resource of one sampled trajectory is shown in Figure 5, where we can see that the resource of the closed-loop scheme tends to ramp up when the output is already around the reference and tends to decrease when the reference signal changes. Meanwhile, we can see that the resource consumption in the open-loop scheme is much higher than that of the closed-loop scheme, which means that the open-loop is less desirable as it requires more frequent triggers to maintain the system performance.

V. Conclusion

This paper proposes a novel resource-aware stochastic self-triggered MPC, which generalizes resource-aware self-triggered MPC to an uncertain environment. The discrete-time covariance dynamics of a discrete-time feedback control law is derived to accommodate a continuous-time uncertain disturbance. This feedback law is intentionally designed to be compatible with a self-triggered control scheme. Finally, the proposed scheme is validated through a numerical example.

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VI. Appendix

A. Appendix to Section III-A

Proof of Lemma 1. First recall the definition $\mathcal{F}_{k,k+1} := \sigma(\mathcal{F}_{k+1} \setminus \mathcal{F}_k)$. According to the independence property of a Wiener process, $\mathcal{F}_{i,i+1} \perp \mathcal{F}_{j,j+1}$ holds for all $i \neq j$, therefore we have $\sigma(\cup_{i=0}^{N-1} \mathcal{F}_{i,i+1}) \subset \mathcal{F}_N$. Then, we show the equality holds by contradiction. If $\mathcal{F}_N \neq \sigma(\cup_{i=0}^{N-1} \mathcal{F}_{i,i+1})$, by definition of $\mathcal{F}_{i,i+1}$, there exists $i \in \mathbb{Z}_0^{N-1}$ such that $\mathcal{F}_{t_i} \neq \mathcal{F}_{t_i^+} := \sigma(\cap_{t>t_i} \mathcal{F}_t)$, which violates the continuity of a Wiener process [18]. Hence, the proof concludes.

Remark 2 *This lemma holds for any càdlàg Lévy process [14], which is practical for real-world applications as all the analysis is established from the current time step or, in particular, the sigma fields accumulated up to the current time instant.*

Remark 3 *Equation* (8) *holds due to product topology given by the Lemma 1, which reflects the fact that the conditional* *covariance matrix $P_k(t) := \mathbb{E}(P(t)|\,\mathcal{F}_{k,k+1})$ is a projection onto the $\mathcal{L}^2$ space of the progressively measurable process on $\mathcal{F}_{k,k+1}$.*

B. Appendix to Section III-B

To prove Theorem 1, we recall the Itô's Lemma [18].

Lemma 2 (Itô's Lemma) *For a given drift-diffusion process $dx = adt + bdW$, if function $f(\cdot)$ is twice-differentiable, Iôo's formula holds as*

$$df = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + b\frac{\partial f}{\partial x}dW\,.$$

Proof of Theorem 1. This proof consists of two steps:
1) Derive the decomposed covariance dynamics $\{P_k(\cdot)\}_{k=0}^{N-1}$.
2) Reconstruct the ensemble conariance dynamics $P(t)$.

Decomposed covariance dynamics

Conditioning on the sigma field $\mathcal{F}_{k,k+1}$, the control law is

$$\mathbb{E}(u(t)\mid \mathcal{F}_{k,k+1}) = \tag{16}$$
$$\begin{cases} v_i & t\in(t_i,t_{i+1}],\ i\in\mathbb{Z}_0^k \\ v_i + K\,\mathbb{E}\{x(t)-\mu(t)|\,\mathcal{F}_{k,k+1}\} & t\in(t_i,t_{i+1}],\ i\in\mathbb{Z}_{k+1}^{N-1}. \end{cases}$$

Notice that under a predictive control scheme, $\{v_i\}_{i=0}^{N-1}$ are determined at t_0, hence $\{v_i\}_{i=0}^{N-1}$ are $\mathcal{F}_0$ measurable and furthermore $\mathcal{F}_{k,k+1}$ is measurable. Before t_{k+1}, none of the triggers can generate feedback with respect to the events in $\mathcal{F}_{k,k+1}$ because $\mathcal{F}_{k,k+1}$ happens later than t_k. These facts conclude the conditional control law in (16). Based on the system dynamics (6), the mean dynamics (11) and the conditional control inputs (16), the SDE of the conditional deviation dynamics of $x(t)-\mu(t)$ is

$$\mathbb{E}\{d(x(t)-\mu(t))|\,\mathcal{F}_{k,k+1}\} = \tag{17}$$
$$\begin{cases} 0 & t\in[t_0,t_k] \\ A(\mathbb{E}\{x(t)-\mu(t)|\,\mathcal{F}_{k,k+1}\})dt + dW & t\in(t_k,t_{k+1}] \\ [A(\mathbb{E}\{x(t)-\mu(t)|\,\mathcal{F}_{k,k+1}\}) + B & t\in(t_i,t_{i+1}] \\ \quad\cdot K(\mathbb{E}\{d(x(t_i)-\mu(t_i))|\,\mathcal{F}_{k,k+1}\})]\,dt & i\in\mathbb{Z}_{k+1}^{N-1}. \end{cases}$$

More specifically, this dynamic means that when $t\le t_k$, there is no uncertainty generated by $\mathcal{F}_{k,k+1}$, and $\mathbb{E}(x(t)-\mu(t)|\mathcal{F}_{k,k+1}) = 0$. After that, the stochastic events within interval $(t_k,t_{k+1}]$ do not generate any feedback before t_{k+1} and the deviation evolves in an open-loop form. After t_{k+1}, no new $\mathcal{F}_{k,k+1}$-measurable events can happen anymore and the feedback control law comes into effect.

As $P_k(t) = \mathbb{E}\{\mathbb{E}\{(x(t)-\mu(t))(x(t)-\mu(t))^\top|\,\mathcal{F}_{k,k+1}\}\}$, we can apply Itô's Lemma (Lemma 2) to the deviation dynamics. As a result, we have $\frac{dP_k(t)}{dt} = 0$ for all $t\in[t_0,t_k]$ and thus,

$$\begin{aligned} dP_k(t) &= (AP_k(t)+P_k(t)A^\top+Q)dt \\ &\quad + \underbrace{\mathbb{E}\{\mathbb{E}\{x(t)-\mu(t)|\,\mathcal{F}_{k,k+1}\}dW^\top\}}_{(a)} \\ &\quad + \underbrace{\mathbb{E}[dW\,\mathbb{E}(x(t)-\mu(t)|\,\mathcal{F}_{k,k+1})^\top]}_{(b)} \end{aligned}$$

holds for all $t\in(t_k,t_{k+1}]$, where $(a)=0$ and $(b)=0$ as $\mathbb{E}\{\mathbb{E}\{x(t)-\mu(t)|\,\mathcal{F}_{k,k+1}\}\}=0$. We thus, conclude

$$\frac{dP_k(t)}{dt} = AP_k(t)+P_k(t)A^\top+Q\,, \forall\, t\in(t_k,t_{k+1}]\,,$$

which shares a form similar to (7b).

The last piece is the intervals in which the feedback control law takes effect. Without loss of generality, we consider one interval $(t_i,t_{i+1}]$ with $i\in\mathbb{Z}_{k+1}^{N-1}$, where we have

$$\frac{dP_k(t)}{dt} = AP_k(t)+P_k(t)A^\top+BKP_{i,t,k}(t)+P_{t,i,k}(t)(BK)^\top$$

with $P_{t,i,k}(t) = P_{i,t,k}(t)^\top :=$

$$\mathbb{E}\{\mathbb{E}\{(x(t)-\mu(t))(x(t_i)-\mu(t_i)^\top\}|\,\mathcal{F}_{k,k+1}\}\,.$$

Applying Itô's Lemma again, we have

$$\frac{dP_{t,i,k}}{dt} = AP_{t,i,k} + BKP_{t,i,k}(t_i) = AP_{t,i,k} + BKP_k(t_i)\,,$$

where the second equality holds by definition.

Reconstruct general dynamics $P(t)$

Considering interval $(t_i,t_{i+1}]$, we have following facts:

1) $\forall\,k\ge i+1$, we have $P_k(t)=0$.
2) Because the feedback is not active for the sigma-fields $\mathcal{F}_{k,k+1}$, $\forall\,k\ge i$, we have $P_{t,i,k}=0$, $\forall\,k\ge i$.

Based on the previous derivation, we have

$$\frac{dP(t)}{dt} \overset{(a)}{=} \sum_{k=0}^{N-1}\frac{dP_k(t)}{dt} = \underbrace{\sum_{k=0}^{i-1}\frac{dP_k(t)}{dt}}_{①} + \underbrace{\frac{dP_i(t)}{dt}}_{②} + \underbrace{\sum_{k=i+1}^{N-1}\frac{dP_k(t)}{dt}}_{③}$$
$$= \underbrace{\sum_{k=0}^{i}(AP_i(t)+P_i(t)A^\top)}_{(b)} + Q + \underbrace{\sum_{k=0}^{i-1}BKP_{k,t,i}(t)+P_{t,k,i}(BK)^\top}_{(c)},$$

where (a) holds by Lemma 1. ① corresponds to the components whose feedback is active, ② incorporates the stochastic event happening in the current interval $(t_i,t_{i+1}]$, while ③ are the future stochastic events which have no effect yet. The first aforementioned fact allows the reformulation of (b) as

$$(b) = \sum_{k=0}^{N-1}(AP_k(t)+P_k(t)A^\top)\,.$$

Similarly, the second aforementioned fact rewrites (c) as

$$(c) = \sum_{k=0}^{N-1}BKP_{i,t,k}(t)+P_{t,i,k}(t)(BK)^\top\,.$$

Hence, by equation (8), we conclude

$$\frac{dP(t)}{dt} = AP(t)+P(t)A^\top+BKP_{k,t}(t)+P_{t,k}(t)(BK)^\top+Q.$$

In a similar approach, we have

$$\frac{dP_{t,k}(t)}{dt} = AP_{t,k}(t)+BKP(t_k),$$

which concludes the proof. ■

Remark 4 *The left-open right-closed time intervals used in this paper stress the continuity and guarantee a unique strong solution [18]*