Leverage Point Identification Method for LAV-Based State Estimation

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***Abstract*—In this paper we enunciate and rigorously demonstrate a new lemma that, based on a previously proposed theorem, proves the identifiability of leverage points in state estimation with specific reference to the Least Absolute Value (LAV) estimator. In this context, we also propose an algorithm for the leverage point identification in LAV estimators whose performance is validated by means of extensive numerical simulations and compared against the well-known approach of Projection Statistics (PS). The obtained results confirm that the proposed method outperforms PS and represents a significant enhancement for LAV-based state estimators as it correctly identifies all the leverage points in the measurement set.**

***Index Terms*—Leverage Points, State Estimation, Bad Data, Least Absolute Value, System Identification, Measurement Error**

I. Introduction

Modern power systems are undergoing a significant innovation characterized by an ever-increasing integration of distributed generation and renewable energy sources [1], [2]. In this challenging scenario, state estimation becomes a crucial functionality for several monitoring and control applications in both transmission and distribution power systems [3]–[6].

The need for robust and reliable state estimates has pushed the development of several estimation approaches, e.g. [7]–[10]. Moreover, the recent literature has investigated the influence of several factors, like measurement accuracy [11], [12], meter placement [13], [14], and pseudo-measurements statistical properties [15]. However, a still open issue is represented by the correct identification of leverage points [16]–[18]. Indeed, even a single leverage point may affect the state estimation process with detrimental effects [19], [20].

In terms of state estimators, there exist several methods: Weighted Least Squares (WLS) [16], standard and extended Kalman Filter [21], [22], or Sparse State Recovery [23]. In this context, the Least Absolute Value (LAV) estimator [24] represents an effective solution as its estimate is obtained by minimizing the sum of the absolute errors and guarantees a significant rejection of bad data by proper measurement scaling [25]. However, even the LAV estimator is vulnerable to leverage points which ought to be identified and transformed in order to maintain robustness of the LAV estimator.

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From a state estimation perspective, it is important to recall the difference between outliers and leverage points. An outlier is a measurement that does not belong to the same statistical distribution of the other measurements. Typically, an outlier is given by a gross error in the measurement acquisition process and produces a measurement value that is inconsistent with the (known) statistical properties of the measurement noise (also referred to as bad data). A leverage point, instead, is a measurement that affects the state estimator in such a way that the state estimate residuals will satisfy closely that particular measurement [26]. A leverage point is not necessarily associated with a wrong measurement value, but if it coincides with a bad data, it will significantly affect the estimation results [27], [28].

The recent literature has proposed several methods for outliers' and leverage points' detection [29]. Among the former ones, the Chi-Square (χ^2) and the Largest Normalized Residuals (LNR) tests are the most widely employed [30], [31]. By assuming the residuals are normally distributed, the χ^2 test determines a probability to decide whether a single measurement value belongs to the same distribution as the other measurements. On the other hand, the LNR test proves to be effective in the presence of either single or multiple interacting but non-conforming[1] bad data [32]–[34]. Unfortunately, though, both χ^2 and LNR tests fail in detecting leverage points due to their inherent reduced residuals [18].

On the other hand, there exist several methods specifically designed for the detection of leverage points, but most of them are tailored to the adopted state model [35]–[37]. In general, such methods attempt to quantify the influence of each measurement by means of statistical tests, e.g. projection statistics [39], or residual analysis, e.g. generalized Cook's distance [40]. In this paper, instead, a more general solution, independent of the number or type of measurements, is presented. Given the $\boldsymbol{H}$ matrix, that defines the relationship between measurements and states, the leverage point identification routine can be carried out offline, independently of the estimated states or the measurement accuracy. Indeed, the proposed method relies on three main assumptions:

- the model (1) is either linear or a linearized version of the relationship between measurement and states;
- the $\boldsymbol{H}$ matrix is known exactly *a priori* and has full rank, i.e. all its columns are linearly independent;

[1] In this context, two bad data are said to be conforming if their residuals are consistent. For instance, in the power system state estimation scenario, if a set of bad data on power flows or power injections (nearly) satisfy the Kirchhoff power law, they are said to be conforming bad data [18].

- the measurement uncertainty consists of additive white Gaussian noise, uncorrelated with the state vector and the time information.

Based on the theorem presented in [16] (Theorem 1 at page 142), we enunciate and rigorously demonstrate a new lemma that proves the identifiability of leverage points in LAV-based state estimation. We also propose an algorithm that effectively implements the aforementioned theoretical results. In this sense, it is worth noticing that the objective is not the improvement of the accuracy of the state estimates, but the correct identification of all leverage points.

Furthermore, by means of extensive numerical simulations, we thoroughly characterize the algorithm performance and compare its results against the traditional PS approach.

It should be noticed that the proposed method shall be applied to LAV-based state estimators only. An extension to other estimators, e.g. based on the WLS method, requires a rigorous theoretical foundation, equivalent to the aforementioned theorem, that goes beyond the scope of this paper.

The paper is organised as follows. Section II introduces the theoretical framework of the leverage point identification problem. In Section III, we present the traditional approach for leverage points' identification and discuss its possible drawbacks in a simple application example. Section IV introduces the lemma and describes in detail the proposed method. In Section V we validate its performance by means of numerical simulations inspired by state estimation applications in power systems. Section VI provides some closing remarks.

II. Leverage Point Definition

Let us define the standard measurement model as a system of linear equations[2]:

$$\boldsymbol{H}\boldsymbol{\theta} + \boldsymbol{\varepsilon} = \boldsymbol{z} \tag{1}$$

$$\boldsymbol{\theta} = \{\theta_k | k = 1, \dots N\} \quad \boldsymbol{z}, \boldsymbol{\varepsilon} = \{z_i, \varepsilon_i | i = 1, \dots M\}$$

where $\boldsymbol{z}$ and $\boldsymbol{\theta}$ are the vectors of the M measurements and the N unknown states, respectively. The matrix $\boldsymbol{H}$, of dimension $M \times N$, defines the relationship between measurements and states, whereas $\boldsymbol{\varepsilon}$ models the measurement uncertainty in terms of an additive random variable.

In LAV-based estimation, leverage points are associated with the factor space $\boldsymbol{A}$ that is defined as the vector space spanned by the rows $\boldsymbol{h}_i$ of the $\boldsymbol{H}$ matrix:

$$span(\boldsymbol{A}) = \left\{ \sum_{i=1}^{M} \alpha_i \boldsymbol{h}_i \middle| \alpha_i \in \mathbb{R}, \quad i = 1, \dots M \right\} \tag{2}$$

In this context, let us introduce the formal definition of leverage point [16] (Chapter 6.3 at page 130).

Definition 1. *Given a $\boldsymbol{H}$ matrix row $\boldsymbol{h}_i$ that is an outlier in the factor space $\boldsymbol{A}$, the corresponding measurement z_i has an undue influence on the state estimate and is referred to as leverage point.*

[2]The assumption of linear measurement model does not represent a limitation of the proposed method. In case the relationship between measurement and states is modeled by a non linear function $h(\boldsymbol{\theta})$, it is always possible to consider the linearized model where $\boldsymbol{H} = \partial h(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$.

From a statistical point of view, an outlier is typically defined as an observation that lies an abnormal distance from other values in a random sample from a population [41]. In the present case, a geometrical interpretation might provide a clearer understanding of this concept. In the factor space, the $\boldsymbol{H}$ matrix rows are expected to be located within a given restricted and well-defined area. As a consequence, also the corresponding measurements are characterized by a similar influence on the final state estimate $\hat{\boldsymbol{\theta}}$. However, if a row $\boldsymbol{h}_i$ proves to be an outlier in the factor space, i.e. it is located far from the cluster of the other $\boldsymbol{H}$ matrix rows or its subspace does not intersect with any other subspace in $\boldsymbol{A}$, the corresponding measurement z_i will have a much higher influence on the final state estimate and has to be treated as a leverage point. The present definition is independent of the measurement accuracy, thus leverage points can be detected offline and identified whether they carry bad data or not.

III. Traditional Approach for the Identification of Leverage Points

In this Section, we summarize the traditional approach towards leverage points' identification, namely projection statistics, and discuss its performance, advantages and still open issues with specific reference to an application example.

Projection statistics has been introduced in [17] as an alternative method to compute the distance in the factor space of each point (i.e. a single row of the $\boldsymbol{H}$ matrix) with respect to the cloud of the other points (i.e. all the other rows). In detail, each point is associated with the maximum value of the corresponding standardized projection, also referred to as *projection statistic* (PS). The points and the corresponding PS values are expected to follow a Gaussian and a χ^2 distribution, respectively, but this does not apply to outliers [35]. Therefore, $\chi^2_{d,0.975}$ is used as cutoff value, where d is the number of degrees of freedom and 0.975 is the quantile of the χ^2 distribution. Any point exceeding this threshold is considered as an outlier, and thus identified as a leverage point. Thanks to its straightforward implementation, the PS method has been proven to be computationally efficient and compatible with real-time state estimation [38].

In Algorithm 1, we report the main steps of the PS method. Given the $\boldsymbol{H}$ matrix and the degrees of freedom d, the algorithm identifies the set $\mathcal{S}$ of $\boldsymbol{H}$ rows corresponding to leverage points.

First, we normalize the $\boldsymbol{H}$ matrix by its covariance (line 1). By multiplying the result by its transpose, we get a symmetric matrix $\boldsymbol{\Omega}$ of dimensions $M \times M$ (line 2). Then, we initialize the projection statistic vector $\boldsymbol{PS}$, three support vectors $\boldsymbol{x0}$, $\boldsymbol{x1}$, $\boldsymbol{x2}$, as well as the pointer variable j (line 3-4). For each $\boldsymbol{\Omega}$ column, $\boldsymbol{x2}$ contains the absolute sums of any two non-coincident entries (line 5). Two consecutive lower median operations, $lomed(\cdot)$, allow us to define the vectors $\boldsymbol{x1}$ and $\boldsymbol{x0}$ (line 6-7). Given the i-th row $\boldsymbol{\omega}_i$ of the $\boldsymbol{\Omega}$ matrix, the PS metric is defined as the maximum value of the division between $\boldsymbol{\omega}_i$ and $\boldsymbol{x0}$ (line 8). If such value exceeds the cutoff value, the corresponding $\boldsymbol{H}$ matrix row is included in the the set $\mathcal{S}$ and identified as a leverage point (line 9).

Algorithm 1 Projection Statistics

Input: $\boldsymbol{H}$, d
Output: $\mathcal{S}$
1. covariance weighing: $\boldsymbol{C} = cov(\boldsymbol{H})^{-1/2} \cdot \boldsymbol{H}$
2. symmetric matrix: $\boldsymbol{\Omega} = \boldsymbol{C} \cdot \boldsymbol{C}^T$
3. vector initializations: $\boldsymbol{x0}$ = $\boldsymbol{x1}$ = $\boldsymbol{x2}$ = $\boldsymbol{PS}$ = $[\cdot]$
  for $i0 = 1, \ldots M$
    for $i1 = 1, \ldots M$
      for $i2 = 1, \ldots M$
4. pointer initialization: $j = 0$
        if $i2 \neq i1$
5. $\boldsymbol{x2}$ vector update: $j = j + 1$, $x2_j = |\boldsymbol{\Omega}_{i1,i0} + \boldsymbol{\Omega}_{i2,i0}|$
        end
      end
6. $\boldsymbol{x1}$ vector update: $x1_{i1}$ = $lomed(\boldsymbol{x2})$
    end
7. $\boldsymbol{x0}$ vector update: $x0_{i0}$ = $1.1926 \cdot lomed(\boldsymbol{x1})$
  end
  for $i3 = 1, \ldots M$
8. projection statistic: PS_{i3} = $\max(\boldsymbol{\omega}_{i3}/\boldsymbol{x0})$
    if $PS_{i3} > \chi^2_{d,0.975}$
9. identified leverage point: $\boldsymbol{h}_{i3} \in \mathcal{S}$
    end
  end

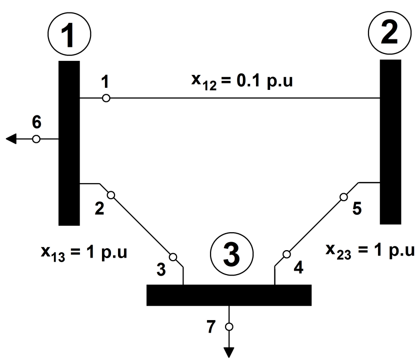

Fig. 1. Schematic of the 3-Bus system as introduced in [17].

The main advantage of the PS method consists of its reduced computational cost. On the other hand, a frequent drawback is the sparsity of the $\boldsymbol{H}$ matrix that might cause the differences between simple outliers and actual leverage points to be almost negligible. In this sense, the definition of the cutoff value typically relies on preliminary Monte-Carlo analysis that not only requires a prior knowledge of the system, but also risks to overfit the identification technique.

In order to evaluate the effectiveness of PS-based identification of leverage points, we consider a reproducible numerical example relying on the 3-Bus system shown in Fig. 1 originally introduced in [17], where we assume the power flow equations being governed by the DC approximation. By means of this approximation, the system states are only the phase angles φ_i (for the i^{th} node), as the node voltages are assumed to have unitary amplitude. The lines are assumed to have a null resistance and a pure longitudinal reactance. The

TABLE I
PROJECTION STATISTICS RESULTS

Meas. #	PS	$\chi^2_{d,0.975}$	d
1	16.77	7.378	2
2	0.839	5.024	1
3	0.839	5.024	1
4	0.839	5.024	1
5	0.839	5.024	1
6	17.609	7.378	2
7	1.677	7.378	2

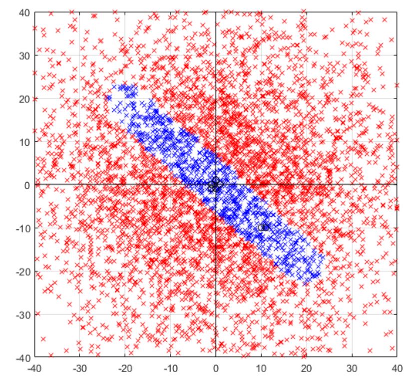

Fig. 2. Successful (blue) and unsuccessful (red) estimations with a gross error of 10 p.u. The x and y axes refer to h_i1 and h_i2, respectively.

measurements consist of 5 power flows and 2 power injections. It is then possible to compute the system $\boldsymbol{H}$ matrix using the DC-flow approximation:

$$\boldsymbol{H} = \begin{bmatrix} 10 & 1 & -1 & 0 & 0 & 11 & -1 \\ -10 & 0 & 0 & -1 & 1 & -10 & -1 \end{bmatrix}^T \tag{3}$$

where the phase φ_3 is taken as the reference.

In Table I, we report the PS results in terms of projection statistic, χ^2 test and the corresponding degrees of freedom.

These results show that PS identifies as leverage points the $\boldsymbol{H}$ matrix rows associated to $P_{flow1,2}$ and P_{inj1}. As a consequence, a gross error in these measurements would deviate the estimates from the true states. However, the LAV-based estimates do not deviate from the true states, irrespective of the gross error level, and this for any combination of true state values. Therefore, these $\boldsymbol{H}$ matrix rows should not be identified as leverage points.

In order to understand what makes a measurement a leverage point, the following numerical test has been conducted. An additional row $\boldsymbol{h}_8$ is added to the $\boldsymbol{H}$ matrix of the initial 3-Bus system, whose parameters follow a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 30$. The true states $\boldsymbol{\theta}^{true}$ follow a normal distribution with $\mu = 0$ and $\sigma^2 = 1$, and the measurements are computed based on these states. A gross error of 10 p.u. is then added to the measurement of this random row. Then, the LAV estimator is ran multiple times for random values of the last row and states.

In Fig. 2, the result of the state estimation is expressed as a function of the coordinates of the $\boldsymbol{H}$ matrix last row within the factor space (the x- and y-axis represent the first and second column value, respectively). In particular, the blue points correspond to configurations where the LAV estimator converges to the true states, whereas the red points correspond to leverage points that yield biased estimates, independently of the added gross errors. There is a clear geometrical boundary between the rows of the $\boldsymbol{H}$ matrix that yield biased and unbiased state estimates. Such boundaries in the factor space depend on the nature of the LAV estimator and, thus, can be mathematically derived from its definition.

IV. Proposed Method

Describing the above observed boundaries requires a mathematical analysis of the LAV objective function. The Least Absolute Value (LAV) estimator determines the N unknown states θ_k by minimizing the sum of the absolute values of the residuals r_i of M measurements z_i:

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \sum_i^M |r_i| \qquad s.t. \quad \boldsymbol{z} = \boldsymbol{H} \cdot \hat{\boldsymbol{\theta}} + \boldsymbol{r} \tag{4}$$

where the residual r_i depends on the measurement z_i and the corresponding row $\boldsymbol{h}_i$ of the $\boldsymbol{H}$ matrix:

$$r_i = z_i - \boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}} = z_i - h_{i1} \cdot \hat{\theta}_1 - \cdots - h_{iN} \cdot \hat{\theta}_N \tag{5}$$

Then, the objective function in (4) can be written as follows:

$$\sum_i^M |r_i| = \sum_i^M |z_i - \boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}}| = \sum_i^M f_i(\hat{\boldsymbol{\theta}}) \tag{6}$$

The continuous functions $f_i(\hat{\boldsymbol{\theta}}) : \mathbb{R} \to \mathbb{R}$ are proven to be convex based on the Jensen's inequality. Given a weight $t \in [0,1] \in \mathbb{R}$ and two state estimates $\hat{\boldsymbol{\theta}}_1$ and $\hat{\boldsymbol{\theta}}_2$:

$$\begin{aligned} f_i\left(t \cdot \hat{\boldsymbol{\theta}}_1 + (1-t) \cdot \hat{\boldsymbol{\theta}}_2\right) &= \\ = |z_i - \boldsymbol{h}_i \cdot t \cdot \hat{\boldsymbol{\theta}}_1 - \boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}}_2 + \boldsymbol{h}_i \cdot t \cdot \hat{\boldsymbol{\theta}}_2| \\ = |t \cdot z_i - t \cdot z_i + z_i - \boldsymbol{h}_i \cdot t \cdot \hat{\boldsymbol{\theta}}_1 - \boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}}_2 + \boldsymbol{h}_i \cdot t \cdot \hat{\boldsymbol{\theta}}_2| \\ \leq |t \cdot z_i - \boldsymbol{h}_i \cdot t \cdot \hat{\boldsymbol{\theta}}_1| + |z_i - t \cdot z_i - \boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}}_2 + \boldsymbol{h}_i \cdot t \cdot \hat{\boldsymbol{\theta}}_2| \\ t \cdot |z_i - \boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}}_1| + (1-t) \cdot |z_i - \boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}}_2| \\ \Rightarrow f_i\left(t \cdot \hat{\boldsymbol{\theta}}_1 + (1-t) \cdot \hat{\boldsymbol{\theta}}_2\right) \leq t \cdot f_i(\hat{\boldsymbol{\theta}}_1) + (1-t) \cdot f_i(\hat{\boldsymbol{\theta}}_2) \end{aligned} \tag{7}$$

and a similar result holds for the sum F of the M f_i functions:

$$\begin{aligned} F\left(\frac{1}{2} \cdot \hat{\boldsymbol{\theta}}_1 + \frac{1}{2} \cdot \hat{\boldsymbol{\theta}}_2\right) &\underset{t=\frac{1}{2}}{=} \sum_i^M f_i\left(\frac{1}{2} \cdot \hat{\boldsymbol{\theta}}_1 + \frac{1}{2} \cdot \hat{\boldsymbol{\theta}}_2\right) \\ &\leq \sum_i^M \left(\frac{1}{2} \cdot f_i(\hat{\boldsymbol{\theta}}_1) + \frac{1}{2} \cdot f_i(\hat{\boldsymbol{\theta}}_1)\right) \\ = \frac{1}{2} \cdot \left[\sum_i^M f_i(\hat{\boldsymbol{\theta}}_1) + \sum_i^M f_i(\hat{\boldsymbol{\theta}}_2)\right] &= \frac{1}{2} \cdot \left(F(\hat{\boldsymbol{\theta}}_1) + F(\hat{\boldsymbol{\theta}}_2)\right) \end{aligned} \tag{8}$$

where the Jensen's inequality was proven in the case of $t = 1/2$. Indeed, since F is a linear combination of continuous functions, and $\mathbb{R}$ is a convex set, to ensure convexity it is sufficient to check the definition only for the midpoint of the variation range of the weight t [42].

Based on (7) and (8), also the objective function of the LAV estimation problem (4) proves to be convex. This fundamental property will be used in Section III.A.

It is worth noticing that F is convex, but not strictly convex. In fact, there might exist particular combinations of f_i functions that cause the slope of the sum function to be zero. In this case, there might be multiple minima and the identification process will end in one of these points depending on the initial conditions of the optimization problem. This specific condition occurs mostly when there are not enough measurements. Accordingly, it is sufficient to add further measurements and thus guarantee the strict convexity. Nevertheless, as shown in the Application Example, the occurrence of such a condition does not limit the applicability of the proposed method. It means only that one or more $\boldsymbol{H}$ matrix rows lie on the boundary between being considered leverage points and the factor space cluster. To identify them as leverage points represents a conservative approach and guarantees a more accurate and reliable state estimation.

It is also interesting to observe that the zero of $f_i(\hat{\boldsymbol{\theta}})$ is a locus with $N-1$ degrees of freedom. Therefore, any state $\hat{\theta}_k$ can be expressed as a linear function of the other states:

$$\begin{aligned} f_i(\hat{\boldsymbol{\theta}}) &= 0 \\ \Leftrightarrow z_i - h_{i1} \cdot \hat{\theta}_1 - \ldots - h_{iN} \cdot \hat{\theta}_N &= 0 \\ \Leftrightarrow \hat{\theta}_k = \frac{z_i - h_{ij} \cdot \hat{\theta}_j}{h_{ik}}, \quad j \in [1, \ldots N] &\wedge j \neq k \end{aligned} \tag{9}$$

As a consequence, given $M \geq N$ measurements, all the intersections between the functions f_i are defined by at least N of these loci, and are zeros of these functions. Nevertheless, since the objective function is the sum of the functions f_i, the minimum of the sum must coincide with one of these zeros, and must be the global minimum [43].

A. Theoretical Foundations

Hypothesis. *Given the following three assumptions:*

- *The state estimate $\hat{\boldsymbol{\theta}}$ is given by a LAV estimator.*
- *The $\boldsymbol{H}$ matrix is known and refers to a linear or linearized system.*
- *The $\boldsymbol{H}$ matrix does not vary during the application of the method.*

Theorem 1. *If the $\boldsymbol{H}$ matrix column rank is equal to $L \leq N$, then there exists a LAV estimate which satisfies at least L of the measurements z_i exactly (with zero residuals) [16][3].*

As a consequence, if the $\boldsymbol{H}$ matrix is full-rank, there are at least N measurements among M that correspond to the true state values and produce a zero residual. It should be noticed that Theorem 1 does not introduce any assumption regarding the accuracy of measurement sets. In other words, such N state values will satisfy exactly the measurements, independently of the presence of gross errors or bad data.

The following lemma can be thus introduced:

[3]The proof of Theorem 1 is given in [16] and constitutes also the basis for the proof of the following Lemma 1.

Lemma 1. *Let $\boldsymbol{H}$ be the matrix that describes the exact relationship between the N states and M measurements of a linear system, and $\boldsymbol{h}_i$ the i^{th} row of the $\boldsymbol{H}$ matrix. If a row $\boldsymbol{h}_j$ of the H matrix is a leverage point, then there exists a linear combination of $N-1$ other rows that satisfies the following inequality:*

$$s = \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \boldsymbol{v}| \leq |\boldsymbol{h}_j \cdot \boldsymbol{v}| = q \tag{10}$$

where $\boldsymbol{v}$ is the unitary vector that completes the basis $\boldsymbol{B}$ formed by all selected $N-1$ rows.

Given N measurements that satisfy Theorem 1, Lemma 1 guarantees that, if a measurement is a leverage point, it can be correctly identified based on the other $N-1$ measurements[4].

Proof of Lemma 1. Let us consider the case where all measurements z_i satisfy perfectly a set of states $\boldsymbol{\theta}^{true}$, except one measurement z_j corresponding to row $\boldsymbol{h}_j$ that satisfies the set of states $\boldsymbol{\theta}^{err}$, as a result of a measurement gross error.

After running the LAV estimator, if the estimated states $\hat{\boldsymbol{\theta}}$ are equal to $\boldsymbol{\theta}^{true}$, the residuals r_i will be equal to zero, except for the point $\boldsymbol{h_j}$. Conversely, if the estimated states $\hat{\boldsymbol{\theta}}$ are equal to $\boldsymbol{\theta}^{err}$, only the residual r_j will be equal to zero, and $\boldsymbol{h}_j$ has to be identified as a leverage point.

If it is possible to reject the gross error and restore the correct state estimation $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}^{true}$, the residuals' sum yields:

$$\begin{aligned} \sum_{i}^{M} |r_i| &= \sum_{i}^{M} |\boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}} - z_i| \\ &= \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}} - z_i| + |\boldsymbol{h}_j \cdot \hat{\boldsymbol{\theta}} - z_j| \\ &= |\boldsymbol{h}_j \cdot \boldsymbol{\theta}^{true} - \boldsymbol{h}_j \cdot \boldsymbol{\theta}^{err}| = |\boldsymbol{h_j} \cdot \boldsymbol{\Delta\theta}| \end{aligned} \tag{11}$$

If the gross error is not properly neutralized, instead, all the $f_i(\hat{\boldsymbol{\theta}})$ zeros will intersect at the same point, except for the one associated to the leverage point, namely $f_j(\hat{\boldsymbol{\theta}})$. In other words, the minimization problem has converged to the global minimum that contains inevitably the leverage point locus.

Indeed, if $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}^{err}$, the residuals' sum is given by:

$$\begin{aligned} \sum_{i}^{M} |r_i| &= \sum_{i}^{M} |\boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}} - z_i| \\ &= \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \hat{\boldsymbol{\theta}} - z_i| + |\boldsymbol{h}_j \cdot \hat{\boldsymbol{\theta}} - z_j| \\ &= \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \boldsymbol{\theta}^{err} - \boldsymbol{h}_i \cdot \boldsymbol{\theta}^{true}| = \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \boldsymbol{\Delta\theta}| \end{aligned} \tag{12}$$

In this regard, it should be noticed that the LAV estimator minimizes the residuals' sum (in accordance with (4)), but does not guarantee the uniqueness of the obtained minimum[5], as shown by the following inequality:

$$\sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \boldsymbol{\Delta\theta}| \leq |\boldsymbol{h}_j \cdot \boldsymbol{\Delta\theta}| \tag{13}$$

According to Theorem 1, at least N residual terms are equal to zero: one is associated with the leverage point, namely r_j; and the others correspond to $N-1$ rows $\boldsymbol{h}'_i$ such that:

$$r'_i = \boldsymbol{h}'_i \cdot \boldsymbol{\Delta\theta} = 0 \Leftrightarrow \boldsymbol{h}'_i \perp \boldsymbol{\Delta\theta} \tag{14}$$

In other words, there exist at least $N-1$ rows $\boldsymbol{h}'_i$ perpendicular to the state error $\boldsymbol{\Delta\theta}$, that is defined in a N dimensional vector space. Hence, $\boldsymbol{\Delta\theta}$ is also collinear with the unitary vector $\boldsymbol{v}$ that completes the basis $\boldsymbol{B}$ formed by all $N-1$ vectors $\boldsymbol{h}'_i$:

$$\boldsymbol{v} = null(\boldsymbol{h}'_1, \dots \boldsymbol{h}'_i, \dots \boldsymbol{h}'_{N-1}), \quad \|\boldsymbol{v}\| = 1 \tag{15}$$

Consequently, the state error $\boldsymbol{\Delta\theta}$ can be also expressed as:

$$\boldsymbol{\Delta\theta} = \boldsymbol{v} \cdot \|\boldsymbol{\Delta\theta}\| \tag{16}$$

By substituting (15) and (16) in the inequality (13), we get:

$$\begin{aligned} \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \boldsymbol{\Delta\theta}| &\leq |\boldsymbol{h}_j \cdot \boldsymbol{\Delta\theta}| \\ \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \boldsymbol{v}| \cdot ||\boldsymbol{\Delta\theta}|| &\leq |\boldsymbol{h}_j \cdot \boldsymbol{v}| \cdot ||\boldsymbol{\Delta\theta}|| \\ \sum_{i \neq j}^{M} |\boldsymbol{h}_i \cdot \boldsymbol{v}| &\leq |\boldsymbol{h}_j \cdot \boldsymbol{v}| \end{aligned} \tag{17}$$

□

It is worth noticing that the proved lemma guarantees full-identifiability of any leverage point, independently of the actual state estimates provided by the LAV estimator. In this sense, its application does not introduce any constraints on the estimator accuracy or on the number of leverage points, and can be run offline (even before the state estimation).

Lemma 1 descends directly from Theorem 1, i.e. it is valid only for LAV-based state estimation. Its extension to other estimation approaches, e.g. WLS, is not rigorously guaranteed. Indeed, WLS relies on the same $\boldsymbol{H}$ matrix and it is reasonable to assume that similar considerations might be derived also in that case. However, there exists no equivalent formulation of Theorem 1 in the context of WLS state estimation. For this reason, a generalized approach towards the identification of leverage points requires first a consistent theoretical foundation that is independent of the adopted estimator. Such an effort goes beyond the scope of this paper, but represents the future step of this research.

B. Method Implementation

Based on Lemma 1, we developed an algorithm for the identification of leverage points, whose main processing stages are summarized in Algorithm 2. In particular, given the system $\boldsymbol{H}$ matrix, the algorithm identifies the set $\mathcal{S}$ of rows $\boldsymbol{h}_j$ that correspond to leverage points and bias the state estimation.

[4] In this sense, the Lemma 1 provides a quantitative criterion for the identification of leverage points, that is compatible with their formal definition in LAV-based state estimation, as stated by Definition 1.

[5] As aforementioned, based on (9) and (10), the objective function in (5) is proved to be convex, but not strictly convex. Nevertheless, the unlikely condition of multiple minimum points is associated to the presence of leverage points on the boundary of the subspace spanned by the other $\boldsymbol{H}$ matrix rows and has to be suitably addressed, as shown by the example in Section IV.

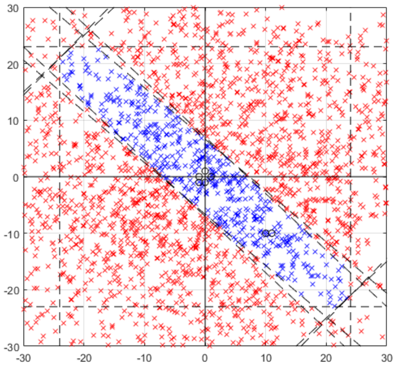

Fig. 3. Factor for successful (in blue) and unsuccessful (in red) estimations with a gross error of 10 p.u. with the boundaries described by Lemma 6. The x and y axes refer to h_i1 and h_i2, respectively.

First, we select a set of $N-1$ rows $\boldsymbol{h}_i$, that constitute a vector basis $\boldsymbol{B}$ (line 1). Then, we complete the basis by including a unitary vector $\boldsymbol{v}$ that is orthogonal to all the selected rows (line 2). Given a generic row index j, we compute the linear combination s (line 3). If the inequality (10) is verified, the corresponding row $\boldsymbol{h}_j$ is identified as a leverage point, otherwise we proceed to the next iteration (line 4). Once all the possible combinations are processed, all the identified leverage points are collected together in the set $\mathcal{S}$ (line 5). This information can be employed to reject the corresponding measurements z_j and get a refined state estimate.

Algorithm 2 Leverage Point Identification

Input: $\boldsymbol{H}$
Output: $\mathcal{S}$
1. subspace with $N-1$ dof: $\boldsymbol{B} = \bigcup \boldsymbol{h}_i$, $i = 1, \dots N-1$
2. normalized perpendicular vector: $\boldsymbol{v} \perp \boldsymbol{B}, ||v|| = 1$
   for $j = 1, \dots M$
3. projection sum: $s = \sum |\boldsymbol{h}_i \cdot \boldsymbol{v}|$, $i \neq j$
   if $s < |\boldsymbol{h}_j \cdot \boldsymbol{v}|$
4. leverage point identification: $\boldsymbol{h}_j \to L$
   end
   end
5. identified leverage points: $\forall \boldsymbol{h}_j \in \mathcal{S}$

The proposed algorithm scans sequentially all $\boldsymbol{H}$ matrix rows. As a consequence, it will identify all leverage points in a single run. A rigorous validation of this property is provided in Section V.

Taking into account the 3-Bus system example presented in Section III, (6) allows us to determine the geometric boundaries between leverage and non-leverage points in the factor space. In this context, Fig. 3 shows that all $\boldsymbol{H}$ matrix rows that spawned a successful estimation despite their measurement being altered by a gross error lie inside the boundaries described by (6), and all others lie outside these boundaries. If we consider the initial $\boldsymbol{H}$ matrix from 3, it is now possible to determine which rows are leverage points with regards to all other measurements. In Table II, we report the two sides s and q of the inequality (10) in Lemma 1 for all the rows and 5 different realizations of $\boldsymbol{v}$ using 2. The results show that no leverage points are detected, which matches the previous observations.

In this instance, the $\boldsymbol{H}$ matrix is a linear approximation of the power flow expression. To show that this approximation does not affect the accuracy of the Lemma 1, the case of PMU measurements is considered, namely the systems states are the voltage phasors V_i at each node. The measurements are the equivalent current phasor flows and injections of the previously measured powers. By means of a simple conversion from polar to rectangular coordinates, the complex phasor values can be expressed in terms of their real ($\Re$) and imaginary ($\Im$) parts. In this case, the link between phasor voltages and currents is linear and does not need to be approximated, therefore the $\boldsymbol{H}$ matrix is not a linearization but the exact link between states and measurements. If we consider the same line impedances as in Fig. 1, the resulting $\boldsymbol{H}$ matrix when computing the load flow equations can be decoupled in two submatrices:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_1 & 0 \\ 0 & \boldsymbol{H}_2 \end{bmatrix} \tag{18}$$

$$\boldsymbol{H}_1^T = \begin{array}{cc} & \begin{array}{cccccccccc} V_1^{\Im} & V_2^{\Im} & V_3^{\Im} & I_{12}^{\Re} & I_{13}^{\Re} & I_{31}^{\Re} & I_{32}^{\Re} & I_{23}^{\Re} & I_1^{\Re} & I_3^{\Re} \end{array} \\ \begin{array}{c} V_3^{\Im} \\ V_2^{\Im} \\ V_1^{\Im} \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 10 & 0 & 0 & 1 & -1 & 10 & 1 \\ 1 & 0 & 0 & -10 & -1 & 1 & 0 & 0 & -11 & 1 \end{bmatrix} \end{array}$$

$$\boldsymbol{H}_2^T = \begin{array}{cc} & \begin{array}{cccccccccc} V_1^{\Re} & V_2^{\Re} & V_3^{\Re} & I_{12}^{\Im} & I_{13}^{\Im} & I_{31}^{\Im} & I_{32}^{\Im} & I_{23}^{\Im} & I_1^{\Im} & I_3^{\Im} \end{array} \\ \begin{array}{c} V_3^{\Re} \\ V_2^{\Re} \\ V_1^{\Re} \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & -10 & 0 & 0 & -1 & 1 & -10 & -1 \\ 1 & 0 & 0 & 10 & 1 & -1 & 0 & 0 & 11 & -1 \end{bmatrix} \end{array}$$

These submatrices are very similar to the DC approximation $\boldsymbol{H}$ matrix, in the sense that all current measurement rows have the same parameters, only the sign changes in one of them which does not affect the leverage point detection method. Furthermore, they are completed by an identity matrix of the voltage measurements. Therefore, the leverage point identification method is applicable to any model type, approximation or real link, as long as the $\boldsymbol{H}$ matrix describes a linear system.

C. Computational Complexity

The computational complexity of the proposed method can be assessed in terms of the number of v vectors to compute. This value depends on the number of states and measurements, i.e. on the dimensions of the $\boldsymbol{H}$ matrix. In this regard, Fig. 5 represents the vector number as a function of N for four different values of M. Given a state number, the computational complexity grows rapidly with the measurement number, due to the increasing number of possible subspaces combinations. In a similar way, given a measurement number, the computational complexity grows exponentially with the state number. In this sense, the only exception is represented by the case in which the number of states is nearly equal or coincident with the measurement number.

TABLE II
PROPOSED METHOD APPLIED TO THE $\boldsymbol{H}$ MATRIX OF THE 3 BUS SYSTEM

	(0.707;0.707)		(0,1)		(1,0)		(0.673;0.74)		(-0.707;0.707)	
	s	q	s	q	s	q	s	q	s	q
$\boldsymbol{h_1}$	4.95	0	13	10	14	10	4.24	0.672	1.768	1.141
$\boldsymbol{h_2}$	4.24	0.707	23	0	23	1	4.24	0.672	3.111	0.707
$\boldsymbol{h_3}$	4.24	0.707	23	0	23	1	4.24	0.672	3.111	0.707
$\boldsymbol{h_4}$	4.24	0.707	22	1	24	0	4.17	0.74	3.111	0.707
$\boldsymbol{h_5}$	4.24	0.707	22	1	24	0	4.17	0.74	3.111	0.707
$\boldsymbol{h_6}$	4.24	0.707	13	10	13	11	1.485	0	4.911	1.697
$\boldsymbol{h_7}$	3.536	1.414	22	1	23	1	3.498	1.413	3.182	0

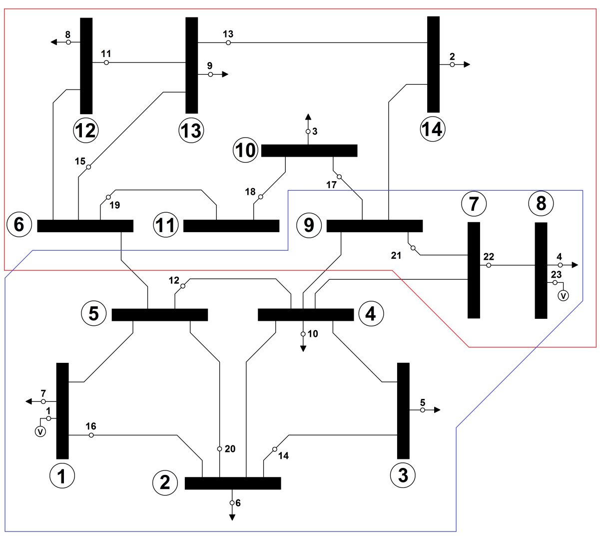

Fig. 4. One-line diagram of the IEEE 14 Bus Test Case [44].

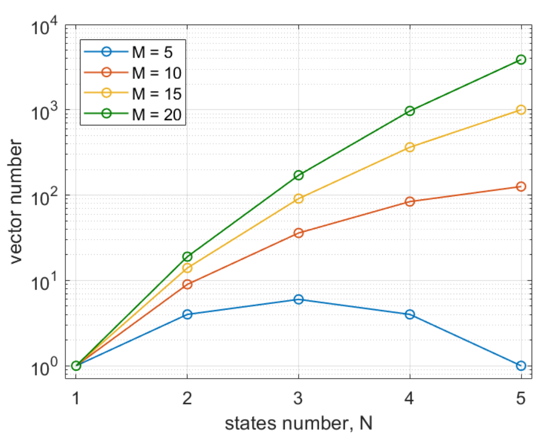

Fig. 5. Computation complexity of the proposed algorithm in terms of number of v vectors to compute, as function of states N and measurements M.

V. APPLICATION EXAMPLE

In order to validate the accuracy and reliability of the proposed method, we carried out extensive numerical simulations on the IEEE 14 Bus Test Case [44]. In this case, the measurement model is derived from the linearized load flow equations and thus fully satisfies the hypothesis in Section III.A. As shown in Fig. 4, this benchmark system represents a portion of the American Electric Power System (in the Midwestern US) as of February, 1962. It consists of 14 buses, 5 generators, and 11 loads, and is equipped with 42 active and reactive power and 2 voltage measurement points, i.e. $M = 44$. The system state vector of dimension $N \times 1$ contains the bus voltage magnitudes and phase angles for each bus, resulting in $N = 28$.

If bus 1 is taken as the angle reference, the number of unitary vectors to compute per measurement is equal to:

$$Card\left(\bigcup \boldsymbol{v}\right) = \frac{(M-1)!}{((M-1)-(N-1))! \cdot (N-1)!} = \frac{(44-1)!}{((44-1)-(27-1))! \cdot (27-1)!} \simeq 420 \cdot 10^9 \tag{19}$$

Such a high number of iterations is hardly tractable and would take a prohibitively long time to be computed. In comparison, PS requires just $170 \cdot 10^{66}$ operations to complete the identification process. However, as shown in Table III and further discussed in the following, the identified leverage points are inconsistent with the ground truth results.

In order to reduce the computational complexity of the proposed method, the system is divided into multiple subsystems. An effective partitioning of the system is beyond the scope of this paper as it has been largely discussed in recent literature [45], [46]. Nevertheless, the basic idea is to consider pairs of separate subsystems without any state in common. In this way, the corresponding state vectors will not intersect in the factor space, and will have no influence on each other. In the present case, Fig. 4 presents a plausible partition where the system is split in two subsystems delimited within red and blue contours, respectively. This enables to reduce the number of unitary vectors to $1.35 \cdot 10^6$, which represents an improvement of nearly five orders of magnitude with respect to the case of unpartitioned system. Furthermore, the computational complexity proves to be nearly hundred times smaller than the one of PS and, as shown in the following, the identified leverage points are coherent with the ground truth.

In addition to the improved identification accuracy and reduced computational complexity, it is worth underlying that both the system partitioning and the leverage point identification are not required to be online processes, as they can run offline without affecting the computational resources allocated to the real-time state estimation.

Table III shows the identification results with respect to the IEEE 14 Bus Test Case for both the proposed method

[6]The operation number of PS approach has been determined analytically based on the state vector and $\boldsymbol{H}$ matrix dimensions.

TABLE III
IEEE 14 BUS TEST CASE - LEVERAGE POINT IDENTIFICATION

Meas.	Added GE	Proposed Blue	Proposed Red	PS
$\|V_1\|$	Unbiased	-	∅	-
P_{inj3}	**Biased**	LP	∅	LP
Q_{inj3}	**Biased**	LP	∅	-
P_{inj2}	**Biased**	LP	∅	-
Q_{inj2}	**Biased**	LP	∅	LP
P_{inj1}	Unbiased	-	∅	-
Q_{inj1}	Unbiased	-	∅	-
P_{inj4}	**Biased**	LP	∅	-
Q_{inj4}	**Biased**	LP	∅	LP
$P_{flow5-4}$	**Biased**	LP	∅	LP
$Q_{flow5-4}$	**Biased**	LP	∅	-
$P_{flow2-3}$	Unbiased	-	∅	-
$Q_{flow2-3}$	Unbiased	-	∅	LP
$P_{flow1-2}$	Unbiased	-	∅	-
$Q_{flow1-2}$	Unbiased	-	∅	-
$P_{flow2-5}$	Unbiased	-	∅	-
$Q_{flow2-5}$	Unbiased	-	∅	LP
P_{inj14}	**Biased**	∅	LP	LP
Q_{inj14}	**Biased**	∅	LP	LP
P_{inj10}	Unbiased	∅	-	-
Q_{inj10}	Unbiased	∅	-	LP
P_{inj8}	**Biased**	LP	LP	LP
Q_{inj8}	Unbiased	-	-	LP
P_{inj12}	**Biased**	∅	LP	LP
Q_{inj12}	**Biased**	∅	LP	LP
P_{inj13}	Unbiased	∅	-	-
Q_{inj13}	Unbiased	∅	-	-
$P_{flow12-13}$	Unbiased	∅	-	-
$Q_{flow12-13}$	Unbiased	∅	-	LP
$P_{flow13-14}$	Unbiased	∅	-	-
$Q_{flow13-14}$	Unbiased	∅	-	-
$P_{flow6-13}$	Unbiased	∅	-	-
$Q_{flow6-13}$	Unbiased	∅	-	LP
$P_{flow10-9}$	Unbiased	∅	-	LP
$Q_{flow10-9}$	Unbiased	∅	-	-
$P_{flow11-10}$	Unbiased	∅	-	LP
$Q_{flow11-10}$	Unbiased	∅	-	LP
$P_{flow6-11}$	**Biased**	∅	LP	LP
$Q_{flow6-11}$	**Biased**	∅	LP	-
$P_{flow9-7}$	**Biased**	LP	LP	-
$Q_{flow9-7}$	**Biased**	LP	LP	LP
$P_{flow7-8}$	Unbiased	LP	LP	LP
$Q_{flow7-8}$	Unbiased	-	-	LP
$\|V_8\|$	Unbiased	-	LP	-

and PS. The first column lists the measured variables. Based on the prior knowledge of the initial state values, we run the state estimation and we define whether each variable behaves as a leverage point or not. To this end, we corrupt each measurement with a gross error and we check whether the final state estimate is biased or not, as reported in the second column. In this context, biased estimates correspond to leverage points and should be accordingly detected by the considered identification techniques.

In the following columns, Table III reports the identification results provided by the proposed method and PS. In the first case, the results are separated in two columns, corresponding to the two subsystems defined in Fig. 4. Each measured variable is denoted by the acronym LP or by a horizontal dash ($-$), to distinguish the identified leverage points. If a measured variable does not belong to the considered partition, and thus is not involved by the identification process, it is denoted by the empty set symbol ($\varnothing$).

As evident from these results, the proposed method successfully identifies all the leverage points, whereas PS provides both false positive and false negative cases. An interesting case is represented by $P_{flow7-8}$ that is identified as a leverage point despite not being affected by a gross error. Even if this might seem a deficiency of the identification technique, in practice there exists a clear motivation for such result. In the factor space, the measurement $P_{flow7-8}$ lies on the boundary between being a leverage point and belonging to the cloud of the other $\boldsymbol{H}$ matrix rows. Its physical counterpart P_{inj8} is corrupted by a gross error, whereas $P_{flow7-8}$ itself is error free. In this limit condition, the convergence of the LAV estimator is susceptible to its initialization and even measurements that are not strictly leverage points might produce a significant deviation of the final state estimate. It is thus reasonable to classify $P_{flow7-8}$ as a leverage point and neutralize its effect in the LAV estimator as a precautionary action.

It is also worth pointing out that not all the partitionings guarantee full identifiability. For instance, let us consider the voltage magnitude measurements $|V_8|$ and $|V_1|$. If they both belong to the same subsystem, they will be correctly classified as unaffected measurements. However, if they are assigned to different subsystems, the method might fail and identify one of the two as a leverage point. This case suggests that the analysis should be repeated with different system partitioning and the results must be compared in case of inconsistent measurement classifications.

Based on all these observations, the proposed method proves to be an effective and reliable solution for the identification of leverage points. In particular, all the leverage points have been correctly identified and the two false positives can be explained by boundary effects or inefficient system partitioning.

VI. CONCLUSIONS

Based on a previously proposed theorem, in this paper, we presented and rigorously demonstrated a new lemma for the full identifiability of leverage points in LAV-based state estimation. Given this theoretical foundation, we developed an effective identification method that does not introduce any constraints on estimator accuracy, measurement gross errors or number of leverage points.

The proposed method has been thoroughly characterized by means of numerical simulations inspired by real-world state estimation applications taken from power system measurements' scenario. In this context, we discussed also the unlikely conditions associated to leverage points lying in a boundary region or inefficient system partitioning. The obtained results proved the proposed method is an effective and reliable solution as it correctly identifies all the leverage points, based only on the knowledge of the $\boldsymbol{H}$ matrix.

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