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Nonperturbative Mellin Amplitudes

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Abstract

Conformal field theory lies at the heart of two central topics in theoretical high energy physics: the study of quantum gravity and the mapping of quantum field theories through the renormalization group. In this thesis we explore a technique to study conformal field theories called *Mellin amplitudes*, which are essentially the Mellin transforms of conformal correlation functions.

The thesis is divided into two parts. In the first part we study the fundamental properties of Mellin amplitudes. We clarify the conditions for the existence of Mellin amplitudes nonperturbatively. We formulate a conjecture for the analytic properties of Mellin amplitudes and partially prove it. Finally, we discuss Polyakov conditions, which are nonperturbative zeros of Mellin amplitudes.

In the second part of the thesis we consider applications of Mellin amplitudes. We apply Mellin amplitudes to: the Wilson-Fisher model in $4-\epsilon$ dimensions, three dimensional conformal field theories with slightly broken higher spin symmetry, two dimensional minimal models and loop diagrams in Anti-de-Sitter space. Our main result is the derivation of nonperturbative sum rules that constrain effective field theories in Anti-de-Sitter space.

Keywords: Conformal field theory, Holography, Quantum Gravity, Quantum field theory

Résumé

La théorie conforme des champs est au cœur de deux sujets centraux de la physique théorique des hautes énergies : la gravité quantique et la représentation des théories quantiques des champs à travers le groupe de renormalisation. Dans cette thèse, nous explorons une technique d'étude des théories conformes des champs : les *amplitudes de Mellin*, qui sont essentiellement les transformées de Mellin des fonctions de corrélation conformes.

La thèse est divisée en deux parties. Dans la première partie, nous étudions les propriétés fondamentales de ces amplitudes. Nous clarifions leurs conditions d'existence non perturbatives. Nous formulons une conjecture sur leurs propriétés analytiques, que nous prouvons partiellement. Finalement, nous discutons des conditions de Polyakov, qui sont des zéros non perturbatifs des amplitudes de Mellin.

La deuxième partie de la thèse porte sur les applications des amplitudes de Mellin. Nous appliquons la technique des amplitudes de Mellin : au modèle de Wilson-Fisher dans $4-\epsilon$ dimensions, aux théories conformes des champs tridimensionelles avec symétrie de spin plus large presque brisée, aux modèles minimaux bidimensionnels et aux diagrammes de loops dans l'espace Anti-de Sitter. Notre résultat principal concerne la déduction de nouvelles règles de sommes non perturbatives qui contraignent certaines théories des champs effectives dans l'espace Anti-de Sitter.

Mots clés : Théorie des Champs Conformes, Holographie, Gravité Quantique, Théorie Quantiques des Champs

Foreword

This thesis is based on the paper

Paper I Joao Penedones, Joao Silva, Alexander Zhiboedov, *Nonperturbative Mellin Amplitudes: Existence, Properties, Applications*, *JHEP* **08** (2020) 031 [1912.11100].

And on the following preprints that are under review in scientific journals:

Paper II Dean Carmi, Joao Penedones, Joao Silva, Alexander Zhiboedov, *Applications of dispersive sum rules: ϵ-expansion and holography* , [2009.13506].

Paper III Joao Silva, *Four point functions in CFT's with slightly broken higher spin symmetry*, [2103.00275]

Chapters 1, 2, 3, 4, 5 and 8 are based on **Paper I**. Chapter 4 also contains some elements of **Paper II**. Chapters 6 and 7 are based on **Paper II**. Chapter 9 is based on **Paper III**.

Contents

Introduction

Conformal field theory

A conformal field theory (CFT) is a quantum field theory with a special symmetry called conformal symmetry, which means invariance through transformations that preserve angles. CFT's are interesting because generically they are strongly interacting, but conformal symmetry makes it possible to make progress on their study. Quantum field theories that are strongly interacting are notoriously hard to study and the usual approach based on Feynman diagrams fails for them. Thanks to their conformal symmetry, conformal field theories obey the so called bootstrap equations. The bootstrap equations are extremely complicated, but they are completely explicit and well defined. Their study by both numerical and analytical methods is one of the most important topics of research in theoretical physics.

CFT's can be applied to: study statistical systems at the critical point; study the fixed point of renormalization group flows of quantum field theories; study quantum gravity theories in Anti-de-Sitter space (AdS), through the holographic principle. The latter is the applicant's main motivation to study conformal field theories.

The research contained in this thesis is part of a vast and ambitious scientific program called the conformal bootstrap. The goal of the conformal bootstrap is to find and solve all CFT's. In $d = 2$ dimensions this program was a great success, due to the groundbreaking 1984 paper of Belavin, Polyakov and Zamolodchikov [1]. In this paper, an infinite set of interacting conformal field theories called "minimal models" were discovered and solved. This led to much progress in the study of two dimensional conformal field theory. Nevertheless, even in $d = 2$ we are still very far from having a complete classification of CFT's.

The conformal bootstrap program in $d > 2$ dimensions was stagnant for many years, due to the fact that conformal symmetry in $d > 2$ dimensions is finite dimensional, whereas in $d = 2$ it is infinite dimensional. This situation was changed by the 2008 paper of Rattazzi, Rychkov, Tonni and Vichi [2]. They were able to derive bounds on the operator spectrum of $\mathbb{Z}_2$ symmetric conformal field theories in 4 dimensions through a novel numerical method they proposed. More important than the obtained result, this paper showed that progress on the conformal bootstrap program in $d > 2$ dimensions is possible. Ever since 2008 the conformal bootstrap is one of the most active topics of research in theoretical physics. Investigations in the conformal bootstrap typically consist in studies on how to obtain constraints on the operator spectrum of CFT's by careful treatment of the bootstrap equations. There are two main research directions, depending on whether the bootstrap equations are studied numerically (numerical bootstrap) or analytically (analytic bootstrap).

Mellin amplitudes

The topic of this thesis is an analytic bootstrap technique called Mellin amplitudes[1]. A four point function of scalar primaries $\langle\phi\phi\phi\phi\rangle$ in a CFT can be written as

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}} + \frac{1}{x_{13}^{2\Delta}x_{24}^{2\Delta}} + \frac{1}{x_{14}^{2\Delta}x_{23}^{2\Delta}} + \frac{f(u,v)}{x_{13}^{2\Delta}x_{24}^{2\Delta}}, \tag{1}$$
$$x_{ij} \equiv |x_i - x_j|\,, u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2},\ v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},$$

$$f(u,v) = \int_{-i\infty}^{+i\infty}\frac{d\gamma_{12}}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{d\gamma_{14}}{2\pi i}\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{14})\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})M(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}, \tag{2}$$

where Δ is a number that characterizes the field ϕ called the conformal dimension. $M(\gamma_{12},\gamma_{14})$ is called the Mellin amplitude. The Mellin amplitude is essentially the Mellin transform of the connected piece of the correlation function. The factor $\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{14})\Gamma^2(\Delta - \gamma_{12} - \gamma_{14})$ basically ensures that the Mellin amplitude is polynomially bounded, as we demonstrate in this thesis. Crossing symmetry is the statement that

$$M(\gamma_{12},\gamma_{14}) = M(\gamma_{14},\gamma_{12}) = M(\gamma_{12},\Delta-\gamma_{12}-\gamma_{14}). \tag{3}$$

In this thesis, we establish the main properties of Mellin amplitudes, namely:

- their existence for nonperturbative CFT's and the subtractions necessary to have a Mellin representation
- Mellin amplitudes are meromorphic functions
- Mellin amplitudes are polynomially bounded at infinity

[1] Mellin amplitudes will hopefully also be useful numerically.

- Polyakov conditions. These are statements about zeros of the Mellin amplitude for nonperturbative CFT's

We prove some of these properties from CFT axioms. When we fail to give proofs, we try to write our assumptions as clearly as possible.

Mellin amplitudes were introduced in physics in 2009 in a paper by Gerhard Mack [3]. He introduced the definition of a Mellin amplitude (2), with the Γ functions in the prefactor. He also pointed out the above properties of Mellin amplitudes (except for Polyakov conditions). Furthermore, he computed the residues of the Mellin amplitude, which are essentially kinematic polynomials, nowadays named "Mack polynomials".

The first use of Mellin amplitudes was made in a 2010 paper by João Penedones [4]. He pointed out that tree level Witten diagrams have a very simple form in Mellin space. Furthermore, he conjectured a formula connecting the Mellin amplitude of diagrams in AdS with diagrams in flat space and checked it in some examples. After this work many papers about the Mellin amplitudes of AdS diagrams appeared, see for example [5, 6]. Recently, Mellin amplitudes were found to be very useful to calculate correlation functions of supergravity theories in AdS, see for example [7].

Another important development was the "Mellin-Polyakov" bootstrap, pioneered in the papers [8, 9]. The main idea proposed there was to write correlation functions as sums of crossing symmetric Witten exchange diagrams. Afterwards, one imposes the absence of double twist operators, and uses such constraints to calculate the CFT data. These are the so called Polyakov conditions, which played a crucial role in subsequent developments and are nowadays believed to hold nonperturbatively.

In practice, the computations in [8,9] were done using Mellin space. The Mellin-Polyakov boostrap had great success perturbatively, namely in the ϵ-expansion, where new predictions were derived using this scheme. However, to our understanding, it is unclear whether such a bootstrap scheme holds nonperturbatively[2].

Nonperturbative sum rules

In this thesis we advocate the use of Mellin amplitudes nonperturbatively. We show that existence and Regge boundedness of Mellin amplitudes, along with Polyakov conditions and crossing symmetry, give rise to nonperturbative sum rules that constrain CFT data nontrivially. Let us explain the logic behind such sum rules, using as an example a functional α that we develop in the thesis in chapter 5. Consider a four point function of scalar primaries $\langle \phi\phi\phi\phi \rangle$ and consider the exchange of operators of twist τ and spin l,

[2]The idea of writing an explicitly crossing symmetric ansatz for the correlation function was recently revisited in [10].

where the twist is defined as the conformal dimension minus the spin. We then have the equation

$$\sum_{\tau,l} C^2_{\tau,l}\alpha(\tau,l) = 0, \tag{4}$$

where $C^2_{\tau,l}$ is an OPE coefficient and we sum over all operators exchanged in a given channel. The functional α produces a number for each primary operator exchanged in the four point function $\langle\phi\phi\phi\phi\rangle$. Equation (4) is a sum rule that constrains the CFT data, namely the dimension of the external operator ϕ, as well as the dimension, spin and OPE coefficients of the exchanged operators.

It turns out to be very interesting to apply (4) to a CFT with large central charge with an AdS dual. In that case the sum rule can be written as

$$\underbrace{\sum_{\tau<\tau_0,l} C^2_{\tau,l}\alpha(\tau,l)}_{\text{IR contribution. Evaluated using AdS EFT.}} + \underbrace{\sum_{\tau>\tau_0,l} C^2_{\tau,l}\alpha(\tau,l)}_{\text{UV contribution.}} = 0. \tag{5}$$

Given an AdS effective field theory (EFT), we can evaluate the first term of (5) using standard techniques. (5) serves to constrain the possible UV (ultraviolet) completion of such an EFT.

Some α have the curious property that the UV contribution always comes with a positive sign. Let us suppose that we evaluate the IR (infrared) contribution for some EFT and it turns out to have a positive sign. In that case, equation (5) cannot possibly be satisfied and so we exclude any UV completion of such an EFT. In other words, nonperturbative facts about CFT's, namely the existence of Mellin amplitudes and Regge boundedness, might lead to the exclusion of UV completions of effective field theories.

We were not able to materialise such a dramatic proposition in any example. In other words, we were not able to exclude any effective field theory. However, we can bound interaction couplings. We view nonperturbative sum rules, along with the idea of using them to constrain the UV completion of effective field theories in AdS, as the main result of this thesis[3].

Besides this we also apply Mellin amplitudes to perturbative CFT's, like the Wilson-Fischer model in $d = 4 - \epsilon$ dimensions and three dimensional CFT's with slightly-broken higher spin symmetry, and derive new conformal data.

[3] Similar approaches were developped at about the same time in [11], [12], in coordinate space. The work [13] unifies different approaches to dispersive CFT sum rules.

Preliminaries

Some basic facts about CFT's and Mellin amplitudes

Conformal field theories are quantum field theories invariant under the conformal algebra[4]

$$\begin{aligned}[D,P_\mu]=P_\mu,\ \ [D,K_\mu]=-K_\mu,\ \ [K_\mu,P_\nu]=2\delta_{\mu\nu}D-2iM_{\mu\nu},\\ [M_{\mu\nu},P_\alpha]=i(\delta_{\mu\alpha}P_\nu-\delta_{\nu\alpha}P_\mu),\quad [M_{\mu\nu},K_\alpha]=i(\delta_{\mu\alpha}K_\nu-\delta_{\nu\alpha}K_\mu),\\ [M_{\alpha\beta},M_{\mu\nu}]=i(\delta_{\alpha\mu}M_{\beta\nu}+\delta_{\beta\nu}M_{\alpha\mu}-\delta_{\beta\mu}M_{\alpha\nu}-\delta_{\alpha\nu}M_{\beta\mu}).\end{aligned} \tag{6}$$

P_μ and $M_{\mu\nu}$ generate translations and rotations, like in the usual Poincaré group. D and K_μ generate dilatations and special conformal transformations. Dilatations are transformations of the type $x_\mu \to \lambda x_\mu$, where $\lambda > 0$. Special conformal transformations are compositions of an inversion, a translation and another inversion, where an inversion is the transformation $x_\mu \to \frac{x_\mu}{x^2}$.

The basic observables of conformal field theories are local operators. Local operators $\mathcal{O}$ can be divided in two types: primaries and descendants. Primary operators are annihilated by the special conformal transformations acting at the origin. Furthermore they are eigenvalues of the dilatation operator. In summary, they obey

$$[K_\mu,\mathcal{O}(0)]=0,\qquad [D,\mathcal{O}(0)]=\Delta O(0),\qquad [M_{\mu\nu},\mathcal{O}_A(0)]=[M_{\mu\nu}]_A^B\mathcal{O}_B(0) \tag{7}$$

The matrix $[M_{\mu\nu}]_A^B$ is related to the irreducible representation of the rotation group under which the conformal primaries transform. Descendants are derivatives of primaries.

The basic observables of conformal field theories are correlation functions of conformal primaries. In this thesis we will be mostly concerned with scalar conformal primaries. Conformal symmetry fixes the form of the one, two and three point functions of conformal primaries.

The one point function always vanishes, except for the identity operator, in which case it

[4] We used Euclidean signature to write the conformal algebra.

is equal to 1. Two point functions are completely fixed. Three point functions are fixed, up to a finite a number of coefficients. For scalar primaries, we have

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \rangle = \frac{\delta_{p_1,p_2}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}, \tag{8}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{9}$$

C_{123} is called an Operator Product Expansion (OPE) coefficient.

Four point functions of conformal primaries are constrained by conformal symmetry, but they are not completely fixed by it. A four point function of scalar conformal primaries takes the form

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = x_{13}^{-2\Delta_1} x_{23}^{\Delta_1-\Delta_2-\Delta_3+\Delta_4} x_{24}^{-\Delta_1-\Delta_2+\Delta_3-\Delta_4} x_{34}^{\Delta_1+\Delta_2-\Delta_3-\Delta_4} f(u,v), \tag{10}$$

$$x_{ij} \equiv |x_i - x_j|, \;\; u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \;\; v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},$$

where f is a function of two variables that is not determined by conformal symmetry.

A basic property of conformal field theories is the Operator Product Expansion (OPE). The idea is that the product of sufficiently close operators can be reproduced by a differential operator acting on only one such operator. For example, for equal scalar primaries

$$\mathcal{O}(x)\mathcal{O}(0) = \sum_{\mathcal{O}_p} \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}}{|x|^{2\Delta-\Delta_p}} \left(1 + \frac{1}{2} x^\mu \partial_\mu + ...\right) \mathcal{O}_p(0) \tag{11}$$

$C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}$ is an OPE coefficient. In the previous expression we sum over all conformal primaries and the ... are fully determined by conformal symmetry. The OPE coefficients are equal to three point function coefficients due to the diagonal property (8) of two point functions. The set of OPE coefficients and conformal dimensions determines all conformal correlation functions in a CFT.

Associativity of the OPE places nontrivial constraints on correlation functions. For example, at the level of a four point function, we can choose to take the OPE between operators at positions 1 and 2 and also between the operators at positions 3 and 4. Alternatively, we can take the OPE between operators at positions 1 and 3 and also between the operators at positions 2 and 4. These two ways of expressing a four point function must lead to the same result. This leads to the equation

$$\sum_p C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_p} C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_p} G^{(12)(34)}_{\Delta_p,l_p}(x_1,...,x_4) = \sum_p C_{\mathcal{O}_1\mathcal{O}_3\mathcal{O}_p} C_{\mathcal{O}_2\mathcal{O}_4\mathcal{O}_p} G^{(13)(24)}_{\Delta_p,l_p}(x_1,...,x_4), \tag{12}$$

which places non trivial constraints on the OPE coefficients and conformal dimensions of conformal primaries. $G^{(12)(34)}_{\Delta_p,l_p}(x_1,...,x_4)$ and $G^{(13)(24)}_{\Delta_p,l_p}(x_1,...,x_4)$ are called conformal blocks, which are kinematical functions fully determined by conformal symmetry.

Mellin amplitudes are defined for n-point correlation functions of scalar conformal primaries as[5]

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \int [\frac{d\gamma_{ij}}{2\pi i}] \prod_{i=1}^{n}\prod_{j=i+1}^{n} |x_i - x_j|^{-2\gamma_{ij}}\Gamma(\gamma_{ij}) \underbrace{M(\{\gamma_{ij}\})}_{\text{Mellin amplitude}} \ , \quad \sum_{j\neq i}\gamma_{ij} = \Delta_i. \tag{13}$$

Nonperturbatively one might be forced to perform subtractions to the lhs of (13) in order to have a Mellin representation. For sufficiently light scalar conformal primaries, the correct definition at the level of the four point function is given by equations (1) and (2)[6].

Mellin amplitudes have poles at locations predicted by the OPE, namely $\gamma_{ij} = \Delta - \frac{\tau}{2} - n$, where $n \in \mathbb{N}_0$. We prove that for some regions of $(\gamma_{12}, \gamma_{14}) \in \mathbb{C}^2$ these are the only singularities of the Mellin amplitude. This leads us to conjecture that such a fact extends to all of $\mathbb{C}^2$, i.e. that all singularities of the Mellin amplitude correspond to physical operators.

As explained in [16, 17], the twist spectrum of conformal field theories in $d \geqslant 3$ has accumulation points in the following sense: given two primary operators of twists τ_1 and τ_2 in the spectrum, there is a sequence of operators labelled by their spin s such that when $s \to \infty$ their twist approaches $\tau_1 + \tau_2$. This implies that the Mellin amplitude has accumulation points of poles. Near such accumulation points, the Mellin amplitude has a branch point like behaviour. Sometimes the Mellin amplitude vanishes at such accumulation points. The precise conditions are equations (4.29), (4.30) and (4.31). These are called the Polyakov conditions and they are very important for the development of dispersive sum rules.

In this thesis we apply Mellin amplitudes to effective field theories in AdS, as well as the Wilson-Fischer model in $4-\epsilon$ dimensions. We also apply Mellin amplitudes to two dimensional minimal models and three dimensional CFT's with slightly broken higher spin symmetry. We review both classes of such theories in what follows.

[5]Mellin amplitudes can also be defined for spinning correlation functions, see [15].

[6]The precise condition is that $\frac{d-2}{2} < \Delta < \frac{3(d-2)}{4}$. If this condition is not met, one needs to perform more subtractions, see equation (2.19).

Minimal Models

The conformal group is larger in $d=2$ than in $d>2$. For $d=2$, the number of conformal generators is infinite. They can be divided in holomorphic conformal generators L_n and antiholomorphic conformal generators $\bar{L}_n$. Each of the two classes of generators obeys a Virasoro algebra:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [L_n, \bar{L}_m] &= 0. \end{aligned} \tag{14}$$

We note that $\{L_{-1}, L_0, L_1, \bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$ form a closed subalgebra, which does not depend on the central charge. This is called the global conformal group. For $d>2$, all the conformal generators belong to the global conformal group.

The Hilbert space of two dimensional conformal field theories is made up of highest weight representations of the Virasoro algebra. These are formed out of a primary state $|h\rangle$ obeying

$$L_n|h\rangle = 0, \text{if } n>0, \quad L_0|h\rangle = h|h\rangle. \tag{15}$$

Descendant states have the schematic form $L_{-n_1}...L_{-n_m}|h\rangle$. The representations involved in minimal models are special, since they involve descendant states that are singular and thus decouple from the Hilbert space. This is similar to the situation concerning conserved operators.

All Virasoro primaries in minimal models are scalars[7] and a given minimal model only contains a finite number of them.

Minimal models are labelled by two coprime integers (p,q). These determine the operator content of the theory. The central charge is given by

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}. \tag{16}$$

We label a Virasoro primary $\phi_{(r,s)}$ by the two integer numbers r and s, where $1 \leqslant r \leqslant q-1$ and $1 \leqslant s \leqslant p-1$. The conformal dimension of a Virasoro primary is given by

$$\Delta_{r,s} = \frac{(pr-qs)^2-(p-q)^2}{2pq}. \tag{17}$$

[7] In this thesis we apply Mellin amplitudes to the so called A-series minimal models, which only contain scalar Virasoro primaries. There are other minimal models that contain spinning Virasoro primaries. Modular invariance is an important consistency condition that minimal models need to obey.

There are $\frac{(p-1)(q-1)}{2}$ Virasoro primaries in the (p, q) minimal model due to an equivalence between the primaries $\phi_{(r,s)}$ and $\phi_{(q-r,p-s)}$.

Many important statistical systems have a description in terms of minimal models, like the 2d Ising model ($\mathcal{M}(4,3)$), or the tricritical Ising model ($\mathcal{M}(5,4)$). Not all minimal models are unitary, in fact they are unitary only if $p = q + 1$. Minimal models have a Coulomb gas formulation, which we explain and use in chapter 8.

CFT's with slightly broken higher spin symmetry

A CFT with higher spin symmetry is one that has a conserved primary $j_{\mu_1 \ldots \mu_s}$ of spin $s > 2$. An important result proven by Maldacena and Zhiboedov in [18] is that in $d > 2$ such theories are free, i.e. they are either the theory of free bosons or of free fermions. This result can be seen as an extension of the Coleman-Mandula theorem to CFT's. The Coleman-Mandula theorem states that the maximum spacetime symmetry of an interacting QFT with an S-matrix is the super-Poincare group. This theorem does not directly apply to CFT's, since they do not have an S-Matrix.

Let us give a flavour of how this result is derived. We will set $d = 3$, so that it is easier to phrase the results. Consider a CFT with a higher spin conserved current j_s. We can define a charge $Q_s = \int dS^\mu j_{\mu\mu_2\ldots\mu_s}$, which can be used to constrain correlation functions:

$$\langle [Q_s, O_1] O_2 O_3 \rangle + \langle O_1 [Q_s, O_2] O_3 \rangle + \langle O_1 O_2 [Q_s, O_3] \rangle = 0. \tag{18}$$

A priori, we do not know the algebra of the symmetries, i.e. we do not know $[Q_s, j_{s'}] =?$. So, we must constrain the correlation functions and the algebra at the same time. At the level of correlation functions, conformal symmetry implies that three point functions of conserved currents have to be linear combinations of known structures:

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = c_b \langle j_{s_1} j_{s_2} j_{s_3} \rangle_{FreeBoson} + c_f \langle j_{s_1} j_{s_2} j_{s_3} \rangle_{FreeFermion} + c_{odd} \langle j_{s_1} j_{s_2} j_{s_3} \rangle_{odd}. \tag{19}$$

Two of the structures can be found in the theory of free bosons and in the theory of free fermions. The odd structure is also fully known explicitly [19].

Furthermore, unitarity constrains $[Q_s, j_{s'}]$. A conserved current in $d = 3$ has twist 1. The twist is defined as the conformal dimension minus the spin. Thus, the charge Q_s has twist 0. We conclude that $[Q_s, j_{s'}]$ has twist 1. Let us introduce coordinates $ds^2 = -dx^+ dx^- + dy^2$ and let us study $[Q_s, j_{s'}]$ setting all indices in the minus directions. We conclude that

$$[Q_s, j_{s'}] = \sum_{s''} c_{s,s',s''} (\partial_-)^{s+s'-s''-1} j_{s''} \tag{20}$$

Notice that derivatives along the minus directions do not change the twist. (19) and (20) form a system of equations, from which it is possible to conclude that $c_{odd} = 0$ and c_b and c_f cannot be both nonzero at the same time. This proves the result at the level of three point functions. The result can be generalised to n-point correlation functions using bilocal operators [18].

CFT's with slightly broken higher spin symmetry are large N 3d CFT's where higher spin symmetry is broken by $1/N$ effects [20]. There are two types of theories of this kind: the quasi-boson and the quasi-fermion. They depend on two parameters, $\tilde{N}$ and $\tilde{\lambda}$. $\tilde{\lambda}$ plays the role of a t'Hooft like coupling. These theories have a Lagrangian description in terms of N massless particles (bosons or fermions) coupled to a Chern-Simons field in the fundamental representation of the gauge group ($O(N)$ or $U(N)$).

The spectrum of these theories is composed of single trace operators of even spin $s = 2, 4, 6, etc$. They are the quasi-conserved currents. In the quasi-fermion theory there is a scalar single trace operator of dimension 2, whereas in the quasi-boson theory there is a scalar single trace operator of dimension 1. The dimensions of all operators receive corrections at order $\frac{1}{\tilde{N}}$. Three point functions of all single trace operators were computed in [20].

Part I

Existence and Basic Properties of Mellin amplitudes

1 Existence of Mellin amplitudes

1.1 Introduction

In this chapter we argue that a four-point correlation function of light scalar operators in a generic CFT admits a Mellin representation in the sense of (2). We start by reviewing basic facts about the multi-dimensional Mellin transform and a natural space of functions associated with it. These are functions analytic and polynomially bounded in a sectorial domain (see below for the precise definition).

We then consider the Mellin transform of the four-point function of identical scalar primary operators in a generic CFT. We show that the Mellin transform has a physical interpretation of an integral over the principal Euclidean sheet, a connected space of conformally non-equivalent configurations for which all x_{ij}^2 are space-like. Equipped with this understanding, we use the OPE to identify the sectorial domain of analyticity of the physical correlator. We then argue (not fully rigorously) that upon appropriate subtractions the physical correlator also satisfies the required polynomial boundedness and therefore admits the Mellin representation (2).

For simplicity, we consider the four-point function $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ of equal scalar primary operators. Conformal symmetry restricts it as follows

$$\begin{aligned} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle &= \frac{F(u,v)}{(x_{13}^2 x_{24}^2)^{\Delta}}, \\ u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}\,, \quad v &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})\,, \end{aligned} \tag{1.1}$$

where Δ is the scaling dimension of $\mathcal{O}$ and the arbitrary function of cross ratios $F(u,v)$ satisfies crossing relations

$$F(u,v) = F(v,u) = v^{-\Delta} F\left(\frac{u}{v}, \frac{1}{v}\right)\,. \tag{1.2}$$

Unitarity implies that $\Delta \geqslant \frac{d-2}{2}$.

1.2 Two-dimensional Mellin Transform

Here we review inversion theorems of the two-dimensional Mellin transform relevant for the four-point function. The generalization to an arbitrary number of dimensions is straightforward and the corresponding theorems and proofs can be found for example in [21].

Consider a two-variable function $g(u,v)$. Its Mellin transform is defined as

$$\mathcal{M}[g](\gamma_{12},\gamma_{14}) \equiv \int_0^\infty \frac{dudv}{uv}\, u^{\gamma_{12}} v^{\gamma_{14}} g(u,v). \tag{1.3}$$

Similarly, consider a two-variable function $\hat{M}(\gamma_{12},\gamma_{14})$. We define the inverse Mellin transform as

$$\mathcal{M}^{-1}[\hat{M}](u,v) = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}\, u^{-\gamma_{12}} v^{-\gamma_{14}} \hat{M}(\gamma_{12},\gamma_{14}). \tag{1.4}$$

There are two natural vector spaces of functions associated with these transforms: M_Θ^U and W_U^Θ. Let us define them.

We say that a function $g(u,v)$ belongs to the vector space of functions M_Θ^U if two conditions are satisfied. First, it is holomorphic in a sectorial domain $(\arg[u],\arg[v]) \in \Theta \subset \mathbb{R}^2$ which we assume to be open and bounded, as well as to include the origin $(0,0) \in \Theta$. Second, in the region of holomorphy $g(u,v)$ obeys

$$|g(u,v)| \leqslant C(c_u,c_v)\frac{1}{|u|^{c_u}}\frac{1}{|v|^{c_v}}, \quad (\arg[u],\arg[v]) \in \Theta, \quad (c_u,c_v) \in U\,, \tag{1.5}$$

for some open region $U \in \mathbb{R}^2$. A typical example will be $a_u < c_u < b_u$ and $a_v < c_v < b_v$.

Similarly, we say that $\hat{M}(\gamma_{12},\gamma_{14})$ belongs to a vector space of functions W_U^Θ if two conditions are satisfied. First, $\hat{M}(\gamma_{12},\gamma_{14})$ is holomorphic in a tube $U + i\mathbb{R}^2$ for some open region $U \in \mathbb{R}^2$. Second, in the holomorphic tube it decays exponentially fast in the imaginary directions

$$|\hat{M}(\gamma_{12},\gamma_{14})| \leqslant K(\mathrm{Re}[\gamma_{12}],\mathrm{Re}[\gamma_{14}])e^{-\sup_\Theta(\arg[u]\mathrm{Im}[\gamma_{12}]+\arg[v]\mathrm{Im}[\gamma_{14}])} \tag{1.6}$$
$$(\gamma_{12},\gamma_{14}) \in U + i\mathbb{R}^2.$$

Having introduced M_Θ^U and W_U^Θ, we then have the following theorems [21,22]:

Theorem I: Given $F(u,v) \in M_\Theta^U$, its Mellin transform $\mathcal{M}[F](\gamma_{12},\gamma_{14})$ exists and is in W_U^Θ. Moreover, $\mathcal{M}^{-1}\mathcal{M}[F](u,v) = F(u,v)$ for any $(\arg[u],\arg[v]) \in \Theta$.

Theorem II: Given $\hat{M}(\gamma_{12},\gamma_{14}) \in W_U^\Theta$, its inverse Mellin transform $\mathcal{M}^{-1}[\hat{M}](u,v)$ exists and is in M_Θ^U. Moreover, $\mathcal{M}\mathcal{M}^{-1}[\hat{M}](\gamma_{12},\gamma_{14}) = \hat{M}(\gamma_{12},\gamma_{14})$ for any $(\gamma_{12},\gamma_{14}) \in U + i\mathbb{R}^2$.

Note that we are not saying that the function has to be in M_Θ^U to admit a Mellin representation[1], but we will see that M_Θ^U and W_U^Θ are indeed the relevant classes for the physical CFT correlators.

Since the discussion might look a little too abstract let us consider a couple of one-dimensional examples that illustrate the application of these theorems. The first example is $g(u) = e^{-u}$. This is an entire function which is polynomially bounded for $-\frac{\pi}{2} < \arg[u] < \frac{\pi}{2}$ for $c_u > 0$. According to the theorem I, the Mellin transform of $g(u)$ exists and decays as $e^{-\frac{\pi}{2}|\mathrm{Im}[\gamma_{12}]|}$, where we used (1.6). Indeed, the Cahen-Mellin integral takes the form

$$e^{-u} = \int_{c_u - i\infty}^{c_u + i\infty} \frac{d\gamma_{12}}{2\pi i} \Gamma(\gamma_{12}) u^{-\gamma_{12}}, \quad c_u > 0,\ |\arg[u]| < \frac{\pi}{2}\,. \tag{1.7}$$

It is a well-known fact that in the complex plane $\Gamma(\gamma_{12}) \sim e^{-\frac{\pi}{2}|\mathrm{Im}[\gamma_{12}]|}$ in agreement with the theorem above.

The second example is $g(u) = \frac{1}{(1+\sqrt{u})^\alpha}$. Note that it is analytic for $|\arg[u]| < 2\pi$ and satisfies (1.5) for $0 < c_u < \frac{\alpha}{2}$. Using theorem I, we conclude that the Mellin transform of $g(u)$ has to decay as $e^{-2\pi|\mathrm{Im}[\gamma_{12}]|}$. The explicit formula takes the form

$$\frac{1}{(1+\sqrt{u})^\alpha} = \int_{c_u - i\infty}^{c_u + i\infty} \frac{d\gamma_{12}}{2\pi i} \frac{2\Gamma(2\gamma_{12})\Gamma(\alpha - 2\gamma_{12})}{\Gamma(\alpha)} u^{-\gamma_{12}}, \quad 0 < c_u < \frac{\alpha}{2},\ |\arg[u]| < 2\pi. \tag{1.8}$$

Again one can explicitly check that the Mellin amplitude decays with an expected exponential rate dictated by the analyticity region of the original function.

We show below that the physical CFT correlators are indeed analytic in a certain sectorial domain. However, they do not satisfy (1.5) and therefore the Mellin transform of the full correlation function does not exist. We claim, however, that it does exist upon doing simple subtractions.[2] These subtractions are designed in such a way that they do not spoil the analytic structure of the correlator and at the same time they make it polynomially bounded in the sense described above. The result of this analysis is (2). We

[1] As an example consider $u^a v^b \theta(0 < u < 1)\theta(0 < v < 1)$ which is obviously not holomorphic but it has a well-defined Mellin amplitude. We will use this loophole to define the intermediate objects that we call the K-functions in section 2.1.

[2] This is true in a generic, interacting CFT. We will discuss some special cases like $2d$ minimal models and higher d free theories separately.

will see that performing the subtractions beyond the disconnected part is equivalent[3] to introducing a deformed Mellin contour $\mathcal{C}$.

1.3 Principal Euclidean Sheet

Let us understand better the physical meaning of the integration region in (1.3). By conformal transformations, we can put the four points on a 2d Lorentzian plane with coordinates (t^M, x^M). Furthermore, by performing conformal transformations, we put point 1 at $(0,0)$, point 3 at $(0,1)$ and point 4 at $(0,\infty)$. The position of point 2 is not fixed and we denote it by (t_2^M, x_2^M).

We map the plane to the Lorentzian cylinder. On the cylinder, we use coordinates (t,ϕ), which are related to coordinates on the plane by

$$t^M = \frac{\sin t}{\cos t + \cos\phi}, \quad x^M = \frac{\sin\phi}{\cos t + \cos\phi}. \tag{1.9}$$

The metric on the cylinder is related to the metric on the plane by $-(dt^M)^2 + (dx^M)^2 = \frac{-dt^2+d\phi^2}{(\cos t+\cos\phi)^2}$, so they are indeed conformal to one another. The cross ratios are given by

$$u = -(t_2^M)^2 + (x_2^M)^2 = \frac{\cos t_2 - \cos\phi_2}{\cos t_2 + \cos\phi_2}, \tag{1.10}$$

$$v = -(t_2^M)^2 + (x_2^M - 1)^2 = 2\frac{\cos t_2 - \sin\phi_2}{\cos t_2 + \cos\phi_2}. \tag{1.11}$$

Points 1, 3 and 4 are mapped respectively to $(0,0)$, $(0,\frac{\pi}{2})$ and $(0,\pi)$ on the cylinder. Note that cross ratios are invariant under $t_2 \to -t_2$.

A simple observation is that the region $0 \leqslant u, v \leqslant \infty$ is the region of Minkowski space (or of Lorentzian cylinder) of spacetime dimension $d \geq 3$, for which the point x_2 is space-like separated from the three other points. Indeed, it is a well-known fact that for $u, v > 0$ and four points in Euclidean space, cross ratios satisfy $(1-\sqrt{u})^2 \leqslant v \leqslant (1+\sqrt{u})^2$, see figure 1.1. The rest of the quadrant $u, v > 0$ is covered by the fully spacelike configurations on the 2d Lorentzian cylinder depicted in figure 1.2. This fact allows us to use the OPE and analyze convergence of the integral in different regions. It will be natural for us to split the integral into three regions, each of which containing the region between the light cones emitted from points x_1, x_3 and x_4 (see figure 1.2) and part of the Euclidean domain. The regions are defined as follows

$$\text{Region I}: \qquad 0 \leqslant u, v \leqslant 1\,,$$

[3] Up to some very special situations that we analyze separately.

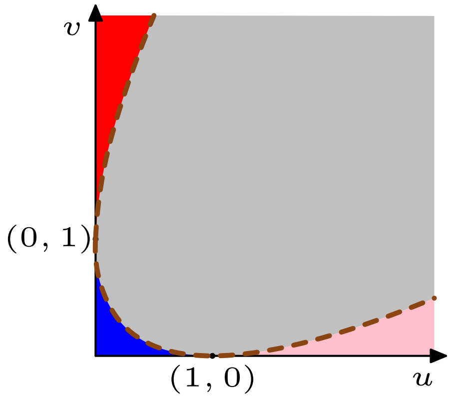

Figure 1.1. We divide the region $u, v > 0$ into 4 regions coloured in blue, pink, red and grey. The regions are separated by the curves $v = (1-\sqrt{u})^2$ and $v = (1+\sqrt{u})^2$ or, equiavelntly, $z = \bar{z}$. In the grey region z and $\bar{z}$ are the complex conjugate of each other. In the colored regions z and $\bar{z}$ are real and independent variables. In the red region we have that $z, \bar{z} \in (-\infty, 0)$. In the blue and pink regions we have that $z, \bar{z} \in (0, 1)$ and $z, \bar{z} \in (1, \infty)$ respectively.

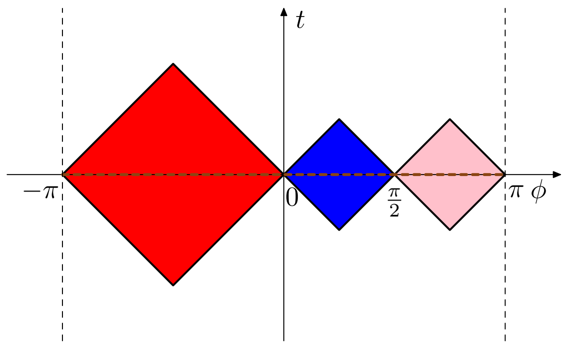

Figure 1.2. Cylinder picture of points 1, 3 and 4. The point at $(t = 0, \phi = \pi)$ should be identified with the point at $(t = 0, \phi = -\pi)$. The colored area signifies regions where point 2 is spacelike separated from three other points and no light-cones has been crossed. Note that the colored region is a double cover in the cross ratio space. Indeed changing $t_2 \to -t_2$ does not change the cross ratios (1.10).

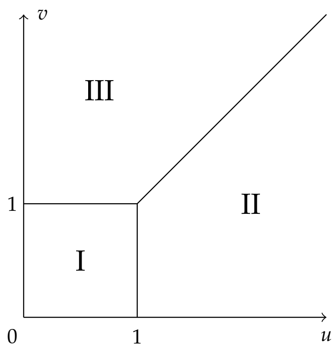

Figure 1.3. Different regions in the (u, v) plane that we will find convenient to consider. Different regions are mapped into each other by crossing.

$$
\begin{aligned}
\text{Region II}: &\quad 1 \leqslant u,\ 0 \leqslant v \leqslant u\ , \\
\text{Region III}: &\quad 1 \leqslant v,\ 0 \leqslant u \leqslant v\ .
\end{aligned} \tag{1.12}
$$

These three regions are mapped to each other via crossing transformations (1.2).

1.4 Analyticity in a Sectorial Domain

Let us study the analytic properties of $F(u,v)$ in the (u,v) plane. Analyticity for real and positive $u,v > 0$ is obvious from the discussion above. Indeed, for such cross ratios the correlation function describes a generic configuration of space-like separated operators, whereas non-analyticities of correlation functions can only occur when "something happens" [23], say a pair of two points become light-like separated. A more rigorous argument relies on the exponential convergence of the OPE in CFTs which makes analyticity manifest [24].

To discuss Mellin amplitudes however we need to understand analytic properties of correlation functions in a sectorial domain, namely we would like to allow for $\arg[u], \arg[v] \neq 0$. In other words, consider the correlation function $F(|u|e^{i\arg[u]}, |v|e^{i\arg[v]})$ with $|u|, |v| \in (0, \infty)$.

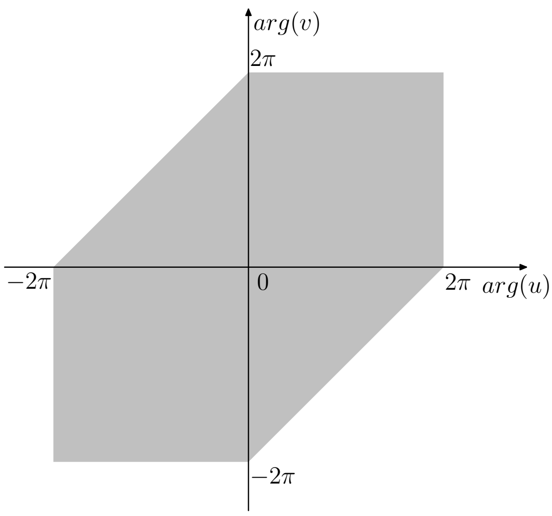

Figure 1.4. Sectorial domain Θ_{CFT} of analyticity of a generic CFT correlation function $F(u,v)$.

Claim: Any physical correlator $F(u,v)$ is analytic in the convex sectorial domain $\Theta_{CFT} = (\arg[u], \arg[v])$ defined as the union of the following 3 regions

$$
\begin{aligned}
\Theta_{CFT} : |\arg[u]| + |\arg[v]| &< 2\pi, \\
0 < \arg[u], \arg[v] &< 2\pi, \\
-2\pi < \arg[u], \arg[v] &< 0,
\end{aligned}
$$

where $\arg[u] = \arg[v] = 0$ corresponds to the principal Euclidean sheet. Θ_{CFT} is depicted on 1.4.

Let us briefly outline the argument for our claim. We present all the details in appendix F. As we analytically continue $u \to |u|e^{i\arg[u]}$, $v \to |v|e^{i\arg[v]}$ we are sure that we do not encounter any singularity provided there is an OPE channel that converges. Indeed, we can then use the Cauchy-Schwarz inequality to bound the continued correlator by its value at $\arg[u] = \arg[v] = 0$ which then ensures analyticity. The rhombus of figure 1.4 is precisely the union of the regions where at least one OPE channel converges, see figure 1.5.

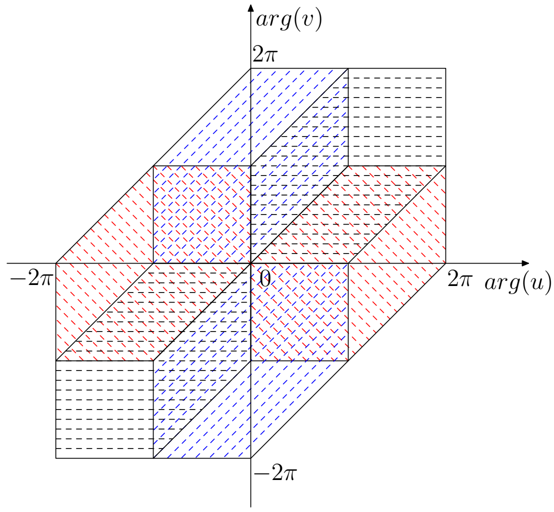

Figure 1.5. We use red, blue and grey to colour the regions where the $\mathcal{O}(x_1) \times \mathcal{O}(x_2)$, $\mathcal{O}(x_1) \times \mathcal{O}(x_4)$ and $\mathcal{O}(x_1) \times \mathcal{O}(x_3)$ OPE channels converge respectively.

We expect that generic correlation functions will have a non-analyticty at the boundary of the Θ_{CFT} for some values of $|u|, |v|$. One example of such a non-analyticity is the bulk point singularity [23,25].[4] In special cases this singularity may be absent. An example is free field theory.

1.5 Dangerous Limits

To analyze the convergence of the integrals above let us understand what are the relevant regions when we integrate in $0 \leqslant u, v \leqslant 1$. There are several different regions involved:

a) **Euclidean OPE** region, which corresponds to $u \to 0$ and $v \to 1$. The correlator in this

[4]Note that in terms of (u, v) the bulk point locus $z = \bar{z}$ can lead to a non-analyticity of the correlator in terms of (u, v) even in $2d$ CFTs. This does not contradict [23] which established analytic properties of the correlator as a function of $(z, \bar{z})$ which are fully analytic at $z = \bar{z}$ and is related to the singular character of the Jacobian when going from $(z, \bar{z})$ to (u, v). See appendix F for more on that.

limit behaves as

$$\lim_{u\to 0,\frac{1-v}{\sqrt{u}}-\text{ fixed}} F(u,v) \sim u^{\frac{\Delta_{min}}{2}-\Delta}, \tag{1.13}$$

where Δ_{min} is the minimal scaling dimension of the operators that appear in the OPE of $\mathcal{O}\times\mathcal{O}$. More generally, we can use the Euclidean OPE to bound the correlator in the vicinity of $u=0$ and $v=1$, not necessarily along the directions (1.13).

b) **Lorentzian OPE** region, which corresponds to $u\to 0$ and v fixed and finite. The correlator in this limit behaves as

$$\lim_{u\to 0} F(u,v) \sim u^{\frac{\tau_{min}}{2}-\Delta}, \tag{1.14}$$

where τ_{min} is the minimal twist[5] of the operator that appears in the OPE of $\mathcal{O}\times\mathcal{O}$.

In unitary theories, the identity is the lowest dimension operator exchanged in $\mathcal{O}\times\mathcal{O}$, thus $\Delta_{min}=\tau_{min}=0$.

c) **Double light-cone limit**, which corresponds to $u,v\to 0$[6]. In general we do not know what is the behavior of the correlator in this limit. For the moment, we just bound the correlator in this limit.

As explained for example in [28], the correlator can be expanded as

$$F(u,v) = u^{-\Delta}\sum_{h,\bar{h}} a_{h,\bar{h}} z^h \bar{z}^{\bar{h}}, \tag{1.15}$$

where $h=\frac{\Delta_{ex}\pm J}{2}$, $\bar{h}=\frac{\Delta_{ex}\mp J}{2}$ and we sum over all exchanged operators in the $\mathcal{O}\times\mathcal{O}$ OPE, both primaries and descendants. Δ_{ex} is the conformal dimension of the exchanged operator and J is its spin. The coefficients $a_{h,\bar{h}}$ are non-negative in a unitary theory.

Suppose now that we are in the Lorentzian region $\sqrt{u}+\sqrt{v}<1$[7]. In this region, z and $\bar{z}$ are independent and real positive variables. Using unitarity and the OPE expansion (1.15) we conclude that

$$F(z_1,\bar{z}_1)(z_1\bar{z}_1)^{\Delta} < F(z_2,\bar{z}_1)(z_2\bar{z}_1)^{\Delta}, \tag{1.16}$$

provided that we pick z_2 such that $z_1<z_2<1$.

The previous inequality can also be stated in the following manner. Pick z_1 and $\bar{z}_1$ as independent real variables between 0 and 1. Define u_1 and v_1 in the usual manner. Then,

[5]We define the twist as $\tau=\Delta-J$, where J is the spin of the operator.
[6]This limit was discussed for example in [26,27].
[7]This region is given by a Lorentzian correlator.

for any z_2 such that $z_1 < z_2 < 1$,

$$F(u_1, v_1) < F\big(u_2, (1-\bar{z}_1)(1-z_2)\big)\frac{u_2^\Delta}{u_1^\Delta}, \tag{1.17}$$

where $u_2 = \bar{z}_1 z_2$. Now let us take the double light-cone limit $z_1 \to 0, \bar{z}_1 \to 1$. Since z_2 is fixed we can use the lightcone limit in the RHS to get

$$F(u,v) < \frac{c_0}{u^\Delta v^\Delta}, \quad 0 < u, v < c, \tag{1.18}$$

where $c < 1$ and c_0 are some constants. This bound is saturated in the 2d Ising model (see (E.7)). So, we cannot improve it without making further assumptions.

1.6 Subtractions and Polynomial Boundedness

By combining the small u, v analysis of the previous section with crossing (1.2) we can find the power-like bounds and asymptotics of the full correlator $F(u,v)$ for any $0 < u, v < \infty$. Plugging them in the definition of the Mellin transform (1.3) it is clear that the integral diverges for any γ_{12} and γ_{14}.

To improve the convergence of the Mellin transform we consider the subtracted correlator

$$\begin{aligned} F_{sub}(u,v) &\equiv F(u,v) - (1 + u^{-\Delta} + v^{-\Delta}) \\ &- \sum_{\tau_{gap} \leqslant \tau \leqslant \tau_{sub}} \sum_{J=0}^{J_{max}} \sum_{m=0}^{\left[\frac{\tau_{sub}-\tau}{2}\right]} C^2_{\tau,J} \left(u^{-\Delta+\frac{\tau}{2}+m} g^{(m)}_{\tau,J}(v) + v^{-\Delta+\frac{\tau}{2}+m} g^{(m)}_{\tau,J}(u) + v^{-\frac{\tau}{2}-m} g^{(m)}_{\tau,J}\left(\frac{u}{v}\right)\right), \end{aligned} \tag{1.19}$$

where $g^{(m)}_{\tau,J}(v)$ is defined as the m-th term in the small u expansion of a single conformal block $g_{\tau,J}(u,v)$

$$F(u,v) = u^{-\Delta} \sum_{\tau, J-\text{even}} C^2_{\tau,J} g_{\tau,J}(u,v), \quad g_{\tau,J}(u,v) = u^{\frac{\tau}{2}} \sum_{m=0}^{+\infty} u^m g^{(m)}_{\tau,J}(v). \tag{1.20}$$

These functions satisfy the following useful identity $v^{-\frac{\tau}{2}-m} g^{(m)}_{\tau,J}(\frac{1}{v}) = (-1)^J g^{(m)}_{\tau,J}(v)$, see e.g. [29]. The subtraction (1.19) makes every OPE limit of the correlator less singular. More precisely, we explicitly subtract the contribution of operators with twists $\tau \leqslant \tau_{sub} < \tau_*$ in every channel. Here τ_* is the smallest twist accumulation point that is exchanged in the OPE of $\mathcal{O} \times \mathcal{O}$. On general grounds [16,17] $\tau_* \leqslant 2\Delta$. The choice of our twist cut-off $\tau_{sub} < \tau_*$ guarantees that in (1.19) $J_{max} < \infty$.

The subtracted correlator $F_{sub}(u,v)$ still satisfies crossing (1.2) and it is still analytic in the sectorial domain Θ_{CFT} due to the analytic properties of the $g^{(m)}_{\tau,J}$ functions. However, on top of that we claim that $F_{sub}(u,v)$ is polynomially bounded as follows

$$|F_{sub}(u,v)| \leqslant C(\gamma_{12},\gamma_{14})\frac{1}{|u|^{\gamma_{12}}}\frac{1}{|v|^{\gamma_{14}}}, \quad (\arg[u],\arg[v]) \in \Theta_{CFT}, \quad (\mathrm{Re}[\gamma_{12}],\mathrm{Re}[\gamma_{14}]) \in U_{CFT}\,, \tag{1.21}$$

where U_{CFT} is given by

$$U_{CFT}: \qquad \Delta - \frac{\tau_{sub}}{2} > \mathrm{Re}[\gamma_{12},\gamma_{13},\gamma_{14}] > \Delta - \frac{\tau'_{sub}}{2}\,, \tag{1.22}$$

where τ'_{sub} is the next twist after τ_{sub} appearing in the OPE $\mathcal{O}\times\mathcal{O}$ and recall that $\gamma_{13} = \Delta - \gamma_{12} - \gamma_{14}$. The condition $\mathrm{Re}[\gamma_{12}] > \Delta - \frac{\tau'_{sub}}{2}$ follows trivially from the light-cone limit $F_{sub}(u,v) \sim u^{-\Delta+\tau'_{sub}/2}$ when $u \to 0$ with fixed $v > 0$. The condition $\gamma_{12} + \gamma_{14} = \Delta - \gamma_{13} > \tau_{sub}/2$ follows from the behaviour of the last subtraction term in (1.19) in double light-cone limit $u \sim v \to 0$. Finally, the remaining conditions in (1.22) are obtained from these two by crossing symmetry. The domain U_{CFT} is depicted in figure 1.6. One may also think of the domain U_{CFT} as the region of analyticity surrounding the crossing symmetric point $\gamma_{12} = \gamma_{13} = \gamma_{14} = \frac{\Delta}{3}$.

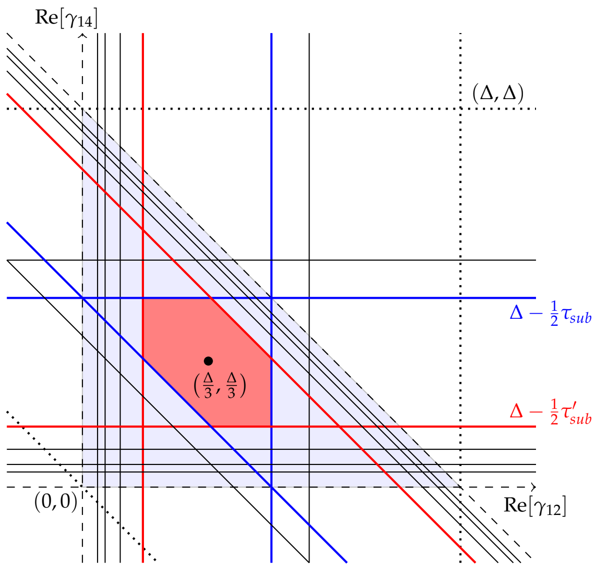

Figure 1.6. Domain U_{CFT} is shown in red with the crossing symmetric point $\gamma_{12} = \gamma_{13} = \gamma_{14} = \frac{\Delta}{3}$ at the center. In this picture we assumed that the first twist accumulation point is $\tau_* = 2\Delta$ (marked with dashed lines). The *Mellin-Mandelstam triangle* $\mathrm{Re}(\gamma_{12},\gamma_{13},\gamma_{14}) > 0$ is depicted in light blue. The blue lines correspond to $\mathrm{Re}(\gamma_{1j}) = \Delta - \tau_{sub}/2$ for $j = 2,3,4$. Similarly, the red lines correspond to $\mathrm{Re}(\gamma_{1j}) = \Delta - \tau'_{sub}/2$. The black lines correspond to $\mathrm{Re}(\gamma_{1j}) = \Delta - \tau/2$ for other twists $\tau < \tau_*$ in the spectrum. The dotted lines corresponds to the identity operator with $\tau = 0$.

We conclude that $F_{sub}(u,v) \in M^{U_{CFT}}_{\Theta_{CFT}}$ and we can write its Mellin representation with the straight contour

$$F_{sub}(u,v) = \int_{U_{CFT}-i\infty}^{U_{CFT}+i\infty} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}\, u^{-\gamma_{12}}v^{-\gamma_{14}}\hat{M}(\gamma_{12},\gamma_{14}). \tag{1.23}$$

In section 2.1 we show that (1.23) implies (2), if we use a deformed contour $\mathcal{C}$ in (2).

A few comments are in order. First of all a necessary condition for the non-emptiness of U_{CFT} (1.22) is $\frac{\Delta}{3} > \Delta - \frac{\tau_*}{2}$ or equivalently

$$\Delta < \frac{3}{4}\tau_* \,. \tag{1.24}$$

In an interacting CFT we expect that $\tau_* = 2\tau_{lightest}$, where $\tau_{lightest}$ is the scaling dimension of the lightest operator present in the theory. It is in this precise sense that our construction concerns only the correlation functions of the light operators in the theory. In section 2.7, we will present a different argument that allows us to generalize this construction beyond (1.24). Unfortunately, we do not establish (1.21) rigorously. Nevertheless we believe that it is a true property of physical correlators (see appendix G). Proving (1.21) rigorously is an important missing step in our analysis.

At this point we also introduce the notion of *Mellin-Mandelstam triangle* (see figure 1.6). Consider a CFT where the lightest primary operator is a scalar with $\Delta_{lightest} < d-2$. In this case, if we consider the Mellin amplitude for the lightest scalar, the first twist accumulation point that appears in the OPE is $\tau_* = 2\Delta_{lightest}$. Therefore, in the region

$$\mathrm{Re}[\gamma_{12},\gamma_{13},\gamma_{14}] > 0\,, \tag{1.25}$$

there are no twist accumulation points. This is the analog of the Mandelstam triangle in the context of flat space scattering amplitudes. The twist accumulation point at $\gamma_{12} = \Delta - \tau_*/2 = 0$ is the analogue of the two-particle branch point (in the 12 channel) of the flat space scattering amplitude.

According to theorem I $\hat{M}(\gamma_{12},\gamma_{14})$ decays exponentially fast in the complex plane

$$|\hat{M}(\gamma_{12},\gamma_{14})| \leqslant K(\mathrm{Re}[\gamma_{12}],\mathrm{Re}[\gamma_{14}])e^{-\sup_{\Theta_{CFT}}(\arg[u]\mathrm{Im}[\gamma_{12}]+\arg[v]\mathrm{Im}[\gamma_{14}])},\quad (\gamma_{12},\gamma_{14}) \in U_{CFT}+i\mathbb{R}^2. \tag{1.26}$$

Note that the Mellin amplitude $M(\gamma_{12},\gamma_{14})$ is defined by

$$\hat{M}(\gamma_{12},\gamma_{14}) = \left[\Gamma(\gamma_{12})\Gamma(\gamma_{14})\Gamma(\Delta-\gamma_{12}-\gamma_{14})\right]^2 M(\gamma_{12},\gamma_{14}). \tag{1.27}$$

One can check that

$$\begin{aligned}\left[\Gamma(\gamma_{12})\Gamma(\gamma_{14})\Gamma(\Delta-\gamma_{12}-\gamma_{14})\right]^2 &\sim e^{-\pi(|\mathrm{Im}[\gamma_{12}]|+|\mathrm{Im}[\gamma_{14}]|+|\mathrm{Im}[\gamma_{12}+\gamma_{14}]|)}\\ &= e^{-\sup_{\Theta_{CFT}}(\arg[u]\mathrm{Im}[\gamma_{12}]+\arg[v]\mathrm{Im}[\gamma_{14}])},\end{aligned} \tag{1.28}$$

and therefore $M(\gamma_{12}, \gamma_{14})$ is polynomially bounded. Moreover, as we will show below its maximal power growth is controlled by the Regge limit.

2 Analytic properties of Mellin amplitudes

2.1 Introduction

In chapter 1 we analyzed the conditions for the existence of the Mellin amplitude of the correlator and put forward the subtractions necessary to define it. Next we would like to understand the analytic properties of the Mellin amplitude $M(\gamma_{12}, \gamma_{14})$. This is the main purpose of this chapter.

To attack this problem we find it useful to develop a different approach to define Mellin amplitudes, namely we split the integral over cross ratios in the definition of the Mellin transform into 3 regions mapped into each other by crossing. The integral of the correlator over a sub-region is manifestly convergent for certain values of γ_{12} and γ_{14}. We then define the full Mellin amplitude by bringing the contributions from different pieces together, see (2.8) below. As a result the subtractions we postulated in the previous chapter arise very naturally. It is also clear that they can be absorbed into the definition of the contour in (1), up to non-essential subtleties that we discuss below.

We then analyse the analytic properties of Mellin amplitudes and argue that the only singularities of Mellin amplitudes are the ones that correspond to physical operators. Concretely, we claim that the only singularities of nonperturbative Mellin amplitudes are simple poles at

$$\gamma_{ij} = \Delta - \frac{\tau}{2} - n, \quad n \in \mathbb{N}_0 \tag{2.1}$$

where τ is the twist of the exchanged conformal primary in the relevant channel. This is discussed in more detail in appendix D.

2.2 Auxillary K-functions

We split the integral in (1.3) into three regions as shown in figure 1.3. We define

$$K(\gamma_{12},\gamma_{14}) \equiv K_{\mathrm{I}}(\gamma_{12},\gamma_{14}) \equiv \int_0^1 \frac{dudv}{uv}\, u^{\gamma_{12}} v^{\gamma_{14}} F(u,v)\,. \tag{2.2}$$

Notice that $K_{\mathrm{I}}(\gamma_{12},\gamma_{14}) = K_{\mathrm{I}}(\gamma_{14},\gamma_{12})$ as the result of crossing (1.2). The lightcone behavior (1.14) and the double lightcone bound (1.18) imply that the integral converges for $\mathrm{Re}\,\gamma_{12} > \Delta$ and $\mathrm{Re}\,\gamma_{14} > \Delta$. Therefore, $K_{\mathrm{I}}(\gamma_{12},\gamma_{14})$ is analytic in this region.

Similarly, we have

$$K_{\mathrm{II}}(\gamma_{12},\gamma_{14}) \equiv \int_1^\infty \frac{du}{u}\int_0^u \frac{dv}{v}\, u^{\gamma_{12}} v^{\gamma_{14}} F(u,v)\,. \tag{2.3}$$

Let us do a change of variables $u \to \frac{1}{u}$ and $v \to \frac{v}{u}$. The measure stays invariant and using crossing $F(\frac{1}{u},\frac{v}{u}) = u^{\Delta}F(u,v)$ we get

$$K_{\mathrm{II}}(\gamma_{12},\gamma_{14}) = K_{\mathrm{I}}(\gamma_{13},\gamma_{14}), \qquad \gamma_{13} = \Delta - \gamma_{12} - \gamma_{14}. \tag{2.4}$$

Lastly, we get

$$K_{\mathrm{III}}(\gamma_{12},\gamma_{14}) \equiv \int_1^\infty \frac{dv}{v}\int_0^v \frac{du}{u}\, u^{\gamma_{12}} v^{\gamma_{14}} F(u,v)\,, \tag{2.5}$$

$$K_{\mathrm{III}}(\gamma_{12},\gamma_{14}) = K_{\mathrm{I}}(\gamma_{12},\gamma_{13}), \qquad \gamma_{13} = \Delta - \gamma_{12} - \gamma_{14}, \tag{2.6}$$

where in the last line we again made use of crossing symmetry.

Importantly, splitting the analytic function $F(u,v)$ into three non-analytic pieces [1] leads to a dramatic effect on the convergence properties of the inverse Mellin transform. Instead of converging in the sectorial domain the inverse Mellin transforms above converge only for $\arg[u] = \arg[v] = 0$. Nevertheless we will find it useful to use the K-functions to describe subtractions and analytic properties of the full Mellin amplitude.

We conclude that in any CFT and for arbitrary correlation functions of scalar primaries, we have

$$\begin{aligned} F(u,v) = &\int_{\mathrm{Re}[\gamma_{12},\gamma_{14}]>\Delta} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} K(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} \\ &+ \int_{\mathrm{Re}[\gamma_{13},\gamma_{14}]>\Delta} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} K(\gamma_{13},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} \\ &+ \int_{\mathrm{Re}[\gamma_{13},\gamma_{12}]>\Delta} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} K(\gamma_{13},\gamma_{12}) u^{-\gamma_{12}} v^{-\gamma_{14}}. \end{aligned} \tag{2.7}$$

[1] $F(u,v) = F(u,v)\Theta(1-u)\Theta(1-v) + F(u,v)\Theta(u-1)\Theta(u-v) + F(u,v)\Theta(v-1)\Theta(v-u)$.

where we denoted $K = K_{\rm I}$ and $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$. The contours run parallel to the imaginary axis of γ_{12} and γ_{14} with real parts obeying the inequalities shown under the integral sign.

Next, we would like to bring the three integrals in (2.7) to the same contour. We discuss this procedure below and it will naturally lead to the subtractions that appeared in (1.19). For the moment we can rather formally define the Mellin amplitude[2] as the sum of the three terms analytically continued to the whole complex plane,

$$\hat{M}(\gamma_{12}, \gamma_{14}) = K(\gamma_{12}, \gamma_{14}) + K(\gamma_{13}, \gamma_{14}) + K(\gamma_{12}, \gamma_{13}), \qquad \gamma_{13} = \Delta - \gamma_{12} - \gamma_{14}. \quad (2.8)$$

2.3 Analytic Structure of the K-function

The function $K(\gamma_{12}, \gamma_{14})$ is analytic for $\mathrm{Re}[\gamma_{12}] > \Delta$ and $\mathrm{Re}[\gamma_{14}] > \Delta$. We would like to analytically continue this function to the rest of $\mathbb{C}^2$. Different regions are shown in figure 2.1.

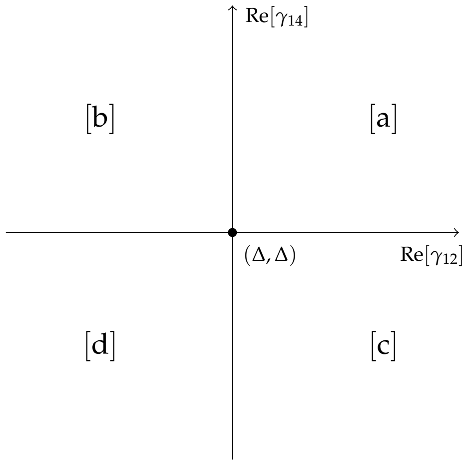

Figure 2.1. We want to analytically continue $K(\gamma_{12}, \gamma_{14})$ into $\mathbb{C}^2$. We break $\mathbb{C}^2$ according to the four regions in the figure. For example, region [a] corresponds to $\mathrm{Re}[\gamma_{12}], \mathrm{Re}[\gamma_{14}] > \Delta$. In region [a], $K(\gamma_{12}, \gamma_{14})$ is completely analytic and is defined by the integral (2.2). In the other regions, it will be defined by analytic continuation.

In appendix B, we explain how such analytic continuation is obtained for single-variable Mellin transforms. We shall see that we can use the same trick at fixed γ_{14} to extend the domain in γ_{12}. The main trick is to *add and subtract* the leading behavior at small u,

$$K(\gamma_{12}, \gamma_{14}) = \int_0^1 \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} \left(\left(F(u,v) - \sum_\tau u^{-\Delta + \frac{\tau}{2}} h_\tau(v) \right) + \sum_\tau u^{-\Delta + \frac{\tau}{2}} h_\tau(v) \right), \quad (2.9)$$

[2] There is an abuse of language here. What we mean is that (2.8) is equal to $\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{14})\Gamma^2(\Delta - \gamma_{12} - \gamma_{14}) M(\gamma_{12}, \gamma_{14})$, where $M(\gamma_{12}, \gamma_{14})$is the Mellin amplitude.

associated to the exchange of operators of twist τ. With this subtraction, we improved the convergence in γ_{12} of the first term, without affecting the convergence in γ_{14}. The second term just gives simples poles at

$$\gamma_{12} = \Delta - \frac{\tau}{2}, \qquad \mathcal{O}_\tau \in \mathcal{O} \times \mathcal{O}. \tag{2.10}$$

These are just the usual OPE poles (2.1) (notice that here we are not distinguishing between primaries and descendants). As we review below, CFT's have accumulation points in the twist spectrum. For this reason one may need infinite subtractions in order to analytically continue in γ_{12} by a finite amount. In appendix D, we explain in detail how to use OPE convergence to overcome this difficulty. The conclusion is that we can analytically continue $K(\gamma_{12}, \gamma_{14})$ into region [b] except for the OPE singularities at $\gamma_{12} = \Delta - \frac{\tau}{2}$. Similarly, we can analytically continue $K(\gamma_{12}, \gamma_{14})$ into region [c] except for the OPE singularities at $\gamma_{14} = \Delta - \frac{\tau}{2}$. However, we cannot use the same strategy to analytically continue $K(\gamma_{12}, \gamma_{14})$ into region [d] because we do not have enough control over the double lightcone limit $u \sim v \to 0$. In appendix (D), we give strong evidence that $K(\gamma_{12}, \gamma_{14})$ can also be extended to region [d], except for the same OPE singularities. Our arguments are based on Bochner's theorem (see appendix C) for analytic functions of two complex variables. See also appendix E.1 for explicit formulas for $K(\gamma_{12}, \gamma_{14})$ in free theories and in the 2d Ising model.

The results of this section strongly support the conjecture that the Mellin amplitude (2.8) has singularities only at the OPE poles (2.1). We call this property *maximal Mellin analyticity* by analogy with the S-matrix. It would be interesting to understand the precise relation between the two.

2.4 Twist spectrum

The analytic structure of Mellin amplitudes (and K-functions) is controlled by the CFT twist spectrum. Let us review its basic properties in $d = 2$ and $d \geq 3$.

It is convenient to organize the CFT spectrum in terms of twist $\tau = \Delta - J$ and spin J.[3] Unitarity implies that $\tau \geqslant d-2$ for $J > 0$, and $\tau \geqslant \frac{d-2}{2}$ for $J = 0$. In other words, the twist spectrum of a unitary CFT is bounded from below. As shown in [30] operators organize themselves in the Regge trajectories $\tau(J)$, at least for $J > 1$. It is interesting to ask what is the structure of the twist spectrum as $J \to \infty$.

In $d \geq 3$ the twist spectrum exhibits additivity at large spin. Given operators with twists τ_1 and τ_2 (we can call them seed operators) there exists an infinite set of Regge

[3]We restrict our discussion to symmetric traceless operators.

trajectories with the property [16,17]

$$\lim_{J\to\infty} \tau^{(n)}_{1,2}(J) = \tau_1 + \tau_2 + 2n. \tag{2.11}$$

The corrections to (2.11) at finite J are given by powers of $\frac{1}{J}$. The existence of such Regge trajectories implies that the twist spectrum of an interacting CFT exhibits an intricate pattern of accumulation points. Indeed, as J approaches infinity (2.11) implies that there is an infinite number of operators with twist $|\tau - \tau_1 - \tau_2| < \epsilon$, where ϵ is an arbitrary positive constant. These are the so-called double-twist Regge trajectories.

A double-twist operator $\tau^{(n)}_{1,2}(J_0)$ can itself serve as a seed that can be paired with another operator to produce triple-twist operators. In this case, (2.11) implies the existence of an infinite set of accumulation points

$$\lim_{J\to\infty} \tau^{(m)}_{\tau^{(n)}_{1,2}(J_0),\tau_3}(J) = \tau^{(n)}_{1,2}(J_0) + \tau_3 + 2m. \tag{2.12}$$

Therefore, already at the second step we obtain an infinite number of accumulation points, for every J_0, in the space of twists (2.12). Moreover, there are also accumulation points of accumulation points at $\tau_1 + \tau_2 + \tau_3 + 2m$.

As we increase twist the number of available seed operators, as well as the number of the multi-twist Regge trajectories, quickly grows and therefore at high enough twist we expect the twist spectrum to become very thinly spaced. It is an open question if it really becomes dense in a finite interval. One can get some intuition from the toy model

$$\tau^{(k+1)}_{\text{toy}}(J_1, \ldots, J_k) = c - \sum_{i=1}^{k} \frac{1}{J_i}, \qquad J_i \in \mathbb{N}, \tag{2.13}$$

for $(k+1)^{\text{th}}$-twist operators. We can think of $\tau^{(k+1)}_{\text{toy}}(J_1, \ldots, J_k)$ as being the twist of $\partial_{J_1}\phi ... \partial_{J_k}\phi\phi$. In this model, the twist spectrum contains many accumulation points but it is not dense in any interval of $\mathbb{R}$.[4] This suggest that the same is true for the twist spectrum of CFTs in $d \geq 3$ dimensions. Clearly this question has important implications for the analytic structure of Mellin amplitudes. Namely, if the twist spectrum becomes continuous then we expect branch cuts in the Mellin amplitudes.

This issue has recently been made much more precise in the context of $d = 2$ generic unitary (irrational, compact) CFTs. Using the Virasoro fusion kernel [31–33], it was argued in [34, 35] that the twist spectrum becomes continuous for $\tau \geq \frac{c-1}{24}$. More precisely, for every $\tau > \frac{c-1}{24}$ there are infinitely many Regge trajectories that end in every interval $\delta\tau$. Moreover, all these Regge trajectories appear in a single four-point correlation function.

[4] The only limit points of the set of twists $\tau^{(k+1)}_{\text{toy}}$ is the set $\tau^{(k)}_{\text{toy}}$ (see math.stackexchange.com).

There are situations where this complicated twist spectrum simplifies dramatically. As we review below one example is planar CFTs[5]. In this case, the (single-trace) twist spectrum of the planar correlators becomes simple. Accumulation points in the twist spectrum are also absent in rational $d = 2$ CFTs. In fact, the twist spectrum in this case is given by a finite set of non-trivial twists plus non-negative integers. It is precisely in these contexts, when the twist spectrum simplifies, that Mellin amplitudes become particularly useful.

2.5 Recovering the Straight Contour

We would like to bring the three integrals in (2.7) to the same straight contour. There are infinitely many ways to do this depending on the choice of the final contour. In the process of doing so we need to know the analytic structure of $K(\gamma_{12}, \gamma_{14})$ on $\mathbb{C}^2$. Above we argued that $K(\gamma_{12}, \gamma_{14})$ is an analytic function, with simple poles at

$$\gamma_{12} = \Delta - \frac{\tau_i}{2} - n_1, \quad \gamma_{14} = \Delta - \frac{\tau_j}{2} - n_2, \tag{2.14}$$

for each primary operator of twist τ_i or τ_j being exchanged in $\mathcal{O} \times \mathcal{O}$ and for each nonnegative integer n_1 and n_2.

At this point, we need to make a choice about the final contour for the Mellin representation. We consider two options: a straight contour and a deformed contour. We examine the second possibility in section 2.7.

Let us reunite the three integrals in (2.7) into a single integral with a straight contour. We will pick the straight contour at $\text{Re}[\gamma_{12}] = \text{Re}[\gamma_{14}] = \frac{\Delta}{3}$. This choice is very natural, since it is completely symmetric in γ_{12}, γ_{13} and γ_{14}.

We need to deform the contours in the integrals (2.7). Let us see how this comes about. Consider the first integral. When we deform its contour we pick up poles. This will tell us the subtractions we need to make to the four point function, so as to have a Mellin representation with a straight contour. See figure 2.2.

From figure 2.2, we conclude that

$$\begin{aligned} &\int_{\text{Re}(\gamma_{12})>\Delta} \frac{d\gamma_{12}}{2\pi i} \int_{\text{Re}(\gamma_{14})>\Delta} \frac{d\gamma_{14}}{2\pi i} K(\gamma_{12}, \gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} \\ = \; &\int_{\text{Re}(\gamma_{12})=\frac{\Delta}{3}} \frac{d\gamma_{12}}{2\pi i} \int_{\text{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{14}}{2\pi i} K(\gamma_{12}, \gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} \end{aligned} \tag{2.15}$$

[5] In planar gauge CFTs the twist of single-trace operators grows like $\log J$ therefore there are no accumulation points (for finite 't Hooft coupling). Double-trace operators do exhibit accumulation points, but they do not give rise to poles of the Mellin amplitude $M(\gamma_{ij})$ at the planar level.

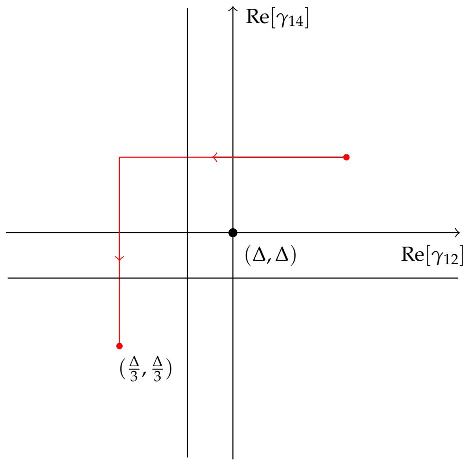

Figure 2.2. Picture of the contour manipulation that corresponds to formula (2.15). The contours run parallel to the imaginary axis of γ_{12} and γ_{14} and therefore correspond to a point in this figure. In red we display the change in the contour. We pick up poles along the way, which we denote by black lines. At the end, we arrive at the point $(\mathrm{Re}(\gamma_{12}), \mathrm{Re}(\gamma_{14})) = (\frac{\Delta}{3}, \frac{\Delta}{3})$.

$$+ \sum_{\tau<\frac{4\Delta}{3}} \sum_J \sum_{n=0}^{[-\frac{\tau}{2}+\frac{2\Delta}{3}]} u^{-\Delta+\frac{\tau}{2}+n} \int_{\mathrm{Re}(\gamma_{14})>\Delta} \frac{d\gamma_{14}}{2\pi i} \hat{K}_{\tau,J}(\gamma_{12} = \Delta - \frac{\tau}{2} - n, \gamma_{14}) v^{-\gamma_{14}}$$

$$+ \sum_{\tau<\frac{4\Delta}{3}} \sum_J \sum_{m=0}^{[-\frac{\tau}{2}+\frac{2\Delta}{3}]} v^{-\Delta+\frac{\tau}{2}+m} \int_{\mathrm{Re}(\gamma_{12})=\frac{\Delta}{3}} \frac{d\gamma_{12}}{2\pi i} \hat{K}_{\tau,J}(\gamma_{12}, \gamma_{14} = \Delta - \frac{\tau}{2} - m) u^{-\gamma_{12}},$$

where $\hat{K}_{\tau,J}(\gamma_{12} = \Delta - \frac{\tau}{2} - n, \gamma_{14})$ denotes the contribution from the operator $\mathcal{O}_{\tau,J}$ to the residue of K at $\gamma_{12} = \Delta - \frac{\tau}{2} - n$. The symbol $[y]$ denotes the biggest integer that is smaller than y.

Let us compute the residues of K. The four point function can be expanded as $F(u, v) = \sum_{\tau,J} C^2_{\tau,J} \sum_{m=0}^{+\infty} u^{-\Delta+\frac{\tau}{2}+m} g^{(m)}_{\tau,J}(v)$, with the sum running over the primary operators $\mathcal{O}_{\tau,J}$ exchanged. Here $g^{(m)}_{\tau,J}(v)$ is what multiplies $u^{\frac{\tau}{2}+m}$ in the small u expansion of the conformal block. For example, $g^{(0)}_{\tau,J}(v)$ is the collinear block

$$g^{(0)}_{\tau,J}(v) = (1-v)^J \, {}_2F_1(\frac{\tau}{2} + J, \frac{\tau}{2} + J, \tau + 2J, 1 - v)\,. \tag{2.16}$$

Therefore, we can write

$$\hat{K}_{\tau,J}(\gamma_{12} = \Delta - \frac{\tau}{2} - m, \gamma_{14}) = C^2_{\tau,J} \int_0^1 \frac{dv}{v} v^{\gamma_{14}} g^{(m)}_{\tau,J}(v). \tag{2.17}$$

This integral converges when $\mathrm{Re}(\gamma_{14}) > 0$. By inverting the Mellin transform, we can compute all the integrals in (2.15). We conclude that

$$\int_{\mathrm{Re}(\gamma_{12})>\Delta} \frac{d\gamma_{12}}{2\pi i} \int_{\mathrm{Re}(\gamma_{14})>\Delta} \frac{d\gamma_{14}}{2\pi i} K(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} \tag{2.18}$$

$$= \int_{\mathrm{Re}(\gamma_{12})=\frac{\Delta}{3}} \frac{d\gamma_{12}}{2\pi i} \int_{\mathrm{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{14}}{2\pi i} K(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}}$$

$$+ \sum_{\tau<\frac{4\Delta}{3}} \sum_{J=0}^{J_{max}} \sum_{m=0}^{\left[-\frac{\tau}{2}+\frac{2\Delta}{3}\right]} C^2_{\tau,J} \left[u^{-\Delta+\frac{\tau}{2}+m} \theta(1-v) g^{(m)}_{\tau,J}(v) + v^{-\Delta+\frac{\tau}{2}+m} \theta(1-u) g^{(m)}_{\tau,J}(u) \right] ,$$

where we assumed that we only subtracted a finite number of operators. In other words, the spin is bounded, $J \leq J_{max}$. Situations where $J_{max} = \infty$ should be analyzed on a case by case basis. We will see such examples below when analyzing the free field theory and minimal models.

A similar exercise can be done to deform the other K functions. We use the identity $u^{-\frac{\tau}{2}-m} g^{(m)}_{\tau,J}(\frac{1}{u}) = g^{(m)}_{\tau,J}(u)$ (see [29]) valid for the exchange of operators of even spin. This way we get rid of the θ functions. We conclude that

$$F(u,v) = 1 + u^{-\Delta} + v^{-\Delta} + \int_{\mathrm{Re}(\gamma_{12})=\frac{\Delta}{3}} \frac{d\gamma_{12}}{2\pi i} \int_{\mathrm{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{14}}{2\pi i} \hat{M}(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}}$$

$$+ \sum_{0<\tau<\frac{4\Delta}{3}} \sum_{J=0}^{J_{max}} \sum_{m=0}^{\left[-\frac{\tau}{2}+\frac{2\Delta}{3}\right]} C^2_{\tau,J} \left[u^{-\Delta+\frac{\tau}{2}+m} g^{(m)}_{\tau,J}(v) + v^{-\Delta+\frac{\tau}{2}+m} g^{(m)}_{\tau,J}(u) + v^{-\frac{\tau}{2}-m} g^{(m)}_{\tau,J}\left(\frac{u}{v}\right) \right], \tag{2.19}$$

where the Mellin integral is to be done with a straight contour. We therefore recovered (1.19) and (1.23).

When we deformed the integration contour as described in figure 2.2 we assumed that moving the real part of the integration contour did not affect convergence of the integral (2.15) at large imaginary values of the Mellin variables. In appendix D.3, we discuss the asymptotic behavior of K-functions at large values of Mellin variables. The main point is that K-functions decay as powers for large imaginary Mellin variables but their sum $\hat{M}$ decays exponentially.

2.6 Subtractions with Unbounded Spin

We do not have a general understanding of the case with an infinite number of subtractions with an unbounded spin. Here we simply present a couple of simple examples of this type: minimal models and free field theory correlators.

In the 2d Ising model, we know explicitly the correlator $\langle\sigma\sigma\sigma\sigma\rangle$, where $\Delta_\sigma = \frac{1}{8}$. It is equal to $\frac{F(u,v)}{|x_1-x_3|^{\frac{1}{4}}|x_2-x_4|^{\frac{1}{4}}}$, where

$$F^{\text{2dIsing}}(u,v) = \frac{\sqrt{\sqrt{u}+\sqrt{v}+1}}{\sqrt{2}\sqrt[8]{uv}}. \tag{2.20}$$

We also know a formula for the function $K(\gamma_{12}, \gamma_{14})$ for this correlator (see appendix E.1.2). So, we can implement the procedure outlined in this section to obtain a Mellin representation with straight contours. This is done in appendix (E.2). Note that in this case we do not need to use conformal blocks, since we know the function $K(\gamma_{12}, \gamma_{14})$.

We conclude that if we define

$$\begin{aligned} F^{\text{2dIsing}}_{sub}(u,v) \;&\equiv\; F^{\text{2dIsing}}(u,v) - \frac{\sqrt{1+\sqrt{u}}+\sqrt{1+\sqrt{v}}+\sqrt{\sqrt{u}+\sqrt{v}}}{\sqrt{2}(uv)^{\frac{1}{8}}} \\ &+\; \frac{u^{-\frac{1}{8}}v^{-\frac{1}{8}}+u^{-\frac{1}{8}}v^{\frac{1}{8}}+u^{\frac{1}{8}}v^{-\frac{1}{8}}}{\sqrt{2}}, \end{aligned} \tag{2.21}$$

then

$$F^{\text{2dIsing}}_{sub}(u,v) = \int_{0<\text{Re}(\gamma_{12})<\frac{1}{8}} \frac{d\gamma_{12}}{2\pi i} \int_{0<\text{Re}(\gamma_{14})<\frac{1}{8}} \frac{d\gamma_{14}}{2\pi i} \hat{M}(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}}, \tag{2.22}$$

where the Mellin integral is evaluated with a straight contour and $\hat{M}(\gamma_{12}, \gamma_{14})$ is given by

$$\hat{M}(\gamma_{12},\gamma_{14}) = -\sqrt{\frac{2}{\pi}}\Gamma\left(2\gamma_{12}-\frac{1}{4}\right)\Gamma\left(2\gamma_{14}-\frac{1}{4}\right)\Gamma(-2\gamma_{12}-2\gamma_{14}). \tag{2.23}$$

Another example is the free scalar theory in which we consider the four-point function of $\mathcal{O} = \frac{1}{\sqrt{2N}}(\vec{\phi})^2$, where $\vec{\phi}$ has N components. In appendix E.1.1, we show that $\hat{M}(\gamma_{12}, \gamma_{14}) = 0$. In this case, (2.18) takes the form

$$\begin{aligned} &\int_{\text{Re}(\gamma_{12})>\Delta} \frac{d\gamma_{12}}{2\pi i} \int_{\text{Re}(\gamma_{14})>\Delta} \frac{d\gamma_{14}}{2\pi i} K(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} \\ =\;& \int_{\text{Re}(\gamma_{12})=\frac{\Delta}{3}} \frac{d\gamma_{12}}{2\pi i} \int_{\text{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{14}}{2\pi i} K(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} + u^{-\Delta}\theta(1-v) + v^{-\Delta}\theta(1-u) \\ +\;& \frac{4}{N}\left(\theta(1-v)(u^{-\Delta/2}+u^{-\Delta/2}v^{-\Delta/2}) + \theta(1-u)u^{-\Delta/2} - \theta(u-1)u^{-\Delta/2}v^{-\Delta/2}\right), \end{aligned} \tag{2.24}$$

where $\Delta = d-2$. Note the presence of the $\theta(u-1)$ term in the last line which is absent in (2.18). This is related to the fact that in this case we have $J_{max} = \infty$. Therefore in this case the whole correlator comes from subtraction terms.

2.7 Deformed Contour

Another option is to have a deformed contour. We will bring the three integrals in (2.7) into a single integral with a deformed contour (we explain this in detail in appendix B, for the single-variable case). The integration contours in (2.7) can be deformed as long as we do not cross any OPE pole of the K-functions. In particular, if there is a deformed contour $\mathcal{C}$ that passes to the right of all OPE poles at $\gamma_{12} = \Delta - \frac{\tau_k}{2}$ in the γ_{12} complex plane and similarly for γ_{13} and γ_{14}, then we can bring the 3 integrals to the same contour. Such a contour exists unless three poles collide at

$$\gamma_{12} = \Delta - \frac{\tau_k}{2}\,, \qquad \gamma_{13} = \Delta - \frac{\tau_j}{2}\,, \qquad \gamma_{14} = \Delta - \frac{\tau_i}{2}\,. \tag{2.25}$$

When this happens, we say that the contour gets pinched. In order to deal with possible pinches, we employ a regularization procedure, that is encapsulated in the formula

$$\begin{aligned} F(u,v) &= \lim_{\epsilon\to 0} \int_{\mathcal{C}} \frac{d\gamma_{12}}{2\pi i}\frac{d\gamma_{14}}{2\pi i} \hat{M}^{\epsilon}(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}}\,, \\ \hat{M}^{\epsilon}(\gamma_{12},\gamma_{14}) &\equiv K(\gamma_{12}+\epsilon,\gamma_{14}+\epsilon) + K(\gamma_{13}+\epsilon,\gamma_{14}+\epsilon) + K(\gamma_{12}+\epsilon,\gamma_{13}+\epsilon)\,. \end{aligned} \tag{2.26}$$

We introduce a regulator $\epsilon > 0$ that allows us to separate poles that are pinched. Here we are assuming that the twist spectrum is discrete.

The contour $\mathcal{C}$ can be described as follows. Firstly, we fix γ_{12} and integrate over γ_{14}. The integrand has poles at

$$\gamma_{14} = -\epsilon + \Delta - \frac{\tau_i}{2}\,, \qquad \gamma_{14} = -\gamma_{12} + \epsilon + \frac{\tau_j}{2}\,, \tag{2.27}$$

where τ_i, τ_j are twists of exchanged operators (including descendants). The second set of poles originates from OPE poles in γ_{13}. The contour $\mathcal{C}$ splits the two set of poles to the left and to the right in the γ_{14} complex plane as shown in figure 2.3. Notice that this is always possible for generic values of γ_{12}. For special values of γ_{12} poles from the left set can collide with poles from the right set, pinching the integration contour and giving rise to poles in γ_{12}. Secondly, we consider the integral in γ_{12}. In this case, the integrand will have poles at

$$\gamma_{12} = -\epsilon + \Delta - \frac{\tau_k}{2}\,, \qquad \gamma_{12} = 2\epsilon - \Delta + \frac{\tau_i + \tau_j}{2}\,, \tag{2.28}$$

where the second set of poles arises from pinching the contour integral over γ_{14}. Again, the contour $\mathcal{C}$ splits the two sets of poles two the left and to the right in the γ_{12} complex plane as shown in figure 2.3. Notice that this is always possible for arbitrarily small ϵ and a discrete twist spectrum. So, with $\epsilon \neq 0$, the integral in (2.26) is always well defined. After evaluating the integral, we take the limit $\epsilon \to 0$.

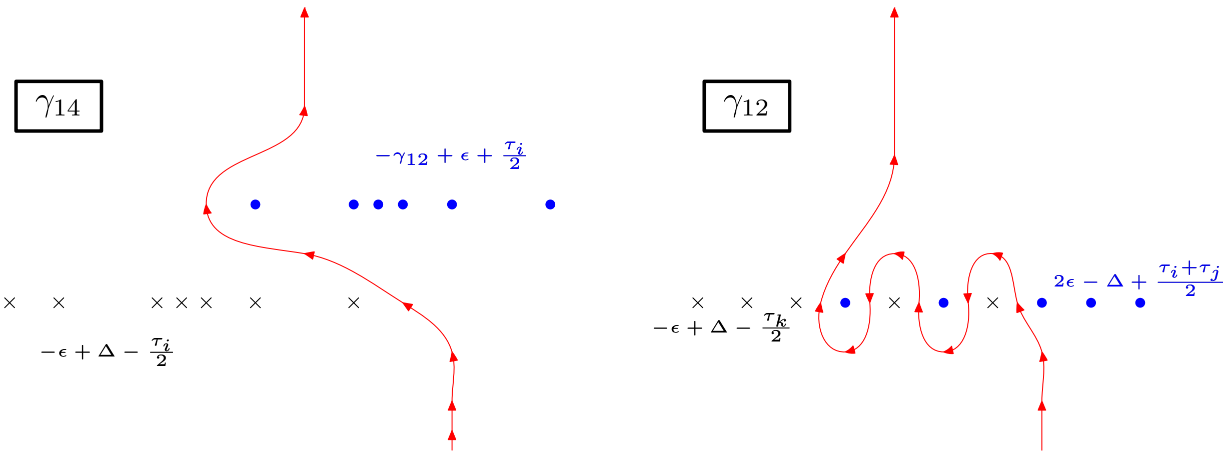

Figure 2.3. Deformed integration contour $\mathcal{C}$. Firstly, we integrate over γ_{14} as shown on the left keeping γ_{12} fixed. Secondly, we integrate over γ_{12} as shown on the right. Pinching occurs if, as $\epsilon \to 0$, a pole marked with a black cross collides with a pole marked with a blue dot on the γ_{12} complex plane.

Unfortunately, equation (2.26) is not very useful, since it involves a regularization procedure. Furthermore, it also involves the function $K(\gamma_{12}, \gamma_{14})$, which we expect to be more complicated than the Mellin amplitude $\hat{M}(\gamma_{12}, \gamma_{14})$. In what follows, we will consider a generic CFT[6] and evaluate the contribution from the pinches in (2.26). We will then set $\epsilon = 0$ and obtain a Mellin representation with a deformed contour, that makes no reference to $K(\gamma_{12}, \gamma_{14})$.

The first step is to understand when there will be pinches in a generic CFT. From (2.28), we conclude that the deformed contour will be pinched as $\epsilon \to 0$, if the condition

$$\tau_i + \tau_j + \tau_k = 4\Delta \tag{2.29}$$

is satisfied. There are 6 pinches that occur in generic CFTs (see figure 2.4). Firstly, we have collisions between the identity pole ($\tau = 0$) and accumulation points ($\tau = 2\Delta$). These correspond to $(\gamma_{12}, \gamma_{13}, \gamma_{14}) = (\Delta, 0, 0)$ and permutations. Secondly, there can be collisions between the pole associated to the exchange of the external operator ($\tau = \Delta$) and accumulation points. These correspond to $(\gamma_{12}, \gamma_{13}, \gamma_{14}) = (\frac{\Delta}{2}, \frac{\Delta}{2}, 0)$ and permutations.

Suppose the contour goes through the rightmost shaded triangle in figure 2.4. We deform the γ_{12} contour to the left and we pick up the pole at $\gamma_{12} = \Delta - \epsilon$:

$$\int_{\mathrm{Re}(\gamma_{12}) > \Delta - \epsilon} \frac{d\gamma_{12}}{2\pi i} \int \frac{d\gamma_{14}}{2\pi i} \hat{M}^{\epsilon}(\gamma_{12}, \gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}} \tag{2.30}$$

[6] By a generic CFT, we have in mind an interacting and non-perturbative CFT$_d$ in $d > 2$, like the 3d Ising model.

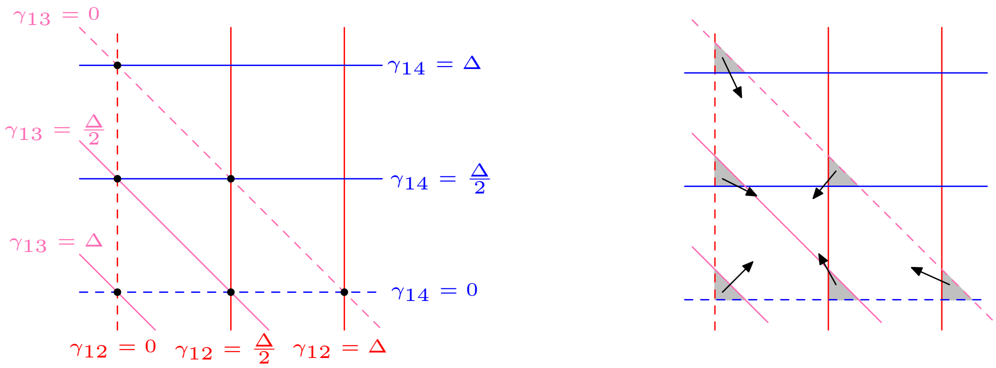

Figure 2.4. Singularities in γ_{12}, γ_{13} and γ_{14} are represented by red, pink and blue lines respectively. We denote poles by continuous lines and accumulation points by dashed lines. It is possible for three singularities to collide at the same point, thus causing a pinch. There are 6 pinches that occur in generic CFTs. These are marked with black dots. Firstly, we have collisions between the identity pole and accumulation points at $(\gamma_{12}, \gamma_{13}, \gamma_{14}) = (\Delta, 0, 0)$ and permutations. Secondly, there can be collisions between the pole associated to the exchange of the external operator and accumulation points at $(\gamma_{12}, \gamma_{13}, \gamma_{14}) = (\frac{\Delta}{2}, \frac{\Delta}{2}, 0)$ and permutations. On the right we consider the case in which we introduce the ϵ regulator. This removes the pinches and the contour can go through the gray triangles. To resolve the pinches we first move the contour out the gray triangles as indicated by the arrows and then send $\epsilon \to 0$.

$$
\begin{aligned}
&= u^{-\Delta+\epsilon} \int \frac{d\gamma_{14}}{2\pi i} \hat{M}^{\epsilon}(\gamma_{12} = \Delta - \epsilon, \gamma_{14}) v^{-\gamma_{14}} \\
&+ \int_{\mathrm{Re}(\gamma_{12})<\Delta-\epsilon} \frac{d\gamma_{12}}{2\pi i} \int \frac{d\gamma_{14}}{2\pi i} \hat{M}(\gamma_{12}, \gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}},
\end{aligned}
$$

where $\hat{M}^{\epsilon}(\gamma_{12} = \Delta - \epsilon, \gamma_{14})$ is the residue of the Mellin amplitude at $\gamma_{12} = \Delta - \epsilon$. In the second integral, we can drop the regularization. To evaluate $\hat{M}^{\epsilon}(\gamma_{12} = \Delta - \epsilon, \gamma_{14})$, consider the contribution of the identity to $K(\gamma_{12}, \gamma_{14})$,

$$
\int_0^1 \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} u^{-\Delta} = \frac{1}{\gamma_{14}(\gamma_{12} - \Delta)}. \tag{2.31}
$$

Similarly, $K(\gamma_{12}, \gamma_{13})$ also has a pole at $\gamma_{12} = \Delta$. We conclude that the regularised Mellin amplitude $\hat{M}^{\epsilon}(\gamma_{12}, \gamma_{14})$ has a pole at $\gamma_{12} = \Delta - \epsilon$ with residue given by

$$
\hat{M}^{\epsilon}(\gamma_{12}, \gamma_{14}) \approx \frac{3\epsilon}{(\gamma_{12} - \Delta + \epsilon)(\gamma_{14} + \epsilon)(-\gamma_{14} + 2\epsilon)}, \qquad \gamma_{12} \to \Delta - \epsilon\,. \tag{2.32}
$$

Notice that this residue goes to 0 in the limit $\epsilon \to 0$. We expected this from the fact that $F(u, v)$ cannot actually diverge due to a pinch. Let us evaluate the finite contribution to the four point function given by the pinch:

$$
\lim_{\epsilon \to 0} u^{-\Delta+\epsilon} \int_{-i\infty}^{+i\infty} \frac{d\gamma_{14}}{2\pi i} \frac{3\epsilon}{(\gamma_{14} + \epsilon)(-\gamma_{14} + 2\epsilon)} v^{-\gamma_{14}} = u^{-\Delta}. \tag{2.33}
$$

Notice that the integrand goes to 0 when ϵ goes to 0, but at the same time the contour gets pinched between a pole from the left with a pole from the right. For this reason, the integration gives a finite result.

To summarize, we saw that the pinch at $\tau_i = 2\Delta$, $\tau_j = 2\Delta$ and $\tau_k = 0$ gives a finite contribution $u^{-\Delta}$ to the four point function. After taking $\epsilon \to 0$, the pole at $\gamma_{12} = \Delta$ disappears and the contour of the Mellin amplitude does not get pinched at all. By crossing symmetry, a similar discussion holds for permutations of the previous pinch condition. In the absence of other pinches, we conclude that

$$\begin{aligned} F_{conn}(u,v) &\equiv F(u,v) - (1 + u^{-\Delta} + v^{-\Delta}) \\ &= \int \int_{\mathcal{C}'} \frac{d\gamma_{12}}{2\pi i} \frac{d\gamma_{14}}{2\pi i} \hat{M}(\gamma_{12}, \gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}}. \end{aligned} \tag{2.34}$$

The Mellin integral computes the connected part of the four point function. The integral is to be taken with a deformed contour $\mathcal{C}'$ that differs from $\mathcal{C}$ as indicated by the arrows in figure 2.4.

Generically, $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle \propto C_{\mathcal{O}\mathcal{O}\mathcal{O}} \neq 0$ and we also need to deal with the pinches at $(\gamma_{12}, \gamma_{13}, \gamma_{14}) = (\frac{\Delta}{2}, \frac{\Delta}{2}, 0)$ and permutations. In order to remove these pinches, we consider the following function

$$\tilde{F}(u,v) \equiv F(u,v) - (1 + u^{-\Delta} + v^{-\Delta}) - C^2_{\mathcal{O}\mathcal{O}\mathcal{O}} \left[u^{-\frac{\Delta}{2}} g^{(0)}_{\Delta,0}(v) + v^{-\frac{\Delta}{2}} g^{(0)}_{\Delta,0}(u) + v^{-\frac{\Delta}{2}} g^{(0)}_{\Delta,0}\left(\frac{u}{v}\right) \right]. \tag{2.35}$$

The corresponding K-function $\tilde{K}(\gamma_{12}, \gamma_{14})$ does not have poles at $\gamma_{12} = \Delta/2$ or $\gamma_{14} = \Delta/2$. This means that the contour $\mathcal{C}'$ can be shifted as indicated by the arrows in figure 2.4. Thus, we can write

$$\tilde{F}(u,v) = \int \int_{\mathcal{C}'} \frac{d\gamma_{12}}{2\pi i} \frac{d\gamma_{14}}{2\pi i} \left(\hat{M}(\gamma_{12}, \gamma_{14}) + \delta\hat{M}(\gamma_{12}, \gamma_{14}) \right) u^{-\gamma_{12}} v^{-\gamma_{14}} \tag{2.36}$$

where $\delta\hat{M}$ is obtained from

$$\begin{aligned} \delta K(\gamma_{12}, \gamma_{14}) \equiv &- \int_0^1 \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} \left[1 + u^{-\Delta} + v^{-\Delta} \right] \\ &- C^2_{\mathcal{O}\mathcal{O}\mathcal{O}} \int_0^1 \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} \left[u^{-\frac{\Delta}{2}} g^{(0)}_{\Delta,0}(v) + v^{-\frac{\Delta}{2}} g^{(0)}_{\Delta,0}(u) + v^{-\frac{\Delta}{2}} g^{(0)}_{\Delta,0}\left(\frac{u}{v}\right) \right]. \end{aligned} \tag{2.37}$$

The first integral is elementary. The second integral converges when $\mathrm{Re}[\gamma_{12}, \gamma_{14}] > \frac{\Delta}{2}$. Using

$$g^{(0)}_{\Delta,0}(v) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} v^{-s} \frac{\Gamma(s)^2 \Gamma(\frac{\Delta}{2} - s)^2 \Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^4}, \tag{2.38}$$

with $0 < c < \frac{\Delta}{2}$, we obtain

$$\delta K(\gamma_{12}, \gamma_{14}) = -\frac{1}{\gamma_{12}\gamma_{14}} - \frac{1}{(\gamma_{12} - \Delta)\gamma_{14}} - \frac{1}{\gamma_{12}(\gamma_{14} - \Delta)} - C^2_{\mathcal{O}\mathcal{O}\mathcal{O}} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \tag{2.39}$$
$$\times \frac{\Gamma(s)^2\Gamma(\frac{\Delta}{2} - s)^2\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^4} \left(\frac{1}{\gamma_{12} - \frac{\Delta}{2}} \frac{1}{\gamma_{14} - s} + \frac{1}{\gamma_{14} - \frac{\Delta}{2}} \frac{1}{\gamma_{12} - s} + \frac{1}{\gamma_{12} - s} \frac{1}{\gamma_{14} + s - \frac{\Delta}{2}} \right),$$

with $\text{Re}[\gamma_{12}, \gamma_{14}] > \frac{\Delta}{2}$. We can then explicitly compute [7]

$$\delta \hat{M}(\gamma_{12}, \gamma_{14}) = \delta K(\gamma_{12}, \gamma_{14}) + \delta K(\gamma_{12}, \gamma_{13}) + \delta K(\gamma_{13}, \gamma_{14}) = 0\,. \tag{2.40}$$

We conclude that the subtractions do not affect the Mellin amplitude but only the integration contour. In particular, the subtractions in (2.35) did not remove the poles at $\gamma_{1i} = \frac{\Delta}{2}$ (for $i = 2, 3, 4$) from the Mellin amplitude. This happens as follows. The function $\tilde{K}(\gamma_{12}, \gamma_{14}) = K(\gamma_{12}, \gamma_{14}) + \delta K(\gamma_{12}, \gamma_{14})$ does not have poles at $\gamma_{12} = \frac{\Delta}{2}$ nor at $\gamma_{14} = \frac{\Delta}{2}$ like the original K-function $K(\gamma_{12}, \gamma_{14})$. However, $\tilde{K}(\gamma_{12}, \gamma_{14})$ has a pole at $\gamma_{13} = \frac{\Delta}{2}$ that was not present in $K(\gamma_{12}, \gamma_{14})$. The same mechanism happens for the other subtractions in (1.19). The exception being the exchange of the identity operator (or disconnected piece) that does not give rise to any poles in the Mellin amplitude.

In the end, we can simply write

$$\tilde{F}(u, v) = \int \int_{C'} \frac{d\gamma_{12}}{2\pi i} \frac{d\gamma_{14}}{2\pi i} \hat{M}(\gamma_{12}, \gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}}\,. \tag{2.41}$$

Therefore most of the subtractions that we encountered in the straight contour formula (1.19) can be neatly absorbed into the deformation of the contour of integration. Moreover, the argument in this subsection is valid even if the straight contour formula requires an infinite number of subtractions with unbounded spin. In section 8.1, we show examples of such deformed contours for correlators in 2d minimal models. If there are special relations among the scaling dimensions of the theory such that $\tau_i + \tau_j + \tau_k = 4\Delta$ for some trio of operators, then there are extra pinches in the limit $\epsilon \to 0$ that must be analysed. This is relevant for perturbative CFTs. In appendix E.3, we confirm that our general discussion works in the critical ϕ^3 theory in $d = 6 + \epsilon$ spacetime dimensions to first order in ϵ.

[7] To see this write $s = \frac{\Delta}{4} + i\,x$ in (2.39) and integrate over real x. Then decrease the real part of γ_{12} and γ_{14} from bigger than $\frac{\Delta}{2}$ to the neighbourhood of the crossing symmetric point $\gamma_{12} = \gamma_{14} = \frac{\Delta}{3}$. This can be done without any pole of the integrand crossing the s-integration contour. Finally, sum the 3 δK's and observe that the total integrand is an odd function of x.

3 Unitarity properties of Mellin amplitudes

3.1 Introduction

In this chapter we analyze constraints on the Mellin amplitude coming from unitarity (or the OPE expansion) and boundedness of the correlator in the Regge limit [30, 36]. The OPE expansion dictates the form of the residues of the Mellin amplitude $M(\gamma_{12}, \gamma_{14})$ which are given by the Mack polynomials [3]. Bounds on the Regge limit restrict the rate of growth of the Mellin amplitude as one of its arguments becomes large.

3.2 OPE expansion

The OPE expansion states that we can write the correlation function as a sum of conformal blocks with positive coefficients

$$F(u,v) = u^{-\Delta} \sum_{\tau, J-\text{even}} C^2_{\tau,J} g_{\tau,J}(u,v). \tag{3.1}$$

As before it is convenient to write each conformal block as a sum in the powers of u [37]

$$\begin{aligned} g_{\tau,J}(u,v) &= u^{\frac{\tau}{2}} \sum_{m=0}^{+\infty} u^m g^{(m)}_{\tau,J}(v), \\ g^{(0)}_{\tau,J}(v) &= (-1)^J (1-v)^J \, {}_2F_1\left(\frac{\tau}{2}+J, \frac{\tau}{2}+J, \tau+2J, 1-v\right). \end{aligned} \tag{3.2}$$

From the definition of the Mellin transform (1.3) it is clear that powers of u will lead to the poles in γ_{12} dictated by the twists of the exchanged operators. In particular, a single primary operator with twist τ introduces an infinite series of poles at

$$\gamma_{12}, \gamma_{14}, \gamma_{13} = \Delta - \frac{\tau}{2} - m, \quad m = 0, 1, 2, \dots, \tag{3.3}$$

where m is precisely the same m that appears in (3.2). Our next task is to fix the residue of the Mellin amplitude at a given OPE pole so that it reproduces the OPE expansion (3.1).

To make contact with [29], where this question was investigated in great detail we introduce *Mellin-Mandelstam variables*

$$\begin{aligned} t &= 2\Delta - 2\gamma_{12}, \\ s &= 2(\gamma_{12} + \gamma_{14} - \Delta) = -2\gamma_{13}. \end{aligned} \tag{3.4}$$

In terms of these variables the residue of the pole takes the following form

$$\begin{aligned} M(s,t) &\simeq \frac{C_{\tau,J}^2 \mathcal{Q}_{J,m}^{\tau,d}(s)}{t-(\tau+2m)} + \dots, \quad m = 0,1,2,\dots, \\ \mathcal{Q}_{J,m}^{\tau,d}(s) &= -K(\tau,J,m)\, Q_{J,m}^{\tau,d}(s), \end{aligned} \tag{3.5}$$

where $Q_{J,m}^{\Delta,\tau,d}(s)$ are *Mack polynomials* in s of degree J. $K(\Delta,J,m)$ is a non-negative kinematical pre-factor[1]

$$K(\tau,J,m) = \frac{2\Gamma(\tau+2J)(\tau+J-1)_J}{2^J \Gamma(\frac{\tau+2J}{2})^4} \frac{1}{m!(\tau+J-\frac{d}{2}+1)_m \Gamma\left(\Delta-\frac{\tau}{2}-m\right)^2}\,. \tag{3.6}$$

One feature worth mentioning is that $K(\tau,J,m)$ has double zeros at the position of double twist operators $\tau = 2\Delta + 2n$. This will play an important role in the consideration of dispersion relations in Mellin space below.

3.3 Properties of Mack Polynomials

Mack polynomials have many remarkable properties. Let us review some of them (we are largely following [29]). Similarly to Legendre polynomials Mack polynomials satisfy [29,38]

$$Q_{J,m}^{\tau,d}(s) = (-1)^J Q_{J,m}^{\tau,d}(-s-\tau-2m)\,. \tag{3.7}$$

For $m = 0$ they are related to continuous Hahn polynomials

$$Q_{J,0}^{\tau,d}(s) = \frac{2^J \left(\frac{\tau}{2}\right)_J^2}{(\tau+J-1)_J}\, {}_3F_2(-J, J+\tau-1, -\frac{s}{2}; \frac{\tau}{2}, \frac{\tau}{2}; 1). \tag{3.8}$$

Higher m polynomials can be computed recursively, see formulas (J.4) and (J.5).

[1]Note a difference by a factor of 2^J compared to [29] due to the normalization of conformal blocks that we adopt in this thesis.

In the large s limit Mack polynomials behave as follows

$$\lim_{s\to\infty} Q^{\Delta,\tau,d}_{J,m}(s) = s^J + O(s^{J-1}). \tag{3.9}$$

This goes along well with the flat space scattering and s being the usual Mandelstam variable. Indeed, given an exchange Witten diagram in AdS, its asymptotic behavior for large Mellin s is controlled by the spin of the exchanged operator.

Another limit which is relevant to the flat space limit is $s, \tau, m \gg 1$ with J fixed. In this case we get

$$Q^{\tau,d}_{J,m}(s) \approx m^{\frac{J}{2}} (m+\tau)^{\frac{J}{2}} C_J^{(\frac{d-2}{2})} \left(\frac{\frac{\tau}{2}+m+s}{\sqrt{m(m+\tau)}} \right) + \dots . \tag{3.10}$$

This asymptotic of Mack polynomials is relevant for recovering the flat space scattering amplitudes.

Mack polynomials have interesting positivity properties. We observed that for even J and for general s, τ, m Mack polynomials are non-negative for

$$\left| \frac{\tau}{2} + m + s \right| > \sqrt{m(m+\tau)} \, , \tag{3.11}$$

which again generalizes the familiar property of Gegenbauer polynomials that emerge in the flat space limit (3.10). We would like this to hold for any m and any τ that satisfies the unitarity bound. Indeed, we observed that

$$\partial_s^n Q^{\tau,d}_{J,m}(s)|_{s\geqslant 0} \geqslant 0 \, , \quad n \geqslant 0. \tag{3.12}$$

This again parallels a famous property of Gegenbauer polynomials familiar from the S-matrix bootstrap considerations [39]. We observed it by studying many particular examples. It would be of course better to prove it rigorously. We will use this property in our considerations of Mellin space dispersion relations below. It would be also interesting to understand more refined positivity properties of the Mack polynomials in the spirit of [40].

Another interesting limit is $J \gg 1$ with all other parameters being fixed[2]

$$\mathcal{Q}^{\tau,d}_{J,0}(s) = \frac{2^{2J+\tau}}{\sqrt{\pi}\Gamma^2(\Delta - \frac{\tau}{2})} \left(\frac{J^{s+\frac{1}{2}}}{\Gamma^2(\frac{s+\tau}{2})} + (-1)^J \frac{J^{-s-\tau+\frac{1}{2}}}{\Gamma^2(-\frac{s}{2})} + O(J^{-1}) \right) . \tag{3.13}$$

We will use this asymptotic below in our considerations of the double twist operators in Mellin space. See appendix J for further notes on this.

[2] This limit is very different from the corresponding limit of Legendre polynomials that appear in the description of the flat space physics. It is responsible for intrinsically AdS effects, see appendix B in [41].

3.4 Boundedness at Infinity and the Regge limit

Let us understand the behaviour of Mellin amplitudes at infinity. The relevant limit to consider is the Regge limit $s \to \infty$, t - fixed. As explained in [29] this limit of the Mellin amplitude controls the Regge limit of the correlation function. Thanks to the OPE it is very easy to bound the Regge behaviour of the CFT correlation functions both nonperturbatively [30] and in the planar limit [36]. This leads to bounds on the Mellin amplitude $M(s,t)$ that we review in this section.

To describe the Regge limit consider a Lorentzian time-ordered four-point function

$$F(t',\rho) = \langle \mathrm{T}\left[V(x_1)V(x_2)W(x_3)W(x_4)\right]\rangle, \tag{3.14}$$

where we restrict points to a Lorentzian plane and choose the following light-cone coordinates ($x^{\pm} \equiv t \pm x$)

$$x_1^{\pm} = \pm 1,\; x_2^{\pm} = \mp 1,\; x_3^{\pm} = \mp e^{\rho \pm t'}, x_4^{\pm} = \pm e^{\rho \pm t'}\,. \tag{3.15}$$

see figure 9.1.

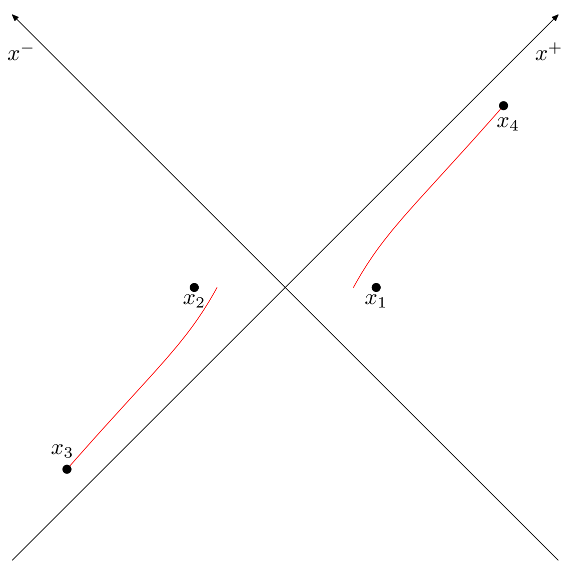

Figure 3.1. Kinematics (3.15). The Regge limit corresponds to taking $t' \to \infty$. In this limit $x_{34}^2 \to 0$.

In the Regge limit the cross ratios take the following values

$$u = \sigma^2, \quad v \simeq 1 - 2\sigma\cosh\rho, \quad \sigma = 4e^{-t'}. \tag{3.16}$$

As we increase t', x_{14}^2, x_{23}^2 become time-like and $\sigma \to 0$. All other distances are space-like. The ordering of operators implies that we analytically continue $v \to ve^{2\pi i}$ around $v = 0$.

Unitarity and the Euclidean OPE then imply that in any CFT [30]

$$\lim_{\tilde{\sigma}\to 0} \frac{F(\tilde{\sigma},\rho)}{F_{disc}(\tilde{\sigma},\rho)} \leqslant 1, \tag{3.17}$$

where $F_{disc} = \langle \mathrm{T}\,[V(x_1)V(x_2)]\rangle\langle \mathrm{T}\,[W(x_3)W(x_4)]\rangle$.

In the context of the Regge limit in large N CFT, [36] considered $f(t,\rho) \equiv \frac{F(\tilde{\sigma},\rho)}{F_{disc}(\tilde{\sigma},\rho)}$ and showed that the correlator obeys

$$\frac{|\partial_t f(t,\rho)|}{1-f} \leqslant 1 + O(e^{-2(t-t_0)}) \tag{3.18}$$

for $t > t_0 = O(1)$.

Let us see how this comes about from the conformal Regge theory [29]. Assuming that the leading Regge behavior comes from a pole,[3] we get the following behavior of the correlator in the Regge limit

$$f(t,\rho) \quad = 1 - 2\pi \int_{-\infty}^{\infty} d\nu\, \hat{\alpha}(\nu) e^{[j(\nu)-1]t}\Omega_{i\nu}(\rho) + ..., \tag{3.19}$$

where $\hat{\alpha}(\nu)$ is related in a specific way to the product of the three-point couplings $c_{VV\mathcal{O}_J}c_{WW\mathcal{O}_J}$ with $\mathcal{O}_J$ being the operators of the leading Regge trajectory and $j(\nu) = j(-\nu)$. The integral over ν is then evaluated via a saddle point at $\nu = 0$. The location of the saddle at $\nu = 0$ follows from convexity properties of the Regge trajectories.[4]

In the language of the Regge trajectory $j(\nu)$ the bounds (3.17) and (3.18) imply that

$$\begin{aligned} j_{full}(0) &\leqslant 1\,, \\ j_{planar}(0) &\leqslant 2\,, \end{aligned} \tag{3.20}$$

where $j_{full}(0)$ is the leading Regge trajectory in the finite N CFT, whereas $j_{planar}(0)$ is the Regge trajectory of the single trace operators in the planar theory. The bounds can be also generalized for non-zero complex ν's, see [42–44].

Let us now consider the Regge limit $s \to \infty$ of the reduced Mellin amplitude $M(s,t)$. For simplicity we consider identical operators $\Delta_V = \Delta_W = \Delta$. The relation between the correlator in the Regge limit and the Mellin amplitude in the strip of holomorphy was worked out in [29] with the following result

$$f_{sub}(\tilde{\sigma},\rho) = \int_{U_{CFT}} \frac{dt}{4i}\tilde{\sigma}^t\Gamma\left(\Delta - \frac{t}{2}\right)^2 \int^{\infty} dx M(\mathrm{Re}[s]+ix,t)\left(\frac{x}{2}\right)^{t-2} e^{-x\tilde{\sigma}\cosh\rho} + ... \,, \tag{3.21}$$

[3] Having a more complicated singularity in the J-plane, say a cut, does not change a discussion since it only affects the sub-exponential terms.

[4] Convexity of $j_{full}(\nu)$ has been proven in [42]. Convexity of $j_{planar}(\nu)$ is simply assumed here.

where the integration contour is the straight line along the imaginary axis with

$$\begin{aligned} U_{CFT}: \quad \tau_{sub} < \ \mathrm{Re}[t] \ &< \ \tau'_{sub}\,, \\ \tau_{sub} - 2\Delta < \ \mathrm{Re}[s] \ &< \ \tau'_{sub} - 2\Delta\,, \\ \tau'_{sub} - 2\Delta < \ \mathrm{Re}[s] + \mathrm{Re}[t] \ &< \ 2\Delta - \tau_{sub}\,. \end{aligned} \tag{3.22}$$

We also suppressed terms that are subleading in the Regge limit.

Consider next the leading Regge pole contribution to the Mellin amplitude [29]

$$M(s,t) \simeq \int_{-\infty}^{\infty} d\nu\, \beta(\nu)\omega_{\nu,j(\nu)}(t)\frac{s^{j(\nu)} + (-s)^{j(\nu)}}{\sin\left(\pi j(\nu)\right)} + ..., \tag{3.23}$$

where $\beta(\nu)$ is related in a known way to $\hat{\alpha}(\nu)$ in (3.19). The function $\omega_{\nu,J}(t)$ is given by

$$\begin{aligned} \omega_{\nu,J}(t) &= \frac{\Gamma(\frac{2\Delta_V+J+i\nu-\frac{d}{2}}{2})\Gamma(\frac{2\Delta_W+J+i\nu-\frac{d}{2}}{2})\Gamma(\frac{2\Delta_V+J-i\nu-\frac{d}{2}}{2})\Gamma(\frac{2\Delta_W+J-i\nu-\frac{d}{2}}{2})}{8\pi\Gamma(i\nu)\Gamma(-i\nu)} \\ &\times \frac{\Gamma(\frac{\frac{d}{2}+i\nu-J-t}{2})\Gamma(\frac{\frac{d}{2}-i\nu-J-t}{2})}{\Gamma(\Delta_V-\frac{t}{2})\Gamma(\Delta_W-\frac{t}{2})}. \end{aligned} \tag{3.24}$$

As reviewed in the previous section as we vary t the Mellin amplitude should exhibit poles at the positions of the physical operators. Let us review how they come about in (3.23) for the operators on the leading Regge trajectory. This expression has poles whenever $j(\nu) = 2\mathbb{Z}$. Recall that $j(\nu) = j(-\nu)$ describes the leading Regge trajectory $\Delta(J)$ and is defined via

$$\nu^2 + \left(\Delta(j(\nu)) - \frac{d}{2}\right)^2 = 0. \tag{3.25}$$

Therefore, say for $j(\nu) = 2$ which corresponds to the stress tensor we have $\Delta(2) = d$ and $\nu = \pm i\frac{d}{2}$. Furthermore $\beta(\nu)$ has poles at the locations (3.25).

At the same time $\omega_{\nu,j(\nu)}(t)$ has poles at $t = 2\mathbb{Z}_{\geqslant 0} + \frac{d}{2} - j(\nu) \pm i\nu$. These poles collide with the poles of $\beta(\nu)$ when t crosses $\tau + 2m$, where τ is the twist of a physical operator with spin j. In this way (3.23) generates the expected poles in t.

Let us next plug (3.23) into (3.21). We get

$$\begin{aligned} f_{sub}(\tilde{\sigma},\rho) &= \pi\int_{-\infty}^{\infty} d\nu\beta(\nu)\frac{2^{j(\nu)}}{\sin\frac{\pi j(\nu)}{2}}\tilde{\sigma}^{1-j(\nu)} \\ &\int_{U_{CFT}} \frac{dt}{2\pi i}\Gamma\left(\Delta_V - \frac{t}{2}\right)\Gamma\left(\Delta_W - \frac{t}{2}\right)\frac{\Gamma(j(\nu)+t-1)}{(2\cosh\rho)^{j(\nu)+t-1}}\omega_{\nu,J}(t). \end{aligned} \tag{3.26}$$

We next deform the t contour to $\mathrm{Re}[t] = 0$. As explained above in doing so the ν-contour develops pinches at the position of the physical operators. These are precisely the operators with $0 < \tau \leqslant \tau_{sub}$ which cancel the subtractions that we made in defining f_{sub}.[5] After deforming the contour to $\mathrm{Re}[t] = 0$ and doing the t integral we arrive at (3.19) as shown in [29]. We should again contrast the Regge behavior of the Mellin amplitude which is controlled by the large s behavior and the subtractions that originate from the poles in t.

After reviewing the relation to the coordinate space Regge limit, let us come back to the expression in Mellin space (3.23). As usual we assume that in the Regge limit the ν integral is dominated by the region $\nu = 0$,[6] and therefore we can use bounds on $j(0)$ (3.20) to bound the growth of the Mellin amplitude.

An important question is for which values of $\mathrm{Re}[t]$ this argument holds? Above we made it for $\mathrm{Re}[t] = 0$. As we increase $\mathrm{Re}[t]$ the integral over ν can develop a pinch as we reviewed above and will not be dominated by $\nu = 0$ anymore. The relevant pinch corresponds to $J = 2$ operator on the leading twist Regge trajectory, which for identical operators is the stress tensor, namely $\mathrm{Re}[t] = d - 2$.

Taking into accounts bounds on $j(0)$, in that way we get the following conditions on the Mellin amplitude

$$\begin{aligned} \lim_{|\mathrm{Im}[s]| \to \infty} |M_{full}(s,t)| &\leqslant c|s|, \quad \mathrm{Re}[t] < d-2, \\ \lim_{|\mathrm{Im}[s]| \to \infty} |M_{planar}(s,t)| &\leqslant c|s|^2, \quad \mathrm{Re}[t] < d-2, \end{aligned} \tag{3.27}$$

where in writing the planar bound we implicitly allowed for slower than a power growing corrections. In terms of γ_{ij} the bound corresponds to $\mathrm{Re}[\gamma_{12}] > \Delta - \frac{d-2}{2}$ as we send $\mathrm{Im}[\gamma_{14}] \to \infty$.

3.5 Extrapolation

The bounds in the previous section were derived only in the limit $\mathrm{Im}[s] \to +\infty$, or equivalently $\arg[s] = \pm\frac{\pi}{2}$, however we would like to relax this condition and apply the Regge bounds for any $\arg[s]$.[7] As we change $\arg[s]$ away from $\pm\frac{\pi}{2}$ in principle some sort of Stokes phenomenon might occur. Here we assume that for physical correlators

[5] Strictly speaking this only refers to the operators on the leading Regge trajectory in the analyticity in spin $J > J_0^{Regge}$ region.

[6] This is not always true. For example, in the minimal models the trajectories are exactly linear and the integral is dominated by the closest pole in the upper half-plane.

[7] For $\arg[s] = 0, \pm\pi$ we understand them in the tauberian [24,45] or averaged sense. Indeed for these values of the arguments the Mellin amplitude has poles, however if we average the Mellin amplitude over s then we assume that the Regge bound still hold.

this does not happen and the Regge bound that we found along the imaginary axis holds everywhere in the complex s-plane.[8] At the same time as we observed above it is important to keep $\mathrm{Re}[t]$ in the region of holomorphy.

In this way we arrive at the following Regge bounds for the Mellin amplitude

$$\lim_{|\gamma_{12}|\to\infty} |M_{full}(\gamma_{12},\gamma_{14})| \leqslant c|\gamma_{12}|, \quad \mathrm{Re}[\gamma_{14}] > \Delta - \frac{d-2}{2},$$
$$\lim_{|\gamma_{12}|\to\infty} |M_{planar}(\gamma_{12},\gamma_{14})| \leqslant c|\gamma_{12}|^2, \quad \mathrm{Re}[\gamma_{14}] > \Delta - \frac{d-2}{2}\,. \tag{3.28}$$

Again we will assume that for $\arg[\gamma_{12}] = 0, \pm\pi$ the Regge limit is still bounded in the averaged sense that the corresponding dispersion relations will converge. Imposing that the Regge bound holds at the crossing symmetric point leads to $\frac{\Delta}{3} > \Delta - \frac{d-2}{2}$, which gives us *the Regge lightness condition*

$$\Delta < \frac{3}{4}(d-2)\,, \tag{3.29}$$

for the four-point function of identical operators. If we were to consider heavier operators we expect the Regge limit, for γ_{14} around the crossing symmetric point, to be dominated by subtractions. For simplicity below we will restrict our bootstrap analysis to the case $\Delta < \frac{3}{4}(d-2)$.

[8]This is consistent with all examples that we know, but of course it requires a proof.

4 Polyakov Conditions

4.1 Introduction

At this point the reader might be perplexed by the following two facts. On one hand, we have the crossing relation for the full correlator

$$F(u,v) = u^{-\Delta} \sum_{\tau,J-\text{even}} C^2_{\tau,J} g_{\tau,J}(u,v) = v^{-\Delta} \sum_{\tau,J-\text{even}} C^2_{\tau,J} g_{\tau,J}(v,u) = F(v,u), \tag{4.1}$$

where the full correlator is reproduced by either the s-channel exchanges, or by the t-channel exchanges. On the other hand, we have the formulas (1), (2), where only the connected correlator is represented via a Mellin transform. Moreover, as we explained in the previous chapter, the Mellin amplitude has poles at the position of all operators (except the identity operator) designed in precisely such a way to reproduce the OPE expansion (4.1). Therefore, we seem to be running into a paradox: closing the, say γ_{12}, integration contour in (1), (2) would produce the full correlator, instead of producing the connected correlator only!

This confusion is related to the subtle nature of the twist accumulation points that we are bound to cross when trying to run into the paradox above. As we will see in the end everything is consistent, however the fact that the Mellin amplitude correctly reproduces the full correlator does lead to some subtle and nontrivial conditions on nonperturbative Mellin amplitudes, which we call Polyakov conditions [46]. Indeed, in spirit they are the same familiar conditions from the Mellin-Polyakov bootstrap program [8, 9, 47]. However, we will see that the nonperturbative nature of the Mellin amplitude makes them much more subtle.

4.2 Reproducing the identity

Let us try to run into the paradox described above. We assume that the lowest twist τ_{gap} after the identity obeys $\frac{4\Delta}{3} < \tau_{gap} < 2\Delta$, so that the only subtractions in (1.19) are the disconnected parts of the correlator. Consider then the straight contour formula

$$F(u,v) = \left(1 + u^{-\Delta} + v^{-\Delta}\right) + \int_{\mathrm{Re}(\gamma_{12})=\mathrm{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{12}}{2\pi i}\frac{d\gamma_{14}}{2\pi i}\hat{M}(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}, \tag{4.2}$$

Now keep the γ_{14} contour fixed and deform the γ_{12} contour to the left picking up the poles. Assuming that the resulting sum over residues converges and exchanging the sum and the γ_{14} integration we get

$$\begin{aligned} F(u,v) &= \left(1 + u^{-\Delta} + v^{-\Delta}\right) \\ &+ \sum_{\tau}\sum_{m=0}^{\infty} u^{-\Delta+\frac{\tau}{2}+m}\int_{\mathrm{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{14}}{2\pi i} v^{-\gamma_{14}} \mathrm{Res}_{\gamma_{12}=\Delta-\frac{\tau}{2}-m}\hat{M}(\gamma_{12},\gamma_{14}), \end{aligned} \tag{4.3}$$

where τ are the twists of the primary operators and the sum over m is a sum over descendants. As we explained in the previous section the residue is given in terms of OPE coefficients and Mack polynomials (3.5)

$$\mathrm{Res}_{\gamma_{12}=\Delta-\frac{\tau_i}{2}-m}\hat{M}(\gamma_{12},\gamma_{14}) = -\frac{1}{2}C^2_{\tau_i}\mathcal{Q}^{\tau_i,d}_{J_i,m}(\gamma_{14})\Gamma^2(\gamma_{14})\Gamma^2\left(\frac{\tau_i}{2}+m-\gamma_{14}\right)\Gamma^2\left(\Delta-\frac{\tau_i}{2}-m\right). \tag{4.4}$$

A factor of $-\frac{1}{2}$ in front comes from the fact that we take the residue in γ_{12}, see (3.5).

Notice that the contour $\mathrm{Re}(\gamma_{14}) = \frac{\Delta}{3}$ falls in between the series of poles produced by the Gamma functions (since $\frac{\tau_{gap}}{2} - \mathrm{Re}(\gamma_{14}) + m > \frac{\Delta}{3} > 0$). Therefore the Mellin integral just reproduces the collinear blocks as required by the OPE. Thus, we find

$$F(u,v) = \sum_{\tau,J} C^2_{\tau,J} g_{\tau,J}(u,v) + \left(1 + v^{-\Delta}\right) + \sum_{n=0}^{\infty} u^n \int_{\mathrm{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{14}}{2\pi i} v^{-\gamma_{14}} R_n(\gamma_{14}), \tag{4.5}$$

where

$$R_n(\gamma_{14}) \equiv \mathrm{Res}_{\gamma_{12}=-n}\hat{M}(\gamma_{12},\gamma_{14}). \tag{4.6}$$

Given that the first sum already gives the full correlator, the other two terms must cancel. This gives the Polyakov conditions

$$\left(1 + v^{-\Delta}\right) + \int_{\mathrm{Re}(\gamma_{14})=\frac{\Delta}{3}} \frac{d\gamma_{14}}{2\pi i} v^{-\gamma_{14}} R_0(\gamma_{14}) = 0 \tag{4.7}$$

and $R_n = 0$ for $n > 0$. However, clearly (4.7) is impossible to satisfy. Indeed, $1 + v^{-\Delta}$ does not admit the usual Mellin representation as required by (4.7).

The resolution of this apparent paradox lies in the fact that our assumption about the convergence of the sum over γ_{12} residues for $\mathrm{Re}[\gamma_{14}] = \frac{\Delta}{3}$ does not hold. To see this we note that the relevant divergence comes from the large J fixed τ operators which are controlled by the light-cone bootstrap [16,17]. To leading order we can therefore simply use the mean field theory OPE data together with (3.13) to get [1]

$$\begin{aligned} &-\frac{1}{2}(C^{GFF}_{\tau=2\Delta,J})^2 \mathcal{Q}^{\tau,d}_{J,0}(\gamma_{14})\Gamma^2(\gamma_{14})\Gamma^2\left(\frac{\tau}{2}-\gamma_{14}\right)\Gamma^2\left(\Delta-\frac{\tau}{2}\right) \\ &= \frac{4}{\Gamma^2(\Delta)}\frac{1}{J}\left(J^{2(\Delta-\gamma_{14})}\Gamma^2(\gamma_{14}) + J^{2\gamma_{14}}\Gamma^2(\Delta-\gamma_{14}) + ...\right), \qquad J\to\infty,\ \ \tau\to 2\Delta, \end{aligned} \tag{4.8}$$

where we omitted the terms that are suppressed at large spin J. Note that the sum over J of the first term in the brackets in the second line of (4.8) converges only for $\mathrm{Re}\,\gamma_{14} > \Delta$, while the second for $\mathrm{Re}\,\gamma_{14} < 0$. Therefore, if we try to evaluate the Mellin integral by closing the γ_{12} contour we run into a divergent sum for any value of γ_{14}.

The resolution is that we should first deform (4.8) into the region where the sum converges. Indeed, let us first deform the contour in the first term (4.8) to $\mathrm{Re}\,\gamma_{14} > \Delta$ and in the second term to $\mathrm{Re}\,\gamma_{14} < 0$. This makes the sum over J convergent and thus we can exchange the order of the sum and the integral. One way to do it is to split the Mellin amplitude back into the K-functions such that two powers of J in (4.8) appear in different K-functions. We then first deform the γ_{14} contour before closing the γ_{12} contour, essentially going back to (2.7). In doing so we encounter extra poles which cancel the disconnected piece and in this way the double counting is avoided.

A simpler way to see it is to note the following. To ensure the convergence of the integral we would like to deform the γ_{14} for each of the two terms in (4.8) to the region, where the sum over J converges. Let us start with the second term. In this case we would like to deform the Mellin integral to the region $\mathrm{Re}\,\gamma_{14} < 0$. It is easy to see that in doing so we encounter a pole. Indeed, $\sum_{J-\text{even}} \frac{1}{J} J^{2\gamma_{14}} \sim -\frac{1}{4\gamma_{14}}$. The residue of this pole produces -1. Similarly, in the first term when continuing to the region $\mathrm{Re}\,\gamma_{14} > \Delta$ we encounter the pole at $\gamma_{14} = \Delta$ with the residue being precisely $-v^{-\Delta}$. We see that by deforming the γ_{14} region so that the sum over the γ_{12} residues converges we precisely canceled the disconnected piece.

Let us quickly check that the deformed integral indeed correctly reproduces the expected light-cone singularity in the dual channel

$$\begin{aligned} \delta f(u,v) = &\int_{\mathrm{Re}(\gamma_{14})<0} \sum_{J>J_0, J-\text{even}}^{\infty} \frac{4}{\Gamma^2(\Delta)}\frac{1}{J}\frac{d\gamma_{14}}{2\pi i} J^{2\gamma_{14}}\Gamma^2(\Delta-\gamma_{14})v^{-\gamma_{14}} \\ &+ \int_{\mathrm{Re}(\gamma_{14})>\Delta} \sum_{J>J_0, J-\text{even}}^{\infty} \frac{4}{\Gamma^2(\Delta)}\frac{1}{J}\frac{d\gamma_{14}}{2\pi i} J^{2(\Delta-\gamma_{14})}\Gamma^2(\gamma_{14})v^{-\gamma_{14}} + ... \end{aligned}$$

[1] Since we are looking at poles accumulating at $\gamma_{12} = 0$ when $J \to \infty$ we can use $\gamma_{13} = \Delta - \gamma_{14}$.

Both integrals converge. They give

$$\delta f(u,v) = \sum_{J>J_0, J-\text{even}}^{\infty} \frac{8J^{-1+2\Delta}}{\Gamma^2(\Delta)} \left(v^{-\Delta} K_0\left(\frac{J}{\sqrt{v}}\right) + K_0(2J\sqrt{v}) \right) + \dots \,. \tag{4.9}$$

Since we are interested in the small v asymptotic we can turn the sum into an integral. The fact that we sum over even J produces an extra factor of $\frac{1}{2}$ and we get

$$\begin{aligned} \delta f(u,v) &= 4\int_0^{\infty} dJ \frac{J^{-1+2\Delta}}{\Gamma^2(\Delta)} \left(v^{-\Delta} K_0\left(\frac{2J}{\sqrt{v}}\right) + K_0(2J\sqrt{v}) \right) + \dots \\ &= 1 + v^{-\Delta} + \dots \end{aligned} \tag{4.10}$$

as expected.

Below we devise a toy model which demonstrates the issue discussed above in a simpler and more controlled setting.

4.3 Toy Model

We can illustrate the general ideas discussed above in a specific example. Consider the following function

$$f(u,v) = e^{-u} \sum_{J=1}^{\infty} u^{-\gamma(J)} e^{-Jv}, \tag{4.11}$$

where $\lim_{J\to\infty} \gamma(J) = 0$. This mimicks the accumulation point in the $u \to 0$ OPE channel. In the dual channel we have the following asymptotic

$$f(u,v) = \frac{e^{-u}}{v} + \dots, \qquad v \to 0 \,. \tag{4.12}$$

We can now compute the Mellin amplitude

$$\hat{M}(\gamma_{12}, \gamma_{14}) \equiv \int_0^{\infty} \frac{du dv}{uv} u^{\gamma_{12}} v^{\gamma_{14}} f(u,v) = \Gamma(\gamma_{14}) \sum_{J=1}^{\infty} J^{-\gamma_{14}} \Gamma(\gamma_{12} - \gamma(J)), \tag{4.13}$$

For $\text{Re}[\gamma_{12}] > \gamma(J)$, $\forall J \in \mathbb{N}$ and $\text{Re}[\gamma_{14}] > 1$. As expected the Mellin amplitude has a pole at $\gamma_{14} = 1$, namely $\hat{M} \sim \frac{\Gamma(\gamma_{12})}{\gamma_{14}-1}$ which is of course consistent with (4.12).

We can also easily write down the expression for the analytic continuation of the Mellin amplitude to $\text{Re}[\gamma_{14}] > 0$

$$\hat{M}(\gamma_{12}, \gamma_{14}) = \Gamma(\gamma_{12})\Gamma(\gamma_{14})\zeta(\gamma_{14}) + \Gamma(\gamma_{14}) \sum_{J=1}^{\infty} J^{-\gamma_{14}} \left(\Gamma(\gamma_{12} - \gamma(J)) - \Gamma(\gamma_{12})\right), \tag{4.14}$$

where $\zeta(x)$ is the Riemann zeta function. Here we assumed that $\gamma(J) \to 0$ at large J not slower than $\frac{1}{J}$.

We can write the inverse Mellin representation

$$\begin{aligned} f(u,v) &= \int_{\mathrm{Re}[\gamma_{12}]>\gamma(J)} \frac{d\gamma_{12}}{2\pi i} \int_{\mathrm{Re}[\gamma_{14}]>1} \frac{d\gamma_{14}}{2\pi i} u^{-\gamma_{12}} v^{-\gamma_{14}} \hat{M}(\gamma_{12},\gamma_{14}) \\ &= \frac{e^{-u}}{v} + \int_{\mathrm{Re}[\gamma_{12}]>\gamma(J)} \frac{d\gamma_{12}}{2\pi i} \int_{0<\mathrm{Re}[\gamma_{14}]<1} \frac{d\gamma_{14}}{2\pi i} u^{-\gamma_{12}} v^{-\gamma_{14}} \hat{M}(\gamma_{12},\gamma_{14}), \end{aligned} \tag{4.15}$$

where in the second line we deformed the contour to extract the leading singularity in the dual channel (which is analogous to the disconnected piece of a CFT correlator).

Now let us try to evaluate the Mellin integral by closing the γ_{12}-contour. As above we can write the contribution of the physical operators as follows

$$\Gamma(\gamma_{14}) \sum_{J=1}^{\infty} J^{-\gamma_{14}} u^{-\gamma(J)}. \tag{4.16}$$

If we are to blindly exchange the sum and the γ_{14} integral we would run into the double counting paradox as in the section above. The resolution of course is that the sum (4.16) converges only for $\mathrm{Re}[\gamma_{14}] > 1$. Therefore, we can only close the γ_{12} contour in the first line of (4.15) and the double counting problem does not arise.

We can also use (4.14) to write for the residues as we deform the γ_{12} contour to $\mathrm{Re}[\gamma_{12}] < 0$

$$\Gamma(\gamma_{14}) \left(\zeta(\gamma_{14}) + \sum_{J=1}^{\infty} J^{-\gamma_{14}} \left(u^{-\gamma(J)} - 1 \right) \right) . \tag{4.17}$$

Note that the expression above is formal in the sense that strictly speaking as we deform the contour we separately get $\sum_{J=1}^{\infty} J^{-\gamma_{14}} u^{-\gamma(J)}$ and $-\sum_{J=1}^{\infty} J^{-\gamma_{14}}$. However for $0 < \mathrm{Re}[\gamma_{14}] < 1$ only the combined sum is well-defined.

Plugging (4.17) into (4.15) and expanding it up to $O(u)$ we get

$$f(u,v) = \frac{1}{v} + \left(\frac{1}{e^v - 1} - \frac{1}{v} \right) + \sum_{J=1}^{\infty} \left(u^{-\gamma(J)} - 1 \right) e^{-vJ} + ... \tag{4.18}$$

$$= \sum_{J=1}^{\infty} u^{-\gamma(J)} e^{-vJ} + ..., \tag{4.19}$$

where $\frac{1}{e^v-1} - \frac{1}{v}$ is the Mellin transform of $\Gamma(\gamma_{14})\zeta(\gamma_{14})$. Thus, we see that again the double counting problem does not arise.

We can now ask what is the behavior of the Mellin amplitude close to the accumulation point. To this extent following the analogy to the light-cone bootstrap let us set $\gamma(J) = \frac{\alpha}{J^\beta}$,

$0 < \alpha < 1$ and $\beta > 0$. For simplicity we can also consider an integral instead of the sum to get

$$\tilde{M} = \int_1^\infty dJ\, J^{-\gamma_{14}} \frac{1}{\gamma_{12} - \frac{\alpha}{J^\beta}} = \frac{{}_2F_1(1, \frac{\gamma_{14}-1}{\beta}, \frac{\gamma_{14}+\beta+1}{\beta}, \frac{\alpha}{\gamma_{12}})}{\gamma_{12}(\gamma_{14}-1)}, \qquad \mathrm{Re}[\gamma_{14}] > 1\,. \tag{4.20}$$

This has the following behavior close to the accumulation point (we can approach it from the regular direction which is $\arg[\gamma_{12}] \neq 0$)

$$\lim_{|\gamma_{12}|\to 0,\ \arg[\gamma_{12}]\neq 0} \tilde{M} = \frac{\pi}{\beta\gamma_{12}} \left(-\frac{\alpha}{\gamma_{12}}\right)^{\frac{1-\gamma_{14}}{\beta}} \frac{1}{\sin\frac{\pi}{\beta}(\gamma_{14}-1)} + ..., \quad \mathrm{Re}[\gamma_{14}] > 1, \tag{4.21}$$

where we suppressed regular terms. Therefore the accumulation point behaves like a branch point with the asymptotic controlled by the "large spin OPE data". However, for $\mathrm{Re}[\gamma_{14}] > 1$, it is not a branch point because there is no monodromy, *i.e.* we can do a contour integral around it (going in between the poles for $\gamma_{12} > 0$) and the result is just the convergent sum of the residues of the enclosed poles. Note also that in the region of convergence, namely $\mathrm{Re}[\gamma_{14}] > 1$ we have

$$\lim_{|\gamma_{12}|\to 0,\ \arg[\gamma_{12}]\neq 0} \gamma_{12}\tilde{M} = 0, \quad \mathrm{Re}[\gamma_{14}] > 1. \tag{4.22}$$

which states that we do not have a double trace operator at $\gamma_{12} = 0$. Below we will see that this is the relevant condition for the nonperturbative Mellin amplitudes.

This complicated behavior has to be contrasted with the perturbative behavior. Indeed, if we think of $\alpha = \frac{1}{c_T} \sim \frac{1}{N^2} \to 0$ we get order by order a very simple expansion

$$\tilde{M} = \frac{1}{\gamma_{12}(\gamma_{14}-1)} + \frac{\alpha}{\gamma_{12}^2(\gamma_{14}+\beta-1)} + \frac{\alpha^2}{\gamma_{12}^3(\gamma_{14}+2\beta-1)} + ...\,, \tag{4.23}$$

which is much simpler than the "non-perturbative" limit (4.21). Notice also that the condition (4.22) is genuinely nonperturbative. If we try to plug the "large N" expansion (4.23) in (4.22) we see that only the leading term produces a finite result $\frac{1}{\gamma_{14}-1}$, whereas the higher terms in $\alpha \sim \frac{1}{N^2}$ produce infinity.

4.4 Nonperturbative Polyakov Conditions

We are now ready to formulate the Polyakov conditions for nonperturbative amplitudes. In light of the discussion above we consider the derivative of the correlator

$$-u\partial_u F(u,v) \;=\; \Delta u^{-\Delta} + \int_{\mathcal{C}} \frac{d\gamma_{12}}{2\pi i}\frac{d\gamma_{14}}{2\pi i} \gamma_{12}\hat{M}(\gamma_{12},\gamma_{14}) u^{-\gamma_{12}} v^{-\gamma_{14}}. \tag{4.24}$$

Note that since $-u\partial_u(1+v^{-\Delta}) = 0$ if we are to close the γ_{12} contour we will not run into the double counting problem described above. Correspondingly, close to the first twist accumulation point $\gamma_{12} = \Delta - \frac{\tau_{[\mathcal{O},\mathcal{O}]_{0,J}}}{2} = -\frac{\gamma_{[\mathcal{O},\mathcal{O}]_{0,J}}}{2} \sim \frac{1}{J^{\tau_{gap}}}$ which improves the convergence of the sum over spins in (4.8).[2] In particular, the sum over J of both terms in (4.8) multiplied by $\frac{1}{J^{\tau_{gap}}}$ converges for

$$\frac{\tau_{gap}}{2} > \text{Re}[\gamma_{14}] > \Delta - \frac{\tau_{gap}}{2}. \tag{4.25}$$

Notice that our original assumption $\Delta < \frac{3}{4}\tau_{gap}$ guarantees that there are allowed values of γ_{14} compatible with this condition.

To analyze the behavior of the Mellin amplitude close to the branch point we can use the toy model from the previous section. As in the toy model example above, we conclude that the presence of the double-twist trajectory makes $\gamma_{12} = 0$ look like a branch point, see (4.21). Similarly, we conclude that

$$\lim_{|\gamma_{12}|\to 0,\ \arg[\gamma_{12}]\neq 0} \gamma_{12}\left(\gamma_{12}\hat{M}(\gamma_{12},\gamma_{14})\right) = 0, \qquad \frac{\tau_{gap}}{2} > \text{Re}[\gamma_{14}] > \Delta - \frac{\tau_{gap}}{2}. \tag{4.26}$$

In other words, the higher spin tail produces a contribution which is softer than a pole.

Let us now translate this condition to a statement about the Mellin amplitude $M(\gamma_{12},\gamma_{14})$ itself. Recall that due to the pre-factor $\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{13})\Gamma^2(\gamma_{14})$ that relates $\hat{M}$ to M and which has a double pole at $\gamma_{12} = 0$ we can rewrite the condition above as follows

$$M(\gamma_{12}=0,\gamma_{14}) = 0, \qquad \frac{\tau_{gap}}{2} > \text{Re}[\gamma_{14}] > \Delta - \frac{\tau_{gap}}{2}, \tag{4.27}$$

where we set $\gamma_{12} = 0$ by approaching the accumulation point from any direction with $\arg[\gamma_{12}] \neq 0$. The condition (4.27) is the central result of this section. Note that the non-perturbative Polyakov condition is very subtle. In particular, we cannot argue that the Mellin amplitude has a double zero at $\gamma_{12} = 0$ and similarly we cannot simply go to the accumulation points with twists $2\Delta + 2n$. Finally, the condition is a genuinely nonperturbative (finite N) condition.

4.5 More Polyakov conditions

With the same methods as in the previous section, one can see that close to the accumulation point $\gamma_{12} = -p$,

$$M(\gamma_{12},\gamma_{13}) \sim (\gamma_{12}+p)^{1+2\frac{\gamma_{13}-\Delta}{\tau_{gap}}} + (\gamma_{12}+p)^{1+2\frac{\gamma_{14}-\Delta}{\tau_{gap}}} + O(\gamma_{12}+p)^2\,. \tag{4.28}$$

[2] By $[\mathcal{O},\mathcal{O}]_{n,J}$ we as usual denote a family of the double-twist operators that approach twist $2\Delta+2n$ at infinite spin.

We conclude that the Polyakov conditions

$$M(\gamma_{12} = -p, \gamma_{13}) = 0 \qquad \text{and} \qquad \partial_{\gamma_{12}} M(\gamma_{12} = -p, \gamma_{13}) = 0 \tag{4.29}$$

can be imposed, respectively, in

$$\Delta - \frac{\tau_{gap}}{2} < \operatorname{Re}\gamma_{13} < p + \frac{\tau_{gap}}{2}, \tag{4.30}$$

$$\Delta < \operatorname{Re}\gamma_{13} < p. \tag{4.31}$$

These correspond to the red and green regions of figure 4.1.

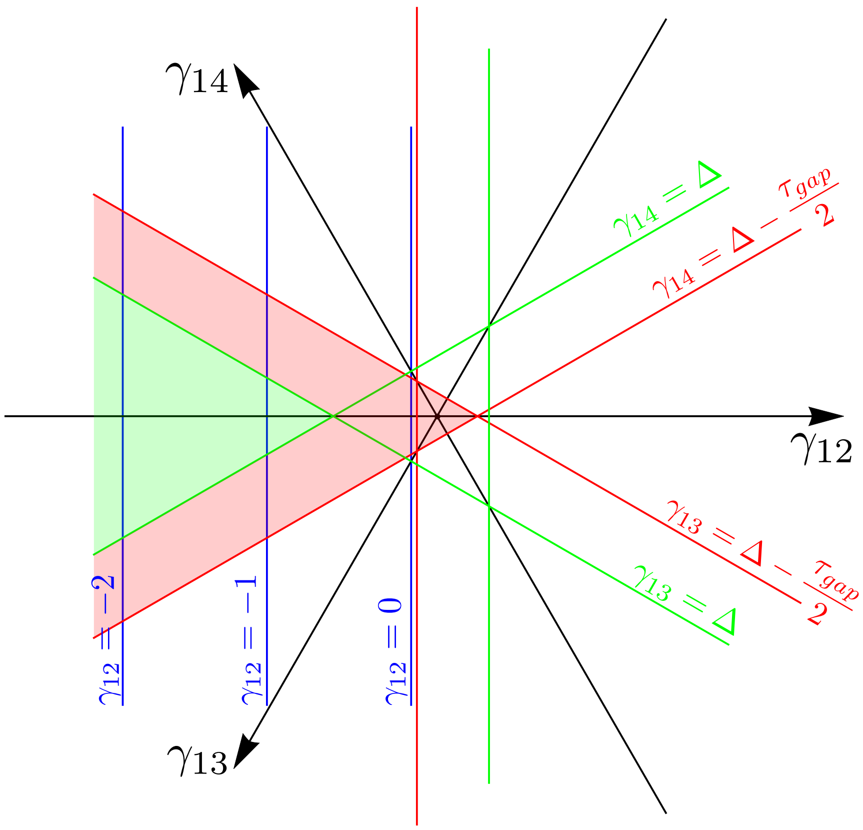

Figure 4.1. The Mellin-Mandelstam plane. The axis at 120° ensure that every point on the plane satisfies $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$. The accumulation points $\gamma_{12} = -n$ for $n = 0, 1, \ldots$ are shown in blue. The region of convergence for sum rules with a function F with a simple/double pole at one of these points is shown in red/green. Notice that the red region contains the green region.

Exact double twist operators We tacitly assumed that the s-channel OPE, or $x_{12}^2 \to 0$, expansion of the correlator does not contain operators of twist $\tau = 2\Delta$. This is the case for the sum rules that we analyze in this thesis and in a generic CFT. More generally, we can have

$$M(\gamma_{12} = -p, \gamma_{13}) = a_p(\gamma_{13}) \qquad \text{and} \qquad \partial_{\gamma_{12}} M(\gamma_{12} = -p, \gamma_{13}) = b_p(\gamma_{13}) \tag{4.32}$$

if the s-channel OPE contains the derivative of the conformal block $\partial_\Delta G_{2\Delta+2n+\ell,\ell}(u,v)$ with $n \leqslant p$, which contribute to the former, and conformal blocks $G_{2\Delta+2n+\ell,\ell}(u,v)$ also with $n \leqslant p$, which contribute to the latter. This fact was recently discussed in detail in [13].

Imagine we have an operator of twist $\tau = 2\Delta + 2n$ with $n \geqslant 0$ and spin ℓ that appears in the OPE. It is convenient to take τ slightly away from this value and then take the limit $\tau \to 2\Delta + 2n$. We can compute the behaviour of the factor $M(\gamma_{12}, \gamma_{13})\Gamma^2(\gamma_{12})$ close to the pole at $\gamma_{12} = \Delta - \frac{\tau}{2} - (p-n)$ where $p \geqslant n$. This gives

$$\begin{aligned} M(\gamma_{12}, \gamma_{13})\Gamma^2(\gamma_{12}) &\approx \frac{-\frac{1}{2} C^2_{\tau,\ell} \mathcal{Q}^{\tau,d}_{\ell,p-n}(-2\gamma_{13})}{\gamma_{12} - (\Delta - \frac{\tau}{2} - (p-n))} \Gamma^2\left(\Delta - \frac{\tau}{2} - (p-n)\right) \\ &\xrightarrow[\tau \to 2\Delta+2n]{} \frac{-\frac{1}{2} C^2_{2\Delta+2n,\ell} \tilde{\mathcal{Q}}^{2\Delta+2n,d}_{\ell,p-n}(-2\gamma_{13})}{\gamma_{12} + p}, \end{aligned} \tag{4.33}$$

where

$$\tilde{\mathcal{Q}}^{2\Delta+2n,d}_{\ell,p}(-2\gamma_{13}) \equiv \lim_{\tau \to 2\Delta+2n} \mathcal{Q}^{\tau,d}_{\ell,p}(-2\gamma_{13}) \Gamma^2\left(\Delta - \frac{\tau}{2} - (p-n)\right). \tag{4.34}$$

This means that an exact double twist operator with $\tau = 2\Delta + 2n$ gives rise to simple poles in $\hat{M}(\gamma_{12}, \gamma_{14})$ at $\gamma_{12} = -p$ for all integer $p \geq n$. From this fact together with $\lim_{\gamma_{12} \to -p} \Gamma^2(\gamma_{12})(\gamma_{12} + p)^2 = \frac{1}{p!^2}$ we conclude that the contribution of such an operator to (4.32) is

$$a_p(\gamma_{13}) = 0, \quad b_p(\gamma_{13}) = -\frac{(p!)^2}{2} C^2_{2\Delta+2n,\ell} \tilde{\mathcal{Q}}^{2\Delta+2n,d}_{\ell,p-n}(-2\gamma_{13}), \qquad p \geq n. \tag{4.35}$$

Applications of Mellin amplitudes Part II

5 Dispersion relations

5.1 Introduction

After establishing the basic properties of nonperturbative CFT Mellin amplitudes we would like to consider some applications. The strategy we adopt relies on all the properties that we established in the previous chapters. We use analyticity and polynomial boundedness of Mellin amplitudes to write down subtracted dispersion relations. We then impose crossing to simplify them and we finally impose the nonperturbative Polyakov condition (4.27). The result of all this is a set of linear functionals that act on the OPE data and give zero.

These functionals have some interesting properties. They annihilate generalized free field theory. They are non-negative for heavy operators and have double zeros at the positions of the double twist operators $[\mathcal{O}, \mathcal{O}]_{n,J}$. We find that in this way they are particularly suitable for studies of large N holographic CFTs.

We check the overall consistency of our construction by applying the functionals to the OPE data of the 3d Ising model. After this successful and nontrivial test we move on and apply them to some simple holographic theories.

5.2 Subtractions and Polyakov condition

For simplicity in this section we consider a limited class of CFTs in $d > 2$ for which the analysis is particularly simple. We assume that the theory admits a scalar primary operator with dimension Δ and the twist gap τ_{gap} such that $\Delta < \frac{3}{4}\tau_{gap}$. As we argued above in this case we can write down the Mellin representation for the connected correlator with the straight contour

$$F_{conn}(u, v) = \int_{\mathcal{C}} \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} \, u^{-\gamma_{12}} v^{-\gamma_{14}} \Gamma(\gamma_{12})^2 \Gamma(\gamma_{14})^2 \Gamma(\Delta - \gamma_{12} - \gamma_{14})^2 M(\gamma_{12}, \gamma_{14})$$

$$\mathcal{C}: \Delta - \frac{\tau_{gap}}{2} < \mathrm{Re}(\gamma_{12}), \mathrm{Re}(\gamma_{14}), \quad \mathrm{Re}(\gamma_{12}) + \mathrm{Re}(\gamma_{14}) < \frac{\tau_{gap}}{2} \,. \tag{5.1}$$

Furthermore, we shall assume that $\tau_{gap} = d - 2$ corresponds to the stress tensor operator.

Let us consider dispersion relations for the Mellin amplitude $M(\gamma_{12}, \gamma_{13}, \gamma_{14})$, where $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$, at fixed γ_{13}. According to (3.28), the Mellin amplitude is bounded by the linear growth $|\gamma_{12}|$ for fixed $\mathrm{Re}(\gamma_{13}) > \Delta - \tau_{gap}/2$. In particular, this includes a neighbourhood of $\gamma_{13} = \frac{\Delta}{3}$. Therefore we can write the fixed γ_{13} subtracted dispersion relation as follows

$$\begin{aligned}
\frac{M(\gamma_{12}, \gamma_{13}, \gamma_{14})}{(\gamma_{12} - \frac{\Delta}{3})(\gamma_{13} - \frac{\Delta}{3})(\gamma_{14} - \frac{\Delta}{3})} &= \oint_{\gamma_{12}} \frac{d\gamma}{2\pi i} \frac{1}{\gamma - \gamma_{12}} \frac{M(\gamma, \gamma_{13}, \Delta - \gamma_{13} - \gamma)}{(\gamma - \frac{\Delta}{3})(\gamma_{13} - \frac{\Delta}{3})(\Delta - \gamma_{13} - \gamma - \frac{\Delta}{3})} \\
&= -\frac{M(\frac{\Delta}{3}, \gamma_{13}, \frac{2\Delta}{3} - \gamma_{13})}{(\gamma_{13} - \frac{\Delta}{3})^2} \left(\frac{1}{\gamma_{12} - \frac{\Delta}{3}} + \frac{1}{\gamma_{14} - \frac{\Delta}{3}} \right) \\
-\frac{1}{2} \sum_{\tau, J, m} & \frac{C^2_{\tau,J} \mathcal{Q}^{\tau,d}_{J,m}(\gamma_{13})}{(\Delta - \frac{\tau}{2} - m - \frac{\Delta}{3})(\gamma_{13} - \frac{\Delta}{3})(\Delta - \gamma_{13} - (\Delta - \frac{\tau}{2} - m) - \frac{\Delta}{3})} \\
& \times \left(\frac{1}{\gamma_{12} - \Delta + \frac{\tau}{2} + m} + \frac{1}{\gamma_{14} - \Delta + \frac{\tau}{2} + m} \right) ,
\end{aligned} \tag{5.2}$$

where the last expression was obtained by opening up the contour integral in the first line and picking up all the poles in the γ complex plane. In the last expression, we used the fact that $M(\gamma_{12}, \gamma_{13}, \gamma_{14}) = M(\gamma_{14}, \gamma_{13}, \gamma_{12})$. Crossing symmetry further implies that

$$M\left(\frac{\Delta}{3}, \gamma_{13}, \frac{2\Delta}{3} - \gamma_{13}\right) = G\left(\left(\gamma_{13} - \frac{\Delta}{3}\right)^2\right) . \tag{5.3}$$

We can solve for derivatives of $G(x^2)$ at $x^2 = 0$ in terms of the OPE data. To do it we evaluate the formula above as $M(\frac{\Delta}{3} - x, \frac{\Delta}{3}, \frac{\Delta}{3} + x) = G(x^2)$, expand in x and solve for derivatives of G. We should remark at this point that as usual doing subtractions in dispersion relations is a matter of choice. We will comment on other choices below.

We next consider the nonperturbative Polyakov condition (4.27). More precisely, we expand the condition $M(0, \gamma_{13}, \Delta - \gamma_{13}) = 0$ around $\gamma_{13} = \frac{\Delta}{3}$. In this way, we get an infinite set of equations the simplest of which takes the following form

$$\sum_{\tau,J,m} C^2_{\tau,J} \alpha_{\tau,J,m} = 0, \;\; \alpha_{\hat{1}} = \alpha_{0,0,m} = 0 \tag{5.4}$$

$$\alpha_{\tau,J,m} = -\frac{16\Delta}{3(\tau - \frac{2\Delta}{3} + 2m)(\tau - \frac{4\Delta}{3} + 2m)} \left(\frac{(\tau + 2m - \Delta)\mathcal{Q}^{\tau,d}_{J,m}(\frac{\Delta}{3})}{(\tau - \frac{2\Delta}{3} + 2m)(\tau - \frac{4\Delta}{3} + 2m)} - \frac{\Delta}{3} \frac{\mathcal{Q}^{\tau,d}_{J,m}(\frac{\Delta}{3})'}{\tau + 2m - 2\Delta} \right) ,$$

where in the last line we explicitly wrote that the identity operator does not contribute to (5.4). In other words, we arrived at a particular set of linear functionals that act on the OPE data. They have very interesting properties that we describe below.

First, note that $\mathcal{Q}^{\tau,d}_{J,m}$ have double zeros at the position of double twist operators $[\mathcal{O},\mathcal{O}]_{n,J}$. This directly translates to the fact that $\alpha_{\tau,J,m}$ have double zeros for $\tau = 2\Delta + 2n$ with $n \geqslant 1$. For $n = 0$ and $J \neq 0$, $\alpha_{2\Delta,J,m}$ has a single zero due to the extra pole in the second term in (5.4) for $m = 0$. Therefore, generalized free fields automatically satisfy the sum rule (5.4).

Second, we find that

$$\alpha_{\tau,J,m} \geqslant 0, \qquad \tau \geqslant 2\Delta\,, \quad J \in 2\mathbb{N}\,, \quad m \geq 0\,, \tag{5.5}$$

where the only zeros of $\alpha_{\tau,J,m}$ for $\tau \geqslant 2\Delta$ are the ones at the positions of the double twist operators described above. We elaborate on evidence for this claim (that we do not prove) below in section 5.2.1. Therefore we conclude that the functionals above are extremal (in the sense of [48, 49]) for the following bootstrap problem: *Find the maximal value of τ_0 for which there exist a unitary solution to the crossing equations with all twists $\tau \geq \tau_0$ for all spins J (apart from the identity operator).*

From the properties of the functionals described above it immediately follows that there are no nontrivial solutions to crossing equations that satisfy $\tau_0 > 2\Delta$. For $\tau_0 = 2\Delta$ the only solution to crossing with this property is GFF. Note that in CFTs presence of the stress tensor in the spectrum and unitarity bound for scalar operators immediately imply this, see e.g. [17]. However, the claim above applies as well for non-local CFTs, say AdS QFTs, which do not have a stress tensor.

We have to emphasize that we have not proven (5.5). But we did exhaustive tests to the best of our knowledge. These include $m = 0$ and any spin and arbitrary m for low spin. It would be helpful to prove (5.5) rigorously to put our results on a more solid ground.

Third, let us comment on the convergence of the sum (5.4). Using the standard results of the light-cone bootstrap we can explicitly check the convergence at large J. We can also check the convergence of the sum at large Δ and fixed J using the results of [45]. Note that the contribution of the heavy operators to the sum rule (5.4) is suppressed like a power of τ independent of Δ, see appendix H.

Finally, let us mention that in the derivation above we can also find functionals that do not rely on the Polyakov conditions, solely from crossing symmetry. As an example we get

$$\sum_{\tau,J,m} C^2_{\tau,J}\beta_{\tau,J,m} = 0, \tag{5.6}$$

$$\beta_{\tau,J,m} = 6\frac{\mathcal{Q}^{\tau,d}_{J,m}(\frac{\Delta}{3})'}{(\tau - \frac{4\Delta}{3} + 2m)^4} + \frac{\mathcal{Q}^{\tau,d}_{J,m}(\frac{\Delta}{3})''}{(\tau - \frac{4\Delta}{3} + 2m)^3} ,$$
$$\beta_{\hat{1}} = \beta_{0,0,m} = 0 .$$

One can easily check, however, that this functional does not have the crucial positivity property (5.5). For that reason below we use (5.4).

5.2.1 Positivity of $\alpha_{\tau,J}$

Here we elaborate on our claim (5.5) above. We restrict our consideration only to the relevant case of $d \geqslant 3$. Let us emphasize that we do not prove (5.5) but present evidence for it to the best of our current knowledge. The reason being that computing high spin Mack polynomials up to arbitrary spin J and checking positivity in the three-dimensional parameter space of Δ, τ, m is a computationally difficult task. Therefore, we could only analyze (5.5) explicitly for low spins $0 \leqslant J \leqslant 40$, as well as for arbitrary spins in some limits, namely the flat space limit and for collinear Mack polynomials $m = 0$.

Strictly speaking, what we care about is only the positivity properties of $\alpha_{\tau,J} = \sum_{m=0}^{\infty} \alpha_{\tau,J,m}$ and not positivity of each descendant $\alpha_{\tau,J,m}$ separately. In practice however we found it much easier to analyze $\alpha_{\tau,J,m}$ for fixed m. It would be very interesting to improve our analysis in this regard.

We start by analyzing (5.5) for low spins, namely $J = 0, ..., 26$. The simplest way to check positivity is to fix the external dimension to some particular value. Foreseeing our holographic consideration below we can fix $\Delta = \frac{5}{8}(d-2)$. Then setting $d = 3 + \delta d$ and $\tau = 2\Delta + \delta\tau$ we checked that all $\alpha_{\tau,J,m}$ are polynomials in $\delta d, \delta\tau, m \geqslant 0$ with positive coefficients.[1] Similarly, as is relevant for our case setting $\Delta = c_\delta(d-2)$ with $\frac{1}{2} < c_\delta < \frac{3}{4}$ we checked that the same property holds for $c_\delta = 0.51, 0.52, ..., 0.74$, for spins $J = 0, ..., 16$. Keeping Δ general and writing $\Delta = \frac{d-2}{2} + \delta$ we observed that for $J = 0, 2, 4$ the same manifest positivity holds (this time polynomial also includes powers of δ). However, starting from $J = 6$ the polynomial is not manifestly positive. Restricting to particular low values of Δ we have not observed any violations of positivity but the simple analytic argument that we presented above does not hold in this case. One simple analytic check in this more general case is to consider the $m \gg 1$ limit. We computed such a limit for the cases $J = 0, ..., 12$ and found that the functional is positive for any $\delta d, \delta\tau \geqslant 0$ and $\Delta \geqslant \frac{d-2}{2}$.

Another test of (5.5) is the flat space limit. Indeed, we consider $m, \tau \gg 1$ with $\frac{\tau}{m}$ fixed. In this case we can use (3.10) to evaluate the functional. The result is that to leading order in the large τ, m it is proportional to $C_J^{(\frac{d-2}{2})}\left(\frac{\frac{\tau}{2}+m}{\sqrt{m(m+\tau)}}\right) \geqslant 0$.

[1] In $d = 4$ and $\Delta = \frac{5}{8}$ we checked positivity of the functional up to spin 40.

Finally, we set $m = 0$ and used collinear Mack polynomials to perform the large J tests. At large J, the leading contribution to the functional is given by

$$\frac{\Delta^2 J^{\frac{1}{2}-\frac{2\Delta}{3}} \log(J) 2^{2\Delta+\delta\tau+2J+5}}{\sqrt{\pi}\delta\tau(2\Delta+3\delta\tau)(4\Delta+3\delta\tau)\Gamma(-\frac{\delta\tau}{2})^2\Gamma(\frac{2\Delta}{3}+\frac{\delta\tau}{2})^2}, \tag{5.7}$$

where we set $\tau = 2\Delta + \delta\tau$. If $\delta\tau > 0$, the functional is positive.

5.3 3d Ising

One example of the situation above is given by the correlator $\langle\sigma\sigma\sigma\sigma\rangle$ in the 3d Ising model, which has scaling dimension $\Delta_\sigma \approx 0.518$ and where the twist gap is controlled by the stress tensor $\tau_{gap} = 1$, so that as required we have $\Delta_\sigma < \frac{3}{4}$. We conclude that

$$\begin{aligned} \mathcal{C} \quad : \quad & \Delta_\sigma - \frac{1}{2} < \mathrm{Re}(\gamma_{12}), \mathrm{Re}(\gamma_{14}), \\ & \mathrm{Re}(\gamma_{12}) + \mathrm{Re}(\gamma_{14}) < \frac{1}{2}\,. \end{aligned} \tag{5.8}$$

and that the connected part of $\langle\sigma\sigma\sigma\sigma\rangle$ in the 3d Ising model admits the Mellin representation with a straight contour

$$\begin{aligned} F^{3d}_{Ising}(u,v) &= 1 + u^{-\Delta_\sigma} + v^{-\Delta_\sigma} \\ &+ \int_{\mathcal{C}} \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2}\, u^{-\gamma_{12}} v^{-\gamma_{14}} \Gamma(\gamma_{12})^2 \Gamma(\gamma_{14})^2 \Gamma(\Delta_\sigma - \gamma_{12} - \gamma_{14})^2 M^{3d}(\gamma_{12},\gamma_{14}). \\ \mathcal{C} \quad &: \quad \Delta_\sigma - \frac{1}{2} < \mathrm{Re}(\gamma_{12}), \mathrm{Re}(\gamma_{14}), \quad \mathrm{Re}(\gamma_{12}) + \mathrm{Re}(\gamma_{14}) < \frac{1}{2}\,. \end{aligned} \tag{5.9}$$

Next we analyze the sum rule (5.4). We can rewrite the sum rule (5.4) as follows

$$\begin{aligned} -\sum_{\tau<2\Delta_\sigma, J>0, m} C^2_{\tau,J}\alpha_{\tau,J} &= \sum_{\tau>2\Delta_\sigma, J>0, m} C^2_{\tau,J}\alpha_{\tau,J} + \sum_{\tau, J=0} C^2_{\tau,J}\alpha_{\tau,J}, \\ \alpha_{\tau,J} &= \sum_{m=0}^{\infty} \alpha_{\tau,J,m}\,, \end{aligned} \tag{5.10}$$

where m is a sum over descendants. We see that the leading twist Regge trajectory is mapped to the rest of the spectrum.

Using the results from [50] we get the following numerical values for some terms in the relation above

$$\begin{aligned} 0.0924 &= 0.028968_{T_{\mu\nu}} + 0.012122_{J=4} + 0.029107_{6\leqslant J\leqslant 30} + 0.0222_{J>30} \\ &= 0.084569_{\epsilon} + 0.0018_{[\sigma,\sigma]_1^{0\leqslant J\leqslant 30}} + 0.0016_{[\epsilon,\epsilon]_0^{4\leqslant J\leqslant 30}} + 0.0014_{[\epsilon,\epsilon]_0^{J\geqslant 32}} + \ldots \end{aligned} \tag{5.11}$$

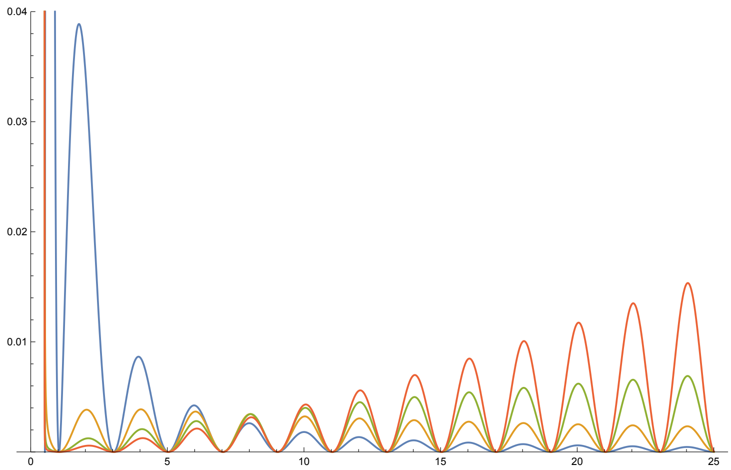

Figure 5.1. Functionals $\alpha_{\tau,0,m}$ as a function of the twist τ. They are non-negative with double zeros at the position of double trace operators $\tau_n = 2\Delta_\sigma + 2n$. Different colors correspond to the contribution of descendants labeled by m. The external dimension is set to its numerical value in the 3d Ising model $\Delta_\sigma \simeq 0.518$.

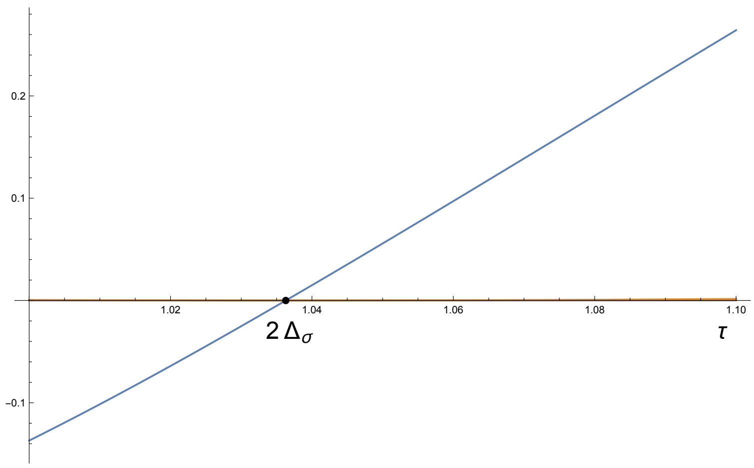

Figure 5.2. When acting on operators with spin the functionals $\alpha_{\tau,J,m}$ are negative for operators with twist $\tau < 2\Delta_\sigma$ and non-negative for $\tau > 2\Delta_\sigma$ with double zeros at the position of double trace operators $\tau_n = 2\Delta_\sigma + 2n$ with $n \geqslant 1$. Here we plot the result for $J = 2$. Different colors correspond to the contribution of descendants labeled by m. The external dimension is set to its numerical value in the 3d Ising model $\Delta_\sigma \simeq 0.518$.

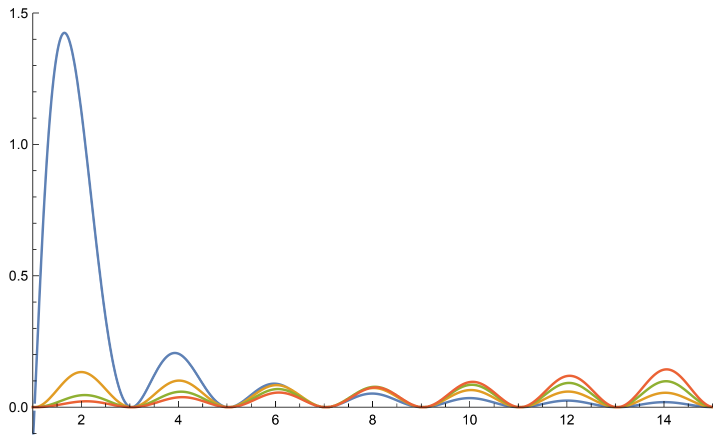

Figure 5.3. Same as figure 5.2 but with an extended range of twists τ plotted.

where we indicated explicitly the contribution of which operators we took into account. In the first line we computed the contribution of $J > 30$ currents using the light-cone bootstrap formulae from [50]. Similarly, in the second line for the higher spin tail of $[\epsilon, \epsilon]_0$ we used the formulae from [51] and the contribution of descendants (terms with $m \geqslant 1$ in (5.4)). All dropped operators in the second line of (5.11) contribute positively. Note also that the contribution of the heavy operators is only suppressed by a power of Δ. We consider therefore a 5% difference between the LHS and the RHS for the included operators to be reasonable. It would be great to check the sum rule above in the 3d Ising model with a greater precision by including more operators in the RHS of (5.10).

Similarly, we checked that the β functionals (5.6) that do not receive contributions from the scalar operators lead to reasonable numbers. We also observed that the β functional sum rules are more sensitive to higher spin operators.

5.4 Bounds on holographic CFTs

Let us now apply (5.4) to holographic CFTs, namely a CFT with large central charge $c_T \gg 1$ [52, 53]. As the simplest example we can consider a free massive scalar in AdS coupled to another field dual to a single trace operator $\tilde{\mathcal{O}}_{st}$ (for example, another scalar field or graviton). We restrict our consideration to external scalars which satisfy $\Delta < \frac{3}{4}\tau_{\tilde{\mathcal{O}}_{st}}$.

If we simply consider a free massive scalar in AdS the sum rule (5.4) is trivially satisfied. Indeed, as we emphasized above $\alpha_{2\Delta+2n,J,m} = 0$. However, as we weakly couple our free scalar field to another field it is not at all obvious that (5.4) is satisfied. As we emphasized several times above the sum rule (5.4) is essentially nonperturbative in c_T. For example, in deriving it we used the nonperturbative Regge bound as well as Polyakov conditions. Neither holds in perturbation theory in c_T. This is in a stark contrast with [54] where perturbation theory in AdS was mapped to solutions to crossing perturbative in $1/c_T$.

Due to the nonperturbative nature of (5.4) we cannot simply expand it in $\frac{1}{c_T}$. However, we can isolate some parts of it which can be safely computed using the low-energy physics from those sensitive to the details of the UV completion. To that extent we write the sum rule as follows

$$C^2_{\tilde{\mathcal{O}}_{st}} \alpha_{\tau_{\tilde{\mathcal{O}}_{st}}, J_{\tilde{\mathcal{O}}_{st}}} + \sum_{J>0} C^2_{[\mathcal{O},\mathcal{O}]_{0,J}} \alpha_{\tau_{[\mathcal{O},\mathcal{O}]_{0,J}}, J} + \text{rest}_{UV} = 0\,, \tag{5.12}$$

where the details of the UV completion are in rest_{UV} which is non-negative due to (5.5).

We are, thus, left with computing the contribution due to the leading twist double traces. Note that due to a single zero of the functional at $\tau = 2\Delta$, to leading order in $C^2_{\tilde{\mathcal{O}}_{st}} \sim \frac{1}{c_T}$ we get the result $\sim (C^{GFF}_{[\mathcal{O},\mathcal{O}]_{0,J}})^2 \gamma_{[\mathcal{O},\mathcal{O}]_{0,J}} \sim \frac{1}{c_T}$ with $J > 0$, where $\gamma_{[\mathcal{O},\mathcal{O}]_{0,J}}$ is the anomalous dimension of double trace operators [2]. The anomalous dimensions $\gamma_{[\mathcal{O},\mathcal{O}]_{0,J}}$ are observables that can be reliably computed using the low-energy theory in AdS.

Let us first consider an example when the external scalar is coupled to another scalar, namely $J_{\tilde{\mathcal{O}}_{st}} = 0$. The contribution from the scalar exchange to the double trace operators can be found for example in [55]. In this paper the relevant OPE data was computed for all J using the Lorentzian inversion formula instead of computing the relevant Witten diagrams. In this case we numerically observed that the contributions exactly cancel to leading order in $\frac{1}{c_T}$

$$C^2_{\tilde{\mathcal{O}}_{st}} \alpha_{\Delta_{\tilde{\mathcal{O}}_{st}}, 0} + \sum_{J>0} C^2_{[\mathcal{O},\mathcal{O}]_{0,J}} \alpha_{\tau_{[\mathcal{O},\mathcal{O}]_{0,J}}, J} + O\left(\frac{1}{c_T^2}\right) = 0\,. \tag{5.13}$$

We provide more details on this calculation in appendix I.

Next we consider a scalar minimally coupled to gravity. In this case we can use the results of [53, 56] for the anomalous dimensions of double trace operators. One subtlety in this case is that there is a contribution at $J = 0, 2$ which is non-analytic in spin. In this case, (experimenting across d and $\frac{d-2}{2} < \Delta < \frac{3}{2}\frac{d-2}{2}$) we find that

$$C^2_{T_{\mu\nu}} \alpha_{d-2,2} + \sum_{J>0} C^2_{[\mathcal{O},\mathcal{O}]_{0,J}} \alpha_{\tau_{[\mathcal{O},\mathcal{O}]_{0,J}}, J} = -\frac{a(d,\Delta)}{c_T} + O\left(\frac{1}{c_T^2}\right)\,, \qquad a(d,\Delta) > 0\,. \tag{5.14}$$

The results for $d = 4$ and $d = 3$ are presented on fig. 5.4 and fig. 5.5 correspondingly. See appendix I for more details on this. Therefore, we conclude that the rest of the sum

[2] The notation $C^2_{\tilde{\mathcal{O}}_{st}} \sim \frac{1}{c_T}$ is only precise if the exchanged single-trace operator is the stress tensor. In the other cases, we think of $1/c_T$ as the square of the small cubic coupling in AdS.

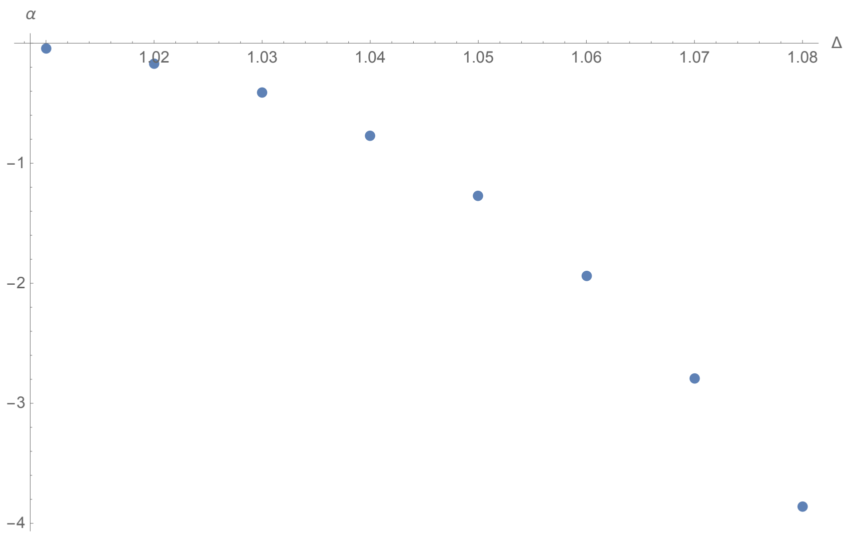

Figure 5.4. We consider a scalar minimally coupled to gravity in AdS_5 or CFT_4. We imagine that the gravitational coupling is weak, or, equivalently, $c_T \gg 1$. The α-functional (5.4) can be applied to $1 < \Delta < 1.5$. We plot the sum given by (5.14). We find that the sum is always negative within the region of applicability of the functional.

in (5.4) must give

$$\text{rest}_{UV} = \sum_{J,\tau>2\Delta} C^2_{\tau,J}\alpha_{\tau,J} = \frac{a(d,\Delta)}{c_T} + O\left(\frac{1}{c_T^2}\right). \tag{5.15}$$

It would be very interesting to understand what operators produce this contribution.

Notice that the IR contribution is always negative. If it were positive in some window, it would lead to the dramatic conclusion that a scalar weakly coupled to gravity is an inconsistent theory in that window. In section (7.5) we explain why the sum rule gives 0 for scalar exchange, but returns a nonzero result for graviton exchange.

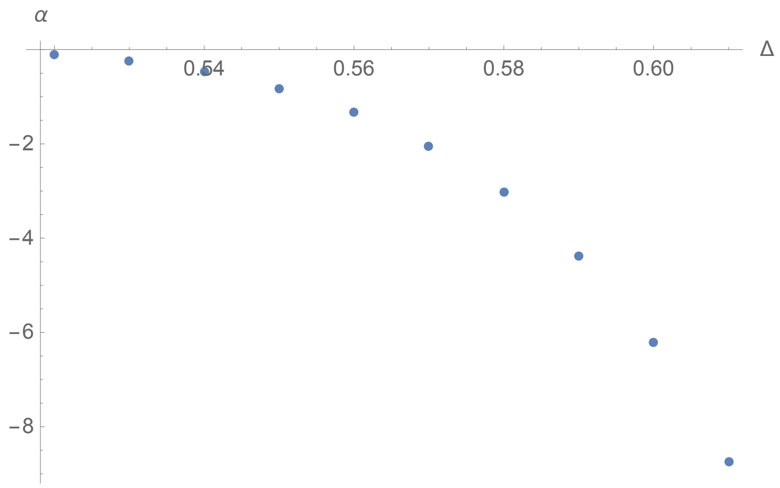

Figure 5.5. We consider a scalar minimally coupled to gravity in AdS_4 or CFT_3. We imagine that the gravitational coupling is weak, or, equivalently, $c_T \gg 1$. The α-functional (5.4) can be applied to $\frac{1}{2} < \Delta < \frac{3}{4}$. We plot the sum given by (5.14). We find that the sum is always negative within the region of applicability of the functional.

6 ϵ expansion

6.1 Introduction

In this chapter we develop more systematically functionals derived using dispersion relations in Mellin space and use them to study the four-point function of the fundamental scalar field ϕ in the Wilson-Fisher (WF) model in $d = 4 - \epsilon$ dimensions [57]. We reproduce and confirm previously known results up to order ϵ^4, as well as derive new results.

The ϵ-expansion of the Wilson-Fisher model has been studied using conformal bootstrap techniques previously [58–61]. Most notably, in [8, 9, 47] a bootstrap scheme based on the sum of Polyakov blocks in the three channels was proposed. This method was used to obtain CFT data up to order ϵ^3. It is unclear whether such a method holds nonperturbatively, or if it can be used in the Wilson-Fisher model to extract predictions to higher order in ϵ [13]. In [62] the ϵ-expansion was studied to order ϵ^4 using the Lorentzian inversion formula and large spin re-summation.

We use dispersive Mellin sum rules to derive the OPE data of low twist operators perturbatively in the ϵ-expansion. We confirm the predictions of [8, 62]. Our procedure is systematic and no assumptions of analyticity down to spin 0 were made. Furthermore, we make some new predictions. These are

- The averaged OPE coefficients of twist 4 operators at order ϵ^3, see table 6.2.
- The coefficient of ϕ^2 in the $\phi \times \phi$ OPE at order ϵ^4, see table 6.3.

It would be very interesting to develop these methods further. This can potentially enable the computation of CFT data at order ϵ^5 and higher. Doing this requires a better handle on the various sums and integrals that involve Mack polynomials that we discuss below. We leave this for future work.

6.2 Family of Functionals

We explore dispersion relations of the type

$$\omega_F \equiv \oint_{\mathcal{C}_\infty} \frac{d\gamma_{12}}{2\pi i} M(\gamma_{12}, \gamma_{13}) F(\gamma_{12}, \gamma_{13}) = 0\,, \tag{6.1}$$

where the contour $\mathcal{C}_\infty$ encircles $\gamma_{12} = \infty$ and the rational function F decays at least as fast as $1/\gamma_{12}^3$ at large γ_{12}. Different choices of the function F and different values of γ_{13} lead to different sum rules after closing the integration contour using Cauchy's theorem.

It is convenient to choose F to be a rational function with poles at special locations where the Mellin amplitude vanishes (see chapter 4 on the Polyakov conditions). Such choices lead to sum rules that only involve the CFT data (τ and $C_{\tau,\ell}$) and not the Mellin amplitude itself.

All such sum rules have double zeros at the position of double trace operators $\tau = 2\Delta + 2n$, $n \in \mathbb{Z}_{\geqslant 0}$ above certain twist τ_0 which depends on the choice of F in (6.1). Sum rules with this property were called "dispersive" in [13].

Let us introduce a family of functionals that will be useful in the present chapter. They are specified by a simple rational function $F(\gamma_{12}, \gamma_{13})$ that enters into the sum rule (6.1) and takes the following form

$$F_{p_1,p_2,p_3}(\gamma_{12}, \gamma_{13}) \equiv \frac{2}{(\gamma_{12}+p_1)(\gamma_{12}+p_2)(\Delta-\gamma_{12}-\gamma_{13}+p_3)}, \quad p_i \in \mathbb{Z}_{\geqslant 0}\,. \tag{6.2}$$

For fixed γ_{13} and large γ_{12} we have $F_{p_1,p_2,p_3} \sim \frac{1}{\gamma_{12}^3}$ and arcs at infinity indeed do not contribute. We denote the corresponding functional $\omega_{p_1,p_2,p_3} \equiv \omega_{F_{p_1,p_2,p_3}}$.

Using the Polyakov conditions (4.29) we get the following sum rule

$$\omega_{p_1,p_2,p_3} = \sum_{\tau>0,\ell,m=0}^{\infty} C^2_{\tau,\ell}\, \omega^{\tau,\ell,m}_{p_1,p_2,p_3} = 0, \tag{6.3}$$

where $\omega^{\tau,\ell,m}_{p_1,p_2,p_3}$ denotes the contribution of a given collinear family of descendants from the primary operator with twist τ and spin ℓ into the sum rule,

$$\begin{aligned}\omega^{\tau,\ell,m}_{p_1,p_2,p_3} \equiv\; & \frac{\mathcal{Q}^{\tau,d}_{\ell,m}(-2\gamma_{13})}{\prod_{i=1}^2(\gamma_{13}-m-p_i-\frac{\tau}{2})(\Delta-m+p_3-\frac{\tau}{2})} \\ & -\frac{\mathcal{Q}^{\tau,d}_{\ell,m}(-2\gamma_{13})}{\prod_{i=1}^2(\Delta-m+p_i-\frac{\tau}{2})(-\gamma_{13}+m+p_3+\frac{\tau}{2})}\,.\end{aligned} \tag{6.4}$$

Let us discuss properties of these functionals. First of all, these functionals are not all independent. Obviously, $\omega_{p_1,p_2,p_3} = \omega_{p_2,p_1,p_3}$. Moreover, it is easy to check that

$$\omega_{p_1,p_2,p_2} = \omega_{p_1,p_2,p_1}. \tag{6.5}$$

Next using figure 4.1 we can understand convergence properties of the functionals ω_{p_1,p_2,p_3}. The dangerous contribution comes from the large spin double twist operators. To discuss this it is convenient to distinguish two cases $p_1 = p_2$ and, without loss of generality, $p_1 > p_2$.

When $p_1 = p_2$ we use the subleading Polyakov condition in γ_{12} which converges in the green region in figure 4.1, we also use the leading Polyakov condition in γ_{14} which converges in the crossing transformation of the red region in figure 4.1. As a result we conclude that

$$\begin{aligned} &\omega_{p,p,p_3} \text{ converges when } p \geqslant 1,\ p_3 \geqslant 0, \quad \gamma_{13} \in \text{green region} \\ &\Leftrightarrow \quad \Delta < \operatorname{Re}\gamma_{13} < \min\left(p, p_3 + \frac{\tau_{gap}}{2}\right) \end{aligned} \tag{6.6}$$

When $p_1 > p_2$ we use the leading Polyakov condition both in γ_{12} and γ_{14} which converges in the red region (and its crossing transformation) in figure 4.1 so no extra constraints arise and we have

$$\omega_{p_1,p_2,p_3} \text{ converges when } \quad p_1 > p_2, \quad \gamma_{13} \in \text{red region}, \tag{6.7}$$

$$\Leftrightarrow \quad \Delta - \frac{\tau_{gap}}{2} < \operatorname{Re}\gamma_{13} < \min(p_2, p_3) + \frac{\tau_{gap}}{2} \tag{6.8}$$

The defining property of ω_{p_1,p_2,p_3} functional is sensitivity to the double twist operators with twists $\tau = 2\Delta + 2n$, where $n \leqslant p_i$. Indeed, the Mack polynomials $\mathcal{Q}^{\tau,d}_{\ell,m}(-2\gamma_{13})$ have double zeros at $\tau = 2\Delta + 2n$, whereas the brackets in (6.4) have poles at $\tau = 2\Delta + 2(p_i - m)$ which enhances the contribution of the corresponding double twist families. See figures 6.1 and 6.2.

In our analysis of the Wilson-Fisher model in $d = 4 - \epsilon$ dimensions below we will only study the properties of the leading, $n = 0$, and the first sub-leading, $n = 1$, double twist family of operators. This naturally restrict our attention to the functionals with $p_i \leqslant 1$. Together with linear dependence and convergence properties explained above it leaves us with two functionals

$$\omega_{1,0,0}, \quad \omega_{1,1,1}. \tag{6.9}$$

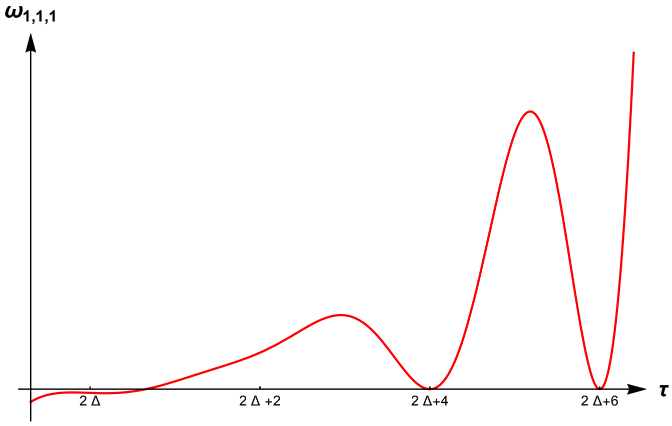

Figure 6.1. A plot of the $\omega_{1,1,1}^{\tau,\ell} \equiv \sum_{m=0}^{\infty} \omega_{1,1,1}^{\tau,\ell,m}$ functional as a function of the twist τ of the exchanged operator. We picked $\Delta = \frac{3}{5}$, $\gamma_{13} = \frac{3}{4}$, $\ell = 2$ and $d = 3$. The plot is qualitatively the same for other values. We sum in m from 0 to 50, since this is enough to have an accurate plot of the functional. Notice that the functional does not vanish for twists $\tau = 2\Delta$ and $\tau = 2\Delta + 2$. However, it has double zeros for all $\tau = 2\Delta + 2n$, for $n \geqslant 2$.

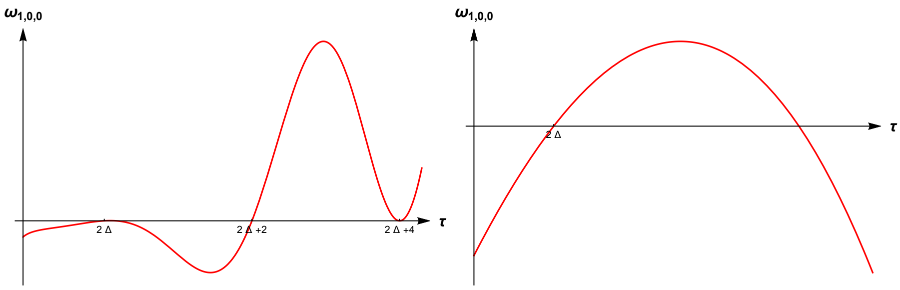

Figure 6.2. On the left, a plot of the $\omega_{1,0,0}$ functional as a function of the twist τ of the exchanged operator. We picked $\Delta = 1.1$, $\gamma_{13} = \frac{1}{3}$, $\ell = 2$ and $d = 4$. The plot is qualitatively similar for other values. We sum in m from 0 to 50, since this is enough to have an accurate plot of the functional. Notice that the functional has single zeros for twists $\tau = 2\Delta = 2.2$ (though it is not very clear from the left plot) and $\tau = 2\Delta + 2 = 4.2$. However, it has double zeros for all $\tau = 2\Delta + 2n$, where $n \geqslant 2$. On the right, we zoom in to the region around $\tau = 2\Delta$, so that we can observe the single zero at $\tau = 2\Delta$.

6.3 Check with Mean Field Theory

In mean field theory (MFT), there are only exact double twist operators in the OPE $\phi \times \phi$, where ϕ is the gaussian field. Therefore, the functionals ω_{p_1,p_2,p_3} annihilate every term of the sum in (6.3) because the functions $\mathcal{Q}_{\ell,m}^{\tau,d}(-2\gamma_{13})$ have double zeros at $\tau - 2\Delta \in \mathbb{Z}_{\geq 0}$. This conclusion is not correct if we choose $p_1 = p_2 \in \mathbb{Z}_{\geq 0}$ because the double pole of F at $\gamma_{12} = p_1$ cancels the double zero.

For concreteness consider the functional $\omega_{1,1,p}$. Using the MFT OPE coefficients, we obtain

$$\omega_{1,1,p} = -\frac{2^{2+2\Delta}}{\sqrt{\pi}\,\Gamma^2(\Delta)(1+\Delta-\gamma_{13}+p)} \sum_{\substack{\ell=0 \\ even}}^{\infty} \frac{2^\ell \Gamma\left(\frac{1}{2}+\Delta+\ell\right) P_\ell(\gamma_{13})}{(d+2\ell)(2-d+4\Delta+2\ell)\ell!\,\Gamma(\Delta+\ell+1)}\,, \tag{6.10}$$

where

$$P_\ell(\gamma_{13}) = (d+2\ell)(\ell+\Delta) Q_{\ell,1}^{\tau=2\Delta,d}(-2\gamma_{13}) + (2-d+2\Delta)(2\Delta+2\ell+1) Q_{\ell,0}^{\tau=2\Delta+2,d}(-2\gamma_{13})\,. \tag{6.11}$$

Here we used the (non-calligraphic) Mack polynomials $Q_{\ell,m}^{\tau,d}$ defined in appendix J.1. Notice that only the double-twist operators with twist $\tau = 2\Delta$ and $\tau = 2\Delta+2$ contribute to this sum rule. The polynomial $P_\ell(\gamma_{13})$ inherits the symmetry $P_\ell(\gamma_{13}) = P_\ell(1+\Delta-\gamma_{13})$ from the Mack polynomials.

The large ℓ behaviour of the summand in (6.10) is given by

$$\sim \ell^{\max(2\Delta-2\gamma_{13}-1,2\gamma_{13}-3)}\,, \tag{6.12}$$

which implies convergence of the sum over ℓ for $\Delta < \gamma_{13} < 1$. In figure 6.3, we plot the partial sums

$$S_J(\gamma_{13}) = \sum_{\substack{\ell=0 \\ even}}^{J} \frac{2^\ell \Gamma\left(\frac{1}{2}+\Delta+\ell\right) P_\ell(\gamma_{13})}{(d+2\ell)(2-d+4\Delta+2\ell)\ell!\,\Gamma(\Delta+\ell+1)}\,, \tag{6.13}$$

for several values of J. One can see that $S_J(\gamma_{13})$ tends to zero when $J \to \infty$ if $\Delta < \gamma_{13} < 1$ and it diverges otherwise. Moreover, one can also check the large J behaviour

$$\log S_J(\gamma_{13}) \approx \max\left(2\Delta - 2\gamma_{13}, 2\gamma_{13} - 2\right) \log J\,, \tag{6.14}$$

in agreement with (6.12).

The knowledgeable reader may ask: how can we get a sum rule for MFT using Mellin amplitudes? Indeed, the Mellin amplitude for MFT vanishes identically [63]. One way to understand the success of the exercise above is as follows. Consider an interacting theory with a continuous coupling λ, such that at $\lambda = 0$ we obtain MFT. The Mellin amplitude is non-trivial and leads to the sum rules (6.3) for any $\lambda > 0$. Then, one obtains the sum rule (6.10) for MFT in the limit $\lambda \to 0$.

Let us now consider the functional

$$\omega_{0,0,p} = -\frac{2^{1+2\Delta}}{\sqrt{\pi}\,\Gamma^2(\Delta)(\Delta-\gamma_{13}+p)} \sum_{\substack{\ell=0 \\ even}}^{\infty} \frac{2^\ell \Gamma\left(\frac{1}{2}+\Delta+\ell\right)}{\ell!\,\Gamma(\Delta+\ell)} Q_{\ell,0}^{\tau=2\Delta,d}(-2\gamma_{13})\,. \tag{6.15}$$

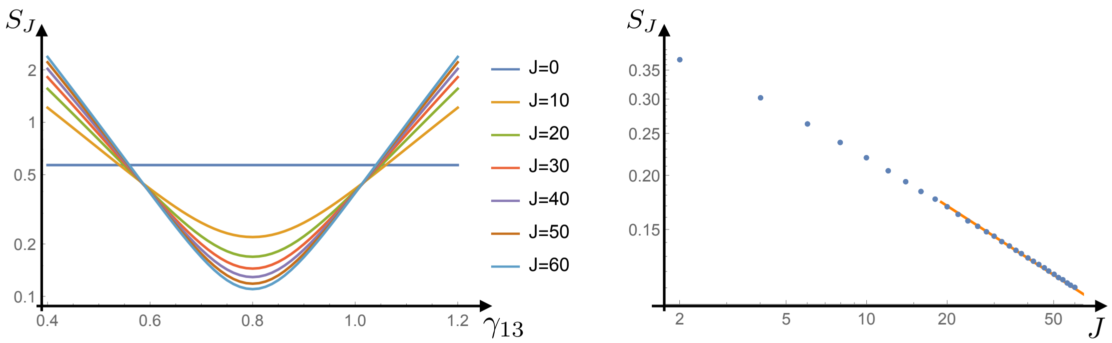

Figure 6.3. Partial sum $S_J(\gamma_{13})$ defined in (6.13) for $d = 3$ and $\Delta = \frac{3}{5}$. On the left, one can see that only for $\Delta < \gamma_{13} < 1$ the partial sum converges to zero as expected. On the right, we fix $\gamma_{13} = \frac{4}{5}$ and use a log-log plot to exhibit the large J behavior predicted by (6.14). The straight orange line is a fit (to the points $20 \leq J \leq 60$) with slope given by (6.14).

One can check that this sum vanishes for $\Delta < \operatorname{Re}\gamma_{13} < 0$ in agreement with the general formula (6.7). Notice that in unitary CFTs there is no convergence region for this sum rule because $\Delta < 0$ is forbidden. Nevertheless, we shall use the functional

$$\omega_{0,0,0}$$

in the ϵ–expansion by applying it to the difference between the CFT data in the interacting theory and in MFT. This trick will give rise to a finite region of convergence for the sum rule $\omega_{0,0,0}$.

6.4 Setup for the ϵ-expansion

The Wilson-Fisher fixed point in $d = 4 - \epsilon$ spacetime dimensions contains the lightest scalar operator ϕ of dimension Δ_ϕ and we consider the four-point $\langle\phi\phi\phi\phi\rangle$ and the associated CFT data which includes the following operators:

- The lightest scalar that appears in the OPE $\phi \times \phi$. We denote this operator by ϕ^2 with dimension Δ_{ϕ^2} and OPE coefficient C_{ϕ^2}.
- The leading twist operators j_ℓ (also called twist-two) with twist τ_ℓ and the three-point function C_{j_ℓ}. These are non-degenerate (there is a single operator for every even spin $\ell \geqslant 2$) and can be identified with the double twist operators with $n = 0$.
- Twist-four operators of even spin $\ell \geqslant 0$. These are non-degenerate for $\ell = 0, 2$ [64], and degenerate for $\ell \geqslant 4$. They also include the double twist family with $n = 1$. We will study their OPE data on average.
- Higher twist operators. These have twist six and higher when $\epsilon = 0$. We do not say anything about these operators. They will not appear in the dispersive Mellin

sum rules to the perturbative order that we analyze them.

From the bootstrap point of view this model can be defined as follows. We start with $\langle\phi\phi\phi\phi\rangle$ being the mean field theory correlator in $d = 4-\epsilon$ dimensions. Next we assume that the CFT data (scaling dimensions and OPE coefficients) depends on ϵ as a power series.

We then study the crossing equations perturbatively in ϵ. Such perturbative solutions to crossing were analyzed in [54] and they include infinitely many ambiguities due to contact interactions in AdS. It is reasonable to conjecture that these are completely fixed by requiring that at every order in ϵ the correlator satisfies the nonperturbative Regge bound and that the stress energy tensor is conserved. In particular, this means that we can use the dispersive Mellin sum rules order by order in ϵ.

Let us state our definitions. The known results for the OPE data in the Wilson-Fisher fixed point in $d = 4-\epsilon$ and the relevant references can be found in appendix K.

We follow [62] and define the expansion parameter g to be the anomalous dimension of ϕ^2,

$$\Delta_{\phi^2} = 2\Delta_\phi + g, \tag{6.16}$$

Then, the spacetime dimensionality d

$$d = 4 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4 + \dots. \tag{6.17}$$

Using this equation one can find g as a function of ϵ and vice versa. For the conformal dimension of ϕ we write

$$\Delta_\phi = \frac{d-2}{2} + \gamma_1(\phi)g + \gamma_2(\phi)g^2 + \gamma_3(\phi)g^3 + \gamma_4(\phi)g^4 + \dots. \tag{6.18}$$

We will derive that $\gamma_1(\phi) = 0$. So, the correction to the dimension of ϕ starts at order g^2. Similarly, we write for the twist of the twist-two operators

$$\tau_\ell = 2\Delta_\phi + \gamma_1(j_\ell)g + \gamma_2(j_\ell)g^2 + \gamma_3(j_\ell)g^3 + \gamma_4(j_\ell)g^4 + \dots, \quad \ell \geqslant 2. \tag{6.19}$$

We will derive that $\gamma_1(j_\ell) = 0$. The twist of the stress energy tensor is protected $\tau_2 = d-2$. For the twist-four operators due to the degeneracy we only compute the averaged values of the relevant OPE data

$$\tau_{4,i}(\ell)) = 2\Delta_\phi + 2 + \gamma_1(\tau_{4,i}(\ell))g + \dots, \tag{6.20}$$

where i denotes various degenerate operators at $g = 0$. We will only compute averaged moments of the twist-four anomalous dimensions as follows

$$\langle \gamma(\tau_4(\ell))^n \rangle \equiv \frac{\sum_i C^2_{\tau_{4,i}(\ell)} \gamma^n(\tau_{4,i}(\ell))}{\sum_i C^2_{\tau_{4,i}(\ell)}}, \tag{6.21}$$

where the sum is over the degenerate operators. Note that $\ell = 0, 2$ operators are not degenerate.

Concerning OPE coefficients, we define them relative to the mean field theory. In MFT, the square of the OPE coefficient of the operator of dimension $2\Delta_\phi + 2n + \ell$ and spin ℓ is equal to

$$C^2_{n,\ell} = \frac{(1+(-1)^\ell)\left(\Delta - \frac{d}{2} + 1\right)^2_n (\Delta)^2_{\ell+n}}{\Gamma(\ell+1)\Gamma(n+1)\left(\frac{d}{2}+\ell\right)_n (2\Delta - d + n + 1)_n (2\Delta + \ell + 2n - 1)_\ell \left(2\Delta - \frac{d}{2} + \ell + n\right)_\ell} \tag{6.22}$$

where Δ_ϕ is the dimension of the fundamental field and d is the spacetime dimension, both of which are nontrivial functions of g.

We parametrise the OPE coefficients of ϕ^2, of the leading twist operators j_ℓ and the sum of squares of OPE coefficients over degenerate twist 4 operators $\tau_4(l)$ as

$$\begin{aligned} C^2_{\phi^2} &= C^2_{0,0} \times \left(1 + c_1(\phi^2) g + c_2(\phi^2) g^2 + c_3(\phi^2) g^3 + c_4(\phi^2) g^4 + \ldots\right), \\ C^2_{j_\ell} &= C^2_{0,\ell} \times \left(1 + c_1(j_\ell) g + c_2(j_\ell) g^2 + c_3(j_\ell) g^3 + c_4(j_\ell) g^4 + \ldots\right), \\ \sum_i C^2_{\tau_{4,i}(\ell)} &= C^2_{1,\ell} \times \left(b_2(\ell) + b_3(\ell) g + \ldots\right) \sim O(g^2), \end{aligned} \tag{6.23}$$

respectively, where we plug Δ_ϕ and d as functions of g as above. In the last line we emphasized that since in the free field theory $C^2_{n\geqslant 1,\ell}|_{\Delta=\frac{d-2}{2}} = 0$ the twist four operators, or $n = 1$, first contribute at order g^2. Higher twist operators, or $n \geqslant 2$, first contribute at order g^4.

The quantities $c_i(\phi^2)$, $c_i(j_\ell)$, and $b_i(\ell)$ are to be determined using the dispersive Mellin functionals below.

6.5 Order g^0

Let us discuss the action of the functionals $\omega_{1,0,0}$, $\omega_{1,1,1}$ and $\omega_{0,0,0}$ at order g^0. When $g = 0$ the correlator $\langle\phi\phi\phi\phi\rangle$ is the one of the free scalar field in 4 dimensions. The relevant action of the functionals on mean field theory was discussed in section 6.3 to which we refer the reader for the relevant formulas. In the following applications, we always consider the difference between the action of the functionals on the Wilson-Fischer fixed point and their action on mean field theory.

6.6 Order g^1

Let us discuss the action of $\omega_{1,0,0}$ and $\omega_{0,0,0}$. $\omega_{1,0,0}$ has single zeros for the leading and subleading twist trajectories, whereas it has double zeros for the other twist trajectories. For this reason, the contribution of the twist 2 and twist 4 operators comes proportional to their anomalous dimensions, whereas the contribution of twist 6 or higher operators comes proportional to their anomalous dimensions squared. For this reason, twist 6 operators do not contribute at this order. Furthermore, the OPE coefficients of twist 4 operators vanish at order g^0. Thus, twist 4 operators do not contribute to $\omega_{1,0,0}$ to first order in g.

So, only twist 2 operators contribute at first order in g. Their contribution is given by

$$\begin{aligned}\omega_{1,0,0}\big|_{g^1} = &\frac{2}{(-2+\gamma_{13})}\left(Q^{\tau=2,d=4}_{\ell=0,m=0}(-2\gamma_{13}) - Q^{\tau=2,d=4}_{\ell=0,m=1}(-2\gamma_{13})\right) \\ &+ \sum_{\substack{\ell=2\\ \text{even}}}^{\infty} \frac{2^{\ell+2}\Gamma(\ell+\frac{3}{2})}{(-2+\gamma_{13})\sqrt{\pi}\Gamma(\ell+1)^2}\left(Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13}) - Q^{\tau=2,d=4}_{\ell,m=1}(-2\gamma_{13})\right)\gamma_1(j_\ell)\end{aligned} \tag{6.24}$$

The first line corresponds to the contribution of ϕ^2 and the second line corresponds to the contribution of the operators in the leading twist trajectory with even spin $\ell \geqslant 2$. It turns out that the contribution of ϕ^2 vanishes identically since $Q^{\tau=2,d=4}_{\ell=0,m=0}(-2\gamma_{13}) = Q^{\tau=2,d=4}_{\ell=0,m=1}(-2\gamma_{13}) = 1$.

We will want to apply the orthogonality relation (J.8) to (6.24). The decomposition

$$Q^{\tau=2,d=4}_{\ell_1,m=1}(s) = Q^{\tau=2,d=4}_{\ell_1,m=0}(s) + \sum_{\ell_2=0}^{\ell_1-1} \frac{2^{-\ell_1+\ell_2+1}\Gamma(\ell_1)\Gamma(\ell_1+1)\Gamma\left(\ell_2+\frac{3}{2}\right)}{\Gamma\left(\ell_1+\frac{1}{2}\right)\Gamma(\ell_2+1)^2} Q^{\tau=2,d=4}_{\ell_2,m=0}(s) \tag{6.25}$$

will be important. Furthermore, let us define

$$\zeta_1 = \sum_{\substack{\ell=2\\ \text{even}}}^{\infty} \frac{\ell+\frac{1}{2}}{\ell}\gamma_1(j_\ell). \tag{6.26}$$

Then, by permuting the order of the sums, (6.24) can be rewritten as

$$(-2+\gamma_{13})\omega_{1,0,0}\big|_{g^1} = -\sum_{\ell=0}^{\infty} \frac{2^{\ell+3}}{\sqrt{\pi}}\frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell+1)^2}\left(\zeta_1 - \sum_{\substack{\ell_1=2\\ \text{even}}}^{\ell} \frac{\ell_1+\frac{1}{2}}{\ell_1}\gamma_1(\ell_1)\right) Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13})\,. \tag{6.27}$$

Since the Mack polynomials are orthogonal (see (J.8)), every term in the sum over ℓ in

(6.27) must vanish. Then, the term with $\ell = 0$ implies that $\zeta_1 = 0$. The term with $\ell = 2$ implies $\gamma_1(j_{\ell=2}) = 0$. The term with $\ell = 4$ implies $\gamma_1(j_{\ell=4}) = 0$ and so on. We therefore conclude that

$$\boxed{\gamma_1(j_\ell) = 0} \tag{6.28}$$

Conservation of the stress tensor implies that $2\Delta_\phi + \gamma(j_{\ell=2}) = d - 2$. Expanding this to first order in g and using $\gamma_1(j_{\ell=2}) = 0$, we obtain

$$\boxed{\gamma_1(\phi) = 0} \tag{6.29}$$

Let us consider now the action of $\omega_{0,0,0}$. This functional does not vanish for the leading twist trajectory, but it has double zeros for all the subleading twist trajectories. For this reason only the twist 2 operators contribute to first order in g. We have that

$$\begin{aligned}\omega_{1,0,0}\big|_{g^1} = &-\sum_{\substack{\ell=2\\ \text{even}}}^{\infty} \frac{2^{\ell+3}\Gamma(\ell+\frac{3}{2})}{(-1+\gamma_{13})\sqrt{\pi}\Gamma(\ell+1)^2} c_1(j_\ell) Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13}) \\ &- \frac{4(1+c_1(\phi^2))}{(-1+\gamma_{13})} Q^{\tau=2,d=4}_{\ell=0,m=0}(-2\gamma_{13}).\end{aligned} \tag{6.30}$$

The orthogonality relation (J.8) applied to $(-1+\gamma_{13})\times$(6.30) immediately implies that

$$\boxed{c_1(\phi^2) = -1, \quad c_1(j_\ell) = 0} \tag{6.31}$$

6.7 Order g^2

Let us consider the action of the $\omega_{1,0,0}$ functional. At order g^2, only ϕ^2 and operators in the leading Regge trajectory j_ℓ contribute. The twist four and higher operators do not contribute, since their OPE coefficients start at order g^2 and the functional $\omega_{1,0,0}$ vanishes for the exchange of sub-leading twists.

The contribution of ϕ^2 is equal to

$$\sum_{m=0}^{\infty} C^2_{\phi^2} \omega^{\Delta_{\phi^2},0,m}_{1,0,0}\Bigg|_{g^2} = \frac{-2 + a_1(-1+\gamma_{13}) + \gamma_{13}}{(-2+\gamma_{13})(-1+\gamma_{13})}. \tag{6.32}$$

The contribution of the leading twist trajectory is equal to

$$\sum_{\substack{\ell=2\\ \text{even}}}^{\infty} C^2_{j_\ell} \omega^{\tau_\ell,\ell,0}_{1,0,0}\big|_{g^2} = \sum_{\substack{\ell=2\\ \text{even}}}^{\infty} \frac{2^{\ell+2}\Gamma(\ell+\frac{3}{2})\big((\ell+1)Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13}) - Q^{\tau=2,d=4}_{\ell,m=1}(-2\gamma_{13})\big)\gamma_2(j_\ell)}{(-2+\gamma_{13})(\ell+1)\sqrt{\pi}\Gamma(\ell+1)^2}. \tag{6.33}$$

Recall that $\gamma_2(j_\ell)$ is the anomalous dimension of the leading twist operators.

In order to extract the anomalous dimensions $\gamma_2(j_\ell)$ we will use the orthogonality relation (J.8) among $m = 0$ Mack polynomials. We will also use equation (6.25) to decompose $m = 1$ Mack polynomials into $m = 0$ Mack polynomials. From the sum rule (6.32) + (6.33) = 0 we can obtain several equations by multiplying it by $(-2+\gamma_{13})\Gamma^2(\gamma_{13})\Gamma^2(1-\gamma_{13})\, Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13})$, integrating over γ_{13} and using the orthogonality relation (J.8). For even ℓ we obtain

$$
\begin{aligned}
&-\frac{\sqrt{\pi}2^{1-\ell}\ell\gamma_2(j_\ell)\Gamma(\ell+1)^2}{(\ell+1)\Gamma\left(\ell+\frac{1}{2}\right)}+\frac{\sqrt{\pi}2^{1-\ell}\Gamma(\ell+1)^2\left(\zeta-\sum_{\ell_1=2}^{\ell}\left[\frac{(2\ell_1+1)\gamma_2(j_{\ell_1})}{\ell_1(\ell_1+1)}\right]\right)}{\Gamma\left(\ell+\frac{1}{2}\right)}\\
&=\int_{-1-i\infty}^{-1+i\infty}\frac{ds}{4\pi i}\left(a_1+\frac{4+s}{2+s}\right)\Gamma\left(-\frac{s}{2}\right)^2\Gamma\left(\frac{s}{2}+1\right)^2 Q^{\tau=2,d=4}_{\ell,m=0}(s) \qquad (6.34)\\
&=-\frac{\sqrt{\pi}2^{-\ell-1}\Gamma(\ell+1)^2\left((\ell+1)^2\psi^{(1)}\left(\frac{\ell}{2}+1\right)-(\ell+1)^2\psi^{(1)}\left(\frac{\ell+3}{2}\right)-4\right)}{(\ell+1)^2\Gamma\left(\ell+\frac{1}{2}\right)}+\delta_{\ell,0}(1+a_1),
\end{aligned}
$$

where $s = -2\gamma_{13}$, $\psi^{(1)}(x)$ is the order 1 polygamma function and we compute the integral above in appendix M. We also introduced the quantity

$$
\zeta \equiv \sum_{\substack{\ell=2\\ \text{even}}}^{\infty}\frac{(2\ell+1)}{\ell(\ell+1)}\gamma_2(j_\ell). \qquad (6.35)
$$

We treat ζ as an independent parameter from the anomalous dimensions, that we will compute.

Furthermore, the orthogonality relation with respect to odd spin Mack polynomials is also very useful. It is given by

$$
-\frac{\sqrt{\pi}2^{1-\ell}\Gamma(\ell+1)^2\left(\zeta-\sum_{\ell_1=2}^{\ell-1}\left[\frac{(2\ell_1+1)\gamma_2(j_{\ell_1})}{\ell_1(\ell_1+1)}\right]\right)}{\Gamma\left(\ell+\frac{1}{2}\right)} \qquad (6.36)
$$

$$
=-\frac{\sqrt{\pi}2^{-\ell-1}\Gamma(\ell+1)^2\left((\ell+1)^2\psi^{(1)}\left(\frac{\ell}{2}+1\right)-(\ell+1)^2\psi^{(1)}\left(\frac{\ell+3}{2}\right)-4\right)}{(\ell+1)^2\Gamma\left(\ell+\frac{1}{2}\right)}. \qquad (6.37)
$$

We used the spin 1 equation to determine $\zeta = \frac{\pi^2}{12} - 1$.

Knowing ζ, then equations (6.34) can be solved in the following manner. The spin 0 and spin 2 equations determine a_1 and $\gamma_2(2)$. The spin 4 equation determines $\gamma_2(4)$ and so on. More generically,

$$
\boxed{a_1 = -3, \quad \gamma_2(j_\ell) = -\frac{1}{\ell(\ell+1)}} \qquad (6.38)
$$

This agrees with known results.

Since we already know the anomalous dimensions of the leading twist operators, we can fix the dimension of ϕ by demanding that the stress tensor has twist $d-2$

$$2\gamma_2(\phi)+\gamma_2(2)=0 \;\Rightarrow\; \boxed{\gamma_2(\phi)=\frac{1}{12}} \tag{6.39}$$

In order to compute the corrections to the OPE coefficients of the operators in the leading twist trajectory, let us consider the action of $\omega_{0,0,0}$. At order g^2 only ϕ^2 and the leading twist trajectory contribute. The contribution of ϕ^2 is

$$\sum_{m=0}^{\infty} C^2_{\phi^2}\omega_{0,0,0}^{\Delta_{\phi^2},0,m}|_{g^2}=2\left(\frac{2a_1}{\gamma_{13}-1}-\frac{2c_2(\phi^2)}{\gamma_{13}-1}+\frac{4(\gamma_{13}-2)\gamma_{13}+(\gamma_{13}-1)^2\psi^{(1)}(2-\gamma_{13})+5}{2(\gamma_{13}-1)^3}\right), \tag{6.40}$$

The contribution of the leading twist trajectory is

$$\begin{aligned}\sum_{\substack{\ell=2\\ \text{even}}}^{\infty} C^2_{j_\ell}\omega_{0,0,0}^{\tau_\ell,\ell,0}|_{g^2} &= -2\sum_{\substack{\ell=2\\ \text{even}}}^{\infty}\frac{2^{\ell+2}\Gamma\left(\ell+\frac{3}{2}\right)}{\sqrt{\pi}(\gamma_{13}-1)\Gamma(\ell+1)^2} \\ &\times\left(Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13})\left(c_2(j_\ell)\gamma_2(j_\ell)\left(2S_1(2\ell)-3S_1(\ell)+\frac{1}{2(\ell+\frac{1}{2})}\right)\right)\right.\\ &\left.+\gamma_2(j_\ell)\frac{d}{d\tau}Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13})\right),\end{aligned} \tag{6.41}$$

The above series converges for $0<Re(\gamma_{13})<\frac{1}{2}$. We determined $c_2(\ell)$ and $c_2(\phi^2)$ in the following manner. The series (6.41) contains a part proportional to $c_2(\ell)$ and another part proportional to $\gamma_2(\ell)$. We have already determined $\gamma_2(\ell)$ in the preceding section. So, we can sum the part of the series (6.41) that is proportional to $\gamma_2(\ell)$. In practice, we used, from spin 2 to 100, the exact expressions for Mack polynomials and, from spin 102 to infinity, we used the approximation (J.21), (J.22) to Mack polynomials to sum the tails.

After doing this summation we can apply the orthogonality relation (J.8). We evaluate such integrals numerically. Proceeding in this manner we obtained

$$\boxed{c_2(\phi^2)=-1} \tag{6.42}$$

We also obtained $c_2(j_\ell)$ for low spins. It precisely matches the formula[1]

$$\boxed{c_2(j_\ell) = \frac{S_1(2\ell) - S_1(\ell) + \frac{1}{\ell+1}}{\ell(\ell+1)}} \tag{6.43}$$

derived in [8], where

$$S_n(\ell) = \sum_{m=1}^{\ell} \frac{1}{m^n}. \tag{6.44}$$

Thus far we have deduced the conformal data of ϕ, ϕ^2 and the leading twist trajectory to order g^2. Let us compute the OPE coefficients of the twist 4 operators at order g^2, using the $\omega_{1,1,1}$ functional. The contribution of ϕ^2 is equal to

$$\begin{aligned}\sum_{m=0}^{\infty} C^2_{\phi^2} \omega_{1,1,1}^{\Delta_{\phi^2},0,m}|_{g^2} = 2\bigg(&\frac{a_1}{\gamma_{13}-3} - \frac{2c_2(\phi^2)}{\gamma_{13}-3} \\ &+ \frac{2(\gamma_{13}-5)\gamma_{13}((\gamma_{13}-5)\gamma_{13}+13)+85}{2(\gamma_{13}-3)^3(\gamma_{13}-2)^2} + \frac{\psi^{(1)}(4-\gamma_{13})}{2(\gamma_{13}-3)}\bigg).\end{aligned} \tag{6.45}$$

Of course we already know the values of a_1 and $c_2(\phi^2)$, we are just exhibiting which CFT data matters for the action of $\omega_{1,1,1}$.

The contribution of the leading twist trajectory is given by

$$\begin{aligned}\sum_{\substack{\ell=2\\ \text{even}}}^{\infty} C^2_{j_\ell} \omega_{1,1,1}^{\tau_\ell,\ell,0}|_{g^2} &= -2\sum_{\substack{\ell=2\\ \text{even}}}^{\infty} \frac{2^{\ell+2}\Gamma(\ell+\frac{3}{2})}{\sqrt{\pi}(\gamma_{13}-3)\Gamma(\ell+1)\Gamma(\ell+2)} \\ \times\bigg(Q^{\tau=2,d=4}_{\ell,m=1}(-2\gamma_{13})\Big(&c_2(j_\ell) + \gamma_2(j_\ell)(2S_1(2\ell) - 3S_1(\ell) + \frac{1}{2(\ell+\frac{1}{2})} - \frac{1}{\ell+1} + 1)\Big) \\ &+\gamma_2(j_\ell)\frac{d}{d\tau}Q^{\tau=2,d=4}_{\ell,m=1}(-2\gamma_{13})\bigg)\end{aligned} \tag{6.46}$$

The contribution of the subleading twist trajectory is equal to

$$\sum_{\substack{\ell=0\\ \text{even}}}^{\infty}\sum_i C^2_{\tau_{4,i}(\ell)} \omega_{1,1,1}^{\tau_{4,i}(\ell),\ell,0}|_{g^2} = \sum_{\substack{\ell=0\\ \text{even}}}^{\infty}\left(- \frac{2\gamma_2(\phi)2^{\ell+3}(b_2(\ell)-1)\Gamma\left(\ell+\frac{5}{2}\right) Q^{\tau=4,d=4}_{\ell,m=0}(-2\gamma_{13})}{\sqrt{\pi}(\gamma_{13}-3)(\ell+1)(\ell+2)\Gamma(\ell+1)\Gamma(\ell+2)}\right) g^2. \tag{6.47}$$

[1] At low orders in g, our work consists in rederiving known formulas, so we do not keep track of error bars when executing our numerical procedure. However, when making new predictions, we were careful with errors and we do present our results with error bars.

Our goal is to compute the averaged OPE coefficients of the twist 4 operators, which we call $b_2(\ell)$. For this reason we will use the orthogonality relation with respect to $m = 0$ twist 4 Mack polynomials. (6.46) is a complicated expression, however we can in principle evaluate it, since we already know all of the conformal data that enters the expression. This involves summing tails, like before. After doing such a summation and using the orthogonality relation (numerically), we managed to obtain $b_2(\ell)$ at low spins. Our results agree with the analytical expression

$$\boxed{b_2(\ell) = 1 + \frac{6}{(\ell+1)(\ell+2)}} \tag{6.48}$$

derived in [62].

6.8 Order g^3

Let us compute the order g^3 corrections to the OPE coefficients of the leading twist trajectory. We assume the formulas for a_2 and $\gamma_3(j_\ell)$ [62], see appendix K.

We will use the $\omega_{0,0,0}$ functional. The numerical procedure to extract CFT data is the same as before. The action of the $\omega_{0,0,0}$ functional allows us to calculate the OPE coefficients $c_3(\phi^2)$ and $c_3(j_l)$ at low spins. This precisely matches the analytic formulas [9]

$$\boxed{c_3(j_\ell) = \frac{3(S_1(\ell)-S_1(2\ell)+S_2(2\ell))-2\left(S_1^2(\ell)-S_1(\ell)S_1(2\ell)+S_2(\ell)\right)}{\ell(\ell+1)} + \frac{3(\ell+\frac{1}{2})(S_1(2\ell)-\frac{\ell-1}{\ell+1})-(\ell+\frac{3}{2})S_1(\ell)}{\ell^2(\ell+1)^2}} \tag{6.49}$$

$$\boxed{c_3(\phi^2) = \frac{5}{6} + \frac{7\zeta(3)}{4}} \tag{6.50}$$

Let us turn our attention to the twist 4 operators. Let us apply the $\omega_{1,0,0}$ functional, in order to obtain their first order correction to the anomalous dimensions. Since their OPE coefficients start at order g^2, this means we need to study the $\omega_{1,0,0}$ functional to order g^3.

There are three types of contributions: from ϕ^2, from the leading twist operators and from the subleading twist operators. The contribution of ϕ^2 is equal to

$$\sum_{m=0}^{\infty} C^2_{\phi^2}\omega_{1,0,0}^{\Delta_{\phi^2},0,m}|_{g^3} = -\frac{3S_1(-\gamma_{13})}{2\left(\gamma_{13}^2-3\gamma_{13}+2\right)} + \frac{\gamma_E(6\gamma_{13}-3)}{\gamma_{13}^2-3\gamma_{13}+2} + \frac{2\gamma_{13}(6(\gamma_{13}-3)\gamma_{13}+19)-19}{2\left(\gamma_{13}^2-3\gamma_{13}+2\right)^2}$$

where we used the Euler Gamma constant γ_E.

$\gamma_1(\tau_4(\ell))$	Numerical Result	Analytic Expression (6.53)
$\ell = 0$	$3.000000 \pm (2 \times 10^{-6})$	$3 = 3.000000$
$\ell = 2$	$1.33334 \pm (4 \times 10^{-5})$	$\frac{4}{3} = 1.33333$
$\ell = 4$	$0.6667 \pm (2 \times 10^{-4})$	$\frac{2}{3} = 0.6667$
$\ell = 6$	$0.3872 \pm (6 \times 10^{-4})$	$\frac{12}{31} = 0.3871$
$\ell = 8$	$0.250 \pm (2 \times 10^{-3})$	$\frac{1}{4} = 0.250$
$\ell = 10$	$0.175 \pm (5 \times 10^{-3})$	$\frac{4}{23} = 0.174$
$\ell = 12$	$0.13 \pm (2 \times 10^{-2})$	$\frac{6}{47} = 0.13$

Table 6.1. Numerical results for the averaged anomalous dimensions of the twist-four operators $\gamma_1(\tau_4(\ell))$ at order g^1, as well as the suggested analytic expression.

The contribution of the leading twist operators is equal to

$$\sum_{\substack{\ell=2\\ \text{even}}}^{\infty} C^2_{j_\ell} \omega^{\tau_\ell,\ell,0}_{1,0,0}|_{g^3} = \sum_{\substack{\ell=2\\ \text{even}}} \Bigg(\frac{2^\ell \Gamma(\ell+\frac{3}{2})}{\sqrt{\pi}(\gamma_{13}-2)^2\ell^2\Gamma(\ell+2)^2} \times (r_1(\ell) Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13}) \\ + r_2(\ell) Q^{\tau=2,d=4}_{\ell,m=1}(-2\gamma_{13})) + \frac{3\, 2^{\ell+3}\Gamma\left(\ell+\frac{3}{2}\right)}{\sqrt{\pi}(4-2\gamma_{13})\ell\Gamma(\ell+2)^2} \Big(\partial_d Q^{\tau=2,d=4}_{\ell,m=1}(-2\gamma_{13}) \\ - (1+\ell)\partial_\tau Q^{\tau=2,d=4}_{\ell,m=0}(-2\gamma_{13}) + \partial_\tau Q^{\tau=2,d=4}_{\ell,m=1}(-2\gamma_{13}) \Big) \Bigg), \tag{6.51}$$

where $r_1(\ell)$ and $r_2(\ell)$ are written in appendix L.1.

The contribution of the subleading twist operators is

$$\sum_{\substack{\ell=0\\ \text{even}}}^{\infty} \sum_i C^2_{\tau_{4,i}(\ell)} \omega^{\tau_{4,i}(\ell),\ell,0}_{1,0,0}|_{g^3} = - \sum_{\substack{\ell=0\\ \text{even}}} \frac{2^{\ell+1}(\ell(\ell+3)+8)\Gamma\left(\ell+\frac{5}{2}\right)}{3\sqrt{\pi}(\gamma_{13}-2)(\ell+1)\Gamma(\ell+3)^2} \gamma_1(\tau_4(\ell)) Q^{\tau=4,d=4}_{\ell,m=0}(-2\gamma_{13}). \tag{6.52}$$

Expression (6.51) is cumbersome and for this reason we did not manage to compute $\gamma_1(\tau_4(\ell))$ analytically. We computed $\gamma_1(\tau_4(\ell))$ through the following numerical procedure: we can sum the series (6.51) and evaluate the orthogonality integral for $Q^{\tau=4,d=4}_{\ell,m=0}(-2\gamma_{13})$ numerically. The obtained results can be found in table 6.1. The integrals can be evaluated extremely efficiently and the reported errors are not due to the evaluation of the integrals. The errors are due to using the approximation (J.21) for Mack polynomials at large spins, so as to sum the series (6.51). The preceding table agrees with the expression [65]

$$\boxed{\gamma_1(\tau_4(\ell)) \equiv \langle \gamma(\tau_4(\ell)) \rangle|_{g^1} = \frac{24}{(\ell+1)(\ell+2)+6}} \tag{6.53}$$

$b_3(\ell)$	Numerical Result
$\ell = 0$	$-29.99998 \pm (6 \times 10^{-5})$
$\ell = 2$	$-3.7541 \pm (3 \times 10^{-4})$
$\ell = 4$	$-1.3029 \pm (8 \times 10^{-4})$
$\ell = 6$	$-0.628 \pm (1.5 \times 10^{-3})$
$\ell = 8$	$-0.358 \pm (3 \times 10^{-3})$
$\ell = 10$	$-0.226 \pm (5 \times 10^{-3})$
$\ell = 12$	$-0.152 \pm (7 \times 10^{-3})$
$\ell = 14$	$-0.107 \pm (5 \times 10^{-3})$

Table 6.2. Numerical results for the averaged corrections to the three-point functions of the twist four operators at order g^3.

for the averaged anomalous dimensions of twist 4 operators.

Finally, let us consider corrections to the OPE coefficients of twist 4 operators. We computed the function $b_3(\ell)$ for low spins numerically using the functional $\omega_{1,1,1}$. As before to find $b_3(\ell)$ we use the orthogonality property of Mack polynomials (J.8). By integrating $\omega_{1,1,1}|_{g^3}$ against $(8-2\gamma_{13})\Gamma^2(\gamma_{13})\,\Gamma^2(2-\gamma_{13})\,Q^{\tau=4,d=4}_{\ell,m=0}(-2\gamma_{13})$ we get an equation that expresses $b_3(\ell)$ in terms of the previously found OPE data. The results that we found are presented in table 6.2. These predictions are new. We could not guess an analytic formula for $b_3(\ell)$. It is clear from the numerical results that $b_3(0) = -30$.

6.9 Order g^4

Let us outline the computation of the OPE coefficients of the leading twist trajectory and of ϕ^2 at order g^4. We assume the formulas for $\gamma_4(j_l)$ and a_3.[2] We will use the $\omega_{0,0,0}$ functional. The computation is numerically more involved than at lower orders, since there are more tails to sum, as we will see next.

Let us take into account the dependence on the quantum number m. The sum rule can be written as $\sum_{\tau,\ell}\sum_m C_{\tau,\ell}\,\omega^{\tau,\ell,m}_{0,0,0} = 0$. For nonzero m, $\omega^{\tau,\ell,m}_{0,0,0}$ has double zeros for every double twist operator. For $m=0$, $\omega^{\tau,\ell,m}_{0,0,0}$ has double zeros at the subleading twist trajectories, $\tau = 2\Delta+2, 2\Delta+4, \ldots$ and it is nonzero for the leading twist trajectory at $\tau = 2\Delta$.

The OPE coefficients of operators of twist 6 or of higher twist are at most of order g^4. Since $\omega_{0,0,0}$ vanishes for the subleading twists, their contribution comes proportional to the anomalous dimensions squared, which are generically of order g^1. We conclude that

[2] Expression (3.21) in [66] for $\gamma_4(j_\ell)$ contains a typo. There should be $-65/96$ instead of $-65/81$. We are very grateful to Apratim Kaviraj for letting us know about it and for his precious help in navigating the ϵ expansion literature.

Three-point function	Numerical Result	Analytic Expression
$c_4(\phi^2)$	$-15.830116 \pm (2\times 10^{-6})$	?
$c_4(2)$	$0.0814153 \pm (3\times 10^{-7})$	$\frac{6037}{10368} - \frac{5\zeta(3)}{12} = 0.08141533356$
$c_4(4)$	$0.05753436 \pm (5\times 10^{-8})$	$\frac{1964452177}{11854080000} - \frac{9\zeta(3)}{100} = 0.05753437588$
$c_4(6)$	$0.05416485 \pm (5\times 10^{-8})$	$\frac{30173094509693}{298200051072000} - \frac{23\zeta(3)}{588} = 0.05416483628$

Table 6.3. Numerical and analytic results for the three-point functions of the twist two operators and ϕ^2 at order g^4.

twist 6 operators or higher do not contribute at order g^4. So, only twist 2 and twist 4 operators will contribute to the $\omega_{0,0,0}$ sum rule at order g^4.

Concerning twist 4 operators, their contribution at a given spin l will come proportional to $\sum_i C^2_{\tau_{4,i}(\ell)}\gamma^2(\tau_{4,i}(\ell))|_{g^4}$, where the index i denotes the degeneracy of twist 4 operators at spin i. We obtained the value of this quantity from [62] (combine equations 3.6 and 3.10 in [62])

$$\sum_i C^2_{\tau_{4,i}(\ell)}\gamma^2(\tau_{4,i}(\ell))|_{g^4} = \frac{\sqrt{\pi}2^{-2\ell-1}(\ell(\ell+3)+6)\Gamma(\ell+1)}{(\ell+1)(\ell+2)^2\Gamma(\ell+\frac{3}{2})}. \tag{6.54}$$

For this calculation, we implement a numerical scheme that is a little different from the previous cases. We consider the functional $f_4(s) = (2+s)^3(4+s)^2\omega_{0,0,0}(s)$. Afterwards, we use the orthogonality relation (J.8) for each spin ℓ, even and odd. This will give us nontrivial equations that determine the CFT data. $f_4(s)$ has the advantage that it allows to compute the $m=0$ contributions of the operators in the leading and subleading twist trajectories easily. Such operators contribute polynomially to the sum rule. So, we can just decompose their contribution into Mack polynomials, without needing to do the integrals (J.8) explicitly. We computed the OPE coefficients of twist 2 operators with low spins and the results that we obtained are presented in table 6.3. The errors come from not taking into account large spin tails. These are hard to compute because of difficulties in evaluating Mack polynomials at large spins. $c_4(\ell)$ for $\ell \geqslant 2$ were computed exactly in [62]. Our numerical estimates agree with the exact values which can be computed using the explicit formula presented in appendix K. The prediction for $c_4(\phi^2)$ is new.

Based on the structure of the perturbative expansion in g we expect the analytic answer for $c_4(\phi^2)$ to contain π^4, $\zeta(3)$, $\zeta(5)$ with some simple rational coefficients. For example we can consider

$$1 - \frac{6\pi^4}{5} + \frac{31\zeta(3)}{24} + 95\zeta(5) = -15.83011567...$$

Our precision is not good enough to exclude or confirm various possibilities of this type.

Let us compare $c_4(\phi^2)$ that we obtained with the values from the 3d Ising model [50, 67]. In that case the relevant three-point function is $C^2_{\sigma\sigma\epsilon} = 1.1063962(92)$. This should be matched to the results of the ϵ-expansion at $\epsilon = 1$

$$\begin{aligned} C^2_{\phi\phi\phi^2} &= 2 - \frac{2\epsilon}{3} - \frac{34}{81}\epsilon^2 + \frac{1863\zeta(3) - 611}{4374}\epsilon^3 \\ &+ \left(- \frac{6859}{472392} + \frac{323\zeta(3)}{729} - \frac{80\zeta(5)}{81} + \frac{\pi^4}{405} + \frac{2}{81}c_4(\phi^2) \right)\epsilon^4|_{\epsilon=1} \\ &= 2.0000000 - 0.6666667 - 0.4197531 + 0.3722981 - 0.656398 \\ &= 0.62948 \end{aligned} \tag{6.55}$$

where we used that $c_4(\phi^2) = -15.830116$. Notice that the order ϵ^4 correction makes the agreement with 3d Ising model worse. At order ϵ^2 the deviation between $C^2_{\phi\phi\phi^2}$ and $C^2_{\sigma\sigma\epsilon}$ is -0.1927. At order ϵ^3 it is 0.179518. At order ϵ^4 it is -0.47688. This is a manifestation of the asymptotic nature of the ϵ-expansion. [3]

[3] Padé approximations are often used in this context (see for example [68]). We checked that a rational ansatz for $C^2_{\phi\phi\phi^2}$ with a degree 3 polynomial of ϵ in the numerator and a degree 2 denominator, which is completely fixed by the Taylor expansion (6.55) and the condition $C^2_{\phi\phi\phi^2} = \frac{1}{4}$ for $\epsilon = 2$, does not have poles for $0 < \epsilon < 2$. This approximation gives $C^2_{\phi\phi\phi^2} \approx 1.15$ for $\epsilon = 1$. This result is relatively insensitive to the ϵ^4 term.

7 Holographic Applications

7.1 Introduction

It is interesting to consider a broader family of functionals than the ones that we discussed so far. In particular, we consider

$$\omega_{p_2,p_3} = \frac{2}{(\gamma_{12}+p_2)(\gamma_{14}+p_3)} M(\gamma_{12},\gamma_{13}) + (\text{OPE data}) = 0, \tag{7.1}$$

where p_2, p_3 are positive integers. By "OPE data" we mean the dispersive representation of $\frac{2}{(\gamma_{12}+p_2)(\gamma_{14}+p_3)} M(\gamma_{12},\gamma_{13})$, which can be expressed in terms of the OPE data and Mack polynomials. It is then convenient to define

$$\omega^{p_2,p_3}(\gamma_{12},\gamma_{13}) \equiv \frac{(\gamma_{12}+p_2)(\gamma_{14}+p_3)}{2} \omega_{p_2,p_3}. \tag{7.2}$$

To get a sum rule that involves only the OPE data and not the Mellin amplitude itself we use crossing symmetry. More precisely, $\gamma_{13} \leftrightarrow \gamma_{14}$ crossing symmetry of the Mellin amplitude $M(\gamma_{12},\gamma_{13})$ implies that

$$\omega^{p_2,p_3}(\gamma_{12},\gamma_{13}) - \omega^{p_2,p_3}(\gamma_{12},\gamma_{14}) = \sum_{\tau,\ell,m} C^2_{\tau,\ell} \Lambda^{p_2,p_3}_{\tau,\ell,m}(\gamma_{13},\gamma_{14}) = 0. \tag{7.3}$$

Since $M(\gamma_{12},\gamma_{13})$ cancels in (7.3), this sum rule involves only the OPE data. This family of functionals depends on two integer parameters p_2 and p_3, and two continuous parameters γ_{13} and γ_{14}. The function $\Lambda^{p_2,p_3}_{\tau,\ell,m}(\gamma_{13},\gamma_{14})$ has zeros at the double twist locations $\tau = 2\Delta + 2n$. It has single zeros when when $m+n = p_2$ or $m+n = p_3$, where $m \geqslant 0$ and $m \in \mathbb{Z}$, and otherwise the zeros are double zeros.

We will find it useful below to consider $p_2 = p_3 = 0$. In this case the functional has a single zero at the leading double twist trajectory and double zero otherwise. An example of the functional of this type was considered in chapter 5, namely

$$\partial_x \left[\omega^{0,0}(\frac{\Delta}{3}, \frac{\Delta}{3} - x) - \omega^{0,0}(\frac{\Delta}{3}, \frac{\Delta}{3} + x) \right] |_{x=0} = 0. \tag{7.4}$$

This functional was found to possess interesting positivity properties, namely all operators with $\tau \geqslant 2\Delta$ produce a nonnegative contribution.

We can similarly consider

$$\tilde{\omega}_{p_1,p_2} = \frac{2}{(\gamma_{12}+p_1)(\gamma_{12}+p_2)} M(\gamma_{12}, \gamma_{13}) + \text{OPE} = 0. \tag{7.5}$$

Again defining

$$\tilde{\omega}^{p_1,p_2}(\gamma_{12}, \gamma_{13}) \equiv \frac{(\gamma_{12}+p_1)(\gamma_{12}+p_2)}{2} \tilde{\omega}_{p_1,p_2}, \tag{7.6}$$

and using crossing symmetry of the Mellin amplitude we can get sum rules in terms of the OPE data only

$$\tilde{\omega}^{p_1,p_2}(\gamma_{12}, \gamma_{13}) - \tilde{\omega}^{p_1,p_2}(\gamma_{12}, \gamma_{14}) = 0. \tag{7.7}$$

A new feature here compared to (7.3) is that by setting $p_1 = p_2 = p$ we are probing the sub-leading Polyakov condition. In this case the functional has zeros at the position of double twist operators $\tau = 2\Delta + 2n$ except at $n + m = p$, where $m \geqslant 0$ and $m \in \mathbb{Z}$. Next we consider a few examples of applications of these functionals.

7.2 $\lambda\phi^4$ in AdS at one loop

Let us consider a scalar field in AdS of dimension Δ and introduce a quartic interaction

$$\delta S_E = \frac{\lambda}{4!} \int d^d x \sqrt{g} \phi^4. \tag{7.8}$$

At tree-level this interaction leads to anomalous dimension of spin zero double twist operators [69]

$$\gamma^{(1)}_{n,\ell=0} = \lambda \frac{2^{-d-1}\pi^{-d/2}\Gamma\left(\frac{d}{2}+n\right)\Gamma(\Delta+n)\Gamma\left(\Delta-\frac{d}{2}+n+\frac{1}{2}\right)\Gamma\left(2\Delta-\frac{d}{2}+n\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(n+1)\Gamma\left(\Delta+n+\frac{1}{2}\right)\Gamma\left(\Delta-\frac{d}{2}+n+1\right)\Gamma(2\Delta-d+n+1)}. \tag{7.9}$$

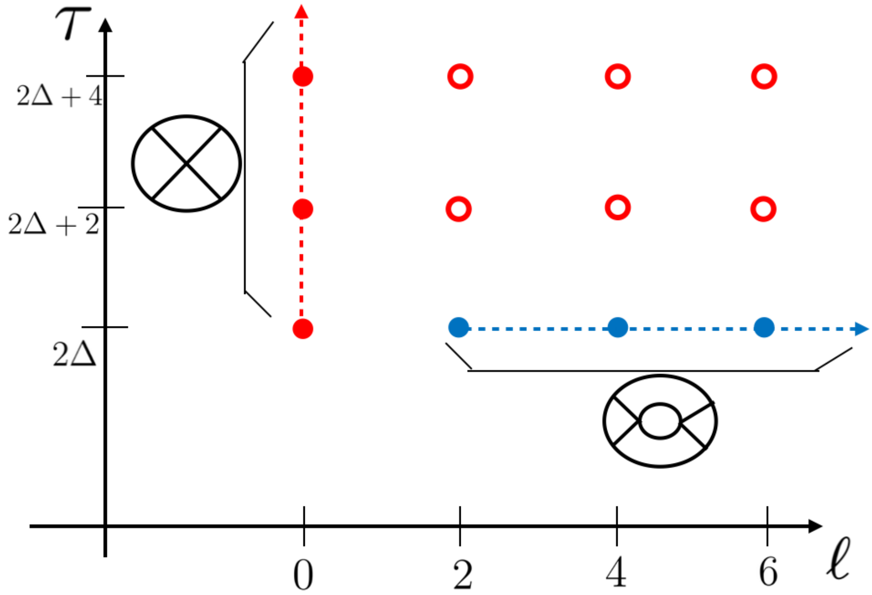

Figure 7.1. The functional $\Lambda_{\tau,\ell,m}$ given in (7.10), and its contribution to the sum rule (7.11). The red dots signify double zeros of the functional, $\Lambda_{\tau,\ell,m}$, and the blue dots signify *single* zeros of the functional. At subleading order, the sum rule (7.11) receives two types of contributions. A contribution at $\ell = 0$ and $n = 0, 1, 2, \ldots$ coming from contact diagrams in AdS. And a contribution from the leading tracjectory $\tau = 2\Delta$ and $\ell = 2, 4, 6, \ldots$ coming from the 1-loop bubble diagrams in AdS.

We next consider the action of the functional (7.3) with $p_2 = p_3 = 0$ which gives [1]

$$\begin{aligned}\Lambda_{\tau,\ell,m} =& 8\frac{\gamma_{14}(\gamma_{13}+\gamma_{14}-\Delta)(\tau+2m-\gamma_{13}-\Delta)\mathcal{Q}^{\tau,d}_{\ell,m}(-2\gamma_{13})}{(\tau+2m-2\gamma_{13})(\tau+2m+2\gamma_{14}-2\Delta)(\tau+2m-2\Delta)(\tau+2m-2\gamma_{13}-2\gamma_{14})}\\ &-8\frac{\gamma_{13}(\gamma_{13}+\gamma_{14}-\Delta)(\tau+2m-\gamma_{14}-\Delta)\mathcal{Q}^{\tau,d}_{\ell,m}(-2\gamma_{14})}{(\tau+2m-2\gamma_{14})(\tau+2m+2\gamma_{13}-2\Delta)(\tau+2m-2\Delta)(\tau+2m-2\gamma_{13}-2\gamma_{14})}.\end{aligned} \tag{7.10}$$

Expanding to the leading order in λ, we find

$$\sum_{m,n=0}^{\infty}\frac{1}{2}C^2_{n,\ell=0}(\gamma^{(1)}_{n,\ell=0})^2\partial^2_\tau\Lambda_{\tau=2\Delta+2n,\ell=0,m}+\sum_{\substack{\ell=2\\ \text{even}}}^{\infty}C^2_{n=0,\ell}\gamma^{(2)}_{0,\ell}\partial_\tau\Lambda_{\tau=2\Delta,\ell,m=0}=0, \tag{7.11}$$

where $\gamma^{(2)}_{0,\ell}$ is the $O(\lambda^2)$ anomalous dimension of the leading double twist trajectory operators. The situation is depicted on fig. 7.1.

We then plug the formula (7.10) into (7.11) and note that the sum rule factorizes. In other words, it takes the form $f(\gamma_{13}) = f(\gamma_{14})$ for arbitrary γ_{13} and γ_{14}. The solution to

[1] From now on, we do not write the superscripts $p_2 = p_3 = 0$ and the arguments γ_{13}, γ_{14} of the function $\Lambda_{\tau,\ell,m}$ to avoid cluttering.

this of course is that $f(\gamma_{13}) = \text{const}$. Next we project this result to the collinear Mack polynomial of given spin $\ell \geqslant 2$ and $\tau = 2\Delta$ using (J.8). Note that the unknown constant is projected out. In this way we get the following equation

$$\gamma^{(2)}_{0,\ell} = -\frac{1+(-1)^\ell}{2}\sum_{n=0}^{\infty} C^2_{n,\ell=0}(\gamma^{(1)}_{n,\ell=0})^2 \mathcal{I}_{n,\ell}\,, \quad \ell \geqslant 2, \tag{7.12}$$

$$\mathcal{I}_{n,\ell} = \sum_{m=0}^{\infty} \frac{2^{4\Delta+\ell+2n-4}\Gamma(m+n+1)^2\Gamma\left(\Delta+n+\frac{1}{2}\right)\Gamma\left(2\Delta-\frac{d}{2}+2n+1\right)}{\Gamma(m+1)\Gamma(\Delta+n)^3\Gamma\left(2\Delta-\frac{d}{2}+m+2n+1\right)}$$

$$\times \frac{\Gamma(\Delta)^2\Gamma\left(\Delta+\ell-\frac{1}{2}\right)}{\pi\Gamma(\Delta+\ell)\Gamma(2\Delta+\ell-1)}\int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\frac{Q^{2\Delta,d}_{\ell,0}(s)}{\frac{s}{2}+m+n+\Delta}\Gamma^2\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s+2\Delta}{2}\right).$$

It is possible to compute $\mathcal{I}_{n,\ell}$ precisely using the results of [55] as we explain in the next section, see equations (7.18)-(7.19). More precisely we get that

$$\mathcal{I}_{n,\ell} = -\frac{1}{C^2_{0,\ell}}\frac{(\delta hP|_{\text{pert}}+\delta hP|_{\text{nonpert}})}{(\Delta+n-\frac{\tau_\chi}{2})^2}\Big|_{\tau_\chi=2\Delta+2n} = -\frac{1}{2C^2_\chi}\frac{\gamma^{\tau_\chi,\text{exch}}_{\ell}}{(\Delta+n-\frac{\tau_\chi}{2})^2}\Big|_{\tau_\chi=2\Delta+2n}, \tag{7.13}$$

where $\delta hP|_{\text{pert}}$ is given by (2.36) and $\delta hP|_{\text{nonpert}}$ by (2.37) in [55] upon substituting $h_i = \frac{\Delta}{2}$, $h_{\mathcal{O}} = \frac{\tau_\chi}{2}$ and $\bar{h} = \Delta + J$. By $\gamma^{\tau_\chi,\text{exch}}_{\ell}$ we denote the anomalous dimension induced by the tree level exchange of a scalar field χ in AdS, analyzed in the next section. The constant C_χ is the coefficient of the three point function $\langle\phi\phi\chi\rangle$ and it is proportional to the corresponding bulk cubic coupling. Eqs (7.12) and (7.13) compute the 1-loop anomalous dimension in $\lambda\phi^4$ theory, in terms of an infinite sum of tree-level anomalous dimensions in $g\phi^2\chi$ theory.[2]

We checked numerically that the formulas above agree with the conformal data from [70], who computed the 1-loop diagrams in AdS_4, see (5.16) there. They computed double trace anomalous dimensions for the leading twist $n=0$ trajectory and spin J, for two values of external $\Delta = 1, 2$:

$$\begin{aligned}
\gamma^{(2)}_{0,\ell} &= -\left[\frac{4}{2\ell+1}\psi^{(1)}(\ell+1)+\frac{2}{\ell(\ell+1)}\right]\gamma^2, \quad \ell \geqslant 2, \quad \Delta = 1,\\
\gamma^{(2)}_{0,\ell} &= -\frac{6}{\ell(\ell+1)(\ell+2)(\ell+3)}\gamma^2, \qquad\qquad \ell \geqslant 2, \quad \Delta = 2,
\end{aligned} \tag{7.14}$$

where γ is the tree level anomalous dimension $\gamma \equiv \gamma^{(1)}_{n,J=0} = \frac{1}{8\pi^2}$, which turns out to be independent of n for $\Delta = 1, 2$ and $d = 3$.

[2]This is reminiscent of the identity $[G_\Delta(X,Y)]^2 = \sum_{n=0}^{\infty} a_n(\Delta)G_{2\Delta+2n}(X,Y)$ that expresses the square of the AdS bulk-to-bulk scalar propagator $G_\Delta(X,Y)$ as a sum of propagators with double-trace dimensions [69]. This identity can be used to write the 1-loop diagrams of $\lambda\phi^4$ theory in AdS in terms of tree level exchange diagrams, which is what formulas (7.12) and (7.13) achieve.

It should be possible to determine one-loop anomalous dimensions of higher twist operators $\gamma^{(2)}_{k,\ell}$ using the functional (7.3) with $p_2 + p_3 > 0$ but we do not pursue this here.

7.3 Tree-level scalar exchange in AdS

Consider the interaction $g\phi^2\chi$ in AdS, where ϕ is a scalar with scaling dimension Δ and χ is a scalar field with twist τ_χ. At tree level, the sum rule is:

$$C_\chi^2 \sum_{m=0}^{\infty} \Lambda_{\tau_\chi,0,m} + \sum_{\substack{\ell=2 \\ \text{even}}}^{\infty} C^2_{n=0,\ell}\gamma_\ell^{\tau_\chi,\text{exch}} \partial_\tau \Lambda_{\tau=2\Delta,\ell,m=0} = 0 \tag{7.15}$$

where the functional $\Lambda_{\tau,\ell,m}$ is defined in (7.10). The first term above is the single trace exchange of χ, the second term is the double trace contribution from the leading trajectory $n=0$, and $\gamma_\ell^{\tau_\chi,\text{exch}}$ are the double-trace anomalous dimensions with $n=0$, arising from exchange of a bulk field χ. The second term above can be written explicitly as:

$$\partial_\tau \Lambda_{\tau=2\Delta,\ell,m=0} = -\frac{2^{-1-\ell}\Gamma(2\Delta+2\ell)\Gamma(2\Delta+2\ell-1)}{\Gamma(\Delta+\ell)^4\Gamma(2\Delta+\ell-1)}\left(Q^{2\Delta,d}_{\ell,0}(-2\gamma_{13}) - Q^{2\Delta,d}_{\ell,0}(-2\gamma_{14})\right) \tag{7.16}$$

The equation above depends on γ_{13} only through $Q^{2\Delta,d}_{\ell,0}(-2\gamma_{13})$. Thus, we can extract $\gamma_\ell^{\tau_\chi,\text{exch}}$ by integrating (7.15) against $\int \frac{ds}{4\pi i}\Gamma^2(-\frac{s}{2})\Gamma^2(\frac{s+\tau}{2})Q^{2\Delta,d}_{\ell,0}(s)$ with $s=-2\gamma_{13}$, and using orthogonality of the Mack polynomials. This gives:

$$\gamma_\ell^{\tau_\chi,\text{exch}} = \frac{-C_\chi^2\,\Gamma^2(\Delta)\Gamma(\Delta+\ell-\frac{1}{2})}{\sqrt{\pi}2^{2-\ell-2\Delta}\Gamma(\Delta+\ell)\Gamma(2\Delta+\ell-1)} \tag{7.17}$$
$$\times \int_{-i\infty}^{i\infty}\frac{ds}{4\pi i}\Gamma^2\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s+2\Delta}{2}\right)Q^{2\Delta,d}_{\ell,0}(s)\sum_{m=0}^{\infty}\Lambda_{\tau_\chi,0,m}$$

where $\ell \geqslant 2$. This equation computes the $n=0$ double trace anomalous dimension arising from a tree level exchange of a scalar χ. Plugging Eq. (7.10) inside (7.17) gives:

$$\gamma_\ell^{\tau_\chi,\text{exch}} = \frac{C_\chi^2 2^{\ell+2\Delta-2}\Gamma^2(\Delta)\Gamma(\Delta+\ell-\frac{1}{2})}{\pi^{\frac{5}{2}}\Gamma(\Delta+\ell)\Gamma(2\Delta+\ell-1)}\frac{\Gamma(\tau_\chi)\sin^2(\pi(\Delta-\frac{\tau_\chi}{2}))}{\Gamma^4(\frac{\tau_\chi}{2})}\sum_{m=0}^{\infty}\frac{\Gamma^2(\frac{2m-2\Delta+\tau_\chi+2}{2})}{\Gamma(m+1)(\tau_\chi-\frac{d}{2}+1)_m}$$
$$\times(-1)\int_{-i\infty}^{i\infty}\frac{ds}{4\pi i}Q^{2\Delta,d}_{\ell,0}\,\Gamma^2\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s+2\Delta}{2}\right)\left(\frac{1}{s+2m+\tau_\chi}+\frac{1}{-s+2m-2\Delta+\tau_\chi}\right) \tag{7.18}$$

In principle, this computation can be generalized to exchanges of operators with spin $J>0$. The main difference is that the associated tree-level Mellin amplitude grows like γ_{12}^{J-1} in the Regge limit. This means that we need to take into account the contribution from the arcs at infinity in (6.1) or choose a function F decaying faster than $1/\gamma_{12}^{J+1}$ at infinity.

Note that the RHS of (7.17) has double zeros at $\tau_\chi = 2\Delta + 2n$, due to the factor $\sin^2(\pi(\Delta - \frac{\tau_\chi}{2}))$. We now take the limit $\tau_\chi \to 2\Delta + 2n$ of Eq. (7.18), and notice that we get:

$$-\frac{1}{2C_\chi^2}\frac{\gamma_\ell^{\tau_\chi,\mathrm{exch}}}{(\Delta+n-\frac{\tau_\chi}{2})^2}\Bigg|_{\tau_\chi\to 2\Delta+2n} = \mathcal{I}_{n,\ell} \tag{7.19}$$

where $\mathcal{I}_{n,\ell}$ is given in (7.12). In other words, the coefficients of the double zeros are proportional to $\mathcal{I}_{n,\ell}$. We used this result in equation (7.13).

We could also study ϕ^3 theory at 1-loop level. Notice that this involves Witten diagrams, like the box diagram in AdS, that are not present in ϕ^4 theory. Using methods similar to the ones in section 7.2, one should be able to derive formulas for the one-loop anomalous dimensions of the leading twist operators $\gamma_{0,\ell}^{(2)}$.

7.4 $(\partial\phi)^4$ in AdS

Let us next consider a weakly coupled theory in AdS with the following low energy action

$$S_E = \int d^{d+1}x\sqrt{g}\left[\frac{1}{2}(\partial_\mu\phi)^2 + \lambda(\partial_\mu\phi\partial^\mu\phi)^2 + ...\right], \tag{7.20}$$

where λ is a small parameter and ... stands for terms that are higher order in λ.

We can study this theory with the functional (7.4) which was also considered in [63]. It takes the following form

$$\partial_x\left[\omega^{0,0}(\frac{\Delta}{3},\frac{\Delta}{3}-x) - \omega^{0,0}(\frac{\Delta}{3},\frac{\Delta}{3}+x)\right]|_{x=0} = \sum_{\tau,\ell,m} C_{\tau,\ell}^2\alpha_{\tau,\ell,m} = 0, \tag{7.21}$$

$$\alpha_{\tau,\ell,m} = -\frac{16\Delta}{3(\tau-\frac{2\Delta}{3}+2m)(\tau-\frac{4\Delta}{3}+2m)}\left(\frac{(\tau+2m-\Delta)\mathcal{Q}_{\ell,m}^{\tau,d}(\gamma_{13}=\frac{\Delta}{3})}{(\tau-\frac{2\Delta}{3}+2m)(\tau-\frac{4\Delta}{3}+2m)} - \frac{\Delta}{3}\frac{\partial_{\gamma_{13}}\mathcal{Q}_{\ell,m}^{\tau,d}(\gamma_{13}=\frac{\Delta}{3})}{\tau+2m-2\Delta}\right).$$

At tree level, the contact diagram contributes only to the OPE data of the double twist operators with spin $\ell = 0$ and $\ell = 2$. Since the functional (7.21) has double zeros at the position of double twist operators for $\ell = 0$, the only contribution at order λ comes from $\ell = 2$ and $\tau = 2\Delta$ (i.e $n = 0$). Operators with $n \geqslant 1$ contribute at order $O(\lambda^2)$. The sum rule at order $O(\lambda)$ therefore takes the following form

$$\frac{2\Delta(\Delta+1)(2\Delta+1)\Gamma(2\Delta+4)}{3\Gamma(\Delta+2)^4}C_{n=0,\ell=2}^2\gamma_{n=0,\ell=2}^{(1)} + \sum_{\tau\geqslant 2\Delta,\ell,m} C_{\tau,\ell}^2\alpha_{\tau,\ell,m}\big|_{\lambda^1} = 0. \tag{7.22}$$

Non-negativity of the functional $\alpha_{\tau\geqslant 2\Delta,\ell,m}$ then implies non-positivity of the anomalous dimension

$$\boxed{\gamma^{(1)}_{n=0,\ell=2} \leqslant 0} \tag{7.23}$$

This matches the previously derived causality constraint from [28,71]. The argument above only applies to $\frac{d-2}{2} < \Delta < \frac{3(d-2)}{4}$, where our functional converges.

Let us now briefly discuss a possible mechanism how (7.22) can be satisfied. In order for it to work, there must be a cancelation between the tree level result $\sim \gamma^{(1)}_{n=0,\ell=2}$ and heavy operators $\tau \geqslant 2\Delta$. We want to estimate the contribution of the heavy operators into the sum rule. At large energies the theory becomes non-perturbative, we can estimate this scale by looking at the anomalous dimensions $\gamma^{(1)}_{n,\ell=2}$ at large n and demanding that $\gamma^{(1)}_{n,\ell} \sim 1$. Using dimensional analysis we can immediately write[3]

$$\gamma^{(1)}_{n,\ell=0,2} \sim \lambda(\text{energy})^{d+1} \sim \lambda n^{d+1}. \tag{7.24}$$

Therefore, the value of n above which non-perturbative effects become important is:

$$n_* \sim \lambda^{\frac{-1}{d+1}} \qquad \rightarrow \qquad \tau_* \sim 2n_* \sim \lambda^{\frac{-1}{d+1}} \tag{7.25}$$

We can estimate the expected contribution to the sum rule at large τ and fixed ℓ as

$$\sum_{\tau\geqslant 2\Delta,m} C^2_{\tau,\ell}\alpha_{\tau,\ell,m}|_{\lambda^1} \sim \int_{\tau_*}^{\infty} \frac{d\tau}{\tau^{d+2}} \sim \int_{\lambda^{\frac{-1}{d+1}}}^{\infty} \frac{d\tau}{\tau^{d+2}} \sim \left.\frac{1}{\tau^{d+1}}\right|^{\infty}_{\lambda^{\frac{-1}{d+1}}} \sim \lambda. \tag{7.26}$$

This estimate is therefore consistent with the structure of the sum rule (7.22). In the estimate above we assumed that the universal OPE asymptotic derived in [45] for fixed spin OPE data does not change in the presence of $\sin^2\frac{\pi(\tau-2\Delta)}{2}$.

7.5 UV Complete Holographic Theories

More generally, we can consider a theory with a weakly coupled gravity dual. As in the previous section we can compute the tree-level Mellin amplitude and we will find that

$$\lim_{\gamma_{12}\to\infty} M_{\text{tree}}(\gamma_{12},\gamma_{13}) = \gamma_{12}^2 f_{\text{IR}}(\gamma_{13})(1+O(\gamma_{12}^{-1})) \sim O\left(\frac{1}{c_T}\right), \tag{7.27}$$

where c_T is the central charge of the CFT and it is inversely proportional to the AdS gravitational coupling. In the formula above $f_{\text{IR}}(\gamma_{13})$ receives contributions from the exchanges by spin two particles, e.g. graviton exchange, as well as tree-level higher derivative interactions $(\partial_\mu\phi\partial^\mu\phi)^2$ considered in the previous section, as well as from

[3]The coupling constant has mass dimension $[\lambda] = -d-1$, and $\gamma_{n,\ell}$ is dimensionless.

$\phi^2\phi_{;\mu\nu\sigma}\phi^{;\mu\nu\sigma}$, see [54], which together contribute as $f_{\text{hd}}(\gamma_{13}) = c_1 + c_2\gamma_{13}$. On the other hand, spin zero and spin one particle exchanges will not contribute to (7.27).

Let us also present for completeness the result for the graviton exchange, see formula (164) in [56],

$$f_{\text{grav}}(\gamma_{13}) = C^2_{T_{\mu\nu}} \frac{(d-1)d\Gamma(d+2)\,{}_3F_2\left(\frac{d}{2}-\Delta, \frac{d}{2}-\Delta, \frac{d}{2}+\gamma_{13}-\Delta-1; \frac{d}{2}+1, \frac{d}{2}+\gamma_{13}-\Delta; 1\right)}{32\Gamma\left(\frac{d}{2}+1\right)^4 \Gamma\left(-\frac{d}{2}+\Delta+1\right)^2 (d-2\Delta+2\gamma_{13}-2)}. \tag{7.28}$$

If we now consider a function

$$F(\gamma_{12}, \gamma_{13}) = \frac{1}{\gamma_{12}^3}\left(1 + O(\gamma_{12}^{-1})\right) \tag{7.29}$$

in the functional (6.1) we will find

$$\oint_{\mathcal{C}_\infty} \frac{d\gamma_{12}}{2\pi i} M_{\text{tree}}(\gamma_{12}, \gamma_{13}) F(\gamma_{12}, \gamma_{13}) = f_{\text{IR}}(\gamma_{13}). \tag{7.30}$$

On the other hand, by closing the contour on the singularities of M_{tree} and $F(\gamma_{12}, \gamma_{13})$ we will get the type of sum rules that we analyzed in this thesis, namely

$$C^2_{T_{\mu\nu}} \sum_{m=0}^{\infty} \omega_F^{d,2,m} + \sum_{n=0}^{n_{\max}} \sum_{\ell,m} C^2_{\tau(n,\ell),\ell}\, \omega_F^{\tau(n,\ell),\ell,m} = f_{\text{IR}}(\gamma_{13}), \tag{7.31}$$

where the second term in the LHS of (7.31) represents the contribution of the double trace operators. For simple choices of meromorphic F that we consider here only a finite number of double trace families contribute at leading order in $\frac{1}{c_T}$. This is signified by $n_{\max}$ in the sum above.

Consider now a UV complete theory which at low energies is given by $M_{\text{tree}}(\gamma_{12}, \gamma_{13})$. We can apply to this theory a functional ω_F to get a sum rule

$$\omega_F = \sum_{\tau,\ell,m} C^2_{\tau,\ell}\omega_F^{\tau,\ell,m} = 0. \tag{7.32}$$

Imagine we now want to analyze this sum rule to leading order in $\frac{1}{c_T}$. The contribution of M_{tree} to the sum rule (7.32) can be conveniently computed using (7.31). Of course, we can alternatively sum over the relevant single and double trace operator OPE data that contributes at order in $\frac{1}{c_T}$ in (7.32) however it is much simpler to use (7.31) instead. In this way we get that (7.32) becomes

$$f_{\rm IR}(\gamma_{13}) + \sum_{\tau,\ell,m}{}' C^2_{\tau,J} \omega_F^{\tau,J,m}|_{\frac{1}{c_T}} = 0, \tag{7.33}$$

where by $\sum'$ we denoted the contribution of all operators that are perturbatively suppressed and are responsible for the fact that the sum rule is satisfied in the non-perturbative theory. We can think of these operators as the UV completion of the theory. For example, in the theory of a scalar minimally coupled to gravity $f_{\rm IR}(\gamma_{13}) = f_{\rm grav}(\gamma_{13})$. The argument above can be repeated for more general $F(\gamma_{12}, \gamma_{13}) \sim \frac{1}{\gamma_{12}^{2k+1}}$. This would lead to similar sum rules which are sensitive to the $\frac{1}{c_T^k}$ terms in the large c_T expansion.

An interesting situation is when each term in $\sum'$ is non-negative. We discussed some examples of such functionals in the present thesis, see e.g. section 7.4. The sum rule (7.33) then is an interesting prediction about the UV completion of a given theory. It would be very interesting to understand a detailed mechanism how such sum rules are satisfied in the UV completion of gravitational theories and if it imposes any new nontrivial constraint on the consistent low energy effective theories.

8 Minimal Models

8.1 Introduction

In this chapter, we compute Mellin amplitudes in minimal models.[1] Minimal models were discovered in [1] and formulas for correlation functions were found in [72] and [73]. These theories provide a context where we can explicitly compute Mellin amplitudes in interacting non-perturbative CFT's. We will see that in minimal models any correlator of scalar Virasoro primaries has a well defined Mellin amplitude. We will also check in several examples that Mellin amplitudes only have the singularities (2.1) dictated by the OPE.

This chapter is structured as follows. In section 8.2, we briefly review the Coulomb gas formalism of minimal models. The Coulomb gas technique is a way to determine correlation functions that is very suitable for computing the associated Mellin amplitudes. In section 8.3, we compute the Mellin amplitude of $\langle \Phi_{1,2} O \Phi_{1,2} O \rangle$, where $\Phi_{1,2}$ is a second-order degenerate Virasoro primary and O is any Virasoro primary. This is a simple example of a Mellin amplitude that serves as warmup for section 8.4, where we compute the Mellin amplitude of any correlation function of scalar Virasoro primaries. In section 8.5, we consider the Mellin amplitude associated to $\langle \Phi_{1,3} O \Phi_{1,3} O \rangle$ and check that it is a meromorphic function with poles at the locations given by the OPE. This is a non-trivial check that Mellin amplitudes only have the OPE poles, since the Mellin amplitude of $\langle \Phi_{1,3} O \Phi_{1,3} O \rangle$ is given by 5 Mellin-Barnes integrals in our setup, so its pole structure is not apparent. In section 8.6, we compute the bulk point limit of any correlator of Virasoro primaries in minimal models. It agrees with the expectations about the bulk point limit coming from [23].

Upon completion of our work, we learned about the paper [74], whose results overlap with those of this section. To be precise, the idea of using Symanzik's formula in

[1]By minimal models, we mean the 2d CFT's with finite number of Virasoro primaries at $c < 1$ and with diagonal partition function.

conjunction with the Coulomb gas formalism is already present in that paper. It also contains the computation of the Mellin amplitudes of $\langle \Phi_{1,2} O \Phi_{1,2} O \rangle$ and $\langle \Phi_{1,3} O \Phi_{1,3} O \rangle$ and an analysis of the respective pole structure (albeit with a different method). So, our sections 8.3 and 8.5 mostly reproduce results already contained in [74]. By contrast, the other results we present in this chapter are new, to the best of our knowledge.[2]

8.2 The Coulomb gas formalism of minimal models

In this section, we review the Coulomb gas formalism of minimal models, following [76]. Our goal is just to state the formulas we will need, in order to compute Mellin amplitudes. For a detailed review of this technique, see [72,73,76,77].

The Coulomb gas formalism provides a representation of minimal models in terms of the theory of a deformed scalar field. The idea is to associate Virasoro primaries in minimal models with vertex operators in the deformed scalar field theory and thus compute correlation functions in minimal models using the correlation functions of vertex operators in the scalar theory.

To begin with, let us remind ourselves of basic facts about the theory of a massless two dimensional scalar field ϕ. The action is

$$S = \frac{1}{4\pi} \int d^2x \partial\phi \bar{\partial}\phi \tag{8.1}$$

where we used complex coordinates $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. The two point function is given by

$$\langle \phi(x)\phi(x') \rangle = -2 \log |x - x'|^2 . \tag{8.2}$$

Furthermore,

$$T(z) \equiv T_{zz}(z) = -\frac{1}{4} : \partial\phi\partial\phi :, \tag{8.3}$$

from which it follows that $\langle T(z)T(z') \rangle = \frac{1/2}{(z-z')^4}$. Thus, the theory has central charge $c = 1$. For each real number α we can define a vertex operator $V_\alpha \equiv : e^{i\alpha\phi(x)} :$. From the OPE of V_α with $T(z)$, we conclude that V_α is a scalar Virasoro primary, of dimension $2\alpha^2$. Correlation functions of vertex operators are given by

$$\langle V_{\alpha_1}(x_1)...V_{\alpha_n}(x_n) \rangle = \prod_{i<j} |x_i - x_j|^{4\alpha_i\alpha_j}, \tag{8.4}$$

[2] In [75] Mellin amplitudes in minimal models are also studied, but from a different point of view. In particular there it is proposed to define a Mellin amplitude as a transform of a chiral block, whereas we define Mellin amplitudes as transforms of the full correlation function.

if $\sum_{i=1}^{n} \alpha_i = 0$, otherwise the correlation function vanishes.

Now imagine adding to the theory a background charge $-2\alpha_0$, where α_0 is a real number that we can pick. More precisely, correlations functions in the deformed theory are defined by

$$\begin{aligned}\langle V_{\alpha_1}(x_1)...V_{\alpha_n}(x_n)\rangle_{-2\alpha_0} &= \lim_{x_{n+1}\to\infty} |x_{n+1}|^{16\alpha_0^2}\langle V_{\alpha_1}(x_1)...V_{\alpha_n}(x_n)V_{-2\alpha_0}(x_{n+1})\rangle \\ &= \prod_{i<j} |x_i - x_j|^{4\alpha_i\alpha_j},\end{aligned} \tag{8.5}$$

if $\sum_{i=1}^{n} \alpha_i = 2\alpha_0$, otherwise the correlation function vanishes. V_α is still a scalar Virasoro primary, but now with conformal dimension

$$\Delta(\alpha) = 2\alpha^2 - 4\alpha_0\alpha. \tag{8.6}$$

This means that the stress tensor also changed. In fact, the stress energy tensor is

$$T(z) = -\frac{1}{4} : \partial\phi\partial\phi : +i\alpha_0\partial^2\phi. \tag{8.7}$$

As explained in [76], this follows from changing the boundary conditions we impose on ϕ in the derivation of Noether's theorem. We now have that $\langle T(z)T(z')\rangle = \frac{1-24\alpha_0^2}{2(z-z')^4}$, so the deformed theory has central charge

$$c = 1 - 24\alpha_0^2. \tag{8.8}$$

Formula (8.5) is still too simple to represent correlation functions in minimal models. So, besides introducing a background charge, we still need to modify the free field theory further. We notice that equation (8.6) allows for the existence of vertex operators of dimension 2, which we denote by V_+ and V_-. The corresponding α's obey

$$\alpha_+ + \alpha_- = 2\alpha_0\,, \qquad \alpha_+\alpha_- = -1. \tag{8.9}$$

We modify action (8.1) by introducing interacting terms

$$S = \frac{1}{4\pi}\int d^2x\partial\phi\bar{\partial}\phi - \int d^2x\,(V_-(x) + V_+(x))\,. \tag{8.10}$$

We will use theory (8.10) defined with a background charge $-2\alpha_0$ to represent minimal models. In the theory (8.10), correlation functions of vertex operators are given by

$$\langle V_{\alpha_1}(x_1)...V_{\alpha_n}(x_n)\rangle = \frac{1}{l!k!}\prod_{j=1}^{l}\int d^2w_j\prod_{i=1}^{k} d^2y_i \tag{8.11}$$

$$\times \langle V_{\alpha_1}(x_1) ... V_{\alpha_n}(x_n) V_-(y_1) ... V_-(y_l) V_+(w_1) ... V_+(w_k) \rangle_{-2\alpha_0},$$

if we can find non-negative integers l and k such that $\sum_{i=1}^{n} \alpha_i = 2\alpha_0 - l\alpha_- - k\alpha_+$. Otherwise, the correlation function is equal to 0. In the second line of (8.11) we can use (8.5).

Given a minimal model at central charge c, we associate to it a scalar field theory with background charge according to (8.8) and to the Virasoro primaries in that theory, we associate vertex operators according to (8.6), by matching the respective conformal dimensions. Since equation (8.6) is quadratic in α, to the same Virasoro primary of dimension Δ, we can associate two vertex operators V_α and $V_{2\alpha_0-\alpha}$. In minimal models, both the central charge and the dimensions of operators are discretized. For the minimal model $\mathbb{M}(p,q)$ (with $p > q$), we have

$$\alpha_+^2 = \frac{q}{p}. \tag{8.12}$$

This fixes the central charge according to (8.8) and (8.9). Minimal models are composed of Virasoro primaries $\Phi_{m,n}$ that are degenerate, in the sense that they have (null) descendants that are themselves Virasoro primaries. The Virasoro primaries $\Phi_{m,n}$ have conformal dimension given by (8.6), with

$$\alpha_{m,n} = \frac{1-m}{2}\alpha_- + \frac{1-n}{2}\alpha_+, \tag{8.13}$$

where $1 \leqslant m \leqslant p-1$ and $1 \leqslant n \leqslant q-1$. Correlation functions of Virasoro primaries in a minimal model can then be computed using the vertex operator representation

$$\Phi_{m,n}(x) = \frac{1}{N_{m,n}} V_{\alpha_{m,n}}(x) = N_{m,n} V_{2\alpha_0 - \alpha_{m,n}}(x)\,, \tag{8.14}$$

where $N_{m,n}$ is a normalization constant given in equation (A.33) in appendix (A.4). Notice that the representations in terms of $V_{\alpha_{m,n}}$ and $V_{2\alpha_0-\alpha_{m,n}}$ are equivalent but in a given correlation function one choice will typically lead to simpler computations. For example, the two-point function is trivial to compute as follows

$$\langle \Phi_{m,n}(x)\Phi_{m,n}(y) \rangle = \langle V_{\alpha_{m,n}}(x) V_{2\alpha_0-\alpha_{m,n}}(y) \rangle = \frac{1}{|x-y|^{2\Delta(\alpha_{m,n})}}\,. \tag{8.15}$$

On the other hand, the non-trivial computation (see [73])

$$\langle \Phi_{m,n}(x)\Phi_{m,n}(y) \rangle = \frac{1}{N_{m,n}^2} \langle V_{\alpha_{m,n}}(x) V_{\alpha_{m,n}}(y) \rangle = \frac{1}{|x-y|^{2\Delta(\alpha_{m,n})}}\,, \tag{8.16}$$

determines the normalization constant $N_{m,n}$.

8.3 Example: the $\langle\Phi_{1,2}O\Phi_{1,2}O\rangle$ correlator

We consider the four point function $\langle\Phi_{1,2}O\Phi_{1,2}O\rangle$, where $\Phi_{1,2}$ is a second order degenerate Virasoro primary, O is an arbitrary Virasoro primary $\Phi_{m,n}$ and we consider such a correlator for general central charge.[3] The point of considering such a correlator is that it has a simple Mellin amplitude. We will see that its pole structure is dictated by the operator product expansion.

We associate each Virasoro primary to vertex operators in the following way:

$$\Phi_{1,2} \to \frac{1}{N_{1,2}} V_{\alpha_{1,2}}, \qquad O \to \frac{1}{N_{m,n}} V_{\alpha}, \qquad \Phi_{1,2} \to \frac{1}{N_{1,2}} V_{\alpha_{1,2}}, \qquad O \to N_{m,n} V_{2\alpha_0-\alpha}, \tag{8.17}$$

where $\alpha = \alpha_{m,n}$. With these choices, we only need to insert one positive screening charge to compute the correlator

$$\langle\Phi_{1,2}O\Phi_{1,2}O\rangle = \frac{1}{N_{1,2}^2} \prod_{i<j}^{4} |x_i - x_j|^{4\alpha_i\alpha_j} \int d^2w \prod_{k=1}^{4} |x_i - w|^{4\alpha_i\alpha_+}. \tag{8.18}$$

We would like to compute the Mellin amplitude of the correlator (8.18). In our work, the following formula due to Symanzik (see [78]) will prove very useful:

$$\frac{1}{\pi^{\frac{d}{2}}} \int d^d u \prod_{i=1}^{n} \frac{\Gamma(y_i)}{|x_i - u|^{2y_i}} = \prod_{i<j} \int [d\gamma_{ij}] \Gamma(\gamma_{ij}) |x_i - x_j|^{-2\gamma_{ij}}, \tag{8.19}$$

where the Mellin integral is constrained to $\sum_{j\neq i} \gamma_{ij} = y_i$ and the formula is only valid if $\sum_i y_i = d$[4]. Since $4(\sum_{i=1}^4 \alpha_i)\alpha_+ = 4\alpha_-\alpha_+ = -4$, then we can apply (8.19) in (8.18).

[3] For the Yang-Lee and the Ising model, the Mellin amplitude of $\langle\Phi_{1,2}\Phi_{1,2}\Phi_{1,2}\Phi_{1,2}\rangle$ was computed in [27].

[4] Formula (8.19) was derived in [78] under the additional assumptions $0 < Re(y_i) < \frac{d}{2}$. The condition $Re(y_i) < \frac{d}{2}$ ensures that the integral (8.19) converges when $u \to y_i$. The condition $Re(y_i) > 0$ enables us to use the formula

$$\frac{\Gamma(y_i)}{|x_i - u|^{2y_i}} = \int_0^\infty \frac{d\sigma_i}{\sigma_i} \sigma_i^{y_i} e^{-\sigma_i |x_i - u|^2} \tag{8.20}$$

which is used in deducing (8.19). If $Re(y_i) > 0$, then the rhs of (8.20) converges. The assumptions $0 < Re(y_i) < \frac{d}{2}$ ensure that on the rhs of (8.19) the contour of the Mellin variables γ_{ij} can be straight and parallel to the imaginary axis. In this chapter, we will use formula (8.19) in contexts where the assumptions $0 < Re(y_i) < \frac{d}{2}$ are not met. In that case, we can think of the rhs of (8.19) as an analytic continuation in y_i of its lhs. Indeed, consider the integral on the rhs of (8.19) and suppose we start shifting continuously the positions of the poles. At some point, it is no longer possible to use a straight contour and so the contour should bend in order to account for this. Notice that even with a bent contour the integral the rhs is perfectly well defined, since it has the same asymptotics when $\gamma_{ij} \to i\infty$ as before.

Further doing a change of variables, we get

$$\langle \Phi_{1,2}(x_1)O(x_2)\Phi_{1,2}(x_3)O(x_4)\rangle = C_0 \prod_{i<j} \int [d\gamma_{ij}]\Gamma(\gamma_{ij} + 2\alpha_i\alpha_j)|x_i - x_j|^{-2\gamma_{ij}}. \tag{8.21}$$

The Mellin constraints are $\sum_{j\neq i} \gamma_{ij} = 2\alpha_i^2 - 4\alpha_i\alpha_0 = \Delta(\alpha_i)$, which indeed is the dimension of the operator inserted at the position x_i. We associate an α_i to each operator according to (8.17). C_0 is a constant given by

$$C_0 = \frac{\pi}{N_{1,2}^2 \prod_{i=1}^n \Gamma(-2\alpha_i\alpha_+)} = \frac{\Gamma\left(2 - 2\alpha_+^2\right)}{\Gamma^2\left(1-\alpha_+^2\right)\Gamma\left(2\alpha_+^2 - 1\right)\Gamma(-2\alpha\alpha_+)\Gamma\left(2 - 2\alpha_+^2 + 2\alpha\alpha_+\right)}. \tag{8.22}$$

From (8.21) we can read off the Mellin amplitude

$$\begin{aligned}\hat{M}(\gamma_{12}, \gamma_{14}) = {}& C_0\Gamma\left(\gamma_{13} - \frac{2\Delta(\alpha_{1,2}) - \Delta(\alpha_{1,1})}{2}\right)\Gamma\left(\gamma_{13} - \frac{2\Delta(\alpha_{1,2}) - \Delta(\alpha_{1,3})}{2}\right) \\ & \Gamma\left(\gamma_{12} - \frac{\Delta(\alpha_{1,2}) + \Delta(\alpha) - \Delta(\alpha - \frac{1}{2}\alpha_+)}{2}\right)\Gamma\left(\gamma_{12} - \frac{\Delta(\alpha_{1,2}) + \Delta(\alpha) - \Delta(\alpha + \frac{1}{2}\alpha_+)}{2}\right) \\ & \Gamma\left(\gamma_{14} - \frac{\Delta(\alpha_{1,2}) + \Delta(\alpha) - \Delta(\alpha - \frac{1}{2}\alpha_+)}{2}\right)\Gamma\left(\gamma_{14} - \frac{\Delta(\alpha_{1,2}) + \Delta(\alpha) - \Delta(\alpha + \frac{1}{2}\alpha_+)}{2}\right),\end{aligned} \tag{8.23}$$

where $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta(\alpha_{1,2})$. We conclude that the Mellin amplitude is a meromorphic function. The position of its simple poles is dictated by the OPE as in (2.1). To see that, recall that the OPE of an arbitrary Virasoro primary $\Phi_{m,n}$ with $\Phi_{1,2}$ only contains 2 Virasoro primaries,

$$\Phi_{1,2} \times \Phi_{m,n} = \Phi_{m,n+1} + \Phi_{m,n-1}\,. \tag{8.24}$$

Each Γ–function in (8.23) encodes the poles associated to each Virasoro primary exchanged in a given channel. In appendix A.2, we compare the Mellin representation of this four-point function with its expansion in Virasoro conformal blocks. In particular, we check that the overall normalization is correct by matching the contribution of the identity block.

The correlator $\langle\phi_{2,1}O\phi_{2,1}O\rangle$ can be similarly analysed. In the Coulomb gas formalism, we now insert a negative screening charge instead of a positive one. An expression analogous to (8.21) holds for the correlator, with

$$C_0 = \frac{\Gamma(2 - \frac{2}{\alpha_+^2})}{\Gamma^2(1 - \frac{1}{\alpha_+^2})\Gamma(-1 + \frac{2}{\alpha_+^2})\Gamma(2 - \frac{2}{\alpha_+^2} - 2\frac{\alpha}{\alpha_+})\Gamma(\frac{2\alpha}{\alpha_+})}. \tag{8.25}$$

The correlator $\langle \Phi_{1,2} O \Phi_{1,2} O \rangle$ obeys a BPZ differential equation, in virtue of the fact that $\Phi_{1,2}$ is a degenerate operator. This implies that the Mellin amplitude (8.23) obeys a recursion relation. We confirm this in appendix A.1.

Consider now the more general correlator $\langle \Phi_{1,2} O_2 O_3 O_4 \rangle$, where O_2, O_3 and O_4 are arbitrary Virasoro scalar primaries. Given expression (8.23), it is simple to guess a Mellin amplitude for this correlator:

$$\hat{M}(\gamma_{12}, \gamma_{14}) = C_0 \times \tag{8.26}$$
$$\times \prod_{j=2}^{4} \Gamma\left(\gamma_{1j} - \frac{\Delta(\alpha_{1,2}) + \Delta(\alpha_j) - \Delta(\alpha_j - \frac{1}{2}\alpha_+)}{2} \right) \Gamma\left(\gamma_{1j} - \frac{\Delta(\alpha_{1,2}) + \Delta(\alpha_j) - \Delta(\alpha_j + \frac{1}{2}\alpha_+)}{2} \right)$$

where $\gamma_{13} + \gamma_{12} + \gamma_{14} = \Delta(\alpha_{1,2})$ and α_2, α_3 and α_4 can be computed from the conformal dimensions of O_2, O_3 and O_4. We confirm this guess in appendix (A.1), where we check that the correlator $\langle \Phi_{1,2} O_2 O_3 O_4 \rangle$ defined with the Mellin amplitude (8.26) obeys the appropriate BPZ differential equation. A formula for C_0 can be found in appendix (A.3), see formula (A.30).

8.4 Mellin amplitude of any correlation function

In this section, we write the Mellin amplitude of a general n point correlator in minimal models. Our procedure to do so will be a generalisation of the computation in section 8.3. The key ingredients are the Coulomb gas formalism and Symanzik formula (8.19).

Consider then a general n point correlator $\langle \Phi_1(x_1) ... \Phi_n(x_n) \rangle$. To each operator Φ_i, we associate a vertex operator: $\Phi_i \to \frac{1}{N_i} V_{\alpha_i}$. We will assume that all operators are degenerate, i.e. each α_i is of the form (8.13) as it is the case in minimal models. [5] In the Coulomb gas formalism suppose we need to insert p_1 positive screening charges and q_1 negative screening charges to compute $\langle V_{\alpha_1}(x_1) ... V_{\alpha_n}(x_n) \rangle$. Then, this correlator has a Mellin representation, with Mellin amplitude given by

$$\hat{M}(\gamma_{ij}) = C_0 \int \prod_{r=1}^{z-1} \left[\prod_{k_r < l_r}^{n+r} [d\xi^r_{k_r, l_r}] \, \Gamma(\xi^r_{k_r, l_r}) \prod_{s_r=1}^{n+r-1} \frac{1}{\Gamma(-2\alpha_{s_r} \alpha_{n+r} + \sum_{v=r}^{z-1} \xi^v_{s_r, n+r})} \right] \tag{8.27}$$
$$\times \prod_{i<j}^{n} \Gamma(\gamma_{ij} - \sum_{r=1}^{z-1} \xi^r_{ij} + 2\alpha_i \alpha_j),$$

where $\alpha_{n+1} = ... = \alpha_{n+p_1} = \alpha_+$ and $\alpha_{n+p_1+1} = ... = \alpha_{n+p_1+q_1} = \alpha_-$ and $z = p_1 + q_1$ is

[5] Here, we have used the shorter notation $\Phi_i \equiv \Phi_{m_i, n_i}$ and $N_i \equiv N_{m_i, n_i}$.

the total number of screening charges.[6] The measure is constrained by

$$\sum_{\substack{j=1\\ j\neq i}}^{n+r} \xi_{ij}^{r} = -2\alpha_i\alpha_{n+r+1} + \sum_{s=r+1}^{z-1} \xi_{i,n+r+1}^{s}\,, \qquad\qquad i = 1,2,\ldots,n+r, \tag{8.28}$$

for $r = 1, ..., z-1$. The constant C_0 can be determined from the normalisations,

$$C_0 = \frac{\pi^z}{\prod_{i=1}^{n} N_i \prod_{j=1}^{n+z-1} \Gamma(-2\alpha_j\alpha_{n+z})}. \tag{8.29}$$

Formula (8.27) can be deduced by iteratively applying (8.19). We show how to do this for the case in which we insert two screening charges in appendix A.5. The complexity of (8.27) grows very quickly with the total number z of screening charges.[7] Nevertheless, this explicit Mellin-Barnes representation is useful to derive several general properties of the Mellin amplitude.

Formula (8.27) is correct in every CFT for which the Coulomb gas technique of computing correlation functions applies. This is the case in diagonal minimal models (whether they are unitary or not). There are other CFT's that can be viewed as limits of minimal models, like generalised minimal models and Liouville theory (see [79] and [80]).[8] So, perhaps formula (8.27) can be useful in that context. See also [81] and [82] for attempts to generalise the Coulomb gas formalism to higher dimensions. Identities of Dotsenko-Fateev integrals were used in [83] and [84] to derive properties of 3d supersymmetric theories, so maybe our formulas can be useful in that context.

8.5 Analyticity of Mellin amplitudes

We conjecture that the Mellin amplitudes in 2D minimal models are meromorphic functions with only poles at the locations predicted by the OPE as in (2.1). Meromorphicity follows from the Mellin-Barnes representation (8.27). However, we were not able to show in that the only singularities are the ones predicted by the OPE. In this section, we

[6] We assumed $p \geqslant 1$.

[7] Namely, it involves

$$\sum_{r=1}^{z-1} \frac{(n+r)(n+r-3)}{2} = (z-1)\frac{3n(n+z-3)+z(z-5)}{6} \tag{8.30}$$

integrals of an integrand with

$$\sum_{r=1}^{z-1}\left[\frac{(n+r)(n+r-1)}{2} + (n+r-1)\right] + \frac{n(n-1)}{2} = \frac{z^3+3nz(n+z)-6n-7z+6}{6} \tag{8.31}$$

Γ-functions.

[8] We thank Sylvain Ribault for discussions on this.

check this statement for a class of correlators in minimal models.

We consider the correlation function $\langle \Phi_{1,3} O \Phi_{1,3} O \rangle$, where O is an arbitrary Virasoro primary. This correlation function can be computed in the Coulomb gas formalism by the prescription

$$\Phi_{1,3} \to \frac{1}{N_{1,3}} V_{\alpha_{1,3}}, \qquad O \to \frac{1}{N_{m,n}} V_\alpha, \qquad \Phi_{1,3} \to \frac{1}{N_{1,3}} V_{\alpha_{1,3}}, \qquad O \to N_{m,n} V_{2\alpha_0 - \alpha}, \tag{8.32}$$

with $\alpha = \alpha_{m,n}$ and inserting two positive screening charges.

Its Mellin amplitude can be obtained fom the general formulas (8.27) and (8.29). We can write the Mellin amplitude in the form

$$\begin{aligned}
\hat{M}(\gamma_{12}, \gamma_{14}) = C_0 \int [d\xi_{12}] \int [d\xi_{15}] \int [d\xi_{24}] \int [d\xi_{34}] \int [d\xi_{35}] \Gamma(\xi_{12}) \Gamma(\xi_{15}) \\
\Gamma(-2\alpha\alpha_+ + \gamma_{12} - \xi_{12}) \Gamma(\xi_{24}) \Gamma(\xi_{15} - \xi_{24} - \xi_{34} + 1) \Gamma\left(-2\alpha_+^2 + 2\alpha\alpha_+ + \gamma_{12} - \xi_{34} + 2\right) \\
\Gamma(\xi_{34}) \Gamma(-2\alpha\alpha_+ - \xi_{12} - \xi_{15} + \xi_{34} - 1) \Gamma(-2\alpha\alpha_+ + \gamma_{14} - \xi_{15} + \xi_{24} + \xi_{34} - 1) \\
\Gamma\left(-2\alpha_+^2 + 2\alpha\alpha_+ + \gamma_{14} + \xi_{12} + \xi_{24} - \xi_{35} + 1\right) \Gamma\left(2\alpha_+^2 - \xi_{15} + \xi_{24} - \xi_{35} - 1\right) \\
\Gamma\left(\xi_{12} - \xi_{34} - \xi_{35} + 2\alpha\alpha_+ - 2\alpha_+^2 + 1\right) \Gamma(\xi_{35}) \Gamma(-\xi_{12} - \xi_{24} + \xi_{35} + 1) \\
\frac{\Gamma\left(-\gamma_{12} - \gamma_{14} + \xi_{15} - \xi_{24} + \xi_{35} + 4\alpha_+^2 - 1\right)}{\Gamma\left(2\alpha_+^2 + \xi_{15}\right) \Gamma(-4\alpha\alpha_+ - \xi_{12} - \xi_{15} + \xi_{34} - 1)} \\
\frac{\Gamma(-\gamma_{12} - \gamma_{14} - \xi_{24})}{\Gamma\left(2\alpha_+^2 + \xi_{35}\right) \Gamma\left(-4\alpha_+^2 + 4\alpha\alpha_+ + \xi_{12} - \xi_{34} - \xi_{35} + 3\right)}.
\end{aligned} \tag{8.33}$$

The above formula is a complicated integral that the reader should not read carefully. Our point is just to consider an application of formula (8.27). In the rest of this section, we will check that (8.33) only gives rise to singularities at the locations (2.1) predicted by the OPE.

In the OPE of Φ_{13} and Φ_α, there are three Virasoro primaries exchanged:

$$\Phi_{1,3} \times \Phi_\alpha = \Phi_{\alpha+\alpha_+} + \Phi_\alpha + \Phi_{\alpha-\alpha_+}. \tag{8.34}$$

Each Virasoro family contains many global primaries, whose twists differ by even integers. So, we expect the Mellin amplitude to have 9 sequences of poles: 3 sequences of poles in γ_{12}, another 3 in γ_{14} and another 3 in γ_{13}.

Let us now outline how we obtained the singularity structure of (8.33). We used the technique introduced in [85] to compute the singularities of Mellin-Barnes integrals. The reader interested in understanding this technique can read appendix C of that paper. In the next sentences, we briefly describe the method. Suppose we have a multiple Mellin-

Barnes integral and we ask what are its singularities. Given just one Mellin-Barnes integral, it diverges whenever its contour gets pinched by two poles of the integrand.

When we have multiple integrals, we must take a more global perspective. Let us illustrate this with an example taken from [85]. Consider the integral

$$\int \frac{dxdy}{(2\pi i)^2}\Gamma(a_1+x)\Gamma(a_2-x)\Gamma(b_1+x-y)\Gamma(b_2-x+y)\Gamma(c_1+y)\Gamma(c_2-y). \tag{8.35}$$

We might expect expect to obtain its singularities in the following manner. Suppose we first integrate in x and later in y. For a single Melin-Barnes integral, we know how to predict its singularities. So, we might expect to determine the singularities of a multiple Mellin Barnes integral by applying the one dimensional technique in succession. It turns out that if we do that we end up with a lot of fake poles.

In this example, let us list the poles that we would obtain by iteratively applying the one dimensional technique. We can integrate first in x and later in y, or vice versa. The set of singularities thus obtained is:

$$\begin{aligned}\Gamma_{\{1,2\}}(a_1+a_2),\Gamma_{\{3,4\}}(b_1+b_2),\Gamma_{\{5,6\}}(c_1+c_2),\Gamma_{\{1,4,6\}}(a_1+b_2+c_2),\Gamma_{\{2,3,5\}}(a_2+b_1+c_1)\\ \Gamma_{\{1,2,3,4\}}(a_1+a_2+b_1+b_2),\Gamma_{\{1,2,5,6\}}(a_1+a_2+c_1+c_2),\Gamma_{\{3,4,5,6\}}(b_1+b_2+c_1+c_2),\\ \Gamma_{\{1,2,3,4,5,6\}}(a_1+a_2+b_1+b_2+c_1+c_2).\end{aligned} \tag{8.36}$$

Here, the use of the Γ function is symbolic. We just mean that the integral is singular whenever the argument of the Γ functions above is a nonpositive integer. We use a subscript to denote the position of the poles that gave rise to the singularity in the original integrand. For example, $\Gamma_{\{1,2\}}(a_1+a_2)$ came from the collision of $\Gamma(a_1+x)$ and $\Gamma(a_2-x)$.

Consider the Γ functions in (8.36). We will say that the poles coming from a certain Γ function are *composite* whenever the corresponding subscript contains as a subset the subscript of another Γ function in (8.36). The poles from $\Gamma_{\{1,2,3,4\}}$, $\Gamma_{\{1,2,5,6\}}$, $\Gamma_{\{3,4,5,6\}}$ and $\Gamma_{\{1,2,3,4,5,6\}}$ are composite. The rest of the poles are non composite.

All composite poles are fake. This is a very important statement that allows to generate a fast algorithm to determine the real poles in a multiple Mellin-Barnes integral. [85] contains some general arguments in favour of this statement, although it is not clear to the present authors that we can conclude from them that such a statement is rigorously proved. Sill, to the best of our knowledge the algorithm of [85] works in every example.

With this technique, we predict the Mellin amplitude to have singularities whenever any of the following expressions is equal to 0:

$$\{\gamma_{12} + 2\alpha\alpha_+ - 2\alpha_+^2 + 2, \gamma_{13} - 4\alpha_+^2 + 2, \gamma_{14} + 2\alpha\alpha_+ - 2\alpha_+^2 + 2,$$
$$\gamma_{12} - 2\alpha_+^2 + 1, \gamma_{13} - 2\alpha_+^2 + 1, \gamma_{14} - 2\alpha_+^2 + 1,$$
$$\gamma_{12} - 2\alpha\alpha_+, \gamma_{13} + 2\alpha_+^2, \gamma_{14} - 2\alpha\alpha_+\}, \tag{8.37}$$

with $\gamma_{13} = -\gamma_{12} - \gamma_{14} + 4\alpha_+^2 - 2$. This precisely matches the expectations from (8.34). In particular, the first three poles correspond to the exchange of $\Phi_{\alpha+\alpha_+}$, the next three to the exchange of Φ_α and the final three poles to the exchange of $\Phi_{\alpha-\alpha_+}$.

8.6 The bulk point limit

Correlation functions in Lorentzian signature can have divergences due to the existence of a point that is null separated from the points where we insert the external operators (see [23]).

Such singularities can arise from pertubation theory in the boundary or from Landau diagrams in the bulk. In a fully nonperturbative CFT those singularities are expected to be absent. This was proven in $d = 2$ in [23].

We will show that the divergence in the bulk point limit is controlled by the limit of large Mellin variables of the Mellin amplitude. For simplicity we will consider the case of a four point function of equal primaries of conformal dimension Δ.

We have already shown in chapter 1 that the Mellin amplitude is polynomially bounded at infinity. Consider the limit where $\gamma_{12} = \beta s$, $\gamma_{14} = (1-\beta)s$, where $\beta \in [0,1]$ and we take s very large. Suppose that in that limit the Mellin amplitude $M(\gamma_{12} = \beta s, \gamma_{14} = (1-\beta)s) \sim s^b$. Suppose also that the four point function diverges in the bulk point limit like $\frac{1}{(z-\bar{z})^{2a}}$ when $z \to \bar{z}$. In the next subsection we will demonstrate the formula

$$a = -\frac{3}{2} + 2\Delta + b. \tag{8.38}$$

Even if the correlation function is regular in the bulk point limit this still constrains the power with which the Mellin amplitude can grow. We carry out this analysis for the case of minimal models.

8.6.1 Relating the behaviour of the Mellin amplitude for large Mellin variables and the bulk point limit

Let us start by briefly reminding the reader about the bulk point limit with an example taken from [23]. Consider the following configuration of four point points in $d \geqslant 3$[9], see figure (8.1):

$$x_1 = (-t, 0, 1, \vec{0}), x_2 = (t, 1, 0, \vec{0}), x_3 = (-t, 0, -1, \vec{0}), x_4 = (t, -1, 0, \vec{0}). \tag{8.39}$$

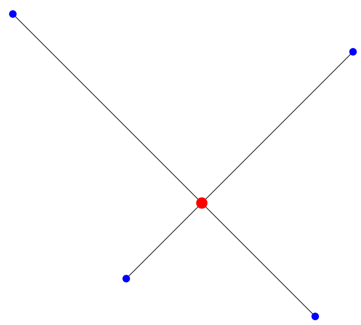

Figure 8.1. Example of a boundary Landau diagram.

We employ the $i\epsilon$ prescription $t = \tilde{t} - i\epsilon$, where $\tilde{t} \in [0, 1]$. When $\tilde{t} = 0$ this configuration is Euclidean, but when $\tilde{t} = 1$ the point $(0, 0, 0, \vec{0})$ is null separated from the external points. This signals a divergence in the correlation function.

The cross ratios u and v are given by

$$u = \frac{(-(2t)^2 + 2)^2}{16}, v = \frac{(-(2t)^2 + 2)^2}{16}. \tag{8.40}$$

When we vary $\tilde{t}$ from 0 to 1 the cross ratios u and v circle around the origin in the complex plane.

Let us see how the Mellin representation of the correlation function changes as we vary $\tilde{t}$ from 0 to 1. We write

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \frac{f(u,v)}{x_{13}^{2\Delta} x_{24}^{2\Delta}}, \tag{8.41}$$

where

$$f(u,v) = \int \frac{d\gamma_{12}}{2\pi i} \int \frac{d\gamma_{14}}{2\pi i} M(\gamma_{12}, \gamma_{14}) \Gamma(\gamma_{12})^2 \Gamma(\gamma_{14})^2 \Gamma(\Delta - \gamma_{12} - \gamma_{14})^2 u^{-\gamma_{12}} v^{-\gamma_{14}} \tag{8.42}$$

[9]The above configuration is an example of a Landau diagram. In $d = 2$ there are no boundary Landau diagrams. However, there are bulk diagrams. It was shown in [23] that in $d = 2$ for the full nonperturbative theory there are no singularities except for the lightcone singularities.

$$\to \int \frac{d\gamma_{12}}{2\pi i} \int \frac{d\gamma_{14}}{2\pi i} M(\gamma_{12},\gamma_{14})\Gamma(\gamma_{12})^2\Gamma(\gamma_{14})^2\Gamma(\Delta-\gamma_{12}-\gamma_{14})^2 u^{-\gamma_{12}} v^{-\gamma_{14}} e^{-2\pi i\gamma_{12}} e^{-2\pi i \gamma_{14}}. \tag{8.43}$$

Notice that the factors $e^{-2\pi i\gamma_{12}}e^{-2\pi i\gamma_{14}}$ grow exponentially in certain directions at infinity and cancel the exponential decay of the Γ functions in the prefactor.

Let us suppose that $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ diverges in the bulk point limit and let us see how this comes about from (8.43). Such a divergence can only come from the region where both γ_{12} and γ_{14} have a very big positive imaginary part. We basically follow section (3.1) of [4].

Let us analyse the integral (8.43) in that region. Let us write $\gamma_{12} = \alpha_{12} + im_1$ and $\gamma_{14} = \alpha_{14} + im_2$, where $m_1, m_2 \gg 1$. α_{12} and α_{14} depend on the contour that we pick and we consider them arbitrary. Let us write furthermore $m_1 = \beta s$, $m_2 = (1-\beta)s$, where $\beta \in [0,1]$ and $s \gg 1$. We can use Stirling's formula to approximate the Γ functions in (8.43). Furthermore we assume that the Mellin amplitude goes like $g(\beta)s^b$ for large s, where $g(\beta)$ is a function that we do not know and b is a power that will control the bulk point limit divergence. In this regime, the integral (8.43) can be written as

$$\int_{s_0}^{\infty} \frac{ds}{s} \int_0^1 d\beta g(\beta) s^{-1+2\Delta+b}(-8i)e^{-i\pi\Delta}\pi^3 u^{-\alpha_{12}-is\beta} v^{-\alpha_{14}-is(1-\beta)} \tag{8.44}$$
$$(1-\beta)^{-1+2\alpha_{14}-2is(-1+\beta)}\beta^{-1+2\alpha_{12}+2is\beta},$$

where we take s_0 very large.

The integrand goes like $\exp\Big(-is\big(\beta\log(u) + (1-\beta)\log(v) - 2(1-\beta)\log(1-\beta) - 2\beta\log(\beta)\big)\Big)$. The integral in β is dominated by the saddle point $\beta_s = \frac{\sqrt{u}}{\sqrt{u}+\sqrt{v}}$. At the saddle point the previous exponential becomes $e^{2is\log(\sqrt{u}+\sqrt{v})}$. The integral can only diverge when $\sqrt{u}+\sqrt{v} = 1$. This happens precisely at the bulk point limit. Indeed, we have that $\log(\sqrt{u}+\sqrt{v}) \approx -\frac{(z-\bar{z})^2}{16(1-\bar{z})\bar{z}}$ when $z \to \bar{z}$ and $0 \leqslant z \leqslant 1, 0 \leqslant \bar{z} \leqslant 1$. So, expression (8.45) becomes

$$\int_{s_0}^{\infty} \frac{ds}{s} g(\beta_s) s^{-1+2\Delta+b} \frac{1}{\sqrt{-is}}(-4\sqrt{2}i)e^{-i\pi\Delta}\pi^{\frac{7}{2}} u^{-\alpha_{12}} v^{-\alpha_{14}} \tag{8.45}$$
$$(1-\beta)^{-1+2\alpha_{14}}\beta^{-1+2\alpha_{12}} e^{is\frac{(z-\bar{z})^2}{8(1-\bar{z})\bar{z}}}$$

Equation (8.45) enables us to relate the rate of divergence of the correlator with the polynomial growth of the Mellin amplitude:

$$M(s\gamma_{ij}) \sim s^b \implies f(u,v) \sim \frac{1}{(z-\bar{z})^{-3+4\Delta+2b}} \tag{8.46}$$

Let us consider the case in which the correlation function does not diverge in the bulk point limit. The previous analysis gives us the bound $b < \frac{3}{2} - 2\Delta$. Furthermore it is reasonable to assume that also the derivatives with respect to z of the correlation function should be analytic at the bulk point. Thus, let us assume that the correlation function is regular in $z - \bar{z}$. Then b can only take the values $b = \frac{3}{2} - 2\Delta - \frac{n}{2}$ where n is a positive integer. Indeed if it were not to take such values this would generate a divergence in some derivative of the correlation function. A similar but more general analysis in general d was recently performed in [86] with similar conclusions (which are identical in $d = 2$). It would be interesting to establish analyticity of the bulk point locus in higher d rigorously. It would be also interesting to understand further implications of the bulk point analyticity for the high energy limit of the flat space scattering amplitudes.

8.6.2 The bulk point limit in minimal models

In minimal models we expect no divergence in the bulk point limit. As we saw in the last section this constrains the Mellin amplitude of a four point function of equal scalar primaries to behave like $M(\gamma_{12} = \beta s, \gamma_{14} = (1-\beta)s) \sim s^{\frac{3}{2}-2\Delta-\frac{n}{2}}$ where n is a positive integer. In this subsection we will use our general formula (8.27) to confirm this prediction.

The general formula (8.27) drastically simplifies in the limit of large Mellin variables $\gamma_{ij} \to i\infty$. Let us divide (8.27) by $\prod_{i<j} \Gamma(\gamma_{ij})$. We can use Stirling's approximation to get

$$\frac{\Gamma(\gamma_{ij} - \sum_{r=1}^{z-1} \xi_{ij}^r + 2\alpha_i\alpha_j)}{\Gamma(\gamma_{ij})} = \gamma_{ij}^{-\sum_{r=1}^{z-1} \xi_{ij}^r + 2\alpha_i\alpha_j} \left[1 + O(\gamma_{ij}^{-1})\right]. \tag{8.47}$$

If we consider the case where all Mellin variables are proportional to some parameter s that is much bigger than any number in our system, then the Mellin amplitude is proportional to

$$s^{-\sum_{r=1}^{z-1} \sum_{i<j}^{n} \xi_{ij}^r + 2\sum_{i<j}^{n} \alpha_i\alpha_j}. \tag{8.48}$$

The sum in the exponent of (8.48) simplifies drastically due to the measures (8.28) and in fact does not depend on the integration variables ξ_{ij}. We do this sum in appendix (A.6) and obtain that the Mellin amplitude grows with

$$s^{1-\frac{1}{2}\sum_i \Delta_i}, \tag{8.49}$$

for a general n point function of scalar Virasoro primaries in minimal models with conformal dimensions Δ_i. For the case of a four point function of equal scalar primaries we find $s^{1-2\Delta}$ in agreement with what we predicted in the preceding subsection.

9 CFT's with slightly broken higher spin symmetry

9.1 Introduction and summary of results

The dualities between conformal field theories and higher spin gravity theories in AdS are one of the most intriguing topics in the AdS/CFT correspondence. Potentially, these dualities should allow for an improved understanding of the AdS/CFT correspondence, since both sides of the duality are simple, at least when compared to the more standard case of $\mathcal{N} = 4$ SYM and type IIB superstring theory[1]. Of particular interest are CFT's with slightly broken higher spin symmetry, that were studied most notably in the paper by Maldacena and Zhiboedov [20], where all three point functions of single trace operators at the planar level were computed at finite t'Hooft coupling. In our thesis, we compute some four point functions of spinning single trace operators at the planar level at finite t'Hooft coupling. The formulas we obtain are very simple and our formalism, which is based on pure CFT arguments in which Mellin space plays an important role, potentially paves the way for the computation of all spinning four point functions.

CFT's with slightly broken higher spin symmetry are large N CFT's where higher spin symmetry is broken by $1/N$ effects. There are two such theories, the quasi-boson theory and the quasi-fermion theory, which are defined in 3 dimensions. We will focus on the quasi-fermion theory. This theory depends on two parameters, $\tilde{N}$ and $\tilde{\lambda}$ (we follow the notation of [20]). We will study the theory at the planar level, i. e. at leading order in $\tilde{N}$. In that case the theory interpolates between the free fermion theory at $\tilde{\lambda} = 0$ and the critical point of the $O(N)$ model (critical boson) at $\tilde{\lambda} = \infty$.

Being a large N theory, the spectrum of the quasi-fermion theory organises into single and multitrace primary operators. Let us describe the single trace operators. There is one single trace operator for each even spin $s = 0, 2, ...$ The scalar primary, which we will denote by $j_{\tilde{0}}$, has dimension $2 + O(\frac{1}{\tilde{N}})$ [91]. The spin 2 primary j_2 is exactly

[1] See [87] (which builds on the works [88–90]) for recent progress, where the path integral for critical $O(N)$ models was written in terms of higher spin gauge fields defined in the bulk of AdS.

conserved. A higher spin primary j_s of spin $s > 2$ has dimension $s+1$ and acquires anomalous dimensions of $O(\frac{1}{\tilde{N}})$ [92], [93].

This theory is believed to be solvable in the planar limit. In [20] three point functions of single trace operators were computed at the planar level and for finite $\tilde{\lambda}$ through the use of slightly broken higher spin Ward identities[2]. In [94] four point functions of scalar operators were computed using the Lorentzian inversion formula and Schwinger-Dyson equations. In [95] the four point function $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ was computed using the pseudo-conservation equations[3].

We obtain a formula for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ for generic spin $s \geqslant 4$:

$$\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle = \frac{1}{\tilde{N}\sqrt{1+\tilde{\lambda}^2}} \langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} + \frac{\tilde{\lambda}}{\tilde{N}\sqrt{1+\tilde{\lambda}^2}} \langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}, \tag{9.1}$$

where $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff}$ is the correlator in the free fermion theory (which is fully known) and $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}$ is the corresponding correlator in the critical boson theory. The critical boson theory is the IR fixed point of the theory of $\tilde{N}$ free real scalar fields perturbed by $(\phi_i \phi_i)^2$. We obtain that

$$\begin{aligned} \langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} &= |x_1 - x_3|^{-4s-2} |x_2 - x_3|^{2s-1} |x_2 - x_4|^{-2s-3} |x_3 - x_4|^{2s-1} \\ \times \sum_{k=0}^{s} \int\int & \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} M(\gamma_{12}, \gamma_{14}; s, k) u^{-\gamma_{12}} v^{-\gamma_{14}} V(1;2,3)^{s-k} V(1;3,4)^k, \end{aligned} \tag{9.2}$$

where $V(i;j,k)$ is a conformal structure (see (9.6)) and u and v are the usual conformal cross ratios. $M(\gamma_{12}, \gamma_{14}; s, k)$ is equal to

$$\begin{aligned} M(\gamma_{12}, \gamma_{14}; s, k) &= \Gamma(-k + \gamma_{14} - 1) \Gamma\left(-k + \gamma_{14} + \frac{1}{2}\right) \Gamma(s - \gamma_{12} - \gamma_{14}) \\ \times \Gamma\left(s - \gamma_{12} - \gamma_{14} + \frac{3}{2}\right) & \Gamma(k - s + \gamma_{12} - 1) \Gamma\left(k - s + \gamma_{12} + \frac{1}{2}\right) p(\gamma_{12}, \gamma_{14}; s, k), \end{aligned} \tag{9.3}$$

where $p(\gamma_{12}, \gamma_{14}; s, k)$ is a polynomial in γ_{12} and γ_{14}. This polynomial is fully determined by crossing, pseudo-conservation and Regge boundedness, see equations (9.11) and (9.12), see (9.14) and see also (9.35), (9.36) and (9.37).

We explain in section 9.2 how formula (9.1) solves the crossing and pseudo-conservation equations and correctly accounts for the exchange of single trace operators with the OPE coefficients derived in [20]. In section 9.3 we show that formula (9.1) is the unique solution to the pseudo-conservation and crossing equations which is consistent with the bound on chaos. In particular we analyse AdS contact diagrams for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ and we

[2]This calculation was reproduced using higher spin techniques in [19], where also the parity odd structures were given.

[3]Correlators in ABJ theory were computed using slightly broken higher spin symmetry in [96].

conclude that such diagrams violate the bound on chaos, provided $s \geqslant 4$. In appendix N we study the bulk point limit of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. In appendix O we calculate $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ in position space for spins $s = 2, ..., 14$. This calculation agrees with the Mellin space result. In appendix P we recompute $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ by solving the higher spin Ward identities.

9.2 The bootstrap of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$

We will compute $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. Let us start by examining the $\tilde{N}$ and $\tilde{\lambda}$ dependence. It is expected that the quasi-fermion theory interpolates between a theory of $\tilde{N}$ free fermions at $\tilde{\lambda} = 0$ and the critical boson theory at $\tilde{\lambda} = \infty$.

We will work in a normalization where $\langle j_s j_s \rangle \sim 1$, i.e. two point functions of single trace operators do not depend on $\tilde{N}$ or $\tilde{\lambda}$. We use the $\sim$ sign to mean that we do not keep track of numerical factors, but we do keep track of the $\tilde{N}$ and $\tilde{\lambda}$ dependence. Thus, $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle \sim \frac{1}{\tilde{N}}$. At this order, we can only have exchanges of single trace operators or double trace operators $[j_{\tilde{0}}, j_{\tilde{0}}]$ or $[j_s, j_{\tilde{0}}]$.

Let us consider exchanges of single trace operators. The relevant three point functions are $\langle j_s j_{\tilde{0}} j_{s'} \rangle$ and $\langle j_{s'} j_{\tilde{0}} j_{\tilde{0}} \rangle$, with $s' \geqslant 2$. Note that $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle = 0$ [20]. From [20] we see that $\langle j_s j_{\tilde{0}} j_{\tilde{0}} \rangle \sim \frac{1}{\sqrt{\tilde{N}}}$. There are two possible structures for $\langle j_s j_{\tilde{0}} j_{s'} \rangle$, the fermion and the odd structure. We have that $\langle j_s j_{\tilde{0}} j_{s'} \rangle_{fermion} \sim \frac{1}{\sqrt{\tilde{N}}\sqrt{1+\tilde{\lambda}^2}}$ and $\langle j_s j_{\tilde{0}} j_{s'} \rangle_{odd} \sim \frac{\tilde{\lambda}}{\sqrt{\tilde{N}}\sqrt{1+\tilde{\lambda}^2}}$.

Based on this we propose the following ansatz

$$\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle = \frac{1}{\tilde{N}\sqrt{1+\tilde{\lambda}^2}} \langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} + \frac{\tilde{\lambda}}{\tilde{N}\sqrt{1+\tilde{\lambda}^2}} \langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}, \tag{9.4}$$

where $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff}$ is the four point function in the free fermion theory, whose form can be read in [97]. To the best of our knowledge, $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}$ has not yet been computed and it will be the subject of this section to do precisely that. We attached the subscript $_{cb}$ since it is expected that it corresponds to a four point function in the critical boson theory.

We can write parity even and parity odd structures for the correlator $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. The parity odd structures are realised in the free fermion theory. This is because $j_{\tilde{0}}$ is parity odd in the free fermion theory. The parity even structures are realised in the quasi-boson theory. Thus, we write

$$\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} = \sum_{k=0}^{s} f_k(x_{ij}) V(1;2,3)^{s-k} V(1;3,4)^k, \tag{9.5}$$

where $V(i;j,k)$ is a conformal structure which is given in embedding space [98] by

$$V(i;j,k) = \frac{(Z_i \cdot P_j)(P_i \cdot P_k) - (Z_i \cdot P_k)(P_i \cdot P_j)}{P_j \cdot P_k}. \tag{9.6}$$

P_i and Z_i are null vectors on $\mathbb{R}^{3,2}$. Z_i encodes the spinning indices. $f_k(x_{ij})$ is a function of the distances between the points, with appropriate weights on each of the points. We find it advantageous to consider the Mellin representation

$$\begin{gathered} f_k(x_{ij}) = \int [\frac{d\gamma_{ij}}{2\pi i}] \hat{M}(\gamma_{ij}; s, k) x_{ij}^{-2\gamma_{ij}}, \\ \sum_{j\neq 1} \gamma_{1j} = 2s+1, \ \sum_{j\neq i} \gamma_{ij} = 2, \ \ i = 2,3,4. \end{gathered} \tag{9.7}$$

(9.5) can be rewritten as

$$\begin{aligned} \langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} &= |x_1 - x_3|^{-4s-2} |x_2 - x_3|^{2s-1} |x_2 - x_4|^{-2s-3} |x_3 - x_4|^{2s-1} \\ \times \sum_{k=0}^{s} \int \int & \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14}; s, k) u^{-\gamma_{12}} v^{-\gamma_{14}} V(1;2,3)^{s-k} V(1;3,4)^k. \end{aligned} \tag{9.8}$$

We will call $\hat{M}(\gamma_{12}, \gamma_{14}; s, k)$ the Mellin amplitude.

The location of the poles of the Mellin amplitude is related to the operator product expansion of the external operators. Let us make this point explicitly. Consider two external operators O_1, O_2 of dimensions Δ_1, Δ_2 and spins s_1, s_2 and suppose they exchange an operator of dimension Δ and spin s. Then the most singular term in the lightcone OPE is

$$\mathcal{O}_{\mu_1 \ldots \mu_{s_1}}(x) \mathcal{O}_{\nu_1 \ldots \nu_{s_2}}(0) \supset \frac{\mathcal{O}_{\rho_1 \ldots \rho_s}(0) x^{\rho_1} \ldots x^{\rho_s}}{(x^2)^{\frac{\Delta_1+\Delta_2+s_1+s_2}{2} - \frac{\tau}{2}}} x_{\{\mu_1 \ldots \mu_{s_1}\}} x_{\{\nu_1 \ldots \nu_{s_2}\}} \big(1 + O(x^2)\big), \tag{9.9}$$

where $\tau = \Delta - s$. From this logic we expect the Mellin amplitude to have poles at $\gamma_{12} = \frac{\Delta_1+\Delta_2+s_1+s_2}{2} - \frac{\tau}{2} - n$, where n is a nonnegative integer.

For $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ all OPE channels are equal. To order $\frac{1}{N}$ there can be exchanges of higher spin currents and double traces $[j_s, j_{\tilde{0}}]$ and $[j_{\tilde{0}}, j_{\tilde{0}}]$, which have twist 1, 3 and 4 respectively. This motivates the following ansatz

$$\begin{aligned} \hat{M}(\gamma_{12}, \gamma_{14}; s, k) &= \Gamma(-k + \gamma_{14} - 1) \Gamma\left(-k + \gamma_{14} + \frac{1}{2}\right) \Gamma(-s + \gamma_{13} - 1) \\ \times \Gamma\left(-s + \gamma_{13} + \frac{1}{2}\right) & \Gamma(k - s + \gamma_{12} - 1) \Gamma\left(k - s + \gamma_{12} + \frac{1}{2}\right) p(\gamma_{12}, \gamma_{14}; s, k), \end{aligned} \tag{9.10}$$

where $\gamma_{13} = 2s + 1 - \gamma_{12} - \gamma_{14}$. The Γ functions contain all the poles implied by the OPE.

For this reason we assume that $p(\gamma_{12}, \gamma_{14}; s, k)$ is a polynomial in the Mellin variables.

The bound on chaos [23] bounds the degree of the polynomial $p(\gamma_{12}, \gamma_{14}; s, k)$. This is worked out in section (9.3), see (9.35), (9.36) and (9.37) for the precise formulas. Furthermore, $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ is constrained by invariance under interchange of points $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$. This crossing symmetry implies the equations

$$p(\gamma_{12}, \gamma_{14}; s, k) = \sum_{k_2=k}^{s} (-1)^{k_2} \binom{k_2}{k} p(2s + 1 - k_2 - \gamma_{12} - \gamma_{14}, \gamma_{14} - k + k_2; s, k_2), \quad (9.11)$$

$$p(\gamma_{12}, \gamma_{14}; s, k) = p(\gamma_{14}, \gamma_{12}; s, s - k). \quad (9.12)$$

$\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ is constrained by pseudoconservation of j_s. We implement this condition in embedding space. The differential operator for conservation is $\frac{\partial}{\partial P_1^A} D_A$, where

$$D_A = (\frac{d}{2} - 1 + Z_1 \cdot \frac{\partial}{\partial Z_1}) \frac{\partial}{\partial Z_1^A} - \frac{1}{2} (Z_1)_A \frac{\partial^2}{\partial Z_1 \cdot \partial Z_1}. \quad (9.13)$$

Since $\partial \cdot j_s$ is a primary operator of spin $s - 1$ and dimension $s + 2$, then $\langle \partial \cdot j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ is a conformal four point function of primary operators. $\langle \partial \cdot j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ factorizes into products of a two point function times a three point function. Such a four point function is made up of powers of u and of v and so its Mellin amplitude vanishes.

Four point functions of scalars with vanishing Mellin amplitudes were analysed in [63], see in particular section E.E.1. A similar analysis can be performed for the spinning case, though we will not pursue it here. The important conclusion is that in Mellin space pseudoconservation is the same as conservation. In other words, $\langle \partial \cdot j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ has a vanishing Mellin amplitude.

Pseudoconservation implies the equation

$$\sum_{i_1=-1}^{1} \sum_{i_2=-1}^{1} \sum_{i_3=-1}^{2} a_{i_1, i_2, i_3}(\gamma_{12}, \gamma_{14}) p(\gamma_{12} + i_1, \gamma_{14} + i_2; s, k + i_3) = 0. \quad (9.14)$$

The coefficients are written in the appendix L.2, see formula (L.3).

The crossing equations (9.11), (9.12), the pseudoconservation equation (9.14) and Regge boundedness (9.35), (9.36) and (9.37) determine $p(\gamma_{12}, \gamma_{14}; s, k)$ up to a multiplicative constant. This has to do with the fact that we have not picked a normalization for the higher spin current j_s. It is simple to solve this set of equations in a computer algebra system for each spin s. We find that the solution always has the form

$$p(\gamma_{12},\gamma_{14};s,k)=\sum_{k_1=0}^{k}\sum_{k_2=0}^{s-k}b(s,k;k_1,k_2)\gamma_{12}^{k_2}\gamma_{14}^{k_1},\quad k\leqslant\frac{s}{2} \tag{9.15}$$

$$p(\gamma_{12},\gamma_{14};s,k)=p(\gamma_{14},\gamma_{12};s,s-k),\quad k>\frac{s}{2}. \tag{9.16}$$

$p(\gamma_{12},\gamma_{14};s,k)$ turns out to have degree s. Using a laptop we generated solutions up to spin 40. Picking a normalization in which $p(\gamma_{12},\gamma_{14};s,k=0)\supset 1$, we find as an example that for $s=4$ we have

$$p(\gamma_{12},\gamma_{14};s=4,k=0)=1-\frac{19\gamma_{12}}{20}+\frac{119\gamma_{12}^2}{360}-\frac{\gamma_{12}^3}{20}+\frac{\gamma_{12}^4}{360}, \tag{9.17}$$

$$p(\gamma_{12},\gamma_{14};s=4,k=1)=-\frac{8}{15}+\frac{4\gamma_{12}}{9}-\frac{11\gamma_{12}^2}{90}+\frac{\gamma_{12}^3}{90}+\left(\frac{2}{5}-\frac{11\gamma_{12}}{30}+\frac{\gamma_{12}^2}{9}-\frac{\gamma_{12}^3}{90}\right)\gamma_{14},$$

$$p(\gamma_{12},\gamma_{14};s=4,k=2)=\frac{1}{5}-\frac{4\gamma_{12}}{15}+\frac{\gamma_{12}^2}{15}+\left(-\frac{4}{15}+\frac{11\gamma_{12}}{36}-\frac{13\gamma_{12}^2}{180}\right)\gamma_{14}$$
$$+(\frac{1}{15}-\frac{13\gamma_{12}}{180}+\frac{\gamma_{12}^2}{60})\gamma_{14}^2,$$

$$p(\gamma_{12},\gamma_{14};s=4,k=3)=-\frac{8}{15}+\frac{4\gamma_{14}}{9}-\frac{11\gamma_{14}^2}{90}+\frac{\gamma_{14}^3}{90}+\left(\frac{2}{5}-\frac{11\gamma_{14}}{30}+\frac{\gamma_{14}^2}{9}-\frac{\gamma_{14}^3}{90}\right)\gamma_{12},$$

$$p(\gamma_{12},\gamma_{14};s=4,k=4)=1-\frac{19\gamma_{14}}{20}+\frac{119\gamma_{14}^2}{360}-\frac{\gamma_{14}^3}{20}+\frac{\gamma_{14}^4}{360}.$$

In appendix O we implement an algorithm to compute $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$ in position space. We managed to determine $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$ in position space for spins $2,...,14$ using this algorithm. Taking the Mellin transform we get precisely the same as we get with the procedure in Mellin space. The advantage of Mellin space is that it allows to write equations (9.11), (9.12) and (9.14) that determine the solution for generic s.

Let us mention some checks on our solution. One such check is compatibility of the pseudo-conservation equations with conformal symmetry. $\partial\cdot j_s$ is a conformal primary at leading order in $\frac{1}{\tilde{N}}$. $\partial\cdot j_s$ can have contributions coming from $[j_{s_1},j_{\tilde{0}}]$ and $[j_{s_1},j_{s_2}]$. Only the former matter since we are interested in $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$. More precisely,

$$\partial\cdot j_s\supset\sum_{s_1=2}^{s-2}\sum_{m=0}^{s-s_1-1}c_m\partial^m j_{s_1}\partial^{s-s_1-1-m}j_{\tilde{0}}. \tag{9.18}$$

The coefficients c_m are fixed by conformal symmetry (see formula (O.7)). When we run our algorithm in position space we do not need to input the values of c_m, we prefer to keep them unknown. It turns out that our algorithm fixes c_m in agreement with (O.7). This is an important check on our results.

We also checked that the short distance limit of our expression for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}$ agrees with the correct three point structures for the exchange of higher spin currents. Let us take $s = 4$ for concreteness. The short distance limit $u \to 0$ captures the exchange of the higher spin currents in the s-channel. If afterwards we take $v \to 1$, we find that the correlator behaves as

$$\lim_{v \to 1} \lim_{u \to 0} \langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} \sim \sum_{J=2}^{\infty} \frac{1}{u^5} \frac{x_{34}^7}{x_{13}^7 x_{14}^{11} x_{23}^4} (1-v)^J V(1;2,3)^4 \tag{9.19}$$

The $\sim$ sign means that we just keep track of the conformal structure that appears, but we do not keep track of numerical coefficients. (9.19) is matched by the behaviour of conformal blocks of higher spin currents in the same limit.

Formula (9.1) correctly accounts for the exchange of single trace operators in $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. However, it is not obvious that it correctly accounts for the exchange of double trace operators. Indeed, one can imagine adding to (9.1) AdS contact diagrams, which are solutions to crossing that only involve the exchange of double trace operators. By taking linear combinations of AdS contact diagrams one can furthermore obtain solutions to the conservation equations. However, in the next section we consider such linear combinations and show that they always violate the bound on chaos. For this reason, it is not legal to add them to (9.1).

9.3 Bound on chaos for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$

The bound on chaos [36] constrains the Regge limit of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. In this section we review the bound on chaos and derive its consequences for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. There are two possible structures one can write for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. One structure involves the ϵ tensor and the other one does not. We examine the two cases separately in sections (9.3.2) and (9.3.4) and derive bounds on the Regge growth of the Mellin amplitude for both of these cases.

Solutions to crossing that only involve the exchange of double twist operators are given by AdS contact diagrams. This was proven in [54], for the special case of four point functions of external scalars. We will assume that such a result holds for any n-point function of spinning conformal primaries. We study AdS contact diagrams in sections (9.3.3) and (9.3.4). Our main conclusion is that AdS contact diagrams for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ are incompatible with the bound on chaos, provided $s \geqslant 4$. For $s = 2$ we construct the contact diagrams that are compatible with the bound on chaos, see formulas (9.56) and (9.68). This completes the proof of formula (9.1).

9.3.1 Review of the bound on chaos and Rindler positivity

Conformal field theories are constrained by the Regge behaviour of Lorentzian correlators. For nonperturbative CFT's, correlators in the Regge limit are bounded by the Euclidean OPE in the first sheet. For large N CFT's one needs to use the bound on chaos to bound correlators in the Regge limit. In this subsection we review the bound on chaos [36].

We will consider the following kinematics for a four point function, in which we set all four points on the same plane ($x^{\pm} = t \pm x$)

$$x_1^{\pm} = \pm 1,\ x_2^{\pm} = \mp 1,\ x_3^{\pm} = \mp e^{\rho \pm t}, x_4^{\pm} = \pm e^{\rho \pm t}, \tag{9.20}$$

see figure 9.1.

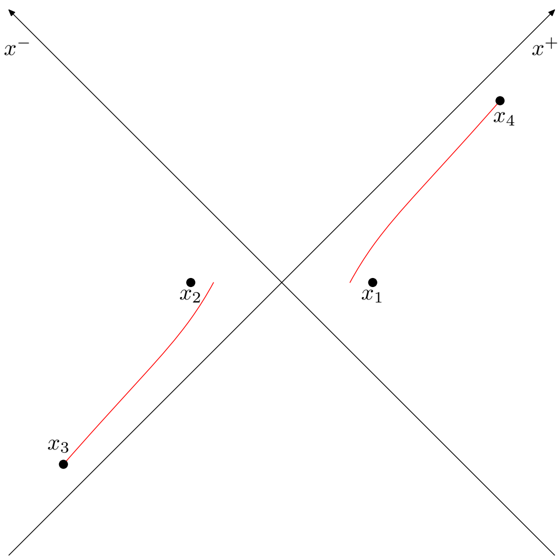

Figure 9.1. The Regge limit corresponds to taking $t \to \infty$ in (9.20).

The bound on chaos applies for systems at finite temperature with a large number of degrees of freedom. For the case of a large N conformal field theory, a correlation function of single trace primaries $\langle V(x_1)V(x_2)W(x_3)W(x_4)\rangle$ obeys

$$\langle V(x_1)V(x_2)W(x_3)W(x_4)\rangle \approx \langle V(x_1)V(x_2)\rangle\langle W(x_3)W(x_4)\rangle(1 + \alpha\frac{e^{\lambda_L t}}{N}), \tag{9.21}$$

where the Lyapunov exponent λ_L obeys the bound $\lambda_L \leqslant 2\pi T$, where T is the temperature of the system. The proportionality constant α does not depend on t. The bound on chaos can be applied to large N CFT's in Minkowski space, in which case we should consider the temperature $T = \frac{1}{2\pi}$ of the Rindler horizon.

We cannot apply directly (9.21) to $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$. However, we can use Rindler positivity [99] to bound $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ by $\langle j_s j_s j_{\tilde{0}} j_{\tilde{0}} \rangle$ and $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ and use the bound on chaos to bound the latter two quantities, as we will explain next.

The Rindler conjugate $\bar{O}$ of an operator O is defined as $\bar{O}_{\mu,\nu...}(t, x, \vec{y}) = O^{\dagger}_{\mu,\nu,...}(-t, -x, \vec{y})$, where $\vec{y}$ refers to a transverse coordinate relative to the plane of figure 9.1. Furthermore we have that $\overline{O_1 O_2} = \bar{O}_1 \bar{O}_2$. Rindler positivity and Cauchy-Schwarz inequalities imply that

$$|\langle \bar{A} B \rangle|^2 \leqslant \langle \bar{A} A \rangle \langle \bar{B} B \rangle. \tag{9.22}$$

where A and B are operators (that might be composite) defined on a single Rindler wedge.

Let us define $A = j_{\tilde{0}}(x_3) j_s(x_2)$, $B = j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3)$. Then, the time-ordered correlation function in the configuration (9.20) is given by

$$\begin{aligned} \langle T[j_s(x_1) j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3) j_{\tilde{0}}(x_4)] \rangle &= \langle \bar{A} B \rangle \\ \leqslant \sqrt{\langle j_{\tilde{0}}(x_4) j_s(x_1) j_{\tilde{0}}(x_3) j_s(x_2) \rangle \times \langle j_{\tilde{0}}(x_1) j_{\tilde{0}}(x_4) j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3) \rangle} \end{aligned} \tag{9.23}$$

The bound on chaos on the rhs of the previous expression implies a bound on $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. In terms of $\sigma = e^{-t}$:

$$\lim_{t \to \infty} \langle T[j_s(x_1) j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3) j_{\tilde{0}}(x_4)] \rangle \sim \frac{\sigma^{\lambda_1}}{N} + O(\frac{1}{N^2}), \tag{9.24}$$

where $\lambda_1 \geqslant -1$.

9.3.2 Consequences for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}$

Let us work out the consequences of the bound on chaos for the Mellin amplitudes of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. In the critical boson theory,

$$\begin{aligned} \langle j_s(x_1) j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3) j_{\tilde{0}}(x_4) \rangle_{cb} &= |x_1 - x_3|^{-4s-2} |x_2 - x_3|^{2s-1} |x_2 - x_4|^{-2s-3} |x_3 - x_4|^{2s-1} \\ &\times \sum_{k=0}^{s} \int \int \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14}; s, k) u^{-\gamma_{12}} v^{-\gamma_{14}} V(1; 2, 3)^{s-k} V(1; 3, 4)^k. \end{aligned} \tag{9.25}$$

where $V(i; j, k)$ was defined in (9.6) and

$$\begin{aligned} \hat{M}(\gamma_{12}, \gamma_{14}; s, k) &= \Gamma(\gamma_{12}) \Gamma(\Delta_1 - \gamma_{12} - \gamma_{14}) \Gamma(\gamma_{14}) \Gamma(\gamma_{12} + \frac{\Delta_3 + \Delta_4 - \Delta_1 - \Delta_2}{2}) \\ \Gamma(\frac{\Delta_1 + \Delta_2 - \Delta_3 + \Delta_4}{2} &- \gamma_{12} - \gamma_{14}) \Gamma(\gamma_{14} + \frac{\Delta_2 + \Delta_3 - \Delta_1 - \Delta_4}{2}) M(\gamma_{12}, \gamma_{14}; s, k), \end{aligned} \tag{9.26}$$

$$\Delta_1 = 2s+1, \quad \Delta_2 = 2, \quad \Delta_3 = 2, \quad \Delta_4 = 2.$$

We call $M(\gamma_{12}, \gamma_{14}; s, k)$ a Mellin amplitude. The arguments of the Γ functions are just the Mellin variables defined in (9.7).

In the limit $t \to \infty$ of the kinematics (9.20), the conformal cross-ratio v acquires a monodromy $v \to ve^{2\pi i}$. Furthermore

$$u \approx 16\sigma^2 + O(\sigma^3), \quad v \approx 1 - 8\sigma\cosh\rho + O(\sigma^2), \;\; \sigma \to 0. \tag{9.27}$$

The polynomial growth of the Mellin amplitude is related to the Regge limit, in a manner that we explain next, following appendix C of [29]. Let us consider the limit

$$\begin{aligned} \lim_{\sigma\to 0} \int\int \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} M(\gamma_{12}, \gamma_{14}; s, k)\Gamma(\gamma_{12})\Gamma(\Delta_1 - \gamma_{12} - \gamma_{14}) \qquad (9.28)\\ \Gamma(\gamma_{14})e^{-2\pi i\gamma_{14}}\Gamma(\gamma_{12} + \frac{\Delta_3+\Delta_4-\Delta_1-\Delta_2}{2})\Gamma(\gamma_{14} + \frac{\Delta_2+\Delta_3-\Delta_1-\Delta_4}{2}) \\ \Gamma(-\gamma_{12} - \gamma_{14} + \frac{\Delta_1+\Delta_2-\Delta_3+\Delta_4}{2})\sigma^{-2\gamma_{12}}(1-8\sigma\cosh\rho)^{-\gamma_{14}}. \end{aligned}$$

The factor $e^{-2\pi i\gamma_{14}}$ becomes very large in the regime $\gamma_{14} \to i\infty$. This is cancelled by the exponential decay of the Γ functions. Let us suppose that the Mellin amplitude grows polynomially as $\gamma_{14}^{\alpha(s,k)} f(\gamma_{12})$, when γ_{14} is large and imaginary and γ_{12} is fixed. In this regime we can rewrite (9.28) as

$$\begin{aligned} \sim \int \frac{d\gamma_{12}}{2\pi i}\Gamma(\gamma_{12})\Gamma(\gamma_{12} + \frac{\Delta_3+\Delta_4-\Delta_1-\Delta_2}{2})\sigma^{-2\gamma_{12}} f(\gamma_{12}) \qquad (9.29)\\ \int_{M1}^{\infty} \frac{dm_1}{2\pi} m_1^{-2-2\gamma_{12}+\Delta_1+\Delta_2+\alpha(s,k)} e^{im_1(8\sigma\cosh\rho + O(\sigma^3))}, \end{aligned}$$

where M_1 is an irrelevant large number. If we substitute $m_1 \to \frac{m_1}{\sigma}$ we get that the integral (9.28) scales like $\sigma^{1-\Delta_1-\Delta_2-\alpha(s,k)}$. In order to compare (9.25) with (9.24), we should furthermore take into account the prefactor and the structures in (9.25), which scale with σ. Our conclusion is that $\alpha(s,k) = 1 - \lambda_1 - k \leqslant 2 - k$.

We can use the crossing symmetry equations

$$\hat{M}(\gamma_{12}, \gamma_{14}; s, k) = \sum_{k_2=k}^{s} (-1)^{k_2} \binom{k_2}{k} \hat{M}(2s+1-k_2-\gamma_{12}-\gamma_{14}, \gamma_{14}-k+k_2; s, k_2), \tag{9.30}$$

$$\hat{M}(\gamma_{12}, \gamma_{14}; s, k) = \hat{M}(\gamma_{14}, \gamma_{12}; s, s-k). \tag{9.31}$$

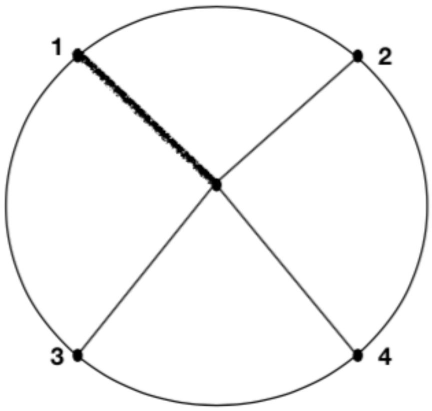

Figure 9.2. AdS contact diagram for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$.

to derive the following bounds on the polynomial growth of the Mellin amplitude

$$\boxed{\begin{aligned} \lim_{\beta\to\infty} M(\gamma_{12}, \beta\gamma_{14}; s, k) &\sim \beta^{\alpha_1(s,k)}, \quad \alpha_1(s,k) \leqslant 2-k \qquad (9.32)\\ \lim_{\beta\to\infty} M(\beta\gamma_{12}, \gamma_{14}; s, k) &\sim \beta^{\alpha_2(s,k)}, \quad \alpha_2(s,k) \leqslant 2-s+k \qquad (9.33)\\ \lim_{\beta\to\infty} M(i\beta+\gamma_{12}, -i\beta+\gamma_{14}; s, k) &\sim \beta^{\alpha_3(s,k)}, \quad \alpha_3(s,k) \leqslant 2+s. \qquad (9.34) \end{aligned}}$$

We can apply these bounds to the ansatz (9.10). We conclude that

$$\boxed{\begin{aligned} \lim_{\beta\to\infty} p(\gamma_{12}, \beta\gamma_{14}; s, k) &\sim \beta^{\eta_1(s,k)}, \quad \eta_1(s,k) = 2+2k+\alpha_1(s,k) \leqslant 4+k \qquad (9.35)\\ \lim_{\beta\to\infty} p(\beta\gamma_{12}, \gamma_{14}; s, k) &\sim \beta^{\eta_2(s,k)}, \quad \eta_2(s,k) = 2+2s-2k+\alpha_2(s,k) \leqslant 4-k+s \qquad (9.36)\\ \lim_{\beta\to\infty} p(i\beta+\gamma_{12}, -i\beta+\gamma_{14}; s, k) &\sim \beta^{\eta_3(s,k)}, \quad \eta_3(s,k) \leqslant 4+s. \qquad (9.37) \end{aligned}}$$

The solution that we found respects this bound.

9.3.3 The Regge limit of AdS contact diagrams for the parity even structure in $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$

We will study the Regge limit of a generic AdS contact diagram for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ (see figure (9.2)), using the methods of [56]. We use vectors P_i and Z_i in embedding space to describe the position and polarization vectors of an operator O_i defined on the boundary of AdS. For tensor fields defined on the bulk of AdS, we use vectors X_i and W_i to denote the position and the polarization. The following identities are obeyed:

$$Z_i^2 = P_i^2 = Z_i \cdot P_i = X_i^2 + 1 = W_i^2 = X_i \cdot W_i = 0. \qquad (9.38)$$

We denote the bulk to boundary propagator of a dimension Δ and spin J field by $\Pi_{\Delta,J}(X,P;W,Z)$. Its formula is

$$\Pi_{\Delta,J}(X,P;W,Z) = \mathcal{C}_{\Delta,J}\frac{((-2P\cdot X)(W\cdot Z)+2(W\cdot P)(Z\cdot X))^J}{(-2P\cdot X)^{\Delta+J}}, \tag{9.39}$$

where $\mathcal{C}_{\Delta,J}$ is a proportionality constant (whose value will not be relevant for us).

An important class of contact diagrams contributing to the parity even structure in $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$ is given by

$$\begin{aligned}\int_{AdS} dX\, \Pi_{\Delta_1=s+1,s_1=s}(X,P_1,K,Z_1)(W\cdot\nabla)^{s_2}\Pi_{\Delta_2=2,s_2=0}(X,P_2) \qquad (9.40)\\ (W\cdot\nabla)^{s_3}\Pi_{\Delta_3=2,s_3=0}(X,P_3)\Pi_{\Delta_4=2,s_4=0}(X,P_4),\end{aligned}$$

where $s_1=s_2+s_3$. There are other contact diagrams one can write by contracting more derivatives among the propagators, but such diagrams will diverge more in the Regge limit, which is the issue we wish to discuss here. The covariant derivative is given by

$$\nabla_A = \frac{\partial}{\partial X^A}+X_A(X\cdot\frac{\partial}{\partial X})+W_A(X\cdot\frac{\partial}{\partial W}). \tag{9.41}$$

The operator K is given by

$$\begin{aligned}K_A = \frac{d-1}{2}\Big(\frac{\partial}{\partial W^A}+X_A(X\cdot\frac{\partial}{\partial W})\Big)+(W\cdot\frac{\partial}{\partial W})\frac{\partial}{\partial W^A} \qquad (9.42)\\ +X_A(W\cdot\frac{\partial}{\partial W})(X\cdot\frac{\partial}{\partial W})-\frac{1}{2}W_A\Big(\frac{\partial^2}{\partial W\cdot\partial W}+(X\cdot\frac{\partial}{\partial W})(X\cdot\frac{\partial}{\partial W})\Big),\end{aligned}$$

where for our purposes $d=3$.

The following identity

$$\begin{aligned}\Pi_{\Delta_1,s_1}(X,P_1,K,Z_1)(W\cdot\nabla)^{s_2}\Pi_{\Delta_2,s_2}(X,P_2)(W\cdot\nabla)^{s_3}\Pi_{\Delta_3,s_3}(X,P_3) \quad (9.43)\\ = C(\Delta_1,\Delta_2,\Delta_3,s_1,s_2,s_3)D_{12}^{s_2}D_{13}^{s_3}\Big(\Pi_{\Delta_1,0}(X,P_1)\Pi_{\Delta_2+s_2,0}(X,P_2)\Pi_{\Delta_3+s_3,0}(X,P_3)\Big).\end{aligned}$$

is useful for us. D_{ij} is an operator that only acts on the external points. It increases the spin at position i by 1 and it decreases the conformal dimension at position j by 1. $C(\Delta_1,\Delta_2,\Delta_3,s_1,s_2,s_3)$ is a constant of proportionality, which will not be relevant for us. The precise definition of D_{ij} is

$$D_{ij} = (P_j\cdot Z_i)Z_i\cdot\frac{\partial}{\partial Z_i}-(P_j\cdot Z_i)P_i\cdot\frac{\partial}{\partial P_i}+(P_j\cdot P_i)Z_i\cdot\frac{\partial}{\partial P_i}. \tag{9.44}$$

We confirmed the identity (9.43) for a few values of the external spins using *Mathematica*.

So, with the help of identity (9.43) we can perform the integration in (9.40) using only scalar propagators and afterwards we act with the differential operators D_{12} and D_{13}. The AdS integral with only scalar propagators corresponds to a contact quartic scalar diagram, whose Mellin amplitude is a constant. Afterwards we act with the differential operators and obtain an expression in the form of (9.8).

Let us exemplify what we mean for the case of $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. Let us take $s_2 = 1$ and $s_3 = 1$ in (9.40). Up to a proportionality constant, the contact diagram is given by

$$
\begin{aligned}
&D_{12} D_{13} \int_{AdS} dX\, \Pi_{\Delta_1=3,s_1=0}(X,P_1) \Pi_{\Delta_2=3,s_2=0}(X,P_2) \Pi_{\Delta_3=3,s_3=0}(X,P_3) \Pi_{\Delta_4=2,s_4=0}(X,P_4) \qquad (9.45)\\
&\sim D_{12} D_{13} \frac{x_{34}}{x_{23} x_{13}^6 x_{24}^5} \int\int \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} \Gamma(\gamma_{12}) \Gamma(3-\gamma_{12}-\gamma_{14}) \Gamma(\gamma_{14})\\
&\qquad \Gamma(\gamma_{12} - \frac{1}{2}) \Gamma(\frac{5}{2} - \gamma_{12} - \gamma_{14}) \Gamma(\gamma_{14} + \frac{1}{2}) u^{-\gamma_{12}} v^{-\gamma_{14}},
\end{aligned}
$$

where the $\sim$ symbol means that we neglected a numerical factor. We now act with the differential operators D_{12} and D_{13} and reorganise the result into the form (9.25), (9.26)[4]. For this contact diagram, we conclude that

$$
\begin{aligned}
M(\gamma_{12}, \gamma_{14}, s=2, k=0) &= \frac{(-4+\gamma_{12})(3-8\gamma_{14}+4\gamma_{14}^2)}{(-4+\gamma_{12}+\gamma_{14})} \qquad (9.46)\\
M(\gamma_{12}, \gamma_{14}, s=2, k=1) &= \frac{-2(-2+\gamma_{12})(-3+2\gamma_{12})(-3+2\gamma_{14})}{(-4+\gamma_{12}+\gamma_{14})}\\
M(\gamma_{12}, \gamma_{14}, s=2, k=2) &= \frac{\gamma_{12}(3-8\gamma_{12}+4\gamma_{12}^2)}{(-4+\gamma_{12}+\gamma_{14})}
\end{aligned}
$$

This contact diagram obeys the chaos bounds (9.32), (9.33) and (9.34). We found that contact diagrams of the type (9.40) obey the bound on chaos for spin 2, but violate the bound on chaos for spin $s \geqslant 4$.

Our goal is to investigate if there are extra solutions to crossing, conservation and Regge boundedness for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. AdS contact diagrams are solutions to the crossing equations, however they are not necessarily conserved, nor Regge bounded. To see that contact diagrams are not necessarily conserved, let us consider a generic contact diagram

$$
\int_{AdS} dX\, \Pi_{\Delta=s+1,s}(X,P_1,W,Z_1)\, J(X,P_i,K,Z_i) \qquad (9.47)
$$

where we denoted by $J(X,P_i,W,Z_i)$ the dependence on the other AdS fields. It turns out that the action of the conservation operator (9.13) on $\Pi_{\Delta=s+1,s}$ gives a pure gauge

[4] The step where we gather different terms into the same contour may give rise to subtractions. These do not change our main conclusion, which is that any finite linear combination of AdS contact diagrams for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ with $s \geqslant 4$ does not obey the bound on chaos.

expression

$$\frac{\partial}{\partial P} \cdot \mathcal{D}_Z \Pi_{\Delta=s+1,s}(X,P,W,Z) \quad (9.48)$$

$$= -2^{-2-s}s^2 W \cdot \nabla_X \left((-P \cdot X)^{-2s-1} \left((-P \cdot X)(W \cdot Z) + (P \cdot W)(X \cdot Z) \right)^{s-1} \right) \quad (9.49)$$

$$\equiv W \cdot \nabla_X F(X,P,W,Z). \quad (9.50)$$

Thus,

$$\frac{\partial}{\partial P_1} \cdot \mathcal{D}_{Z_1} \int_{AdS} dX \Pi_{\Delta=s+1,s}(X,P_1,W,Z_1)\, J(X,P_i,W,Z_i) \quad (9.51)$$
$$= -\int_{AdS} dX F(X,P_1,W,Z_1) W \cdot \nabla_X J(X,P_i,K,Z_i)$$

This vanishes only if $J(X,P_i,K,Z_i)$ is conserved in the bulk of AdS, i.e. a contact diagram involving a bulk to boundary propagator is conserved only when the bulk to boundary propagator is coupled to a conserved current. Clearly, this is not the case for a generic contact diagram (9.40).

So, we consider instead linear combinations of AdS contact diagrams. The most economical way of doing this is to notice that the Mellin transform of any contact diagram, or any linear combination of contact diagrams, can be written as

$$\hat{M}(\gamma_{12},\gamma_{14};s,k) = \Gamma(-k+\gamma_{14})\Gamma\left(-k+\gamma_{14}+\frac{1}{2}\right)\Gamma(-s+\gamma_{13}) \quad (9.52)$$
$$\times\Gamma\left(-s+\gamma_{13}+\frac{1}{2}\right)\Gamma(k-s+\gamma_{12})\Gamma\left(k-s+\gamma_{12}+\frac{1}{2}\right) p_{dt}(\gamma_{12},\gamma_{14};s,k).$$

where $p_{dt}(\gamma_{12},\gamma_{14};s,k)$ is a polynomial. Let us explain this important formula. If we act with the differential operators on the scalar contact diagram, they will shift the arguments of the Γ functions by integers. So, the Mellin transform of an AdS contact diagram will involve 6 Γ functions times a polynomial. The arguments of the Γ functions are related to the operators that appear in the OPE of the external operators. Thus, we arrive at (9.52). Notice that $p_{dt}(\gamma_{12},\gamma_{14};s,k)$ will eventually have zeros.

The chaos bound for $p_{dt}(\gamma_{12},\gamma_{14};s,k)$ is

$$\lim_{\beta\to\infty} p_{dt}(\gamma_{12},\beta\gamma_{14};s,k) \sim \beta^{\eta_1(s,k)}, \quad \eta_1(s,k) = 2+2k+\alpha_1(s,k) \leqslant 2+k \quad (9.53)$$

$$\lim_{\beta\to\infty} p_{dt}(\beta\gamma_{12},\gamma_{14};s,k) \sim \beta^{\eta_2(s,k)}, \quad \eta_2(s,k) = 2+2s-2k+\alpha_2(s,k) \leqslant 2-k+s \quad (9.54)$$

$$\lim_{\beta\to\infty} p_{dt}(i\beta+\gamma_{12},-i\beta+\gamma_{14};s,k) \sim \beta^{\eta_3(s,k)}, \quad \eta_3(s,k) \leqslant 2+s. \quad (9.55)$$

We imposed crossing and conservation on (9.52). We find solutions that always violate the chaos bound, for all spins $s \geqslant 4$. For $s = 2$ we find a solution that respects crossing, conservation and Regge boundedness, which is given by

$$p_{dt}(\gamma_{12}, \gamma_{14}; s = 2, k = 0) = \frac{\gamma_{12}^4 \gamma_{14}}{9} + \frac{\gamma_{12}^4}{24} + \frac{\gamma_{12}^3 \gamma_{14}^2}{9} - \frac{5\gamma_{12}^3 \gamma_{14}}{8} - \frac{5\gamma_{12}^3}{12} - \frac{7\gamma_{12}^2 \gamma_{14}^2}{24} \tag{9.56}$$

$$+ \frac{35\gamma_{12}^2 \gamma_{14}}{36} + \frac{35\gamma_{12}^2}{24} + \frac{7\gamma_{12}\gamma_{14}^2}{72} - \frac{5\gamma_{12}\gamma_{14}}{24} - \frac{25\gamma_{12}}{12} + \frac{\gamma_{14}^2}{12} - \frac{\gamma_{14}}{4} + 1,$$

$$p_{dt}(\gamma_{12}, \gamma_{14}; s = 2, k = 1) = -\frac{2\gamma_{12}^3 \gamma_{14}^2}{9} + \frac{5\gamma_{12}^3 \gamma_{14}}{4} - \frac{37\gamma_{12}^3}{36} - \frac{2\gamma_{12}^2 \gamma_{14}^3}{9}$$

$$+ \frac{13\gamma_{12}^2 \gamma_{14}^2}{4} - \frac{331\gamma_{12}^2 \gamma_{14}}{36} + \frac{37\gamma_{12}^2}{6} + \frac{5\gamma_{12}\gamma_{14}^3}{4} - \frac{331\gamma_{12}\gamma_{14}^2}{36} + \frac{77\gamma_{12}\gamma_{14}}{4} - \frac{407\gamma_{12}}{36}$$

$$- \frac{37\gamma_{14}^3}{36} + \frac{37\gamma_{14}^2}{6} - \frac{407\gamma_{14}}{36} + \frac{37}{6},$$

$$p(\gamma_{12}, \gamma_{14}; s = 2, k = 2) = \frac{\gamma_{12}^2 \gamma_{14}^3}{9} - \frac{7\gamma_{12}^2 \gamma_{14}^2}{24} + \frac{7\gamma_{12}^2 \gamma_{14}}{72} + \frac{\gamma_{12}^2}{12} + \frac{\gamma_{12}\gamma_{14}^4}{9} - \frac{5\gamma_{12}\gamma_{14}^3}{8}$$

$$+ \frac{35\gamma_{12}\gamma_{14}^2}{36} - \frac{5\gamma_{12}\gamma_{14}}{24} - \frac{\gamma_{12}}{4} + \frac{\gamma_{14}^4}{24} - \frac{5\gamma_{14}^3}{12} + \frac{35\gamma_{14}^2}{24} - \frac{25\gamma_{14}}{12} + 1.$$

9.3.4 The Regge limit of AdS contact diagrams for the parity odd structure in $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$

The parity odd structure is

$$\langle j_s(x_1) j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3) j_{\tilde{0}}(x_4) \rangle_{odd} = |x_1 - x_3|^{-4s-2} |x_2 - x_3|^{2s-2} |x_2 - x_4|^{-2s-4} |x_3 - x_4|^{2s-2} \tag{9.57}$$

$$\times \sum_{k=0}^{s-1} \int\int \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} \hat{M}_{odd}(\gamma_{12}, \gamma_{14}; s, k) u^{-\gamma_{12}} v^{-\gamma_{14}} \epsilon(Z_1, P_1, P_2, P_3, P_4) V(1; 2, 3)^{s-1-k} V(1; 3, 4)^k.$$

We define the Mellin amplitude $M_{odd}(\gamma_{12}, \gamma_{14}; s, k)$ in the following manner

$$\hat{M}_{odd}(\gamma_{12}, \gamma_{14}; s, k) = \Gamma(\gamma_{12}) \Gamma(\Delta_1 - \gamma_{12} - \gamma_{14}) \Gamma(\gamma_{14}) \Gamma(\gamma_{12} + \frac{\Delta_3 + \Delta_4 - \Delta_1 - \Delta_2}{2}) \tag{9.58}$$

$$\Gamma(\frac{\Delta_1 + \Delta_2 - \Delta_3 + \Delta_4}{2} - \gamma_{12} - \gamma_{14}) \Gamma(\gamma_{14} + \frac{\Delta_2 + \Delta_3 - \Delta_1 - \Delta_4}{2}) M_{odd}(\gamma_{12}, \gamma_{14}; s, k),$$

$$\Delta_1 = 2s + 1, \quad \Delta_2 = 3, \quad \Delta_3 = 3, \quad \Delta_4 = 3.$$

The following equations encapsulate crossing symmetry:

$$\hat{M}_{odd}(\gamma_{12},\gamma_{14};s,k) = \sum_{k_2=k}^{s-1}(-1)^{k_2}\binom{k_2}{k}\hat{M}_{odd}(2s+1-k_2-\gamma_{12}-\gamma_{14},\gamma_{14}-k+k_2;s,k_2), \tag{9.59}$$

$$\hat{M}_{odd}(\gamma_{12},\gamma_{14};s,k) = \hat{M}_{odd}(\gamma_{14},\gamma_{12};s,s-1-k). \tag{9.60}$$

Let us use the bound on chaos to derive a bound on the polynomial growth of the Mellin amplitude. Let us define the exponent $\alpha(s;k)$ such that $\lim_{\beta\to\infty} M(\gamma_{12},\beta\gamma_{14};s,k) \sim \beta^{\alpha(s;k)}$. In the Regge limit, the Mellin integral goes as $\sigma^{-2s-3-\alpha(s;k)}$. The prefactor times the structure goes as σ^{3+2s-k}. So, (9.57) behaves as $\sigma^{-k-\alpha(s;k)}$. By comparing with the bound on chaos (9.24) and using (9.59), (9.60) we conclude that

$$\lim_{\beta\to\infty} M_{odd}(\gamma_{12},\beta\gamma_{14};s,k) \sim \beta^{\alpha_1(s,k)}, \quad \alpha_1(s,k) \leqslant 1-k \tag{9.61}$$

$$\lim_{\beta\to\infty} M_{odd}(\beta\gamma_{12},\gamma_{14};s,k) \sim \beta^{\alpha_2(s,k)}, \quad \alpha_2(s,k) \leqslant 2-s+k \tag{9.62}$$

$$\lim_{\beta\to\infty} M_{odd}(i\beta+\gamma_{12},-i\beta+\gamma_{14};s,k) \sim \beta^{\alpha_3(s,k)}, \quad \alpha_3(s,k) \leqslant s. \tag{9.63}$$

The Mellin amplitude of an AdS contact diagram of the type (9.57), or of a linear combination of contact diagrams, is given by

$$\hat{M}_{odd}(\gamma_{12},\gamma_{14};s,k) = \Gamma(\gamma_{12}+1+k-s)\Gamma(\gamma_{12}+\frac{1}{2}+k-s) \tag{9.64}$$

$$\Gamma(\gamma_{14}-k)\Gamma(\gamma_{14}-k-\frac{1}{2})\Gamma(\gamma_{13}+1-s)\Gamma(\gamma_{13}+\frac{1}{2}-s)p_{dt}(\gamma_{12},\gamma_{14};s,k),$$

where $\gamma_{13} = 2s+1-\gamma_{12}-\gamma_{14}$. The bound on chaos for $p_{dt}(\gamma_{12},\gamma_{14};s,k)$ is

$$\lim_{\beta\to\infty} p_{dt}(\gamma_{12},\beta\gamma_{14};s,k) \sim \beta^{\lambda_1(s,k)}, \quad \lambda_1(s,k) \leqslant 2+k \tag{9.65}$$

$$\lim_{\beta\to\infty} p_{dt}(\beta\gamma_{12},\gamma_{14};s,k) \sim \beta^{\lambda_2(s,k)}, \quad \lambda_2(s,k) \leqslant 1+s-k \tag{9.66}$$

$$\lim_{\beta\to\infty} p_{dt}(i\beta+\gamma_{12},-i\beta+\gamma_{14};s,k) \sim \beta^{\lambda_3(s,k)}, \quad \lambda_3(s,k) \leqslant 1+s. \tag{9.67}$$

$p_{dt}(\gamma_{12},\gamma_{14};s,k)$ can be found by imposing crossing and conservation. We found that for $s \geqslant 4$ all solutions violate the bound on chaos.

However, for $s=2$ there is one solution that respects the bound on chaos. This solution is

$$p_{dt}(\gamma_{12}, \gamma_{14}; s = 2, k = 0) = \frac{\gamma_{12}^2}{4} + \frac{\gamma_{12}\gamma_{14}}{2} - \frac{5\gamma_{12}}{4} - \frac{\gamma_{14}}{2} + 1, \tag{9.68}$$

$$p_{dt}(\gamma_{12}, \gamma_{14}; s = 2, k = 1) = \frac{\gamma_{12}\gamma_{14}}{2} - \frac{\gamma_{12}}{2} + \frac{\gamma_{14}^2}{4} - \frac{5\gamma_{14}}{4} + 1. \tag{9.69}$$

In the conclusion of the thesis we discuss open problems in CFT's with slightly broken higher spin symmetry.

Conclusion

Main results of the PhD thesis

- *Analyticity and polynomial boundedness of nonperturbative four point functions in CFT's imply the existence of Mellin amplitudes for light external operators.*

  See formulas (1.22), (1.23). Analyticity follows from the OPE. We did not manage to prove polynomial boundedness for subtracted four point functions. This requires a better handle on the double lightcone limit.

- *Analyticity in a sectorial domain Θ_{CFT} of the four-point function justifies the inclusion of the Γ-functions in the definition of the Mellin amplitude (see definition (2)).*

- *Meromorphicity of Mellin amplitudes $M(\gamma_{12}, \gamma_{14})$ is proven in a subset of $\mathbb{C}^2$, see figure D.1.*

  The positions of the poles are given by the twist spectrum of the theory (3.3). Due to the presence of the twist accumulation points in every CFT, the nonperturbative Mellin amplitude has an infinite number of pole accumulation points whose position is known, see section 2.4. We conjecture that the only singularities of the Mellin amplitude correspond to physical operators (3.3).

- *Non-perturbative Mellin amplitude satisfy Polyakov conditions.*

  This is encapsulated by formulas (4.29), (4.30) and (4.31).

- *Existence of Mellin amplitudes, Regge boundedness, crossing symmetry and Polyakov conditions lead to dispersive CFT sum rules.*

  See formula (5.6), for which we performed some checks using the 3d Ising model. This sum rule vanishes for the exchange of exact double twists $\tau = 2\Delta + 2n$ for $n \geqslant 0$. Heavy operators contribute to this sum rule with a definite sign. Sum rules of this type constrain the UV completion of effective field theories in AdS. Potentially, they might lead to the exclusion of a seemingly healthy effective field theory, though we did not manage to prove a result of this kind in this thesis.

- *Mellin dispersive sum rules were used to rederive most of the known CFT data of the Wilson-Fischer model in $d = 4 - \epsilon$ dimensions to order ϵ^4 and were furthermore used to* *derive new results, namely the OPE coefficient $C^2_{\phi\phi\phi^2}$ at order ϵ^4, see table 6.3.*

- *Mellin dispersive sum rules were applied to derive CFT data from AdS Witten diagrams.*

  Our best result is formula (7.12) for the one loop anomalous dimensions in ϕ^4 theory.

- *The pseudoconservation equations, crossing symmetry and Regge boundedness allow the determination of spinning four point functions in CFT's with slightly broken higher spin symmetry.*

  More precisely, we computed $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ in the quasi-fermionic theory for $s \geqslant 4$ at finite t'Hooft like coupling.

Main open directions

We believe that the most important problems unlocked by our thesis are:

1. **Prove a nontrivial swampland conjecture**. A concrete goal is to show that the only conformal field theory with $\mathcal{N} = 4$ supersymmetry in $d = 4$ dimensions is Super Yang Mills. In order to do that, we propose to set up the bootstrap equations for correlators with $\mathcal{N} = 4$ supersymmetry and apply to them our dispersive functionals, like we did for the ϵ expansion. The papers [103], [104], [105] set up the bootstrap equations and should be very useful in our analysis.

2. **Study the 3d Ising model semi-numerically**. What we have in mind is to follow the paper [50], but to use dispersive sum rules, instead of the lightcone bootstrap. In [50] a wealth of CFT data about low twist operators in the 3d Ising model was uncovered using the lightcone bootstrap and the numerical bootstrap. It would be interesting to use the dispersive sum rules to try to improve on that analysis. Improving the numerical studies of the 3d Ising model is important, since it might shed light on how to solve the theory analytically.

3. **Merge the analytic functionals with the numerical bootstrap**. The numeric conformal bootstrap is at present the best tool to study conformal field theories nonperturbatively and it would be great to improve it using analytic methods. A concrete target is to reproduce the results in the early conformal bootstrap papers, namely [2], [106], using the dispersive sum rules.

Perturbative applications of dispersive sum rules

In this thesis we provide evidence of the usefulness of dispersive sum rules in the context of perturbative conformal field theories. Similar problems are:

1. **Improve on our discussion on the ϵ expansion**. The following two improvements could be made. One improvement is to make the discussion completely analytic. At the moment we need to resort to numerics in order to calculate the CFT data, but this step is likely avoidable. Secondly, it would be very good to obtain CFT data at order ϵ^5, since very little is known at this order. A concrete target are the anomalous dimensions and OPE coefficients of the operators in the leading Regge trajectory at this order.

2. **Study the critical O(N) model in $d = 4 - \epsilon$ dimensions.** The theory of N scalar fields interacting quartically has a fixed point in $d = 4 - \epsilon$ dimensions. This fixed point has been studied to order ϵ^4 in [65]. The dispersive sum rules should give access to more perturbative CFT data, like what happened in [107].

3. **Study the critical O(N) model in a large N expansion in $2 < d < 4$.** The idea would be to improve on the work [108].

4. **Loops in AdS**. Deriving OPE data from AdS diagrams is an old problem in AdS/CFT. A promising approach is the use of the analytic bootstrap, see the papers [52, 53]. It is likely that the dispersive sum rules will be useful for this.

Open problems in CFT's with slightly broken higher spin symmetry

The methods developed in chapter 9 potentially pave the way to compute all four point functions in conformal field theories with slightly broken higher spin symmetry. We believe that the next steps in this program are the following:

1. *Compute $\langle j_s j_0 j_0 j_0 \rangle$ in the quasi-boson theory.* The conformal structures involved are the same as in this thesis, so the calculation should be very similar.

2. *Demonstrate that AdS contact diagrams are not present in $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ and $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ in the quasi-fermion theory using pure CFT arguments.* The chaos bound allows for contact diagrams in $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ and $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. Their absence for $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ was demonstrated in [94] using Feynman diagrams. It should be possible to give a pure CFT demonstration of this fact. The idea is to write down the higher spin Ward identity that connects $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ and $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$, plug the AdS contact diagrams multiplied by arbitrary functions of the t'Hooft coupling and obtain that the only way for the Ward identity to be satisfied is if such functions vanish.

Let us mention some more ambitious problems:

1. *Develop a code that computes all spinning four point functions.* Such a code should:

- generate the structures involved for a given four point function
- generate an ansatz for the Mellin transform, which should be a product of 6 Gamma functions (whose arguments are determined by the lightcone OPE, which is known) times polynomials
- impose crossing, pseudo-conservation and Regge boundedness to fix all the undetermined coefficients in the polynomials.

What differs from what we did here is that for generic spins we should not use embedding space, since the conformal structures in embedding space are generically linearly dependent on each other. It is best to use conformal frame techniques instead. Concretely, one would need the 3 dimensional version of [100] (see also [101]).

2. *Demonstrate that AdS contact diagrams are not present in four point functions in CFT's with slightly broken higher spin symmetry*. As above, the hurdle should be in adapting our formalism to use the 3d conformal frame.

Recently, a new formalism for correlators of conserved currents was proposed in [102]. The idea is to write the conformal structures in a helicity basis. It would be very interesting to apply this idea to correlators in CFT's with slightly broken higher spin symmetry.

Ultimately, one would like to understand higher spin symmetry from the point of view of the bulk of AdS. We hope that our CFT computations can be of some utility for this ultimate goal.

Appendices Part

A 2D CFT calculations

A.1 Mellin amplitudes from BPZ differential equations

Let us consider the four point function $\langle \Phi_{1,2}(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) O_3(z_3, \bar{z}_3) O_4(z_4, \bar{z}_4) \rangle$, where O_2, O_3 and O_4 are arbitrary scalar Virasoro primaries and $\Phi_{1,2}$ is a degenerate Virasoro primary. $\langle \Phi_{1,2}(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) O_3(z_3, \bar{z}_3) O_4(z_4, \bar{z}_4) \rangle$ is annihilated when acted upon by the differential operator $\mathcal{L}_{sing}$:

$$\mathcal{L}_{sing} = \sum_{i=2}^{4} \frac{1}{z_1 - z_i} \partial_{z_i} + \frac{h_i}{(z_1 - z_i)^2} - \frac{3}{2(2h_1 + 1)} \partial_{z_1}^2, \tag{A.1}$$

where we used the usual notation in 2d: $\Delta_i = h_i + \bar{h}_i, l_i = h_i - \bar{h}_i$ and in (A.1) we assume $\Phi_{1,2}$ to be at position 1.

The application of $\mathcal{L}_{sing}$ to $\Phi_{1,2}$ increases h and leaves $\bar{h}$ fixed. Thus, it will be useful for us to apply both $\mathcal{L}_{sing}$ and $\bar{\mathcal{L}}_{sing}$ to $\langle \Phi_{1,2}(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) O_3(z_3, \bar{z}_3) O_4(z_4, \bar{z}_4) \rangle$, so as to get a null scalar Virasoro primary, which we denote by $\xi_{1,2}$. In equations,

$$\bar{\mathcal{L}}_{sing} \mathcal{L}_{sing} \langle \Phi_{1,2} O_2 O_3 O_4 \rangle = \langle \xi_{1,2} O_2 O_3 O_4 \rangle = 0. \tag{A.2}$$

In (A.2), we use the Mellin representation:

$$\langle \Phi_{1,2} O_2 O_3 O_4 \rangle = \int \left(\prod_{i<j} [d\gamma_{ij}] (X_{ij}^2)^{-\gamma_{ij}} \right) M_{\Phi_{(1,2)}}(\{\gamma_{ij}\}). \tag{A.3}$$

We see that the differential operators only act on $(X_{ij}^2)^{-\gamma_{ij}}$, which can be factorized in holomorphic and antiholomorphic parts. Afterwards, we use u and v variables and do a change of variables in order to get a recursion relation for the Mellin amplitude:

$$\sum_{p=0}^{2}\sum_{q=0}^{2} c_{p,q}(\gamma_{12},\gamma_{14})M_{\Phi_{1,2}}(\gamma_{12}+p,\gamma_{14}+q)=0. \tag{A.4}$$

Expressions for the coefficients $c_{p,q}$ are not very important, but let us register one such expression for concreteness:

$$c_{1,1}=\frac{1}{9}\Big((2+4h_1)h_2+(-1+4h_1-3\gamma_{12})\gamma_{12}\Big)\Big((2+4h_1)h_4+(-1+4h_1-3\gamma_{14})\gamma_{14}\Big). \tag{A.5}$$

Equation (A.4) is solved by

$$\begin{aligned}M_{\Phi_{(1,2)}}(\gamma_{12},\gamma_{14})=C_0\Gamma(-\frac{\Delta(\alpha_{12})+\Delta(\alpha_2)-\Delta(\alpha_2-\frac{1}{2}\alpha_+)}{2}+\gamma_{12})\\
\Gamma\Big(-\frac{\Delta(\alpha_{12})+\Delta(\alpha_2)-\Delta(\alpha_2+\frac{1}{2}\alpha_+)}{2}+\gamma_{12}\Big)\Gamma(-\frac{\Delta(\alpha_{12})+\Delta(\alpha_3)-\Delta(\alpha_3-\frac{1}{2}\alpha_+)}{2}+\gamma_{13})\\
\Gamma(-\frac{\Delta(\alpha_{12})+\Delta(\alpha_3)-\Delta(\alpha_3+\frac{1}{2}\alpha_+)}{2}+\gamma_{13})\Gamma(-\frac{\Delta(\alpha_{12})+\Delta(\alpha_4)-\Delta(\alpha_4-\frac{1}{2}\alpha_+)}{2}+\gamma_{14})\\
\Gamma(-\frac{\Delta(\alpha_{12})+\Delta(\alpha_4)-\Delta(\alpha_4+\frac{1}{2}\alpha_+)}{2}+\gamma_{14}),\end{aligned} \tag{A.6}$$

where $\gamma_{13}=\Delta(\alpha_{12})-\gamma_{12}-\gamma_{14}$ and $\Delta(\alpha)$ is given by (8.6). C_0 is not fixed by equation (A.4). We compute it in A.3 (see formula (A.30)). Equation (A.6) is a simple generalisation of (8.23). We just wrote a product of six Γ functions, with poles prescribed by the OPE $\Phi_{1,2}\times\Phi_{\alpha_i}=\Phi_{\alpha_i-\frac{1}{2}\alpha_+}+\Phi_{\alpha_i+\frac{1}{2}\alpha_+}$.

A.2 Comparison with conformal block expansion

We start by establishing some notation. Consider 4 scalar Virasoro primaries. We write their four point function in the Mellin representation as

$$\begin{aligned}\langle O_1O_2O_3O_4\rangle=|x_1-x_3|^{-2\Delta_1}|x_2-x_3|^{\Delta_1-\Delta_2-\Delta_3+\Delta_4}|x_2-x_4|^{-\Delta_1-\Delta_2+\Delta_3-\Delta_4}\\
|x_3-x_4|^{\Delta_1+\Delta_2-\Delta_3-\Delta_4}\int_{C_1}\frac{d\gamma_{12}}{2\pi i}\int_{C_2}\frac{d\gamma_{14}}{2\pi i}M(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}},\end{aligned} \tag{A.7}$$

where $M(\gamma_{12},\gamma_{14})$ is the Mellin amplitude. Let us consider the usual kinematics:

$$G^{21}_{34}(z,\bar z)=\lim_{z_1\to\infty,\bar z_1\to\infty} z_1^{2h_1}\bar z_1^{2h_1}\langle\mathcal{O}_1(z_1,\bar z_1),\mathcal{O}_2(1,1),\mathcal{O}_3(z,\bar z),\mathcal{O}_4(0,0)\rangle, \tag{A.8}$$

where we use the 2d notation $h_i = \frac{\Delta_i}{2}$. So,

$$G^{21}_{34}(z,\bar{z}) = v^{h_1-h_2-h_3+h_4}u^{h_1+h_2-h_3-h_4}\int_{C_1}\frac{d\gamma_{12}}{2\pi i}\int_{C_2}\frac{d\gamma_{14}}{2\pi i}M(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}. \quad \text{(A.9)}$$

We write the four point function as a sum over Virasoro blocks in the s-channel:

$$G^{21}_{34}(z,\bar{z}) = \sum_p C^p_{34}C^p_{12}\mathcal{F}^{21}_{34}(p|z)\bar{\mathcal{F}}^{21}_{34}(p|\bar{z}), \quad \text{(A.10)}$$

where C^p_{12} denotes the OPE coefficient for a Virasoro primary exchanged in $O_1 \times O_2$. $\mathcal{F}^{21}_{34}(p|z)$ is a kinematical function that can be expressed as a power series.

$$\mathcal{F}^{21}_{34}(p|z) = z^{h_p-h_3-h_4}\sum_{k=0}^{\infty}\mathcal{F}_k z^k, \quad \text{(A.11)}$$

where h_p is half the conformal dimension of the exchanged primary. An analogous expansion exists for $\bar{\mathcal{F}}^{21}_{34}(p|\bar{z})$. Expression for the first three coeficients are

$$\mathcal{F}_0 = 1, \quad \text{(A.12)}$$

$$\mathcal{F}_1 = \frac{(h_p+h_3-h_4)(h_p+h_2-h_1)}{2h_p}, \quad \text{(A.13)}$$

$$\mathcal{F}_2 = \frac{A}{B}+\frac{C}{B}, \quad \text{(A.14)}$$

where

$$A = (h_p+h_2-h_1)(h_p+h_2-h_1+1) \quad \text{(A.15)}$$

$$\times\Big((h_p+h_3-h_4)(h_p+h_3-h_4+1)(4h_p+\frac{c}{2})-6h_p(h_p+2h_3-h_4)\Big),$$

$$B = 4h_p(2h_p+1)(4h_p+\frac{c}{2})-36h_p^2, \quad \text{(A.16)}$$

$$C = (h_p+2h_2-h_1)\Big(4h_p(2h_p+1)(h_p+2h_3-h_4) \quad \text{(A.17)}$$

$$-6h_p(h_p+h_3-h_4)(h_p+h_3-h_4+1)\Big).$$

Let us now specialise to the case $\langle\Phi_{1,2}\Phi_{1,2}OO\rangle$, where O is an arbitrary Virasoro primary. Its Mellin amplitude is given by

$$M(\gamma_{12},\gamma_{14}) = C_0\Gamma(\gamma_{13}-\alpha\alpha_+)\Gamma(\gamma_{13}+\alpha\alpha_+-\alpha_+^2+1)\Gamma(\gamma_{12}+\frac{\alpha_+^2}{2}) \quad \text{(A.18)}$$

$$\Gamma(\gamma_{12}+1-\frac{3\alpha_+^2}{2})\Gamma(\gamma_{14}-\alpha\alpha_+)\Gamma(\gamma_{14}+\alpha\alpha_+-\alpha_+^2+1),$$

where $\gamma_{13} = \frac{3\alpha_+^2}{2} - 1 - \gamma_{12} - \gamma_{14}$. We obtained this expression from considering (8.23) and doing $\gamma_{12} \leftrightarrow \gamma_{13}$.

In order to match the Mellin representation with the conformal block expansion, we need to expand the rhs of (A.9) for small z and $\bar{z}$. In particular, we want to match with the contribution of the identity

$$(z\bar{z})^{-h_3-h_4} \subset G_{34}^{21}(z,\bar{z}). \tag{A.19}$$

For the case of $\langle \Phi_{1,2}\Phi_{1,2}OO\rangle$, the rhs of (A.9) is

$$u^{\frac{3}{2}\alpha_+^2-1-2\alpha^2+4\alpha_0\alpha} \int_{C_1} \frac{d\gamma_{12}}{2\pi i} \int_{C_2} \frac{d\gamma_{14}}{2\pi i} C_0 \Gamma(\gamma_{13}-\alpha\alpha_+)\Gamma(\gamma_{13}+\alpha\alpha_+-\alpha_+^2+1) \tag{A.20}$$
$$\Gamma(\gamma_{12}+\frac{\alpha_+^2}{2})\Gamma(\gamma_{12}+1-\frac{3\alpha_+^2}{2})\Gamma(\gamma_{14}-\alpha\alpha_+)\Gamma(\gamma_{14}+\alpha\alpha_+-\alpha_+^2+1)u^{-\gamma_{12}}v^{-\gamma_{14}},$$

We want to reproduce the term $(z\bar{z})^{-h_3-h_4} = u^{-2\alpha^2+4\alpha_0\alpha}$. In order to do so, let us consider the limit $u \to 0$ and $v \to 1$ in (A.20). We take the residue of the integrand at $\gamma_{12} = \frac{3}{2}\alpha_+^2 - 1$ and set $v = 1$ to compute the remaining integral:

$$u^{-2\alpha^2+4\alpha_0\alpha}\Gamma(-1+2\alpha_+^2)C_0 \int [\frac{d\gamma_{14}}{2\pi i}]\Gamma(\gamma_{14}-\alpha\alpha_+)\Gamma(\gamma_{14}+\alpha\alpha_+-\alpha_+^2+1) \tag{A.21}$$
$$\Gamma(-\gamma_{14}-\alpha\alpha_+)\Gamma(-\gamma_{14}+\alpha\alpha_+-\alpha_+^2+1)$$
$$= u^{-2\alpha^2+4\alpha_0\alpha}C_0\Gamma(-1+2\alpha_+^2)\Gamma(-2\alpha\alpha_+)\Gamma^2(1-\alpha_+^2)\frac{\Gamma(2-2\alpha_+^2+2\alpha\alpha_+)}{\Gamma(2-2\alpha_+^2)}.$$

Equating this to $u^{-2\alpha^2+4\alpha_0\alpha}$ fixes the value of C_0 according to (8.22).

A.3 Normalisation of $\langle \Phi_{1,2}O_2O_3O_4\rangle$

In A.1 we computed the Mellin amplitude of $\langle \Phi_{1,2}O_2O_3O_4\rangle$, up to a constant C_0 (see A.6). In this appendix, we determine the value of C_0, which is in formula (A.30). Conformal Virasoro primaries are normalised so as to have a two point function "equal" to 1.

This problem was already analysed in [109]. In that paper, following a technique explained in [110], an analytic continuation of three point functions for general central charge from the ones in minimal models is proposed. Here, we just transcribe that result into an expression for C_0.

We direct the reader interested in understanding the details of this analytic continuation to [109]. In what follows, we just define some conventions and then write formula (A.30) for C_0.

In this appendix, we change notation, so as to match the one in [109]. Conformal dimensions are given by

$$h_\alpha = \alpha(\alpha - q), \tag{A.22}$$

where

$$q = \frac{1}{\beta} - \beta. \tag{A.23}$$

Note that h_α is invariant under $\alpha \to q - \alpha$. The central charge is related to β by

$$c = 1 - 6q^2. \tag{A.24}$$

In the notation used in the rest of the thesis, $\beta = -\alpha_+$.

Our formula for C_0 will depend on a Υ function. Its properties can be found in [109]. We just briefly remind some basic facts. A representation for the Υ function is

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t}\Big((\frac{Q}{2} - x)^2 e^{-t} - \frac{\sinh^2((\frac{Q}{2} - x)t)}{\sinh(\frac{\beta t}{2}) \sinh(\frac{t}{2\beta})} \Big). \tag{A.25}$$

This representation is valid for $0 < Re(x) < Q$. We note that $Q = \beta + \beta^{-1}$. For values of x outside this representation, we need to use the shift relations

$$\Upsilon(x + \beta) = \gamma(\beta x)\beta^{1-2\beta x}\Upsilon(x), \tag{A.26}$$

$$\Upsilon(x + \frac{1}{\beta}) = \gamma(\frac{x}{\beta})\beta^{2\frac{x}{\beta}-1}\Upsilon(x). \tag{A.27}$$

Also note the identities

$$\Upsilon(x) = \Upsilon(Q - x), \tag{A.28}$$

$$\Upsilon(\frac{Q}{2}) = 1. \tag{A.29}$$

Now, the formula for C_0.

$$C_0 = \prod_{k=2}^{4} \frac{\tilde{\Upsilon}(\frac{\beta}{2} + \bar{\alpha} - 2\alpha_k)}{\sqrt{\Upsilon(\beta + 2\alpha_k)\Upsilon(-\frac{1}{\beta} + 2\beta + 2\alpha_k)}} \tilde{\Upsilon}(-\frac{1}{\beta} + \frac{3}{2}\beta + \bar{\alpha}) \tag{A.30}$$

$$\frac{\gamma\left(\frac{1}{\beta^2} - 1\right)\gamma\left(\beta^2\right)}{\sqrt{\gamma\left(\frac{1}{\beta^2} - 2\right)}\sqrt{\gamma\left(2\beta^2\right)}\Upsilon(\beta)},$$

where $\bar{\alpha} = \sum_{k=2}^{4} \alpha_k$ and

$$\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}, \tag{A.31}$$

$$\tilde{\Upsilon}(x) = \frac{\Upsilon(x)}{\Gamma(1-\beta x)} \beta^{-\beta x + \frac{1}{4\beta^2} + \frac{3}{4}}. \tag{A.32}$$

(A.30) is symmetric in the transformation $\alpha_i \to q - \alpha_i$, for each $i = 2, 3, 4$, where $q = \frac{1}{\beta} - \beta$.

A.4 Normalisations in the Coulomb gas formalism

Normalisations in the Coulomb gas formalism are computed in section 9 of [76]. The result is

$$N_{m,n}^2 = \frac{(\alpha_+^2 - 1)^2 \pi^{m+n} \gamma(1 - \frac{1}{\alpha_+^2})^m \alpha_+^{4m-4n-2} \gamma(1 - \alpha_+^2)^n \gamma(\frac{m}{\alpha_+^2} - n)(-1)^{4m-4n-2+1}}{\left(\pi^2 \alpha_+^2\right) \gamma\left(m - \alpha_+^2 n\right)}, \tag{A.33}$$

where $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$.

A simple check can be done on formula (A.33). Consider expression (A.30) for C_0 in the correlator $\langle \Phi_{1,2} \Phi_{\alpha_2} \Phi_{\alpha_3} \Phi_{\alpha_4} \rangle$. Consider now the case in which $\alpha_2 + \alpha_3 + \alpha_4 = 2\alpha_0 - \alpha_{12} - \alpha_+$ and eliminate the variable α_4. This corresponds to inserting one positive screening charge in the Coulomb gas model. In that case, we can simplify the expression for C_0 using the Υ identities (A.26) and (A.28), in such a way that the expression for C_0 depends only on Γ functions:

$$\begin{aligned} C_0 = & \frac{1}{\Gamma(-2\alpha_2\alpha_+)\sqrt{\gamma(\frac{1}{\alpha_+^2} + \frac{2\alpha_2}{\alpha_+})}\sqrt{\gamma(-1 - 2\alpha_2\alpha_+ + \alpha_+^2)}} \\ & \frac{1}{\Gamma(-2\alpha_3\alpha_+)\sqrt{\gamma(\frac{1}{\alpha_+^2} + \frac{2\alpha_3}{\alpha_+})}\sqrt{\gamma(-1 - 2\alpha_3\alpha_+ + \alpha_+^2)}} \\ & \frac{1}{\Gamma(2 - \alpha_+^2 + 2\alpha_2\alpha_+ + 2\alpha_3\alpha_+)\sqrt{\gamma(1 - \frac{1}{\alpha_+^2} - \frac{2\alpha_2}{\alpha_+} - \frac{2\alpha_3}{\alpha_+})}\sqrt{\gamma(1 + 2\alpha_+\alpha_2 + 2\alpha_+\alpha_3)}} \\ & \frac{\gamma\left(\frac{1}{\alpha_+^2} - 1\right)\gamma\left(\alpha_+^2\right)}{\sqrt{\gamma\left(\frac{1}{\alpha_+^2} - 2\right)}\sqrt{\gamma\left(2\alpha_+^2\right)}\Gamma\left(1 - \alpha_+^2\right)}. \end{aligned} \tag{A.34}$$

For that same case, it follows from (8.29) that

$$C_0 = \frac{\pi}{\prod_{i=1}^{n} \Gamma(-2\alpha_i\alpha_+) N(\alpha_i)}. \tag{A.35}$$

Let us now compare expressions (A.35) and (A.34). The value of the normalisation of $N(\alpha_{1,2})$ is given in (A.33). We also use the shift symmetry $N(2\alpha_0 - \frac{\alpha_+}{2} - \alpha_2 - \alpha_3) = \frac{1}{N(\alpha_2+\alpha_3+\frac{1}{2}\alpha_+)}$. We conclude that

$$\frac{N(\alpha_2)N(\alpha_3)}{N(\alpha_2+\alpha_3+\frac{1}{2}\alpha_+)} = \sqrt{\gamma(\frac{1}{\alpha_+^2}+\frac{2\alpha_2}{\alpha_+})}\sqrt{\gamma(\frac{1}{\alpha_+^2}+\frac{2\alpha_3}{\alpha_+})}\sqrt{\gamma(1-\frac{1}{\alpha_+^2}-\frac{2\alpha_2}{\alpha_+}-\frac{2\alpha_3}{\alpha_+})} \tag{A.36}$$
$$\sqrt{\gamma(-1-2\alpha_2\alpha_+ + \alpha_+^2)}\sqrt{\gamma(-1-2\alpha_3\alpha_+ + \alpha_+^2)}\sqrt{\gamma(1+2\alpha_+\alpha_2+2\alpha_+\alpha_3)}$$
$$\frac{\sqrt{\gamma\left(\frac{1}{\alpha_+^2}-2\right)}\sqrt{\gamma\left(2\alpha_+^2\right)}}{\gamma(\frac{1}{\alpha_+^2}-1)\gamma(\alpha_+^2)\sqrt{\gamma(-1+2\alpha_+^2)}}\sqrt{\pi}.$$

We can now put formula (A.33) into the left hand side of (A.36) and see if we obtain the right hand side. We checked this for several cases and indeed it is so.

Formula (A.33) is correct when m and n are both positive. We also need to consider normalisations for cases in which n and m are both negative. In that case, formula (A.33) does not apply. However, when n and m are both negative we can still find the correct normalisations. Since we normalise two point functions such that $\langle OO \rangle = 1$, then it follows that $N(\alpha_{-m,-n}) = \frac{1}{N(\alpha_{m,n})}$.

A.5 Two screening charges

In this section, we consider a general n point correlator $\langle \Phi_{\alpha_1} ... \Phi_{\alpha_n} \rangle$ for which we need to insert two positive screening charges in the Coulomb gas formalism and compute its Mellin amplitude.

As usual in the Coulomb gas formalism, we associate to each operator Φ_{α_i} a vertex operator: $V_{\alpha_i} = N_{\alpha_i}\Phi_{\alpha_i}$, where N_{α_i} is a normalisation that for degenerate operators is equal to (A.33). Thus,

$$\langle \Phi_{\alpha_1} ... \Phi_{\alpha_n} \rangle = \prod_{i<j}^{n} |x_i - x_j|^{4\alpha_i\alpha_j} \int d^2x_{n+1} \int d^2x_{n+2} \tag{A.37}$$
$$\prod_{i=1}^{n} \frac{|x_i - x_{n+1}|^{4\alpha_i\alpha_+}|x_i - x_{n+2}|^{4\alpha_i\alpha_+}}{N_{\alpha_i}} |x_{n+1} - x_{n+2}|^{4\alpha_+^2}$$

We use formula (8.19) to transform the integral in x_{n+2} into a Mellin integral. Notice that we can apply that formula since the integral in x_{n+2} is conformal. Indeed, $4\sum_{i=1}^{n}\alpha_i\alpha_+ + 4\alpha_+^2 = 4(-\frac{1}{\alpha_+} - \alpha_+)\alpha_+ + 4\alpha_+^2 = -4$. So, (A.37) is equal to

$$\frac{\pi^{\frac{d}{2}}}{\Gamma(-2\alpha_+^2)}\prod_{i=1}^{n}\frac{1}{\Gamma(-2\alpha_i\alpha_+)N_{\alpha_i}}\int d^2x_{n+1}\prod_{j=1}^{n}|x_j - x_{n+1}|^{4\alpha_j\alpha_+} \tag{A.38}$$
$$\prod_{i_1<j_1}^{n+1}\left[d\xi^1_{i_1,j_1}\right]\Gamma(\xi^1_{i_1,j_1})|x_{i_1} - x_{j_1}|^{-2\xi^1_{i_1,j_1}},$$

where we used the measure $\sum_{j_1\neq i_1}\xi^1_{i_1,j_1} = -2\alpha_{i_1}\alpha_+$.

The integral in x_{n+1} can also be done by use of (8.19), since $4\sum_{i=1}^{n}\alpha_i\alpha_+ - 2\sum_{i=1}^{n}\xi^1_{i,n+1} = -4 - 4\alpha_+^2 + 4\alpha_+^2 = -4$. Further doing a shift of integration variables, we obtain

$$\langle\Phi_{\alpha_1}...\Phi_{\alpha_n}\rangle = \frac{\pi}{\Gamma(-2\alpha_+^2)}\frac{1}{\prod_{k=1}^{n}\Gamma(-2\alpha_k\alpha_+)N(\alpha_k)}\prod_{i<j}^{n}\int[d\gamma_{ij}]\prod_{i_1<j_1}^{n+1}\int\left[d\xi^1_{i_1,j_1}\right] \tag{A.39}$$
$$\Gamma(\xi^1_{i_1,j_1})\frac{1}{\prod_{k_1=1}^{n}\Gamma(-2\alpha_{k_1}\alpha_+ + \xi^1_{k_1,n+1})}\Gamma(\gamma_{ij} + 2\alpha_i\alpha_j - \xi^1_{ij})|x_i - x_j|^{-2\gamma_{ij}}, \tag{A.40}$$

where the measure is $\sum_{j\neq i}\gamma_{ij} = \Delta_{\alpha_i}$ and $\sum_{j_1\neq i_1}^{n+1}\xi^1_{i_1j_1} = -2\alpha_{i_1}\alpha_+$. Formula (A.39) is a particular case of (8.27).

A.6 Sums in exponent of the Mellin amplitude

In this subsection we work out the sums in formula (8.48), using the measure (8.28).

The first term can be rewritten as

$$-\sum_{r=1}^{z-1}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\xi^r_{ij} = -\frac{1}{2}\sum_{r=1}^{z-1}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\xi^r_{ij}. \tag{A.41}$$

Notice that ξ_{ij} with $i > j$ does not exist as an integration variable, so when we write the expression above we mean that $\xi_{ij} = \xi_{ji}$ when $i > j$. It is useful to write things as (A.41) in order to use the measure (8.28). Using the measure (8.28) we get that

$$\sum_{j\neq i}^{n}\xi^r_{ij} = -\sum_{j=n+1}^{n+r}\xi^r_{ij} - 2\alpha_i\alpha_{n+r+1} + \sum_{s=r+1}^{z-1}\xi^s_{i,n+r+1}. \tag{A.42}$$

Let us plug (A.42) into (A.41). We have that

$$\sum_{r=1}^{z-1}\sum_{i=1}^{n}\alpha_i\alpha_{n+r+1} = (2\alpha_0 - p_1\alpha_+ - q_1\alpha_-)\big((p_1-1)\alpha_+ + q_1\alpha_-\big) \tag{A.43}$$

and furthermore

$$-\frac{1}{2}\sum_{r=1}^{z-1}\sum_{i=1}^{n}\Big(-\sum_{j=n+1}^{n+r}\xi^r_{ij} + \sum_{s=r+1}^{z-1}\xi^s_{i,n+r+1}\Big) = \frac{1}{2}\sum_{r=1}^{z-1}\sum_{i=1}^{n}\xi^r_{i,n+1} \tag{A.44}$$

$$= \frac{1}{2}\sum_{r=1}^{z-1}\Big(-\cancel{\sum_{j=n+2}^{n+r}\xi^r_{n+1,j}} - 2\alpha_{n+1}\alpha_{n+r+1} + \cancel{\sum_{s=r+1}^{z-1}\xi^s_{n+1,n+r+1}}\Big)$$

$$= -\alpha_+\big((p_1-1)\alpha_+ + q_1\alpha_-\big), \tag{A.45}$$

where we used $\alpha_{n+1} = \alpha_+$. We conclude that (A.41) is equal to

$$-\big((p_1-1)\alpha_+ + q_1\alpha_-\big)\big(p_1\alpha_+ + (q_1-1)\alpha_-\big). \tag{A.46}$$

Now let us work out the second sum in (8.48).

$$\sum_{i=1}^{n}\sum_{j\neq i}^{n}\alpha_i\alpha_j = \sum_{i=1}^{n}(2\alpha_0\alpha_i - \alpha_i^2) + \sum_{i=1}^{n}\alpha_i(-p_1\alpha_+ - q_1\alpha_-)$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\Delta(\alpha_i) - (p_1\alpha_+ + q_1\alpha_-)\big((1-p_1)\alpha_+ + (1-q_1)\alpha_-\big). \tag{A.47}$$

(A.46) + (A.47) is equal to

$$1 - \frac{1}{2}\sum_i \Delta(\alpha_i), \tag{A.48}$$

like we wanted to show.

B Single-variable Mellin transform

The standard definition of the Mellin transform $\varphi(s)$ of a function $f(z)$ is given by

$$\varphi(s) = \int_0^\infty dz\, f(z)\, z^{s-1}\,, \qquad f(z) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \varphi(s)\, z^{-s}\,. \tag{B.1}$$

Notice that this definition of $\varphi(s)$ only makes sense if the first integral converges for at least some values of s. Assuming that $f(z)$ does not have any (non-integrable) singularity for $z > 0$, the convergence region is determined by the asymptotic behavior [1]

$$z \to 0 : \qquad f(z) = A_1 z^{-a_1} + A_2 z^{-a_2} + \dots\,, \qquad a_1 > a_2 > \dots \tag{B.2}$$

$$z \to \infty : \qquad f(z) = B_1 z^{-b_1} + B_2 z^{-b_2} + \dots\,, \qquad b_1 < b_2 < \dots \tag{B.3}$$

Clearly, the first integral converges in the strip $a_1 < \operatorname{Re} s < b_1$. In this case, for the contour of the second integral we can pick any c such that $a_1 < c < b_1$.

What if $b_1 < a_1$ and therefore the first integral in (B.1) never converges? We shall now argue that even in this case the Mellin transform can still be defined by allowing a bent contour in the second integral in (B.1). The idea is very simple. We just split the first integral in two parts

$$\psi(s) = \int_0^1 dz\, f(z)\, z^{s-1}\,, \qquad \tilde{\psi}(s) = \int_1^\infty dz\, f(z)\, z^{s-1}\,. \tag{B.4}$$

The asymptotics (B.2) imply that $\psi(s)$ is defined and analytic for $\operatorname{Re} s > a_1$ and the asymptotics (B.3) imply that $\tilde{\psi}(s)$ is defined and analytic for $\operatorname{Re} s < b_1$. Therefore, we

[1] We focus on power-like asymptotic behavior because that is the most relevant in the case of Mellin amplitudes. However, the discussion easily generalizes for more general asymptotics. For example, exponential decay as $z \to \infty$ leads to convergence for $\operatorname{Re} s > a_1$ and logarithmic behavior like $z^{-a}(\log z)^n$ does not change the convergence region but leads to higher order poles of $\varphi(s)$.

can write

$$f(z) = \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{ds}{2\pi i} \psi(s)\, z^{-s} + \int_{c_2 - i\infty}^{c_2 + i\infty} \frac{ds}{2\pi i} \tilde{\psi}(s)\, z^{-s} \tag{B.5}$$

with $c_1 > a_1$ and $c_2 < b_1$. The next step is to deform the contours of these two integrals to the same bent contour C without crossing any singularity of the respective integrands. This is depicted in figure B.1. If this is possible then we can write

$$f(z) = \int_C \frac{ds}{2\pi i} \varphi(s)\, z^{-s}\,, \qquad \varphi(s) = \psi(s) + \tilde{\psi}(s)\,. \tag{B.6}$$

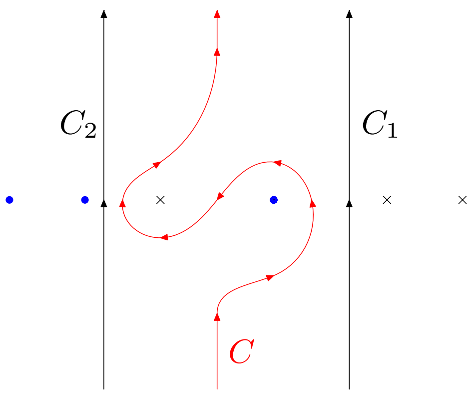

Figure B.1. Singularities in the complex plane of s. Blue balls represent the poles of $\psi(s)$, black crosses represent the poles of $\tilde{\psi}(s)$. Notice that we can gather the two straight contours C_1 and C_2 into a bent contour C, that separates poles to the left from poles to the right.

Bending the contours requires analytic continuation of ψ and $\tilde{\psi}$ beyond the region of convergence of the integrals (B.4). This is easily done by adding and subtracting the asymptotic behaviour of $f(z)$. For example,

$$\psi(s) = \int_0^1 dz\, z^{s-1} \left[f(z) - A_1 z^{-a_1} + A_1 z^{-a_1} \right] = \int_0^1 dz\, z^{s-1} \left[f(z) - A_1 z^{-a_1} \right] + \frac{A_1}{s - a_1} \tag{B.7}$$

where the last integral converges in the larger region $\operatorname{Re} s > a_2$. By adding and subtracting more terms in the asymptotic expansion of $f(z)$ we can further analytically continue ψ and $\tilde{\psi}$. Furthermore, we conclude that the asymptotic behaviour (B.2) and (B.3) gives rise to simple poles of the Mellin transform $\varphi(s)$ at $s = a_i$ and $s = b_i$ as shown in figure B.1. Thus, it is possible to bend the contours without crossing any singularity if and only if none of the poles of ψ (at $s = a_i$) coincides with a pole of $\tilde{\psi}$ (at $s = b_i$). If there is a pole coincidence ($a_i = b_j$ for some i and j) one can introduce a small parameter ϵ and define

$$f(z) = \lim_{\epsilon \to 0} \int_C \frac{ds}{2\pi i} \varphi_\epsilon(s)\, z^{-s}\,, \qquad \varphi_\epsilon(s) = \psi(s + \epsilon) + \tilde{\psi}(s - \epsilon)\,. \tag{B.8}$$

Notice that the limit does not commute with the integral when the contour is pinched by two poles that collide in the limit $\epsilon \to 0$. In fact, in this case, we obtain

$$f(z) = \int_{C'} \frac{ds}{2\pi i} \varphi(s)\, z^{-s} + \sum_i A_i z^{-a_i} \tag{B.9}$$

where the sum runs over the set of colliding poles. The contour C' passes to the left of all poles of $\tilde{\psi}$ and to the right of all poles of ψ, except those that are common.

Consider the simple example $f(z) = z^{-r}$. Then

$$\psi(s) = \frac{1}{s-r}\,, \qquad \tilde{\psi}(s) = \frac{1}{r-s}\,, \qquad \varphi(s) = 0\,, \qquad \varphi_\epsilon(s) = \frac{2\epsilon}{\epsilon^2 - (s-r)^2}\,, \tag{B.10}$$

and indeed

$$\begin{aligned} f(z) &= \lim_{\epsilon\to 0} \int_{r-i\infty}^{r+i\infty} \frac{ds}{2\pi i} \frac{2\epsilon}{\epsilon^2 - (s-r)^2}\, z^{-s} = z^{-r} \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{2\epsilon}{\epsilon^2 + y^2}\, z^{-iy} \\ &= z^{-r} \int_{-\infty}^{\infty} dy \delta(y)\, z^{-iy} = z^{-r}\,. \end{aligned} \tag{B.11}$$

C Bochner's theorem

For completeness we reproduce the statement and proof of Bochner's theorem from the book [111]. Suppose Ω_0 is a non-empty, connected, open set in $\mathbb{R}^n$, with $n > 1$, not necessarily convex. Let f be a holomorphic function of n complex variables, defined on $\Omega_0 + i\mathbb{R}^n$. Suppose that f does not grow too much at infinity, i. e. that for $x \in \Omega_0$ there is an $N \in \mathbb{N}$ such that

$$|f(x+iy)| \ll e^{|y|^{2N}}, \tag{C.1}$$

where $|y|^2 = y_1^2 + \ldots + y_n^2$. Then, f can be holomorphically entended to $\Omega + i\mathbb{R}^n$, where Ω is the convex hull of Ω_0.

The proof consists of an application of Cauchy's residue theorem in 1 dimension. Let x and ξ be two points in Ω_0. Consider the complex plane parametrized by $j(s) \equiv x + s(\xi - x)$ with $s \in \mathbb{C}$. We can define a rectangle R with sides $s = it$ and $s = 1 - it$ for $-t_0 \leqslant t \leqslant t_0$ and t_0 is a positive number that we will eventually take to ∞. The top and bottom of the rectangle are given by $s = \pm it_0 + u$ with $0 < u < 1$. Notice that the rectangle R passes through $x = j(0)$ and $\xi = j(1)$. Now, we consider a point $j(\zeta) = x + \zeta(\xi - x)$ inside this rectangle. In other words, ζ is a complex number with $0 < \operatorname{Re}\zeta < 1$. Then we can write

$$\begin{aligned} W\left(j(\zeta)\right) f\left(j(\zeta)\right) &= \lim_{t_0\to\infty} \int_R \frac{ds}{2\pi i} \frac{W\left(j(s)\right) f\left(j(s)\right)}{\zeta - s} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \frac{W\left(j(it)\right) f\left(j(it)\right)}{\zeta - it} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \frac{W\left(j(1-it)\right) f\left(j(1-it)\right)}{\zeta - 1 + it}, \end{aligned} \tag{C.2}$$

where W is an auxiliary analytic function that ensures that the top and the bottom of the rectangle do not contribute in the limit of large t_0. For example, we can take

$$W(x) = e^{x^{2Q}} \tag{C.3}$$

for Q odd and $Q > N$. In fact, the condition (C.1) could be weakened by choosing another W. The first line of (C.2) is valid if the rectangle is contained in $\Omega_0 + i\mathbb{R}^n$. Remarkably, the second line of (C.2) is valid more generally. It is sufficient that both $x, \xi \in \Omega_0$ because the limit $t_0 \to \infty$ removed the top and the bottom of the rectangle. In this way we extended the domain of analyticity to the convex hull of Ω_0.

D Analytic continuation of $K(\gamma_{12}, \gamma_{14})$

We would like to show that

$$K(\gamma_{12}, \gamma_{14}) \equiv \int_0^1 \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f(u,v)\,, \tag{D.1}$$

can be defined for all γ_{12}, γ_{14} in the complex plane, except at the OPE singularities: $\gamma_{12}, \gamma_{14} = \Delta - \frac{\tau}{2} - m$, where τ is the twist of an exchanged primary and m is a nonnegative integer. The integral above is well-defined for $\mathrm{Re}(\gamma_{12}) > \Delta$ and $\mathrm{Re}(\gamma_{14}) > \Delta$. Our task is to analytically continue (D.1) beyond this region. Firstly, we will show that this can be done for regions [b] and [c] of figure 2.1. Secondly, we will discuss the case of region [d] where we do not have a rigorous proof. Finally, we discuss the asymptotic behaviour of K-functions.

D.1 Regions [b] and [c]

It is convenient to use the following expansion of the four point function

$$f(u,v) = \sum_{\tau,l} a_{\tau,l} u^{-\Delta+\frac{\tau}{2}} (z^l + \bar{z}^l) \tag{D.2}$$

that holds almost everywhere in the integration region $(u,v) \in [0,1] \times [0,1]$. We sum over exchanged operators (both primaries and descendants) of twist τ and spin l (see [28]) and the variables $z, \bar{z}$ are defined by the usual relations $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$. The only points in that square where the expansion does not work are at $v = 0$ and $u = 1$. The coefficients $a_{\tau,l}$ are positive. When it converges, the series (D.2) converges absolutely in each point.

Equation (D.2) will be an essential ingredient in our argument for analyticity of K functions and consequently of Mellin amplitudes. So, we can say that analyticity of Mellin amplitudes follows from the fact that CFT correlation functions enjoy an operator product expansion, whose coefficients have a definite sign for unitary theories. We expect this to be also true for non-identical operators though.

Let us suppose that $\text{Re}(\gamma_{14}) > \Delta$ and let us attempt to analytically continue in γ_{12}. The first step consists in dividing the integration region into two regions.

$$\begin{aligned} K(\gamma_{12}, \gamma_{14}) \equiv & \int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f(u,v) \\ & + \int_{\frac{1}{2}}^1 \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f(u,v). \end{aligned} \tag{D.3}$$

The second integral is completely analytic in γ_{12}. From now on, we will refer ourselves only to the first integral. The usefulness of using these regions will be clear in a moment.

In order to analytically continue beyond the region $\text{Re}(\gamma_{12}) > \Delta$ we add and subtract the twist contributions up to to some twist τ_{max}:

$$\int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f(u,v) = \int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f_{sub}(u,v) \tag{D.4}$$

$$+ \int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} \sum_l \sum_{\tau < \tau_{max}} a_{\tau,l} \, u^{-\Delta + \frac{\tau}{2}} (z^l + \bar{z}^l) \, , \tag{D.5}$$

where

$$f_{sub}(u,v) = f(u,v) - \sum_l \sum_{\tau < \tau_{max}} a_{\tau,l} \, u^{-\Delta + \frac{\tau}{2}} (z^l + \bar{z}^l) \, . \tag{D.6}$$

Let us show that the first term in the rhs of (D.4) is analytic in $\text{Re}(\gamma_{12}) > \Delta - \frac{\tau_{max}}{2}$ and $\text{Re}(\gamma_{14}) > \Delta$. In order to do this, we need to bound $f_{sub}(u,v)$ in the lightcone limits $u \to 0$, $v \to 0$ and the double lightcone limit $u, v \to 0$.

Start by noticing that $f_{sub}(u,v) \sim u^{-\Delta + \frac{\tau_{max}}{2}}$ in the lightcone limit $u \to 0$, due to the subtractions that we made. In the lightcone limit $v \to 0$, the function $f_{sub}(u,v)$ cannot be more singular than $f(u,v)$. This is because in that limit $f(u,v)$ is a sum of positive terms (see (D.2)) and to get $f_{sub}(u,v)$ we just subtracted some of these terms. Thus, in the limit $v \to 0$, $f_{sub}(u,v)$ cannot be more singular than $v^{-\Delta}$.

Finally, we need to bound $f_{sub}(u,v)$ in the double lightcone limit. $f_{sub}(u,v)$ has the following series expansion

$$f_{sub}(u,v) = \sum_l \sum_{\tau > \tau_{max}} a_{\tau,l} u^{-\Delta + \frac{\tau}{2}} (z^l + \bar{z}^l). \tag{D.7}$$

We proceed like in section (1.5). Let us switch to $z, \bar{z}$ coordinates. We find that

$$(z_1\bar{z}_1)^{\Delta-\frac{\tau_{max}}{2}} f_{sub}(z_1,\bar{z}_1) < (z_2\bar{z}_1)^{\Delta-\frac{\tau_{max}}{2}} f_{sub}(z_2,\bar{z}_1), \tag{D.8}$$

if $0 < z_1 < z_2$. Now let us suppose $z_1 \sim 0$. This corresponds to $u \to 0$ on the lhs. Furthermore, let us take the limit $\bar{z} \to 1$. This correponds to the double lightcone on the lhs and to the lightcone limit $v \to 0$ on the rhs. We conclude that

$$f_{sub}(u,v) \sim u^{-\Delta+\frac{\tau_{max}}{2}} v^{-\Delta}, \tag{D.9}$$

where by $\sim$ above we mean that the lhs is not more singular than the rhs. Thus, the rhs of (D.4) is analytic in $\mathrm{Re}(\gamma_{12}) > \Delta - \frac{\tau_{max}}{2}$ and $\mathrm{Re}(\gamma_{14}) > \Delta$.

Let us consider now the second term in (D.5). It is clear that this term is well-defined for $\mathrm{Re}(\gamma_{12})$, $\mathrm{Re}(\gamma_{14}) > \Delta$. We will show that we can commute the sum with the integral. Afterwards, we will analytically continue into the region $\mathrm{Re}(\gamma_{12}) > \Delta - \frac{\tau_{max}}{2}$, $\mathrm{Re}(\gamma_{14}) > \Delta$, except at the points where $\gamma_{12} = \Delta - \frac{\tau}{2}$, for any twist τ exchanged. Those are the OPE singularities.

We make use of the Fubini-Tonelli theorem, which says that commuting the sum with the integral is allowed in case of absolute convergence

$$\int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} \sum_l \sum_{\tau<\tau_{max}} a_{\tau,l}\, u^{-\Delta+\frac{\tau}{2}} |z^l + \bar{z}^l| < \infty\,. \tag{D.10}$$

We divide the integral into two parts, the Lorentzian region $v \leqslant (1-\sqrt{u})^2$, and the Euclidean region $(1-\sqrt{u})^2 \leqslant v \leqslant 1$ (see figure 1.1).

In the Lorentzian region, z and $\bar{z}$ are real and positive. So, the modulus in (D.10) does nothing. Thus, we can commute the sum with the integral over there. Consider now the Euclidean region. In the Euclidean region, z and $\bar{z}$ are the complex conjugate of each other. Note that $|z^l + \bar{z}^l|_{(u,v)} \leqslant (z^l + \bar{z}^l)_{(u,(1-\sqrt{u})^2)}$. This is because (u,v) and $(u,(1-\sqrt{u})^2)$ have the same value of $|z|$, but in the second case z and $\bar{z}$ are positive real numbers. In the Euclidean region

$$\int_0^{\frac{1}{2}} \frac{du}{u} \int_{(1-\sqrt{u})^2}^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} \sum_{\tau,l} a_{\tau,l}\, u^{-\Delta+\frac{\tau}{2}} |z^l + \bar{z}^l| \leqslant \int_0^{\frac{1}{2}} \frac{du}{u} \int_{(1-\sqrt{u})^2}^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f(u,(1-\sqrt{u})^2)$$
$$= \int_0^{\frac{1}{2}} \frac{du}{u} u^{\gamma_{12}} f(u,(1-\sqrt{u})^2) \frac{1}{\gamma_{14}} (1-(1-\sqrt{u})^{2\gamma_{14}}). \tag{D.11}$$

If $\mathrm{Re}(\gamma_{12}) > \Delta$, then this integral is well defined. Note that it was important that the integral in (D.11) did not go up to $u = 1$. This was why we made the separation (D.3).

We conclude that

$$\int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} \sum_l \sum_{\tau<\tau_{max}} a_{\tau,l}\, u^{-\Delta+\frac{\tau}{2}} (z^l + \bar{z}^l) \tag{D.12}$$
$$= \sum_l \sum_{\tau<\tau_{max}} a_{\tau,l} \kappa_l (\gamma_{12} - \Delta + \frac{\tau}{2}, \gamma_{14}),$$

where

$$\kappa_l(\gamma_{12},\gamma_{14}) = \int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} (z^l + \bar{z}^l). \tag{D.13}$$

We conclude that when $\mathrm{Re}(\gamma_{12}), \mathrm{Re}(\gamma_{14}) > \Delta$, then $K(\gamma_{12},\gamma_{14})$ can be written as

$$K(\gamma_{12},\gamma_{14}) = \int_{\frac{1}{2}}^1 \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f(u,v) + \int_0^{\frac{1}{2}} \frac{du}{u} \int_0^1 \frac{dv}{v} u^{\gamma_{12}} v^{\gamma_{14}} f_{sub}(u,v) \tag{D.14}$$
$$+ \sum_l \sum_{\tau<\tau_{max}} a_{\tau,l}\, \kappa_l \left(\gamma_{12} - \Delta + \frac{\tau}{2}, \gamma_{14} \right) .$$

Let us consider analytic continuation of (D.14) in γ_{12}, keeping $\mathrm{Re}(\gamma_{14}) > \Delta$ fixed. The first two terms in (D.14) are analytic for $\mathrm{Re}(\gamma_{12}) > \Delta - \frac{\tau_{max}}{2}$. We will now argue that the analytic continuation of the last sum in (D.14) is analytic for all γ_{12} in the complex plane as long as

$$\mathrm{Re}(\gamma_{12}) > \Delta - \frac{\tau_{max}}{2}, \qquad \left| \gamma_{12} - \Delta + \frac{\tau}{2} + m \right| > \epsilon, \tag{D.15}$$

where $\epsilon > 0$ is a small regulator to avoid the OPE poles, τ is the twist of any exchanged operator and m is a non-negative integer. Firstly, we will discuss the analytic continuation of each term $\kappa_l \left(\gamma_{12} - \Delta + \frac{\tau}{2}, \gamma_{14} \right)$. Secondly, we will discuss the convergence of the (infinite) sum in (D.14).

Notice that when l is a positive integer, then $z^l + \bar{z}^l$ is a polynomial of degree l in u and v. In fact, one can write

$$\kappa_l(\gamma_{12},\gamma_{14}) = 2^{-\gamma_{12}} \sum_{m=0}^{l} \frac{r_{l,m}(\gamma_{14})}{\gamma_{12}+m}, \tag{D.16}$$

where $r_{l,m}(\gamma_{14})$ is a rational function. This shows that the analytic continuation of each term $\kappa_l \left(\gamma_{12} - \Delta + \frac{\tau}{2}, \gamma_{14} \right)$ generates only OPE singularities at $\gamma_{12} = -\frac{\tau}{2} + \Delta - m$, where τ is the twist of an exchanged primary operator and m is a nonnegative integer.

Now we would like to show that the sum over twists and spin in (D.14) converges for any γ_{12} as long as (D.15) is satisfied. We start by writing an upper bound

$$\left|\sum_l \sum_{\tau<\tau_{max}} a_{\tau,l}\,\kappa_l\left(\gamma_{12}-\Delta+\frac{\tau}{2},\gamma_{14}\right)\right| \leq \sum_l \sum_{\tau<\tau_{max}} a_{\tau,l}\left|\kappa_l\left(\gamma_{12}-\Delta+\frac{\tau}{2},\gamma_{14}\right)\right| \tag{D.17}$$

$$\leq \sum_l \sum_{\tau<\tau_{max}} a_{\tau,l} \max_{\substack{\mathrm{Re}\,x>(\tau-\tau_{max})/2\\ |x+m|>\epsilon}} \left|\kappa_l\left(x,\gamma_{14}\right)\right|$$

$$\leq \sum_l A_l \max_{\substack{\mathrm{Re}\,x>-\tau_{max}/2\\ |x+m|>\epsilon}} \left|\kappa_l\left(x,\gamma_{14}\right)\right| \tag{D.18}$$

where $A_l = \sum_{\tau<\tau_{max}} a_{\tau,l}$. It is clear from (D.16) that the maximum is finite for every value of the spin l. Therefore, convergence of the sum follows from the large l behaviour of the summand. To understand this it is convenient to make a small detour into the Lorentzian region.

Consider the convergent sum (for $\mathrm{Re}(\gamma_{12}),\mathrm{Re}(\gamma_{14})>\Delta$)

$$W \equiv \sum_l \sum_{\tau<\tau_{max}} a_{\tau,l}\,\kappa_l^{\mathrm{Lor}}\left(\gamma_{12}-\Delta+\frac{\tau}{2},\gamma_{14}\right) < \int_0^{\frac{1}{2}}\frac{du}{u}\int_0^{(1-\sqrt{u})^2}\frac{dv}{v}u^{\gamma_{12}}v^{\gamma_{14}}f(u,v)\,, \tag{D.19}$$

where

$$\kappa_l^{\mathrm{Lor}}(\gamma_{12},\gamma_{14}) = \int_0^{\frac{1}{2}}\frac{du}{u}\int_0^{(1-\sqrt{u})^2}\frac{dv}{v}u^{\gamma_{12}}v^{\gamma_{14}}(z^l+\bar{z}^l). \tag{D.20}$$

Since all terms are positive and $u\leq\frac{1}{2}$ in the integration region, we have a lower bound

$$W > \sum_l A_l\,\kappa_l^{\mathrm{Lor}}(\gamma_{12}-\Delta+\frac{\tau_{max}}{2},\gamma_{14})\,. \tag{D.21}$$

Convergence of this sum implies a bound on the asymptotic growth of A_l at large spin. Let us compute the large spin behavior of κ_l^{Lor}. When $l\to\infty$, we find

$$\begin{aligned}\kappa_l^{\mathrm{Lor}}(\gamma_{12},\gamma_{14}) &= \int_0^{\frac{1}{2}}\frac{du}{u}\int_0^{(1-\sqrt{u})^2}\frac{dv}{v}u^{\gamma_{12}}v^{\gamma_{14}}(z^l+\bar{z}^l)\\ &= \int_0^1 dz\int_0^{\min\{1,\frac{1}{2z}\}} d\bar{z}|z-\bar{z}|\left(z^{\gamma_{12}-1+l}(1-z)^{\gamma_{14}-1}\bar{z}^{\gamma_{12}-1}(1-\bar{z})^{\gamma_{14}-1}\right)\\ &\approx \frac{1}{l^{\gamma_{14}}}\Gamma(\gamma_{14})B_{\frac{1}{2}}\left(\gamma_{12},\gamma_{14}+1\right),\end{aligned} \tag{D.22}$$

where we used the incomplete β-function. Therefore,

$$\sum_l A_l\,\frac{1}{l^{\gamma_{14}}} < \infty\,, \qquad \mathrm{Re}(\gamma_{14}) > \Delta\,. \tag{D.23}$$

It turns out that this is sufficient to prove convergence of (D.18) for $\mathrm{Re}(\gamma_{14})>\Delta$. The reason is that

$$\kappa_l(\gamma_{12}, \gamma_{14}) \approx \kappa_l^{\mathrm{Lor}}(\gamma_{12}, \gamma_{14}) , \qquad l \to \infty , \tag{D.24}$$

up to exponential corrections of order $2^{-l/2}$ coming from the Euclidean region. When $l \to \infty$, the integral in (D.13) is dominated by the region near $v = 0$, since z (or $\bar{z}$ depending on our choice) achieves its maximum value 1 there.

In this way we established analyticity in region [b] up to OPE poles. The same analyticity in region [c] follows from crossing symmetry $K(\gamma_{12}, \gamma_{14}) = K(\gamma_{14}, \gamma_{12})$.

D.2 Region [d]

Let us use the trick of Bochner's theorem as reviewed in C. We introduce a complex plane parametrized by s embedded in $\mathbb{C}^2$ as

$$\vec{\gamma}(s) = \vec{\gamma}(0) + s(\vec{\gamma}(1) - \vec{\gamma}(0)) , \qquad \vec{\gamma}(0) = (\gamma_{12}^{(0)}, \gamma_{14}^{(0)}) , \qquad \vec{\gamma}(1) = (\gamma_{12}^{(1)}, \gamma_{14}^{(1)}) \tag{D.25}$$

First we choose $\gamma_{12}^{(1)} > \gamma_{12}^{(0)} > \Delta$ and $\gamma_{14}^{(0)} > \gamma_{14}^{(1)} > \Delta$ so that both $\vec{\gamma}(0)$ and $\vec{\gamma}(1)$ are in region [a]. This allows us to write the following representation for the K-function,

$$KW(\vec{\gamma}(\zeta)) = \int \frac{dt}{2\pi} \frac{KW(\vec{\gamma}(it))}{\zeta - it} + \int \frac{dt}{2\pi} \frac{KW(\vec{\gamma}(1-it))}{\zeta - 1 + it} \tag{D.26}$$

where $0 < \mathrm{Re}\zeta < 1$ and KW denotes the product of the K-function by an holomorphic function W that decays fast along the imaginary direction as explained in C.

Notice that the function $K(\vec{\gamma}(s))$ has poles at

$$s = s_\tau^{(12)} \equiv \frac{\Delta - \tau/2 - \gamma_{12}^{(0)}}{\gamma_{12}^{(1)} - \gamma_{12}^{(0)}} , \tag{D.27}$$

and

$$s = s_\tau^{(14)} \equiv \frac{\Delta - \tau/2 - \gamma_{14}^{(0)}}{\gamma_{14}^{(1)} - \gamma_{14}^{(0)}} = 1 + \frac{\Delta - \tau/2 - \gamma_{14}^{(1)}}{\gamma_{14}^{(1)} - \gamma_{14}^{(0)}} . \tag{D.28}$$

For our choice of $\vec{\gamma}(0)$ and $\vec{\gamma}(1)$, the poles obey $s_\tau^{(12)} < 0$ and $s_\tau^{(14)} > 1$ as required by the conditions of Bochner's theorem.

Now let us move $\vec{\gamma}(0)$ into region [b] and $\vec{\gamma}(1)$ into region [c] as depicted in figure D.1. In other words we decrease $\gamma_{12}^{(0)}$ and $\gamma_{14}^{(1)}$ below Δ. Under this deformation, there are poles that cross the contours along $s = it$ and $s = 1 - it$ for $t \in \mathbb{R}$. This will change

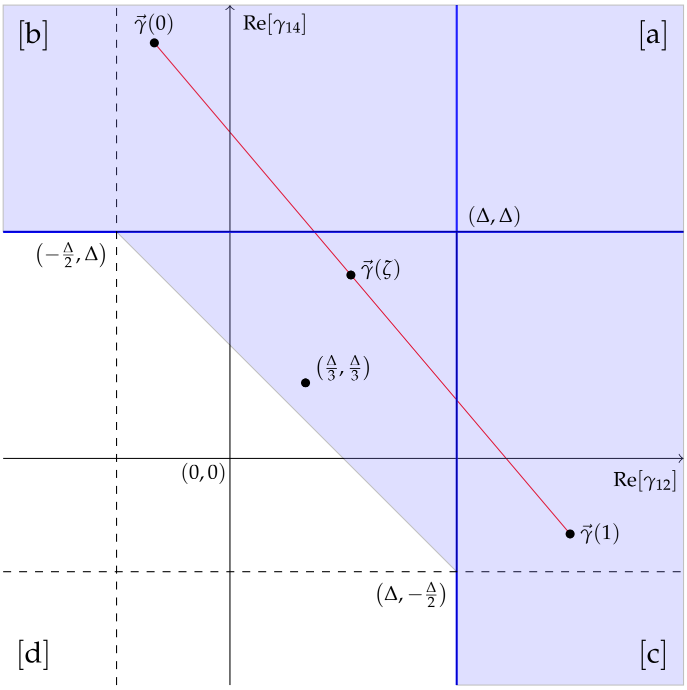

Figure D.1. Construction that leads to equations (D.29) and (D.30). We represent the case $\Delta = \tau_{lightest}$ and therefore $\tau_* = 2\Delta$. We can establish analyticity in the shaded domain without crossing the accumulation point of accumulation points of triple-twist operators (marked with dashed lines).

equation (D.26) into

$$KW\left(\vec{\gamma}(\zeta)\right) = \int \frac{dt}{2\pi}\frac{KW\left(\vec{\gamma}(it)\right)}{\zeta - it} + \int \frac{dt}{2\pi}\frac{KW\left(\vec{\gamma}(1-it)\right)}{\zeta - 1 + it} + \sum_{s_\tau^{(12)}>0} \frac{\text{Res}_{s=s_\tau^{(12)}} KW\left(\vec{\gamma}(s)\right)}{\zeta - s_\tau^{(12)}} + \sum_{s_\tau^{(14)}<1} \frac{\text{Res}_{s=s_\tau^{(14)}} KW\left(\vec{\gamma}(s)\right)}{\zeta - s_\tau^{(14)}} \tag{D.29}$$

Notice that if we denote $\vec{\gamma}(\zeta) = (\gamma_{12}, \gamma_{14})$ then the last equation can be written as

$$\begin{aligned} KW\left(\gamma_{12}, \gamma_{14}\right) = &\int \frac{dt}{2\pi}\frac{KW\left(\vec{\gamma}(it)\right)}{\zeta - it} + \int \frac{dt}{2\pi}\frac{KW\left(\vec{\gamma}(1-it)\right)}{\zeta - 1 + it} \\ &+ \sum_{\tau<2(\Delta-\gamma_{12}^{(0)})} \frac{\text{Res}_{\gamma_{12}=\Delta-\tau/2} KW\left(\gamma_{12}, \gamma_{14}(s_\tau^{(12)})\right)}{\gamma_{12} - \Delta + \tau/2} \\ &+ \sum_{\tau<2(\Delta-\gamma_{14}^{(1)})} \frac{\text{Res}_{\gamma_{14}=\Delta-\tau/2} KW\left(\gamma_{12}(s_\tau^{(14)}), \gamma_{14}\right)}{\gamma_{14} - \Delta + \tau/2} \end{aligned} \tag{D.30}$$

Equations (D.29) or (D.30) imply analyticity of the K-function at $\vec{\gamma}(\zeta) = (\gamma_{12}, \gamma_{14})$ as long as: **i.** we place $\vec{\gamma}(0)$ and $\vec{\gamma}(1)$ in an analytic domain inside region [b] and region [c], respectively; **ii.** the sums converge. Condition **i.** is easy to satisfy if the twist spectrum is not continuous. Although this is an open question for high twist in $\text{CFT}_d > 2$, it is clear that at least until the first accumulation point of accumulation points (triple-twist operators) the spectrum is discrete. Condition **ii.** is more non-trivial. Let us consider several cases of increasing difficulty:

- The sums converge because they contain finite number of terms. This is the case if we do not cross any accumulation point, *i.e.* for $\Delta - \tau_*/2 < \gamma_{12}^{(0)}, \gamma_{14}^{(1)} < \Delta$.
- We cross only one twist accumulation point τ_*. For example, take $\gamma_{12}^{(0)} < \Delta - \tau_*/2$. In this case, the infinite sum over twists accumulating at τ_* only converges if the intersection of $\gamma_{12} = \Delta - \tau_*/2$ with the straight line through $\vec{\gamma}(0)$ and $\vec{\gamma}(1)$ has $\gamma_{14} = \gamma_{14}(s_{\tau_*}^{(12)}) > \Delta$ because of (D.23). Fortunately, this last condition can be relaxed by choosing a function W that vanishes like $(\gamma_{12} - \Delta + \tau_*/2)^p$ at $\gamma_{12} = \Delta - \tau_*/2$ for some integer $p > 0$. This makes the residues in (D.30) smaller as we approach the accumulation point and therefore extends the convergence of the sum to $\gamma_{14}(s_{\tau_*}^{(12)}) > \Delta - p\,\tau_{\text{gap}}$.[1]
- We cross only a finite number of double-twist accumulation points $\tau_* + 2n$ in both region [b] and region [c]. This case can be treated similarly to the previous one. It is enough to choose a function W that vanishes sufficiently fast at $\gamma_{12} = \Delta - \tau_*/2 - n$ and $\gamma_{14} = \Delta - \tau_*/2 - n$ for a finite set of integers n. Notice that if $\Delta = \tau_{lightest}$ then $\tau_* = 2\Delta$ and this allows us to prove analyticity in the corner of region [d] with $\gamma_{13} < \frac{\Delta}{2}$ (see figure D.1). This region contains the crossing symmetric point $\gamma_{12} = \gamma_{13} = \gamma_{14} = \frac{\Delta}{3}$.
- We cross an infinite number of twist accumulation points. For example, we cross the triple-twist accumulation point of accumulation points at $\gamma_{12} = \Delta - \frac{1}{2}\tau_{triple}$. Conservatively, the sums converge as long as the intersection $\gamma_{14}(s_{3\tau_*/2}^{(12)}) > \Delta$. This is not sufficient to extend the region of analyticity beyond the previous case.

D.3 Asymptotic behavior of K functions and of the Mellin amplitude

K functions decay polynomially at infinity. This can be seen starting from their definition:

$$K(\gamma_{12}, \gamma_{14}) = \int_0^1 \frac{du}{u} \int_0^1 \frac{dv}{v} F(u,v) u^{\gamma_{12}} v^{\gamma_{14}}. \tag{D.31}$$

When $\gamma_{12}, \gamma_{14} \to i\infty$ this integral is dominated by $u = v = 1$ and so we obtain $K(\gamma_{12}, \gamma_{14}) \sim \frac{1}{\gamma_{12}\gamma_{14}}$. Subleading corrections to this behaviour can be computed by

[1] Here we used the large spin behaviour $\tau_* - \tau(l) \sim l^{-\tau_{gap}}$ from the lightcone bootstrap.

expanding the correlation function close to the crossing symmetric point $u = v = 1$.

By contrast, the Mellin amplitude

$$M(\gamma_{12}, \gamma_{14}) = K(\gamma_{12}, \gamma_{14}) + K(\gamma_{13}, \gamma_{14}) + K(\gamma_{12}, \gamma_{13}) \tag{D.32}$$

decays exponentially at infinity. We proved this for the cases in which the theorem of section (1.1) applies. This also happens in every example.

Since the Mellin amplitude can be written as a sum of functions that decay polynomially, it is not obvious how come it can decay exponentially from the point of view of K functions. Let us see that crossing symmetry implies that it does not decay polynomially. We check this in some simple examples but not in full generality in the sense that we will see next. Indeed consider expression (D.31) and expand

$$F(u, v) = \sum_{n,m=0}^{N} a_{n,m}(1-u)^n(1-v)^m \tag{D.33}$$

where N is some positive integer. Crossing relates different $a_{n,m}$ to each other. If we plug the function $K(\gamma_{12}, \gamma_{14})$ thus obtained into (D.32), we seem to obtain that $M(\gamma_{12}, \gamma_{14})$ decays polynomially. However notice that the coefficients $a_{n,m}$ are not all arbitrary and they are constrained by crossing symmetry[2]. For this reason many cancellations occur and one obtains that $M(\gamma_{12}, \gamma_{14}) \sim \frac{1}{\gamma_{12}^N \gamma_{14}^N}$. We checked this up to $N = 10$ and we believe that it holds for arbitrary N.

[2] A similar idea was pursued in [112].

E Examples

E.1 Examples of K-functions

E.1.1 Free fields

Consider a free scalar field ϕ of conformal dimension Δ. Then, for $\langle\phi\phi\phi\phi\rangle$ we have

$$F(u,v) = 1 + u^{-\Delta} + v^{-\Delta}, \tag{E.1}$$

$$K(\gamma_{12},\gamma_{14}) = \frac{1}{\gamma_{12}\gamma_{14}} + \frac{1}{\gamma_{12}(\gamma_{14}-\Delta)} + \frac{1}{(\gamma_{12}-\Delta)\gamma_{14}}. \tag{E.2}$$

The corresponding Mellin amplitude is 0.

For the case $O = \frac{1}{\sqrt{2N}}\sum_{i=1}^{N}\phi^i\phi^i$ in free scalar theory, we have

$$F(u,v) = 1 + u^{-\Delta} + v^{-\Delta} + \frac{4}{N}\left(u^{-\frac{\Delta}{2}} + v^{-\frac{\Delta}{2}} + u^{-\frac{\Delta}{2}}v^{-\frac{\Delta}{2}}\right), \tag{E.3}$$

$$K(\gamma_{12},\gamma_{14}) = \frac{1}{\gamma_{12}\gamma_{14}} + \frac{1}{\gamma_{12}(\gamma_{14}-\Delta)} + \frac{1}{(\gamma_{12}-\Delta)\gamma_{14}} \tag{E.4}$$

$$+ \frac{4}{N}\left(\frac{1}{(\gamma_{12}-\frac{\Delta}{2})(\gamma_{14}-\frac{\Delta}{2})} + \frac{1}{\gamma_{12}(\gamma_{14}-\frac{\Delta}{2})} + \frac{1}{(\gamma_{12}-\frac{\Delta}{2})\gamma_{14}}\right). \tag{E.5}$$

The corresponding Mellin amplitude is 0.

E.1.2 The correlator $\langle\sigma\sigma\sigma\sigma\rangle$ in 2D Ising

In the 2d Ising model,

$$\langle\sigma\sigma\sigma\sigma\rangle = \frac{1}{x_{13}^{\frac{1}{4}}x_{24}^{\frac{1}{4}}}F(u,v) = \frac{1}{x_{13}^{\frac{1}{4}}x_{24}^{\frac{1}{4}}}\int\frac{d\gamma_{12}}{2\pi i}\int\frac{d\gamma_{14}}{2\pi i}\hat{M}(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}, \tag{E.6}$$

with

$$F(u,v) = \frac{\sqrt{\sqrt{u}+\sqrt{v}+1}}{\sqrt{2}\sqrt[8]{uv}}, \tag{E.7}$$

$$\hat{M}(\gamma_{12},\gamma_{14}) = -\sqrt{\frac{2}{\pi}}\Gamma\left(2\gamma_{12}-\frac{1}{4}\right)\Gamma\left(2\gamma_{14}-\frac{1}{4}\right)\Gamma(-2\gamma_{12}-2\gamma_{14}). \tag{E.8}$$

We will compute $K(\gamma_{12},\gamma_{14})$ in two ways. The first way is the following. Consider

$$Q(\gamma_{12},\gamma_{14}) = \int_0^1 \frac{dv}{v}\int_0^v \frac{du}{u} u^{\gamma_{12}} v^{\gamma_{14}} F(u,v). \tag{E.9}$$

Then, $F(u,v) = F(v,u)$ implies

$$K(\gamma_{12},\gamma_{14}) = Q(\gamma_{12},\gamma_{14}) + Q(\gamma_{14},\gamma_{12}). \tag{E.10}$$

Our goal is to compute $Q(\gamma_{12},\gamma_{14})$ by expanding $F(u,v)$ in a power series expansion around $u=0$.

For a generic CFT, we would proceed in the following manner. We write [1]

$$F(u,v) = \sum_k C^2_{OOO_k} u^{\frac{\tau_k}{2}-\Delta} \sum_{m=0}^{\infty} u^m g_m(v), \tag{E.11}$$

where Δ is the conformal dimension of the external scalar, τ_k is the twist of an exchanged primary, $C^2_{OOO_k}$ is an OPE coefficient and finally $g_m(v)$ is a collinear block. Suppose we put equation (E.11) into (E.9). Notice that the integral (E.9) does not involve $v=0$, which is where the expansion (E.11) should fail. We find

$$\begin{aligned} K(\gamma_{12},\gamma_{14}) = \sum_k C^2_{OOO_k} \sum_{m=0}^{\infty} \Big(&\frac{1}{\gamma_{12}+\frac{\tau_k}{2}-\Delta+m} \\ &+ \frac{1}{\gamma_{14}+\frac{\tau_k}{2}-\Delta+m}\Big) f_m^{\tau_k}(\gamma_{13}), \end{aligned} \tag{E.12}$$

where

$$f_m^{\tau_k}(x) = \int_0^1 \frac{dy}{y} y^{-x+\frac{\tau_k}{2}+m} g_m(y) \tag{E.13}$$

is a kinematical function. In practice, it was difficult to compute it for general m. We

[1] Presumably, expansion (E.11) converges on the square $(u,v)\in[0,1]\times[0,1]$, except for $v=0$. But we could not prove it.

register the result for $m = 0$:

$$g_0(v) = (v-1)^J \, {}_2F_1\left(\frac{1}{2}(2J+\tau), \frac{1}{2}(2J+\tau); 2J+\tau; 1-v\right), \tag{E.14}$$

$$f_0^{\tau_k}(\gamma_{13}) = (-1)^J 2^{2J+\tau-1}\Gamma(J+1)\Gamma\left(J+\frac{\tau}{2}\right)\Gamma\left(J+\frac{\tau}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\tau}{2}-\gamma_{13}\right) \tag{E.15}$$
$${}_3F_2\left(J+1, J+\frac{\tau}{2}, J+\frac{\tau}{2}; J+\frac{\tau}{2}-\gamma_{13}+1, 2J+\tau; 1\right)\frac{1}{\sqrt{\pi}\Gamma(2J+\tau)\Gamma\left(J-\gamma_{13}+\frac{\tau}{2}+1\right)}.$$

In the case of the 2d Ising model, $F(u,v)$ is simple enough to admit a power series expansion in u. From (E.7), we find

$$F(u,v) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\Gamma(n-\frac{1}{2})\left(\sqrt{v}+1\right)^{\frac{1}{2}-n} u^{n/2}}{2\sqrt{2\pi}\Gamma(n+1)(uv)^{\frac{1}{8}}} \tag{E.16}$$

where the expansion converges in $|\sqrt{u}| < |1+\sqrt{v}|$.

We can now do the integrals to find a series expression for $Q(\gamma_{12},\gamma_{14})$ and $K(\gamma_{12},\gamma_{14})$. We find

$$K(\gamma_{12},\gamma_{14}) = \sum_{n=0}^{\infty}\left(\frac{1}{2\gamma_{12}+n-\frac{1}{4}}+\frac{1}{2\gamma_{14}+n-\frac{1}{4}}\right) \tag{E.17}$$
$$\frac{2^{\frac{5}{2}-n}(-1)^{n+1}\Gamma(-\frac{1}{2}+n)\,{}_2F_1\left(1, \frac{5}{4}-2\gamma_{13}; n-2\gamma_{13}+\frac{3}{4}; -1\right)}{\sqrt{2\pi}\left(-2\gamma_{13}+n-\frac{1}{4}\right)\Gamma(1+n)},$$

where $\gamma_{13} = \frac{1}{8}-\gamma_{12}-\gamma_{14}$. This series converges exponentially fast[2]. We checked for several values of γ_{12} and γ_{14} that using (E.17), then $K(\gamma_{12},\gamma_{14}) + K(\gamma_{12}, \frac{1}{8}-\gamma_{12}-\gamma_{14}) + K(\frac{1}{8}-\gamma_{12}-\gamma_{14},\gamma_{14})$ numerically matches (E.8).

Let us outline another way to compute $K(\gamma_{12},\gamma_{14})$. We first do the u integral. Then, we attempt to do the v integral. It is more complicated, since the integrand is also more complicated. So, we series expand the integrand around $v = 0$. We spare the reader the details. We find

$$K(\gamma_{12},\gamma_{14}) = \sum_{n=0}^{\infty} g_n f_n(\gamma_{12}) f_n(\gamma_{14}), \tag{E.18}$$

[2]Note that (E.17) has no poles in γ_{13}, but Q would have poles in γ_{13}. Each term term of the series (E.17) does have poles in γ_{13}, but the residues of successive terms cancel.

where

$$g_n = -\frac{\Gamma\left(n-\frac{1}{2}\right)}{2^{2n-\frac{3}{2}}\sqrt{\pi}\Gamma(n+1)}, \tag{E.19}$$

$$\begin{aligned} f_n(\gamma_{12}) = & \ {}_2\tilde{F}_1\left(1, n-\frac{1}{2}; 2\gamma_{12}+n+\frac{3}{4}; \frac{1}{2}\right)\Gamma\left(2\gamma_{12}+n-\frac{1}{4}\right) \\ \equiv & \ {}_2F_1\left(1, n-\frac{1}{2}; 2\gamma_{12}+n+\frac{3}{4}; \frac{1}{2}\right)\frac{4}{8\gamma_{12}+4n-1}. \end{aligned} \tag{E.20}$$

This expansion also converges exponentially fast and with it we can obtain the correct value of $\hat{M}(\gamma_{12}, \gamma_{14})$.

E.2 Mellin representation with a straight contour for $\langle\sigma\sigma\sigma\sigma\rangle$ in the 2d Ising model

We provide a Mellin amplitude with a straight contour for $\langle\sigma\sigma\sigma\sigma\rangle$ in the 2d Ising model. The four point function is given by $\frac{1}{x_{13}^{\frac{1}{4}}x_{24}^{\frac{1}{4}}}F(u,v)$, with $F(u,v)$ given by (E.7). For this correlator, we have explicit expressions for the function $K(\gamma_{12}, \gamma_{14})$, see (E.17) and (E.18).

According to the discussion in section (2.1), we have that

$$\begin{aligned} F(u,v) = & \int_{Re(\gamma_{12})=\frac{1}{8}+0^+}\frac{d\gamma_{12}}{2\pi i}\int_{Re(\gamma_{14})=\frac{1}{8}+0^+}\frac{d\gamma_{14}}{2\pi i}K(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}} \\ & + \int_{Re(\gamma_{12})=\frac{1}{8}+0^+}\frac{d\gamma_{12}}{2\pi i}\int_{Re(\gamma_{13})=\frac{1}{8}+0^+}[d\gamma_{13}]K(\gamma_{12},\gamma_{13})u^{-\gamma_{12}}v^{-\gamma_{14}} \\ & + \int_{Re(\gamma_{13})=\frac{1}{8}+0^+}[d\gamma_{13}]\int_{Re(\gamma_{14})=\frac{1}{8}+0^+}\frac{d\gamma_{14}}{2\pi i}K(\gamma_{13},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}. \end{aligned} \tag{E.21}$$

Each of these three integrals should be done with a straight contour. Consider the first integral. We can deform its contour until we reach the crossing symmetric point $Re(\gamma_{12}) = \frac{1}{24}$, $Re(\gamma_{14}) = \frac{1}{24}$. In the process, we pick up some poles, which will give us the appropriate subtractions to perform to the correlator. We do the same procedure for all three integrals on the rhs of (E.21). At the end, we reunite the three integrals into a single integral with a straight contour at the crossing symmetric point $Re(\gamma_{12}) = \frac{1}{24}$, $Re(\gamma_{14}) = \frac{1}{24}$.

Let us work out this procedure for the first integral on the rhs of (E.21).

$$\begin{aligned} & \int_{Re(\gamma_{12})=\frac{1}{8}+0^+}\frac{d\gamma_{12}}{2\pi i}\int_{Re(\gamma_{14})=\frac{1}{8}+0^+}\frac{d\gamma_{14}}{2\pi i}K(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}} \\ & = \int_{Re(\gamma_{12})=\frac{1}{24}}\frac{d\gamma_{12}}{2\pi i}\int_{Re(\gamma_{14})=\frac{1}{24}}\frac{d\gamma_{14}}{2\pi i}K(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}} \end{aligned} \tag{E.22}$$

$$+u^{-\frac{1}{8}}\int_{Re(\gamma_{14})=\frac{1}{8}+0^+}\frac{d\gamma_{14}}{2\pi i}\hat{K}(\gamma_{12}=\frac{1}{8},\gamma_{14})v^{-\gamma_{14}}$$
$$+v^{-\frac{1}{8}}\int_{Re(\gamma_{12})=\frac{1}{24}}\frac{d\gamma_{12}}{2\pi i}\hat{K}(\gamma_{12},\gamma_{14}=\frac{1}{8})u^{-\gamma_{12}},$$

where we used hats to denote the residues.

Let us now evaluate the integrals.

$$u^{-\frac{1}{8}}\int_{Re(\gamma_{14})=\frac{1}{6}}\frac{d\gamma_{14}}{2\pi i}\hat{K}(\gamma_{12}=\frac{1}{8},\gamma_{14})v^{-\gamma_{14}}$$
$$=u^{-\frac{1}{8}}\int_{Re(\gamma_{14})=\frac{1}{6}}\frac{d\gamma_{14}}{2\pi i}\frac{{}_2F_1(1,-\frac{1}{2},2\gamma_{14}+\frac{3}{4},\frac{1}{2})}{\gamma_{14}-\frac{1}{8}}v^{-\gamma_{14}}=\theta(1-v)\frac{\sqrt{1+\sqrt{v}}}{\sqrt{2}u^{\frac{1}{8}}v^{\frac{1}{8}}}$$

and

$$v^{-\frac{1}{8}}\int_{Re(\gamma_{12})=\frac{1}{24}}\frac{d\gamma_{12}}{2\pi i}\hat{K}(\gamma_{12},\gamma_{14}=\frac{1}{8})u^{-\gamma_{12}} \tag{E.23}$$
$$=\theta(1-u)\frac{\sqrt{1+\sqrt{u}}}{\sqrt{2}u^{\frac{1}{8}}v^{\frac{1}{8}}}-\frac{1}{\sqrt{2}}\frac{1}{u^{\frac{1}{8}}v^{\frac{1}{8}}}.$$

We proceed similarly concerning the other two integrals in (E.21). We conclude that if we define

$$F_{sub}(u,v)=F(u,v)-\frac{\sqrt{1+\sqrt{u}}+\sqrt{1+\sqrt{v}}+\sqrt{\sqrt{u}+\sqrt{v}}}{\sqrt{2}(uv)^{\frac{1}{8}}}+\frac{u^{-\frac{1}{8}}v^{-\frac{1}{8}}+u^{-\frac{1}{8}}v^{\frac{1}{8}}+u^{\frac{1}{8}}v^{-\frac{1}{8}}}{\sqrt{2}},$$

then

$$F_{sub}(u,v)=\int_{Re(\gamma_{12})=\frac{1}{24}}\frac{d\gamma_{12}}{2\pi i}\int_{Re(\gamma_{14})=\frac{1}{24}}\frac{d\gamma_{14}}{2\pi i}M(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}, \tag{E.24}$$

where the Mellin integral is evaluated with a straight contour at $Re(\gamma_{12})=Re(\gamma_{14})=\frac{1}{24}$ and $M(\gamma_{12},\gamma_{14})$ is given by (E.8). We checked equation (E.24) for several values of u and v by performing the Mellin integral numerically. $F_{sub}(u,v)$ is crossing symmetric and is softer than $F(u,v)$ in the lightcone limit as well as in the double lightcone limit.

E.3 ϕ^3 in $6+\epsilon$ dimensions

In this section we check some statements in section 2.1 for the example of ϕ^3 theory in $6+\epsilon$ dimensions at the critical point.

Consider the contribution to $\langle\phi\phi\phi\phi\rangle$ given by the three diagrams in figure E.1.

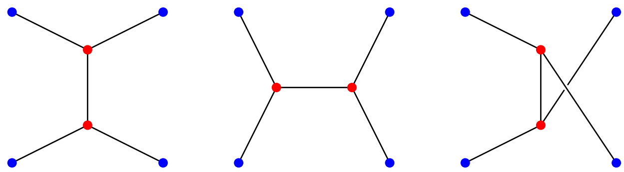

Figure E.1. The connected piece of $\langle\phi\phi\phi\phi\rangle$ at tree level. To first order in ϵ, the scalar ϕ has dimension $\Delta = 2 + \frac{5}{9}\epsilon$.

The main results of section 2.1 are equations (2.19) and (2.41). Our goal in this section is to show that such equations are correct for the four point function in figure E.1.

ϕ^3 in $6+\epsilon$ dimensions was studied in [113] using the skeleton expansion. It was found in [113] that the first diagram in figure E.1 is equal to

$$C^2_{\phi\phi\phi}\frac{\Gamma(\Delta)\Gamma(\frac{d-\Delta}{2})^2}{\pi^{\frac{d}{2}}\Gamma(\frac{\Delta}{2})^2\Gamma(\frac{d-2\Delta}{2})}\frac{1}{x_{12}^{3\Delta-d}x_{34}^{\Delta}}\int\frac{d^dx_5}{x_{12}^{d-\Delta}x_{25}^{d-\Delta}x_{35}^{\Delta}x_{45}^{\Delta}} \tag{E.25}$$
$$= C^2_{\phi\phi\phi}\frac{\Gamma(\Delta)}{\Gamma(\frac{d-2\Delta}{2})\Gamma(\frac{\Delta}{2})^4}\frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}u^{\frac{d-\Delta}{2}}\bar{D}_{\frac{d-\Delta}{2},\frac{d-\Delta}{2},\frac{\Delta}{2},\frac{\Delta}{2}}(u,v),$$

where $C_{\phi\phi\phi}$ is an OPE coefficient.

We can obtain a Mellin representation for (E.25) using Symanzik's trick (8.19). Expression (E.25) is equal to

$$C^2_{\phi\phi\phi}\frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^4\Gamma(\frac{d-2\Delta}{2})}\frac{1}{x_{13}^{2\Delta}x_{24}^{2\Delta}}\int_{\mathrm{Re}(\gamma_{12})=\frac{2}{3}\Delta}\frac{d\gamma_{12}}{2\pi i}\int_{\mathrm{Re}(\gamma_{14})=\frac{1}{6}\Delta}\frac{d\gamma_{14}}{2\pi i}M_{diag}(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}, \tag{E.26}$$

where

$$M_{diag}(\gamma_{12},\gamma_{14}) = \Gamma(\gamma_{12}-\frac{\Delta}{2})\Gamma(\frac{d}{2}-\frac{3}{2}\Delta+\gamma_{12})\Gamma(\gamma_{13})^2\Gamma(\gamma_{14})^2. \tag{E.27}$$

If we set $\epsilon = 0$, then

$$M_{diag}(\gamma_{12},\gamma_{14}) = \Gamma(\gamma_{12}-1)\Gamma(\gamma_{12})\Gamma^2(\gamma_{13})\Gamma(\gamma_{14})^2. \tag{E.28}$$

So, in expression (E.26) the contour is straight and can be placed anywhere in the shaded triangle in figure E.2.

In order to make contact with formula (2.19) we would like to displace the contour in (E.26) to $(\mathrm{Re}(\gamma_{12}) = \frac{\Delta}{3}, \mathrm{Re}(\gamma_{14}) = \frac{\Delta}{3})$. In order to do this we need to pick up the pole at $\gamma_{12} = \frac{\Delta}{2}$. Expression (E.26) is equal to

$$C^2_{\phi\phi\phi}\frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^4\Gamma(\frac{d-2\Delta}{2})}\frac{1}{x_{13}^{2\Delta}x_{24}^{2\Delta}}\int_{Re(\gamma_{12})=\frac{\Delta}{3}}\frac{d\gamma_{12}}{2\pi i}\int_{Re(\gamma_{14})=\frac{\Delta}{3}}\frac{d\gamma_{14}}{2\pi i}M_{diag}(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}$$

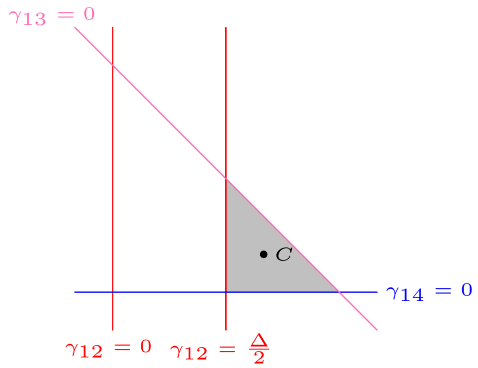

Figure E.2. According to expression (E.28), the contour must be placed to the right of $\mathrm{Re}(\gamma_{12})=1$, above $\mathrm{Re}(\gamma_{14})=0$ and to the bottom of $Re(\gamma_{13})=0$. We are thus led to the shaded triangle in this figure.

$$+\,C^2_{\phi\phi\phi}\frac{u^{-\frac{\Delta}{2}}}{x_{13}^{2\Delta}x_{24}^{2\Delta}}{}_2F_1(\frac{\Delta}{2},\frac{\Delta}{2};\Delta,1-v). \tag{E.29}$$

We see that the subtraction is precisely the collinear block, like we expected.

The full expression for the connected piece of $\langle\phi\phi\phi\phi\rangle$ at tree level is

$$\begin{aligned}\langle\phi\phi\phi\phi\rangle = C^2_{\phi\phi\phi}\frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^4\Gamma(\frac{d-2\Delta}{2})}\frac{1}{x_{13}^{2\Delta}x_{24}^{2\Delta}}\Bigg(&\int\int_{C_1}\frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}M_{diag}(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}\\ &+\int\int_{C_2}\frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}M_{diag}(\gamma_{13},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}\\ &+\int\int_{C_3}\frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}M_{diag}(\gamma_{14},\gamma_{13})u^{-\gamma_{12}}v^{-\gamma_{14}}\Bigg).\end{aligned} \tag{E.30}$$

Each of the 3 Mellin integrals has a different integration contour as in figure E.3..

In order to gather all three integrals in (E.30) into a single integral there are two equivalent ways of proceeding. One way is to introduce an ϵ regularization in order to write a deformed contour, pick up some poles and then set $\epsilon=0$. Another way is to use (E.29) (and its equivalent for the other) diagrams. Our final formula is

$$\begin{aligned}\langle\phi\phi\phi\phi\rangle - \frac{C^2_{\phi\phi\phi}}{x_{13}^{2\Delta}x_{24}^{2\Delta}}\Bigg(&u^{-\frac{\Delta}{2}}{}_2F_1(\frac{\Delta}{2},\frac{\Delta}{2};\Delta,1-v) \\ &+v^{-\frac{\Delta}{2}}{}_2F_1(\frac{\Delta}{2},\frac{\Delta}{2};\Delta,1-u)+v^{-\frac{\Delta}{2}}{}_2F_1(\frac{\Delta}{2},\frac{\Delta}{2};\Delta,1-\frac{u}{v})\Bigg)\end{aligned} \tag{E.31}$$

$$=\frac{C^2_{\phi\phi\phi}\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^4\Gamma(\frac{d-2\Delta}{2})}\frac{1}{x_{13}^{2\Delta}x_{24}^{2\Delta}}\int_{\mathrm{Re}(\gamma_{12})=\frac{\Delta}{3}}\frac{d\gamma_{12}}{2\pi i}\int_{\mathrm{Re}(\gamma_{14})=\frac{\Delta}{3}}\frac{d\gamma_{14}}{2\pi i}M(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}, \tag{E.32}$$

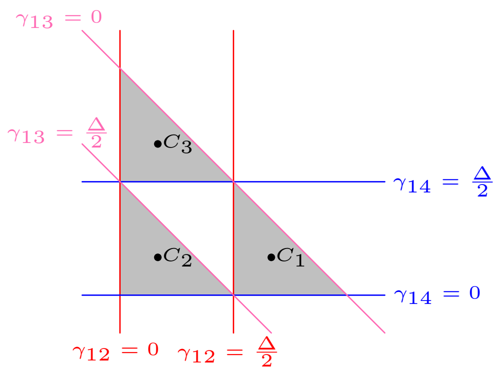

Figure E.3. The shaded triangles represent the regions where we can put the integration contours for each of the integrals in (E.26). If we want to gather all three contours into a single deformed contour, then we run into the problem of having pinches. For example, consider the point in the picture where $\mathrm{Re}(\gamma_{14}) = 0$ and $\mathrm{Re}(\gamma_{14}) = \frac{\Delta}{2}$. In order to have a deformed contour, the contour must pass to the right of $\mathrm{Re}(\gamma_{12}) = \frac{\Delta}{2}$, above $\mathrm{Re}(\gamma_{14}) = 0$ and below $\mathrm{Re}(\gamma_{13}) = \frac{\Delta}{2}$. This is impossible without introducing some regularization of the integrals.

for the connected piece of $\langle\phi\phi\phi\phi\rangle$ at tree level, where

$$\begin{aligned} M(\gamma_{12}, \gamma_{14}) &= \Gamma^2(\gamma_{12})\Gamma^2(\gamma_{13})\Gamma^2(\gamma_{14}) \\ &\times \left(\frac{\Gamma(-\frac{\Delta}{2} + \gamma_{12})\Gamma(\frac{d}{2} - \frac{3\Delta}{2} + \gamma_{12})}{\Gamma^2(\gamma_{12})} + (\gamma_{12} \leftrightarrow \gamma_{13}) + (\gamma_{12} \leftrightarrow \gamma_{14}) \right) \end{aligned} \tag{E.33}$$

Formula (E.31) agrees with the equations (2.19) and (2.41) in the main text.

F Analyticity in a Sectorial Domain Θ_{CFT}

In this appendix we establish the claim made in the main text about the region of analyticity of the correlator. The idea is to use the convergent OPE to bound the analytically continued correlator and its derivatives. We start by stating some preliminaries. Afterwards we give a proof that the correlation function is analytic inside the rhombus, see figure 1.4. Finally, we comment on the case of $\langle\sigma\sigma\sigma\sigma\rangle$ in the 2d Ising model to illustrate our claims.

F.1 Preliminaries

It is convenient to introduce the standard $(z, \bar{z})$ coordinates for the cross ratios

$$
\begin{aligned}
u &= z\bar{z}, \\
v &= (1-z)(1-\bar{z}) \, .
\end{aligned}
\tag{F.1}
$$

Let us briefly discuss the relationship between the two coordinates. We first consider the principal Euclidean sheet which corresponds to $u, v \geqslant 0$. It is convenient to split this in two regions, see figure 1.1.

In the gray region $(z, \bar{z})$ coordinates are complex conjugate

$$
\text{Grey region}: \quad \bar{z} = z^*, \quad \text{Im}[z] \neq 0. \tag{F.2}
$$

In addition to this, in the colored regions we have for $z, \bar{z} \in \mathbb{R}$

$$
\begin{aligned}
\text{Red region} &: \quad z, \bar{z} \leqslant 0, \\
\text{Blue region} &: \quad 0 \leqslant z, \bar{z} \leqslant 1, \\
\text{Pink region} &: \quad 1 \leqslant z, \bar{z} < \infty.
\end{aligned}
\tag{F.3}
$$

In going from $(z,\bar{z})$ to (u,v) there is a square root ambiguity and we have to choose a branch of the continuation

$$\begin{aligned} z &= \tfrac{1}{4}\left(\sqrt{(1+\sqrt{u})^2-v}+\sqrt{(1-\sqrt{u})^2-v}\right)^2, \\ \bar{z} &= \tfrac{1}{4}\left(\sqrt{(1+\sqrt{u})^2-v}-\sqrt{(1-\sqrt{u})^2-v}\right)^2. \end{aligned} \tag{F.4}$$

The branch point is located at $z=\bar{z}$.

It is also useful to recall $(\rho,\bar{\rho})$ variables [24]

$$\rho(z)=\frac{z}{(1+\sqrt{1-z})^2}, \tag{F.5}$$

which map the $[1,\infty)$ cut z-plane into a unit disc. Using the ρ variable we can expand the correlator as follows [24,28]

$$F(z,\bar{z})=\sum_{h,\bar{h}} b_{h,\bar{h}}\rho(z)^h\bar{\rho}(\bar{z})^{\bar{h}}, \quad b_{h,\bar{h}}\geqslant 0. \tag{F.6}$$

This expansion converges for $|\rho|,|\bar{\rho}|<1$ and makes the analytic structure in the z-plane manifest. The correlator has branch points at $z,\bar{z}=0$ which correspond to crossing the light-cone. Moreover, analytic continuation around the origin simply introduces phases in the expansion (F.6). Similarly, we can use unitarity to bound the analytically continued correlator by its value on the principal Euclidean sheet

$$\text{OPE bound}: \quad |F(e^{i\alpha}z,e^{i\beta}\bar{z})|\leqslant F(z^*,\bar{z}^*), \tag{F.7}$$

where $|\rho\left(e^{i\alpha}z\right)|\equiv\rho(z^*)$ and $|\bar{\rho}\left(e^{i\beta}\bar{z}\right)|\equiv\rho(\bar{z}^*)$. Analogous statements apply for analytic continuation around $z=1$ (if we use the same argument in the crossed channel). We, however, do not have a corresponding argument in $d>2$ for analytic continuation simultaneously around $z=0$ and $z=1$. In $d=2$ the analytic structure of the correlation function is the same on every sheet as was shown in [23], thanks to the convergence properties of the so-called q-expansion. In $d>2$ we expect to have extra singularities, a full classification of which is not known. One simple example discussed in [23,25] is the $z=\bar{z}$ singularity, where continuation on the second sheet in one of the variables is implicitly assumed. The $z=\bar{z}$ singularity corresponds to a very simple Landau graph where external points lie on a light-cone emanating from a point, two in the past and two in the future. It is also a singularity of an individual conformal block in $d>2$, see [23] for more details.

F.2 Analyticity from OPE

For $\arg(u) = \arg(v) = 0$ the s-channel OPE converges for all $u > 0$ and $v > 0$, except for the region in which $v \leqslant (1-\sqrt{u})^2$ and $u \geqslant 1$, this is the pink region on figure 1.1. Let us consider $u, v > 0$ inside the region where the s-channel OPE converges and consider the analytic continuation $u \to |u|e^{i\alpha}$, $v \to |v|e^{i\beta}$, where we are interested in $0 < |u|, |v| < \infty$. We ask for which values of α and β does the s-channel OPE still converges.

The s-channel OPE will cease to converge whenever we have either z or $\bar{z}$ bigger or equal to 1 and real. Without loss of generality, suppose that $\bar{z} \geqslant 1$ and real. Then we have the following relation

$$|v|e^{i\beta} = (1-\bar{z})(1-\frac{|u|e^{i\alpha}}{\bar{z}}). \tag{F.8}$$

Taking the real and imaginary part of this equation, we find

$$(|u|, |v|) = \big(-\bar{z}\frac{\sin(\beta)}{\sin(\alpha-\beta)}, (1-\bar{z})\frac{\sin(\alpha)}{\sin(\alpha-\beta)}\big). \tag{F.9}$$

We should read equation (F.9) in the following manner. Given (α, β) it tells us for which values of $|u|$ and of $|v|$ does the s-channel OPE cease to converge. As explained above, $|u|$ and $|v|$ must be positive. So, if equation (F.9) implies that $|u|$ and $|v|$ are negative, then the s-channel OPE converges for such values of α and β (as long as to get to such values the s-channel OPE converged along the way of the analytic continuation). We thus obtain the following conditions for convergence of the s-channel OPE

$$\begin{aligned}\big(\sin(\alpha)\sin(\beta)\big) < 0 \vee \Big(\sin(\alpha-\beta) > 0 \wedge \sin(\alpha) > 0 \wedge \sin(\beta) > 0\Big) \\ \vee\Big(\sin(\alpha-\beta) < 0 \wedge \sin(\alpha) < 0 \wedge \sin(\beta) < 0\Big),\end{aligned} \tag{F.10}$$

see figure F.1. At $\alpha = \beta + n\pi$, with $n \in \mathbb{N}$, condition (F.8) does not hold, unless at possible special points where $\alpha = m\pi$, where $m \in \mathbb{N}$.

Our argument shows that the s-channel OPE converges in the grey region of figure (F.1) for all $0 < |u|, |v| < \infty$. Indeed, in this region both $|\rho|, |\bar{\rho}| < 1$ and since the OPE converges exponentially fast [24], both $F(u,v)$ and any of its ∂_u, ∂_v derivatives are finite. This establishes analyticity of $F(u,v)$ inside the grey region. The situation is slightly different on a boundary of the grey region. Consider for example $\arg(u) = \arg(v) = 0$. As explained above in this case the s-channel OPE converges only for $|v| \leqslant (1-\sqrt{|u|})^2$ and $|u| \geqslant 1$.

Next we combine the argument above with crossing. Applying crossing symmetry to

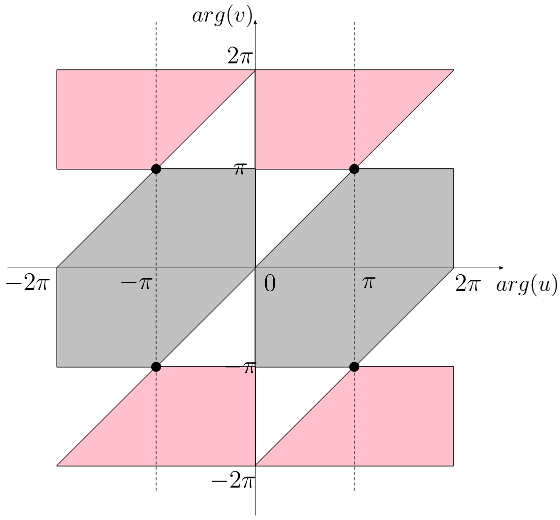

Figure F.1. Conditions (F.10) are verified in the pink and grey region. They are not verified in the white region. The s-channel OPE does not converge at special points that connect the grey and the pink region at $(-\pi, \pi)$, $(-\pi, -\pi)$, $(\pi, -\pi)$, (π, π). This reflects the fact that when continuing from the grey to the pink region we necessarily cross the $[1, \infty]$ cut in the z or $\bar{z}$ plane. Therefore, the s-channel OPE cannot be used in the pink region.

the grey region of figure (F.1) we find the correlation function is analytic in the whole sectorial domain given by the rhombus of figure 1.4. The rhombus in figure 1.4 is the minimal crossing symmetric region that contains the grey regions in figure F.1.

One comment is in order regarding the special point $\arg(u) = \arg(v) = 0$ which is the common boundary point of analyticity of all three OPE channels. In this case as we mentioned above each of the channels converges only in some subspace of the sectorial domain $0 < |u|, |v| < \infty$. However, the union of them covers it fully and thus we have established the desired analyticity in Θ_{CFT}.

F.3 2d Ising

For $\langle \sigma\sigma\sigma\sigma \rangle$ in the 2d Ising model we have that $F(u,v) = \frac{\sqrt{\sqrt{u}+\sqrt{v}+1}}{\sqrt{2}\sqrt[8]{uv}}$. Suppose that we start analytically continuing $u \to |u|e^{i\alpha}$, $v \to |v|e^{i\beta}$. We reach the boundary of the region of analyticity when $1 + |u|^{1/2}e^{i\frac{\alpha}{2}} + |v|^{1/2}e^{i\frac{\beta}{2}} = 0$. Given α and β the boundary of the region of analyticity is achieved when we start at $u, v > 0$ such that

$$(\sqrt{u}, \sqrt{v}) = \left(\frac{\sin(\frac{\beta}{2})}{\sin(\frac{\alpha-\beta}{2})}, -\frac{\sin(\frac{\alpha}{2})}{\sin(\frac{\alpha-\beta}{2})}\right). \tag{F.11}$$

We are only interested in the situations where the RHS is positive. It is not possible for the RHS to be positive inside the rhombus of figure 1.4. So we conclude that $\langle \sigma\sigma\sigma\sigma \rangle$ is analytic inside the rhombus.

Let us also comment on the branch point above in relation to the results of [23]. It was argued in [23] that 2d CFT correlators has only branch point singularities at $z, \bar{z} = 0, 1, \infty$ on every sheet. The branch point above on the other hand is at $z = \bar{z}$. By switching to the $(z, \bar{z})$ one can indeed check that the correlator is fully analytic at this point. The branch point originates from going between the $(z, \bar{z})$ and (u, v) variables, see (F.4), in full agreement with the results of [23].

G Polynomial Boundedness in a Sectorial Domain Θ_{CFT}

In this section we present some arguments in favor of the bound on the double light-cone limit that we used in the main text. Recall that the double light-cone limit is defined as $u, v \to 0$ with $\frac{u}{v}$ fixed (or some more general path of approaching the origin in the (u, v) plane). This limit is not controlled by the OPE and therefore an extra analysis is required.[1]

G.1 Subtractions and a Bound on the Double Light-Cone Limit

Here we use the asymptotic light-cone expansion on the second sheet to derive a better bound on the double light-cone limit of the double discontinuity of the correlator.

Consider the full correlation function

$$F(u,v) = u^{-\Delta} \sum_{h,\bar{h}} a_{h,\bar{h}} z^h \bar{z}^{\bar{h}}. \tag{G.1}$$

We will be interested in the double discontinuity of the connected correlator

$$\text{dDisc}_u[F](u,v) \equiv F(u,v) - \frac{1}{2}\Big(F(ue^{2\pi i}, v) + F(ue^{-2\pi i}, v)\Big). \tag{G.2}$$

Recall that the connected correlator is equal to $F_{conn}(u,v) = F(u,v) - (1 + u^{-\Delta} + v^{-\Delta})$. Therefore we get

$$\text{dDisc}_u[F_{conn}](z,\bar{z}) = 2u^{-\Delta} \sum_{h,\bar{h} \geqslant \frac{\tau_{gap}}{2}} \sin^2 \pi(h-\Delta) a_{h,\bar{h}} z^h \bar{z}^{\bar{h}}, \tag{G.3}$$

where we used that $\text{dDisc}_u[1 + v^{-\Delta}] = 0$ and wrote explicitly that sum goes only over

[1] See [26] for discussion of this limit in the planar gauge theories and [27] for the corresponding limit in the vector model.

operators above the vacuum. Using unitarity we then get

$$\frac{\mathrm{dDisc}_u[F_{conn}](z_1,\bar{z}_1)}{(z_1\bar{z}_1)^{\frac{\tau_{gap}}{2}-\Delta}} \leqslant \frac{\mathrm{dDisc}_u[F_{conn}](z_2,\bar{z}_1)}{(z_2\bar{z}_1)^{\frac{\tau_{gap}}{2}-\Delta}}, \quad 0<z_1<z_2<1\,. \tag{G.4}$$

We know take the limit $\bar{z}_1 \to 1$ and use in the RHS asymptotic light-cone expansion on the second sheet. As reviewed in detail in [114] for a general four-point correlator it is an assumption. For the case of identical scalars at hand, however, it was argued for in [28]. We can then take double discontinuity block by block. Since a double discontinuity of each individual block is zero the leading effect will come from the first twist accumulation point at some

$$\tau_{gap} \leqslant \tau_* \leqslant 2\Delta. \tag{G.5}$$

Using this fact we can write

$$\frac{\mathrm{dDisc}_u[F_{conn}](z_1,\bar{z}_1)}{(z_1\bar{z}_1)^{\frac{\tau_{gap}}{2}-\Delta}} \leqslant c\frac{[(1-z_2)(1-\bar{z}_1)]^{\frac{\tau_*}{2}-\Delta}}{(z_2\bar{z}_1)^{\frac{\tau_{gap}}{2}-\Delta}}. \tag{G.6}$$

In terms of (u,v) the bound becomes

$$\mathrm{dDisc}_u[F_{conn}](u,v) \leqslant \frac{c}{u^{\Delta-\frac{\tau_{gap}}{2}}v^{\Delta-\frac{\tau_*}{2}}}, \quad 0<u,v<c_0. \tag{G.7}$$

Of course, by applying crossing we can write an analogous bound for $\mathrm{dDisc}_v[F_{conn}](u,v) \leqslant \frac{c}{v^{\Delta-\frac{\tau_{gap}}{2}}u^{\Delta-\frac{\tau_*}{2}}}$.

Note that if we try to remove dDisc the argument above fails. Indeed, in this case we cannot write (G.4) because taking the connected part is equivalent to introducing terms with negative coefficients in the sum (G.1). These are due to double twist operators present in the OPE decomposition of the disconnected piece. Without non-negativity of the expansion (G.3) we cannot write (G.4). Instead we can use similar arguments to derive the bound for the full correlator

$$F(u,v) \leqslant \frac{c}{(uv)^{\Delta}}. \tag{G.8}$$

Given the bound on $\mathrm{dDisc}_u[F_{conn}]$ we would like to derive a bound on F_{conn} itself. A natural guess which is consistent with (G.7)

$$|F_{conn}(u,v)| \leqslant \frac{c}{(uv)^{\Delta-\frac{\tau_{gap}}{2}}}, \quad 0<u,v<c_0. \tag{G.9}$$

Two examples where this bound is saturated are minimal models in $d = 2$ and free field theories in $d > 2$. In both of these cases $\tau_{gap} = \tau_*$.

This, however, does not immediately follow from (G.7). As a way to violate (G.9), while having (G.7) being satisfied, we can imagine that the spectrum contains special operators with twist $2\Delta + 2m$ which will contribute to $F(u,v)$ but will not contribute to the double discontinuity. We will see a nontrivial example of such a function below. In an interacting CFT we, however, do not expect such a problem to occur and therefore we believe that (G.9) is a correct bound.

To avoid this problem and to generalize the argument above we consider subtractions. Assuming the number of low lying twist operators is finite we can improve the argument by considering additional subtractions. Indeed, let us consider

$$F_{sub}(u,v) = F(u,v) - (1 + u^{-\Delta} + v^{-\Delta}) \tag{G.10}$$
$$- \sum_{\tau_{gap} \leqslant \tau \leqslant \tau_{sub}} \sum_{J=0}^{J_{max}} \sum_{m=0}^{\left[\frac{\tau_{sub}-\tau}{2}\right]} C^2_{\tau,J} \left(u^{-\Delta+\frac{\tau}{2}+m} g^{(m)}_{\tau,J}(v) + v^{-\Delta+\frac{\tau}{2}+m} g^{(m)}_{\tau,J}(u) + v^{-\frac{\tau}{2}-m} g^{(m)}_{\tau,J}(\frac{u}{v}) \right),$$

where the role of the first subtraction term in the brackets is to make the sum (G.3) to start from $\sum_{h,\bar{h} \geqslant \frac{\tau'_{sub}}{2}}$. Going through the same argument (and the same assumptions) we conclude that

$$|\mathrm{dDisc}_u F_{sub}(u,v)|, |\mathrm{dDisc}_v F_{sub}(u,v)| \leqslant \frac{c}{(uv)^{\Delta - \frac{\tau'_{sub}}{2}}}, \quad 0 < u,v < c_0. \tag{G.11}$$

Note that since the number of subtractions is finite, we have necessarily $\tau_{sub} < \tau_* < 2\Delta$. In writing (G.11) we used the fact that the second and the third subtraction terms in the brackets in (G.10) trivially have double discontinuity which satisfies (G.11).

An interesting example of the function with zero double discontinuity in the s and t-channel but singular in the double light-cone limit is provided by the u-channel subtractions above. Consider for example $v^{-\frac{\tau}{2}} g^{(0)}_{\tau,J}(\frac{u}{v})$ which corresponds to a collinear conformal block exchanged in the u-channel. It is easy to check that $\mathrm{dDics}_u \left(v^{-\frac{\tau}{2}} g^{(0)}_{\tau,J}(\frac{u}{v}) \right) = \mathrm{dDics}_v \left(v^{-\frac{\tau}{2}} g^{(0)}_{\tau,J}(\frac{u}{v}) \right) = 0$, whereas its double light-cone limit is given by

$$\lim_{u,v \to 0, \frac{u}{v} - \text{ fixed}} v^{-\frac{\tau}{2}} g^{(0)}_{\tau,J}(\frac{u}{v}) \sim v^{-\frac{\tau}{2}}. \tag{G.12}$$

This is to be contrasted with a much more regular behavior of the same function in the light-cone limit

$$\begin{aligned}\lim_{u\to 0, v\ -\ \text{fixed}} v^{-\frac{\tau}{2}} g^{(0)}_{\tau,J}(\frac{u}{v}) &\sim \log u,\\ \lim_{v\to 0, u\ -\ \text{fixed}} v^{-\frac{\tau}{2}} g^{(0)}_{\tau,J}(\frac{u}{v}) &\sim \log v.\end{aligned} \tag{G.13}$$

In the argument above we expect that in an interacting CFT only subtraction terms provide examples of such functions. Therefore we conclude that (G.11) implies

$$|F_{sub}(u,v)| \leqslant \frac{c}{u^{\gamma_{12}} v^{\gamma_{14}}}, \quad 0 < u, v < c_0, \tag{G.14}$$

where $\gamma_{12}, \gamma_{14} > \Delta - \frac{\tau'_{sub}}{2}$ and $\gamma_{12} + \gamma_{14} > \tau_{sub}$ due to subtractions (G.12). In a sense we would like to say that by improving the light-cone limits in the u- and v- channels we have also improved the corresponding double light-cone limit in the s- and t-channels but we make it worse due to the subtractions in the u-channel.

To summarize, at the moment we were not able to rigorously prove (G.14) and leave it as an assumption hoping to improve on that in the future.

G.2 Dangerous Limits in the Sectorial Domain

In the subsections above we considered different small $0 < u, v < c_0$ limits with u and v being real. For the purpose of deriving the Mellin amplitude we would like however to generalize this argument for analytically continued u and v. As usual we would like to use the OPE to bound the correlator. However due to subtractions we do not have the OPE expansion representation of the correlator with positive coefficients. Therefore we cannot simply bound the analytically continued correlator by its value on the principal sheet using the Cauchy-Schwarz argument. Nevertheless we believe that our polynomial bounds on the double light-cone limit still apply in the region of analyticity of the correlator. In some sense this is a generalization of the idea that we can use the light-cone OPE on the second sheet as an asymptotic expansion.

To sum up, we would like now to say that with the region of analyticity $F_{sub}(u,v)$ satisfies the same bound as above

$$|F_{sub}(u,v)| \leqslant C(\gamma_{12}, \gamma_{14}) \frac{1}{|u|^{\gamma_{12}}} \frac{1}{|v|^{\gamma_{14}}}, \quad (u,v) \in \Theta_{CFT}, \quad (\gamma_{12}, \gamma_{14}) \in U_{CFT}\ , \tag{G.15}$$

H Heavy Tails in Dispersion Relations

In the main text we discussed several functionals. Here we would like to comment on the convergence of the corresponding OPE sums at large Δ. The relevant formulae for the asymptotic of the OPE coefficients can be found in [45].

For example, let us fix $J = 0$. In this case we have

$$\alpha_{\tau,0} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{2\Gamma(\tau)}{\Gamma(\frac{\tau}{2})^4 \Gamma(\Delta - m - \frac{\tau}{2})^2 (\tau - \frac{d}{2} + 1)_m} \frac{\tau - \Delta + 2m}{(\tau - \frac{4\Delta}{3} + 2m)^2 (\tau - \frac{2\Delta}{3} + 2m)^2}. \tag{H.1}$$

We are interested in the asymptotic of this sum when $\tau \to \infty$.

At large τ the density of primaries multiplied by their three-point couplings asymptotes to [45] (strictly speaking this asymptotic is only true on average)

$$\lim_{\tau \to \infty} \rho_J^{primary} C_J^2 \sim 4^{-\tau} \tau^{4\Delta - \frac{3d}{2}}. \tag{H.2}$$

Combining this with $\alpha_{\tau,0}$ we find that

$$\lim_{\tau \to \infty} \rho_0^{primary} C_0^2 \alpha_{\tau,0} \sim \frac{1}{\tau^{d+2}}, \tag{H.3}$$

and therefore the sum over the heavy scalar tail converges. Note that the power does not depend on the dimension of the external operator. Repeating the exercise for $J = 2$ we get the same power law behavior. We expect that the same holds for any finite J.

I Holographic Calculations

In this appendix we spell out the terms that enter into (5.13) and (5.14). The functionals $\alpha_{\tau,J,m}$ were written explicitly in (5.4), up to the expressions for the Mack polynomials. In the cases of interest,

$$\mathcal{Q}^{\tau,d}_{J=0,m}(\gamma_{13}) = -\frac{2\Gamma(\tau)}{\Gamma(m+1)\Gamma\left(\frac{\tau}{2}\right)^4 \left(-\frac{d}{2}+\tau+1\right)_m \Gamma\left(-m+\Delta-\frac{\tau}{2}\right)^2}, \tag{I.1}$$

$$\begin{aligned}\mathcal{Q}^{\tau,d}_{J=2,m}(\gamma_{13}) = -\Bigg(4\gamma_{13}^2 - 2\gamma_{13}(2m+\tau) \\ + \frac{m^2\left(d^2(\tau+1) - d\left(\tau^2+5\tau+4\right)+(\tau+2)^2\right)}{d(\tau+1)(d-\tau-3)} + \frac{\tau^2(\tau+2)}{4(\tau+1)}\Bigg) \\ \times \frac{(\tau+1)(\tau+2)\Gamma(\tau+4)}{2\Gamma(m+1)\Gamma\left(\frac{\tau+4}{2}\right)^4\left(-\frac{d}{2}+\tau+3\right)_m \Gamma\left(-m+\Delta-\frac{\tau}{2}\right)^2}.\end{aligned} \tag{I.2}$$

Regarding the leading Regge trajectory, we can use

$$\alpha_{\tau_{[\mathcal{O},\mathcal{O}]_{0,J}},J} = \frac{d\alpha_{\tau=2\Delta,J}}{d\tau}\gamma_{[\mathcal{O},\mathcal{O}]_{0,J}} + O(1/c_T^2). \tag{I.3}$$

Furthermore, only the $m=0$ term contributes to $\frac{d\alpha_{\tau=2\Delta,J}}{d\tau}$. For $m=0$ a general expression for Mack polynomials is given by (3.8). In this manner one obtains

$$\frac{d\alpha_{\tau=2\Delta,J}}{d\tau} = -\frac{\Gamma(2(J+\Delta))_3F_2^{(\{0,1,0\},\{0,0\},0)}(-J,\frac{\Delta}{3},2\Delta+J-1;\Delta,\Delta;1)}{\Gamma(\Delta)^2\Gamma(J+\Delta)^2}, \tag{I.4}$$

where the superscript in the hypergeometric function means a derivative with respect to the appropriate entry.

As to the anomalous dimensions, for the exchange of a scalar we used expressions (2.35), (2.36) and (2.37) in [55].[1] For the stress tensor exchange in $d = 4$, [53]

$$\gamma_{[\mathcal{O},\mathcal{O}]_{0,J}} = -C^2_{\mathcal{O}\mathcal{O}T}\frac{60(\Delta-1)^2}{(J+1)(2\Delta+J-2)}, \quad J > 2, \tag{I.5}$$

$$\gamma_{[\mathcal{O},\mathcal{O}]_{0,J=2}} = -C^2_{\mathcal{O}\mathcal{O}T}\frac{10\left(-4\Delta^4+9\Delta^2+7\Delta-12\right)}{\Delta(2\Delta+1)(2\Delta+3)}, \tag{I.6}$$

where $C^2_{\mathcal{O}\mathcal{O}T}$ is the OPE coefficient between the two external scalars and the stress tensor.[2] In general d we use the results of [56][3]

$$\begin{aligned}\gamma_{[\mathcal{O},\mathcal{O}]_{0,J}} = &-\int_{-i\infty+c_1}^{+i\infty+c_1}\frac{dt}{4\pi i}M(s=0,t)\Gamma(\frac{t}{2})^2\Gamma(\frac{-t}{2}+\Delta)^2 \\ &\times {}_3F_2(-J,J+2\Delta-1,\frac{t}{2};\Delta,\Delta;1), \;\; 0<c_1<2\Delta.\end{aligned} \tag{I.7}$$

To get the complete result the graviton exchange diagrams in the three channels should be added. These can be easily obtained by applying crossing to the result $(164-166)$ in [56]. Two of the exchange diagrams produce results that are identical and analytic in spin. The third one only contributes to the anomalous dimension $\gamma_{[\mathcal{O},\mathcal{O}]_{0,J}}$ for $J = 0, 2$.

For a generic exchange of a single trace operator of twist τ and spin J, our sum rule is valid for $\frac{d-2}{2} < \Delta < \frac{3\tau}{4}$. It is interesting to study the behaviour of the sum rule when we take $\Delta \to \frac{3\tau}{4}$. There are two terms that diverge like $\frac{1}{(\Delta-\frac{3\tau}{4})^2}$. One term comes from the direct exchange of the single trace operator. The other comes from the tail of the leading Regge trajectory. The sum of the two terms is equal to

$$\frac{1}{(\Delta-\frac{3}{4}\tau)^2}\frac{9\Gamma(2J+\tau)}{2\Gamma\left(\frac{\tau}{4}\right)^2\Gamma\left(\frac{\tau}{2}\right)^2\Gamma\left(J+\frac{\tau}{2}\right)^2}\left[{}_3F_2\left(-J,\frac{\tau}{4},J+\tau-1;\frac{\tau}{2},\frac{\tau}{2};1\right)-1\right] \tag{I.8}$$

When $J = 0$, (I.8) vanishes. This agrees with the fact that a scalar exchange in AdS does not contribute to the sum rule. When $J > 0$ we numerically find that (I.8) is negative. Furthermore we checked this analytically for spins $J = 2, 4, \ldots\ 50$ and any positive τ. Note that this implies that when $\Delta \to \frac{3\tau}{4}$ there is a UV contribution to the sum rule which is divergent and positive. It would be very interesting to understand the origin of this.

[1]See also [115,116] for a similar discussion for a generic exchange of a spin J primary.

[2]This OPE coefficient is fixed by Ward identities, but actually it is not necessary to know it explicitly in order to determine the sign of the functional, since it enters both in the direct exchange and in the anomalous dimensions of the double twist operators.

[3]Formula (I.7) differs from (172) in [56] by a factor of $\frac{1}{2}$. This comes from the fact that we consider the exchange of identical scalar operators.

J Notes on Mack Polynomials

J.1 Basic definitions

We denote the residue of the Mellin amplitude $M(\gamma_{12}, \gamma_{13})$ as

$$M(\gamma_{12}, \gamma_{13}) \approx -\frac{1}{2} \frac{C^2_{\tau,\ell}\mathcal{Q}^{\tau,d}_{\ell,m}(-2\gamma_{13})}{\gamma_{12} - (\Delta - \frac{\tau}{2} - m)}. \tag{J.1}$$

The calligraphic $\mathcal{Q}^{\tau,d}_{\ell,m}(s)$ is related to the usual Mack polynomial $Q^{\tau,d}_{\ell,m}(s)$ as follows

$$\mathcal{Q}^{\tau,d}_{\ell,m}(s) = -K(\tau, \ell, m) Q^{\tau,d}_{\ell,m}(s), \tag{J.2}$$

where the proportionality factor is given by

$$K(\tau, \ell, m) \equiv \frac{2(\ell + \tau - 1)_\ell \Gamma(2\ell + \tau)}{2^\ell \Gamma\left(\frac{1}{2}(2\ell + \tau)\right)^4 \Gamma(m+1) \Gamma\left(-m + \Delta - \frac{\tau}{2}\right)^2 \left(-\frac{d}{2} + \ell + \tau + 1\right)_m}. \tag{J.3}$$

Note the minus sign in (J.2).

For the Mack polynomial $Q^{\tau,d}_{\ell,m}(s)$ we use the following representation [117]

$$Q^{\tau,d}_{\ell,m}(s) = 4^\ell (-1)^\ell \sum_{n_1=0}^{\ell} \sum_{m_1=0}^{\ell-n_1} (-m)_{m_1} \left(m + \frac{s}{2} + \frac{\tau}{2}\right)_{n_1} \mu(\ell, m_1, n_1, \tau, d), \tag{J.4}$$

where

$$\mu(\ell, m, n, \tau, d) \equiv (2\ell + \tau - 1)_{n-\ell} \tag{J.5}$$
$$\times \frac{2^{-\ell}\Gamma(\ell+1)(-1)^{m+n}\left(\frac{d}{2} + \ell - 1\right)_{-m} \left(\ell - m + \frac{\tau}{2}\right)_m \left(n + \frac{\tau}{2}\right)_{\ell-n} \left(m + n + \frac{\tau}{2}\right)_{\ell-m-n}}{\Gamma(m+1)\Gamma(n+1)\Gamma(\ell - m - n + 1)}$$

$$\times {}_4F_3\left(-m, -\frac{d}{2}+\frac{\tau}{2}+1, -\frac{d}{2}+\frac{\tau}{2}+1, \ell+n+\tau-1; \ell-m+\frac{\tau}{2}, n+\frac{\tau}{2}, -d+\tau+2; 1\right).$$

J.2 Mack Polynomial Projections

We will also find it very useful to consider functionals that are obtained by integrating $\omega_{p_1,p_2,p_3}(\gamma_{13})$ against Mack polynomials. To agree with the standard conventions for Mack polynomials we switch to the s variable

$$s \equiv -2\gamma_{13}. \tag{J.6}$$

We then consider the following projection

$$\omega^{(\tau,\ell)}_{p_1,p_2,p_3} \equiv \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \omega_{p_1,p_2,p_3}(s) Q^{\tau,d}_{\ell,0}(s) \Gamma^2\left(-\frac{s}{2}\right) \Gamma^2\left(\frac{s+\tau}{2}\right). \tag{J.7}$$

where the reduced Mack polynomials $Q^{\tau,d}_{\ell,m}(s)$ were defined above. Note that $Q^{\tau,d}_{\ell,0}(s)$ is d-independent. The collinear Mack polynomials $Q^{\tau,d}_{\ell,0}(s)$ satisfy the orthogonality relation

$$\int_{-i\infty}^{+i\infty} \frac{ds}{4\pi i} Q^{\tau,d}_{\ell,0}(s) Q^{\tau,d}_{\ell',0}(s) \Gamma^2\left(-\frac{s}{2}\right) \Gamma^2\left(\frac{s+\tau}{2}\right) = \frac{\delta_{\ell,\ell'}(-1)^\ell 4^\ell \Gamma(\ell+1) \Gamma\left(\ell+\frac{\tau}{2}\right)^4}{(2\ell+\tau-1)(\ell+\tau-1)^2_\ell \Gamma(\ell+\tau-1)}. \tag{J.8}$$

Another useful identity is the following

$$\begin{aligned} &\int_{-i\infty}^{+i\infty} \frac{ds}{4\pi i} Q^{\tau,d}_{\ell,0}(s) \partial_\tau Q^{\tau,d}_{\ell',0}(s) \Gamma^2\left(-\frac{s}{2}\right) \Gamma^2\left(\frac{s+\tau}{2}\right) \\ &= \begin{cases} 0 & \ell' \leqslant \ell\,, \\ \frac{(-1)^\ell 2^{4-\ell-\ell'-2\tau}\pi}{(\ell'-\ell)(\ell'+\ell+\tau-1)} \frac{\Gamma(\ell+\frac{\tau}{2})\Gamma(\ell'+\frac{\tau}{2})\Gamma(\ell+\tau-1)\Gamma(\ell'+1)}{\Gamma(\ell+\frac{\tau-1}{2})\Gamma(\ell'+\frac{\tau-1}{2})} & \ell' > \ell\,. \end{cases} \end{aligned} \tag{J.9}$$

J.3 Proof of Positivity of $m = 0$ Mack polynomials

The $m = 0$ Mack polynomials are given by the formula

$$Q^{\tau,d}_{\ell,0} = \frac{2^\ell ((\tau/2)_\ell)^2}{(\tau+\ell-1)_\ell} {}_3F_2(-\ell, \ell+\tau-1, -\frac{s}{2}; \frac{\tau}{2}, \frac{\tau}{2}; 1) \tag{J.10}$$

Mack polynomials are related to continuous Hahn polynomials, which obey useful recursion relations to study their positivity properties. A continuous Hahn polynomial is defined by[1]

$$p_n(x;a,b,c,d) = i^n \frac{(a+c)_n(b+d)_n}{\Gamma(n+1)} {}_3F_2(-n, n+a+b+c+d-1, a+ix; a+c, a+d; 1) \tag{J.11}$$

Continuous Hahn polynomials obey a recursion relation. Let us define

$$p_n(x) = p_n(x;a,b,c,d)\frac{\Gamma(n+1)}{(n+a+b+c+d-1)_n}, \tag{J.12}$$

$$A_n = -\frac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)}, \tag{J.13}$$

$$C_n = \frac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \tag{J.14}$$

Then,

$$xp_n(x) = p_{n+1}(x) + i(A_n + C_n + a)p_n(x) - A_{n-1}C_n p_{n-1}(x). \tag{J.15}$$

We have that

$$p_\ell(0) = 2^{-\ell} i^\ell Q^{\tau,d}_{\ell,0}, \tag{J.16}$$

provided we pick

$$a = -\frac{s}{2}, \quad b = -\frac{s}{2}, \quad c = \frac{s+\tau}{2}, \quad d = \frac{s+\tau}{2}. \tag{J.17}$$

Let us define $\tilde{Q}(\ell) = 2^{-\ell} Q^{\tau,d}_{\ell,0}(s)$. Then,

$$-\frac{\ell(-2+\ell+\tau)(-2+2\ell+\tau)^2}{16(-3+2\ell+\tau)(-1+2\ell+\tau)}\tilde{Q}(\ell-1) - \frac{2s+\tau}{4}\tilde{Q}(\ell) + \tilde{Q}(\ell+1) = 0. \tag{J.18}$$

Let us use this recursion relation to demonstrate that $m = 0$ Mack polynomials are positive, i.e. $\tilde{Q}(\ell) \geqslant 0$, provided $s \geqslant -\frac{\tau}{2}$. For spins $\ell = 0, 1, 2$:

$$\tilde{Q}(0) = 1, \quad \tilde{Q}(1) = \frac{2s+\tau}{4}, \quad \tilde{Q}(2) = \frac{1}{16}\left(4s^2 + 4s\tau + \frac{\tau^2(\tau+2)}{\tau+1}\right) \tag{J.19}$$

By mathematical induction the recursion relation implies the positivity at all positive integer ℓ. Furthermore the recursion relation

$$-\frac{1}{2}\tilde{Q}(\ell) - \frac{\ell(-2+\ell+\tau)(-2+2\ell+\tau)^2}{16(-3+2\ell+\tau)(-1+2\ell+\tau)}\partial_s\tilde{Q}(\ell-1) - \frac{2s+\tau}{4}\partial_s\tilde{Q}(\ell) + \partial_s\tilde{Q}(\ell+1) = 0$$

[1] We took formulas for continuous Hahn polynomials from the book [118], see the pages 200 and 201.

together with

$$\partial_s \tilde{Q}(0) = 0, \quad \partial_s \tilde{Q}(1) = \frac{1}{2} \quad \partial_s \tilde{Q}(2) = \frac{1}{4}(2s+\tau) \tag{J.20}$$

implies that $\partial_s \tilde{Q}(\ell)$ is positive for $s \geqslant -\frac{\tau}{2}$. We believe that this argument can be generalized to establish positivity of higher derivatives of Mack polynomials, but we have not tried to do that.

J.4 Limits of Mack polynomials

Let us study Mack polynomials when the spin ℓ is much bigger than all the other quantum numbers. The following expansion

$$Q^{\tau,d}_{\ell,m}(s) \approx \frac{\sqrt{\pi}\ell^{s+\frac{1}{2}}2^{-\ell-m-\tau+2}\Gamma\left(\ell+\frac{\tau}{2}\right)^2\Gamma(\ell+\tau-1)p_m(s)\Gamma\left(-\frac{d}{2}+\ell+m+\tau+1\right)}{\Gamma\left(\ell+\frac{\tau}{2}-\frac{1}{2}\right)\Gamma\left(\ell+\frac{\tau}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{d}{2}+\ell+\tau+1\right)\Gamma\left(\frac{1}{2}(2m+s+\tau)\right)^2} \tag{J.21}$$

$$+\frac{\sqrt{\pi}(-1)^{\ell}2^{-\ell-m-\tau+2}\Gamma\left(\ell+\frac{\tau}{2}\right)^2\Gamma(\ell+\tau-1)\ell^{-2m-s-\tau+\frac{1}{2}}p_m(-2m-s-\tau)\Gamma\left(-\frac{d}{2}+\ell+m+\tau+1\right)}{\Gamma\left(-\frac{s}{2}\right)^2\Gamma\left(\ell+\frac{\tau}{2}-\frac{1}{2}\right)\Gamma\left(\ell+\frac{\tau}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{d}{2}+\ell+\tau+1\right)}$$
$$+O\left(\frac{1}{\ell}\right), \quad \ell \gg 1\,.$$

can be derived [41] from the recursion relation in m, see (J.24). $p_m(s)$ is an m-th degree polynomial in s and it obeys $p_m(s) = s^m + \mathcal{O}(\frac{1}{s})$. We found that

$$p_0(s) = 1\,, \quad p_1(s) = \frac{d-2}{2} + s + \frac{\tau}{2}. \tag{J.22}$$

J.4.1 Some facts about Mack polynomials

Mack polynomials obey the following symmetry property

$$Q^{\tau,d}_{\ell,m}(s) = (-1)^{\ell} Q^{\tau,d}_{\ell,m}(-s-\tau-2m)\,. \tag{J.23}$$

For this reason, at $s = -\frac{\tau}{2} - m$ odd spin Mack polynomials vanish and even spin Mack polynomials have a vanishing derivative. Mack polynomials obey the following recursion relation in m for fixed τ and ℓ [29]

$$Q(s,m)\left(-4dm-4\ell^2-4\ell(\tau-1)+4m^2-4ms+4m\tau-2s^2-2s\tau-\tau^2\right) \tag{J.24}$$
$$+2mQ(s,m-1)(d-2(\ell+m+\tau))+2mQ(s+2,m-1)(d-2(\ell+m+\tau))$$

$$+ s^2 Q(s-2,m) + (2m+s+\tau)^2 Q(s+2,m) = 0.$$

We also found it useful to express $m > 0$ Mack polynomials in terms of $m = 0$ Mack polynomials. In $d = 4$ the relevant formulas are

$$Q^{\tau,d=4}_{\ell,m=1}(s) = (s + \frac{\tau}{2} + 1) Q^{2+\tau,d=4}_{\ell-1,m=0}(s) \tag{J.25}$$

and

$$Q^{\tau,d=4}_{\ell,m=2}(s) = a_1 Q^{4+\tau,d=4}_{\ell-4,m=0}(s) + a_2 Q^{4+\tau,d=4}_{\ell-3,m=0}(s) \tag{J.26}$$

where a_1 and a_2 are given by

$$\frac{(\ell-3)(\ell+\tau-1)(2\ell+\tau-4)^2\left(4s^2(\tau-1)+4s\left(\tau^2+3\tau-4\right)+\tau^3+6\tau^2+8\tau-16\right)}{16(\tau-1)(2\ell+\tau-5)(2\ell+\tau-3)}, \tag{J.27}$$

$$\frac{8s^3(\tau-1)+12s^2\left(\tau^2+3\tau-4\right)+s\left(6\tau^3+40\tau^2+48\tau-96\right)+\tau^4+10\tau^3+32\tau^2+16\tau-64}{8(\tau-1)}, \tag{J.28}$$

respectively. We believe similar recursion relations can be generated at any finite m and in any d.

K OPE data in the $4 - \epsilon$ expansion

For our purposes, it is important to know what has been computed before. The dimensions of ϕ and ϕ^2 are known to the order ϵ^5 [119]. The dimensions of the leading twist trajectory (which starts at spin 2) are known to order ϵ^4 [66]. As to the subleading twist trajectory, twist 4 operators are degenerate for a given spin. This degeneracy was analysed in [64, 120] to order ϵ^2, where anomalous dimensions and OPE coefficients are given for operators with low spins. However, no formulas for arbitrary spin are given.

The OPE coefficient of ϕ^2 with two operators ϕ is known to order ϵ^3 [8]. We compute the order ϵ^4 correction (see table 6.3). The OPE coefficients of the leading twist trajectory are known to the order ϵ^4 [62]. A formula for the averaged OPE coefficients of twist 4 operators is known to the order ϵ [62].

Using our notation from section 6.4, let us register the value of known quantities in the ϵ expansion. The spacetime dimensionality d defined in (6.17) takes the form

$$
\begin{aligned}
d = 4 - 3g + \frac{8}{3}g^2 + \left(-12\zeta(3) - \frac{23}{12}\right) g^3 \\
+ \left(\frac{75\zeta(3)}{2} + 120\zeta(5) - \frac{77}{48} - \frac{3\pi^4}{10}\right) g^4 + a_5 g^5 + \dots,
\end{aligned} \tag{K.1}
$$

where for completeness we also note here a_5 that was computed in [119] even though we did not use it in our thesis

$$
a_5 = -\frac{159\zeta(3)}{2} - 72\zeta(3)^2 - 504\zeta(5) - 1323\zeta(7) + \frac{4175}{576} + \frac{289\pi^4}{240} + \frac{10\pi^6}{21}\,. \tag{K.2}
$$

Similarly for the dimensionality of the scalar defined in (6.18) takes the following form

$$
\Delta_\phi - \frac{d-2}{2} = \frac{1}{12}g^2 + \frac{5}{48}g^3 - \frac{7}{192}g^4 + \left(\frac{7\zeta(3)}{16} + \frac{1}{2304}\right) g^5 + \dots\,. \tag{K.3}
$$

For the twist-two operators the OPE data was defined in (6.19). The anomalous dimensions that are known but were not explicitly written in the main text take the following form

$$\begin{aligned}
\gamma_2(j_\ell) &= -\frac{1}{\ell(\ell+1)}, \quad \gamma_3(j_\ell) = -\frac{2S_1(\ell)}{\ell(\ell+1)} + \frac{3\left(2\ell^2-1\right)}{2\ell^2(\ell+1)^2}, \\
\gamma_4(j_\ell) &= -\frac{3S_2(\ell)+2S_1^2(\ell)}{\ell(\ell+1)} + \frac{\ell\left(\ell^3+2\ell^2-3\ell-4\right)}{4\ell^3(\ell+1)^3}\left(S_2(\frac{\ell-1}{2}) - S_2(\frac{\ell}{2}) + \frac{\pi^2}{3}\right) \\
&\quad + \frac{(53\ell^2+17\ell-18)S_1(\ell)}{6\ell^2(\ell+1)^2} \\
&\quad + \frac{1}{12\ell^3(\ell+1)^3}\left(-58\ell^4+26\ell^3+81\ell^2-15\ell-33\right).
\end{aligned} \tag{K.4}$$

For the three-point functions of the twist-two operators that were defined in (6.23) we have

$$\begin{aligned}
c_2(j_\ell) &= \frac{S_1(2\ell)-S_1(\ell)+\frac{1}{\ell+1}}{\ell(\ell+1)}, \\
c_3(j_\ell) &= \frac{3(\ell+\frac{1}{2})(S_1(2\ell)-\frac{\ell-1}{\ell+1})-(\ell+\frac{3}{2})S_1(\ell)}{\ell^2(\ell+1)^2} + \frac{3(S_1(\ell)-S_1(2\ell)+S_2(2\ell))-2\left(S_1^2(\ell)-S_1(\ell)S_1(2\ell)+S_2(\ell)\right)}{\ell(\ell+1)}, \\
c_4(j_\ell) &= \frac{2\ell\left(29\ell^3-9(\ell+1)^2\left(\ell^2+\ell+4\right)\zeta(3)-48\ell^2-38\ell+24\right)+33}{12\ell^3(\ell+1)^4} \\
&\quad - \frac{2S_1^3(\ell)}{\ell^2+\ell} + \frac{S_1^2(2\ell)}{2\ell^2(\ell+1)^2} + S_1^2(\ell)(\frac{2S_1(2\ell)}{\ell^2+\ell} + \frac{\ell(53\ell+29)-15}{6\ell^2(\ell+1)^2}) \\
&\quad + \frac{(\ell(\ell(\ell+2)-7)-4)S_2(\frac{\ell}{2}))}{2\ell^3(\ell+1)^3} + \frac{(\ell(\ell(\ell(89\ell+130)-10)+45)+48)S_2(\ell)}{12\ell^3(\ell+1)^3} \\
&\quad + S_1(2\ell)\left(\frac{\ell\left(\ell\left(58\ell^2-26\ell-81\right)+27\right)+33}{12\ell^3(\ell+1)^3} + \frac{\left(\ell^2+\ell-4\right)S_2(\frac{\ell}{2})}{2\ell^2(\ell+1)^2} + \frac{2\left(\ell^2+\ell+2\right)S_2(\ell)}{\ell^2(\ell+1)^2}\right) \\
&\quad + \frac{(30-\ell(71\ell+17))S_2(2\ell)}{6\ell^2(\ell+1)^2} + \frac{\left(\ell^2+\ell-4\right)S_3(\frac{\ell}{2})}{4\ell^2(\ell+1)^2} + \frac{(4-7\ell(\ell+1))S_3(\ell)}{\ell^2(\ell+1)^2} + \frac{9S_3(2\ell)}{\ell^2+\ell} \\
&\quad + S_1(\ell)(\frac{\ell(\ell(83-2\ell(29\ell+40))+9)-33}{12\ell^3(\ell+1)^3} + \frac{(12-\ell(53\ell+17))S_1(2\ell)}{6\ell^2(\ell+1)^2} - \frac{\left(\ell^2+\ell-4\right)S_2(\frac{\ell}{2})}{2\ell^2(\ell+1)^2} \\
&\quad + \frac{(-6\ell(\ell+1)-4)S_2(\ell)}{\ell^2(\ell+1)^2} + \frac{6S_2(2\ell)}{\ell^2+\ell}).
\end{aligned}$$

L Auxiliary formulas

L.1 Miscellaneous formulas for the ϵ-expansion

Here we present a few bulky formulas to which we refer from the main part of the thesis.

$$\begin{aligned} r_1(\ell) &= 12\gamma_E(\gamma_{13}-2)\ell(\ell+1) - 14(\gamma_{13}-2)\ell(\ell+1)S_1(\ell) \\ &+ 6(\gamma_{13}-2)\ell(\ell+1)S_1(\ell+\frac{1}{2}) \\ &+ 2\left(6\gamma_{13}\ell^2 + (\gamma_{13}-2)(\ell+1)\ell\log(64) - 3\gamma_{13} - 9\ell^2 + 3\ell + 6\right), \end{aligned} \tag{L.1}$$

$$\begin{aligned} r_2(\ell) &= 14(\gamma_{13}-2)\ell S_1(\ell) - 6(\gamma_{13}-2)\ell S_1(\ell+\frac{1}{2}) - 12\gamma_E(\gamma_{13}-2)\ell \\ &- \frac{2\left(6\gamma_{13}\ell^2 - 3\gamma_{13}\ell + (\gamma_{13}-2)(\ell+1)\ell\log(64) - 3\gamma_{13} - 9\ell^2 + 9\ell + 6\right)}{\ell+1}. \end{aligned} \tag{L.2}$$

L.2 Miscellaneous formulas for higher spin

In this appendix we write some formulas we used in the text. The nonzero coefficients in equation (9.14) are

$$a_{1,-1,-1} = -(\gamma_{14}-1)\left(2\gamma_{14}^2 - \gamma_{14}(4k+5) + 2k^2 + 5k + 2\right)\left(k^2 - 2ks - k + s^2 + s\right), \tag{L.3}$$

$$\begin{aligned} a_{0,0,0} &= -\frac{1}{2}\left(2\gamma_{14}^2 - \gamma_{14}(4k+5) + 2k^2 + 5k + 2\right) \\ &\times(-2\gamma_{12}(k+s) + \gamma_{14}(2k-2s+1) + s(2s+1))(k-s), \end{aligned}$$

$$a_{1,-1,0} = -\frac{1}{2}(2\gamma_{12}^2 + \gamma_{12}(4k-4s-1) + 2k^2 - k(4s+1) + 2s^2 + s - 1)(\gamma_{14}-1)$$

$$\times\left(2k^2-4ks+k+s(2s-1)\right), \quad a_{0,-1,0}=\frac{1}{2}(\gamma_{14}-1)\left(2k^2-4ks+k+s(2s-1)\right)$$
$$\times(2\gamma_{12}^2+\gamma_{12}(4\gamma_{14}-4s-3)+2\gamma_{14}^2-\gamma_{14}(4s+3)+s(2s+3)),$$
$$a_{-1,0,1}=-\frac{1}{2}(\gamma_{12}-1)\left(2k^2+3k+1\right)$$
$$\times(2\gamma_{12}^2+\gamma_{12}(4\gamma_{14}-4s-3)+2\gamma_{14}^2-\gamma_{14}(4s+3)+s(2s+3))$$
$$a_{-1,1,1}=\frac{1}{2}(\gamma_{12}-1)\left(2k^2+3k+1\right)\left(2\gamma_{14}^2-\gamma_{14}(4k+5)+2k^2+5k+2\right),$$
$$a_{0,0,1}=\frac{1}{2}(k+1)\left(2\gamma_{12}^2+\gamma_{12}(4k-4s-1)+2k^2-k(4s+1)+2s^2+s-1\right)$$
$$\times(2\gamma_{12}k+\gamma_{12}-2\gamma_{14}(k-2s+1)-s(2s+1)), \quad a_{-1,1,2}=(\gamma_{12}-1)\left(k^2+3k+2\right)$$
$$\times(2\gamma_{12}^2+\gamma_{12}(4k-4s-1)+2k^2-k(4s+1)+2s^2+s-1).$$

M Computing (6.34)

Consider the integral (6.34). The part proportional to a_1 can be computed using (J.8), since the spin 0 Mack polynomial is equal to 1. Let us deduce an analytic expression for the nontrivial part of the integral (6.34), which is the part not proportional to a_1. The idea will be to consider (6.34) for noninteger ℓ and use the Mellin representation for the Mack polynomial.

$Q^{\tau=2,d=4}_{\ell,m=0}(s)$ has the following Mellin representation

$$Q^{\tau=2,d=4}_{\ell,m=0}(s) = \frac{2^\ell \Gamma(1+\ell)}{(1+\ell)_\ell \Gamma(-\ell)\Gamma(-\frac{s}{2})} \int \frac{ds_1}{2\pi i} \frac{\Gamma(s_1)\Gamma(-\ell-s_1)\Gamma(1+\ell-s_1)\Gamma(-\frac{s}{2}-s_1)(-1)^{-s_1}}{\Gamma(1-s_1)^2}. \tag{M.1}$$

In the formula above ℓ is taken to be non-integer. The contour is bent, in such a way as to pass to the right of the poles of $\Gamma(s_1)$ and to the left of poles of $\Gamma(-\ell-s_1)\Gamma(1+\ell-s_1)\Gamma(-\frac{s}{2}-s_1)$.

We plug this expression in (6.34), exchange the order of integration and evaluate the s integral. This gives

$$\int \frac{ds}{4\pi i}\Gamma(-\frac{s}{2})\Gamma(-\frac{s}{2}-s_1)\Gamma(\frac{s}{2}+1)^2 \frac{4+s}{2+s} = \frac{\pi}{\Gamma(s_1)\sin(\pi s_1)}\left(\frac{\pi^2}{\sin(\pi s_1)^2} - \frac{1}{s_1-1} - \psi^{(1)}(s_1)\right). \tag{M.2}$$

We obtained this formula by deforming the contour to the right, picking up the poles from $\Gamma(-\frac{s}{2})\Gamma(-\frac{s}{2}-s_1)$ and evaluating the infinite sum over residues. The contour of the integral above is supposed to be bent, separating left from right poles.

Thus, the nontrivial part of (6.34) is equal to

$$\frac{2^\ell \Gamma(1+\ell)}{\pi\Gamma(-\ell)(1+\ell)_\ell} \int \frac{ds_1}{2\pi i}\Gamma(-\ell-s_1)\Gamma(1+\ell-s_1)\Gamma(s_1)^2 \tag{M.3}$$

$$\times \sin(\pi s_1)(-1)^{-s_1}\left(\frac{\pi^2}{\sin(\pi s_1)^2} - \frac{1}{s_1-1} - \psi^{(1)}(s_1)\right).$$

We evaluate the s_1 integral by picking the poles at $s_1 = 0, -1, -2, ...$ We thus transform the s_1 integral into an infinite series. Let us take the limit where ℓ becomes an integer again. The infinite series must diverge, in order to cancel the $1/\Gamma(-\ell)$. For each integer ℓ, only ℓ terms out of the infinte sum contribute to the divergence. In fact the terms come from just $\Gamma(-\ell - s_1)$. Thus we conclude that the nontrivial part of (6.34) is equal to

$$\frac{2^\ell \Gamma(1+\ell)^2}{(1+\ell)_\ell} \sum_{n=0}^{\ell} \frac{(-1)^n \Gamma(1+\ell+n)}{\Gamma(1+\ell-n)} \left(\frac{1+n}{\Gamma(2+n)^2} + \frac{\psi^{(1)}(1+n)}{\Gamma(1+n)^2}\right). \tag{M.4}$$

We checked that (M.4) is equal to

$$-\frac{\sqrt{\pi} 2^{-\ell-1} \Gamma(\ell+1)^2 \left((\ell+1)^2 \psi^{(1)}\left(\frac{\ell}{2}+1\right) - (\ell+1)^2 \psi^{(1)}\left(\frac{\ell+3}{2}\right) - 4\right)}{(\ell+1)^2 \Gamma\left(\ell+\frac{1}{2}\right)} + \delta_{\ell,0}. \tag{M.5}$$

that we quoted in the main text.

N Bulk Point Limit in $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$

N.1 Introduction

Correlation functions of conformal field theories in Lorentzian signature may diverge even when none of the distances between the points vanish. At the moment a full classification of the singularity structure of correlation functions in conformal field theories does not exist.

One such singularity is the so called “bulk point singularity". In terms of cross ratios, we can obtain such a singularity in the following manner. In Lorentzian signature z and $\bar{z}$ are independent real numbers. The four point function has branch points. When z and $\bar{z}$ go around the branch points the four point function may develop a divergence when $z = \bar{z}$. More specifically, suppose z goes around the branch point at 1, $\bar{z}$ goes around ∞ and now take $z \to \bar{z}$. We generically expect the four point function to diverge in this limit. A detailed examination of the bulk point limit for a four point function of equal scalars was carried out in [23].

In the bulk point limit a d dimensional conformal block where the external operators are scalars diverges as $\frac{1}{(z-\bar{z})^{d-3}}$ [23]. For this reason it is expected that a generic nonperturbative four point function of scalars diverges as

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle \sim \frac{1}{(z-\bar{z})^{d-3}}. \tag{N.1}$$

However, when the CFT has a local bulk dual, then we expect the divergence to be more severe. For example, a contact quartic diagram in AdS diverges as

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle \sim \frac{1}{(z-\bar{z})^{4\Delta-3}}. \tag{N.2}$$

The plan for this section is the following. In N.1.1 we calculate the bulk point singularity of an AdS contact diagram for a scalar four point function of unequal primaries. The result is a trivial generalisation of (N.2), however to our knowledge its derivation had not appeared before in the literature. We need such a result in order to calculate the bulk point singularity of an AdS contact diagram for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$, which we do in section N.1.2. Finally, in section N.1.3 we calculate the expected bulk point divergence of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ in CFT's with slightly broken higher spin symmetry. We assume that $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ does not diverge more than conformal blocks in the bulk point limit. We conclude that AdS contact diagrams diverge more severely in the bulk point limit than what is expected for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ for $s \geqslant 2$ in CFT's with slightly broken higher spin symmetry. Thus, bulk point softness implies that we cannot add AdS contact diagrams to the solution to the pseudo-conservation equations that we found in section (9.2).

Let us add a caveat. Our result for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ does not rely on assuming bulk point softness and is independent of it. Nevertheless, we choose to keep this appendix, because it was useful for us to think in terms of the bulk point limit in the early stages of our work, and maybe this can be of use to someone else.

N.1.1 Bulk point singularity of an AdS contact diagram for a scalar four point function of unequal primaries

A quartic contact diagram has a Mellin amplitude equal to 1. We will use this to compute the bulk point divergence, proceeding similarly to section 7.5.1 in [63]. Upon analytic continuation, the diagram is given by

$$\begin{aligned}\frac{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{p} &= \int\int \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}\Gamma(\gamma_{12})\Gamma(\gamma_{13})\Gamma(\gamma_{14}) \\ &\times\Gamma(\gamma_{12}+a_{34})\Gamma(\gamma_{13}+a_{24})\Gamma(\gamma_{14}+a_{23})u^{-\gamma_{12}}v^{-\gamma_{14}} \\ &\to \int\int \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}\Gamma(\gamma_{12})\Gamma(\gamma_{13})\Gamma(\gamma_{14}) \\ &\times\Gamma(\gamma_{12}+a_{34})\Gamma(\gamma_{13}+a_{24})\Gamma(\gamma_{14}+a_{23})u^{-\gamma_{12}}v^{-\gamma_{14}}e^{-2\pi i(\gamma_{12}+\gamma_{14})}, \\ p &= |x_1-x_3|^{-2\Delta_1}|x_2-x_3|^{-2a_{23}}|x_2-x_4|^{-2a_{24}-2\Delta_1}|x_3-x_4|^{-2a_{34}}\end{aligned} \tag{N.3}$$

where $a_{ij} = 2(\Delta_i+\Delta_j) - \sum_k \Delta_k$ and $\gamma_{13} = \Delta_1 - \gamma_{12} - \gamma_{14}$. The integral diverges when γ_{12} and γ_{14} have a very big and positive imaginary part. We can use Stirling's approximation for the Γ functions. Indeed suppose we take $\gamma_{12} = is\beta$ and $\gamma_{14} = is(1-\beta)$. Then for very large s we have

$$\begin{aligned}\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle &\approx p\int_{s_0}^{\infty}\frac{ds}{s}\int_0^1 d\beta s^{\frac{\sum_i\Delta_i}{2}-1}f(\beta) \\ \times \exp\Big(is\big(-2(\beta-1)\log(1-\beta)&+2\beta\log(\beta)-\beta\log(u)+(\beta-1)\log(v)\big)\Big),\end{aligned} \tag{N.4}$$

where $f(\beta)$ is a function of β that will not play any role. The integral has a saddle point for $\beta \to \beta_s = \frac{\sqrt{u}}{\sqrt{u}+\sqrt{v}}$. In that case the exponential dependence of the integrand becomes $e^{is\left(\frac{(\sqrt{u}+\sqrt{v})^2}{\sqrt{u}\sqrt{v}}(\beta-\beta_s)^2-2\log\left(\sqrt{u}+\sqrt{v}\right)\right)}$. The integral in β is Gaussian and can be readily evaluated. Furthermore, the phase is stationary when $\sqrt{u}+\sqrt{v}=1$. In that case we have $\log(\sqrt{u}+\sqrt{v}) \sim (z-\bar{z})^2$. So, we conclude that

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle \sim \int_{s_0}^{\infty} \frac{ds}{s} s^{\frac{\sum_i \Delta_i}{2}-\frac{3}{2}} e^{is(z-\bar{z})^2} \sim \frac{1}{(z-\bar{z})^{\sum_i \Delta_i - 3}}. \tag{N.5}$$

N.1.2 Bulk point singularity of AdS contact diagrams for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$

Identity (9.43) allows us to obtain spinning contact AdS diagrams from scalar contact AdS diagrams. So, with the help of identity (9.43) we can perform the integration in (9.40) using only scalar propagators and afterwards we act with the differential operators D_{12} and D_{13}. The scalar propagators cause a divergence like $\frac{1}{(z-\bar{z})^{\sum_i \Delta_i -3+s}}$, see formula (N.5). After acting with the differential operators, we find that the bulk point divergence of the integral (9.40) is $\frac{1}{(z-\bar{z})^{\sum_i \Delta_i -3+3s}} = \frac{1}{(z-\bar{z})^{4s+4}}$.

N.1.3 Bulk point singularity of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$ in CFT's with slightly broken higher spin symmetry

Conformal field theories with slightly broken higher spin symmetry have an infinite number of light single trace operators. For this reason, they are not expected to be dual to a local theory in AdS. Thus, their bulk point singularity should not be enhanced with respect to that of an individual conformal block.

We want to calculate the bulk point divergence of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$. For our discussion, it is useful to introduce the operator

$$d_{11} = (P_1 \cdot P_2) Z_1 \cdot \frac{\partial}{\partial P_2} - (Z_1 \cdot P_2) P_1 \cdot \frac{\partial}{\partial P_2} - (Z_1 \cdot Z_2) P_1 \cdot \frac{\partial}{\partial Z_2} + (P_1 \cdot Z_2) Z_1 \cdot \frac{\partial}{\partial Z_2}, \tag{N.6}$$

where we used embedding space coordinates [98]. This operator acts on conformal blocks where the operator exchanged is symmetric and traceless. It increases the spin of the operator in position 1 by 1 and it decreases its conformal dimension by 1 also. It turns out that $d_{11}^s (z-\bar{z})^a \sim (z-\bar{z})^{a-2s}$, i.e. the action of d_{11}^s increases the divergence by a power of $2s$. For this reason, we expect the divergence of $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$ to be

$$\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle \sim \frac{1}{(z-\bar{z})^{2s}} \tag{N.7}$$

since the scalar conformal block diverges logarithmically. We could have picked other differential operators than d_{11} to create spin from the scalar conformal block. Since such operators only contain first derivatives of P_i (and not higher derivatives), they lead to the same divergence (N.7).

O Algorithm for computing $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ in position space

We will implement an algorithm in position space to calculate $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}$. The results match with the Mellin space calculation.

$\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb}$ is constrained by conformal symmetry, crossing, consistency with OPE and the pseudo-conservation equation that j_s obeys. Conformal symmetry implies that

$$\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} = p \sum_{j=0}^{s} f_j(u,v) w(1;2,3)^j w(1;3,4)^{s-j}, \tag{O.1}$$

where

$$p \equiv \frac{(x_{23}^2 x_{24}^2 x_{34}^2)^{\frac{s}{3}-\frac{5}{6}}}{(x_{12}^2 x_{13}^2 x_{14}^2)^{\frac{2s}{3}+\frac{1}{3}}}, \; u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \; v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \tag{O.2}$$
$$w(i;j,k) \equiv (x_{ij})_\mu \frac{x_{ik}^2}{x_{jk}^2} - (x_{ik})_\mu \frac{x_{ij}^2}{x_{jk}^2}$$

and we use the notation $(x_{ij})_\mu = (x_i)_\mu - (x_j)_\mu$, $x_{ij} = |x_i - x_j|$. The indices are symmetric and traceless. $f_j(u,v)$ is a function of the cross ratios not determined by conformal symmetry.

We write the following ansatz.

$$f_j(u,v) = \frac{u^{a(j)} v^{b(j)}}{(1+\sqrt{u}+\sqrt{v})^s} \sum_{n_j=0}^{N(j)} \sum_{m_j=0}^{M(j)} c_{n_j,m_j} u^{\frac{n_j}{2}} v^{\frac{m_j}{2}}, \tag{O.3}$$

where c_{n_j,m_j} are parameters that will be fixed by crossing and the pseudo-conservation equation. The values of $a(j)$, $b(j)$, $M(j)$ and $N(j)$ will follow from consistency with the operator product expansion.

Let us motivate the preceding ansatz. The spinning four point functions are related to the scalar four point functions by slightly broken higher spin Ward identities. The scalar four point function is a linear combination of powers of u and of v. So, it is natural that $f_j(u,v)$ is made up of powers of u and of v.

We will see below that the contribution to the operator product expansion of a certain operator goes as $\sim u^{\frac{\tau}{2}}$, where τ is the twist, which is defined as the conformal dimension minus the spin. Since all operator dimensions are integers, it is natural that the ansatz involves semi-integer powers of u and of v. The denominator $\frac{1}{(1+\sqrt{u}+\sqrt{v})^s}$ diverges in the bulk point limit as $\frac{1}{(z-\bar{z})^{2s}}$, which agrees with the discussion in N.1.3.

We can fix $a(j), b(j), N(j), M(j)$ by consistency with the lightcone operator product expansion. Let us explain the general idea. Consider two primary operators $O_{\mu_1\ldots\mu_{l_1}}(x)$, $O_{\nu_1\ldots\nu_{l_2}}(0)$ of conformal dimensions Δ_1 and Δ_2 and spins l_1 and l_2 and suppose they exchange a primary operator $O_{\rho_1\ldots\rho_l}$ of dimension Δ and spin l. The most singular term due to $O_{\rho_1\ldots\rho_l}$ that can appear in the lightcone operator product expansion is $\frac{O_{\rho_1\ldots\rho_l} x^{\rho_1}\ldots x^{\rho_l} x_{\{\mu_1}\ldots x_{\mu_{l_1}\}} x_{\{\nu_1}\ldots x_{\nu_{l_2}\}}}{|x|^{\Delta_1+\Delta_2+l_1+l_2+l-\Delta}} \sim (x^2)^{-\frac{\Delta_1+\Delta_2+l_1+l_2}{2}+\frac{\tau}{2}}$, where the μ and ν indices are traceless symmetric and $\tau = \Delta - l$.

For $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ the primary operators exchanged can have twist 1 (higher spin currents), $3+2n$ (double traces $[j_s, j_{\tilde{0}}]$) and $4+2n$ (double traces $[j_{\tilde{0}}, j_{\tilde{0}}]$), where n is a nonnegative integer. There is no primary operator of twist 2 being exchanged. This is an important condition that we impose in our algorithm.

More explicitly

$$j_s(x) j_{\tilde{0}}(0) \sim (x^2)^{-s-1} j_{s'} + (x^2)^{-s}[j_s, j_{\tilde{0}}] + (x^2)^{-s+\frac{1}{2}}[j_{\tilde{0}}, j_{\tilde{0}}], \tag{O.4}$$

$$j_{\tilde{0}}(x) j_{\tilde{0}}(0) \sim (x^2)^{-\frac{3}{2}} j_{s'} + (x^2)^{-\frac{1}{2}}[j_s, j_{\tilde{0}}] + (x^2)^0[j_{\tilde{0}}, j_{\tilde{0}}], \tag{O.5}$$

where we wrote the most singular powers of the distance that can appear in the lightcone operator product expansion. Our ansatz (O.3) needs to be compatible with (O.4), (O.5). This fixes $a(j), b(j), N(j), M(j)$.

The final ingredient is compatibility with pseudo-conservation. $\partial \cdot j_s$ can have contributions coming from $[j_{s_1}, j_{\tilde{0}}]$ and $[j_{s_1}, j_{s_2}]$. Only the former matter since we are interested in $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. More precisely,

$$\partial \cdot j_s \supset \sum_{s_1=2}^{s-2} \sum_{m=0}^{s-s_1-1} c_m \partial^m j_{s_1} \partial^{s-s_1-1-m} j_{\tilde{0}}. \tag{O.6}$$

Since the right-hand side must be a conformal primary, this implies [93]

$$c_m = \frac{-(m-s+s_1)(m-s+s_1-1)}{m(m+2s_1)} c_{m-1}. \tag{O.7}$$

Thus $\langle \partial \cdot j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$ is a linear combination of terms of type $\partial^{n_1} \langle j_{\tilde{0}} j_{\tilde{0}} \rangle \partial^{n_2} \langle j_{s_1} j_{\tilde{0}} j_{\tilde{0}} \rangle$.

Crossing and compatibility with pseudo-conservation fix all coefficients in (O.3) up to a number. This number is related to the normalizaton of j_s. In fact we did not even need to input formula (O.7), we kept the coefficients c_m as unknowns and our algorithm correctly returns (O.7). This serves as a check on our results. We checked that the algorithm fixes the solution for $s = 2, ..., 14$. Afterwards the computation becomes heavy for our laptop.

P Mixed Fourier Transform

We will solve the higher spin Ward identities to compute $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. This is a rederivation of the main result of [95]. Our method involves the use of a mixed Fourier transform, see [102] and [121].

We use the metric $ds^2 = -dx^- dx^+ + dy^2$. We will take all indices lowered and in the minus component. We will study the action of the charge

$$Q = \sqrt{\tilde{N}} \alpha_4 \int_{x^+ = const.} dx^- dy j_{----} \tag{P.1}$$

on the four point function $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$. We make use of equations [20], [94]

$$\partial \cdot j_4 = \alpha \frac{\tilde{\lambda}}{\sqrt{\tilde{N}}\sqrt{1+\tilde{\lambda}^2}} (: \partial_- j_{\tilde{0}} j_2 : -\frac{2}{5} : j_{\tilde{0}} \partial_- j_2 :), \tag{P.2}$$

$$[Q, j_{\tilde{0}}] = \partial_-^3 j_{\tilde{0}} + \frac{\beta}{\sqrt{1+\tilde{\lambda}^2}} (\partial_- \partial_- j_{-y} - \partial_- \partial_y j_{--}). \tag{P.3}$$

α, α_4 and β are numerical coefficients that can be obtained from solving Ward identities at the level of three point functions[1]. We will not need their precise value in what follows.

The scalar four point function obeys the slightly broken spin 4 Ward identity

$$\langle [Q, j_{\tilde{0}}] j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle + ... = \sqrt{\tilde{N}} \alpha_4 \int d^3x \langle \partial \cdot j_4(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle, \tag{P.4}$$

where by ... we mean the permutations (12), (13), (14). Note that

$$\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle = \langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{disc} + \frac{1}{N} \langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff}, \tag{P.5}$$

[1] We normalised the charge such that the coefficient multiplying $\partial_-^3 j_{\tilde{0}}$ in (P.3) is 1.

where $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff}$ denotes the connected piece in the free fermion theory and $\langle j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{disc}$ denotes the disconnected piece. The disconnected piece obeys

$$\langle \partial^3 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{disc} + ... = 0, \tag{P.6}$$

where we summed over all permutations. For this reason the disconnected piece drops out of (P.4). Using our ansatz (9.4) we conclude that

$$\begin{aligned} \langle [Q, j_{\tilde{0}}] j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle + ... = \frac{1}{\tilde{N}} \langle \partial^3 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} + \frac{\beta}{\tilde{N}(1+\tilde{\lambda}^2)} \big(\langle (\partial_- \partial_- j_{-y} - \partial_- \partial_y j_{--}) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} \\ + \tilde{\lambda} \langle (\partial_- \partial_- j_{-y} - \partial_- \partial_y j_{--}) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} \big) + ... \end{aligned} \tag{P.7}$$

From the Ward identities in the free fermion theory this becomes

$$\begin{aligned} \langle [Q, j_{\tilde{0}}] j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle + ... = -\frac{\tilde{\lambda}^2 \beta}{\tilde{N}(1+\tilde{\lambda}^2)} \langle (\partial_- \partial_- j_{-y} - \partial_- \partial_y j_{--}) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} \\ + \frac{\tilde{\lambda} \beta}{\tilde{N}(1+\tilde{\lambda}^2)} \langle (\partial_- \partial_- j_{-y} - \partial_- \partial_y j_{--}) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} + ... \end{aligned} \tag{P.8}$$

Using (P.2) in the right-hand side of (P.4) we get

$$\begin{aligned} \sqrt{\tilde{N}} \alpha_4 \int d^3x \langle \partial \cdot j_4(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle = \alpha \alpha_4 \frac{\tilde{\lambda}}{\sqrt{1+\tilde{\lambda}^2}} \int d^3x \big(\langle \partial_- j_{\tilde{0}}(x) j_{\tilde{0}} \rangle \langle j_2(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle \\ - \frac{2}{5} \langle j_{\tilde{0}}(x) j_{\tilde{0}} \rangle \langle \partial_- j_2(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle + ... \big) \end{aligned} \tag{P.9}$$

We use the decomposition (9.4) to obtain that (P.9) is equal to

$$\begin{aligned} \alpha \alpha_4 \frac{\tilde{\lambda}}{\tilde{N}(1+\tilde{\lambda}^2)} \int d^3x \big(\langle \partial_- j_{\tilde{0}}(x) j_{\tilde{0}} \rangle \langle j_2(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} - \frac{2}{5} \langle j_{\tilde{0}}(x) j_{\tilde{0}} \rangle \langle \partial_- j_2(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} + ... \big) \\ + \alpha \alpha_4 \frac{\tilde{\lambda}^2}{\tilde{N}(1+\tilde{\lambda}^2)} \int d^3x \big(\langle \partial_- j_{\tilde{0}}(x) j_{\tilde{0}} \rangle \langle j_2(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} - \frac{2}{5} \langle j_{\tilde{0}}(x) j_{\tilde{0}} \rangle \langle \partial_- j_2(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} + ... \big) \end{aligned} \tag{P.10}$$

Let us equate (P.8) and (P.10). We see that the dependence on $\tilde{N}$ and $\tilde{\lambda}$ matches on both sides provided

$$\begin{aligned} \beta \langle (\partial_- \partial_- j_{-y} - \partial_- \partial_y j_{--}) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} + ... \\ = -\alpha \alpha_4 \int d^3x \big(\langle \partial_- j_{\tilde{0}} j_{\tilde{0}} \rangle \langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} - \frac{2}{5} \langle j_{\tilde{0}} j_{\tilde{0}} \rangle \langle \partial j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{cb} + ... \big), \end{aligned} \tag{P.11}$$

$$\beta\langle(\partial_-\partial_- j_{-y} - \partial_-\partial_y j_{--}) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle_{cb} + ... \tag{P.12}$$
$$= \alpha\alpha_4 \int d^3x \Big(\langle \partial_- j_{\tilde{0}} j_{\tilde{0}} \rangle \langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} - \frac{2}{5} \langle j_{\tilde{0}} j_{\tilde{0}} \rangle \langle \partial j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle_{ff} + ... \Big).$$

We solved (P.11) and (P.12) using a mixed Fourier transform. We define the mixed Fourier transform of a four point function $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$ as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \to \int \frac{d^3x_2 d^3x_3}{(2\pi i)^2} \langle \mathcal{O}_1(0)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(\infty)\rangle e^{i(p_2\cdot x_2 + p_3 \cdot x_3)}. \tag{P.13}$$

The advantage of the mixed Fourier transform with respect to a usual Fourier transform is that by placing an operator at the origin and another one at ∞ we take advantage of conformal symmetry.

In mixed Fourier space we can get rid of the integrals in equations (P.11) and (P.12). For example, it is simple to see that the mixed Fourier transform of $\int d^3x \langle j_{\tilde{0}}(x) j_{\tilde{0}}\rangle \langle j_2(x) j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$ is equal to

$$\int d^3x \langle j_{\tilde{0}}(x) j_{\tilde{0}}(x_1)\rangle \langle j_2(x) j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3) j_{\tilde{0}}(x_4)\rangle \to \left(\int d^3x \langle j_{\tilde{0}}(x) j_{\tilde{0}}(0)\rangle e^{i(p_2+p_3)\cdot x} \right) \tag{P.14}$$
$$\times \int\int d^3x_2 d^3x_3 e^{i\left(p_2\cdot x_2 + p_3\cdot x_3\right)} \langle j_2(0) j_{\tilde{0}}(x_2) j_{\tilde{0}}(x_3) j_{\tilde{0}}(\infty)\rangle$$

which is just a product of mixed Fourier transforms.

It turns out that $\langle j_2 j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle_{ff}$ is very simple in mixed Fourier space. Let us define $u_p = \frac{p_2^2}{p_1^2}, v_p = \frac{p_3^2}{p_1^2}$, where $p_1 = -p_2 - p_3$. Then,

$$\langle T_{\mu\nu}(0) j_{\tilde{0}}(p_2) j_{\tilde{0}}(p_3) j_{\tilde{0}}(\infty)\rangle_{ff} = \frac{f(u_p, v_p)}{p_1^4} \Big((p_2)_{(\mu}\epsilon_{\nu)\alpha\beta}(p_2)^\alpha (p_3)^\beta \Big) \tag{P.15}$$
$$+ \frac{f(v_p, u_p)}{p_1^4} \Big((p_3)_{(\mu}\epsilon_{\nu)\alpha\beta}(p_3)^\alpha (p_2)^\beta \Big),$$

where $f(u_p, v_p) = \frac{32}{3}\pi^2(-\frac{1}{u_p} + \frac{1}{v_p} - \frac{1}{u_p v_p})$. Plugging this into (P.11) and (P.12) we obtain

$$\langle T_{\mu\nu}(0) j_{\tilde{0}}(p_2) j_{\tilde{0}}(p_3) j_{\tilde{0}}(\infty)\rangle_{cb} = \frac{1}{|p_1|^3} \Big((p_2)_{(\mu}(p_3)_{\nu)} - \frac{p_2\cdot p_3}{3}\eta_{\mu\nu} \Big) f_1(u_p, v_p) \tag{P.16}$$
$$+ \frac{1}{|p_1|^3} \Big((p_2)_\mu (p_2)_\nu - \frac{p_2^2}{3}\eta_{\mu\nu} \Big) f_2(u_p, v_p) + \frac{1}{|p_1|^3} \Big((p_3)_\mu (p_3)_\nu - \frac{p_3^2}{3}\eta_{\mu\nu} \Big) f_2(v_p, u_p),$$

where

$$f_1(u_p,v_p) = \frac{1}{2}\left(\frac{u_p}{v_p}+\frac{v_p}{u_p}\right)+\left(\frac{1}{u_p}+\frac{1}{v_p}\right)-\frac{3}{2u_pv_p},$$
$$f_2(u_p,v_p) = \frac{u_p}{4v_p}+\frac{v_p}{4u_p}+\frac{1}{4u_pv_p}+\frac{3}{2u_p}-\frac{1}{2v_p}. \tag{P.17}$$

Finally, we can transform back to position space to get

$$\begin{aligned}\langle T_{\mu\nu}j_{\tilde{0}}j_{\tilde{0}}j_{\tilde{0}}\rangle_{cb} = p\times\Bigg(& g_1(u,v)\Big(V(1,2,3)_\mu V(1,2,3)_\nu - \frac{V(1,2,3)^2}{3}\eta_{\mu\nu}\Big)\\ &+g_2(u,v)\Big(V(1,2,3)_{(\mu}V(1,3,4)_{\nu)} - \frac{V(1,2,3)\cdot V(1,3,4)}{3}\eta_{\mu\nu}\Big)\\ &+g_3(u,v)\Big(V(1,3,4)_\mu V(1,3,4)_\nu - \frac{V(1,3,4)^2}{3}\eta_{\mu\nu}\Big)\Bigg),\end{aligned} \tag{P.18}$$

where $p = \frac{1}{(x_{12}x_{13}x_{14})^{\frac{10}{3}}(x_{23}x_{24}x_{34})^{\frac{1}{3}}}$, $V(i;j,k) = \frac{x_{ij}^2(x_{ik})_\mu - x_{ik}^2(x_{ij})_\mu}{x_{jk}^2}$ and

$$\begin{aligned}g_1(u,v) &= \frac{u^{2/3}v^{2/3}}{4\pi^3}-\frac{v^{2/3}}{4\pi^3u^{4/3}}+\frac{v^{5/3}}{2\pi^3u^{4/3}}-\frac{v^{8/3}}{4\pi^3u^{4/3}}+\frac{v^{2/3}}{2\pi^3\sqrt[3]{u}}+\frac{v^{5/3}}{2\pi^3\sqrt[3]{u}},\\ g_2(u,v) &= \frac{u^{2/3}v^{2/3}}{2\pi^3}+\frac{u^{2/3}}{2\pi^3\sqrt[3]{v}}+\frac{u^{5/3}}{4\pi^3\sqrt[3]{v}}+\frac{v^{2/3}}{2\pi^3\sqrt[3]{u}}+\frac{v^{5/3}}{4\pi^3\sqrt[3]{u}}-\frac{3}{4\pi^3\sqrt[3]{u}\sqrt[3]{v}},\\ g_3(u,v) &= \frac{u^{2/3}v^{2/3}}{4\pi^3}-\frac{u^{2/3}}{4\pi^3v^{4/3}}+\frac{u^{5/3}}{2\pi^3v^{4/3}}-\frac{u^{8/3}}{4\pi^3v^{4/3}}+\frac{u^{2/3}}{2\pi^3\sqrt[3]{v}}+\frac{u^{5/3}}{2\pi^3\sqrt[3]{v}}.\end{aligned} \tag{P.19}$$

The result agrees with [95]. For correlators of type $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}}\rangle$ with $s \geqslant 4$, the mixed Fourier transform is not so simple, so in practice it was not useful.

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Professional Experience and Education

- **École Polytechnique Fédérale de Lausanne (EPFL)**, Lausanne, Switzerland
  01/09/2016 - Expected: 07/05/2021
  Ph.D. in Theoretical Physics, Advisor : Joao Penedones, co-advisor: Alexander Zhiboedov

- **Universidade do Porto**, Porto, Portugal
  01/01/2016 - 31/08/2016
  Ph.D student in Theoretical Physics, Advisor : Miguel Costa.

- **Universidade do Porto**, Porto, Portugal
  01/09/2013 - 01/10/2015
  Master's degree in Physics. **GPA (18/20)**
  Thesis Title: Black Holes in AdS from Localized Boundary Sources
  Thesis Advisor: Miguel Costa. Co-Advisor: Jorge Santos.

- **Universidade do Porto**, Porto, Portugal
  01/09/2010 - 01/07/2013
  Bachelors' degree, Major Physics, Minor Mathematics. **GPA (18/20)**

Field of Interest

- **Conformal Bootstrap**
- **AdS/CFT**
- **String Theory**
- **Quantum Field Theory**

Preprints and Publications

- Penedones, Joao and Silva, Joao A. and Zhiboedov, Alexander. *Nonperturbative Mellin Amplitudes: Existence, Properties, Applications.* JHEP 08 (2020), 031. 27 citations

- Carmi, Dean and Penedones, Joao and Silva, Joao A. and Zhiboedov, Alexander. *Bootstrap approach to the ϵ-expansion using Mellin sum rules.* arXiv:2009.13506. 6 citations

- Silva, Joao A. *Four point functions in conformal field theories with slightly broken higher spin symmetry.* arXiv:2009.13506. 1 citations

Research Projects

1. **Nonperturbative Mellin amplitudes: existence, properties, applications**. We argue that nonperturbative CFT correlation functions admit a Mellin amplitude representation. We discuss the main properties of nonperturbative CFT Mellin amplitudes: subtractions, analyticity, unitarity, Polyakov conditions and polynomial boundedness at infinity. We then use them to constrain holographic theories.

2. **Bootstrap approach to the ϵ-expansion using Mellin sum rules**. We use Mellin space dispersion relations together with Polyakov conditions to derive a family of sum rules for Conformal Field Theories. The defining property of these sum rules is suppression of the contribution of the double twist operators. Firstly, we apply these sum rules to the Wilson-Fisher model in $d = 4 - \epsilon$ dimensions. We re-derive many of the known results to order ϵ^4 and we make new predictions. No assumptions of analyticity down to spin 0 were made. Secondly, we study holographic CFTs. We use dispersive sum rules to obtain tree-level and one-loop anomalous dimensions. Finally, we briefly discuss the contribution of heavy operators to the sum rules in UV complete holographic theories.

3. **Four point functions in conformal field theories with slightly broken higher spin symmetry**. We compute spinning four point functions in the quasi-fermionic three dimensional conformal field theory with slightly broken higher spin symmetry at finite t'Hooft coupling. More concretely, we obtain a formula for $\langle j_s j_{\tilde{0}} j_{\tilde{0}} j_{\tilde{0}} \rangle$, where j_s is a higher spin current and $j_{\tilde{0}}$ is the scalar single trace operator. Our procedure consists in writing a plausible ansatz in Mellin space and using crossing, pseudo-conservation and Regge boundedness to fix all undetermined coefficients. Our method can potentially be generalised to compute all spinning four point functions in these theories.

4. **Black Holes in AdS from Localized Boundary Sources** (master's thesis). We study a massive scalar field in a spacetime with a negative cosmological constant. We impose as boundary conditions that, near the conformal boundary, spacetime is AdS and the scalar field behaves as a localized defect. For the values of the field strength examined, we provide evidence that no Schwarzschild AdS black hole is formed.

Workshops and Conferences

- 16/01/2020. Iberian Strings 2020, Santiago de Compostela, Spain
- 09/09/2019. Swiss Mathematical Physics General Meeting, Villars-sur-Ollon, Switzerland
- 01/07/2019. Bootstrap Workshop and School, Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada
- 01/02/2019. Young Researchers Integrability School and Workshop, Erwin Schrodinger International Institute for Mathematics and Physics, Vienna, Austria
- 01/09/2018. Bootstrap Workshop and School, California Institute of Technology, Pasadena, California, USA
- 10/05/2018. Project Meeting on Analytic Bootstrap Approaches, Açores, Portugal
- 01/06/2017. Bootstrap Workshop and School, International Centre for Theoretical Physics SAIFR, Sao Paulo, Brazil

Seminars and Conference Talks

- 15.03.2021. “Four Point Functions in CFT’s with Slightly Broken Higher Spin Symmetry”. Talk at the CERN string theory journal club.
- 03.03.2021. “Four Point Functions in CFT’s with Slightly Broken Higher Spin Symmetry”. Talk at the EPFL fields and strings journal club.
- Invitation to give a talk at the “Bootstrapping String Theory” conference at Caltech in june 2020. The conference was cancelled due to the coronavirus pandemic.
- 15/01/2020.“Nonperturbative Mellin Amplitudes”, Iberian Strings 2020
- 11/09/2019.“Four Point Functions in CFT’s with Slightly Broken Higher Spin Symmetry”, SWISS MAP General Meeting
- 22/01/2019.“Bootstrapping $\mathcal{N} = 4$ SYM”, Strings and Fields Journal Club at EPFL
- 10/07/2019.“Conformal Bootstrap With Slightly Broken Higher Spin Symmetry”, Strings and Fields Journal Club at EPFL
- 22/04/2018.“Simplicity in AdS Perturbative Dynamics”, Strings and Fields Journal Club at EPFL

Teaching Experience

- Taught the exercise classes for the following courses at École Polytechnique Fédérale de Lausanne (EPFL) :
  - Quantum Physics I (fall 2017, fall 2018, fall 2019)
  - Quantum Physics II (spring 2017, spring 2018, spring 2019)

Outreach

- Taught a special relativity course to talented high school students at the university of Porto in 2015

Supervision

- Vladimir Schaub, tutored his 4th year project on two dimensional conformal field theory for 2 months
- Charaf-Ed-Dine El Fattahi, tutored him once per week on the basics of conformal field theory for 3 months

Skills

- **Languages:** Portuguese (native), English (fluent), French (intermediate)
- **Programming skills:** Python, Mathematica
- **Typesetting**: LaTeX , TeX,

Scholarships and Awards

- *Prémio de Estímulo à Investigação* prize awarded by the Gulbenkian Foundation, worth 12500 euros, for a project called *Black Holes in AdS from Localized Boundary Sources*, which became the topic of my master's thesis
- Research scholarship worth 12000 euros awarded for the first year of doctoral studies at Porto university