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Filter quotients and non-presentable $(\infty, 1)$-toposes

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A B S T R A C T

We define filter quotients of $(\infty, 1)$-categories and prove that filter quotients preserve the structure of an elementary $(\infty, 1)$-topos and in particular lift the filter quotient of the underlying elementary topos. We then specialize to the case of filter products of $(\infty, 1)$-categories and prove a characterization theorem for equivalences in a filter product.

Then we use filter products to construct a large class of elementary $(\infty, 1)$-toposes that are not Grothendieck $(\infty, 1)$-toposes. Moreover, we give one detailed example for the interested reader who would like to see how we can construct such an $(\infty, 1)$-category, but would prefer to avoid the technicalities regarding filters.

Contents

0. Introduction

0.1. The set-theoretical foundations of algebraic topology

The study of *algebraic topology* has historically strongly relied on a set-theoretical foundation. Many classical results that we would now associate to the field of algebraic topology were proven using some notion of *topological spaces* (manifolds, complexes, ...), which are quite literally sets along with extra data [57,32,84,85].

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Later, an alternative approach towards algebraic topological computations emerged: *simplicial sets* [24, 40,41]. However, early results about simplicial sets relied again on having set-theoretical foundations. It was later observed that both topological spaces and simplicial sets carry *model structures* and they are in fact equivalent [58], which formalized the intuition that both can be used as a foundation for algebraic topology.

The trend established by these structures has continued to dominate the study of algebraic topology until today. For example, while computing homotopy groups, and in particular homotopy groups of spheres, we can observe certain stability phenomena, that lead to *stable homotopy theory* and the study of *spectra* [44,3]. There are various ways of characterizing spectra, but they all involve a collection of spaces with certain relations [21,34,55]. Hence, the study of stable homotopy theory continued the dependence on set-theoretical foundations.

0.2. Elementary toposes as a foundation

In that same time frame a very rich theory of mathematical foundations apart from the standard set theory familiar to algebraic topologists emerged. There was an extensive analysis of *first-order (elementary) languages* (mathematical settings without any set-theoretical assumptions) in the context of *model theory* [28,76,17]. Parallel to that development, there also emerged an alternative to set theory and its model theoretical approach via the notion of *type theory* [67,16]. At that time, such foundational studies seemed very distant from algebraic topological concerns.

A first step towards observing meaningful connections between those two was the development of *category theory* [22]. Although it first emerged in the context of algebraic topology with the goal of studying naturality [23], it very quickly became foundational in other branches of mathematics, and in particular algebraic geometry, where the school of Grothendieck and Bourbaki realized the centrality of the *Grothendieck topos* in order to further study sheaf theory [69–71]. It was then realized by Lawvere and Tierney, building on previous work [43], that this definition can be generalized to an *elementary topos*, which is a category that can itself serve as a foundation to mathematics and be another alternative to set theory [78].

Using an elementary topos (some times with additional assumptions) they, and other authors, construct many interesting foundations for mathematics:

(1) **Models of set theory:** It was proven that *well-pointed Boolean elementary toposes with natural number object* correspond to categories with objects sets that satisfy the *Zermelo-Fraenkel axioms* [54]. This implies that we can construct models of set-theory by constructing certain elementary toposes. As a direct application we can prove that the *continuum hypothesis* is independent of the other axioms of ZF by constructing appropriate categories [77]. Hence, giving an alternative proof to the original one by Cohen, that used the *forcing technique* [18].
(2) **Filter Quotient:** One benefit of using a categorical approach to foundations is the fact that there are many ways to construct new categories and hence many new foundations. For examples, motivated by the study of ultra products in model theory [17], there is a filter quotient construction for elementary toposes [83]. Moreover, Adelman and Johnstone study which properties of the elementary topos will transfer along the filter quotient [4]. One implication is that we can apply the filter construction to construct very unusual models of set theory, which even lack countable coproducts [36, Example A2.1.13].
(3) **Non-Standard Analysis** Applying the ultra product construction to certain sheaves on real numbers, we can develop a non-standard analysis internal to the topos giving us new ways to tackle the concept of infinity and convergence [56].
(4) **Intuitionistic Logic:** As an example related to the previous one, in 1927 Brouwer proved that all real valued functions are continuous [11]! The way to make sense of this fact is that Brouwer used *intuitionistic logic*, which is constructive in nature. Using topos theory, we can in fact construct a topos where this holds [54].

The key fact here is that the definition of an elementary topos is in fact *elementary*, or *first-order*, meaning the definition does not rely on any set-theoretical assumptions. In particular, it manages to avoid any infinite constructions that would necessitate first establishing necessary set-theoretical axioms that allows us to handle the concept of infinity, such as infinite limits and colimits.

We can use the elementary nature of the internal language of an elementary topos to prove statements inside the topos using appropriate syntactic rules, which has been made precise by studying its connection with type theory. We can prove that every elementary topos is a model for certain *higher-order type theories* and so we can prove statements inside the elementary topos by constructing the appropriate type inside the type theory [47]. This equivalence allows us to effectively treat a topos the way we would treat the category of sets. For interesting examples illustrating how this is implemented in practice see [37, D5.1].

0.3. Why have different foundations for algebraic topology?

At a first glance, we might disregard such discussion of foundations as not pertinent to significant questions in algebraic topology. However, a second glance does suggest some interesting examples:

(1) **Natural number object:** In algebraic topology we instinctively order the homotopy groups by the set $\mathbb{N} = \{0, 1, 2, ...\}$. By extension, spectra are then ordered by the integers, which is simply the group completion. This comes from the fact that $\mathbb{N}$ is the *natural number object* inside the standard category of sets, which, as discussed before, has been the foundation of algebraic topology until now.

There are examples of elementary toposes where the natural number object is not standard [37, D5.1.7], which opens the possibility of defining non-standard homotopy groups indexed by the non-standard natural number object. By extension this would also change how we index a spectrum and so also influences the stable homotopy theory in this setting.

However, in order to seriously tackle such questions, we would first need to properly develop algebraic topology in such a general setting.

(2) **Homotopy type theory:** In the 70's Martin-Löf introduced what is now called *Martin-Löf type theory*, introducing the notion of a *identity type* [51,52], giving us a starting point for introducing homotopical thinking into type theory. This was brought to fruition by Voevodsky who introduced *homotopy type theory* or *univalent foundations* and used that to prove classical algebraic topological results in this setting [79]. In the long run Voevodsky was hoping to formalize advanced homotopy theory in a proof assistant in order to prevent the preponderance of mistakes in a theory that is constantly getting more complicated [82]. One side benefit of univalent foundations is that it proves results from algebraic topology in a manner independent from set theory. However, this also poses a challenge. Concretely whenever a proof in algebraic topology explicitly relies on set theoretical assumptions then it cannot be translated immediately into a proof in the homotopy type theoretical setting and thus cannot be formalized.

Hence, in order to further the formalization of homotopy theory we need to be able to develop algebraic topological constructions in different foundations.

(3) **Non-standard models of spaces:** Even if we were solely focused on algebraic topology with standard set-theoretical foundations then taking a broader view can have immense value. For example, while trying to study the homotopy theory of schemes, Voevodsky developed *motivic homotopy theory* [81]. It has found many applications in algebraic geometry, such as solving the *Bloch-Kato conjecture* [75], however, it has also helped us further compute classical homotopy groups of spheres, as can be witnessed in the work of Isaksen [35]. Somewhat ironically, the additional complexity of motivic spheres enables certain computations that would have otherwise been far more challenging. This observation naturally leads to the question whether there are other interesting alternatives to spaces which can help us further our

understanding of the homotopy groups of spheres, which again necessitates studying algebraic topology in different foundations.

(4) **Constructing Kan model structure:** The *Kan model structure* on simplicial sets has been one of the most important model structures in homotopy theory. Until recently it was assumed that any construction of the Kan model structure would require all axioms of set theory [58,26,31].

By taking a constructive approach to mathematics, Gambino-Sattler-Szumilo [29] and Henry [30] construct a model structure on simplicial sets without assuming either the law of excluded middle or the axiom of choice, giving us a possibility to do algebraic topology in a much broader setting.

In order to determine how much further this approach can be taken and which foundations would permit us to construct a version of the Kan model structure on simplicial objects we need to first better understand algebraic topology in alternative foundations.

This leaves us with many important questions:

(1) We can understand sets collectively by studying its category. What is a proper framework to study topological spaces?
(2) Can we find analogous definitions to Grothendieck toposes and elementary toposes?
(3) Can we prove results analogous to what we have observed for elementary toposes in such a setting?

0.4. ∞-toposes

Fortunately many of these questions have already been answered. Concretely, we now consider $(\infty,1)$-*categories* [9] (and sometimes model categories [58,33]) as the appropriate generalizations of categories, suitable for the study of homotopy theories, and in particular the homotopy theory of topological spaces. We also have a working generalization of Grothendieck toposes to this $(\infty,1)$-categorical setting, known as ∞-*toposes* [48] (also called *model topos* [65] in the model categorical setting). Henceforth, we shall call them *Grothendieck* $(\infty,1)$-*toposes* to avoid confusion with other terminology.

Doing algebraic topology inside a Grothendieck $(\infty,1)$-topos is already a great step in the right direction. In particular, it is already established that it models homotopy type theory [72] and we can use that setting to prove classical results in a much more set theory independent manner, as can be seen for the examples of the *Blakers-Massey theorem* [2], *Goodwillie calculus* [1] and *localization theory* [80]. Moreover, Grothendieck higher topos theory has been used as a foundation to develop a lot of *derived* (and *spectral*) *geometry* [48–50].

Following the previous trends, the next step should then be to generalize Grothendieck $(\infty,1)$-toposes to *elementary* $(\infty,1)$-*toposes* and to prove the desired results. A definition does already exist [61], but most desired results remain unsolved. Concretely, here are certain results and remaining work:

(1) We do know that every Grothendieck $(\infty,1)$-topos is in fact an elementary $(\infty,1)$-topos [61]. Moreover, we do have some examples of elementary $(\infty,1)$-toposes that are not Grothendieck [45], however, those examples are sub-categories of the category of spaces and so do not differ from a foundational perspective (for example have the same natural number object).
(2) We can recover some classical algebraic topological constructions in this setting (such as truncations, Blakers-Massey theorem) [60], however, certain results have remained unsettled (for example homotopy groups of spheres, spectral sequence computations, stabilizations).
(3) We do not generally understand their relation with homotopy type theory, except in the cases already covered by Grothendieck $(\infty,1)$-toposes [72].

0.5. Filter quotients of $(\infty, 1)$-categories

The goal of this paper is to take a further step towards better understanding higher topos theory by studying *filter quotients*. Concretely, we will define a filter quotient for $(\infty, 1)$-categories and then prove that elementary $(\infty, 1)$-toposes are closed under this construction (see Subsection 0.6 for more detailed description of the main results). This will then in particular lead to a large class of examples of elementary $(\infty, 1)$-toposes where the underlying logic is quite non-standard (we will carefully analyze one example in Subsection 3.2).

We are here focusing on the benefits of the filter construction to higher topos theory, as our primary goal is to construct non-trivial examples. However, we do expect interesting applications in the opposite direction, and concretely *chromatic homotopy theory*. In chromatic homotopy theory we usually break down spectra in two consecutive steps. First, we choose a prime p and then we choose a height n (corresponding to the various *Morava K-theories*). As the heights form a natural filtration we can then study the limiting behavior as the height increases, as long as we fix a prime.

On the other hand, the various prime numbers do not give us a filtration and thus there is no immediate way to study the limiting behavior as we increase the prime numbers. The insight of Barthel, Schlank and Stapleton [12] was to instead study the ultra product of the various p-local subcategories and show that the ultra product is equivalent to a category that comes from algebraic geometry, at the same time giving meaning and proving the statement "chromatic homotopy theory is asymptotically algebraic at a fixed height".

However, when setting up the theory of ultra products, they observe that general ultra product $(\infty, 1)$-categories are not well-behaved [12, Example 3.19], in the sense that the ultra product construction does not preserve presentability. This presents a serious challenge, as most of the results in stable homotopy theory have been proven for presentable stable $(\infty, 1)$-categories. Hence, they are forced to use an alternative construction, the *compactly generated ultra product*, that does in fact leads to a presentable $(\infty, 1)$-category, however, does not precisely coincide with the original construction anymore [12, Section 3.3].

The power of elementary $(\infty, 1)$-topos theory is that its definition does not rely on presentability. Hence any result proven about them simply transfers to the ultra product and we do not need to make any adjustments. This opens the possibility of studying the ultra product categories that arise in the work of [12], and are of interest to chromatic homotopy theorists, directly.

0.6. Main results

The main result can be summarized as the following theorems:

First, we introduce a general method for constructing a new $(\infty, 1)$-category, namely the *filter quotient*:

Theorem. *Let $\mathcal{C}$ be a finitely complete $(\infty, 1)$-category, which can be:*

(1) a complete Segal space,
(2) a quasi-category,
(3) a Kan enriched category

and Φ a filter (Definition 1.40) of subobjects (Example 1.46).

Then there exists a finitely complete $(\infty, 1)$-category $\mathcal{C}_\Phi$ (Proposition 2.2, Definition 2.10, Theorem 2.13) along with a functor $P_\Phi : \mathcal{C} \to \mathcal{C}_\Phi$ (Definition 2.11, Definition 2.5).

Next we show the construction is well behaved from the perspective of elementary topos theory:

Theorem. *Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos and Φ a filter of subobjects.*

(1) *(Theorem 2.26) The functor P_Φ preserves all defining conditions of an elementary $(\infty,1)$-topos:*

  (a) *finite (co)limits*
  (b) *local Cartesian closure*
  (c) *subobject classifiers*
  (d) *universes*

  and so $\mathcal{E}_\Phi$ is an elementary $(\infty,1)$-topos as well.

(2) *(Theorem 2.29) We have an equivalence of underlying elementary toposes $\tau_0(\mathcal{E}_\Phi) \simeq \tau_0(\mathcal{E})_\Phi$.*

(3) *(Theorem 2.30) So, P_Φ also preserves natural number objects in $\mathcal{E}_\Phi$.*

(4) *(Corollary 2.31) $\mathcal{E}_\Phi$ is not a Grothendieck $(\infty,1)$-topos (Definition 1.20) if $\tau_0(\mathcal{E})_\Phi$ is not a Grothendieck topos.*

This gives us a very effective recipe to construct non-presentable Grothendieck $(\infty,1)$-toposes, namely constructing filter quotients $\mathcal{E}_\Phi$ such that the underlying elementary topos $\tau_0(\mathcal{E})_\Phi$ is not presentable.

In order to find such examples we next focus on a specific class of filter quotients, namely the *filter product*. We first show that in some ways they behave similar to the classical analogue:

Theorem. *Let $\mathcal{C}$ be a finitely complete $(\infty,1)$-category such that the final object has only two subobjects, I be a set and Φ a filter on $P(I)$. Then we can construct the filter product $\prod_\Phi \mathcal{C}$ and we have the following Łoś type results:*

(1) *(Theorem 3.4, Corollary 3.5) Two maps $(f_i)_{i\in I}, (g_i)_{i\in I}$ are equivalent if and only if*

$$\{i \in I : f_i \simeq g_i\} \in \Phi$$

(2) *(Theorem 3.6, Corollary 3.7) A map $(f_i)_{i\in I}$ is an equivalence if and only if*

$$\{i \in I : f_i \text{ is an equivalence}\} \in \Phi$$

Finally, we use the filter product to give a large of examples of elementary $(\infty,1)$-toposes that are not Grothendieck $(\infty,1)$-toposes:

Example. (Example 3.11) There is a class of sets I and filter Φ such that the filter product $\prod_\Phi \mathcal{S}\mathrm{et}$ is not a Grothendieck topos and thus $\prod_\Phi \mathcal{K}\mathrm{an}$ is an elementary $(\infty,1)$-topos that is not a Grothendieck $(\infty,1)$-topos.

0.7. Outline

Section 1 gives a review of some of the concepts we need later on (see also Subsection 0.8 for an overview of the necessary background and Subsection 0.9 for notation). In particular, we review several important features of complete Segal spaces (Subsection 1.1) that play a crucial role in Section 2.

Section 2 breaks down into two subsections. In the first subsection (Subsection 2.1) we define the filter quotient. We give two separate definitions for Kan enriched categories and for complete Segal spaces and then prove these are equivalent. In the second subsection (Subsection 2.2) we prove that the filter quotient construction preserves all the properties of an elementary $(\infty,1)$-topos.

Section 3 focuses on a special filter quotient: *filter products*. In Subsection 3.1 we study general features of filter products and prove some results along the lines of Łoś's theorem. In Subsection 3.2 we finally use filter products to give examples of elementary $(\infty,1)$-toposes that are not Grothendieck $(\infty,1)$-toposes. We will present one example in great detail, for the benefit of any reader who is only interested in understanding one example and wants to skip the technicalities about filters quotients.

Finally, Section 4 mentions some possible future directions.

0.8. Background

We will assume the reader has already some familiarity with the following concepts:

(1) The theory of $(\infty, 1)$-categories, in particular the different models and how they are related.
(2) The theory of elementary $(\infty, 1)$-toposes.

We only give a quick overview in Section 1 and refer the reader to the appropriate sources.

On the other hand, the following concepts require no familiarity and everything we need is covered in Section 1:

(1) Grothendieck $(\infty, 1)$-topos theory,
(2) Filters.

0.9. Notation

We are using various models of $(\infty, 1)$-categories with various levels of strictness. In order to avoid any confusion we will use the following conventions:

(1) $\mathcal{S}$et, s$\mathcal{S}$et and ss$\mathcal{S}$et are the 1-categories of sets, simplicial sets and bisimplicial sets (also called simplicial spaces), respectively.
(2) The notation colim refers to a colimit in a strict 1-category.
(3) The notation $\cong$ refers to an isomorphism of objects in a 1-category.
(4) We will use Kan complexes as our preferred model for the "homotopy theory of spaces" and therefore will avoid using the vague term space throughout.
(5) As we use various models of $(\infty, 1)$-categories we use a (rather arbitrary) convention to help the reader distinguish the various notations.
    (I) $\mathcal{C}$: Arbitrary $(\infty, 1)$-Category
    (II) $\mathcal{E}$: Arbitrary elementary $(\infty, 1)$-topos
    (III) $\mathcal{G}$: Arbitrary Grothendieck $(\infty, 1)$-topos
    (IV) $\mathcal{Q}$: Quasi-Category
    (V) $\mathcal{W}$: Complete Segal space
    (VI) $\mathcal{K}$: Kan enriched category

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I want to thank Asaf Horev for pointing to the connection with the work by Barthel, Schlank and Stapleton [12]. I also want to thank Peter Lumsdaine for making me aware of a more general form of Łoś's theorem that applies to non-ultrafilter, which resulted in the material in Subsection 3.1. Finally, I want to thank the referee for many helpful suggestions which in particular has resulted in a much better introduction and clarified several important proofs.

1. Review of concepts

We will give a minimal overview over the specific concepts required later on and refer the reader to the appropriate sources for more details.

1.1. The theory of $(\infty, 1)$-categories

An $(\infty, 1)$-category is a general idea of a weak category or homotopical category i.e. a category with a weak notion of composition and associativity. Unlike classical category theory, making such an intuition into a precise definition has been quite challenging. Hence, the intuition of $(\infty, 1)$-categories has been captured by various *models*, some of the most important ones are *quasi-categories* [13], *Kan enriched categories* [19,20], and *complete Segal spaces* [63].

Each of these models has their own model structures, namely *Joyal* [38], *Bergner* [8] and *Rezk* [63] model structure respectively. Moreover, they are all Quillen equivalent via various Quillen equivalences. Concretely we have the following diagram of Quillen equivalences

$$\mathcal{sS}et^{Joyal} \underset{i_1^*}{\overset{p_1^*}{\underset{\sim}{\rightleftarrows}}} \mathcal{ssS}et^{Rezk} \underset{t^!}{\overset{t_!}{\underset{\sim}{\rightleftarrows}}} \mathcal{sS}et^{Joyal} \underset{N_\Delta}{\overset{\mathfrak{C}[-]}{\underset{\sim}{\rightleftarrows}}} (\mathcal{C}at_\Delta)^{Bergner} .$$

For the Quillen equivalence $(\mathfrak{C}[-], N_\Delta)$ see [48] and for the other two Quillen equivalences see [39]. Moreover, for an overview about these and other models of $(\infty, 1)$-categories see [9,10].

There are certain aspects of these model structures and their Quillen equivalences that will play an important role in the coming sections and hence will be quickly reviewed here. First of all we benefit from the fact that right Quillen functors preserve fibrant objects.

Remark 1.1. The functors N_Δ and $t^!$ are Quillen right adjoints and thus for any Kan enriched category $\mathcal{K}$, $N_\Delta(\mathcal{K})$ is a quasi-category and $t^! N_\Delta(\mathcal{K})$ is a complete Segal space.

Next, all three models have a notion of *mapping Kan complex*. For a Kan enriched category it is given as part of a data. We will also need the mapping space of a complete Segal space.

Definition 1.2. Let $\mathcal{W}$ be a complete Segal space. For two objects X, Y (i.e. points in $\mathcal{W}_0$) we define the *mapping Kan complex* as the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{W}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{W}_1 \arrow[d, "(s,t)"] \\ * \arrow[r, "(X,Y)"'] & \mathcal{W}_0 \times \mathcal{W}_0 \end{tikzcd} .$$

Moreover, the Quillen equivalences relate the mapping spaces in an appropriate manner.

Remark 1.3. If we have a Kan enriched category $\mathcal{K}$ then the complete Segal space $t^! N_\Delta(\mathcal{K})$ has the same objects as $\mathcal{K}$ and we have an equivalence of mapping Kan complexes

$$\mathrm{Map}_{\mathcal{K}}(X, Y) \simeq \mathrm{Map}_{t^! N_\Delta \mathcal{K}}(X, Y).$$

Next, we observe that all three models have a notion of *underlying* $(\infty, 1)$-*groupoid*, or *core*. Following the convention of classical category theory, for Kan enriched categories it is the subcategory with the same objects and morphisms equivalences. For the other two models we have the following definitions.

Definition 1.4. Let $\mathcal{Q}$ be a quasi-category. Then we call the maximal Kan complex inside $\mathcal{Q}$ the *core of* $\mathcal{Q}$, denoted $\mathcal{Q}^{core}$.

Definition 1.5. Let $\mathcal{W}$ be a complete Segal space. Then we define the underlying $(\infty, 1)$-groupoid $\mathcal{W}$ as $\mathcal{W}^{core} = \mathcal{W}_0$. Hence, we also call $\mathcal{W}_0$ the *Kan complex of objects*.

Remark 1.6. Given that a complete Segal space $\mathcal{W}$ is itself a simplicial space, we might have expected the underlying $(\infty, 1_)$-groupoid to be a simplicial space as well. However, by the completeness condition, a complete Segal space $\mathcal{W}$ is an $(\infty, 1)$-groupoid if and only if it is *homotopically constant*, meaning that for any map $\delta : [n] \to [m]$ the corresponding map of Kan complexes $\delta^* : \mathcal{W}_n \to \mathcal{W}_m$ is a Kan equivalence.

In other words, $(\infty, 1)$-groupoids are precisely the Reedy fibrant simplicial spaces that are weakly equivalent to constant simplicial spaces. Hence, up to equivalence such complete Segal spaces are characterized by their 0-space, and there is no loss of information.

On the other hand, defining the core of a complete Segal space as a simplicial set has the additional benefit that for a quasi-category $\mathcal{Q}$

$$(t^!\mathcal{Q})^{core} = \mathcal{Q}^{core},$$

meaning the definition of the core for complete Segal spaces coincides with the definition for quasi-categories.

Using the core we can construct fibrant objects out of left Quillen functors.

Remark 1.7. For a given quasi-category $\mathcal{Q}$, $p_1^*(\mathcal{Q})$ is not a complete Segal space (unless $\mathcal{Q}$ has no non-trivial automorphisms). So we need an alternative construction. Thus we introduce $\Gamma(\mathcal{Q})$, defined as the simplicial space

$$\Gamma(\mathcal{Q})_n = (\mathcal{Q}^{\Delta^n})^{core}$$

and observe that it is a complete Segal space with the property that $\Gamma(\mathcal{Q})_{n0} = \mathcal{Q}_n$ (for more details see [39]).

There is one $(\infty, 1)$-category that will play an important role later on.

Notation 1.8. We denote by $\mathcal{K}$an the Kan-enriched category of Kan complexes. On the other hand, we use the notation $\mathcal{S}$ for the quasi-category and the complete Segal space of Kan complexes. Note, in particular, $\mathcal{S} = N_\Delta(\mathcal{K}\text{an})$.

Up until now we have not taken a model-independent approach to our treatment of $(\infty, 1)$-categories. There are now in fact various ways to develop $(\infty, 1)$-categories model-independently, either via 2-categories [68] or via homotopy type theory [66]. However, we will not work at this level of generality and rather set the following convention.

Convention 1.9. The term $(\infty, 1)$*-category* henceforth refers to quasi-categories, Kan enriched categories or complete Segal spaces.

This means, whenever a definition is well-defined for all three models, we will apply it to $(\infty, 1)$-categories. This convention will in particular apply to objects, morphism and the core. Here is another important example of a model independent definition.

Definition 1.10. Let $\mathcal{C}$ be an $(\infty, 1)$-category with finite limits. Then $\mathcal{C}$ is *locally Cartesian closed* if for every morphism $f : x \to y$ the pullback functor

$$f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$$

has a right adjoint $f_* : \mathcal{C}_{/x} \to \mathcal{C}_{/y}$.

We will see further model independent definitions in the next sections.

We also need some understanding of the basics of complete Segal spaces (as given in [63]), as well as over-categories and limits in the context of complete Segal spaces [59].

Notation 1.11. In the context of complete Segal spaces we denote the free arrow as $F(1)$, following notation in [63]. Similarly, $F(0)$ denotes the complete Segal space with one object. In particular, both of these are simplicial spaces.

Definition 1.12. Let $\mathcal{W}$ be a complete Segal space and x an object. Then we can define the slice CSS $\mathcal{W}_{/x}$ as the pullback

$$\begin{tikzcd} \mathcal{W}_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{W}^{F(1)} \arrow[d, "t"] \\ F(0) \arrow[r, "\{x\}"'] & \mathcal{W} \end{tikzcd}$$

Definition 1.13. More generally, let I be a simplicial space and $G : I \to \mathcal{W}$ be a fixed diagram in the CSS $\mathcal{W}$. Then we define the *complete Segal space of cones* as

$$\mathcal{W}_{/G} = F(0) \underset{\mathcal{W}^I}{\times} \mathcal{W}^{I \times F(1)} \underset{\mathcal{W}^I}{\times} \mathcal{W}$$

Moreover, we have the following results about the cones.

Lemma 1.14. *[59] Let $G : I \to \mathcal{W}$ be a diagram. Then for an object L in $\mathcal{W}$ the following are equivalent.*

(1) L is the limit of G.
(2) $\mathcal{W}_{/G}$ has a final object, which is a cone with cone point L.
(3) There exists an equivalence of complete Segal spaces $W_{/L} \simeq \mathcal{W}_{/G}$.

There is one particular instance the previous lemma that we will need later on.

Corollary 1.15. *Let $G : F(1) \to \mathcal{W}$ be a diagram. Then, the object $G(0)$ is the limit of G and so there is an equivalence of complete Segal spaces*

$$\mathcal{W}_{/G(0)} \simeq \mathcal{W}_{/G}.$$

Definition 1.16. If $\mathcal{W}$ is a complete Segal space we denote the *complete Segal space of arrows* as $\mathcal{W}^{F(1)}$ (using Notation 1.11). It comes with a target map

$$t : \mathcal{W}^{F(1)} \to \mathcal{W}$$

which is a Cartesian fibration if and only if $\mathcal{W}$ has finite limits. Concretely, this Cartesian fibration models the functor

$$\begin{aligned} \mathcal{W}_{/-} : \mathcal{W}^{op} &\longrightarrow \mathcal{CSS} \\ x &\longmapsto \mathcal{W}_{/x} \end{aligned}$$

where functoriality is given by pullback.

We will require an important sub-fibration of $\mathcal{W}^{F(1)}$ that we will review here.

Definition 1.17. Let

$$\mathcal{O}_{\mathcal{W}} \hookrightarrow \mathcal{W}^{F(1)}$$

be the subcategory with the same objects and morphisms pullback squares. Then, the composition $t : \mathcal{O}_{\mathcal{W}} \to \mathcal{W}$ is in fact a *right fibration* which corresponds to the functor

$$\begin{aligned} (\mathcal{W}_{/-})^{core} : \mathcal{W}^{op} &\longrightarrow \mathcal{S} \\ x &\longmapsto (\mathcal{W}_{/x})^{core} \end{aligned}$$

Remark 1.18. The fibration $\mathcal{O}_{\mathcal{W}}$ has been covered extensively in the context of quasi-categories [48, Section 6].

1.2. Grothendieck $(\infty, 1)$-topos

Grothendieck topos theory is very ubiquitous in algebraic geometry: Grothendieck 1-toposes in classical algebraic geometry and Grothendieck $(\infty, 1)$-topos theory in derived algebraic geometry. However, we only focus on the fact they are special cases of their elementary counterparts and thus we only give a minimal review.

Definition 1.19 *([65, Proposition 2.2]).* A *Grothendieck topos* is a locally presentable 1-category that satisfies weak descent.

Definition 1.20 *([65, Theorem 6.9]).* A *Grothendieck $(\infty, 1)$-topos* is a presentable $(\infty, 1)$-category that satisfies descent.

Example 1.21. The most simple example is $\mathcal{S}$, the $(\infty, 1)$-category of Kan complexes.

We only need the following important observation relating Grothendieck 1-toposes and $(\infty, 1)$-toposes.

Corollary 1.22 *([65, Proposition 11.2]). Let $\mathcal{G}$ be a Grothendieck $(\infty, 1)$-topos. Then the subcategory of 0-truncated objects, denoted $\tau_0\mathcal{G}$, is a Grothendieck 1-topos.*

We will not require (or mention) any further details about Grothendieck topos theory. We refer the interested reader to [54] for Grothendieck 1-topos theory and [48] (using quasi-categories) or [65] (using model categories) for Grothendieck $(\infty, 1)$-topos theory.

1.3. Elementary $(\infty, 1)$-topos

We will assume familiarity with elementary $(\infty, 1)$-topos theory later on and only review basic definitions. We give one detailed example (Example 1.38) with the hope of giving the interested reader an intuition. The main source for elementary $(\infty, 1)$-topos theory is [61].

Definition 1.23 *([48]).* Let $\mathcal{W}$ be a complete Segal space with finite limits. We denote by

$$\mathcal{S}\mathrm{ub}_{\mathcal{W}} : \mathcal{W}^{op} \to \mathcal{S}\mathrm{et}$$

the functor that takes every object x to the set of (isomorphism classes) of subobjects $\mathcal{S}\mathrm{ub}(x)$.

We say Ω is a *subobject classifier* if it represents $\mathcal{S}\mathrm{ub}$, meaning there is a natural bijection

$$\mathcal{S}\mathrm{ub} \cong \mathrm{Map}_{\mathcal{W}}(-, \Omega).$$

Definition 1.24 *([61]).* Let $\mathcal{W}$ be a complete Segal space with finite limits. A map $p_{\mathcal{U}} : \mathcal{U}_* \to \mathcal{U}$ is a *universe* if the induced map of right fibrations

$$\mathcal{W}_{/\mathcal{U}} \hookrightarrow \mathcal{O}_{\mathcal{W}}$$

is an embedding. Here $\mathcal{O}_{\mathcal{W}}$ denotes the right fibration as discussed in Definition 1.17.

Definition 1.25 *([61]).* Let $\mathcal{W}$ be a complete Segal space with finite limits. We say $\mathcal{W}$ has *sufficient universes* if there exists a collection of universes $\{\mathcal{U}\}_{i \in I}$ such that the embeddings $\mathcal{W}_{/\mathcal{U}} \to \mathcal{O}_{\mathcal{W}}$ are jointly surjective.

Remark 1.26. Intuitively this is telling us that for any morphism $f : A \to B$ in $\mathcal{W}$ there exists a universe $\mathcal{U}$ such that f is a pullback of $p_{\mathcal{U}} : \mathcal{U}_* \to \mathcal{U}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}_* \arrow[d, "p_{\mathcal{U}}"] \\ B \arrow[r] & \mathcal{U} \end{tikzcd} .$$

Definition 1.27 *([61]).* Let $\mathcal{W}$ be a complete Segal space. We say $\mathcal{W}$ is an *elementary complete Segal topos* if the following hold:

(1) It has finite limits and colimits (Lemma 1.14).
(2) It is locally Cartesian closed (Definition 1.10).
(3) It has subobject classifier (Definition 1.23).
(4) It has sufficient universes (Definition 1.25).

The definition we gave only applies to complete Segal spaces, however we will need a definition for other models of $(\infty, 1)$-categories as well.

Definition 1.28. A quasi-category $\mathcal{Q}$ is an *elementary quasi topos* if the complete Segal space $\Gamma(\mathcal{Q})$ (or equivalently $t^!(\mathcal{Q})$) is an elementary complete Segal topos. Moreover, a Kan enriched category $\mathcal{K}$ is an *elementary Kan enriched topos* if the quasi-category $N_\Delta(\mathcal{K})$ is an elementary quasi topos.

Remark 1.29. We say $\mathcal{E}$ is an elementary $(\infty, 1)$-topos when we want to refer to the definition in any of those three models, without specifying which model.

We will need one result about elementary $(\infty, 1)$-toposes later on: the *fundamental theorem of topos theory.*

Proposition 1.30 *([61]). Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos and X an object. Then $\mathcal{E}_{/X}$ is an elementary $(\infty, 1)$-topos with subobject classifier $\pi_2 : \Omega \times X \to X$ and universes $\pi_2 : \mathcal{U} \times X \to X$.*

Remark 1.31. The definition of elementary $(\infty, 1)$-topos given here differs slightly from the definition in [45] in the sense that there the universes are assumed to be closed under certain constructions (such as dependent products).

This definition can be seen as a generalization of a Grothendieck $(\infty, 1)$-topos as well as elementary topos.

Definition 1.32 *([54]).* An *elementary topos* is a locally Cartesian closed category with subobject classifier.

Proposition 1.33 *([61]). Every Grothendieck $(\infty, 1)$-topos is an elementary $(\infty, 1)$-topos.*

Proposition 1.34 *([61]). Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos. Then the subcategory of 0-truncated objects, denoted $\tau_0\mathcal{E}$ is an elementary topos. We call it the underlying elementary topos.*

A general elementary $(\infty, 1)$-topos does not have infinite colimits, as it is not an elementary condition. We thus need an elementary alternative to infinite colimits that allow us to still recover some infinite constructions. This is achieved via the natural number object.

Definition 1.35 *([62, Definition 3.2]).* A *natural number object* in an $(\infty, 1)$-category $\mathcal{E}$ is an object $\mathbb{N}$ along with two morphisms $s : \mathbb{N} \to \mathbb{N}$ and $o : 1 \to \mathbb{N}$, such that the triple $(\mathbb{N}, s, o)$ is initial.

Notice a natural number object is only useful in the context where we don't have infinite colimits, as the next example shows.

Example 1.36 *([62, Proposition 6.9]).* If $\mathcal{E}$ is an elementary $(\infty, 1)$-topos with countable colimits then the infinite coproduct $\coprod_{n\in\mathbb{N}} 1$ is the natural number object. This in particular applies to every Grothendieck $(\infty, 1)$-topos.

This example gives us the following valuable recognition principle for elementary $(\infty, 1)$-toposes that are not Grothendieck $(\infty, 1)$-toposes.

Corollary 1.37. *Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos such that its natural number object $\mathbb{N}$ is not the infinite colimit $\coprod_{n\in\mathbb{N}} 1$. Then $\mathcal{E}$ is not a Grothendieck $(\infty, 1)$-topos*

For more details on natural number objects for elementary $(\infty, 1)$-topos see [62].

In order to give a better understanding of the axioms of an elementary $(\infty, 1)$-topos we give one detailed example.

Example 1.38. Let $\mathcal{S}_\bullet$ be the complete Segal space of Kan complexes (we denote it $\mathcal{S}_\bullet$ rather than $\mathcal{S}$ in this example to make it clear it is a simplicial space). We already know that it is an elementary $(\infty, 1)$-topos, as it is a Grothendieck $(\infty, 1)$-topos (Example 1.21). However, we want to use the fact that Kan complexes are well-understood to explain and give a better understanding of the axioms of an elementary $(\infty, 1)$-topos.

The existence of finite limits and colimits is a very standard condition and will not be discussed further. Similarly, it follows from our classical understanding of Kan complexes that $\mathcal{S}_\bullet$ is locally Cartesian closed. For further explanations see [26,42].

Thus we move on to the existence of a subobject classifier. For that we first need to better understand mono maps in the complete Segal space of Kan complexes. By definition a map of Kan complexes $f : X \to Y$ is mono if and only if the square

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is pullback square. This only holds if the map $f : X \to Y$ is a local equivalence i.e. a map f that restricted to each path component in X becomes an equivalence.

Equivalence classes of such maps are completely determined by a choice of path components in Y (namely we choose the ones we want to be in X) and such a choice is determined by a map $Y \to \{0, 1\}$ (where we map the desired path components to 1 and the rest to 0). Thus $\{0, 1\}$ is a subobject classifier in $\mathcal{S}$.

We now want to gain a better understanding of universes in $\mathcal{S}_\bullet$. Fix a large enough cardinal κ and denote the full sub-complete Segal space of κ-small spaces by $\mathcal{S}^\kappa_\bullet$. Then we say a map of Kan complexes is κ-small if each fiber is κ-small. The sub-space $\mathcal{S}^\kappa_0$ is itself an object in $\mathcal{S}_\bullet$. Thus for any Kan complex K we can define the space $\mathrm{Map}_{\mathcal{S}}(K, \mathcal{S}^\kappa_0)$. What are the points in this Kan complex?

A point in $\mathrm{Map}_{\mathcal{S}}(K, \mathcal{S}^\kappa_0)$ is a map of Kan complexes $f : K \to \mathcal{S}^\kappa_0$. Every Kan complex K is also a simplicial set and thus

$$K = \operatorname*{colim}_{\Delta^n \to K} \Delta^n$$

which means the map $f : K \to \mathcal{S}^\kappa_0$ is just a K-indexed family of maps $\Delta^n \to \mathcal{S}^\kappa_0 = \mathcal{S}^\kappa_{0n}$. But n-cells in $\mathcal{S}^\kappa_0$ are just a choice of n homotopic Kan complexes. In particular, a map $\Delta^0 \to \mathcal{S}^\kappa_0$ is just a choice of κ-small Kan complex. Thus a map $K \to \mathcal{S}^\kappa_0$ is a K-indexed diagram of κ-small Kan complexes, which we can denote by a Kan fibration $\int_K f \twoheadrightarrow K$.

We can recover this map as the pullback of Kan complexes

$$\begin{tikzcd} \int_K f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathcal{S}_*)^\kappa_0 \arrow[d] \\ K \arrow[r] & \mathcal{S}^\kappa_0 \end{tikzcd}$$

where $(\mathcal{S}_*)^\kappa_0$ is the 0-level of the complete Segal space of pointed κ-small Kan complexes, which comes with a forgetful map to $\mathcal{S}^\kappa_0$.

This shows that $\mathcal{S}^\kappa_0$ is a universe in $\mathcal{S}_\bullet$. Using the fact that every map of Kan complexes is κ-small for a large enough cardinal κ proves that $\mathcal{S}_\bullet$ has sufficient universes.

Remark 1.39. The example suggests that we should think of a universe as a “small” copy of the core of the $(\infty, 1)$-category sitting inside itself (the way that $\mathcal{S}^\kappa_0$ is sitting inside $\mathcal{S}_\bullet$).

1.4. Filters

They key idea of the coming section is to use filters in the context of $(\infty, 1)$-categories and $(\infty, 1)$-toposes. Hence, we will give a quick review of the history, applications and definitions we need.

Filters were first defined by Cartan in his study of topologies [15,14]. However, it slowly found its way into many other branches of mathematics and in particular model theory, which is a development that happened in several stages.

First, Skolem was trying to construct new models [73,74], which led him to consider an early notion of ultra products. This was made further precise by Łoś [46], who in particular proved the celebrated *Łoś’s* *theorem*, which is a powerful theorem helping us study ultra products of models. A formal definition of ultra products that is still in use came a little later [25]. For further discussions of ultra products see [17].

Given the unusual ability of the ultra product and Łoś's theorem to relate finite concepts with infinite constructions it has found applications in many different and seemingly unrelated branches of mathematics. For examples, it was used to prove the Ax-Kochen isomorphism theorem [5–7]. However, it has also been used by Barthel, Schlank and Stapleton to further our understanding of chromatic homotopy theory [12], using filters in the context of quasi-categories. Our primary interest in filter quotients is from a topos theoretic perspective as has been discussed in Section 0.

We will only need the definition of a filter and so our review here will cover everything we need about a filter in the next sections.

Definition 1.40. Let $(P, \leq)$ be a partially ordered set. A *filter* F is a subset of P that satisfies the following conditions.

(1) $F \neq \emptyset$.
(2) F is downward directed, meaning that for any two object $x, y \in F$ there exists $z \in F$ such that $z \leq x$ and $z \leq y$.
(3) F is upward closed, meaning that if $x \leq y$ and $x \in F$, then $y \in F$.

Remark 1.41. Notice if P has a maximum element, then every filter in P necessarily includes that maximum element. Thus we could replace condition (1) with the condition that the maximum is in F.

We have the following basic but crucial observation about filters that we will need in the next section.

Remark 1.42. If F is a filter (and thus also a category), then the opposite category F^{op} is a filtered category. This has two important implications about a diagram $D : F^{op} \to \mathcal{C}$ (where $\mathcal{C}$ is any cocomplete category) that we will use in the next section:

(1) The colimits of D commutes with finite limits [53].
(2) Let $\mathcal{F}$ be a class of objects in $\mathcal{C}$ characterized by a right lifting property with respect to a set of monomorphisms with compact codomain. If D takes value in $\mathcal{F}$, then the colimit of D is also in $\mathcal{F}$ For more details see the discussion of *small object arguments* in cofibrantly generated model categories [33, 2.1][31, 11.1]. For an explicit argument in the context of simplicial sets see [26, Theorem 4.1].

Definition 1.43. An *ultrafilter* U of a poset P is a filter of P that is maximal, meaning there does not exist a filter F such that $U \not\subset F \not\subset P$.

Example 1.44. Let P be a poset and $x \in P$. Then the subset $\{y \in P : x \leq y\}$ is a filter. A filter of this form is called a *principal filter*.

Example 1.45. Let P be a poset with finite meets. Then any upward closed non-empty subset closed under finite meets is a filter.

There is only one kind of filter we care about in the next section and so will discuss here in more detail.

Example 1.46. Let $\mathcal{E}$ be a finitely complete $(\infty, 1)$-category (for example an elementary $(\infty, 1)$-topos). Then $\mathcal{S}\mathrm{ub}(1)$ has the structure of a poset with finite meets. Here the meet of two subobjects U, V of 1 is given by their product $U \times V \to 1$. Given the explanation above, any subset of $\mathcal{S}\mathrm{ub}(1)$ that includes 1, is closed under finite products and is upward closed is a filter in $\mathcal{S}ub(1)$. We will call any such filter a *product closed filter*.

Example 1.47. There is one interesting example of a filter we will use in Subsection 3.2. Let $\mathbb{N}$ be the set of natural numbers and $P(\mathbb{N})$ be the power set, which is a poset with the inclusion relation. Then cofinite sets (subsets with finite complement) form a product closed filter (commonly called the *Fréchet filter*). Indeed, if two subsets of $\mathbb{N}$ have finite complement, then their intersection also has a finite complement.

2. The filter quotient of $(\infty, 1)$-categories

The goal of this section is to construct a new elementary $(\infty, 1)$-topos using a filter of $\mathcal{S}ub(1)$ (Example 1.46).

In the first subsection we start more generally with a finitely complete $(\infty, 1)$-category $\mathcal{C}$ and use that to construct a new $(\infty, 1)$-category $\mathcal{C}_\Phi$ along with a quotient functor from the original $(\infty, 1)$-category $P_\Phi : \mathcal{C} \to \mathcal{C}_\Phi$. We will give two main constructions depending on the model of $(\infty, 1)$-category:

(1) One for Kan enriched categories, which leaves the collection of objects untouched and thus resembles the definition of the filter quotient for elementary 1-toposes [36, Example A2.1.13] (Proposition 2.2).
(2) Another for complete Segal spaces and quasi-categories (Definition 2.10, Theorem 2.13).

We will then show these definitions agree with each other.

In the second section we prove that if $\mathcal{E}$ has finite limits, finite colimits, Cartesian closure, subobject classifiers, or universes, then P_Φ will preserve them. This then implies that if $\mathcal{E}$ is an elementary $(\infty, 1)$-topos then $\mathcal{E}_\Phi$ is one as well.

Remark 2.1. Throughout this section the model of $(\infty, 1)$-categories for $\mathcal{E}$ will change. The reader is advised to follow the convention introduced in Subsection 0.9. On other hand Φ will constantly denote a product closed filter in $\mathcal{S}ub(1)$.

2.1. The filter quotient construction

In this subsection we present two methods for constructing the filter quotient, one that applies to Kan enriched categories and one for complete Segal spaces and quasi-categories.

First we will describe the filter quotient construction for Kan-enriched categories. Thus, let $\mathcal{K}$ be a finitely complete Kan enriched category. We will construct a new Kan enriched category which we denote by $\mathcal{K}_\Phi$.

The objects of $\mathcal{K}_\Phi$ are just the objects of $\mathcal{K}$. For the morphisms we first need some preliminary observations:

(1) Φ is a full subcategory of $\mathcal{K}$ and in particular there is a canonical functor $\mathcal{I} : \Phi \hookrightarrow \mathcal{K}$.
(2) Let $- \times - : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be the product functor. We can restrict it via $\mathcal{I}$ to a map

$$\mathcal{I}(-) \times - : \Phi \times \mathcal{K} \to \mathcal{K}$$

(3) We can apply opposite categories

$$\mathcal{I}^{op}(-) \times - : \Phi^{op} \times \mathcal{K}^{op} \to \mathcal{K}^{op}$$

(4) We can product this functor with the identity functor

$$(\mathcal{I}^{op}(-) \times -) \times id_{\mathcal{K}} : \Phi^{op} \times \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}^{op} \times \mathcal{K}$$

(5) Using the fact that $\mathcal{K}$ is a Kan-enriched category, we can post-compose the functor above

$$\mathrm{Map}_{\mathcal{K}}(-,-) \circ ((\mathcal{I}^{op}(-) \times -) \times id_{\mathcal{K}}) : \Phi^{op} \times \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}\mathrm{an}$$

(6) We can construct the adjoint of the functor above

$$\mathcal{F} : \mathcal{K}^{op} \times \mathcal{K} \to Fun(\Phi^{op}, \mathcal{K}\mathrm{an})$$

defined as $\mathcal{F}(X, Y)(U) = \mathrm{Map}_{\mathcal{K}}(X \times U, Y)$.

(7) Now we observe that the diagram Φ^{op} is a filtered diagram in Kan complexes and so the colimit must be indeed a Kan complex, as Kan complexes are characterized via a right lifting property with respect to the maps $\Lambda^n_i \to \Delta^n$ (Remark 1.42). Thus we get a functor

$$\mathrm{colim} \circ \mathcal{F} : \mathcal{K}^{op} \times \mathcal{K} \to Fun(\Phi^{op}, \mathcal{K}\mathrm{an}) \to \mathcal{K}\mathrm{an}$$

(8) We define

$$\mathrm{Map}_{\mathcal{K}_\Phi}(-,-) = \mathrm{colim} \circ \mathcal{F}$$

and so in particular

$$\mathrm{Map}_{\mathcal{K}_\Phi}(X, Y) = \mathrm{colim}(\mathrm{Map}(- \times X, Y) : \Phi^{op} \to \mathcal{K}\mathrm{an})$$

First we prove that the Kan complexes $\mathrm{Map}_{\mathcal{K}_\Phi}(X, Y)$ give us indeed a new Kan enriched category.

Proposition 2.2. *$\mathcal{K}_\Phi$ as defined above is a Kan enriched category.*

Remark 2.3. Part of the proof here is a generalization of the argument given in [36, Example A2.1.13].

Proof. We use the observation that a simplicially enriched category is just a simplicial object in categories where each level has the same set of objects. Thus it suffices to prove that for every natural number k, we get a category with object the objects $\mathcal{K}$ and morphisms sets $\mathrm{Map}_{\mathcal{K}_\Phi}(X, Y)_k$ and that composition respects simplicial boundary maps.

First, recall that colimits in the category of simplicial sets are computed level-wise, thus we have

$$\mathrm{Map}_{\mathcal{K}_\Phi}(X, Y)_k = \mathrm{colim}(\mathrm{Map}(- \times X, Y)_k : \Phi^{op} \to \mathcal{S}\mathrm{et})$$

Let X, Y, Z be three objects, we need a composition map

$$\mathcal{C}\mathrm{omp}(X, Y, Z)_k : \mathrm{Map}_{\mathcal{K}_\Phi}(X, Y)_k \times \mathrm{Map}_{\mathcal{K}_\Phi}(Y, Z)_k \to \mathrm{Map}_{\mathcal{K}_\Phi}(X, Z)_k$$

Let $[f]$ in $\mathrm{Map}_{\mathcal{K}_\Phi}(X, Y)_k$ and $[g]$ in $\mathrm{Map}_{\mathcal{K}_\Phi}(Y, Z)_k$ be two morphisms. Then we can represent them as morphisms $f : X \times U \to Y$ and $g : Y \times V \to Z$. We now define the composition as

$$[g] \circ [f] = [g(f \times id_V)]$$

The colimit construction implies that the composition is well-defined. Moreover, as composition in $\mathcal{K}$ is associative this new composition is associative as well.

We have constructed a collection of categories which all have the same objects and where the hom set of the k-th category is given by $\mathrm{Hom}_{\mathcal{K}_\Phi}(X, Y)_k$. We need to show that this collection is a simplicial object in categories. However, this follows immediately from the fact that for every simplicial map $\delta : [k] \to [l]$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{K}_\Phi}(X,Y)_l \times \mathrm{Hom}_{\mathcal{K}_\Phi}(Y,Z)_l \arrow[r, "\mathcal{C}\mathrm{omp}(X,Y,Z)_l"] \arrow[d, "\delta_* \times \delta_*"] & \mathrm{Hom}_{\mathcal{K}_\Phi}(X,Z)_l \arrow[d, "\delta_*"] \\ \mathrm{Hom}_{\mathcal{K}_\Phi}(X,Y)_k \times \mathrm{Hom}_{\mathcal{K}_\Phi}(Y,Z)_k \arrow[r, "\mathcal{C}\mathrm{omp}(X,Y,Z)_k"] & \mathrm{Hom}_{\mathcal{K}_\Phi}(X,Z)_k \end{tikzcd}$$

Thus we have proven that $\mathcal{K}_\Phi$ is a Kan enriched category. □

Definition 2.4. Let $\mathcal{K}_\Phi$ be the category defined above. We call the Kan enriched category $\mathcal{K}_\Phi$ the *filter quotient of the Kan enriched category* $\mathcal{K}$ *with filter* Φ.

Notice the category $\mathcal{K}_\Phi$ comes with a distinguished functor from $\mathcal{K}$.

Definition 2.5. There is a functor $P_\Phi : \mathcal{K} \to \mathcal{K}_\Phi$. It is the identity map on objects and it takes a morphism $f : X \to Y$ to the class $[f]$ of the morphism $f : X \times 1 \to Y$.

We now will give an analogous construction for finitely complete complete Segal spaces (and by extension, quasi-categories). The goal is to construct the filter quotient as a filtered colimit of complete Segal spaces. This means we have to define our diagram and then prove the resulting colimit is a complete Segal space. Let $\mathcal{W}$ be a fixed complete Segal space with product closed filter Φ of $\mathcal{S}\mathrm{ub}(1)$.

Definition 2.6. Let

$$\mathcal{T}_\Phi : \Phi^{op} \to \mathcal{CSS}$$

be the diagram of complete Segal spaces that corresponds to the Cartesian fibration induced by the pullback

$$\begin{tikzcd} \int \mathcal{T}_\Phi \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{W}^{F(1)} \arrow[d] \\ \Phi \arrow[r] & \mathcal{W} \end{tikzcd}$$

Remark 2.7. Concretely, the diagram Φ takes an object V to the over-category $\mathcal{W}_{/V}$ and morphism $W \leq V$ to the product map

$$- \times W : \mathcal{W}_{/V} \to \mathcal{W}_{/W}$$

We want to take the colimit of this diagram, but first we have to confirm that the colimit is a complete Segal space.

Lemma 2.8. *Let Ψ be a filtered category and let $F : \Psi \to ss\mathcal{S}et$ be a diagram in bisimplicial sets such that $F(X)$ is a complete Segal space for each object X in Ψ. Then the filtered colimit in $ss\mathcal{S}et$ is a complete Segal space.*

Proof. As Ψ is filtered, it suffices to observe that a complete Segal space is characterized via right lifting property against a class of monomorphisms with small codomain (Remark 1.42).

A complete Segal space is by definition a *Reedy fibrant* simplicial space that satisfies the *Segal condition* and *completeness condition*. In [63,64] these conditions are explicitly described as right lifting properties with respect to these three sets of monomorphisms with small codomain:

(1) **Reedy Fibrancy [63, 2.4]:** The maps

$$F(n) \times \Lambda^l_k \coprod_{\partial F(n) \times \Lambda^l_k} \partial F(n) \times \Delta^l \to F(n) \times \Delta^l,$$

where $n \geq 0$, $l \geq 0$. Here $F(n)_{ij} = \mathrm{Hom}_\Delta([i],[n])$, $(\Delta^l)_{ij} = \mathrm{Hom}_\Delta([j],[l])$ and Λ^l_k is the horn inside Δ^l.

(2) **Segal Condition [63, 4.1]:** The maps

$$F(n) \times \partial\Delta^l \coprod_{G(n) \times \partial\Delta^l} G(n) \times \Delta^l \to F(n) \times \Delta^l,$$

where $n \geq 2$, $l \geq 0$. Here $G(n)$ is the *spine* inside $F(n)$.

(3) **Completeness Condition [64, Proposition 10.1]:** The maps

$$Z \times \partial\Delta^l \coprod_{\partial\Delta^l} \Delta^l \to Z \times \Delta^l,$$

where $l \geq 0$. Here Z is defined as the pushout

$$Z = F(3) \coprod_{F(1) \coprod F(1)} (F(0) \coprod F(0)). \quad \square$$

Remark 2.9. An analogous (independent) proof for the special case of ultra products can be found in [12, Lemma 3.13].

We now have all the necessary ingredients to define the filter quotient complete Segal space.

Definition 2.10. Let $\mathcal{W}$ be a complete Segal space with finite limits. We define the *filter quotient* $\mathcal{W}_\Phi$ as

$$\mathcal{W}_\Phi = \operatorname*{colim}_{\Phi^{op}} \mathcal{T}_\Phi$$

and observe by the previous lemma that this is indeed a complete Segal space.

The colimit construction also gives us the desired quotient functor.

Definition 2.11. Let $P_\Phi : \mathcal{W} \to \mathcal{W}_\Phi$ be the inclusion map into the filtered colimit, using the fact that $\mathcal{W}_{/1} = \mathcal{W}$ and $1 \in \Phi$.

Notation 2.12. In order to simplify notation we will usually denote the object $P_\Phi(X)$ in $\mathcal{W}_\Phi$ as X again.

We can give an analogous construction for quasi-categories.

Theorem 2.13. *Let $\mathcal{Q}$ be a quasi-category and Φ be a product closed filter on $\mathcal{S}ub(1)$. Then the simplicial set $\mathcal{Q}_\Phi$ defined as the colimit*

$$\mathcal{Q}_\Phi = \operatorname*{colim}_{\Phi^{op}} \mathcal{T}_\Phi$$

has the following properties:

(1) It is a quasi-category.
(2) It is compatible with the definition for complete Segal spaces, in the sense that for a given complete Segal space $\mathcal{W}$ we have equivalence

$$i_1^*(\mathcal{W}_\Phi) \simeq (i_1^*\mathcal{W})_\Phi$$

We call it the filter quotient of the quasi-category $\mathcal{Q}$ with respect to Φ.

Proof. The key observation is that $\Gamma(\mathcal{Q})$ is a complete Segal space with $\Gamma(\mathcal{Q})_{n0} = \mathcal{Q}_n$ (Remark 1.7). Moreover, any filter Φ in $\mathcal{Q}$ is also a filter in $\Gamma(\mathcal{Q})$. Thus, we can apply the filter construction to get a complete Segal space $\Gamma(\mathcal{Q})_\Phi$. Restricting to the first row via $i_1^*(\Gamma(\mathcal{Q}))$ gives us a quasi-category. But, in the construction of $\Gamma(\mathcal{Q})_\Phi$ we constructed colimits level-wise, thus we have

$$i_1^*(\Gamma(\mathcal{Q})_\Phi) = \operatorname*{colim}_{\Phi^{op}} \mathcal{T}_\Phi = \mathcal{Q}_\Phi$$

This proves that $\mathcal{Q}_\Phi$ is a quasi-category and that it is compatible with the definition for complete Segal spaces. □

The question that remains is whether the construction for Kan enriched categories is compatible with the construction for complete Segal spaces (the same way we just showed that the quasi-category and complete Segal space constructions are compatible). Concretely, let $\mathcal{K}$ be a Kan enriched category. Then can we compare $t^! N_\Delta(\mathcal{K}_\Phi)$ and $(t^! N_\Delta(\mathcal{K}))_\Phi$, noticing the fact that in the former we are using the Kan enriched filter quotient construction, whereas in the latter we are using the complete Segal space filter quotient construction.

Thus the final goal of this subsection is to prove that these constructions are indeed equivalent. For that we need an alternative characterization of the filter construction for Kan enriched categories.

Definition 2.14. Define an equivalence relation $\sim$ on the set of objects of $\mathcal{K}$ as follows

$$X \sim Y \Leftrightarrow \text{there exists a } V \in \Phi \text{ such that } X \times V \simeq Y \times V$$

Definition 2.15. Let $Ob(\mathcal{K}_\Phi^{quot})$ be a set of representatives of the equivalence classes of the equivalence relation given in the previous definition. Moreover, define $\mathcal{K}_\Phi^{quot}$ as the full subcategory of $\mathcal{K}_\Phi$ with set of objects in $Ob(\mathcal{K}_\Phi^{quot})$.

Lemma 2.16. *The inclusion functor $\mathcal{E}xt : \mathcal{K}_\Phi^{quot} \hookrightarrow \mathcal{K}_\Phi$ is an equivalence.*

Proof. The functor is fully faithful by definition, thus it suffices to prove that the inclusion functor is essentially surjective. Let X be an arbitrary object in $\mathcal{K}_\Phi$. Then there exists an object Y in the set $Ob(\mathcal{K}_\Phi^{quot})$ and $V \in \Phi$ such that $X \times V \simeq Y \times V$. However, by definition of $\mathcal{K}_\Phi$, the two maps $\pi_1 : X \times V \to X$ and $id_X : X \to X$ are identified in $\operatorname{Map}_{\mathcal{K}_\Phi}(X, X)$, which implies that $X \simeq X \times V$. Similarly $Y \simeq Y \times V$. Thus we get

$$X \simeq X \times V \simeq Y \times V \simeq Y$$

which proves that $\mathcal{E}\mathrm{xt}$ is essentially surjective. □

Lemma 2.17. *Let X, Y be two objects in $\mathcal{K}$. Then we have a bijection*

$$\operatorname{Map}_{\mathcal{K}_\Phi}(X, Y) \cong \operatorname*{colim}_{V \in \Phi^{op}} \operatorname{Map}_{\mathcal{K}}(X \times V, Y \times V)$$

Proof. It suffices to prove that for every V in Φ we have a functorial bijection

$$\mathrm{Map}_{\mathcal{K}}(X \times V, Y \times V) \xrightarrow{\cong} \mathrm{Map}_{\mathcal{K}}(X \times V, Y)$$

We have the following bijections:

$$\mathrm{Map}_{\mathcal{K}}(X \times V, Y \times V) \cong \mathrm{Map}_{\mathcal{K}}(X \times V, Y) \times \mathrm{Map}_{\mathcal{K}}(X \times V, V) \cong \mathrm{Map}_{\mathcal{K}}(X \times V, Y)$$

where the last bijection follows from the fact that $\mathrm{Map}_{\mathcal{K}}(X \times V, V)$ has either no elements or one element as V is a subobject of the final object and it is clearly not empty (we have $\pi_2 : X \times V \to V$). □

We now want to give a construction of $\mathcal{K}_\Phi$ as a colimit analogous to our definition for complete Segal spaces. The problem is that there is no straightforward way to construct over-categories for Kan enriched categories. Thus, we introduce a way to circumvent that problem. The key is the following observation from complete Segal spaces.

Remark 2.18. If V is (-1)-truncated then the over-category $\mathcal{W}_{/V}$ is just a full subcategory of $\mathcal{W}$. Moreover the objects X in $\mathcal{W}_{/V}$ can be characterized by one of the following equivalent conditions:

(1) There exists a map $X \to V$.
(2) The map $\pi_1 : X \times V \xrightarrow{\simeq} X$ is an equivalence.
(3) There is an equivalence $X \simeq Y \times V$ for some object Y.

This observation gives us the following definition:

Definition 2.19. Let $\mathcal{K}$ be a Kan enriched category and V be a (-1)-truncated object. Then we denote by $\mathcal{K}^{S^V}$ the full subcategory of $\mathcal{K}$ consisting of objects which satisfy $X \simeq X \times V$, where X is an object in $\mathcal{K}$.

Definition 2.20. Let

$$\mathcal{T}_\Phi : \Phi^{op} \to \mathcal{C}\mathrm{at}_\Delta$$

be the Kan enriched functor defined as

$$\mathcal{T}_\Phi(V) = \mathcal{K}^{S^V}$$

Remark 2.21. In Definition 1.2 we described the mapping space as a finite limit diagram. Thus using the argument of Lemma 2.8 we see that for any filtered diagram $F : \Psi \to \mathrm{ss}\mathcal{S}\mathrm{et}$ valued in complete Segal spaces we have an isomorphism of mapping Kan complexes

$$\underset{V \in \Psi}{\mathrm{colim}} \mathrm{Map}_{F(V)}(X, Y) \cong \mathrm{Map}_{\underset{V \in \Psi}{\mathrm{colim}} F}(X, Y)$$

We can now compare the filter construction for complete Segal spaces and Kan enriched categories.

Theorem 2.22. *Let $(t^! N_\Delta) \circ \mathcal{T}_\Phi$ be the composition functor*

$$(t^! N_\Delta) \circ \mathcal{T}_\Phi : \Phi^{op} \to \mathcal{C}at_\Delta \to ss\mathcal{S}et$$

Then there is an equivalence of complete Segal spaces

$$\underset{V \in \Phi^{op}}{\mathrm{colim}} (t^! N_\Delta) \circ \mathcal{T}_\Phi \xrightarrow{\simeq} t^! N_\Delta(\mathcal{K}_\Phi).$$

Proof. We have already shown that there is an equivalence $\mathcal{E}\mathrm{xt} : \mathcal{K}_\Phi^{quot} \to \mathcal{K}_\Phi$. Thus it suffices to prove that there is an equivalence

$$\operatorname*{colim}_{V\in\Phi^{op}} (t^! N_\Delta) \circ \mathcal{T}_\Phi \xrightarrow{\simeq} t^! N_\Delta(\mathcal{K}_\Phi^{quot})$$

As a first step we want to define the map above by defining a cocone. This means we have to construct functors $\mathcal{K}^{S_V} \to \mathcal{K}_\Phi^{quot}$ that are consistent with the functors $- \times V$.

Let

$$\mathcal{F}_V : \mathcal{K}^{S_V} \to \mathcal{K}_\Phi^{quot}$$

be defined on objects by $\mathcal{F}_V(X \times V) = [X]$ and on mapping Kan complexes it is the inclusion map into the colimit

$$\iota_V : \mathrm{Map}_{\mathcal{K}^{S_V}}(X \times V, Y \times V) \to \mathrm{Map}_{\mathcal{K}_\Phi^{quot}}([X],[Y]) = \operatorname*{colim}_{W\in\Phi^{op}} \mathrm{Map}_{\mathcal{K}}(X \times W, Y \times W)$$

that takes a map $f : X \times V \to Y \times V$ to the class $[f]$ in the colimit. Here we used the alternative characterization of mapping Kan complexes in $\mathcal{K}_\Phi$ as proven in Lemma 2.17.

We need to show that the collection of functors $\mathcal{F}_V$ give us a cocone over $\mathcal{K}_\Phi^{quot}$. However, this follows immediately from the fact that for any object $W \in \Phi$ with $W \leq V$, we have $[(X \times V) \times W] = [X \times W]$, which proves that $\mathcal{F}_V(- \times W) = \mathcal{F}_W$. Thus the maps $\mathcal{F}_V$ give us a cocone.

Applying the map $t^! N_\Delta$ we get a cocone of complete Segal spaces, which gives us a universal map out of the colimit

$$\operatorname*{colim}_{V\in\Phi^{op}} (\mathcal{F}_V) : \operatorname*{colim}_{V\in\Phi^{op}} t^! N_\Delta(\mathcal{K}^{S_V}) \to t^! N_\Delta(\mathcal{K}_\Phi^{quot})$$

We want to show that this map is an equivalence.

We know that $t^! N_\Delta(\mathcal{K}_\Phi^{quot})$ and $t^! N_\Delta(\mathcal{K}^{S_V})$ are complete Segal spaces (Remark 1.1). Moreover, we prove in Lemma 2.8 that a filtered colimit of complete Segal spaces is a complete Segal space. This proves that the left hand side is also a complete Segal space. Thus it suffices to prove the map is Dwyer-Kan equivalence [63, Theorem 7.7].

Clearly the map is a surjection on objects, thus we need to show we have an equivalence of mapping Kan complexes. Fix two objects X, Y in $\mathcal{K}$. Then we have a diagram of Kan complexes

$$\begin{tikzcd}
\operatorname*{colim}_{V\in\Phi^{op}} \mathrm{Map}_{t^! N_\Delta(\mathcal{K}^{S_V})}(X \times V, Y \times V) \arrow[r] \arrow[d, "(1)"', "\simeq"] & \mathrm{Map}_{t^! N_\Delta(\mathcal{K}_\Phi^{quot})}([X],[Y]) \arrow[d, "(1)"', "\simeq"] \\
\operatorname*{colim}_{V\in\Phi^{op}} \mathrm{Map}_{\mathcal{K}^{S_V}}(X \times V, Y \times V) \arrow[r] \arrow[d, "(2)"', "\cong"] & \mathrm{Map}_{\mathcal{K}_\Phi^{quot}}([X],[Y]) \arrow[d, "(3)"', "\simeq"] \\
\operatorname*{colim}_{V\in\Phi^{op}} \mathrm{Map}_{\mathcal{K}}(X \times V, Y \times V) \arrow[r, "\simeq", "(4)"'] & \mathrm{Map}_{\mathcal{K}_\Phi}(X, Y)
\end{tikzcd}$$

where the numbered morphisms are equivalences for the following reasons:

(1) The map $t^! N_\Delta$ takes mapping Kan complexes to equivalent mapping Kan complexes (Remark 1.3).
(2) $\mathcal{K}^{S_V}$ is a full subcategory $\mathcal{K}$, which gives us a bijection of mapping spaces (Definition 2.19).

(3) The functor $\mathcal{E}\mathrm{xt}$ is an equivalence of Kan enriched categories (Lemma 2.16).
(4) This is an alternative characterization of mapping Kan complexes in $\mathcal{K}_\Phi$ (Lemma 2.17).

This implies that the top horizontal map is also an equivalence of Kan complexes, which proves that we have a Dwyer-Kan equivalence of complete Segal spaces. □

2.2. The filter quotient is an elementary $(\infty, 1)$-topos

The next step is to prove that if $\mathcal{E}$ is an elementary $(\infty, 1)$-topos then $\mathcal{E}_\Phi$ is one as well. The key step of the proof is to show that $\mathcal{E}_\Phi$ also has sufficient universes. For that we need a better understanding of the filter quotient of the complete Segal space of cones.

Lemma 2.23. *Let I be a finite simplicial space (i.e. with finitely many non-degenerate cells) and $\mathcal{W}$ a finitely complete complete Segal space with filter Φ. Let Φ be the induced filter on $\mathcal{W}^I$ consisting of constant functor $I \to \mathcal{W}$ with value $V \in \Phi$. Then, we have*

(1) an equivalence of functor complete Segal spaces

$$(\mathcal{W}^I)_\Phi \simeq (\mathcal{W}_\Phi)^I$$

(2) an equivalence of cocones

$$(\mathcal{W}_{/I})_\Phi \simeq (\mathcal{W}_\Phi)_{/I}$$

Proof. (1) First notice we have an equivalence

$$(\mathcal{W}^I)_{/V} \simeq (\mathcal{W}_{/V})^I$$

as they both are the full subcategory of $\mathcal{W}^I$ consisting of diagrams that take value in the full subcategory $\mathcal{W}_{/V}$. Thus, we get the equivalence

$$(\mathcal{W}^I)_\Phi = \operatorname*{colim}_{V \in \Phi^{op}} (\mathcal{W}^I)_{/V} \simeq \operatorname*{colim}_{V \in \Phi^{op}} (\mathcal{W}_{/V})^I \simeq (\operatorname*{colim}_{V \in \Phi^{op}} \mathcal{W}_{/V})^I = (\mathcal{W}_\Phi)^I$$

where the last equivalence follows from the fact that filtered colimits commute with exponents by finite simplicial spaces.

(2) We will now use our explicit description of cocones, the fact that filtered colimits commute with finite limits and the previous part to get the desired result:

$$(\mathcal{W}_{/I})_\Phi \simeq (\mathcal{W}^{F(1)\times I} \times_{\mathcal{W}^I} *)_\Phi \simeq (\mathcal{W}^{F(1)\times I})_\Phi \times_{(\mathcal{W}^I)_\Phi} * \simeq (\mathcal{W}_\Phi)^{F(1)\times I} \times_{(\mathcal{W}_\Phi)^I} * = (\mathcal{W}_\Phi)_{/I} \quad \square$$

Finally, we need a better understanding of $(\mathcal{W}_\Phi)_{/X}$ and $\mathcal{O}_{\mathcal{W}_\Phi}$.

Lemma 2.24. *Let $\mathcal{W}$ be a complete Segal space and let Φ be a filter of subobjects. Then for any object X, there is an equivalence*

$$(\mathcal{W}_\Phi)_{/X} \simeq \operatorname*{colim}_{V \in \Phi^{op}} (\mathcal{W}_{/X \times V}).$$

Proof. We have

$$
\begin{aligned}
\underset{V\in\Phi^{op}}{\operatorname{colim}}(\mathcal{W}_{/X\times V}) &\simeq \underset{V\in\Phi^{op}}{\operatorname{colim}}(\mathcal{W}_{/V})_{/\pi_2:X\times V\to V} && \text{Corollary 1.15}\\
&= \underset{V\in\Phi^{op}}{\operatorname{colim}}(\mathcal{W}_{/V})^{F(1)} \underset{\mathcal{W}_{/V}}{\times} F(0) && \text{Definition 1.12}\\
&\simeq (\underset{V\in\Phi^{op}}{\operatorname{colim}}\mathcal{W}_{/V})^{F(1)} \underset{\underset{V\in\Phi^{op}}{\operatorname{colim}}\mathcal{W}_{/V}}{\times} F(0) && \text{Lemma 2.8/Lemma 2.23}\\
&\simeq \mathcal{W}_{\Phi}^{F(1)} \underset{\mathcal{W}_{\Phi}}{\times} F(0) = (\mathcal{W}_{\Phi})_{/X} && \text{Definition 2.10} \quad \square
\end{aligned}
$$

Lemma 2.25. *Let $\mathcal{W}$ be a complete Segal space and let Φ be a filter of subobjects. Then there is an equivalence of right fibrations*

$$\mathcal{O}_{\mathcal{W}_\Phi} \simeq \operatorname{colim}_{V\in\Phi^{op}}\mathcal{O}_{\mathcal{W}_{/V}}$$

Proof. By definition the right fibration $\mathcal{O}_{\mathcal{W}_\Phi}$ corresponds to the functor $((\mathcal{W}_\Phi)_{/-})^{core}$ and the right fibration $\operatorname{colim}_{V\in\Phi^{op}}\mathcal{O}_{\mathcal{W}_{/V}}$ corresponds to the functor $\operatorname{colim}_{V\in\Phi^{op}}(\mathcal{W}_{/V\times -})^{core}$, hence it suffices to prove the functors are naturally equivalent. Given the functoriality of colimit it suffices to prove that space $((\mathcal{W}_\Phi)_{/X})^{core}$ is equivalent to $\operatorname{colim}_{V\in\Phi^{op}}(\mathcal{W}_{/X\times V})^{core}$. However, this immediately follows from the previous lemma. □

Theorem 2.26. *Let $\mathcal{W}$ be a complete Segal space with finite limits. Then the functor*

$$P_\Phi : \mathcal{W} \to \mathcal{W}_\Phi$$

preserves

(1) finite limits and colimits
(2) subobject classifiers
(3) locally Cartesian closure
(4) universes

In particular if $\mathcal{E}$ is an elementary $(\infty,1)$-topos, then $\mathcal{E}_\Phi$ is one as well.

Proof. We need to prove that $\mathcal{W}_\Phi$ satisfies the three conditions given in Definition 1.27. We will confirm each separately.

Finite Limits and Colimits: It suffices to prove that $\mathcal{P}_\Phi$ preserves finite limits and the case for finite colimits is analogous. We want to prove that the final object 1 in $\mathcal{W}$ is also the final object in $\mathcal{W}_\Phi$. We have

$$\operatorname{Map}_{\mathcal{W}_\Phi}(X,1) = \underset{V\in\Phi^{op}}{\operatorname{colim}}\operatorname{Map}_{\mathcal{W}}(X\times V,1) \overset{(1)}{\simeq} \underset{V\in\Phi^{op}}{\operatorname{colim}}\Delta[0] \overset{(2)}{=} \Delta[0]$$

where are using the following facts:

(1) 1 is final in $\mathcal{W}$ and thus $\operatorname{Map}_{\mathcal{W}}(X\times U,1)$ is contractible and the colimit construction is homotopy invariant.
(2) filtered colimits of the final object is again final.

We now want to prove that $\mathcal{W}_\Phi$ has I-shaped limits. We have the following diagram

$$\begin{tikzcd} \mathcal{W}_{/I} \arrow[r, "(P_{/I})_\Phi"] \arrow[dr, "(P_\Phi)_{/I}"'] & (\mathcal{W}_{/I})_\Phi \arrow[d, "\simeq"] \\ & (\mathcal{W}_\Phi)_{/I} \end{tikzcd}$$

By the previous lemma the vertical map is an equivalence. By the previous paragraph $(P_{/I})_\Phi$ preserves final objects, which then implies that $(P_\Phi)_{/I}$ also preserves final objects. But a final object in cone category is just the limit and thus P_Φ preserves all finite limits.

Subobject Classifier: Let Ω be the subobject classifier in $\mathcal{W}$. We want to prove that Ω is a subobject classifier in $\mathcal{W}_\Phi$. First, we have to determine the subobjects in $\mathcal{W}_\Phi$. Let $[f]$ be a morphism in $\mathrm{Map}_{\mathcal{W}_\Phi}(X, Y)$. Then $[f]$ is mono if and only if the following is a pullback square in $\mathcal{W}_\Phi$

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "{[f]}"] \\ X \arrow[r, "{[f]}"'] & Y \end{tikzcd}$$

However, we just proved that a pullback in $\mathcal{W}_\Phi$ is evaluated as a pullback of any representative. Thus this is equivalent to the following being a pullback square

$$\begin{tikzcd} X \times U \arrow[r, "id_{X\times U}"] \arrow[d, "id_{X\times U}"'] & X \times U \arrow[d, "f"] \\ X \times U \arrow[r, "f"] & Y \end{tikzcd}$$

where $f : X \times U \to Y$ is any representative of $[f]$. However, this is equivalent to $f : X \times U \to Y$ being mono in $\mathcal{W}$. Thus we just proved that $[f]$ in $\mathcal{W}_\Phi$ is mono if and only if there exists a representative $f : X \times U \to Y$ that is mono in $\mathcal{W}$, which we can state as a bijection

$$\mathcal{S}\mathrm{ub}_{\mathcal{W}_\Phi}(X) \cong \operatorname*{colim}_{V \in \Phi^{op}} \mathcal{S}\mathrm{ub}_{\mathcal{W}}(X \times V)$$

We will use this to prove Ω is a subobject classifier. We have the following bijections

$$\mathrm{Map}_{\mathcal{W}_\Phi}(X, \Omega) = \operatorname*{colim}_{V \in \Phi^{op}} \mathrm{Map}_{\mathcal{W}}(X \times V, \Omega \times V) \cong \operatorname*{colim}_{V \in \Phi^{op}} \mathrm{Map}_{\mathcal{W}}(X \times V, \Omega) \cong \operatorname*{colim}_{V \in \Phi^{op}} \mathcal{S}\mathrm{ub}_{\mathcal{W}}(X \times V) \cong \mathcal{S}\mathrm{ub}_{\mathcal{W}_\Phi}(X)$$

where the first isomorphism follows from Lemma 2.17 and the last step follows from the bijection given in the last paragraph. Thus Ω is also the subobject classifier in $\mathcal{W}_\Phi$.

Locally Cartesian Closure: Let $f : X \times U \to Y \times U$ be a morphism in $\mathcal{W}_\Phi$. We already proved that P_Φ preserves finite limits, hence it suffices to prove that if

$$f^* : \mathcal{W}_{/Y \times U} \to \mathcal{W}_{/X \times U}$$

has a right adjoint, then the induced functor

$$f^* : (\mathcal{W}_\Phi)_{/Y \times U} \to (\mathcal{W}_\Phi)_{/X \times U}$$

has a right adjoint.

By [27, Corollary 2.3] it suffices to prove the following statement: For a morphism $g : Z \times U \to X \times U$ define the complete Segal space $(\mathcal{W}_\Phi)_{/f^*(-)\to Z\times U}$ as the pullback

$$\begin{tikzcd}
(\mathcal{W}_\Phi)_{/f^*(-)\to Z\times U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathcal{W}_\Phi)_{/Z\times U} \arrow[d, "g_*"] \\
(\mathcal{W}_\Phi)_{/Y\times U} \arrow[r, "f^*"] & (\mathcal{W}_\Phi)_{/X\times U}
\end{tikzcd}.$$

Then f^* has a right adjoint if and only if $(\mathcal{W}_\Phi)_{/f^*(-)\to Z\times U}$ has a final object.

Using the fact that filtered colimits commute with pullbacks, this pullback square is the filtered colimit of the pullback squares

$$\begin{tikzcd}
\mathcal{W}_{/f^*(-)\to Z\times U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathcal{W})_{/Z\times U} \arrow[d, "g_*"] \\
\mathcal{W}_{/Y\times U} \arrow[r, "f^*"] & (\mathcal{W})_{/X\times U}
\end{tikzcd}.$$

By assumption $f^* : \mathcal{W}_{/Y\times U} \to \mathcal{W}_{/X\times U}$ has a right adjoint and so $\mathcal{W}_{/f^*(-)\to Z\times U}$ has a final object. Hence, by the previous step, the filtered colimit, $(\mathcal{W}_\Phi)_{/f^*(-)\to Z\times U}$, also has a final object.

Universes: Let $\mathcal{U}$ be a universe in $\mathcal{W}$. We want to prove that $\mathcal{U}$ is a universe in $\mathcal{W}_\Phi$, meaning there is an embedding

$$(\mathcal{W}_\Phi)_{/\mathcal{U}} \hookrightarrow \mathcal{O}_{\mathcal{W}_\Phi}.$$

First, recall that a filtered colimit of embeddings is an embedding (as being an embedding is a finite limit condition, which commutes with filtered colimits Remark 1.42). Next, by Lemma 2.24, we have

$$(\mathcal{W}_\Phi)_{/\mathcal{U}} \simeq \operatorname*{colim}_{V\in\Phi^{op}} \mathcal{W}_{/V\times\mathcal{U}}$$

and, by Lemma 2.25,

$$\mathcal{O}_{\mathcal{W}_\Phi} \simeq \operatorname*{colim}_{V\in\Phi^{op}} \mathcal{O}_{\mathcal{W}_{/V}}.$$

Hence, it suffices to construct an embedding

$$\mathcal{W}_{/V\times\mathcal{U}} \hookrightarrow \mathcal{O}_{\mathcal{W}_{/V}}.$$

However, the existence of this embedding follows from the fact that $\pi_2 : \mathcal{U} \times V \to V$ is a universe in $\mathcal{W}_{/V}$ (Proposition 1.30).

W_Φ is an Elementary $(\infty, 1)$-Topos: Finally, we want to prove that if $\mathcal{W}$ is an elementary complete Segal topos, then $\mathcal{W}_\Phi$ is a complete Segal topos as well. Following the previous steps, all that remains is to prove $\mathcal{W}_\Phi$ has sufficient universes. Let $[f] : X \to Y$ be an arbitrary map in $\mathcal{W}_\Phi$. Then it can be represented by a map $f : X \times U \to Y \times U$ in $\mathcal{W}$. As $\mathcal{W}$ has sufficient universes f is classified by some universe $\mathcal{U}$. Given that pullbacks in $\mathcal{W}_\Phi$ are computed by pullbacks in $\mathcal{W}$, it follows that $[f]$ is classified by $\mathcal{U}$ in $\mathcal{W}_\Phi$ as well.

This proves that if $\mathcal{E}$ is an elementary complete Segal topos then $\mathcal{E}_\Phi$ is one as well. As we defined other models of $(\infty, 1)$-categories to be an elementary $(\infty, 1)$-topos if the corresponding complete Segal space is one, the same result holds for the other two models and hence we are done. □

Remark 2.27. Some parts (such as preservation of limits and colimits and existence of adjoints) restricted to the case of ultra products were already proven in [12, Lemma 3.17].

Notice there are trivial examples of filter quotients.

Example 2.28. Let Φ be the minimal filter (which only includes 1 itself). Then $\mathcal{E}_\Phi = \mathcal{E}$. On the other hand if $\Phi = \mathcal{S}\mathrm{ub}(1)$, then $\mathcal{E}_\Phi$ is the trivial category with one object and identity map. This immediately follows from the fact that in this case the poset Φ^{op} has a final object and thus all colimits are just evaluation at that final object. However, that final object is just 0 and $\mathcal{E}_{/0}$ is the trivial category.

We finish this section by observing that this construction lifts the construction of the underlying toposes.

Theorem 2.29. *Let $\mathcal{E}$ be a locally Cartesian closed $(\infty, 1)$-category with subobject classifier and $\tau_0\mathcal{E}$ the elementary topos of 0-truncated objects. Moreover, let Φ be a product closed filter of $\mathcal{S}ub(1)$. Then there exists a functor*

$$\tau_0(\mathcal{E})_\Phi \to \tau_0(\mathcal{E}_\Phi)$$

induced by $P_\Phi : \mathcal{E} \to \mathcal{E}_\Phi$, which is an equivalence of 1-categories.

Proof. First we determine 0-truncated objects in $\mathcal{E}_\Phi$. Let X be an object in $\mathcal{E}_\Phi$. Then X is 0-truncated if and only if the following commutative square

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\Delta_X"] \\ X \arrow[r, "\Delta_X"'] & X \times X \end{tikzcd}$$

is a pullback in $\mathcal{E}_\Phi$. This is equivalent to

$$\begin{tikzcd} X \times U \arrow[r, "id_{X\times U}"] \arrow[d, "id_{X\times U}"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \times U \arrow[d, "\Delta_X \times id_U"] \\ X \times U \arrow[r, "\Delta_X \times id_U"'] & (X \times X) \times U \end{tikzcd}$$

being a pullback in $\mathcal{E}$ for some object U in Φ. But the map $\Delta_X \times id_U$ is isomorphic to $\Delta_{X\times U}$ (as we have $U \cong U \times U$). This means the square above being a pullback in $\mathcal{E}$ is equivalent to $X \times U$ being 0-truncated in $\mathcal{E}$. Thus we have proven that X is 0-truncated in $\mathcal{E}_\Phi$ if and only if it $X \times U$ is 0-truncated in $\mathcal{E}$ for some U in Φ.

In particular, if X is 0-truncated in $\mathcal{E}$ then $P_\Phi(X)$ is an object in the subcategory $\tau_0(\mathcal{E}_\Phi)$, which means we have the following commutative diagram

$$\begin{tikzcd} \tau_0(\mathcal{E}) \arrow[r, "\tau_0(P_\Phi)"] \arrow[d] & \tau_0(\mathcal{E}_\Phi) \arrow[d] \\ \mathcal{E} \arrow[r, "P_\Phi"] & \mathcal{E}_\Phi \end{tikzcd}$$

Notice Φ is also a filter in $\tau_0(\mathcal{E})$ and so we can apply the filter quotient construction to get a diagram

$$\begin{tikzcd} \tau_0(\mathcal{E}) \arrow[r, "\tau_0(P_\Phi)"] \arrow[d, "P_\Phi"] & \tau_0(\mathcal{E}_\Phi) \\ \tau_0(\mathcal{E})_\Phi \arrow[ur, dashed] & \end{tikzcd}$$

We now want to show that we can lift the diagram to a functor $P_\Phi : \tau_0(\mathcal{E})_\Phi \to \tau_0(\mathcal{E}_\Phi)$.

The map is already defined for objects, but we need to define it for Hom sets, which means we need a map

$$\mathrm{Hom}_{(\tau_0\mathcal{E})_\Phi}(X,Y) \to \mathrm{Hom}_{\tau_0(\mathcal{E}_\Phi)}(X,Y)$$

Using their definition as colimits this means we need a map

$$\operatorname*{colim}_{V\in\Phi^{op}} \mathrm{Hom}_{\tau_0\mathcal{E}}(X\times V,Y) \to \operatorname*{colim}_{V\in\Phi^{op}} \mathrm{Hom}_{\mathcal{E}}(X\times V,Y)$$

But $\tau_0\mathcal{E}$ is a full subcategory of $\mathcal{E}$ and so we can take this map to be the identity.

We now want to prove that this induced functor is an equivalence of categories. The functor is fully faithful by construction, so we only need to prove it is essentially surjective. Let X be an object in $\tau_0(\mathcal{E}_\Phi)$. Then there exists a U such that $X\times U$ is 0-truncated. This means that $X\times U$ is an object in $\tau_0(\mathcal{E})$ and thus also an object in $(\tau_0(\mathcal{E}))_\Phi$. Finally by construction of the filter quotient, the map $U\times X \to X$ is an equivalence. Thus we have proven that every object X is equivalent to an object in the image, namely $X\times U$. $\square$

One interesting implication of this result is the preservation of natural number objects.

Theorem 2.30. *Let $\mathcal{E}$ be an elementary $(\infty,1)$-topos and $\mathbb{N}$ be the natural number object. Moreover, let Φ be a product closed filter on $\mathcal{E}$. Then $P_\Phi(\mathbb{N})$ is the natural number object in $\mathcal{E}_\Phi$.*

Proof. A natural number object $\mathbb{N}$ is always 0-truncated and thus lives in $\tau_0(\mathcal{E})$. By [36] $P_\Phi(\mathbb{N})$ is thus also a natural number object in the filter quotient elementary topos $\tau_0(\mathcal{E})_\Phi$. By the equivalence above, it is thus also a natural number object in $\tau_0(\mathcal{E}_\Phi)$. But by [62, Lemma 3.13, Theorem 5.3] every natural number object in the underlying elementary topos is a natural number object in the whole elementary $(\infty,1)$-topos. Thus $\mathcal{E}_\Phi$ has a natural number object, namely $P_\Phi(\mathbb{N})$. $\square$

We can now combine this theorem with Corollary 1.22 to get a powerful method to construct many elementary $(\infty,1)$-toposes that are not Grothendieck $(\infty,1)$-toposes.

Corollary 2.31. *Let $\mathcal{E}$ be an elementary $(\infty,1)$-topos and Φ be a product closed filter such that the filter quotient elementary topos $\tau_0(\mathcal{E})_\Phi$ is not a Grothendieck topos. Then the filter quotient $\mathcal{E}_\Phi$ is an elementary $(\infty,1)$-topos that is not a Grothendieck $(\infty,1)$-topos.*

Remark 2.32. The conditions given in the previous corollary are sufficient conditions for $\mathcal{E}$ to be elementary without being Grothendieck. It is not known whether the conditions are also necessary i.e. if there is an elementary $(\infty,1)$-topos that is not a Grothendieck $(\infty,1)$-topos, but for which the underlying elementary topos is a Grothendieck topos.

In the next section we will focus on specific filter quotients and then use it, combined with the corollary above, to construct elementary $(\infty,1)$-toposes are not Grothendieck $(\infty,1)$-toposes.

3. Filter products

In this section we want to restrict our attention to specific filter quotients: *filter products*. In the first subsection we study general filter product. In the second subsection we focus on one specific example of a filter product.

3.1. Łoś's theorem for equivalences

Throughout this subsection $\mathcal{C}$ is a fixed finitely complete $(\infty,1)$-category such that 1 only has two subobjects, I is a set and Φ is a product closed filter of $P(I)$, the power set of I. Notice $\mathcal{C}^I$ is also finitely complete. We want to use the filter Φ of $P(I)$ to build a filter on $\mathcal{C}^I$.

First, we observe by assumption there is a bijection

$$\{0,1\} \cong \mathcal{S}ub_{\mathcal{C}}(1)$$

Moreover, we have

$$\mathcal{S}ub_{\mathcal{C}^I}(1) = \mathcal{S}ub_{\mathcal{C}}(1)^I = \{0,1\}^I = P(I)$$

and so a filter Φ on $P(I)$ is automatically a filter on $\mathcal{C}^I$ and we can use it to define the filter quotient $(\mathcal{C}^I)_\Phi$.

Definition 3.1. Let $\mathcal{C}$ be a finitely complete $(\infty,1)$-category such that 1 has only two subobjects, I a set and Φ a product closed filter of $P(I)$. Then the resulting filter quotient $(\mathcal{C}^I)_\Phi$ is called the *filter product* and denoted $\prod_\Phi \mathcal{C}$.

Remark 3.2. If the filter Φ is an ultrafilter, then $\prod_\Phi \mathcal{C}$ is often called the *ultra product*.

Our goal is to prove analogues of Łoś's theorem [46] for homotopies and equivalences in a filter product. We will actually prove a more general statement about filter quotients, from which the desired results for filter products will follow as an immediate corollary.

Lemma 3.3. *Let Φ^{op} be a filtered diagram and $F : \Phi^{op} \to \mathcal{S}$ be a diagram of spaces. Then*

(1) We have isomorphism of sets

$$\pi_0(\mathrm{colim} F) \cong \mathrm{colim}(\pi_0(F))$$

(2) Two points $x, y \in \mathrm{colim}(F)$ are homotopic if and only if there exists $S \in \Phi^{op}$ and $x', y' \in F(S)$ such that $x' \simeq y'$ and $\iota(x') = x$, $\iota(y') = y$, where $\iota : F(S) \to \mathrm{colim}(F)$ is the universal cocone map.

Proof. The first part follows immediately from the fact that π_0 commutes with filtered colimits. Given that the second part focuses on the existence of equivalences we can restrict our attention to π_0 of that diagram.

Thus we have to prove that $x = y$ in $\mathrm{colim}(\pi_0(F))$ if and only if there exists S and $x', y' \in \pi_0(F(S))$ such that $\iota(x') = x, \iota(y') = y$ and $x' = y'$. However, that is just the definition of the colimit. □

Theorem 3.4. *Let $\mathcal{C}$ be a finitely complete $(\infty,1)$-category and Φ a filter. Then for two morphisms $f, g : X \to Y$ in $\mathcal{E}_\Phi$*

$$f \simeq g \ in \ \mathcal{C}_\Phi \Leftrightarrow \exists U \in \Phi(f \times id_U \simeq g \times id_U \ in \ \mathcal{C})$$

Proof. Observe that

$$\mathrm{Map}_{\mathcal{C}_\Phi}(X,Y) = \operatorname*{colim}_{V\in\Phi^{op}} \mathrm{Map}_{\mathcal{C}}(X\times V, Y\times V)$$

So, the result follows immediately from Lemma 3.3 as two morphisms are equivalent if and only if they are in the same path-component. □

Using the result for filter products we get the following corollary.

Corollary 3.5. *Let $(f_i)_{i\in I}, (g_i)_{i\in I} : (X_i)_{i\in I} \to (Y_i)_{i\in I}$ be two maps. Then $(f_i)_{i\in I} \simeq (g_i)_{i\in I}$ in $\prod_\Phi \mathcal{C}$ if and only if*

$$\{i\in I : f_i \simeq g_i \text{ in } \mathcal{C}\} \in \Phi$$

Proof. Assume $\{i\in I : f_i \simeq g_i \text{ in } \mathcal{C}\} \in \Phi$. Then we can use Theorem 3.4 with $U = \{i\in I : f_i \simeq g_i \text{ in } \mathcal{C}\}$ to deduce that $(f_i)_{i\in I} \simeq (g_i)_{i\in I}$.

On the other side, assume that $(f_i)_{i\in I} \simeq (g_i)_{i\in I}$. Then, by Theorem 3.4, there exists a $U\in\Phi$ such that for all $i\in U$, $f_i\simeq g_i$, meaning that

$$U \subseteq \{i\in I : f_i\simeq g_i \text{ in } \mathcal{C}\}.$$

However, by definition of a filter (Definition 1.40), this implies that $\{i\in I : f_i \simeq g_i \text{ in } \mathcal{C}\}\in\Phi$. □

Theorem 3.6. *Let $\mathcal{C}$ be a finitely complete $(\infty,1)$-category, Φ a filter on $\mathcal{C}$. Then a map f in $\mathcal{C}_\Phi$ is an equivalence if and only if there exists $U\in\Phi$ such that $f\times id_U$ is an equivalence in $\mathcal{C}$.*

Proof. Let $\mathcal{C}^{core}$ be the underlying $(\infty,1)$-groupoid of $\mathcal{C}$. Then we have an equivalence

$$(\mathcal{C}_\Phi)^{core} \simeq (\mathcal{C}^{core})_\Phi$$

Indeed, this follows from the fact that in the complete Segal space model, we have $\mathcal{C}^{core} = \mathcal{C}_0$ (Definition 1.5) and the filter quotient is defined as a level-wise colimit (Definition 2.10).

A morphism f in $\mathcal{C}_\Phi$ is an equivalence if and only if it is in $(\mathcal{C}_\Phi)^{core}$, which is equivalent to $(\mathcal{C}^{core})_\Phi$. But, by definition of filtered colimits, this is equivalent to $f\times id_U$ being invertible for some $U\in\Phi$. □

Again, we can restrict our attention to filter products.

Corollary 3.7. *Let $\mathcal{C}$ be a finitely complete $(\infty,1)$-category such that 1 has two subobjects, I a set and Φ a filter on $P(I)$. Then a map $(f_i)_{i\in S}$ in $\prod_\Phi \mathcal{C}$ is an equivalence if and only if*

$$\{i\in I : f_i \textit{ is an equivalence }\}\in\Phi$$

Proof. Assume $\{i\in I : f_i \text{ is an equivalence }\}\in\Phi$. Then we can use Theorem 3.6 with $U = \{i\in I : f_i \text{ is an equivalence }\}$ to deduce that $(f_i)_{i\in I}$ is an equivalence.

On the other side, assume that $(f_i)_{i\in I}$ is an equivalence. Then, by Theorem 3.6, there exists a $U\in\Phi$ such that for all $i\in U$, f_i is an equivalence, meaning that

$$U\subseteq \{i\in I : f_i \text{ is an equivalence }\}.$$

However, by definition of a filter (Definition 1.40), this implies that $\{i\in I : f_i \text{ is an equivalence }\}\in\Phi$. □

Remark 3.8. We can use these results to characterize truncated objects in a filter quotient. See [60, Subsection 6.2] for more details.

Remark 3.9. We only added the restriction that 1 in $\mathcal{C}$ has only two subobject in order to guarantee that Φ is also a filter on $\mathcal{C}^I$. With enough care that condition could possibly be relaxed.

Remark 3.10. Notice in the actual Łoś's theorem, the filter needs to be an ultrafilter, which we did not assume. That is because Łoś's theorem (as used in model theory) holds for all formulas. In particular, it holds for formulas that include negations, which require ultrafilters, as they have a certain closure property under set complements.

On the other hand we prove very particular results, none of which include any negation and that is why these statements hold for any filter. I want to thank Peter Lumsdaine for making me aware of this fact.

We have now gathered enough background to give examples of non-presentable $(\infty, 1)$-toposes.

3.2. Examples of filter-quotients that are not Grothendieck $(\infty, 1)$*-toposes*

In this subsection we finally give examples of elementary $(\infty, 1)$-toposes that are not Grothendieck toposes. First we use our general knowledge to construct a whole class of examples. Then we focus on one specific example and make the construction as explicit as possible.

Example 3.11. As before, let $\mathcal{K}\mathrm{an}$ be the Kan enriched category of Kan complexes, let I be a set and Φ a filter on $P(I)$. Notice that we indeed have $\mathcal{S}\mathrm{ub}(1) = \{0, 1\}$. Thus, we can define the filter product $\prod_\Phi \mathcal{K}\mathrm{an}$ and by Theorem 2.26 it is still an elementary $(\infty, 1)$-topos. By Theorem 2.29 the underlying elementary topos is $\prod_\Phi \mathcal{S}\mathrm{et}$. Based on Corollary 2.31 we only need to show that $\prod_\Phi \mathcal{S}\mathrm{et}$ is not a Grothendieck topos.

For example let I be a set and Φ be a non-principal filter. Then $\prod_\Phi \mathcal{S}\mathrm{et}$ is not a Grothendieck topos [4, Theorem 3.4]. Thus there are at least as many elementary $(\infty, 1)$-toposes that are not Grothendieck $(\infty, 1)$-toposes as there are non-principal filters of sets.

Here is one particular example that satisfies the conditions given in Example 3.11: Let Φ be the filter of cofinite subsets of $\mathbb{N}$, the set of natural numbers (Example 1.47). We can thus apply the filter quotient to the topos $\mathcal{K}\mathrm{an}^{\mathbb{N}}$. The resulting topos $\prod_\Phi \mathcal{K}\mathrm{an}$ is not a Grothendieck $(\infty, 1)$-topos as its subcategory of 0-truncated objects is not a Grothendieck topos [36, Example D5.1.7].

We end this section by giving a detailed description of this particular example without using the language of filter quotients. Thus a reader who only wants to see an example can avoid the technical details of the previous section. We will refrain from giving detailed proofs here and refer the interested reader to the proofs in the previous section.

Let $\mathcal{K}\mathrm{an}$ be the Kan enriched category of Kan complexes. Then $\mathcal{K}\mathrm{an}^{\mathbb{N}}$ is the Kan enriched category with

(1) objects tuples $(X_n)_{n\in\mathbb{N}}$, where X_n is a Kan complex, and
(2) morphisms level-wise morphisms

$$\mathrm{Map}_{\mathcal{K}\mathrm{an}^{\mathbb{N}}}((X_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}}) = \prod_{n\in\mathbb{N}} \mathrm{Map}_{\mathcal{K}\mathrm{an}}(X_n, Y_n)$$

We will now define a new Kan enriched category $\prod_\Phi \mathcal{K}\mathrm{an}$. However for that we need an equivalence relation on the mapping Kan complexes $\mathrm{Map}_{\mathcal{K}\mathrm{an}}(X_n, Y_n)$. We will define an equivalence relation for each level of the Kan complex and then show that the simplicial operator maps respect the equivalence relation, thus giving us a new simplicial set.

Denote the set of natural numbers bigger than m by $\mathbb{N}_{\geq m}$. We define an equivalence relation on the set $\coprod_{m \in \mathbb{N}} (\mathrm{Map}_{\mathcal{K}\mathrm{an}^{\mathbb{N}}}((X_n)_{n\in\mathbb{N}_{\geq m}}, (Y_n)_{n\in\mathbb{N}_{\geq m}}))_k$ as follows. Let

$$(f_n)_{n\in\mathbb{N}_{\geq m_1}} \sim (g_n)_{n\in\mathbb{N}_{\geq m_2}} \Leftrightarrow \text{ there exists } N > m_1, m_2 \text{ such that for all } n > N : (f_n = g_n)$$

Notice if we have two maps

$$f, g : \Delta[k] \times X_n \to Y_n$$

for $n > m_1$ and $f \sim g$ then $f\delta \sim g\delta$ for any simplicial map $\delta : \Delta[l] \to \Delta[k]$, as $f_n\delta = g_n\delta$ for n large enough. Thus imposing the equivalence relation level-wise still gives us a simplicial set, and in fact a Kan complex.

We can use that to define a new category $\prod_\Phi \mathcal{K}\mathrm{an}$ with

(1) objects tuples $(X_n)_{n\in\mathbb{N}}$ (so the same objects as $\mathcal{K}\mathrm{an}^{\mathbb{N}}$) and
(2) morphisms level-wise equivalence classes of morphisms

$$\mathrm{Map}_{\prod_\Phi \mathcal{K}\mathrm{an}}((X_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}}) = \left[\coprod_{m\in\mathbb{N}} \prod_{n\in\mathbb{N}_{\geq m}} \mathrm{Map}_{\mathcal{K}\mathrm{an}}(X_n, Y_n)\right] / \sim .$$

Intuitively a k-cell in $\mathrm{Map}_{\prod_\Phi \mathcal{K}\mathrm{an}}((X_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}})$ is a class

$$[f_n] : (X_n)_{n\in\mathbb{N}_{\geq m}} \times \Delta[k] \to (Y_n)_{n\in\mathbb{N}_{\geq m}}$$

where two morphisms f_n, g_n are in the same class if $f_n = g_n$ for n large enough.

We need to confirm that we actually get a category. It suffices to check that we have a composition. For two morphisms classes $[f_n] : (X_n)_{n\in\mathbb{N}_{\geq m_1}} \to (Y_n)_{n\in\mathbb{N}_{\geq m_1}}$ and $[g_n] : (Y_n)_{n\in\mathbb{N}_{\geq m_2}} \to (Z_n)_{n\in\mathbb{N}_{\geq m_2}}$ we define the composition as

$$[g_n \circ f_n] : (X_n)_{n\in\mathbb{N}_{\geq Max(m_1,m_2)}} \to (Z_n)_{n\in\mathbb{N}_{\geq Max(m_1,m_2)}}.$$

and notice this definition of composition is indeed well-defined.

We now want to prove that $\prod_\Phi \mathcal{K}\mathrm{an}$ is an elementary $(\infty, 1)$-topos. We need to show that it has finite limits and colimits, is locally Cartesian closed, has subobject classifier and universes. We will treat each separately.

First, $\prod_\Phi \mathcal{K}\mathrm{an}$ has a final object, namely $(\Delta[0])_{n\in\mathbb{N}}$. Now, for two morphisms $(f_n)_{n\in\mathbb{N}_{\geq m_1}} : (X_n)_{n\in\mathbb{N}_{\geq m_1}} \to (Z_n)_{n\in\mathbb{N}_{\geq m_1}}$ and $(g_n)_{n\in\mathbb{N}_{\geq m_2}} : (Y_n)_{n\in\mathbb{N}_{\geq m_2}} \to (Z_n)_{n\in\mathbb{N}_{\geq m_2}}$ a routine calculation shows that the pullback is just the level-wise pullback $(X_n \times_{Z_n} Y_n)_{n\in\mathbb{N}_{\geq Max(m_1,m_2)}}$. This proves that $\prod_\Phi \mathcal{K}\mathrm{an}$ has finite limits. The argument for finite colimits is analogous.

Next, observe that $\prod_\Phi \mathcal{K}\mathrm{an}$ is locally Cartesian closed. Again we use the Cartesian closure of Kan complexes level-wise. Concretely, for a map $(p_n)_{n\in\mathbb{N}_{\geq m_1}} : (Z_n)_{n\in\mathbb{N}_{\geq m_1}} \to (X_n)_{n\in\mathbb{N}_{\geq m_1}}$ and $(f_n)_{n\in\mathbb{N}_{\geq m_2}} : (X_n)_{n\in\mathbb{N}_{\geq m_2}} \to (Y_n)_{n\in\mathbb{N}_{\geq m_2}}$, the map f_*p is given by

$$((f_n)_*(p_n))_{n\in\mathbb{N}_{\geq Max(m_1,m_2)}} : ((f_n)_*(Z_n))_{n\in\mathbb{N}_{\geq Max(m_1,m_2)}} \to (Y_n)_{n\in\mathbb{N}_{\geq Max(m_1,m_2)}}$$

which gives us the desired right adjoint to the pullback.

Next we want to find the subobject classifier. We showed in Example 1.38 that the subobject classifier in $\mathcal{K}\mathrm{an}$ is $\{0, 1\}$. We will now show that the constant sequence $(\{0, 1\})_{n\in\mathbb{N}}$ is the subobject classifier in $\prod_\Phi \mathcal{K}\mathrm{an}$. First, notice that a map $(f_n)_{n\in\mathbb{N}_{\geq m}} : (X_n)_{n\in\mathbb{N}_{\geq m}} \to (Y_n)_{n\in\mathbb{N}_{\geq m}}$ is mono if and only if f_n is mono for $n > N$ for some N. But every mono $f_n : X_n \to Y_n$ is uniquely determined by a map $Y_n \to \{0,1\}$ as it is a subobject classifier in $\mathcal{K}an$, This means we get maps $Y_n \to \{0,1\}$ for $n > N$, which is exactly the data of a map $(Y_n)_{n\in\mathbb{N}} \to (\{0,1\})_{n\in\mathbb{N}}$. So, every mono map in $\prod_\Phi \mathcal{K}an$ is determined by a map into $(\{0,1\})_{n\in\mathbb{N}}$ and we can show this assignment is unique, proving it is a subobject classifier.

Finally, we need to show $\prod_\Phi \mathcal{K}an$ has universes. Recall that the universes in $\mathcal{K}an$ are the objects $\mathcal{S}_0^\kappa$, where κ is a cardinal (see Example 1.38 for more details). We want to show that the objects $(\mathcal{S}_0^\kappa)_{n\in\mathbb{N}}$ give us universes in $\prod_\Phi \mathcal{K}an$. Let $(f_n)_{n\in\mathbb{N}_{\geq m}} : (X_n)_{n\in\mathbb{N}_{\geq m}} \to (Y_n)_{n\in\mathbb{N}_{\geq m}}$ be an arbitrary map. Choose a cardinal κ, such that every morphism f_n is κ-small. Then every map f_n is a pullback of a map $Y_n \to \mathcal{S}^\kappa$. However, we just showed that pullbacks in the category $\prod_\Phi \mathcal{K}an$ are evaluated level-wise. Thus $(f_n)_{n\in\mathbb{N}_{\geq m}}$ is the pullback of $(Y_n)_{n\in\mathbb{N}_{\geq m}} \to (\mathcal{S}^\kappa)_{n\in\mathbb{N}_{\geq m}}$, which shows that every map is classified by a universe. We can show that this assignment gives us an equivalence.

In order to finish this example we need to show that $\prod_\Phi \mathcal{K}an$ is not a Grothendieck $(\infty,1)$-topos. Following Corollary 1.37 it suffices to prove that its natural number object is not equivalent to the countable colimit.

The argument we give here is analogous to [36, Example D5.1.7]. The natural number object in $\prod_\Phi \mathcal{K}an$ is the constant sequence $(\mathbb{N})_{n\in\mathbb{N}}$. Let $\Delta : (1)_{n\in\mathbb{N}} \to (\mathbb{N})_{n\in\mathbb{N}}$ be the map that at level n is just the map $\{n\} : 1 \to \mathbb{N}$. We can think of this map as a “diagonal map”. Let $(P_n^m)_{n\in\mathbb{N}}$ be the following pullbacks

$$\begin{tikzcd}
(P_n^m)_{n\in\mathbb{N}} \arrow[r, "\varphi_m"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (1)_{n\in\mathbb{N}} \arrow[d, "\Delta"] \\
(1)_{n\in\mathbb{N}} \arrow[r, "\{m\}"'] & (\mathbb{N})_{n\in\mathbb{N}}
\end{tikzcd}$$

By descent if the cocone formed by the maps $\{\{m\} : (1)_{n\in\mathbb{N}} \to (\mathbb{N})_{n\in\mathbb{N}}\}_{m\in\mathbb{N}}$ is a colimiting cocone, then the cocone formed by $\{\varphi_m : (P_n^m)_{n\in\mathbb{N}} \to (1)_{n\in\mathbb{N}}\}_{m\in\mathbb{N}}$ is also a colimiting cocone. However, the maps $\Delta, \{m\} : (1)_{n\in\mathbb{N}} \to (\mathbb{N})_{n\in\mathbb{N}}$ only coincide when $n = m$ and disagree otherwise. Thus for $n > m$, we have $P_n^m = \emptyset$, which implies that $(1)_{n\in\mathbb{N}}$ is isomorphic to the colimit $\coprod_{n\in\mathbb{N}} \emptyset \cong \emptyset$, which only happens in the topos with one element.

Notice, by the explanation above, that $\prod_\Phi \mathcal{K}an$ also does not have infinite colimits. Thus we cannot use traditional methods to study its homotopy theory (such as define truncations and prove Blakers-Massey theorem). This necessitates developing elementary $(\infty,1)$-topos theory and proving those results in the elementary context. For such an elementary approach and a careful analysis of truncations in $\prod_\Phi \mathcal{K}an$ see [60].

Remark 3.12. For an alternative argument why $\prod_\mathbb{N} \mathcal{K}an$ does not have infinite limits see [12, Example 3.19].

4. Future directions

Up to here we answered the important question on whether there is an elementary $(\infty,1)$-topos that is not a Grothendieck $(\infty,1)$-topos. In this section we want to pose some new questions about elementary $(\infty,1)$-topos theory motivated by the response.

Homotopy Filter Quotient: Although the filter quotient construction enables us to construct new elementary $(\infty,1)$-toposes, it does have certain limitations. In particular, a filter on $\mathcal{E}$ is completely determined by a filter on the underlying elementary topos $\tau_0\mathcal{E}$ as we are just using $\mathcal{S}ub(1)$, which is a subcategory of both categories. This is particularly problematic when we have interesting categorical data as we will illustrate in the following example.

Example 4.1. Let $\mathcal{K}an$ be the Kan enriched category of Kan complexes and let K be a connected Kan complex. Then the slice category $\mathcal{K}an_{/K}$ is also an elementary $(\infty,1)$-topos with final object $id_K : K \to K$, which has only two subobjects, namely $\emptyset \to K$ and $id_K : K \to K$. Thus $\mathcal{K}\mathrm{an}_{/K}$ only has trivial filter quotients (see Example 2.28).

Ideally we would like to have a notion of filter quotient which allows us to take the higher homotopical data of K into account and use that to form a quotient, which should lead us to a notion of *homotopy filter quotient*.

Filter Quotients of Sheaves: We give explicit examples of filter quotients for the simplest of all Grothendieck $(\infty, 1)$-toposes, namely $\mathcal{K}\mathrm{an}^S$, where S is a set (Example 3.11). However, given that the filter construction works for all Grothendieck $(\infty, 1)$-toposes the next step is to construct the filter quotient on the category of sheaves.

Non-standard Models of Spaces: One long standing goal of elementary $(\infty, 1)$-topos theory is to study models for spaces (similar to how elementary toposes are used to study models of set theory) and the filter quotient might be a step in developing such models. Concretely, let S be an infinite set and Φ a non-principal ultrafilter (Definition 1.43) on the power set $P(S)$. Then we can construct the ultra product $\prod_\Phi \mathcal{K}\mathrm{an}$. The underlying elementary topos is the ultra product $\prod_\Phi \mathcal{S}\mathrm{et}$, which has many interesting properties [36, A2.2]:

(1) It is Boolean.
(2) It is generated by the final object.
(3) It doesn't have infinite colimits.

Thus $\prod_\Phi \mathcal{S}\mathrm{et}$ shares many properties with the category of sets. This suggests that $\prod_\Phi \mathcal{K}\mathrm{an}$ should behave similar to the $(\infty, 1)$-category of Kan complexes. An important first step would be to prove that $\prod_\Phi \mathcal{K}\mathrm{an}$ is generated by the final object.

Models of Homotopy Type Theory: One important goal of elementary $(\infty, 1)$-topos theory is to construct models of homotopy type theory. We already know that every Grothendieck $(\infty, 1)$-topos is a model of homotopy type theory [72]. However, as of now there are no other known models. Given that we have an explicit construction of the filter quotient the hope is that we can show the filter quotient of a Grothendieck $(\infty, 1)$-topos is also a model of homotopy type theory.

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