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The influence of trip length distribution on urban traffic in network-level models

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To my grandmother Alla (1938-1999)
who taught me basics of English ...

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Abstract

Modeling urban traffic on the level of network is a wide research area oriented to the development of ITS (Intelligent Transportation Systems). In this thesis properties of models based on MFD (Macroscopic Fundamental Diagram) are studied. The idea behind MFD is to say that the state of the traffic inside an urban zone is fully determined by its accumulation (the number of traveling vehicles) and that the dynamics of accumulation is caused by the inflow of vehicles (flow of vehicles that enter the zone or start their trips from inside). Nowadays, two different philosophies of modeling dynamics of accumulation exist in the literature. The first one (outflow-MFD) postulates that the outflow of vehicles depends on accumulation. The second one (speed-MFD) postulates that the space-mean speed of vehicles depends on accumulation. The second philosophy already has strong empirical support based on observations of traffic inside many big (kilometer-scale) urban areas around the world. Thus, different speed-MFD models are of great scientific interest.

The thesis is mainly devoted to the comparison of so-called PL model (which assumes the existence of both speed-MFD and outflow-MFD) and TB model (which assumes the equality of speeds of vehicles and explicitly assumes the existence of trip length distribution). It was shown that PL model cannot accurately describe the dynamics of accumulation after the jump of inflow. In this case TB model is more preferable. Moreover, it was proven that PL model is a specific case of TB model for the exponential trip length distribution. This makes TB model more attractive than PL model for the practical usage.

TB model can be formulated mathematically either as integral equation or nonlocal PDE (Partial Differential Equation). Thus, the main drawback of TB model that can be an obstacle in practice is its computational complexity. In this thesis it was proposed to approximate TB model with a simpler model which does not require precise information about the trip length distribution. This so-called M model operates with only the mean and the standard deviation of distribution and has a form of ODE (Ordinary Differential Equation). The analytical comparison between PL, TB and M models proved that M model is much closer to TB model than PL model in the case of constant speed-MFD. The more realistic case of decreasing speed-MFD was studied through the numerical tests and also showed the same effect. Thus, the main conclusion of the study is that M model has practical potential as an elegant and computationally cheap approximation of TB model. Also, given that in the case of constant speed TB model is a type of LTI (Linear Time-Invariant) system, it can be expected that M model might be useful for a wide range of problems that are not related to transportation. However, in this thesis the conclusion about the small difference between M and TB models was made for the inflows that are typical for the transportation field. More precisely, only peak hour shaped (smooth and with small jumps) functions were studied.

Despite the simple form of M model it was found that there exists even more

simple ODE approximation of TB model. This so-called α model works quite well for both smooth and jumping inflows except the case of a short period of time following the jump of inflow. Thus, it might be another good alternative to TB model. The main advantage of α model in the case of realistic speed-MFD function is its convex formulation which allows α model to be efficiently used inside optimization frameworks.

Keywords: Urban Traffic, Macroscopic Fundamental Diagram, Trip Length Distribution, Nonlocal Partial Differential Equation, Linear Time-Invariant System

Résumé

La modélisation du trafic urbain au niveau d'un réseau est un large champ de recherche dans le développement des ITS (Systèmes de Transport Intelligents). Dans cette thèse, les propriétés des modèles basés sur le MFD (Diagramme Fondamental Macroscopique) sont étudiées. L'idée du MFD est de supposer que l'état du trafic à l'intérieur d'une zone urbaine est complètement déterminé par son accumulation (le nombre de véhicules en mouvement) et que la dynamique de l'accumulation est causée par le flux de véhicules entrant (le flux de véhicules qui entrent dans la zone ou commencent leurs trajets à l'intérieur de celle-ci). Deux différentes philosophies concernant la modélisation dynamique de l'accumulation existent actuellement dans la littérature. La première (MFD-flux-sortant) suppose que le flux de véhicules sortant dépend de l'accumulation. La seconde (MFD-vitesse) suppose que la vitesse moyenne spatiale des véhicules dépend de l'accumulation. Cette seconde philosophie dispose d'un fort soutien empirique basé sur des observations de trafic à l'intérieur de plusieurs grande (à l'échelle du kilomètre) zones urbaines à travers le monde. Ainsi, les différents modèles MFD-vitesse sont d'un grand intérêt scientifique.

Cette thèse porte principalement sur la comparaison des modèles soi-disant PL (qui supposent l'existence des deux MFD-flux-sortant et MFD-vitesse) et TB (qui supposent l'égalité des vitesses des véhicules et l'existence d'une distribution des longueurs de trajets de façon explicite). Il a été montré que le modèle PL ne peut décrire précisément la dynamique de l'accumulation après un saut du flux entrant. Dans ce cas, le modèle TB est préférable. En outre, il a été prouvé que le modèle PL est un cas particulier du modèle TB pour une distribution exponentielle des longueurs de trajets. Ceci rend le modèle TB plus attractif que le modèle PL pour un usage pratique.

Le modèle TB peut être formule mathématiquement soit comme une équation intégrale, soit comme une PDE (équation aux dérivées partielles) non locale. Le principal inconvénient du modele TB qui peut être un obstacle dans la pratique est donc sa complexité de calcul. Dans cette thèse, il a été proposé d'approximer le modèle TB par un modèle simplifié qui n'a pas besoin d'information précise concernant la distribution des longueurs de trajets. Ce modèle, nommé M, opère seulement avec la moyenne et l'écart-type de la distribution et prend la forme d'une ODE (équation différentielle ordinaire). La comparaison analytique entre les modèles PL, TB et M prouve que le modèle M est plus proche du modèle TB que du modèle PL dans le cas d'un MFD-vitesse constant. Le cas plus réaliste consistant en un MFD-vitesse décroissant a été étudié avec des approches numériques et a montré le même effet. La conclusion principale de cette étude est donc que le modèle M a un potentiel pratique comme approximation élégante et peu couteuse en temps de calcul du modèle TB. De plus, puisque le modèle TB est un type de système LTI (Linéaire Invariant en Temps) dans le cas de vitesse constante, il est pensable

que le modèle M puisse être utile pour une large gamme de problèmes non liés au transport. Cependant, dans cette thèse, la conclusion concernant la petite erreur entre les modèles M et TB a été faite pour des flux entrants typiques du domaine du transport. Plus précisément, seulement des fonctions typiques de l'heure de pointe (lisse et avec de petits sauts) ont été étudiées.

Malgré la forme simple du modèle M, il a été trouvé qu'il existe une ODE encore plus simple approximant le modèle TB. Ce modèle nommé alpha fonctionne bien pour les flux entrants lisse ou avec des sauts sauf pour un court temps après le saut du flux entrant. Il s'agit donc d'une autre bonne alternative au modèle TB. The principal avantage du modèle alpha dans le cas d'une fonction MFD-vitesse réaliste est que sa formulation convexe lui permet d'être efficace dans les processus d'optimisation.

Mots-clés: Trafic Urbain, Diagramme Fondamental Macroscopique, Distribution des Longueurs de Trajets, Équation aux Dérivées Partielles Non Local, Système Linéaire Invariant en Temps

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1 Introduction

Many city centers around the world suffer from congestion on the roads. Patterns of congestion can vary a lot from day to day as they can be caused by random effects. The challenging research question is to build real-time control strategy for the traffic lights that will reduce travel times for most of drivers. Such strategies can be build on models that describe the dynamics of traffic situation on aggregate level. In this work we focus on a single zone case and investigate different dynamic models of "accumulation" (number of vehicles currently traveling inside the zone). The dynamics of accumulation $n(t)$ satisfies

$$\dot{n}(t) = i(t) - o(t)$$

where $i(t)$ is a trip generation rate (flow of vehicles that are entering the zone or starting trips from inside) and $o(t)$ is a trip completion rate (flow of vehicles that are exiting the zone or ending trips inside). In line with the literature, we refer hereafter to $i(t)$ as an "inflow" and to $o(t)$ as an "outflow". We consider models of the following type:

Input. Initial value of accumulation $n(0) = n_0$, time horizon T and inflow function $i(t)$, $t \in (0, T]$.
Output. Dynamics of accumulation $n(t)$, $t \in (0, T]$.

These models are rather approximate. Regardless the initial spatial distribution of accumulation in the zone and how much inflow comes from particular places on the perimeter or from inside, the resulting $n(t)$ depends only on the total initial accumulation n_0 and on the total inflow $i(t)$. We want to study this type of models because we believe that in the real world the strategy that controls inflow can be very efficient in relieving congestion. However, we should be careful, because the model developed for a zone under every-day conditions can lose much of its predictive power when perimeter control changes the inflow rapidly. This question is discussed in Section 2.6. Note also that if inflow control happens at the intersection, it is difficult to not affect the outflow at the same time. This can make the predictive power of the model even less. Another type of inflow control based on these models could be vehicle rerouting (directing vehicles to the roads that do not belong to the zone). For this type of control it is not necessary to change traffic lights regime. To conclude, considered models allow inflow control, but one should check that when implemented in real scenarios the applied control will not destroy the assumptions of the model. Inflow control seems to be a wide and complicated area of research. In this work we do not investigate it directly, but we develop models that can ease future control directions. Our main goal is to understand the properties of developed models and compare them with each other.

1.1 NEF and PL models

A very simple modeling approach was proposed in [4]. That work suggested to model $n(t)$ as a solution of Cauchy problem:

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \mathbf{o}(n(t)),\, t \in [0, T] \end{cases} \tag{1.1}$$

The function $\mathbf{o}(n)$, called NEF (Network Exit Function), was considered to be a unimodal curve (with one or multiple maxima). This model produced multiple theoretical results having potential practical applications. One of them is the optimal control strategy that minimizes the total travel time for a zone with a virtual queue in the entrance ([4]). However, NEF model lacks some physical interpretation. Intuitively, there is no reason why the outflow should depend on the number of vehicles in the zone (only). To explain this [4] postulates two principles:

P1. Each steady state has $o(t) = i(t) = const$ that is fully determined by $n(t) = const$.
P2. In an arbitrary case $o(t)$ is equal to the steady state value for the current $n(t)$.

Steady state is a useful abstraction. It should be understood as a situation when each small segment of road receives and sends the same amount of vehicles during some short period of time. P1 implies the existence of function $\mathbf{o}(n)$ for steady states. It does not necessarily mean that only one steady state is possible for given $n(t) = const$. There can be a set of different steady states for the same accumulation, but P1 implies that all of them have the same value of $i(t) = o(t)$. P2 is viewed as a mathematical trick that produces satisfactory results for the slowly-evolving inflow. It is based on the idea that the system goes through a sequence of states that are almost steady.

In order to connect NEF model with the speed of vehicles (initially there is no speed in NEF model) and to validate P1 some works (e.g. [5]) utilize a concept of "production" (the growth rate of cumulative distance traveled inside the zone). They consider P1 to be a consequence of the following pair of assumptions:

A1. Production $p(t)$ depends on the current accumulation $n(t)$ only: $p(t) = \mathbf{p}(n(t))$.
A2. There exists parameter L such that $o(t) = i(t) = \frac{p(t)}{L}$ in each steady state.

Note that A1 also means that the space-mean speed $\frac{p(t)}{n(t)}$ depends on accumulation only. We denote this dependency as $\mathbf{v}(n) = \frac{\mathbf{p}(n)}{n}$. Nowadays A1 has some empirical support (e.g. [5, 10, 3, 7, 11]). One can also notice here the analogy with the first order traffic models at the point scale. They state that the vehicle flow at some point of a road depends only on the vehicle density at this point. This dependency is called "Fundamental diagram". Similarly the dependency $\mathbf{p}(n)$ is called "production-MFD", where MFD means "Macroscopic Fundamental Diagram" (it is also known as Network Fundamental Diagram). The dependency $\mathbf{v}(n)$ is called "speed-MFD". The concept of production-MFD (speed-MFD) should not be confused with a concept of NEF which is often called "outflow-MFD". A2 is still not validated with real data directly, without additional assumptions. However, there is one promising technique that appeared recently (see [8] for the details). The experiment took place in Athens. It was shown how the system of flying drones can observe the traffic situation in the center of a big city during a long time period. If the streets can be seen from the air and the system of drones is able to track vehicles constantly, then all the trajectories of vehicles that were inside the zone can be reconstructed with very high accuracy. We believe that in the future such an experiment will take place in several cities and will allow to validate a lot of different assumptions related to network-level models. The only difficulty while validating A2 is that in reality there is no pure steady state. However, notice that in a steady state the value $\frac{p(t)}{o(t)}$ is equal to the average distance that vehicle travels inside the zone. Thus, we get an alternative formulation of A2:

A2. Average trip length in each steady state is equal to L.

If one shows with a set of trajectories that average distance traveled inside the zone does not change a lot during the day, it will be a strong argument in favor of a similar value for any steady state.

In this work all considered models are based on A1 and A2. The simplest way to build a model is to keep P2. From A1, A2 and P2 follows that there exists NEF of the form $\mathbf{o}(n) = \frac{\mathbf{p}(n)}{L}$. We refer to this model as "PL model". It states that $n(t)$ is a solution of the following Cauchy problem:

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{p}(n(t))}{L},\ t \in [0, T] \end{cases} \tag{1.2}$$

The main concern about PL model arises when $i(t)$ has a jump discontinuity. Consider the case when $\mathbf{p}(n)$ is differentiable and the inflow starts not from the steady state value. In other words, let $i_+(0) = \frac{\mathbf{p}(n_0)}{L} + \Delta i$, where Δi is some non-zero value. We can easily calculate the right derivative of outflow at time $t = 0$ in PL model:

$$\dot{o}_+(0) = \frac{\mathbf{p}'(n_0))\dot{n}_+(0)}{L} = \frac{\mathbf{p}'(n_0)}{L}\Delta i$$

If we assume that the system was in a non-congested state and $\mathbf{p}'(n_0) > 0$ then the derivative of outflow $\dot{o}_+(0)$ is positive for positive Δi. However, if we assume that the system was in a steady state before $t = 0$, this contradicts common sense in the case when the minimum trip length is higher than zero. In this case incoming vehicles will not produce additional outflow immediately and, as inflow increases, vehicles, that are close to finish their trips, will not do it faster. Thus, the derivative of outflow cannot be positive. The problem appears because NEF and PL models describe the system without considering trips explicitly.

1.2 TB model

In this work we utilize assumptions A1 and A2 to build a consistent model that considers trips (the more general approach can be found in [9]). Instead of postulating P2 we postulate the following principles:

P3. Incoming vehicles have time-independent distribution of trip length (distance that they will travel inside the zone).
P4. All vehicles have equal speeds at any moment of time.

Note that P3 does not imply that percentage of incoming vehicles that takes particular route does not change. Such an assumption would be very strong because congestion can influence the route choice. P3 can be seen as a relaxation of this assumption. It allows congestion to change the route choice the way that percentage of routes of certain length does not change. This does not contradict to the fact that in the presence of congestion vehicles tend to take longer routes (this fact makes sense for the whole city, but no one can say what will happen to a relatively small zone that generally contains some part of the trip of vehicle). P4 means that all the speeds are equal to the space-mean speed, $\mathbf{v}(n(t)) = \frac{\mathbf{p}(n(t))}{n(t)}$. This assumption is rather strong, because in reality every zone has some spatial heterogeneity of speed. However, we believe that for some rather homogeneous zones models based on P4 can be useful. We refer to the model built on A1, A2, P3 and P4 as "TB model" (Trip-Based).

The mathematical formulation of P3 looks as follows: the vehicle that entered the zone at time s is still in the zone at time t if $\int\limits_s^t \mathbf{v}(n(u))du < l$, where l is a trip length of this vehicle, and, therefore, the portion of vehicles that entered the zone at time s and are still in the network at time t is $1-F\left(\int_s^t \mathbf{v}(n(u))du\right)$, where $F(l)$ is a CDF (Cumulative Distribution Function) of trip length distribution. To simplify mathematical derivations, we assume that trip length distribution is continuous and has PDF (Probability Density Function) $f(l)$. This means that we cannot consider distributions with non-zero weights for some l. However, we can approximate such distributions by considering $f(l)$ with very sharp peaks. Notice also that the mean of trip length distribution should be equal to L, otherwise P3 is not consistent with A2. To be rigorous, we should define the prehistory of inflow and accumulation for $t \in (-\infty, 0]$. As we know n_0, the most natural way to do this is to say that the system was in a steady state for all $t \in (-\infty, 0]$. With this assumption TB model states that $n(t)$ is a solution of the following problem:

$$\begin{cases} n(t) = n_0 & , t \in (-\infty, 0] \\ i(t) = \frac{\mathbf{v}(n_0)}{L} n_0 & , t \in (-\infty, 0] \\ n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds & , t \in (-\infty, T] \end{cases} \tag{1.3}$$

Note that, from the physical point of view, the trip length distribution has finite maximum possible value. However, we assume that there is no such restriction and allow distributions with infinite domain as well.

In Section 2.1 we prove that TB model with $f(l) = \frac{1}{L}e^{-l/L}$ is equivalent to PL model. This means that TB model can be viewed, from mathematical point of view, as a generalization of PL model. Thus, for practical applications, if one chooses trip length distribution properly, TB model looks at least as useful as PL.

1.3 M model

From the computational point of view, PL model (1.2) is more convenient to work with: numerical solution can be found with Euler method or similar. TB model (1.3) contains one integral equation which is not easy to solve. In Section 2.2 we show how one can write the necessary conditions for the solution of (1.3) in the form of nonlocal PDE (Partial Differential Equation). We use this representation to solve (1.3) numerically. However, this method is still computationally expensive and requires precise knowledge of $F(l)$. We consider this to be an important problem. We approach it by constructing an approximation of (1.3) which has form of ODE (Ordinary Differential Equation) and utilizes only two parameters describing $F(l)$: L (mean) and σ (standard deviation). This allows to use Euler method or similar while finding the solution numerically. We refer to our approximation as "M model" (first introduced in [1]). It states that $n(t)$ solves the Cauchy problem

$$\begin{cases} n(0) = n_0 \\ M(0) = \frac{L}{\alpha} n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} \left(n(t) + 3\left(n(t) - \frac{\alpha}{L} M(t) \right) \right) & , t \in [0, T] \\ \dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t) & , t \in [0, T] \end{cases} \tag{1.4}$$

where $\alpha = \frac{2L^2}{L^2+\sigma^2} \in (0, 2)$ is dimensionless parameter. We do not investigate the case $\alpha = 2$, because it does not correspond to any PDF. It is rather convenient to use parametrization L, α instead of L, σ. The auxiliary function $M(t)$, $t \in [0, T]$ is an approximation of total distance to be traveled by all the vehicles that are currently traveling inside the zone.

In this work we shed light on the difference between PL, TB and M models (some analytical and numerical comparisons between PL and TB models are done in [6], but we utilize a different framework). In Section 2.1 theoretical properties of PL and TB models are given. In Section 2.2 we derive PDE formulation of TB model. In Section 2.3 we build M model. In Sections 2.4 and 2.5 we compare models using different inflow profiles. In Section 2.6 we discuss results and potential applications.

1.4 α model

While developing M model we assumed that the inflow can make a jump. For the slowly changing inflows we can find simpler model. In Chapter **??** we show that the model

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = \alpha \left(i(t) - \frac{\mathbf{v}(n(t))}{L} n(t) \right), t \in (0, T] \end{cases}$$

is a good approximation of TB model, if $i(t)$ changes slowly. We refer to this model as "α model". It is less accurate than M model but more accurate than PL model. Note that in the case $\alpha = 1$ it becomes equivalent to PL model.

2 Properties of PL, TB and M models

This chapter contains the main results related to PL, TB and M models. It is shown that PL model can be viewed as a specific case of TB model for the exponential trip length distribution. In other words, there exists at least one distribution such that TB model can be solved as an ODE. The development of this idea leads to the formulation of M model which takes parameter α of trip length distribution into account. M model is shown to be equivalent to TB model for some realistic distributions, but all of them have $\alpha \leqslant \frac{4}{3}$ $\left(\sigma \geqslant \frac{L}{\sqrt{2}}\right)$. To cover other cases that are of practical interest, we show analytically and numerically for realistic distributions with $\alpha > 1$ $(\sigma < L)$ that M model is a better approximation of TB model than PL model. Thus, we suggest to use M model as an ODE alternative to TB model.

2.1 Properties of PL and TB models

In this section we define PL and TB models in a rigorous way by introducing constraints on parameters of models and the input (constraints are not strong and cover most of the cases that are of practical interest). We introduce them mainly to ensure the existence and uniqueness of solutions of problems (1.2) and (1.3). We also show some other properties of solutions that hold for these constraints.

2.1.1 Constraints

In Chapter 1 we introduced assumptions A1 and A2 that are common for PL and TB models. They describe the modeled zone with parameters $\mathbf{v}(n)$ and L. In line with empirical studies, we assume that $\mathbf{v}(n)$ is decreasing, with finite value $\mathbf{v}(0)$. For the convenience, we consider speed to be dimensionless and put $\mathbf{v}(0) = 1$ (thus limiting $\mathbf{v}(n(t))$ to the interval $[0, 1]$). This also means that we measure length in time units. More precisely, we say that any length is equal to the time needed to cover it with the speed $\mathbf{v}(0)$. This gives a better intuition about duration of different processes that depend on parameter L, which we interpret hereafter as the mean travel time in empty zone (we also interpret $f(l)$ as the distribution of travel time in empty zone). From the physical point of view, the trick that we use is very easy to understand. Imagine a zone and vehicles moving inside this zone. Then imagine the same zone increased several times. All the distances and speeds were increased the same amount of times, but the dynamics of $i(t)$, $n(t)$ and $o(t)$ was preserved. Thus, if one knows $\frac{L}{\mathbf{v}(0)}$ and wants to model $n(t)$, he can freely choose $\mathbf{v}(0)$. We suggest to take $\mathbf{v}(0) = 1$ to equalize L and $\frac{L}{\mathbf{v}(0)}$. Note that the model is usually used for some practical purposes and the real value of $\mathbf{v}(0)$ in km/h might be useful for the interpretation of speeds and distances (for example, if one wants to know the traveled distance in kilometers). Nevertheless, we can always use $\mathbf{v}(0) = 1$ while modeling, and then multiple resulted speeds and distances by the real $\mathbf{v}(0)$.

To avoid mathematical difficulties, we consider two cases of $\mathbf{v}(n)$:

1. $\mathbf{v}(n) = 1 = const$ (we refer to this case as "constant" $\mathbf{v}(n)$).
2. An arbitrary decreasing Lipschitz $\mathbf{v}(n)$ such that $\mathbf{v}(0) = 1$, $\mathbf{v}(n) > 0$, $n < n_{jam}$ and $\mathbf{v}(n) = 0$, $n \geqslant n_{jam}$ (we refer to this case as "conventional" $\mathbf{v}(n)$).

Constant $\mathbf{v}(n)$ allows to consider cases of low accumulation in a much simpler way. Conventional $\mathbf{v}(n)$ has parameter n_{jam}, we interpret it as maximum possible number of vehicles inside the zone. We make an assumption that the case $n > n_{jam}$ is possible to be able to prove existence of solutions of PL and TB models on the interval $[0, T]$. Also, to be able to prove uniqueness, we assume that conventional $\mathbf{v}(n)$ is Lipschitz (which is rather realistic). Here we illustrate the mathematical importance of this assumption by considering the case where multiple solutions of PL model are possible with non-Lipschitz $\mathbf{v}(n)$. Consider function

$$\mathbf{v}(n) = \begin{cases} 1 & , n \in \left[0, \frac{n_{jam}}{4}\right] \\ \frac{n_{jam}}{4n} & , n \in \left[\frac{n_{jam}}{4}, \frac{n_{jam}}{2}\right] \\ \frac{n_{jam}}{4n}\left(1 - \sqrt{\frac{2n}{n_{jam}} - 1}\right) & , n \in \left[\frac{n_{jam}}{2}, n_{jam}\right] \\ 0 & , n \in [n_{jam}, +\infty) \end{cases}$$

and corresponding

$$\mathbf{p}(n) = \begin{cases} n & , n \in \left[0, \frac{n_{jam}}{4}\right] \\ \frac{n_{jam}}{4} & , n \in \left[\frac{n_{jam}}{4}, \frac{n_{jam}}{2}\right] \\ \frac{n_{jam}}{4}\left(1 - \sqrt{\frac{2n}{n_{jam}} - 1}\right) & , n \in \left[\frac{n_{jam}}{2}, n_{jam}\right] \\ 0 & , n \in [n_{jam}, +\infty) \end{cases}$$

shown in Figure 2.1.

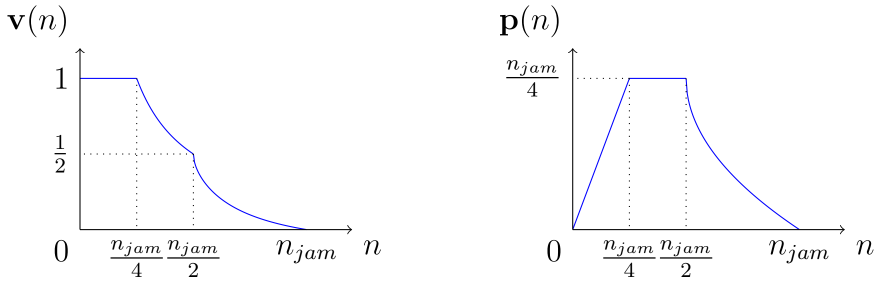


Figure 2.1: Example of non-Lipshitz $\mathbf{v}(n)$ and corresponding $\mathbf{p}(n)$.

Now consider the following input: $n_0 = \frac{n_{jam}}{2}$, $i(t) = \frac{n_{jam}}{4L} = const$. Obviously, $n(t) = \frac{n_{jam}}{2} = const$ is a solution of PL model (1.2). However,

$$n(t) = \begin{cases} \frac{n_{jam}}{2} & , t \in [0, t_0] \\ \frac{n_{jam}}{2} + \frac{n_{jam}}{32}\left(\frac{t-t_0}{L}\right)^2 & , t \in [t_0, t_0 + 4L] \end{cases}$$

where t_0 is an arbitrary moment of time, also solves (1.2). For such a solution the system reaches gridlock at time $t = t_0 + 4L$. In Figure 2.2 we illustrate these two types of solution.

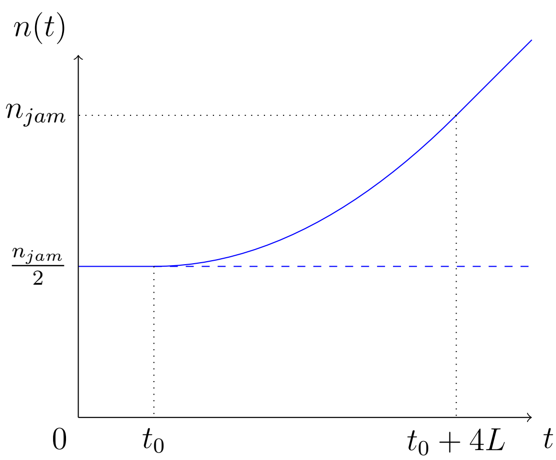


Figure 2.2: Example of bifurcation in PL model for considered $\mathbf{v}(n)$ and the input. Bifurcation can occur at any moment of time. One of solutions (solid) reaches gridlock, while the steady state (dashed) is also a solution.

Hereafter we assume that $\mathbf{v}(n)$ is Lipschitz with constant C, i.e. for any $n_1 < n_2$ the inequality

$$|\mathbf{v}(n_2) - \mathbf{v}(n_1)| \leqslant C|n_2 - n_1|$$

is satisfied. The exact value of C does not influence any further result and we can put it rather big.

As for the parameter $f(l)$ in TB model, we already showed in Chapter 1 that condition $\int\limits_0^{+\infty} f(l)ldl = L$ is necessary to build a consistent model. We also assume that $f(l)$ is a piecewise continuous function bounded on the interval $[0, f_{max}]$. The piecewise continuity property in this work means finite number of jump discontinuities and no removable or essential discontinuities. We use notations $f_-(l) = \lim\limits_{\Delta l \to -0} f(l + \Delta l)$, $l \in (0, +\infty)$ and $f_+(l) = \lim\limits_{\Delta l \to +0} f(l + \Delta l)$, $l \in [0, +\infty)$ for the left and right limits of $f(l)$.

The constraints on input are similar. We assume that $T > 0$ and $i(t)$, $t \in (0, T]$ is a piecewise continuous function bounded on the interval $[0, i_{max}]$. We use notations $i_-(t) = \lim\limits_{\Delta t \to -0} i(t + \Delta t)$, $t \in (0, T]$ and $i_+(t) = \lim\limits_{\Delta t \to +0} i(t + \Delta t)$, $t \in [0, T)$ for the left and right limits of $i(t)$. Moreover, we assume that $n_0 \geqslant 0$, $\mathbf{v}(n_0) > 0$, $\mathbf{p}(n_0) \leqslant Li_{max}$.

2.1.2 Existence and uniqueness of solutions

First, given the assumption of piecewise continuity of $i(t)$, we formulate PL model as

$$\begin{cases} n(0) = n_0 \\ \dot{n}_-(t) = i_-(t) - \frac{\mathbf{v}(n(t))}{L} n(t),\ t \in (0, T] \\ \dot{n}_+(t) = i_+(t) - \frac{\mathbf{v}(n(t))}{L} n(t),\ t \in [0, T) \end{cases} \tag{2.1}$$

which is more rigorous. We do this, because $n(t)$ is not differentiable at points of discontinuity of $i(t)$. However, to simplify the text, we will not write equations for both derivatives, assuming that the equation on the derivative

$$\dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} n(t),\ t \in [0, T]$$

should be understood as two similar equations on left and right derivatives. The first step in proving the existence and uniqueness of solution of (2.1) is

Proposition 1. *The function $\frac{\mathbf{v}(n)}{L} n$ is Lipschitz.*

The proof of this and all other propositions can be found in Chapter 3. After proving Proposition 1 we can prove

Proposition 2. *The solution $n(t)$ of problem*

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} n(t),\ t \in [0, T] \end{cases}$$

exists and is unique.

Note that existence here implies $n(t) \geqslant 0$ and the proof is more difficult than just applying Cauchy-Lipschitz theorem.

The existence and uniqueness property of TB model follows from Propositions 3 and 4.

Proposition 3. *If for some $T_1 \leqslant T$ the solution $n(t)$, $t \in (-\infty, T_1]$ of problem*

$$\begin{cases} n(t) = n_0 & , t \in (-\infty, 0] \\ i(t) = \frac{\mathbf{v}(n_0)}{L} n_0 & , t \in (-\infty, 0] \\ n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds & , t \in (-\infty, T_1] \end{cases}$$

exists then it is continuous.

Proposition 4. *There exists unique continuous solution $n(t)$, $t \in (-\infty, T]$ of problem*

$$\begin{cases} n(t) = n_0 & , t \in (-\infty, 0] \\ i(t) = \frac{\mathbf{v}(n_0)}{L} n_0 & , t \in (-\infty, 0] \\ n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds & , t \in (-\infty, T] \end{cases}$$

Proposition 3 is used while proving Proposition 4 and also guarantees that the unique continuous solution $n(t)$, $t \in (0, T]$ is unique in a wider set of all functions defined on $(0, T]$.

2.1.3 Gridlock situation

We already proved the existence and uniqueness property of PL and TB models. Therefore, if we guess the solution, we can be sure that it is unique. In this subsection we show that in a gridlock situation the solution is to accumulate all the rest of inflow without producing any outflow. In other words, if $n(t_1) = n_{jam}$ in PL or TB model then the rest of solution is $n(t) = n_{jam} + \int\limits_{t_1}^{t} i(s)ds$, $t \in [t_1, T]$. First, notice that for such $n(t)$ the speed $\mathbf{v}(n(t))$ is equal to zero. This means that $n(t)$ corresponds to PL model, because

$$\dot{n}(t) = i(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} n(t)$$

For TB model we also have a correspondence, because

$$
\begin{aligned}
n(t) = \ & n_{jam} + \int_{t_1}^{t} i(s)ds = \\
= & \int_{-\infty}^{t_1} \left(1 - F\left(\int_s^{t_1} \mathbf{v}(n(u))du\right)\right) i(s)ds + \int_{t_1}^{t} i(s)ds = \\
= & \int_{-\infty}^{t_1} \left(1 - F\left(\int_s^{t} \mathbf{v}(n(u))du\right)\right) i(s)ds + \\
& + \int_{t_1}^{t} \left(1 - F\left(\int_s^{t} \mathbf{v}(n(u))du\right)\right) i(s)ds = \\
= & \int_{-\infty}^{t} \left(1 - F\left(\int_s^{t} \mathbf{v}(n(u))du\right)\right) i(s)ds
\end{aligned}
$$

Note that we do not say here that with the same input the time t_1 when the gridlock occurs is the same in PL and TB models. It is also possible that gridlock occurs only in one of two models.

The gridlock situation is not easy to interpret, because it is not realistic. First, in reality vehicles cannot stay with zero speed forever. Sooner or later, an empty space in front of the queue of vehicles in the exit of the zone will appear and they will start moving. Second, $n(t) > n_{jam}$ is not possible by definition of n_{jam}. Recall that we assume inflow to be an arbitrary function from some set. However, in reality we always have an upper bound on the inflow that depend on the situation inside the zone. Thus, gridlock can be interpreted as a situation when the inflow that was given as an input cannot enter the zone and accumulation remains n_{jam}. Anyway, the exact interpretation is not important, in practice, if one uses such a model, gridlock should be understood as a bad situation that should be avoided.

2.1.4 Properties of outflow and equivalence of models

Notice that the function $\mathbf{v}(n(t))$ is continuous in both PL and TB models because $n(t)$ and $\mathbf{v}(n)$ are continuous. Therefore, the outflow in PL model $o(t) = \frac{\mathbf{v}(n(t))}{L} n(t)$ is always continuous. For TB model it also takes place. First we prove

Proposition 5. *The outflow in TB model is equal to*

$$
o(t) = \mathbf{v}(n(t)) \int_{-\infty}^{t} f\left(\int_s^{t} \mathbf{v}(n(u))du\right) i(s)ds
$$

Here we should comment that this is not a trivial result, because $i(t) - o(t)$ cannot be obtained just by differentiating

$$n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds$$

with respect to t using Leibniz's rule. The reason is that the function $i(s)$ is not necessarily continuous. Now we can prove

Proposition 6. *The outflow in TB model is continuous function.*

The main consequence of this proposition is that the jump of $i(t)$ always produces a corner of $n(t)$ and there are no other corners of $n(t)$. Proposition 6 highly depends on the time-independence and continuity of trip length distribution. One can easily find counterexamples when either time-independence or continuity is violated.

We do not know if the result about continuity of $o(t)$ can be significantly improved, but we can show for differentiable $\mathbf{v}(n)$ that $o(t)$ has right derivative at time $t = 0$ and even find its value. These two facts are summarized in

Proposition 7. *If* $i_+(0) = \frac{\mathbf{p}(n_0)}{L} + \vartriangle i$ *and* $\mathbf{v}(n)$ *is differentiable then the outflow in TB model has right derivative*

$$\dot{o}_+(0) = \left(\frac{\mathbf{p}'(n_0)}{L} + (Lf_+(0) - 1)\frac{\mathbf{v}(n_0)}{L}\right) \vartriangle i$$

Recall that PL model gives $\dot{o}_+(0) = \frac{\mathbf{p}'(n_0)}{L} \vartriangle i$ which is positive for $\mathbf{p}'(n_0) > 0$, $\vartriangle i > 0$. As we discussed, this is not realistic in the case when the minimum trip length is not zero. TB model resolves this problem. Notice first that $\mathbf{p}'(n) = \mathbf{v}'(n)n + \mathbf{v}(n) \leqslant \mathbf{v}(n)$. Therefore, $\left(\frac{\mathbf{p}'(n_0)}{L} + (Lf_+(0) - 1)\frac{\mathbf{v}(n_0)}{L}\right) \leqslant 0$ in the case $f_+(0) = 0$. However, if the minimum trip length is equal to zero then TB model might give positive $\dot{o}_+(0)$ for $\vartriangle i > 0$. This highly depends on the value of $f_+(0)$. If, for example, $f_+(0) = \frac{1}{L}$ then TB model gives the same value of $\dot{o}_+(0)$ as PL model.

We have already seen that PL and TB models have many common properties and the reasonable question is if they can be equivalent under some conditions. In Proposition 8 we give an answer.

Proposition 8. *TB model is equivalent to PL model for any input if and only if* $f(l) = \frac{1}{L}e^{-l/L}$.

The proof relies on Proposition 5. The main consequence of Proposition 8 is that PL model can be considered as a special case of TB model.

2.2 PDE formulation of TB model

In this section we show how one can formulate TB model as a nonlocal PDE (Partial Differential Equation). We need this formulation to build a scheme for numerical solution of problem (1.3). We start by introducing function $n(t,a)$ which is equal to the number of vehicles inside the zone at time t that have remaining distance to be traveled greater than a. Hereafter we show that $n(t,a)$ can be found as a unique solution of nonlocal PDE with some boundary condition on $n(0,a)$. If one solves this equation, he knows $n(t) = n(t,0)$.

First, notice that

$$n(t,a) = \int\limits_{-\infty}^{t} \left(1 - F\left(a + \int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds \tag{2.2}$$

Second, find partial derivatives of $n(t,a)$ (Propositions 9, 10) and boundary condition on $n(0,a)$ (Proposition 11):

Proposition 9.

$$\frac{\partial n}{\partial t}(t,a) = (1 - F(a))i(t) - \mathbf{v}(n(t)) \int\limits_{-\infty}^{t} f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds$$

The physical explanation of this proposition is very simple. Consider vehicles with remaining distance to be traveled greater than a. The number of vehicles in this group is $n(t,a)$, the inflow to this group is $(1 - F(a))i(t)$ and the outflow from this group is $\mathbf{v}(n(t)) \int\limits_{-\infty}^{t} f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds$. For $a = 0$ these values are equal to the inflow and outflow.

Proposition 10.

$$\frac{\partial n}{\partial a}(t,a) = - \int\limits_{-\infty}^{t} f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds$$

The physical meaning of $-\frac{\partial n}{\partial a}(t,a)$ is the number (or, more precisely, density) of vehicles with remaining distance to be traveled equal to a.

Proposition 11.

$$n(0,a) = n_0 \int\limits_{0}^{+\infty} \tfrac{1-F(a+l)}{L} dl$$

Finally, combine the results into one problem, assuming $\mathbf{v}(n(t)) = \mathbf{v}(n(t,0))$:

$$\begin{cases} n(0,a) = n_0 \int\limits_{0}^{+\infty} \tfrac{1-F(a+l)}{L} dl & , a \in [0,+\infty) \\ \frac{\partial n}{\partial t}(t,a) - \mathbf{v}(n(t,0))\frac{\partial n}{\partial a}(t,a) = & \\ = (1 - F(a))i(t) & , t \in [0,T],\ a \in [0,+\infty) \end{cases} \tag{2.3}$$

Formally speaking, we found some necessary conditions on the solution of (1.3) and called them problem (2.3). Therefore, we can be sure only in the existence of solution of (2.3), not uniqueness. However, for the most interesting case, when the speed is always positive, uniqueness can be easily proven:

Proposition 12. *If $n(t,a)$ is a solution of problem*

$$\begin{cases} n(0,a) = n_0 \int\limits_0^{+\infty} \frac{1-F(a+l)}{L} dl & , a \in [0,+\infty) \\ \frac{\partial n}{\partial t}(t,a) - \mathbf{v}(n(t,0))\frac{\partial n}{\partial a}(t,a) = (1-F(a))i(t) & , t \in [0,T],\ a \in [0,+\infty) \\ \mathbf{v}(n(t,0)) > 0 & , t \in (0,T] \end{cases}$$

then $n(t,a) = \int\limits_{-\infty}^{t} \left(1 - F\left(a + \int_s^t \mathbf{v}(n(u,0))du\right)\right) i(s)ds$, *where* $n(t,0) = n_0$, $t \in (-\infty, 0]$ *and* $i(t) = \frac{\mathbf{v}(n_0)}{L} n_0$, $t \in (-\infty, 0]$.

Indeed, from Proposition 12 follows that $n(t,0)$ coincides with $n(t)$ in TB model and, therefore, unique. Moreover, $n(t,a)$ for $a > 0$ is also unique, because the formula for it is explicit. We do not know if bifurcation is possible for $n(t,0) = n_{jam}$ (without condition of positive speed), but when we will solve the problem (2.3) numerically, the solution will be, as in TB model, to accumulate all the rest of inflow. Thus, we can say that (2.3) is an alternative formulation of TB model.

2.3 M model

In Section 2.2 we derived an alternative formulation of TB model which can be easily used for building a numerical solution. However, it requires a lot of computations, because $n(t,a)$ includes an additional variable a. Our idea is to build an approximation of TB model that is based on $n(t)$ and $M(t) = \int\limits_0^{+\infty} n(t,a)da$. The physical meaning of $M(t)$ is the total remaining distance to be traveled by all the vehicles that are inside the zone. As we deal with trip length distributions with domain $(0,+\infty)$, we first have to prove that $M(t)$ exists. To start with, we prove

Proposition 13. *For any $\phi > -1$ the following equalities hold:*

$$\int\limits_0^{+\infty} (1-F(l))l^{\phi}dl = \frac{1}{\phi+1} \int\limits_0^{+\infty} f(l)l^{\phi+1}dl$$

$$\int\limits_0^{+\infty}\int\limits_0^{+\infty} (1-F(a+l))l^{\phi}dlda = \frac{1}{(\phi+2)(\phi+1)} \int\limits_0^{+\infty} f(l)l^{\phi+2}dl$$

We will mostly use Proposition 13 for the case $\phi = 0$. In the proofs of propositions of Section 2.4 we will also use cases $\phi = 1$ and $\phi = 2$.

From Propositions 13 and 11 follows that

$$M(0) = \int_0^{+\infty} n(0,a)da = \int_0^{+\infty} n_0 \int_0^{+\infty} \tfrac{1-F(a+l)}{L} dl da = \tfrac{n_0}{L}\tfrac{L^2+\sigma^2}{2} = \tfrac{L}{\alpha}n_0 \qquad (2.4)$$

Prove that $M(t)$ is bounded by the value $M(0) + Li_{max}t$ and, therefore, exists:

$$\begin{aligned}
&\frac{\partial n}{\partial t}(t,a) = (1-F(a))i(t) + \mathbf{v}(n(t,0))\frac{\partial n}{\partial a} \leqslant (1-F(a))i(t)\\
&n(t,a) = n(0,a) + \int_0^t \frac{\partial n}{\partial t}(s,a)ds \leqslant n(0,a) + (1-F(a))i_{max}t\\
&M(t) = \int_0^{+\infty} n(t,a)da \leqslant\\
\leqslant\ &\int_0^{+\infty} n(0,a)da + \left(\int_0^{+\infty}(1-F(a))da\right) i_{max}t = M(0) + Li_{max}t
\end{aligned}$$

Now, when we proved the existence of $M(t)$, we can try to take its derivative. In fact, the result is very simple:

Proposition 14.

$$\dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t)$$

Note that from the physical point of view Proposition 14 is obvious: each unit of inflow adds L to the total distance to be traveled and at the same time each of $n(t)$ vehicles in the zone reduces its distance to be traveled with the rate $\mathbf{v}(n(t))$. We put rigorous mathematical proof to show that physical intuition works and we can really consider two processes separately.

2.3.1 M model with parameters β_1 and β_2

Intuitively, in two different situations with the same $n(t)$ the outflow tends to be higher for the lower values of $M(t)$. Also, the outflow should be proportional to the speed, if it is given exogenously. Thus, we will investigate the approximation of TB model that assumes $o(t) = \mathbf{v}(n(t))(\beta_1 n(t) - \beta_2 M(t))$. We consider β_1 and β_2 to be constants. If later we find that β_1 and β_2 are positive, it will look natural. However, we do not put this constraint explicitly. Combining the approximation of outflow with precise values of $n(0)$ and $M(0)$ and the expression 14 of dynamics of $M(t)$, we get the following Cauchy problem:

$$\begin{cases}
n(0) = n_0\\
M(0) = \frac{L}{\alpha}n_0\\
\dot{n}(t) = i(t) - \mathbf{v}(n(t))(\beta_1 n(t) - \beta_2 M(t)) & , t \in [0,T]\\
\dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t) & , t \in [0,T]
\end{cases} \qquad (2.5)$$

This model requires the same input n_0, T and $i(t)$, $t \in (0,T]$. We will refer to it as “M model”. The question about existence of solution $n(t)$, $t \in (0, T]$ is not trivial. The function $\mathbf{v}(n)$ is defined for non-negative values, but there is no guarantee that the solution $n(t)$ of (2.5) will not reach zero with negative $\dot{n}(t)$ before the time T. Thus we formulate first the result about uniqueness:

Proposition 15. *If the solution of problem*

$$\begin{cases} n(0) = n_0 \\ M(0) = \frac{L}{\alpha} n_0 \\ \dot{n}(t) = i(t) - \mathbf{v}(n(t))(\beta_1 n(t) - \beta_2 M(t)) & , t \in [0, T] \\ \dot{M}(t) = Li(t) - \mathbf{v}(n(t)) n(t) & , t \in [0, T] \end{cases}$$

exists, then it is unique.

Note that the solution of M model might reach gridlock. If this happens then, as for PL or TB models, the solution will be to accumulate all the rest of inflow without producing any outflow. In other words, if $n(t_1) = n_{jam}$ then the rest of solution is

$$\begin{bmatrix} n(t) \\ M(t) \end{bmatrix} = \begin{bmatrix} n_{jam} \\ M(t_1) \end{bmatrix} + \begin{bmatrix} 1 \\ L \end{bmatrix} \int_{t_1}^{t} i(s) ds, \; t \in [t_1, T]$$

Indeed, the derivative of such a solution is

$$\begin{bmatrix} \dot{n}(t) \\ \dot{M}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ L \end{bmatrix} i(t) = \begin{bmatrix} 1 \\ L \end{bmatrix} i(t) - \mathbf{v}(n(t)) \begin{bmatrix} \beta_1 n(t) - \beta_2 M(t) \\ n(t) \end{bmatrix}$$

and satisfies (2.5).

Our main goal is to find β_1, β_2 such that M model is equivalent to TB model. First, because we want to build the model that can be equivalent to TB model for non-exponential distributions. Second, the existence of solution in this case follows automatically. Surprisingly, the question about equivalence is connected with differential equation $f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$. First, we prove one technical fact that comes from the theory of ODE:

Proposition 16. $f(l)$, $l \in (0, +\infty)$ *is a PDF satisfying equation of the form* $f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$ *if and only if it belongs to one of four families:*

F1) $C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l}$

where $\lambda_2 < \lambda_1 < 0$, $C_1 > 0$, $C_2 > 0$, $-\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1$

F2) $C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l}$

where $\lambda_2 < \lambda_1 < 0$, $C_1 > 0$, $C_2 < 0$, $C_1 + C_2 \geqslant 0$, $-\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1$

F3) $C_3 e^{\lambda_3 l}$

where $\lambda_3 < 0$, $C_3 > 0$, $-\frac{C_3}{\lambda_3} = 1$

F4) $(C_1 l + C_2) e^{\lambda_1 l}$

where $\lambda_1 < 0$, $C_1 > 0$, $C_2 \geqslant 0$, $\frac{C_1}{\lambda_1^2} - \frac{C_2}{\lambda_1} = 1$

Proposition 16 is used while proving the main fact:

Proposition 17. *M model with parameters β_1 and β_2 is equivalent to TB model for any input if and only if $f(l)$ is a solution of equation $f''(l)+\beta_1 f'(l)+\beta_2 f(l)=0$. Moreover, for such a solution the equality $1-\beta_1 L+\frac{\beta_2 L^2}{\alpha}=0$ always holds.*

2.3.2 M model with parameter β

As our main goal is to find β_1 and β_2 such that M model is equivalent to TB model, hereafter we consider M models with parameters β_1, β_2 satisfying $1-\beta_1 L+\frac{\beta_2 L^2}{\alpha}=0$. Denote $\beta=\beta_1 L-1=\frac{\beta_2 L^2}{\alpha}$. With this notation $\beta_1=\frac{1+\beta}{L}$ and $\beta_2=\frac{\alpha\beta}{L^2}$. The outflow is equal to $o(t)=\mathbf{v}(n(t))(\beta_1 n(t)-\beta_2 M(t))=\frac{\mathbf{v}(n(t))}{L}\left(n(t)+\beta\left(n(t)-\frac{\alpha}{L}M(t)\right)\right)$ and M model looks as follows:

$$\begin{cases} n(0)=n_0 & \\ M(0)=\frac{L}{\alpha}n_0 & \\ \dot{n}(t)=i(t)-\frac{\mathbf{v}(n(t))}{L}\left(n(t)+\beta\left(n(t)-\frac{\alpha}{L}M(t)\right)\right) & ,\ t\in[0,T] \\ \dot{M}(t)=Li(t)-\mathbf{v}(n(t))n(t) & ,\ t\in[0,T] \end{cases}$$

Hereafter we consider only $\beta\neq 0$ to have a bijection between (β_1,β_2) and (α,β). Indeed, if $\beta\neq 0$ then α and β can be expressed through β_1 and β_2 as $\alpha=\frac{\beta_2 L^2}{\beta_1 L-1}$ and $\beta=\beta_1 L-1$. Moreover, the case $\beta=0$ is not interesting, because M model takes the form of PL model. We should comment here that M model can be still equivalent to PL model for $\beta\neq 0$:

Proposition 18. *M model with parameter $\beta\neq 0$ is equivalent to PL model for any input if and only if $\alpha=1$.*

Note, that this property holds regardless the value of β. Recall that TB model is equivalent to PL model for any input if $f(l)=\frac{1}{L}e^{-l/L}$. This distribution has $\alpha=1$ and, therefore, M model is equivalent to TB model with $f(l)=\frac{1}{L}e^{-l/L}$. This means that we found at least one case when M model with parameter $\beta\neq 0$ is equivalent to TB model. In Proposition 19 we show that there are a lot of pairs of α and β such that M model with parameter $\beta\neq 0$ is equivalent to TB model for some $f(l)$ with parameter α.

Proposition 19. *If $\alpha\in\left(0,\frac{4}{3}\right]$ and*

$$\beta\in\begin{cases} (0,+\infty) & ,\ \alpha\in(0,1) \\ (-\infty,+\infty)\setminus\{0\} & ,\ \alpha=1 \\ \left[2\alpha-1+\sqrt{4\alpha^2-4\alpha},\frac{1}{\alpha-1}\right] & ,\ \alpha\in\left(1,\frac{4}{3}\right] \end{cases}$$

then there exists exactly one $f(l)$ with parameters L and α such that M model with parameter $\beta \neq 0$ is equivalent to TB model for any input. Otherwise, there is no such $f(l)$.

In Figure 2.3 we plot all the feasible pairs (α, β) and indicate the family from Proposition 16 to which $f(l)$ belongs.

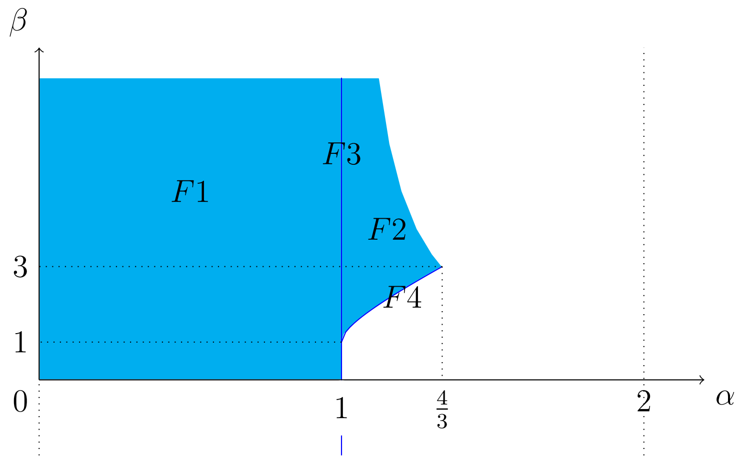


Figure 2.3: Pairs (α, β) that correspond to the case when M model is equivalent to TB model with some $f(l)$. Each feasible point corresponds to unique $f(l)$. All the families $F1$, $F2$, $F3$, $F4$ are presented.

From Proposition 19 follows that M model can be equivalent to TB model only if $\alpha \leqslant \frac{4}{3}$. The distribution that corresponds to $(\alpha, \beta) = \left(\frac{4}{3}, 3\right)$ is $f(l) = \frac{4l}{L^2}e^{-2l/L}$. We consider this distribution to be rather realistic. However, most of the values $\alpha \leqslant \frac{4}{3}$ are not realistic. We believe that all realistic α are greater than 1 and some of them are close to 1.6 $\left(\frac{\sigma}{L} = 0.5\right)$. Therefore, Proposition 19 says that M model cannot be equivalent in principle to some realistic TB models. Even if $\alpha \leqslant \frac{4}{3}$ $\left(\frac{\sigma}{L} \geqslant \frac{1}{\sqrt{2}} \approx 0.7\right)$, we should check first that some of $f(l)$ that give equivalence are realistic.

2.3.3 Realistic trip length distributions

It is very difficult to define, which $f(l)$ are realistic. We start by investigating possible values of moments of distribution with bounded domain. We assume that domain belongs to the interval $(0, \Lambda L)$, where $\Lambda > 1$ is some not very big parameter. For the convenience, we consider central moments

$$
\begin{aligned}
\sigma^2 &= \int_0^{+\infty} f(l)(l-L)^2 dl = \int_0^{+\infty} f(l)l^2 dl - L^2 \\
\rho^3 &= \int_0^{+\infty} f(l)(l-L)^3 dl = \int_0^{+\infty} f(l)l^3 dl - 3L\sigma^2 - L^3
\end{aligned}
\tag{2.6}
$$

In Propositions 20 and 21 we give an answer about all the possible values of σ^2 and ρ^3:

Proposition 20. *If $f(l)$, $l \in (0, \Lambda L)$ is a PDF with mean L and variance σ^2 then all the possible values of $\frac{\sigma^2}{L^2}$ are $(0, \Lambda - 1)$.*

Proposition 21. *If $f(l)$, $l \in (0, \Lambda L)$ is a PDF with mean L, variance σ^2 and third central moment ρ^3 then all the possible values of $\frac{\rho^3}{L^3}$ are*

$$\left(-\frac{\sigma^2}{L^2} + \left(\frac{\sigma^2}{L^2}\right)^2, (\Lambda - 1)\frac{\sigma^2}{L^2} - \frac{1}{\Lambda-1}\left(\frac{\sigma^2}{L^2}\right)^2\right)$$

We believe that in reality there should be very few trips longer than $3L$. This gives us an idea of necessary condition on the realistic distribution. We will assume that for any realistic distribution there should exist distribution defined on $(0, 3L)$ that has the same first three moments. This means that $\frac{\rho^3}{L^3}$ for realistic distribution should satisfy

$$\frac{\rho^3}{L^3} \in \left(-\frac{\sigma^2}{L^2} + \left(\frac{\sigma^2}{L^2}\right)^2, 2\frac{\sigma^2}{L^2} - \frac{1}{2}\left(\frac{\sigma^2}{L^2}\right)^2\right) \tag{2.7}$$

However, not all the values from this set seem to be realistic. Thus, if $\frac{\rho^3}{L^3}$ satisfies this condition, we will say that distribution is "reasonable", but not necessarily realistic. To be able to move forward we make an assumption that all realistic distributions that share the same $\frac{\sigma^2}{L^2}$ also have similar values of $\frac{\rho^3}{L^3}$. We will find these values approximately using rather realistic distributions from gamma family:

$$f(l) = \frac{1}{\Gamma(\phi)}\left(\frac{\phi}{L}\right)^{\phi} e^{-\phi l/L} l^{\phi-1}$$

where $\phi > 1$ is an arbitrary parameter.
Such distributions are unimodal and heavy-tailed. They are responsible for many real processes and appear very often in economic and physical models. Gamma family can be also parametrized with $\alpha \in (1, 2)$. Indeed, the second moment of gamma distribution is equal to $\int\limits_0^{+\infty} f(l) l^2 dl = \frac{\phi+1}{\phi} L^2$. Therefore, $\alpha = \frac{2\phi}{\phi+1}$ and $\phi = \frac{\alpha}{2-\alpha}$. In Figure 2.4 we give two examples of gamma distributions for $\alpha = 1.2$ $\left(\phi = 1.5,\ \frac{\sigma}{L} = \sqrt{\frac{2}{3}} \approx 0.8\right)$ and $\alpha = 1.6$ $\left(\phi = 4,\ \frac{\sigma}{L} = 0.5\right)$:

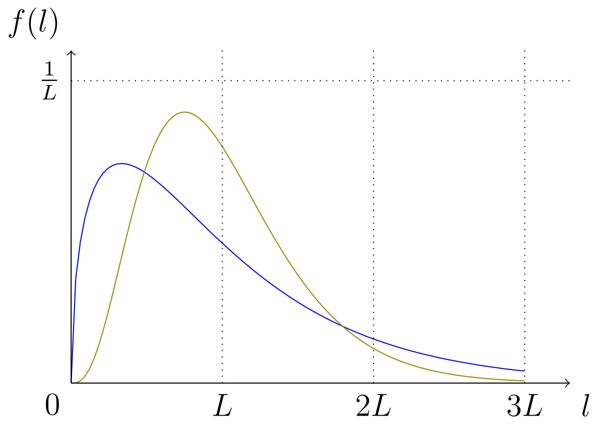


Figure 2.4: Examples of gamma distribution for $\alpha = 1.2$ (blue) and $\alpha = 1.6$ (olive).

Now calculate $\frac{\sigma^2}{L^2}$ and $\frac{\rho^3}{L^3}$ for gamma distribution

$$\frac{\sigma^2}{L^2} = \frac{1}{L^2} \int\limits_0^{+\infty} f(l) l^2 dl - 1 = \frac{1}{\phi}$$

$$\frac{\rho^3}{L^3} = \frac{1}{L^3} \int\limits_0^{+\infty} f(l) l^3 dl - 3\frac{\sigma^2}{L^2} - 1 = \frac{(\phi+1)(\phi+2)}{\phi^2} - \frac{3}{\phi} - 1 = \frac{2}{\phi^2} = 2\left(\frac{\sigma^2}{L^2}\right)^2$$

Hereafter we will say that any distribution that satisfies $\frac{\sigma^2}{L^2} < 1$, $\frac{\rho^3}{L^3} = 2\left(\frac{\sigma^2}{L^2}\right)^2$ is "gamma-like". The equivalent definition is $\alpha \in (1, 2)$, $\frac{\rho^3}{L^3} = 2 - 8\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right)$. In Figure 2.5 we visualize possible moments of reasonable and gamma-like distributions:

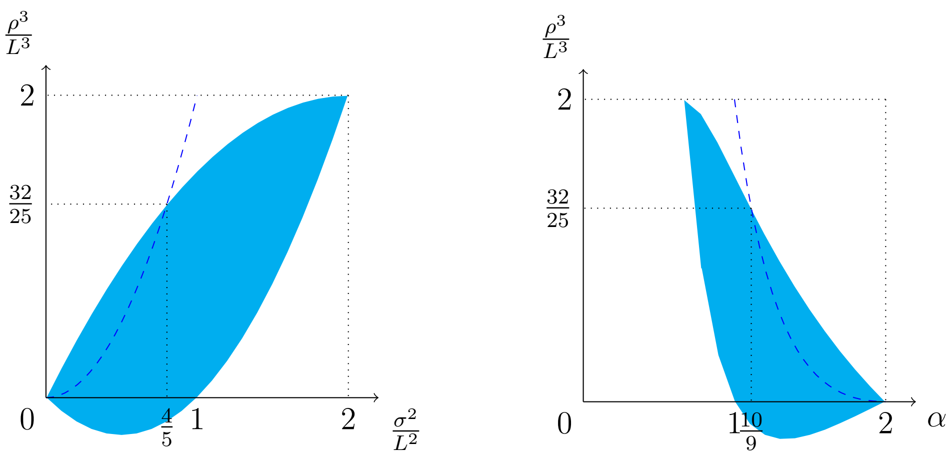


Figure 2.5: The set of feasible $\frac{\sigma^2}{L^2}$ and $\frac{\rho^3}{L^3}$ that correspond to reasonable (cyan area) and gamma-like (blue, dashed) distributions.

Note, that if $\frac{\sigma^2}{L^2} > \frac{4}{5}$, $\left(\alpha < \frac{10}{9}\right)$ then gamma-like distribution cannot be reasonable.
To conclude, we do not define the set of realistic distributions rigorously. For us realistic distribution is reasonable and approximately gamma-like:

$$\alpha \in \left(\frac{10}{9}, 2\right), \ \frac{\rho^3}{L^3} \approx 2 - 8\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right)$$

Now we will find out which pairs (α, β) from Proposition 19 correspond to reasonable and gamma-like distributions. From the proof of Proposition 19 follows that the formula for $f(l)$ (families F1 and F2) is

$$f(l) = \frac{1}{L}\left(\mu_1^2 \frac{\mu_2 - 1}{\mu_2 - \mu_1} e^{-\mu_1 l/L} + \mu_2^2 \frac{1 - \mu_1}{\mu_2 - \mu_1} e^{-\mu_2 l/L}\right) \tag{2.8}$$

where $\mu_1 = \frac{1+\beta}{2} - \sqrt{\psi}$, $\mu_2 = \frac{1+\beta}{2} + \sqrt{\psi}$, $\psi = \frac{(1+\beta)^2}{4} - \alpha\beta$. Note, that this formula is also valid for $\alpha = 1$ (family F3, exponential distribution) and for the case $\beta \in (1, 3]$ and $\alpha = \frac{(1+\beta)^2}{4\beta}$ (family F4) it gives valid result

$$f(l) = \frac{1+\beta}{L}\left(\frac{3-\beta}{4} + \frac{\beta^2 - 1}{8}\frac{l}{L}\right) e^{-\frac{1+\beta}{2} l/L}$$

as a limit when β is constant and $\alpha \to \frac{(1+\beta)^2}{4\beta}$.

Therefore, the ratio $\frac{\rho^3}{L^3}$ for each distribution that gives equivalence between M and TB models is equal to

$$\begin{aligned}
&\frac{\rho^3}{L^3} = \frac{1}{L^3}\int_0^{+\infty} f(l) l^3 dl - \frac{3\sigma^2}{L^2} - 1 = \\
&= 6\left(\frac{1}{\mu_1^4}\mu_1^2 \frac{\mu_2 - 1}{\mu_2 - \mu_1} + \frac{1}{\mu_2^4}\mu_2^2 \frac{1 - \mu_1}{\mu_2 - \mu_1}\right) - 3\left(\frac{2}{\alpha} - 1\right) - 1 = \\
&= \frac{6}{\mu_1^2 \mu_2^2}\left(\frac{\mu_2^3 - \mu_2^2 + \mu_1^2 - \mu_1^3}{\mu_2 - \mu_1}\right) - 3\left(\frac{2}{\alpha} - 1\right) - 1 = \\
&= \frac{6}{\mu_1^2 \mu_2^2}\left((\mu_1 + \mu_2)^2 - \mu_1\mu_2 - \mu_1 - \mu_2\right) - 3\left(\frac{2}{\alpha} - 1\right) - 1 = \\
&= \frac{6}{\alpha^2\beta^2}\left(1 + 2\beta + \beta^2 - \alpha\beta - 1 - \beta\right) - 3\left(\frac{2}{\alpha} - 1\right) - 1 = \\
&= 2 - 6\left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right)
\end{aligned} \tag{2.9}$$

Distribution (2.8) is reasonable if

$$\begin{cases} 2 - 6\left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) > -\left(\frac{2}{\alpha} - 1\right) + \left(\frac{2}{\alpha} - 1\right)^2 \\ 2 - 6\left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) < 2\left(\frac{2}{\alpha} - 1\right) - \frac{1}{2}\left(\frac{2}{\alpha} - 1\right)^2 \end{cases}$$

If $\alpha = 1$ then the second inequality is not satisfied which means that exponential distribution (family F3) is not reasonable. For all other families $\beta > 0$

and the system is equivalent to

$$\begin{cases} \beta > 3(\alpha - 1) \\ \left(\frac{4}{3} - \alpha\right)^2 \beta < \frac{4}{3}(\alpha - 1) \end{cases}$$

Obviously, the second inequality is not satisfied for the family F1 which has $\alpha < 1$. The first inequality is satisfied for families F2 and F4. Therefore, all reasonable distributions (2.8) are defined by

$$\begin{cases} \beta \geqslant 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha} \\ \beta \leq \frac{1}{\alpha - 1} \\ \left(\frac{4}{3} - \alpha\right)^2 \beta < \frac{4}{3}(\alpha - 1) \end{cases}$$

Distribution (2.8) is gamma-like if

$$2 - 6\left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) = 2 - 8\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right)$$

This equation is equivalent to $\beta = 3$. In Figure 2.6 we visualize the sets of pairs (α, β) that correspond to reasonable and gamma-like distributions.

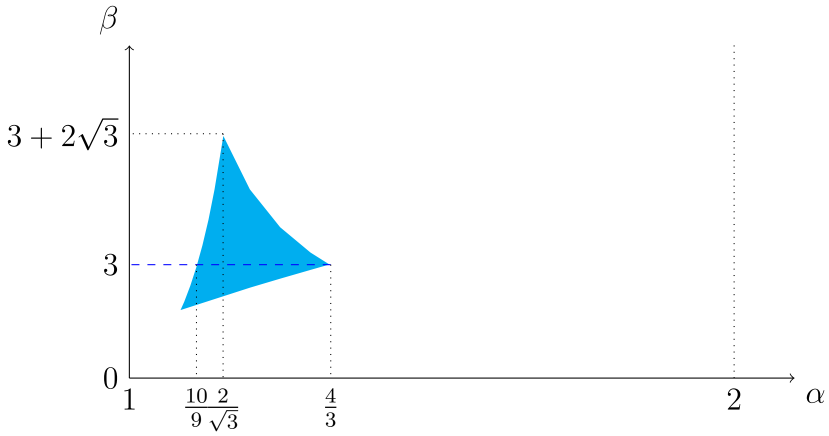


Figure 2.6: Pairs (α, β) from Proposition 19 that correspond to reasonable (cyan area) and gamma-like (blue, dashed) distributions.

We suggest to use $\beta = 3$ as a rule of thumb for M model when $\alpha \in \left(\frac{10}{9}, \frac{4}{3}\right]$, because M model in this case is equivalent to TB model with some realistic distribution. Here we show the density $f(l)$ for this distribution. It is a specific case of formula (2.8) for $\beta = 3$.

$$\begin{aligned} f(l) = \frac{1}{L}\left(\frac{9\alpha - 8}{2\sqrt{4 - 3\alpha}} + \frac{4 - 3\alpha}{2}\right) e^{-\left(2 - \sqrt{4 - 3\alpha}\right) l / L} + \\ + \frac{1}{L}\left(\frac{8 - 9\alpha}{2\sqrt{4 - 3\alpha}} + \frac{4 - 3\alpha}{2}\right) e^{-\left(2 + \sqrt{4 - 3\alpha}\right) l / L} \end{aligned} \tag{2.10}$$

This distribution is very close to gamma not only in the sense of first three moments, but also in the sense of value of $f(l)$ for not very small l. In Figure 2.7 we compare $f(l)$ obtained with the above formula for $\alpha = 1.2$ with $f(l)$ of gamma distribution.

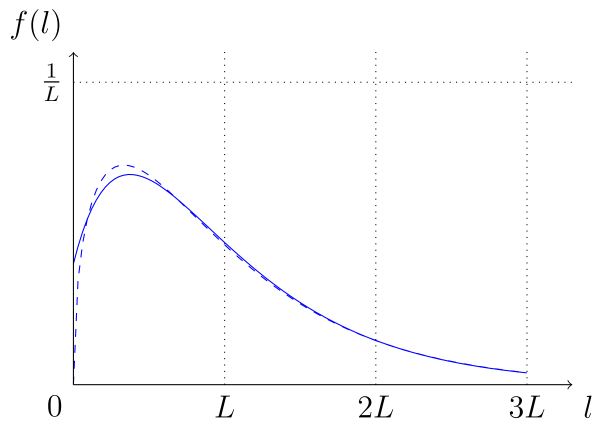


Figure 2.7: Comparison of gamma-like distribution (solid) given by formula (2.10) with gamma distribution (dashed) for $\alpha = 1.2$.

We think that in practice it is rather difficult to gather precise information about the distribution and, probably, only L and σ can be measured accurately. Thus, if we have a rule of thumb that says which β should be used for some α, it will be very useful. In Section 2.4 we show that $\beta = 3$ is also a good rule of thumb for $\alpha \in \left(\frac{4}{3}, 2\right)$. Thus, we suggest to formulate M model only as

$$\begin{cases} n(0) = n_0 \\ M(0) = \frac{L}{\alpha} n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} \left(n(t) + 3\left(n(t) - \frac{\alpha}{L} M(t)\right)\right) & , t \in [0, T] \\ \dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t) & , t \in [0, T] \end{cases}$$

We utilize this formulation in Section 2.5 to show numerically that M model approximates TB model better than PL model.

2.4 Analytical comparison of models

Our main goal is to show that M model is a good approximation of TB model. Hereafter we take TB model as a reference and compare M and PL models as approximations of TB model. We assume that α that describes $f(l)$ is known and only the correct value of α is used in the formulation of M model. Therefore, we cannot claim that M model is a better approximation of TB model than PL model just because PL model is M model for $\alpha = 1$. As we want to investigate realistic $f(l)$, we focus on the case $\alpha \in (1, 2)$. In this Section we show that $\beta = 3$ is still a good rule of thumb for M model when $\alpha \in \left(\frac{4}{3}, 2\right)$. We also show in some sense that M model with $\beta = 3$ works better than PL model as an approximation of TB model.

We investigate only the case of constant $\mathbf{v}(n)$. First, because we want to get analytical expressions for solutions of models. The case of conventional $\mathbf{v}(n)$ is more complicated, we consider it later in Section 2.5, where we compare models numerically. Second, because in the case of constant $\mathbf{v}(n)$ it is easy to investigate stability of models. We say that model is stable if solution does not change a lot under small changes of input. This is not a rigorous definition. However, this definition is enough to find cases when some model is not completely wrong. We did not address the question of stability before, because for conventional $\mathbf{v}(n)$ the answer is very unclear. There is a possibility, for example, that for some input the solution reaches gridlock while this can be avoided by very small change of input. For constant $\mathbf{v}(n)$ gridlock situation is not possible and the analysis becomes very simple.

2.4.1 TB, PL and M models under constant $\mathbf{v}(n)$

As we discussed in Section 2.1, for constant $\mathbf{v}(n)$ trip length is equal to trip time. The solution of TB model is

$$n(t) = \int\limits_{-\infty}^{t} (1 - F(t-s))i(s)ds \tag{2.11}$$

where $i(t) = \frac{n_0}{L}$, $t \in (-\infty, 0]$. Note that TB model is always stable because $1 - F(t-s)$ is decreasing function of t and is bounded on the interval $[0, 1]$. The solution of PL model is

$$n(t) = e^{-t/L} n_0 + \int\limits_{0}^{t} e^{-(t-s)/L} i(s)ds$$

PL model is always stable because exponential rates in the expression are negative. To find the solution of M model we rewrite it as

$$\begin{cases} \begin{bmatrix} n(0) \\ \frac{M(0)}{L} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} n_0 \\ \begin{bmatrix} \dot{n}(t) \\ \frac{\dot{M}(t)}{L} \end{bmatrix} = -\frac{1}{L} A \begin{bmatrix} n(t) \\ \frac{M(t)}{L} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} i(t) \end{cases}$$

where matrix $A = \begin{bmatrix} 1+\beta & -\alpha\beta \\ 1 & 0 \end{bmatrix}$ is time-independent. Thus, M model becomes a Cauchy problem with unique solution

$$\begin{bmatrix} n(t) \\ \frac{M(t)}{L} \end{bmatrix} = e^{-At/L} \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} n_0 + \left(\int\limits_{0}^{t} e^{-A(t-s)/L} i(s)ds \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

There exists an alternative way to calculate the solution, which gives the value of the outflow $o(t)$ instead of $M(t)$. Notice that

$$\begin{bmatrix} Lo(t) \\ n(t) \end{bmatrix} = A \begin{bmatrix} n(t) \\ \frac{M(t)}{L} \end{bmatrix}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} Lo(t) \\ n(t) \end{bmatrix} &= e^{-At/L} A \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} n_0 + \left(\int_0^t e^{-A(t-s)/L} i(s) ds \right) A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \\ &= e^{-At/L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} n_0 + \left(\int_0^t e^{-A(t-s)/L} i(s) ds \right) \begin{bmatrix} 1 + \beta - \alpha\beta \\ 1 \end{bmatrix} \end{aligned}$$

The resulting expression for $n(t)$ depends on the value of $\psi = \frac{(1+\beta)^2}{4} - \alpha\beta$ and takes the following form:

1. $\psi > 0$

$$n(t) = e^{-\frac{1+\beta}{2} t/L} \left(cosh(\sqrt{\psi} t/L) + \frac{\beta - 1}{2\sqrt{\psi}} sinh(\sqrt{\psi} t/L) \right) n_0 +$$
$$+ \int_0^t e^{-\frac{1+\beta}{2}(t-s)/L} \left(cosh(\sqrt{\psi}(t-s)/L) + \frac{(2\alpha-1)\beta-1}{2\sqrt{\psi}} sinh(\sqrt{\psi}(t-s)/L) \right) i(s) ds$$

2. $\psi = 0$

$$n(t) = e^{-\frac{1+\beta}{2} t/L} \left(1 + \frac{\beta-1}{2} \frac{t}{L} \right) n_0 + \int_0^t e^{-\frac{1+\beta}{2}(t-s)/L} \left(1 + \frac{(2\alpha-1)\beta-1}{2} \frac{t-s}{L} \right) i(s) ds$$

3. $\psi < 0$

$$n(t) = e^{-\frac{1+\beta}{2} t/L} \left(cos(\sqrt{-\psi} t/L) + \frac{\beta-1}{2\sqrt{-\psi}} sin(\sqrt{-\psi} t/L) \right) n_0 +$$
$$+ \int_0^t e^{-\frac{1+\beta}{2}(t-s)/L} \left(cos(\sqrt{-\psi}(t-s)/L) + \frac{(2\alpha-1)\beta-1}{2\sqrt{-\psi}} sin(\sqrt{-\psi}(t-s)/L) \right) i(s) ds$$

To derive the formula for $n(t)$ we should first calculate matrix exponential $e^{-At/L}$. We do not show here these technical details. One can find them in the proof of Proposition 27. Now we will try to understand for which pairs of α and β M model is stable. If $\alpha < 1$ then $\psi = \frac{(1+\beta)^2}{4} - \alpha\beta > \frac{(\beta-1)^2}{4} \geqslant 0$ and $\left| \frac{\beta-1}{2\sqrt{\psi}} \right| < 1$. Therefore, M model is stable if and only if $-\frac{1+\beta}{2} + \sqrt{\psi} < 0$ which is the highest exponential rate in the expression for $\psi > 0$. However, this is equivalent to $\alpha\beta > 0$ or $\beta > 0$. If $\alpha = 1$ then M model is stable as it is equivalent to PL model. If $\alpha > 1$ then either $\psi > 0$ or $\psi \leqslant 0$, depending on β. If $\psi > 0$ then $\left| \frac{\beta-1}{2\sqrt{\psi}} \right| > 1$ and M model is stable if and only if $\beta > 0$. If $\psi \leqslant 0$ then $\alpha\beta \geqslant \frac{(1+\beta)^2}{4} \geqslant \beta$ and, therefore, $\beta > 0$. At the same time M model is stable, because with $\beta > 0$ the exponential rate is $-\frac{1+\beta}{2} < 0$. Concluding, M model is stable if and only if $\beta > 0$ or $\alpha = 1$.

Hereafter we focus on the realistic case $\alpha \in (1, 2)$ and assume $\beta > 0$ to consider only stable models. Next important question is when the solution of M model exists (by existence we mean $n(t) \geqslant 0$) for any input in the case of constant $\mathbf{v}(n)$. The answer is given in Proposition 22:

Proposition 22. *If $\alpha \in (1, 2)$ then the solution of M model with $\beta > 0$ exists for any input in the case of constant $\mathbf{v}(n)$ if and only if*

$$\beta \geqslant 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}$$

Interestingly, this fact is also valid in the case of conventional $\mathbf{v}(n)$. This follows from Proposition 23:

Proposition 23. *The solution of M model exists for any input in the case of constant $\mathbf{v}(n)$ if and only if the solution of the same M model exists for any input in the case of conventional $\mathbf{v}(n)$.*

In Figure 2.8 we show combined results of Propositions 19, 22 and 23. We also show the set of pairs (α, β) such that M model is equivalent to TB model with some reasonable $f(l)$.

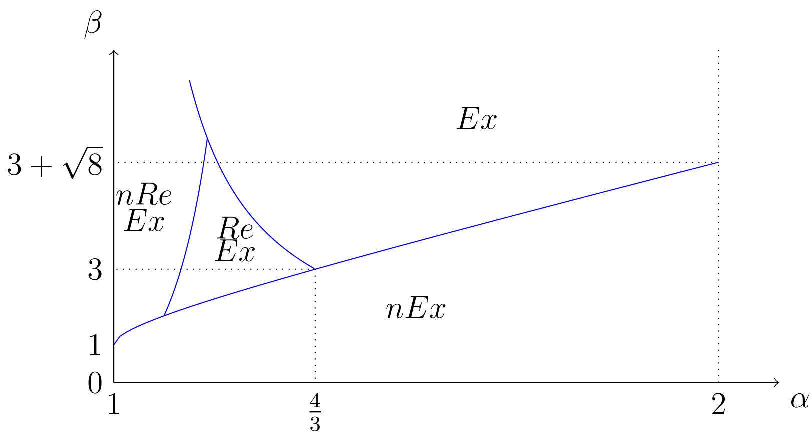


Figure 2.8: Properties of M model. Re - M model is equivalent to TB model with reasonable $f(l)$. nRe - M model is equivalent to TB model with $f(l)$ that is not reasonable. Ex - solution of M model exists for any input. nEx - solution of M model does not exist for some inputs.

Note that $\beta = 3$ does not ensure the existence of solution for any input for $\alpha \in \left(\frac{4}{3}, 2\right)$. In this case $\psi = 4 - 3\alpha < 0$ and the solution does not exist, for example, for rather big T and $i(t) = 0$, $t \in (0, T]$. However, if inflow does not drop fast or does not reach zero, solution usually exists. Later in this Section we show that $\beta = 3$ is a good rule of thumb in such cases.

2.4.2 Comparison of models for big t

In this subsection we assume that $f(l)$ has finite domain $(0, \Lambda L)$ and $T > \Lambda L$. This allows to simplify the solution of TB model for $t \geqslant \Lambda L$ and make it independent from n_0:

$$n^{TB}(t) = \int_0^t (1 - F(t-s))i(s)ds, \; t \in [\Lambda L, T]$$

We will compare models using quadratic $i(t)$:

$$i(t) = i_0 + i_1 t + i_2 t^2, \; t \in (0, T]$$

The constant term i_0 is not necessarily equal to $i(0)$ that we introduce for TB model. Thus, we can consider jump discontinuities at $t = 0$. Also we assume that coefficients i_1 and i_2 are chosen the way that $i(t) \geqslant 0$, $t \in [0, T]$. The reason to take quadratic inflow is that we can easily model non-linear and non-monotonic cases this way. Moreover, for the quadratic inflow the solution of TB model depends only on i_0, i_1, i_2 and the first three moments of $f(l)$. Note that the solution of M model depends on i_0, i_1, i_2 and the first two moments. Thus, we can partially give an answer, how close is M model to TB model under different third moments of $f(l)$, in other words, is it enough to describe $f(l)$ with only L and α to build a precise approximation of TB model.

Proposition 24. *The solution of TB model for $i(t) = i_0 + i_1 t + i_2 t^2$ in the case of constant $\mathbf{v}(n)$ is*

$$n^{TB}(t) = L(i_0 + i_1 t + i_2 t^2) - \tfrac{L^2}{\alpha} i_1 - 2\tfrac{L^2}{\alpha} i_2 t + 2\left(\tfrac{1}{\alpha} - \tfrac{1}{3} + \tfrac{\rho^3}{6L^3}\right) L^3 i_2, \; t \in [\Lambda L, T]$$

From Proposition 24 follows that

$$o^{TB}(t) = i(t) - \dot{n}^{TB}(t) = i_0 + i_1 t + i_2 t^2 - L i_1 - 2L i_2 t + 2\tfrac{L^2}{\alpha} i_2$$

Proposition 25. *The solution of PL model for $i(t) = i_0 + i_1 t + i_2 t^2$ in the case of constant $\mathbf{v}(n)$ is*

$$n^{PL}(t) = L(i_0 + i_1 t + i_2 t^2) - L^2 i_1 - 2L^2 i_2 t + 2L^3 i_2 + \\ + e^{-t/L}\left(n_0 - L i_0 + L^2 i_1 - 2L^3 i_2\right)$$

Proposition 26. *The solution of M model with $\beta > 0$ for $i(t) = i_0 + i_1 t + i_2 t^2$ in the case of constant $\mathbf{v}(n)$ is*

$$\begin{aligned}\begin{bmatrix} Lo^M(t) \\ n^M(t) \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} L(i_0 + i_1 t + i_2 t^2) - \\ &- \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} L^2 i_1 - 2 \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} L^2 i_2 t + 2 \begin{bmatrix} \frac{1}{\alpha} \\ \frac{1}{\alpha^2} + \frac{1}{\alpha^2\beta} - \frac{1}{\alpha\beta} \end{bmatrix} L^3 i_2 + \\ &+ e^{-At/L} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} (n_0 - L i_0) + \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} L^2 i_1 - 2 \begin{bmatrix} \frac{1}{\alpha} \\ \frac{1}{\alpha^2} + \frac{1}{\alpha^2\beta} - \frac{1}{\alpha\beta} \end{bmatrix} L^3 i_2 \right)\end{aligned}$$

where $A = \begin{bmatrix} 1+\beta & -\alpha\beta \\ 1 & 0 \end{bmatrix}$.

From Propositions 25 and 24 follows that if $\alpha \in (1, 2)$ then $n^{PL}(t)$ does not converge to $n^{TB}(t)$ for any quadratic inflow with $i_2 \neq 0$. Moreover, if inflow is linear then $n^{PL}(t) - n^{TB}(t)$ converges to the constant $\left(1 - \frac{1}{\alpha}\right) L^2 i_1$. This constant becomes zero if and only if $i_1 = 0$. Now investigate convergence of M model. From Propositions 26 and 24 follows that $n^M(t) - n^{TB}(t)$ converges to

$$2\left(-\left(1 + \tfrac{1}{\beta}\right)\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3} - \tfrac{\rho^3}{6L^3}\right) L^3 i_2$$

which is constant. If inflow is linear, this constant is equal to zero. Therefore, M model converges to TB model for any linear inflow. If inflow is quadratic, it becomes very important to estimate the value

$$\delta = -\left(1 + \tfrac{1}{\beta}\right)\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3} - \tfrac{\rho^3}{6L^3} \tag{2.12}$$

for realistic distributions. If $\delta = 0$ then M model converges to TB model for any quadratic inflow.

We start by calculating possible values of δ for reasonable distributions. From (2.9) and equality $\frac{\sigma^2}{L^2} = \frac{2}{\alpha} - 1$ follows that

$$\begin{cases} \delta > -\left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) + \frac{1}{12}\left(9 - \frac{12}{\alpha} + \frac{4}{\alpha^2}\right) \\ \delta < -\left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) + \frac{1}{3}\left(\frac{3}{\alpha} - \frac{2}{\alpha^2}\right) \end{cases} \tag{2.13}$$

The value of δ for gamma-like distributions is

$$\delta = -\left(1 + \tfrac{1}{\beta}\right)\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3} - \tfrac{1}{6}\left(2 - 8\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right)\right) = \left(\tfrac{1}{3} - \tfrac{1}{\beta}\right)\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) \tag{2.14}$$

This means that if $\beta = 3$ then $\delta \approx 0$ for realistic distributions regardless the value of α. We show this property in Figure 2.9:

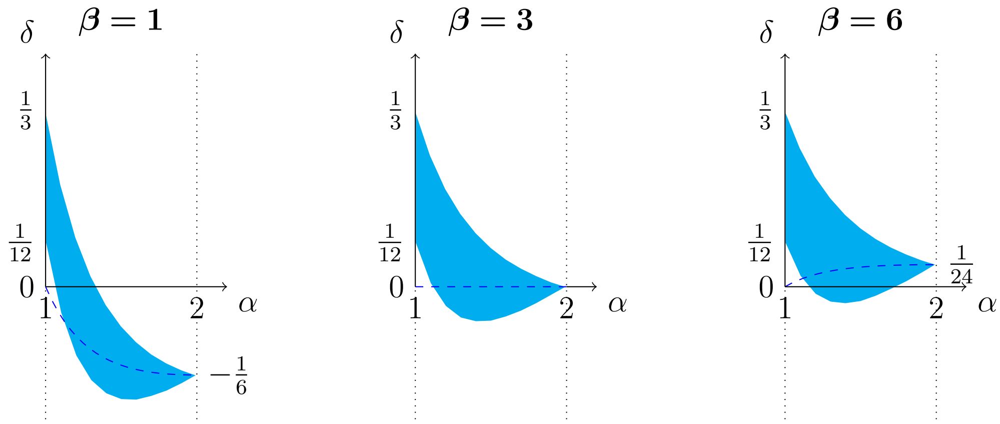


Figure 2.9: Possible values of δ for reasonable (cyan area) and gamma-like (blue, dashed) distributions.

One important question we should address is the speed of convergence of $n^M(t) - n^{TB}(t)$. If the speed is low, we might not see that the difference between M and TB models is almost constant for $t \in [\Lambda L, T]$. To estimate the speed of convergence we make a very rough estimation of the absolute value of matrix exponential $e^{-At/L}$:

Proposition 27. *If $\alpha > 1$ and $\beta \geqslant 1$ then the absolute value of matrix exponential $e^{-At/L}$, where $A = \begin{bmatrix} 1+\beta & -\alpha\beta \\ 1 & 0 \end{bmatrix}$, is less than*

$$e^{-(1-e^{-1})t/L} \begin{bmatrix} \frac{3+\beta}{2} & \alpha\beta \\ 1 & \frac{3+\beta}{2} \end{bmatrix}$$

From Proposition 27 follows that the term that contains $e^{-At/L}$ in the expression of $n^M(t)$ becomes very small after not very long time. For example, if $\beta = 3$ and $t/L > 5$ then

$$e^{-(1-e^{-1})t/L} \begin{bmatrix} \frac{3+\beta}{2} & \alpha\beta \\ 1 & \frac{3+\beta}{2} \end{bmatrix} < e^{-3} \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} < \frac{1}{20} \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$$

This value is already rather small and will decrease rather fast with growing t/L. Thus, if $\beta = 3$ then we can be sure that M model approximates TB model very well for quadratic inflow after $t/L > 5$. Here we assume, of course, that the difference $n_0 - Li_0$ is not extremely large.

2.4.3 Hysteresis effect for big t

The difference between PL and TB models can be clearly seen on the plane $(n(t), o(t))$. While PL model predicts that $o(t)$ is proportional to $n(t)$ with coefficient $\frac{1}{L}$, TB model does not show this behavior in general. This is another drawback of PL model as an approximation of TB model. In this subsection we show for big t that behavior of M model on the plane $(n(t), o(t))$ approximates behavior of TB model much better.

We assume that $i(t) = i_0 + i_1 t + i_2 t^2$, where $i_0 \geqslant 0$, $i_1 > 0$ and $i_2 < 0$ which simulates the peak hour. We also assume that the maximum point $t = -\frac{i_1}{2i_2}$ is less than T.

From Proposition 24 follows that $n^{TB}(t)$ reaches maximum at $t = -\frac{i_1}{2i_2} + \frac{L}{\alpha}$ and $o^{TB}(t)$ reaches maximum at $t = -\frac{i_1}{2i_2} + L$.
For $t > 5L$ the exponential term in the expressions for the solutions of PL and M models can be neglected. Therefore, $n^{PL}(t)$ and $o^{PL}(t)$ reach maximum approximately at $t = -\frac{i_1}{2i_2} + L$. This value is correct for $o(t)$ but not for $n(t)$. However, M model says that $n^M(t)$ and $o^M(t)$ reach maximum approximately at $t = -\frac{i_1}{2i_2} + \frac{L}{\alpha}$ and $t = -\frac{i_1}{2i_2} + L$ which are the correct values.
The other way to see the difference between models is to look at the function $n(t) - Lo(t)$ which is equivalent to zero for PL model. For TB model this function looks as

$$\begin{aligned} & n^{TB}(t) - Lo^{TB}(t) = \\ = & \left(1 - \tfrac{1}{\alpha}\right) L^2 \left(i_1 + 2i_2 t\right) + 2\left(-\tfrac{1}{3} + \tfrac{\rho^3}{6L^3}\right) L^3 i_2,\ t \in [\Lambda L, T] \end{aligned}$$

It is equal to zero if

$$t = -\frac{i_1}{2i_2} + \frac{\frac{1}{3} - \frac{\rho^3}{6L^3}}{1 - \frac{1}{\alpha}} L$$

Interestingly, if $\alpha > 1$ and distribution is reasonable then this value is greater than $-\frac{i_1}{2i_2} + \frac{L}{\alpha}$, in other words, $n^{TB}(t)$ reaches maximum before $n^{TB}(t) - Lo^{TB}(t)$ changes sign. This fact follows from Proposition 28

Proposition 28. *If $\alpha > 1$ and ρ^3 correspond to some distribution defined on $(0, 3L)$ then*

$$\frac{\frac{1}{3} - \frac{\rho^3}{6L^3}}{1 - \frac{1}{\alpha}} > \frac{1}{\alpha}$$

As we assume $\alpha > 1$ and $i_2 < 0$, the sign of $n^{TB}(t) - Lo^{TB}(t)$ changes from positive to negative. Before this change $o^{TB}(t)$ is less than $\frac{n^{TB}(t)}{L}$ and after is greater than $\frac{n^{TB}(t)}{L}$. Such behavior is usually called "counterclockwise hysteresis". In Figure 2.10 we illustrate it for $i(t) = i_0 + 4\frac{t}{T}\left(1 - \frac{t}{T}\right)\Delta i$, where $T = 10L$ and $n_0 = Li_0$. As $i(t)$ is symmetric, its maximum $i_0 + \Delta i$ is reached at $t = \frac{T}{2} = 5L$.

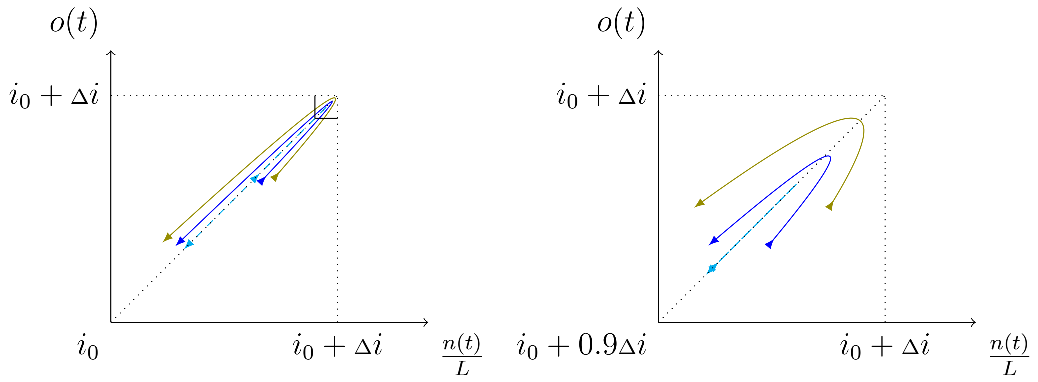


Figure 2.10: Hysteresis effect for TB model. The solutions are given for two gamma-like $f(l)$ defined on $(0, 3L)$ with $\alpha = 1.2$ (blue) and $\alpha = 1.6$ (olive). The solution of PL model (cyan, dashed) does not produce hysteresis. The input satisfies $T = 10L$, $n_0 = Li_0$ and $i(t) = i_0 + 4\frac{t}{T}\left(1 - \frac{t}{T}\right)\Delta i$. The considered time intervals are $t \in [3L, 10L]$ (left) and $t \in [5L, 7L]$ (right).

For $t/L > 5$ the solution of M model satisfies

$$n^M(t) - Lo^M(t) \approx \left(1 - \frac{1}{\alpha}\right) L^2 (i_1 + 2i_2 t) - 2\left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) L^3 i_2$$

The function $n^M(t) - Lo^M(t)$ is equal to zero if $t \approx -\frac{i_1}{2i_2} + \left(1 + \frac{1}{\beta}\right)\frac{L}{\alpha}$. Therefore, M model with $\beta > 0$ also says that $n^M(t) - Lo^M(t)$ changes sign after $n^M(t)$ reaches maximum. Moreover, if $\delta = 0$ then

$$\frac{\frac{1}{3} - \frac{\rho^3}{6L^3}}{1 - \frac{1}{\alpha}} = \left(1 + \frac{1}{\beta}\right)\frac{1}{\alpha}$$

This fact is expected, because if $\delta = 0$ then M model converges to TB model for any quadratic inflow. In the case of realistic $f(l)$ both TB model and M model with $\beta = 3$ say that $n(t) - Lo(t)$ changes sign approximately at $t = -\frac{i_1}{2i_2} + \frac{4L}{3\alpha}$.

It is also interesting to mention that if $\alpha < 1$ then both TB and M models say that the hysteresis is clockwise for $i_2 < 0$. However, the case $\alpha < 1$ is not realistic and we do not consider it in this section.

2.4.4 Comparison of models for small t

From inequalities (2.13) follows that the range of δ for reasonable distributions has width $\frac{1}{4}\left(-3 + \frac{8}{\alpha} - \frac{4}{\alpha^2}\right)$ which does not depend on β. Also from equation (2.12) follows that if we take only $\beta \geqslant 1$ then we can change δ by not more than $\frac{1}{\alpha} - \frac{1}{\alpha^2}$, which is less than $\frac{1}{4}$. This means that the exact value of $\beta \geqslant 1$ does not play a big role for the quality of approximation for big t. Also, from Propositions 22 and 23 follows that the solution of M model with $\beta \geqslant 3 + \sqrt{8}$ exists for any input. Taking these β can be an advantage in practice. However, in this subsection we show that such M model is not very accurate for small t, where $\beta = 3$ still works very well.

The most important question we should answer and which is related to practice is how M model approximates TB model for the inflow that makes a jump. To avoid mathematical difficulties we assume $i(t) = \frac{n_0}{L} + \Delta i$, $t \in (0, T]$. We make an assumption that $i(t)$ is constant after the jump because we expect that in reality $i(t)$ does not change fast during the beginning of the time period $(0, T]$. In this subsection we assume that $t \in (0, T']$, where $T' < 5L$. For bigger t one can use asymptotic results that we already presented. We also assume that $\Lambda < 5$.

From Proposition 26 follows that $o^M(t)$ converges to $o^{TB}(t)$ for any quadratic inflow. Thus, to compare models for big t we look at $n(t)$. However, the difference between models appears in the very beginning of interval $(0, T]$. Even if $\delta = 0$, there is a possibility that models are not very similar for small t. To visualize the difference between models we take $o(t)$ instead of $n(t)$ to see the difference more clearly.

To calculate $o^{TB}(t)$ for $t \in (0, T']$ it is not enough to take only first three moments of $f(l)$ into account. In fact, we should use the whole distribution:

$$o^{TB}(t) = \int\limits_{-\infty}^{t} f(t-s)i(s)ds =$$

$$= \int\limits_{-\infty}^{t} f(t-s)\tfrac{1}{L}n_0 ds + \int\limits_{0}^{t} f(t-s)\Delta i ds = \tfrac{1}{L}n_0 + F(t)\Delta i$$

To compare TB, PL and M models we take two reasonable (but not gamma-like) distributions:

$$\text{D1} = \tfrac{1}{2}U[0, L] + \tfrac{1}{2}U[0, 3L]$$
$$\text{D2} = U\left[\left(1 - \tfrac{\sqrt{3}}{2}\right)L, \left(1 + \tfrac{\sqrt{3}}{2}\right)L\right]$$

Their densities are shown in Figure 2.11.

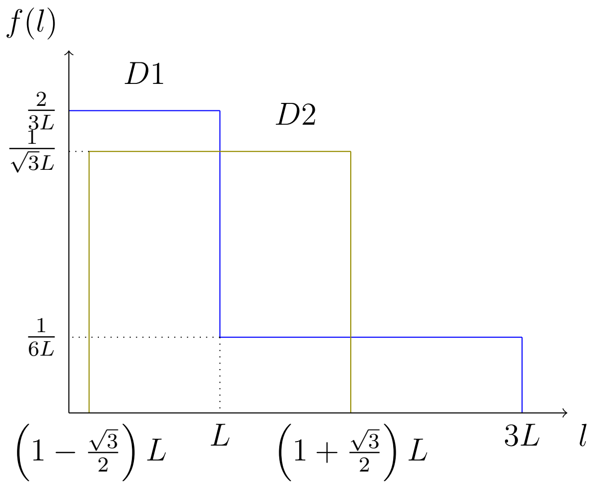


Figure 2.11: Distributions that are used for the analysis. D1 (blue) has $\alpha = 1.2$, D2 (olive) has $\alpha = 1.6$.

The function $o^M(t)$ for the constant $i(t)$ can be easily obtained by differentiating $n^M(t)$. In this case one can make change of variables $u = t - s$ and write $n^M(t)$ as an integral from 0 to t of the function that does not depend on t and depends only on u. This gives the following expressions:

1. $\psi > 0$

$$o^M(t) = \tfrac{n_0}{L} + \left(1 - e^{-\frac{1+\beta}{2}t/L}\left(cosh(\sqrt{\psi}t/L) + \tfrac{(2\alpha-1)\beta-1}{2\sqrt{\psi}} sinh(\sqrt{\psi}t/L)\right)\right)\Delta i$$

2. $\psi = 0$

$$o^M(t) = \tfrac{n_0}{L} + \left(1 - e^{-\frac{1+\beta}{2}t/L}\left(1 + \tfrac{(2\alpha-1)\beta-1}{2}\tfrac{t}{L}\right)\right)\Delta i$$

3. $\psi < 0$

$$o^M(t) = \tfrac{n_0}{L} + \left(1 - e^{-\frac{1+\beta}{2}t/L}\left(cos(\sqrt{-\psi}t/L) + \tfrac{(2\alpha-1)\beta-1}{2\sqrt{\psi}} sin(\sqrt{-\psi}t/L)\right)\right)\Delta i$$

In Table 2.1 we give values of ψ that correspond to $\alpha = 1.2$ and $\alpha = 1.6$. We consider cases $\beta = 1$, $\beta = 3$ and $\beta = 6$ to show that the solution of M model strongly depends on β for small t.

	$\beta = 1$	$\beta = 3$	$\beta = 6$
$\alpha = 1.2$	-0.2	0.4	5.05
$\alpha = 1.6$	-0.6	-0.8	2.65

Table 2.1: Values of ψ that correspond to considered α and β.

The solution of M model for all of these cases is given in Figure 2.12.

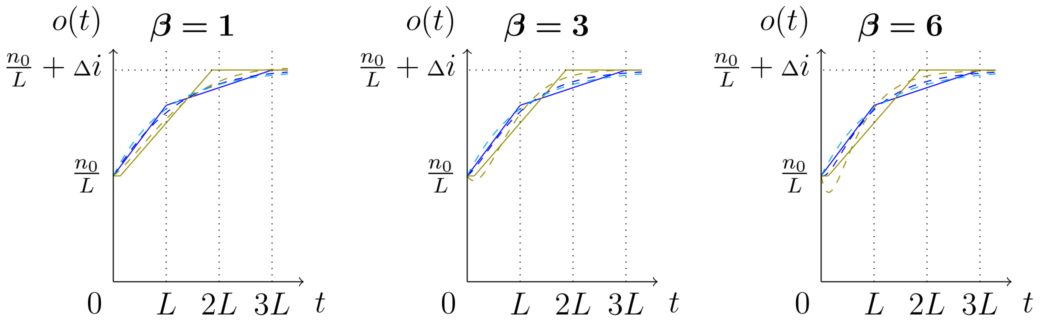


Figure 2.12: Solutions of models for $i(t) = \frac{n_0}{L} + \Delta i$. TB model is given for D1 (blue, solid) and D2 (olive, solid). M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) is given in three different variants. It approximates TB model very well for $\beta = 3$ and not very well for $\beta = 1$ and $\beta = 6$. PL model (cyan, dashed) is also a good approximation of TB model.

2.4.5 Comparison of models for very small t

In this section we proved in some sense that $\beta = 3$ still works well for realistic distributions with $\alpha > \frac{4}{3}$. Thus, we suggest to use $\beta = 3$ for any α. However, the proof is constructed for constant speed and while comparing M and TB models we do not look at the very beginning of time interval $[0, T]$. If we are interested only in the very short interval of time, the good choice of β should be based on $\dot{o}_+(0)$ (to get second-order approximation of $n(t)$). This value can be easily calculated in M model:

Proposition 29. *If $i_+(0) = \frac{\mathbf{p}(n_0)}{L} + \Delta i$ and $\mathbf{v}(n)$ is differentiable then the outflow in M model has right derivative*

$$\dot{o}_+(0) = \left(\frac{\mathbf{p}'(n_0)}{L} + \beta(1-\alpha)\frac{\mathbf{v}(n_0)}{L}\right)\Delta i$$

Recall that $\dot{o}_+(0) = \left(\frac{\mathbf{p}'(n_0)}{L} + (Lf_+(0) - 1)\frac{\mathbf{v}(n_0)}{L}\right)\Delta i$ in TB model. If we assume $f_+(0) = 0$ (this condition makes sense for $\alpha > \frac{4}{3}$) then the correct value of $\dot{o}_+(0)$ can be obtained by taking M model with $\beta = \frac{1}{\alpha-1}$. This value belongs to the interval $(1, 3)$. The effect can be clearly seen in Figure 2.12 for distribution D2. If we take M model with $\beta = 6$ then the value of $\dot{o}_+(0)$ is completely wrong, while M models with $\beta = 1$ and $\beta = 3$ show better correspondence with TB model.

2.5 Numerical comparison of models

In this section we try to understand, like in Section 2.4, how accurate are M and PL models as approximations of TB model. We test different realistic scenarios for the conventional $\mathbf{v}(n)$. In this case finding solutions of models analytically is very problematic and it is easier to compare them numerically.

To compare models numerically we should build a numerical scheme for finding the solution of TB model. We consider discrete moments of time $t = k\Delta t$, where $k = 0, 1, ..., K$ and $K = \frac{T}{\Delta t}$.

Instead of taking continuous distribution $f(l)$ we take its discrete approximation. We assume that l takes only values $m\Delta t$, where $m = 1, 2, ...$ and denote as f_m the probability of $l = m\Delta t$. The procedure of obtaining such an approximation should keep the mean equal to L. While finding the solution of TB model we will utilize numbers $F_m = \sum\limits_{r=1}^{m} f_r$ which is the CDF that corresponds to f_m.

Here we demonstrate one of possible procedures of finding discrete approximation of the mixture of uniform distributions. This class of distributions is very wide and we can say that it covers most of the cases that are of practical interest. We start by considering simple uniform distribution $U[A\Delta t - b, A\Delta t + b]$, where $b \leqslant A\Delta t$. We assume that A is an integer to simplify the procedure. This assumption puts some restrictions on the mixtures that can be considered. However, these restrictions are not strong if Δt is small. Denote the largest integer that is less than $\frac{b}{\Delta t}$ as B. Then we can take

$$f_m = \tfrac{1}{2B+1},\ m = A - B, A - B + 1, ..., A + B$$

This gives an approximation with mean $A\Delta t$. Now consider a mixture of uniforms

$$w_1 U_1[A_1\Delta t - b_1, A_1\Delta t + b_1] + w_2 U_2[A_2\Delta t - b_2, A_2\Delta t + b_2] + ...$$

The mean value of such distribution is equal to $L = (w_1 A_1 + w_2 A_2 + ...)\Delta t$ and does not change with our discretization.

To discretize variable a (remaining distance to be traveled) we assume that $a = m\Delta t$, $m = 0, 1, ...$ We estimate the value of accumulation $n(t, a)$ on the grid $n_{k,m}$, $k = 0, 1, ..., K$, $m = 0, 1, ...,$ where $n_{k,m}$ corresponds to $n(k\Delta t, m\Delta t)$. We assume that $n_{0,0} = n_0$ and inflow is discretized as

$$i_k = \tfrac{1}{\Delta t} \int\limits_{(k-1)\Delta t}^{k\Delta t} i(t)dt,\ k = 1, ..., K$$

This gives the following numerical scheme for solving (2.3):

$$\begin{cases} n_{0,m} = n_0 \frac{\Delta t}{L} \sum\limits_{r=m}^{\infty} (1 - F_r) \\ n_{k+1,m} - n_{k,m} = (1 - F_m) i_{k+1} \Delta t - \mathbf{v}(n_{k,0})(n_{k,m} - n_{k,m+1}) \end{cases}$$

The dependences of this scheme are shown in Figure 2.13

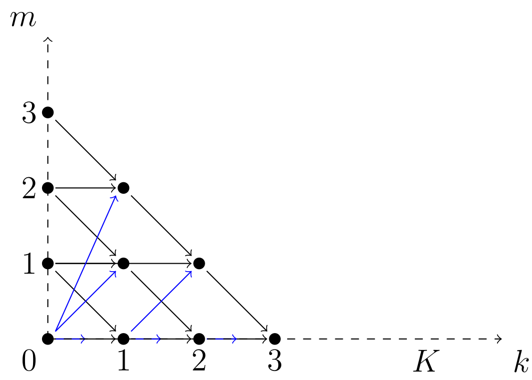


Figure 2.13: The dependences of numerical scheme for finding the solution of TB model. Nonlocal dependences (blue) are caused by $\mathbf{v}(n_{k,0})$.

We should note that

$$\sum_{r=0}^{\infty}(1 - F_r) = \sum_{r=0}^{\infty} \sum_{m=r+1}^{\infty} f_m = \sum_{m=1}^{\infty} (f_m m) = \frac{L}{\Delta t}$$

Therefore, $n_{0,0} = n_0$ which means that the numerical scheme is consistent with the input. Also, such numerical scheme gives correct answer $n_{k,0} = n_0$ for the steady state inflow $i(t) = \frac{\mathbf{v}(n_0)}{L} n_0$. Finally, note that this numerical scheme satisfies CFL (Courant-Friedrichs-Lewy) conditions, because its Courant number is equal to $\frac{\mathbf{v}(n_{k,0})\Delta t}{\Delta t} = \mathbf{v}(n_{k,0}) \leqslant 1$.

To solve PL model we use the following numerical scheme:

$$n_{k+1} - n_k = i_{k+1}\Delta t - \frac{\mathbf{v}(n_k)}{L} n_k \Delta t$$

It gives correct answer $n_k = n_0$ for the steady state inflow $i(t) = \frac{\mathbf{v}(n_0)}{L} n_0$.

To solve M model we use the following numerical scheme:

$$\begin{cases} M_0 = \frac{L}{\alpha} n_0 \\ n_{k+1} - n_k = i_{k+1}\Delta t - \frac{\mathbf{v}(n_k)}{L} \left(n_k + 3\left(n_k - \frac{\alpha}{L} M_k\right)\right) \Delta t \\ M_{k+1} - M_k = L i_{k+1} \Delta t - \mathbf{v}(n_k) n_k \Delta t \end{cases}$$

It also gives correct answer $n_k = n_0$ for the steady state inflow $i(t) = \frac{\mathbf{v}(n_0)}{L} n_0$. Note also that if $\alpha = 1$ then one can easily prove by induction that $M_k = L n_k$

and, therefore, the numerical solution of M model is equivalent to numerical solution of PL model.

In this section we compare models using conventional $\mathbf{v}(n)$. We want to choose rather realistic $\mathbf{v}(n)$. Empirical observations usually show that the function $\mathbf{p}(n) = \mathbf{v}(n)n$ is unimodal and the maximum is reached at some critical point n_{cr} that is less than $\frac{n_{jam}}{2}$. Also, the typical value of $\mathbf{v}(n_{cr})$ is around $\frac{1}{2}$. Thus, we will assume $n_{jam} = 3n_{cr}$ and $\mathbf{v}(n_{cr}) = \frac{1}{2}$. One of possible differentiable functions $\mathbf{v}(n)$ that satisfies these assumptions has the following form:

$$\mathbf{v}(n) = \begin{cases} 1 - \frac{n}{2n_{cr}} & , n \in [0, n_{cr}] \\ \frac{2}{1+\frac{n}{n_{cr}}} - \frac{1}{2} & , n \in [n_{cr}, 3n_{cr}] \\ 0 & , n \in [3n_{cr}, +\infty) \end{cases}$$

In figure 2.14 we show this function and corresponding $\mathbf{p}(n)$.

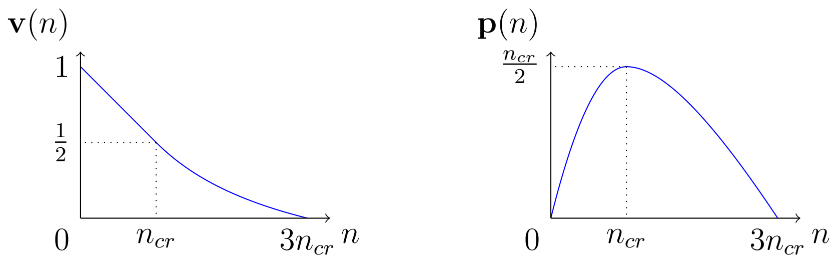


Figure 2.14: Realistic function $\mathbf{v}(n)$ and corresponding $\mathbf{p}(n)$.

We utilize this $\mathbf{v}(n)$ for our numerical simulations as it has very realistic shape. As for the trip length distributions, we want to test $f(l)$ that have α from 1.2 to 1.6 as we consider this range to be the most realistic. Also, we want to test $f(l)$ that are reasonable but not gamma-like, because, as we showed in Section 2.4, in the case of gamma-like distributions M model is a very good approximation of TB model. As it is extremely difficult to say which $f(l)$ should be tested for an arbitrary α, we focus only on extreme cases $\alpha = 1.2$ and $\alpha = 1.6$. We take distributions

$$\begin{aligned} \text{D1} &= \tfrac{1}{2}U[0, L] + \tfrac{1}{2}U[0, 3L] \\ \text{D2} &= U\left[\left(1 - \tfrac{\sqrt{3}}{2}\right) L, \left(1 + \tfrac{\sqrt{3}}{2}\right) L\right] \end{aligned}$$

that we investigated in Section 2.4. They are reasonable, because their domains belong to $[0, 3L]$, and not gamma-like (one can easily calculate that D1 has $\frac{\rho^3}{L^3} = \frac{1}{2}$ and $\delta = \frac{7}{108}$, D2 has $\frac{\rho^3}{L^3} = 0$ and $\delta = \frac{1}{48}$). These δ are not big and we can say that distributions are realistic. We show this fact in Figure 2.14:

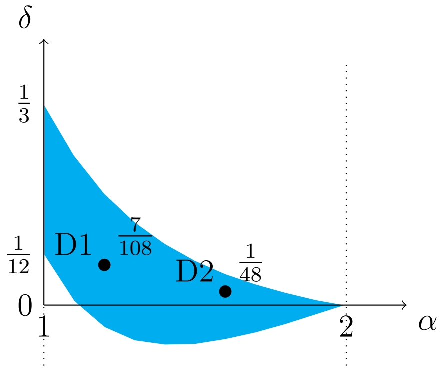


Figure 2.15: Values of δ for distributions D1 and D2 and all reasonable distributions (cyan area).

Define the capacity of the zone as the maximum possible inflow in a steady state. Obviously, it is equal to $c = \frac{\mathbf{p}(n_{cr})}{L} = \frac{n_{cr}}{2L}$. To do numerical comparison of models we should choose the input. We take the same quadratic inflow $i(t) = i_0 + 4\frac{t}{T}\left(1 - \frac{t}{T}\right)\Delta i$ as we used for investigating the hysteresis effect. However, our $\mathbf{v}(n)$ is not constant. Thus, i_0 and Δi play a big role for the result. As we try to simulate the peak hour, we should take reasonable values, such that the difference between the maximum inflow $i_0 + \Delta i$ and c is not very big. We assume $i_0 = \frac{\mathbf{v}(n_0)}{L} n_0$, in other words, $i(t)$ does not jump at $t = 0$. Thus, we can start by taking realistic value of n_0. We take $n_0 = 0.3n_{cr}$. This results in $\mathbf{v}(n_0) = 0.85$ and $i_0 = 0.51c$. These values look rather realistic for the beginning of the peak hour. We expect that with $i_0 + \Delta i > 1.2c$ and realistic T the system should definitely go to gridlock. However, if $i_0 + \Delta i < 0.9c$, the gridlock should definitely not occur. Thus, to compare models under different levels of congestion, we will check $i_0 + \Delta i$ from $0.9c$ to $1.2c$. Finally, we should define realistic value of T. More precisely, as we do not specify L, we should define T/L. Here we can imagine that T is from 1 to 3 hours and L is from 3 to 6 minutes for the zone of several kilometers in diameter. Thus, T/L should be somewhere from 10 to 60. We take the value $T/L = 30$ and do not check other ratios to not overload the analysis. For us it is more important to investigate what will happen if $i(t)$ makes some jumps. First, because real $i(t)$ cannot be very smooth and, probably, its fluctuations can be predicted based on some real-time observations. Second, because we expect to see jumps if one controls the inflow. Big jumps of $i(t)$ are not likely to happen with some optimal control, because they can produce large traffic jams that can block the traffic outside the zone. However, some small interventions might be useful. We will not try to model fluctuations in a non-control scenario. Also, we will not try to model scenarios with the optimal control, because this requires modeling of queues outside the zone and a rigorous formulation of objective function. Instead, we can think what is the typical size of the jump and the typical duration. We assume that the jump is about $0.1c$ and the duration is about L. This gives us the idea that for any continuous $i(t)$ we can consider corresponding jumping inflow $i(t)+(-1)^{[t/L]}0.1c$ to understand approximately what happens under fluctuations or optimal control. We show inflow profiles that we use for simulations in Figure 2.16

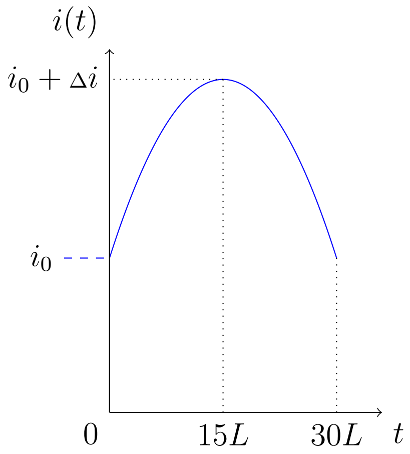


Figure 2.16: Inflow profiles that are used for numerical comparison of models. The peak-hour (left) is simulated as $i(t) = i_0 + 4\frac{t}{T}\left(1-\frac{t}{T}\right)\Delta i$, where $i_0 = 0.51c$ and $i_0 + \Delta i \in [0.9c, 1.2c]$. The corresponding scenario with jumps of inflow (right) is simulated by adding step function $(-1)^{[t/L]}0.1c$.

For the discretization of $i(t)$ and $f(l)$ we choose time step $\Delta t = \frac{L}{100}$, which is very small value. We do this to be as close to the continuous case as possible.

We start our analysis from the case of smooth inflow. In Figure 2.17 we present solutions of models for $i(t)$ with $i_0 + \Delta i = 1.05c$.

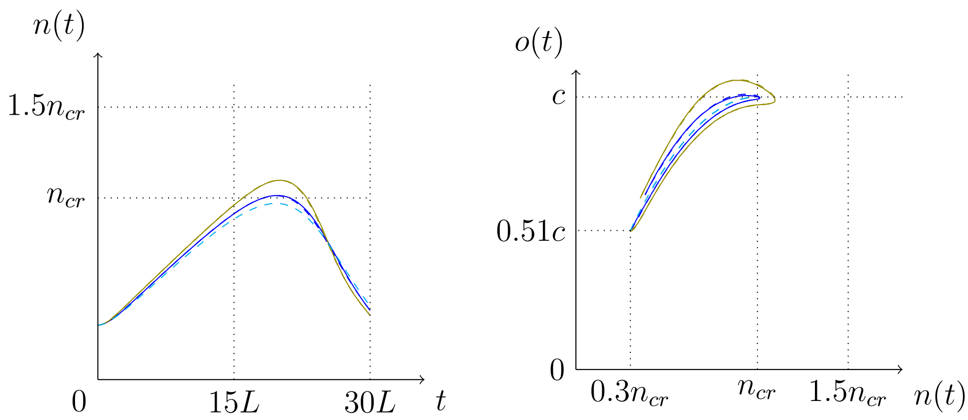


Figure 2.17: Solutions of models for smooth inflow with $i_0 + \Delta i = 1.05c$. TB model is given for D1 (blue) and D2 (olive). M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) approximates TB model very well. PL model (cyan, dashed) approximates TB model not very well.

One can notice that solution of TB model with distributions D1 and D2 exceeds n_{cr}. This effect is captured very well by M model. However, PL model says that $n(t)$ reaches only the value of approximately $0.97n_{cr}$. In Figure 2.18 we present solutions for $i(t)$ with $i_0 + \Delta i = 1.1c$. The small increase of inflow in the case when capacity is exceeded can lead to completely different results. In fact, TB model with D2 reaches gridlock.

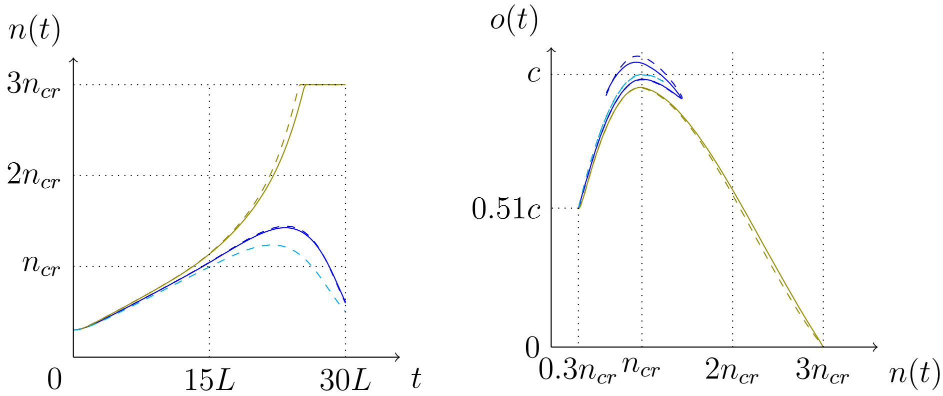


Figure 2.18: Solutions of models for smooth inflow with $i_0 + \Delta i = 1.1c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) approximates TB model very well. PL model (cyan, dashed) is not a good approximation and cannot predict gridlock for D2.

One can see that in this highly congested scenario PL model significantly underestimates accumulation. This can be a drawback while using PL model in practice.

To compare models for an arbitrary Δi we should first choose important values that will show how close are models to each other. We think that the maximum accumulation n_{max} plays a big role in practice, because it indicates the maximum level of congestion that will be reached and, consequently, control measures that might be taken. Thus, we will first look at n_{max}. The results are presented in Figure 2.19.

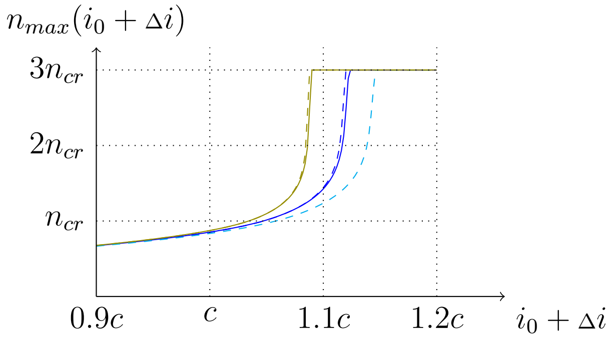


Figure 2.19: Maximum accumulation for smooth inflow depending on $i_0+\Delta i$. TB model is given for D1 (blue) and D2 (olive). M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) shows similar values. PL model (cyan, dashed) shows very different values for $i_0 + \Delta i > c$.

One can see that n_{max} in M model is very similar to n_{max} in TB model, even for the cases close to gridlock. PL model tends to underestimate n_{max} and shows values close to $1.15n_{cr}$ when TB and M models reach gridlock for D1 and close to $1.5n_{cr}$ when TB and M models reach gridlock for D2.

To have more information about the quality of approximation we suggest to look also at the average relative error. If, for example, we consider PL as an approximation of TB model then it is defined as

$$\varepsilon^{PL/TB} = \frac{1}{K}\sum_{k=1}^{K}\left|\frac{n_k^{PL} - n_{k,0}^{TB}}{n_{k,0}^{TB}}\right|$$

Such an error takes the whole time interval into account and will not say that two models are necessarily close to each other if their n_{max} are similar. Theoretically, moments of time corresponding to maximum accumulation can be different. In Figure 2.20 we present $\varepsilon^{PL/TB}$ and $\varepsilon^{M/TB}$ as functions of $i_0+\Delta i$.

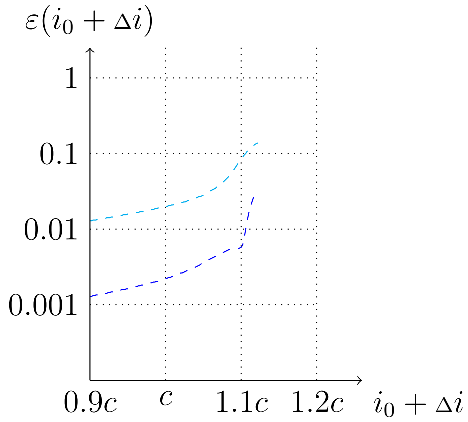


Figure 2.20: Relative error between approximations and TB model for smooth inflow. TB model is given for D1 (left) and D2 (right). The error of M model for $\alpha = 1.2$ (blue, dashed) and M model for $\alpha = 1.6$ (olive, dashed) is much lower than the error of PL model (cyan, dashed).

One can see that the error of M model is approximately one order smaller than the error of PL model. However, if $i_0 + \Delta i$ is smaller than capacity, PL model is also a good approximation of TB model.

Hereafter in this section we do the same analysis for jumping inflow. One can easily see that jumps do not produce big change of results.

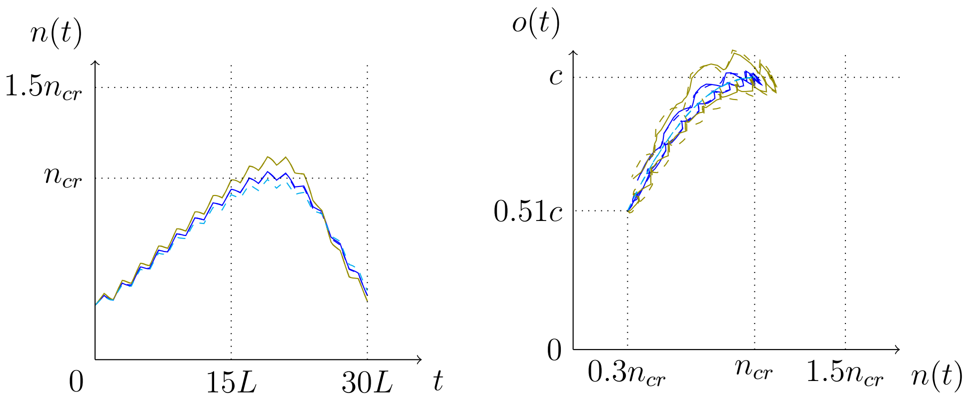


Figure 2.21: Solutions of models for jumping inflow with $i_0 + \Delta i = 1.05c$. TB model is given for D1 (blue) and D2 (olive). M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) approximates TB model very well. PL model (cyan, dashed) approximates TB model not very well.

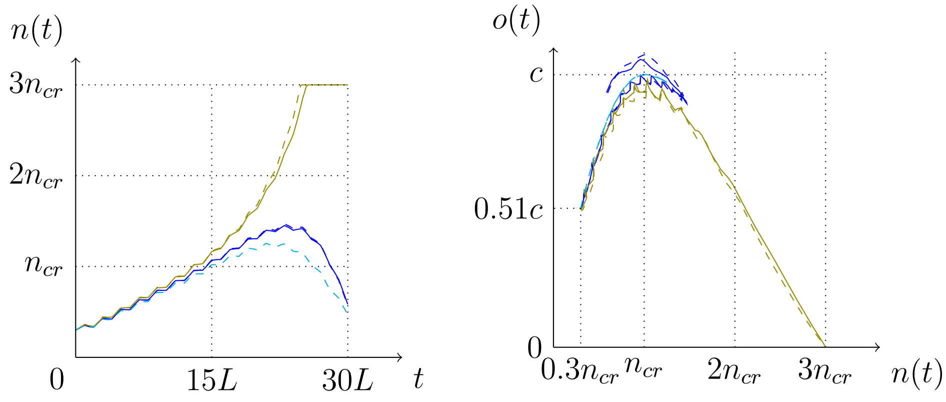


Figure 2.22: Solutions of models for jumping inflow with $i_0 + \Delta i = 1.1c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) approximates TB model very well. PL model (cyan, dashed) is not a good approximation and cannot predict gridlock for D2.

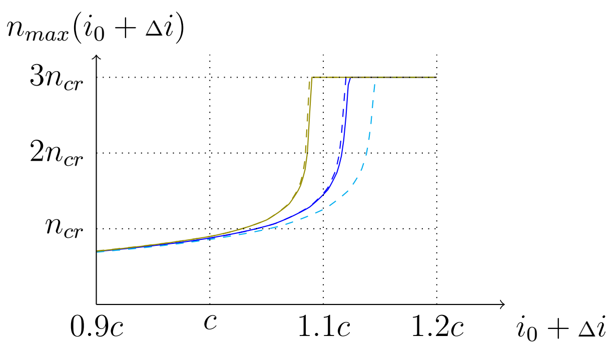
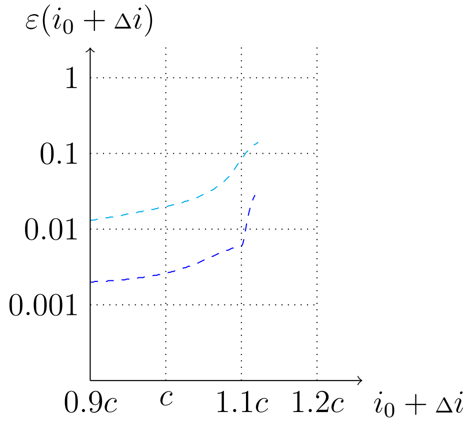


Figure 2.23: Maximum accumulation for jumping inflow depending on $i_0+\Delta i$. TB model is given for D1 (blue) and D2 (olive). M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) shows similar values. PL model (cyan, dashed) shows very different values for $i_0 + \Delta i > c$.



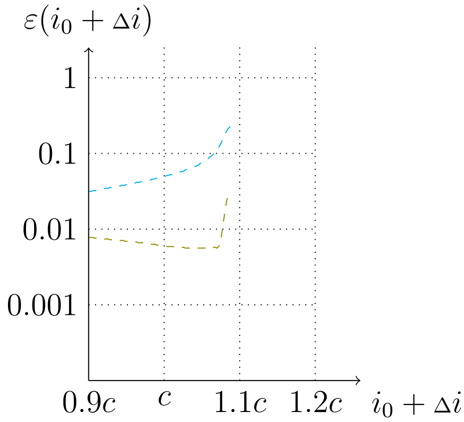


Figure 2.24: Relative error between approximations and TB model for jumping inflow. TB model is given for D1 (left) and D2 (right). The error of M model for $\alpha = 1.2$ (blue, dashed) and M model for $\alpha = 1.6$ (olive, dashed) is much lower than the error of PL model (cyan, dashed).

Interestingly, $\varepsilon^{PL/TB}$ shows very similar values to the case of smooth inflow. However, $\varepsilon^{M/TB}$ tends to be higher for jumping inflow, especially for $i_0 + \Delta i$ from $0.9c$ to c.

2.6 Conclusion and discussion

In this chapter we analyzed properties of popular network-level models that are based on the concept of speed-MFD. PL model is simple to solve numerically, but lacks physical interpretation. TB model has clear physical interpretation, but is difficult to solve numerically. To overcome this problem we suggest an approximation of TB model which has the form of ODE. We refer to this approximation as "M model". To find the solution of M model one needs to know the coefficient $\alpha = \frac{2L^2}{L^2+\sigma^2}$, where L is the mean of trip-length distribution and σ is its standard deviation. We show that for some realistic distributions M model can be equivalent to TB model, but this can happen only if $\alpha \leqslant \frac{4}{3}$ $\left(\sigma \geqslant \frac{L}{\sqrt{2}}\right)$ which does not cover all the cases that are of practical interest. Thus, we try to prove analytically that the difference between M and TB models is small. However, such an analysis is done for the case of constant speed and cannot give an answer about the case of decreasing speed-MFD, where the error can accumulate. Thus, we build numerical schemes for solving PL, TB and M models and show for some realistic conditions that M model is still a good approximation of TB model in the case of decreasing speed-MFD. At least, it approximates TB model much better than PL model. We think that TB model can arise not only in transportation field. The case of constant speed corresponds to LTI (Linear Time-Invariant) system and can appear in many different fields, from economics to pharmacokinetics. In this work we prove that M model is a good approximation of TB model for the parameters and the inputs that are typical for transportation systems. However, we believe that M model can be useful also in other fields. Investigating of this question lies outside of this work. Hereafter in this section we discuss difficulties related to applying PL, TB and M models for transportation systems and mathematical difficulties related to TB and M models.

2.6.1 Difficulties related to perimeter control

In our general framework of modeling the inflow causes the outflow. We do not assume that the parameters, i.e. speed-MFD and trip-length distribution can be changed externally. Therefore, the only natural way to control such a system is to control the inflow. We should be very careful while trying to apply this framework for the real city. First, before the real inflow control is implemented, parameters can be measured for everyday conditions only. We do not think that variety of everyday inflow patterns can cover all the cases that become feasible with inflow control. Thus, to be fully sure that the parameters will not change, one needs to perform real experiments with inflow control before building the final control strategy. We think that big rapid adjustments of inflow might destroy the speed-MFD, but small adjustments should not influence it a lot. This question is not trivial and should be investigated. Second, the modeling approach assumes that dynamics of accumulation does not depend on the exact place where the inflow comes to the zone. This makes more sense in a non-control case, because inflow should change more or less proportionally in different places. However, in a control case it is almost impossible to keep these ratios, because some portion of inflow comes from inside the zone and it is difficult to stop such vehicles separately from other vehicles. Imagine a vehicle that was parked and started its trip in a dense flow. Thus, keeping the ratios is difficult and there is no guarantee that parameters of the model will not change with control. Fortunately, we can imagine some cases (e.g. morning peak hour in a city center area) when the portion of vehicles that start trips from inside is negligibly small. For such a case the ratios of most important inflows can be kept, if there are no spillbacks that reach entrances from inside the zone. Summarizing, the presented modeling approach can be potentially used for the inflow control, but the exact implementation is still unclear and requires additional investigations.

2.6.2 Difficulties related to adjustment of speed-MFD

Though in this work we assume that $\mathbf{v}(n)$ cannot be changed externally, physically it is possible. The shape of $\mathbf{v}(n)$ should depend on the regimes of all traffic lights inside the zone. If one knows how to achieve desired speed by regulating them, he can control the system. For this type of control we can consider a modification of our modeling framework. Instead of assuming that the space-mean speed $v(t)$ is equal to $\mathbf{v}(n(t))$, we can assume that it is given exogenously and corresponding $n(t)$ allows to keep desired $v(t)$. Thus, PL model can be formulated as

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \frac{v(t)}{L} n(t), \ t \in [0,T] \end{cases}$$

and TB model can be formulated as

$$\begin{cases} n(t) = n_0 & , t \in (-\infty, 0] \\ i(t) = \frac{v(0)}{L} n_0 & , t \in (-\infty, 0] \\ n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t v(u)du\right)\right) i(s)ds & , t \in (-\infty, T] \end{cases}$$

Note that existence and uniqueness of PL and TB models holds for any positive, continuous on the interval $[0,T]$ function $v(t)$. The solution of PL model can be found as a solution of linear differential equation and the solution of TB model is given explicitly. Moreover, continuity of $n(t)$ still holds. For PL model it is obvious and for TB model it follows from the proof of Proposition 3 where we did not use the equality $v(t) = \mathbf{v}(n(t))$. M model for exogenous speed takes the form

$$\begin{cases} n(0) = n_0 \\ M(0) = \frac{L}{\alpha} n_0 \\ \dot{n}(t) = i(t) - \frac{v(t)}{L}\left(n(t) + 3\left(n(t) - \frac{\alpha}{L} M(t)\right)\right) & , t \in [0,T] \\ \dot{M}(t) = Li(t) - v(t)n(t) & , t \in [0,T] \end{cases}$$

The case of exogenous speed is very similar to the case of constant speed. Consider the change of variable $y = \int\limits_0^t v(u)du$. Define functions $n^*(y) = n(t)$, $M^*(y) = M(t)$ and $g(y) = \frac{i(t)}{v(t)}$. Denote $Y = \int\limits_0^T v(u)du$. Notice that $\dot{y}(t) = v(t)$. Therefore, the equivalent formulation of PL model is

$$\begin{cases} n^*(0) = n_0 \\ \dfrac{dn^*}{dy}(y) = g(y) - \frac{1}{L} n^*(y), \ y \in [0,Y] \end{cases} \tag{2.15}$$

The equivalent formulation of TB model is

$$\begin{cases} n^*(y) = n_0 & , y \in (-\infty, 0] \\ g(y) = \frac{1}{L} n_0 & , y \in (-\infty, 0] \\ n^*(y) = \int\limits_{-\infty}^{y} (1 - F(y-x))g(x)dx & , y \in (-\infty, Y] \end{cases} \tag{2.16}$$

The equivalent formulation of M model is

$$\begin{cases} n^*(0) = n_0 \\ M^*(0) = \frac{L}{\alpha} n_0 \\ \frac{dn^*}{dy}(y) = g(y) - \frac{1}{L}\left(n^*(y) + 3\left(n^*(y) - \frac{\alpha}{L} M^*(y)\right)\right) \\ \frac{dM^*}{dy}(y) = Lg(y) - n^*(y) \end{cases} \tag{2.17}$$

We can see that equivalent formulations of PL, TB and M models correspond to the case of constant speed. Thus, we can be sure that M model still approximates TB model very well. Summarizing, M model with exogenous speed can be potentially used in a control based on adjustment of speed-MFD.

2.6.3 Difficulties related to MPC

Real transportation systems have some degree of uncertainty. In our work the approach is deterministic. Thus, the analysis that we made ignores the fact that real $i(t)$ cannot be predicted with a good precision for the whole peak hour. The main goal of the analysis was not to show that M model approximates TB model very well on the long time horizon, but to show that it approximates TB model very well on the onset and on the offset of congestion. We think, that in practice M model should be used inside some MPC (Model Predictive Control) framework with a short time horizon that reflects the level of noise or sudden big disturbances in the system. We do not know, what is better in this situation, to use robust values of L and α measured for the whole day during a long time period, to use values of L and α measured for the current time of the day during a long time period, or to always measure L and α during the current day and make a short-time prediction of their values. In any case, applying M model seems to be easier than TB model that requires full knowledge of the trip-length distribution. Also, M model is more preferable than PL model, not because it approximates TB model better than PL model, but because it can be viewed as a generalization of PL model. If we consider M model as a self-standing model, there is no need to measure α based on the trip-length distribution (which can be not well defined). Of course, if there exists information about distribution, this can be useful, but probably, the best value of α does not fully correspond to the trip-length distribution and can be calibrated based on some other observations. Thus, PL model, that is specific case of M model for $\alpha = 1$, lacks one degree of freedom that might be important. There is one question about the most efficient way of using M model that remains unclear. Our formulation of M model assumes that initial value $M(0) = \frac{L}{\alpha} n_0$ corresponds to the steady state. We make this assumption to be able to apply the model in the case when the prehistory of $n(t)$ is not known and only the value n_0 is available. If we do MPC, we should run M model many times. Each new run can use information from previous runs. Thus, it is probably better to start M model not from the steady state value of $M(t)$, but from the value taken from the previous run, or some other run. We think, that the choice of initial value of $M(t)$ in MPC framework is an interesting research question.

2.6.4 Mathematical difficulties

First, we clarify why we took quadratic $i(t)$ for analytical comparison of models. The main goal was to show that M model is a good approximation of TB model for any $\alpha > 1$. We wanted to show this for a wide range of $f(l)$ and $i(t)$. The most difficult problem here is to find classes of $f(l)$ and $i(t)$. Recall that we started to investigate the quality of approximation from the question of equivalence between models. The main results were that M model is equivalent to TB model for any $f(l)$ if $i(t) = \frac{\mathbf{P}(n_0)}{L} = const$ and is equivalent to TB model for any $i(t)$ if $f(l)$ has some specific shape which has $\alpha \leqslant \frac{4}{3}$. Both results are not very interesting from the practical point of view. The first result is about constant $i(t)$ which means that no control is possible, the second result is about very limited range of α that do not cover most of the different values that we expect to have in reality. To solve this problem we took a class of quadratic $i(t)$ in order to relax constraints on $f(l)$. However, this forced us to switch from equivalence to convergence in the case of constant $\mathbf{v}(n)$, because equivalence is a very strong condition. Note, that putting additional constraints on $i(t)$ in order to make M model a good approximation of TB model for a wider range of $f(l)$ seems to be a principle. If one takes class of linear $i(t)$ instead of quadratic he obtains convergence of M model to TB model for all $f(l)$ instead of gamma-like.

Another important question is negative values of accumulation that can appear in M model for $\alpha > \frac{4}{3}$. If this happens during computational process, it might cause big problems in practice. However, our numerical comparison of models never faced this case. In fact, to reproduce such an effect, one needs to perform some actions that are not likely to happen in reality, for example, drop the inflow to zero. The less important drawback than negative accumulation is negative outflow which can also happen for $\alpha > \frac{4}{3}$. It might produce problems if one decides to extend the model and say that the outflow is an input to the neighboring zone. However, negative outflow usually happens if $n(t)$ is close to zero and $i(t)$ makes a big jump. This is something we also do not expect to have in reality. In any case, if the solution of M model does not exist or show some strange behavior like negative outflow or gridlock, one can interpret this situation as a non-realistic input and try to adjust it to get consistent solution.

There is one difficulty related to numerical solution of TB model. In this work we suggest to consider $f(l)$ that are mixtures of uniforms to make the discretization procedure clear. However, if one wants to discretize other type of distribution (e.g. exponential) he needs different procedure. The best approximation of exponential distribution $f(l) = \frac{1}{L}e^{-l/L}$, for our opinion, is

$$f_m = \frac{\Delta t}{L}\left(1 - \frac{\Delta t}{L}\right)^{m-1}, \; m = 1, 2, ...$$

These probabilities correspond to geometric distribution with mean $\frac{L}{\Delta t}$. The reason to take this approximation is that exponential distribution gives equivalence between TB and PL models and we want numerical solutions to be equivalent as well. We can formulate this result as

Proposition 30. *Numerical solution of TB model is equivalent to numerical solution of PL model for any input if and only if f_m, $m = 1, 2, \ldots$ correspond to geometric distribution with mean $\frac{L}{\Delta t}$.*

This example shows, that the best rule of converting continuous distribution into discrete is not obvious. Note, that we do not suggest to put $F_m = F(m\Delta t)$ which seems to be the simplest way. In this case the mean of approximation will be slightly higher than L.

Finally, we should discuss how the size of Δt influences solutions of models. In this work we used value $\Delta t = \frac{L}{100}$. However, in practice this can be computationally expensive. Especially for TB model. The number of computational steps for TB model grows quadratically with $\frac{L}{\Delta t}$, while for PL and M models it grows linearly. To get an intuition about the influence of time step on the numerical solutions of models, in Figures 2.25-2.28 we compare our results for $\Delta t = \frac{L}{100}$ with results for $\Delta t = \frac{L}{10}$.

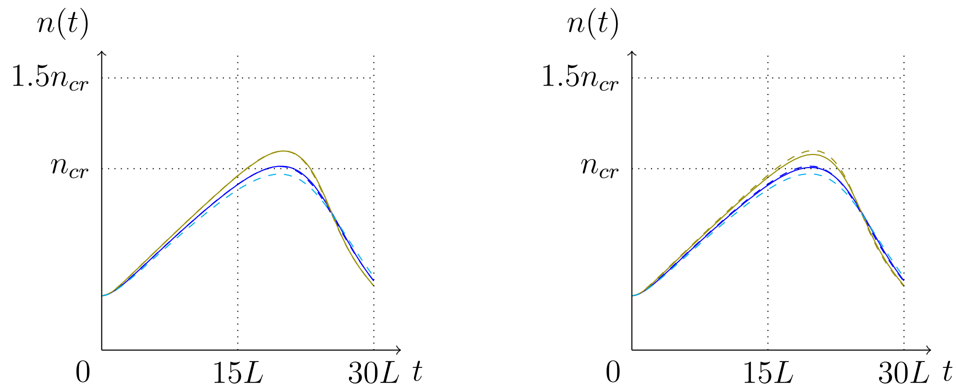


Figure 2.25: Solutions of models for smooth inflow with $i_0 + \Delta i = 1.05c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). TB model shows slightly different results for $\Delta t = \frac{L}{100}$ (left) and $\Delta t = \frac{L}{10}$ (right), while M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) is not very sensitive to the time step. PL model (cyan, dashed) is also not very sensitive to the time step.

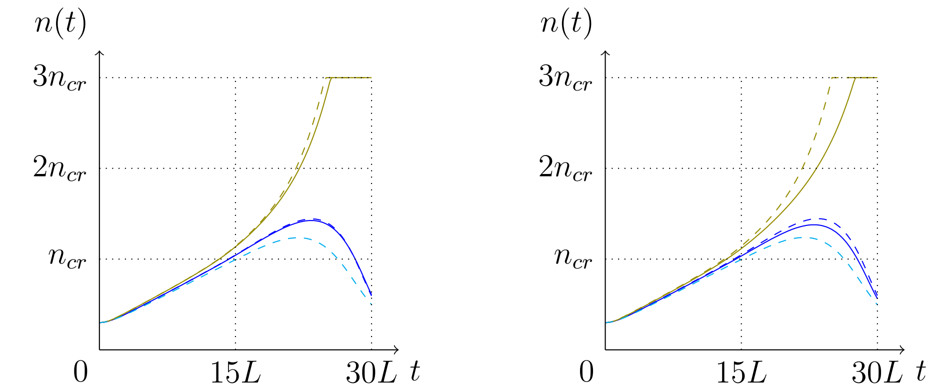


Figure 2.26: Solutions of models for smooth inflow with $i_0 + \triangle i = 1.1c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). TB model shows different results for $\triangle t = \frac{L}{100}$ (left) and $\triangle t = \frac{L}{10}$ (right), while M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) is not very sensitive to the time step. PL model (cyan, dashed) is also not very sensitive to the time step.

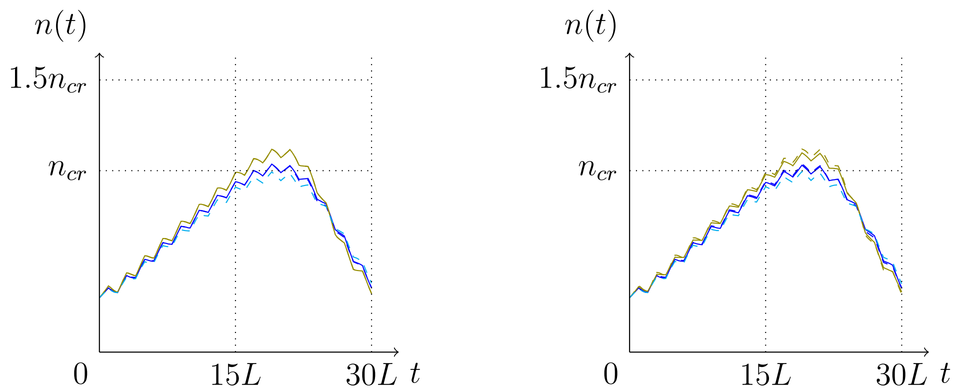


Figure 2.27: Solutions of models for jumping inflow with $i_0 + \triangle i = 1.05c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). TB model shows slightly different results for $\triangle t = \frac{L}{100}$ (left) and $\triangle t = \frac{L}{10}$ (right), while M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) is not very sensitive to the time step. PL model (cyan, dashed) is also not very sensitive to the time step.

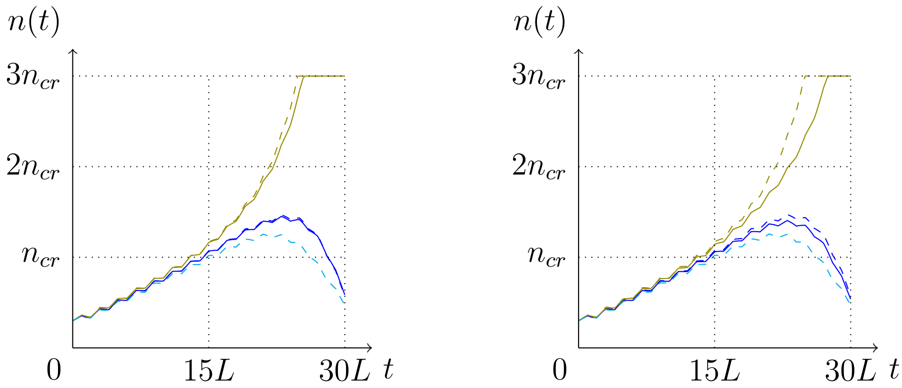


Figure 2.28: Solutions of models for jumping inflow with $i_0 + \triangle i = 1.1c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). TB model shows different results for $\triangle t = \frac{L}{100}$ (left) and $\triangle t = \frac{L}{10}$ (right), while M model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) is not very sensitive to the time step. PL model (cyan, dashed) is also not very sensitive to the time step.

We can see that the solution of M model is not very sensitive to the time step, while the solution of TB model changes a lot. This result is paradoxical, because in the case of small time step TB and M models are very close to each other and M model should be a very good approximation of continuous TB model. Thus, if we want to find the solution of continuous TB model using our numerical scheme with a big time step, M model is more preferable than TB model, even for the inflow with jumps. However, the paradox might disappear if one finds better numerical scheme for solving TB model, but this lies outside of our work.

3 Mathematical proofs

This chapter contains proofs of all propositions from Chapter 2. Here we repeat main definitions:

1. The speed-MFD function $\mathbf{v}(n)$ is a non-negative decreasing continuous function that satisfies $\mathbf{v}(0) = 1$. We assume that either $\mathbf{v}(n) = 1 = const$ (this case is called "constant" $\mathbf{v}(n)$) or $\mathbf{v}(n)$ reaches zero at $n = n_{jam}$ and has Lipschitz constant C (this case is called "conventional" $\mathbf{v}(n)$).

2. The PDF of trip length distribution in TB model $f(l)$, $l \in (0, +\infty)$ is a piecewise continuous function bounded on the interval $[0, f_{max}]$ and satisfying $\int\limits_0^{+\infty} f(l) l dl = L$. The piecewise continuity property in this thesis means finite number of jump discontinuities and no removable or essential discontinuities. We use notations $f_-(l) = \lim\limits_{\Delta l \to -0} f(l + \Delta l)$, $l \in (0, +\infty)$ and $f_+(l) = \lim\limits_{\Delta l \to +0} f(l + \Delta l)$, $l \in [0, +\infty)$ for the left and right limits of $f(l)$. The notations for the second and the third central moments of $f(l)$ are

$$\sigma^2 = \int\limits_0^{+\infty} f(l)(l - L)^2 dl = \int\limits_0^{+\infty} f(l) l^2 dl - L^2$$
$$\rho^3 = \int\limits_0^{+\infty} f(l)(l - L)^3 dl = \int\limits_0^{+\infty} f(l) l^3 dl - 3L\sigma^2 - L^3$$

We also use notation $\alpha = \frac{2L^2}{L^2 + \sigma^2}$ for the convenience. Note that $\alpha \in (0, 2)$.

3. The inflow function $i(t)$, $t \in (0, T]$ is a piecewise continuous function bounded on the interval $[0, i_{max}]$. We use notations $i_-(t) = \lim\limits_{\Delta t \to -0} i(t + \Delta t)$, $t \in (0, T]$ and $i_+(t) = \lim\limits_{\Delta t \to +0} i(t + \Delta t)$, $t \in [0, T)$ for the left and right limits of $i(t)$.

Proposition 1. *The function* $\frac{\mathbf{v}(n)}{L}n$ *is Lipschitz.*

Proof. If $\mathbf{v}(n)$ is constant then $\left|\frac{\mathbf{v}(n_2)}{L}n_2 - \frac{\mathbf{v}(n_1)}{L}n_1\right| = \left|\frac{n_2-n_1}{L}\right|$ and, therefore, $\frac{\mathbf{v}(n)}{L}n$ is Lipschitz with constant $\frac{1}{L}$. If $\mathbf{v}(n)$ is conventional then in the case $n_1,\, n_2 > n_{jam}$ we have $\left|\frac{\mathbf{v}(n_2)}{L}n_2 - \frac{\mathbf{v}(n_1)}{L}n_1\right| = 0$. Otherwise we can put $n_1 \leqslant n_{jam}$ without loss of generality. This gives

$$\begin{aligned}
&\left|\tfrac{\mathbf{v}(n_2)}{L}n_2 - \tfrac{\mathbf{v}(n_1)}{L}n_1\right| = \tfrac{1}{L}|\mathbf{v}(n_2)n_2 - \mathbf{v}(n_2)n_1 + \mathbf{v}(n_2)n_1 - \mathbf{v}(n_1)n_1| \leqslant \\
&\leqslant \tfrac{1}{L}\mathbf{v}(n_2)|n_2 - n_1| + \tfrac{1}{L}|\mathbf{v}(n_2) - \mathbf{v}(n_1)|n_1 \leqslant \\
&\leqslant \tfrac{1}{L}|n_2 - n_1| + \tfrac{1}{L}C|n_2 - n_1|n_{jam}
\end{aligned}$$

Therefore, $\frac{\mathbf{v}(n)}{L}n$ is Lipschitz with constant $\frac{1}{L}(1 + Cn_{jam})$. □

Proposition 2. *The solution* $n(t)$ *of problem*

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L}n(t),\ t \in [0,T] \end{cases}$$

exists and is unique.

Proof. If $\mathbf{v}(n)$ is constant then we get a linear differential equation with unique solution

$$n(t) = e^{-t/L}n_0 + \int\limits_0^t e^{-(t-s)/L}i(s)ds$$

If $\mathbf{v}(n)$ is conventional then we define the function $\mathbf{o}(n)$, $n \in (-\infty, +\infty)$ the way that

$$\mathbf{o}(n) = \begin{cases} 0 & ,\ n < 0 \\ \frac{\mathbf{v}(n)}{L}n & ,\ n \geqslant 0 \end{cases}$$

Obviously, it is Lipschitz with the same constant $\frac{1}{L}(1 + Cn_{jam})$ that Lipschitz function $\frac{\mathbf{v}(n)}{L}n$ has. The function $i(t)$ is piecewise continuous and, therefore, $n(t)$ that satisfies

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \mathbf{o}(n(t)) \end{cases} \tag{3.1}$$

exists and is unique in some small neighborhood $[0, \varepsilon]$ of initial point (by Cauchy-Lipschitz theorem). The uniqueness and existence of $n(t)$ on the whole interval $[0,T]$ follows from the fact that the function $n(t)$, $t \in [0,T]$ is bounded:

$$\begin{aligned} n(t) &\leqslant n_0 + i_{max}t \\ n(t) &\geqslant n_0 - \tfrac{n_{jam}}{L}t \end{aligned}$$

Now prove by contradiction that $n(t) \geqslant 0$ (and consequently $\mathbf{o}(n(t)) = \frac{\mathbf{v}(n(t))}{L} n$). Assume there exists $t_1 : n(t_1) < 0$. As the function $n(t)$ is continuous, there exists $t_2 < t_1 : t_2 = \max(t \mid t \in [0, t_1], n(t) = 0)$. Therefore,

$$0 > n(t_1) - n(t_2) = \int_{t_2}^{t_1} \dot{n}(t)dt = \int_{t_2}^{t_1} i(t)dt \geqslant 0$$

Contradiction. $\square$

Proposition 3. *If for some $T_1 \leqslant T$ the solution $n(t)$, $t \in (-\infty, T_1]$ of problem*

$$\begin{cases} n(t) = n_0 & , t \in (-\infty, 0] \\ i(t) = \frac{\mathbf{v}(n_0)}{L} n_0 & , t \in (-\infty, 0] \\ n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds & , t \in (-\infty, T_1] \end{cases}$$

exists then it is continuous.

Proof. To prove the proposition, we show that

$$\lim_{\Delta t \to 0} (n(t + \Delta t) - n(t)) = 0$$

Denote $v(t) = \mathbf{v}(n(t))$ to simplify formulas. By definition,

$$\begin{aligned} & n(t + \Delta t) - n(t) = \\ = & \int\limits_{-\infty}^{t+\Delta t} \left(1 - F\left(\int_s^{t+\Delta t} v(u)du\right)\right) i(s)ds - \\ & - \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t v(u)du\right)\right) i(s)ds = \\ = & \underbrace{\int\limits_{t}^{t+\Delta t} \left(1 - F\left(\int_s^{t+\Delta t} v(u)du\right)\right) i(s)ds -}_{N_1(t,\Delta t)} \\ & \underbrace{- \int\limits_{0}^{t} \left(F\left(\int_s^{t+\Delta t} v(u)du\right) - F\left(\int_s^t v(u)du\right)\right) i(s)ds -}_{N_2(t,\Delta t)} \\ & \underbrace{- \int\limits_{-\infty}^{0} \left(F\left(\int_s^{t+\Delta t} v(u)du\right) - F\left(\int_s^t v(u)du\right)\right) i(s)ds}_{N_3(t,\Delta t)} \end{aligned}$$

It is obvious that $|N_1(t, \Delta t)| \leqslant |\Delta t| i_{max}$ and $|N_2(t, \Delta t)| \leqslant f_{max} |\Delta t| i_{max} T_1$. As for the $N_3(t, \Delta t)$, we first introduce notation $y(t) = \int\limits_0^t v(u)du$. Then we can rewrite $N_3(t, \Delta t)$ in the following form:

$$
\begin{aligned}
&N_3(t, \Delta t) = \\
&= \int\limits_{-\infty}^{0} i(0) \left(F(y(t+\Delta t) - v(0)s) - F(y(t) - v(0)s)\right) ds = \\
&= \int\limits_{0}^{+\infty} \tfrac{i(0)}{v(0)} \left(F(y(t+\Delta t) + l) - F(y(t) + l)\right) dl = \\
&= \tfrac{n_0}{L} \int\limits_{0}^{+\infty} \int\limits_{y(t)}^{y(t+\Delta t)} f(l+x) dx dl = \tfrac{n_0}{L} \int\limits_{y(t)}^{y(t+\Delta t)} \int\limits_{0}^{+\infty} f(l+x) dl dx = \\
&= \tfrac{n_0}{L} \int\limits_{y(t)}^{y(t+\Delta t)} (1 - F(x)) dx.
\end{aligned}
$$

Therefore, $|N_3(t, \Delta t)| \leqslant \frac{n_0}{L} |\Delta t|$ and

$$|n(t+\Delta t) - n(t)| \leqslant \left(i_{max} + f_{max} i_{max} T_1 + \tfrac{n_0}{L}\right) |\Delta t|$$

which means that

$$\lim_{\Delta t \to 0} (n(t+\Delta t) - n(t)) = 0$$

$\square$

Proposition 4. *There exists unique continuous solution* $n(t)$, $t \in (-\infty, T]$ *of problem*

$$
\begin{cases}
n(t) = n_0 & , t \in (-\infty, 0] \\
i(t) = \frac{\mathbf{v}(n_0)}{L} n_0 & , t \in (-\infty, 0] \\
n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u)) du\right)\right) i(s) ds & , t \in (-\infty, T]
\end{cases}
$$

Proof. We prove the proposition by induction. We take small value $h = \frac{T}{K}$ and show for all $k = 0, 1, 2, ..., K$ that there exists unique pair of continuous on the interval $(-\infty, kh]$ functions $n_k(t)$ and $v_k(t)$ such that

$$
\begin{cases}
n_k(t) = n_0 & , t \in (-\infty, 0] \\
n_k(t) = \int\limits_{-\infty}^{t} i(s) \left(1 - F\left(\int_s^t v_k(u) du\right)\right) ds & , t \in [0, kh] \\
v_k(t) = \mathbf{v}(n_k(t)) & , t \in (-\infty, kh]
\end{cases}
$$

We construct the proof for $K > \left(\frac{n_0}{L} + f_{max} i_{max} T\right) CT$. The reason to take such K is explained later.
Basis. For $k = 0$ there exists unique pair of functions $n_0(t) = n_0$ and $v_0(t) = \mathbf{v}(n_0)$.

Induction step. For $k \geqslant 1$ we assume that $n_{k-1}(t)$ and $v_{k-1}(t)$ are already reconstructed. Now consider the set of functions

$$v_k(t) = \begin{cases} v_{k-1}(t) & , t \in (-\infty, (k-1)h) \\ \tilde{v}_k(t) & , t \in [(k-1)h, kh] \end{cases}$$

The function $\tilde{v}_k(t)$ belongs to the class $C([(k-1)h, kh], [0,1])$ i.e., it is an arbitrary continuous on the interval $[(k-1)h, kh]$ function bounded on the interval $[0,1]$. The set of such functions is complete metric space (in uniform metric). This follows from the fact that $C([(k-1)h, kh])$ is complete metric space and the limit of convergent sequence of bounded on $[0,1]$ functions is bounded on the same interval. For each $v_k(t)$ we consider $n_k(t)$ defined as

$$n_k(t) =$$
$$= \begin{cases} n_{k-1}(t) & , t \in (-\infty, (k-1)h) \\ \tilde{n}_k(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t v_k(u)du\right)\right) i(s)ds & , t \in [(k-1)h, kh] \end{cases}$$

Our goal is to show that the transformation $\tilde{v}_k(t) \to \mathbf{v}(\tilde{n}_k(t))$ is a contraction. If we prove this then, according to Banach fixed-point theorem, there exists unique $\tilde{v}_k(t) = \mathbf{v}(\tilde{n}_k(t))$ and consequently unique $v_k(t) = \mathbf{v}(n_k(t))$. The corresponding function $n_k(t)$ is continuous by Proposition 3 for $T_1 = kh$. Recall that $\mathbf{v}(n)$ is Lipschitz by definition. This means that $v_k(t)$ is also continuous.

To show that the transformation is a contraction we consider two arbitrary functions $\tilde{v}^a_k(t)$, $\tilde{v}^b_k(t)$ from $C([(k-1)h, kh], [0,1])$ and corresponding $\tilde{n}^a_k(t)$, $\tilde{n}^b_k(t)$. Define

$$\begin{aligned} \Delta v &= \max_{t \in [(k-1)h, kh]} \left|\tilde{v}^a_k(t) - \tilde{v}^b_k(t)\right| \\ \Delta n &= \max_{t \in [(k-1)h, kh]} \left|\tilde{n}^a_k(t) - \tilde{n}^b_k(t)\right| \\ \Delta V &= \max_{t \in [(k-1)h, kh]} \left|\mathbf{v}(\tilde{n}^a_k(t)) - \mathbf{v}(\tilde{n}^b_k(t))\right| \end{aligned}$$

The transformation is a contraction if $\Delta V \leqslant \varepsilon \Delta v$ for some $\varepsilon < 1$. For the proof we denote the function $v_k(t)$ that corresponds to $\tilde{v}^a_k(t)$ or $\tilde{v}^b_k(t)$ as $v^a_k(t)$ or $v^b_k(t)$ respectively. Denote also $y^a_k(t) = \int\limits_0^t v^a_k(u)du$, $y^b_k(t) = \int\limits_0^t v^b_k(u)du$. The upper bound on Δn can be found as

$$\Delta n = \max_{t\in[(k-1)h,kh]} \left| \int\limits_{-\infty}^{t} \left(F\left(\int_s^t v_k^a(u)du\right) - F\left(\int_s^t v_k^b(u)du\right)\right) i(s)ds \right| \leqslant$$

$$\leqslant \underbrace{\max_{t\in[(k-1)h,kh]} \left| \int\limits_{-\infty}^{0} \left(F\left(\int_s^t v_k^a(u)du\right) - F\left(\int_s^t v_k^b(u)du\right)\right) i(s)ds \right|}_{n_k^{ab}} +$$

$$+ \underbrace{\max_{t\in[(k-1)h,kh]} \left| \int\limits_{0}^{t} \left(F\left(\int_s^t v_k^a(u)du\right) - F\left(\int_s^t v_k^b(u)du\right)\right) i(s)ds \right|}_{N_k^{ab}}$$

$$n_k^{ab} = \max_{t\in[(k-1)h,kh]} \left| \int\limits_{0}^{+\infty} \left(F\left(l + y_k^a(t)\right) - F\left(l + y_k^b(t)\right)\right) \tfrac{i(0)}{\mathbf{v}(n_0)} dl \right| =$$
$$= \max_{t\in[(k-1)h,kh]} \left| \tfrac{n_0}{L} \int\limits_{0}^{+\infty} \int\limits_{y_k^b(t)}^{y_k^a(t)} f(l+x)dxdl \right| =$$
$$= \max_{t\in[(k-1)h,kh]} \left| \tfrac{n_0}{L} \int\limits_{y_k^b(t)}^{y_k^a(t)} \int\limits_{0}^{+\infty} f(l+x)dldx \right| =$$
$$= \max_{t\in[(k-1)h,kh]} \left| \tfrac{n_0}{L} \int\limits_{y_k^b(t)}^{y_k^a(t)} (1-F(x))dx \right| \leqslant$$
$$\leqslant \max_{t\in[(k-1)h,kh]} \left| \tfrac{n_0}{L} \left(y_k^a(t) - y_k^b(t) \right) \right| \leqslant \tfrac{n_0}{L} \Delta v h$$

$$N_k^{ab} \leqslant f_{max} \Delta v h i_{max} T$$

$$\Delta n \leqslant n_k^{ab} + N_k^{ab} \leqslant \left(\tfrac{n_0}{L} + f_{max} i_{max} T \right) \Delta v h$$

Using Lipschitz property of $\mathbf{v}(n)$, we get

$$\Delta V \leqslant C \Delta n \leqslant \left(\tfrac{n_0}{L} + f_{max} i_{max} T \right) C \Delta v h = \left(\tfrac{n_0}{L} + f_{max} i_{max} T \right) C \Delta v \tfrac{T}{K}$$

Therefore, the transformation is a contraction for any

$$K > \left(\tfrac{n_0}{L} + f_{max} i_{max} T \right) CT$$

□

Proposition 5. *The outflow in TB model is equal to*

$$o(t) = \mathbf{v}(n(t)) \int\limits_{-\infty}^{t} f\left(\int_s^t \mathbf{v}(n(u))du \right) i(s)ds$$

Proof. The function $i(t)$ is not necessarily continuous and, therefore, we have to prove that $\dot{n}_+(t) = i_+(t) - o(t)$ and $\dot{n}_-(t) = i_-(t) - o(t)$, where

$$o(t) = \mathbf{v}(n(t)) \int\limits_{-\infty}^{t} f\left(\int_s^t \mathbf{v}(n(u))du\right) i(s)ds$$

Here we show the proof for the right derivative, the proof for the left one can be obtained the same way. By definition, the right derivative is equal to

$$\lim_{\Delta t \to +0} \tfrac{n(t+\Delta t)-n(t)}{\Delta t}$$

The expression for the difference $n(t+\Delta t) - n(t)$ is already given in the proof of Proposition 3. We will use the same notation for the terms:

$$\lim_{\Delta t \to +0} \tfrac{n(t+\Delta t)-n(t)}{\Delta t} = \underbrace{\lim_{\Delta t \to +0} \tfrac{N_1(t,\Delta t)}{\Delta t}}_{R_1(t)} - \underbrace{\lim_{\Delta t \to +0} \tfrac{N_2(t,\Delta t)}{\Delta t}}_{R_2(t)} - \underbrace{\lim_{\Delta t \to +0} \tfrac{N_3(t,\Delta t)}{\Delta t}}_{R_3(t)}$$

Now calculate each of $R_1(t), R_2(t), R_3(t)$ separately:

$$R_1(t) = \lim_{\Delta t \to +0}\left(\tfrac{1}{\Delta t}\int\limits_{t}^{t+\Delta t}\left(1 - F\left(\int_s^{t+\Delta t} v(u)du\right)\right) i(s)ds\right) =$$
$$= \lim_{\Delta t \to +0}\left(\tfrac{1}{\Delta t}\int\limits_{t}^{t+\Delta t}\left(1 - F\left(\int_s^{t+\Delta t} v(u)du\right)\right) i_+(s)ds\right)$$

According to Mean value theorem, $R_1(t) = 1{\cdot}i_+(t) = i_+(t)$.

$$R_2(t) =$$
$$= \lim_{\Delta t \to +0}\left(\tfrac{1}{\Delta t}\int\limits_{0}^{t}\left(F\left(\int_s^{t+\Delta t} v(u)du\right) - F\left(\int_s^{t} v(u)du\right)\right) i(s)ds\right) =$$
$$= \lim_{\Delta t \to +0}\int\limits_{0}^{t} \tfrac{F\left(\int_s^{t+\Delta t} v(u)du\right) - F\left(\int_s^{t} v(u)du\right)}{\Delta t} i(s)ds$$

According to Mean value theorem,

$$\lim_{\Delta t \to +0} \tfrac{F\left(\int_s^{t+\Delta t} v(u)du\right) - F\left(\int_s^{t} v(u)du\right)}{\Delta t} = \mathbf{v}(n(t)) f_+\left(\int_s^t v(u)du\right) \leqslant f_{max}$$

The function $f_{max} i(s)$ is integrable on the interval $[0, t]$. Therefore, we can use Lebesgue's dominated convergence theorem to put the limit inside the integral:

$$R_2(t) = \lim_{\Delta t \to +0}\int\limits_{0}^{t} \tfrac{F\left(\int_s^{t+\Delta t} v(u)du\right) - F\left(\int_s^{t} v(u)du\right)}{\Delta t} i(s)ds =$$
$$= \int\limits_{0}^{t} \mathbf{v}(n(t)) f_+\left(\int_s^t v(u)du\right) i(s)ds = \mathbf{v}(n(t))\int\limits_{0}^{t} f\left(\int_s^t v(u)du\right) i(s)ds$$

To calculate $R_3(t)$ we take the formula for $N_3(t, \vartriangle t)$ from Proposition 3, where the notation $y(t) = \int\limits_0^t \mathbf{v}(n(u))du$ is used. This gives

$$
\begin{aligned}
R_3(t) &= \lim_{\Delta t \to +0} \tfrac{N_3(t,\Delta t)}{\Delta t} = \\
&= \lim_{\Delta t \to +0} \tfrac{n_0}{L\Delta t} \int\limits_{y(t)}^{y(t+\Delta t)} (1 - F(x))dx = \\
&= \tfrac{n_0 \mathbf{v}(n(t))}{L} \left(1 - F\left(\int_0^t \mathbf{v}(n(u))du\right)\right) = \\
&= \tfrac{i(0)\mathbf{v}(n(t))}{\mathbf{v}(n_0)} \int\limits_0^{+\infty} f\left(\int_0^t \mathbf{v}(n(u))du + x\right) dx = \\
&= i(0)\mathbf{v}(n(t)) \int\limits_{-\infty}^0 f\left(\int_0^t \mathbf{v}(n(u))du - \mathbf{v}(n_0)s\right) ds = \\
&= \mathbf{v}(n(t)) \int\limits_{-\infty}^0 f\left(\int_s^t \mathbf{v}(n(u))du\right) i(0)ds
\end{aligned}
$$

Finally, combining $R_1(t)$, $R_2(t)$, $R_3(t)$, we get

$$
\lim_{\Delta t \to +0} \tfrac{n(t+\Delta t)-n(t)}{\Delta t} = i_+(t) - \mathbf{v}(n(t)) \int\limits_{-\infty}^t f\left(\int_s^t \mathbf{v}(n(u))du\right) i(s)ds
$$

$\square$

Proposition 6. *The outflow in TB model is continuous function.*

Proof. As the function $v(t) = \mathbf{v}(n(t))$ is continuous, we have to prove the continuity of function $\int\limits_{-\infty}^t f\left(\int_s^t \mathbf{v}(n(u))du\right) i(s)ds$. We show that its right limit at point $t + \vartriangle t$ when $\vartriangle t$ tends to 0 is equal to the value at point t. The proof for the left limit can be obtained the same way.

$$\lim_{\Delta t \to +0} \left(\int\limits_{-\infty}^{t+\Delta t} f\left(\int_s^{t+\Delta t} \mathbf{v}(n(u))du \right) i(s)ds - \int\limits_{-\infty}^{t} f\left(\int_s^{t} \mathbf{v}(n(u))du \right) i(s)ds \right) =$$
$$= \lim_{\Delta t \to +0} \left(\int\limits_{t}^{t+\Delta t} f\left(\int_s^{t+\Delta t} \mathbf{v}(n(u))du \right) i(s)ds \right) +$$
$$+ \lim_{\Delta t \to +0} \left(\int\limits_{0}^{t} \left(f\left(\int_s^{t+\Delta t} \mathbf{v}(n(u))du \right) - f\left(\int_s^{t} \mathbf{v}(n(u))du \right) \right) i(s)ds \right) +$$
$$+ \lim_{\Delta t \to +0} \left(\int\limits_{-\infty}^{0} \left(f\left(\int_s^{t+\Delta t} \mathbf{v}(n(u))du \right) - f\left(\int_s^{t} \mathbf{v}(n(u))du \right) \right) i(s)ds \right) =$$
$$= 0 + \int\limits_{0}^{t} \lim_{\Delta t \to +0} \left(f\left(\int_s^{t+\Delta t} \mathbf{v}(n(u))du \right) - f\left(\int_s^{t} \mathbf{v}(n(u))du \right) \right) i(s)ds -$$
$$- \frac{i(0)}{\mathbf{v}(n_0)} \lim_{\Delta t \to +0} \left(F\left(\int_0^{t+\Delta t} \mathbf{v}(n(u))du \right) - F\left(\int_0^{t} \mathbf{v}(n(u))du \right) \right) =$$
$$= 0 + \int\limits_{0}^{t} 0 \cdot i(s)ds - 0 = 0$$

As $\left| f\left(\int_s^{t+\Delta t} \mathbf{v}(n(u))du \right) - f\left(\int_s^{t} \mathbf{v}(n(u))du \right) \right| \leqslant f_{max}$ and the function $f_{max} i(s)$ is integrable on the interval $[0, t]$, we applied here Lebesgue's dominated convergence theorem to put the limit inside the integral. □

Proposition 7. *If $i_+(0) = \frac{\mathbf{p}(n_0)}{L} + \Delta i$ and $\mathbf{v}(n)$ is differentiable then the outflow in TB model has right derivative*

$$\dot{o}_+(0) = \left(\frac{\mathbf{p}'(n_0)}{L} + (L f_+(0) - 1) \frac{\mathbf{v}(n_0)}{L} \right) \Delta i$$

Proof.

$$\dot{o}_+(0) = \lim_{\Delta t \to +0} \tfrac{o(\Delta t) - o(0)}{\Delta t} =$$

$$= \lim_{\Delta t \to +0} \tfrac{\mathbf{v}(n(\Delta t))}{\Delta t} \int\limits_0^{\Delta t} f\left(\int_s^{\Delta t} \mathbf{v}(n(u))du\right) i(s)ds +$$

$$+ \tfrac{\mathbf{v}(n_0)}{L} n_0 \lim_{\Delta t \to +0} \tfrac{1}{\Delta t} \left(\mathbf{v}(n(\Delta t)) \int\limits_{-\infty}^{0} f\left(\int_s^{\Delta t} \mathbf{v}(n(u))du\right) ds - 1\right) =$$

$$= \mathbf{v}(n_0) \lim_{\Delta t \to +0} \tfrac{1}{\Delta t} \int\limits_0^{\Delta t} f_+\left(\int_s^{\Delta t} \mathbf{v}(n(u))du\right) i_+(s)ds +$$

$$+ \tfrac{\mathbf{v}(n_0)}{L} n_0 \lim_{\Delta t \to +0} \tfrac{1}{\Delta t} \left(\tfrac{\mathbf{v}(n(\Delta t))}{\mathbf{v}(n_0)} \left(1 - F\left(\int_0^{\Delta t} \mathbf{v}(n(u))du\right)\right) - 1\right) =$$

$$= \mathbf{v}(n_0) f_+(0) i_+(0) + \tfrac{n_0}{L} \lim_{\Delta t \to +0} \tfrac{\mathbf{v}(n(\Delta t)) - \mathbf{v}(n_0)}{\Delta t} -$$

$$- \tfrac{n_0}{L} \lim_{\Delta t \to +0} \mathbf{v}(n(\Delta t)) \lim_{\Delta t \to +0} \tfrac{1}{\Delta t} F\left(\int_0^{\Delta t} \mathbf{v}(n(u))du\right) =$$

$$= \mathbf{v}(n_0) f_+(0) i_+(0) + \tfrac{n_0}{L} \mathbf{v}'(n_0) \dot{n}_+(0) - \tfrac{n_0}{L} (\mathbf{v}(n_0))^2 f_+(0) =$$

$$= \mathbf{v}(n_0) f_+(0) \left(i_+(0) - \tfrac{\mathbf{p}(n_0)}{L}\right) + \tfrac{n_0}{L} \mathbf{v}'(n_0) \dot{n}_+(0) =$$

$$= \mathbf{v}(n_0) f_+(0) \Delta i + \tfrac{n_0}{L} \mathbf{v}'(n_0) \Delta i = \left(\tfrac{\mathbf{p}'(n_0)}{L} + (L f_+(0) - 1) \tfrac{\mathbf{v}(n_0)}{L}\right) \Delta i$$

$\square$

Proposition 8. *TB model is equivalent to PL model for any input if and only if* $f(l) = \frac{1}{L} e^{-l/L}$.

Proof. First, prove that if $f(l) = \frac{1}{L} e^{-l/L}$ then TB model is equivalent to PL model for any input. To do this we show that the outflow in TB model is always equal to $\frac{\mathbf{v}(n(t))}{L} n(t)$:

$$1 - F(l) = e^{-l/L} = L f(l)$$

$$o(t) = \mathbf{v}(n(t)) \int\limits_{-\infty}^{t} f\left(\int_s^t v(u)du\right) i(s)ds =$$

$$= \mathbf{v}(n(t)) \int\limits_{-\infty}^{t} \tfrac{1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)}{L} i(s)ds = \tfrac{\mathbf{v}(n(t))}{L} n(t)$$

Now prove that if the distribution is not exponential then there is an input such that the outflow in TB model differs from $\frac{\mathbf{v}(n(t))}{L} n(t)$ for some t. First notice that $f(l) - \frac{1-F(l)}{L} \neq 0$ for some l. This function is piecewise continuous and, therefore, exists an interval (l_1, l_2) such that $f(l) - \frac{1-F(l)}{L} > 0$, $l \in (l_1, l_2)$ or $f(l) - \frac{1-F(l)}{L} < 0$, $l \in (l_1, l_2)$. Consider the input

$$n_0 = 0$$

$$i(t) = \begin{cases} j,\ t \in (0, h] \\ 0,\ t \in (h, T] \end{cases}$$

where $h < l_2 - l_1$. The system will not go to a gridlock if we choose $j < \frac{n_{jam}}{h}$. Moreover, the minimum speed is not lower than $\mathbf{v}(jh)$. Now choose $T > \frac{l_2}{\mathbf{v}(jh)}$. This condition allows to find $t < T$ such that $\int\limits_0^t \mathbf{v}(n(u))du = l_2$. Notice that for all $s \in (0, h]$ we have $\int\limits_s^t \mathbf{v}(n(u))du = \int\limits_0^t \mathbf{v}(n(u))du - \int\limits_0^s \mathbf{v}(n(u))du$, where $\int\limits_0^s \mathbf{v}(n(u))du \in (0, l_2 - l_1)$. Therefore, $\int\limits_s^t \mathbf{v}(n(u))du \in (l_1, l_2)$ and

$$\mathbf{v}(n(t)) \int\limits_{-\infty}^t \left(f\left(\int_s^t \mathbf{v}(n(u))du\right) - \frac{1-F\left(\int_s^t \mathbf{v}(n(u))du\right)}{L} \right) i(s)ds$$

has the same sign as $f(l) - \frac{1-F(l)}{L}$ on (l_1, l_2), i.e. $o(t) - \frac{\mathbf{v}(n(t))}{L} n(t) \neq 0$. □

Proposition 9.

$$\frac{\partial n}{\partial t}(t, a) = (1 - F(a))i(t) - \mathbf{v}(n(t)) \int\limits_{-\infty}^t f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds$$

Proof. The function $i(t)$ is not necessarily continuous and the proposition should be understood as

$$\begin{cases} & \lim\limits_{\Delta t \to -0} \frac{n(t+\Delta t, a) - n(t,a)}{\Delta t} = \\ = & i_-(t)(1 - F(a)) - \mathbf{v}(n(t)) \int\limits_{-\infty}^t f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds \\ & \lim\limits_{\Delta t \to +0} \frac{n(t+\Delta t, a) - n(t,a)}{\Delta t} = \\ = & i_+(t)(1 - F(a)) - \mathbf{v}(n(t)) \int\limits_{-\infty}^t f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds \end{cases}$$

The proof is almost the same as the proof of Proposition 5. Thus, we will not repeat it (one should always put $a + \int\limits_s^t \mathbf{v}(n(u))du$ instead of $\int\limits_s^t \mathbf{v}(n(u))du$). □

Proposition 10.

$$\frac{\partial n}{\partial a}(t, a) = -\int\limits_{-\infty}^t f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds$$

Proof.

$$\frac{\partial n}{\partial a}(t, a) = \lim_{\Delta a \to 0} \frac{n(t, a+\Delta a) - n(t,a)}{\Delta a} =$$
$$= \lim_{\Delta a \to 0} \int\limits_0^t \frac{F\left(a + \int_s^t \mathbf{v}(n(u))du\right) - F\left(a + \Delta a + \int_s^t \mathbf{v}(n(u))du\right)}{\Delta a} i(s)ds +$$
$$+ \lim_{\Delta a \to 0} \int\limits_{-\infty}^0 \frac{F\left(a + \int_s^t \mathbf{v}(n(u))du\right) - F\left(a + \Delta a + \int_s^t \mathbf{v}(n(u))du\right)}{\Delta a} i(s)ds$$

Notice that $\left|\frac{F\left(a+\int_s^t \mathbf{v}(n(u))du\right)-F\left(a+\Delta a+\int_s^t \mathbf{v}(n(u))du\right)}{\Delta a}\right| \leqslant f_{max}$ and the function $f_{max}i(s)$ is integrable on the interval $[0, t]$. Therefore, we can use Lebesgue's dominated convergence theorem to put the first limit inside the integral:

$$\lim_{\Delta a \to 0} \int_0^t \frac{F\left(a+\int_s^t \mathbf{v}(n(u))du\right)-F\left(a+\Delta a+\int_s^t \mathbf{v}(n(u))du\right)}{\Delta a} i(s)ds =$$

$$= \int_0^t \lim_{\Delta a \to 0} \frac{F\left(a+\int_s^t \mathbf{v}(n(u))du\right)-F\left(a+\Delta a+\int_s^t \mathbf{v}(n(u))du\right)}{\Delta a} i(s)ds =$$

$$= -\int_0^t f\left(a + \int_s^t \mathbf{v}(n(u))du\right) i(s)ds$$

The second limit can be calculated as follows:

$$\lim_{\Delta a \to 0} \int_{-\infty}^0 \frac{F\left(a+\int_s^t \mathbf{v}(n(u))du\right)-F\left(a+\Delta a+\int_s^t \mathbf{v}(n(u))du\right)}{\Delta a} i(s)ds =$$

$$= -\lim_{\Delta a \to 0} \frac{i(0)}{\mathbf{v}(n_0)\Delta a} \int_{a+y(t)}^{a+\Delta a+y(t)} (1 - F(x))dx =$$

$$= -\frac{i(0)}{\mathbf{v}(n_0)} \left(1 - F\left(a + \int_0^t \mathbf{v}(n(u))du\right)\right) =$$

$$= -\frac{i(0)}{\mathbf{v}(n_0)} \int_0^{+\infty} f\left(a + \int_0^t \mathbf{v}(n(u))du + l\right) dl =$$

$$= -i(0) \int_{-\infty}^0 f\left(a + \int_0^t \mathbf{v}(n(u))du - \mathbf{v}(n_0)s\right) ds =$$

$$= -i(0) \int_{-\infty}^0 f\left(a + \int_s^t \mathbf{v}(n(u))du\right) ds$$

Combining results, we get $\frac{\partial n}{\partial a}(t, a) = -\int_{-\infty}^t i(s)f\left(a + \int_s^t \mathbf{v}(n(u))du\right) ds.$ □

Proposition 11.

$$n(0, a) = n_0 \int_0^{+\infty} \frac{1-F(a+l)}{L} dl$$

Proof. By definition,

$$n(0, a) = \int_{-\infty}^0 \left(1 - F\left(a + \int_s^0 \mathbf{v}(n(u))du\right)\right) i(0)ds =$$

$$= \int_{-\infty}^0 (1 - F(a - \mathbf{v}(n_0)s))\, i(0)ds = \frac{i(0)}{\mathbf{v}(n_0)} \int_0^{+\infty} (1 - F(a + l))dl =$$

$$= n_0 \int_0^{+\infty} \frac{1-F(a+l)}{L} dl$$

□

Proposition 12. *If $n(t,a)$ is a solution of problem*

$$\begin{cases} n(0,a) = n_0 \int\limits_0^{+\infty} \frac{1-F(a+l)}{L} dl & , a \in [0,+\infty) \\ \frac{\partial n}{\partial t}(t,a) - \mathbf{v}(n(t,0)) \frac{\partial n}{\partial a}(t,a) = (1-F(a))i(t) & , t \in [0,T],\ a \in [0,+\infty) \\ \mathbf{v}(n(t,0)) > 0 & , t \in (0,T] \end{cases}$$

then $n(t,a) = \int\limits_{-\infty}^{t} \left(1 - F\left(a + \int_s^t \mathbf{v}(n(u,0))du\right)\right) i(s)ds$, *where* $n(t,0) = n_0$, $t \in (-\infty, 0]$ *and* $i(t) = \frac{\mathbf{v}(n_0)}{L} n_0$, $t \in (-\infty, 0]$.

Proof. Any solution $n(t,a)$ of the above problem has left and right derivatives with respect to t. Therefore, the function $n(t,0)$ is continuous, and $\mathbf{v}(n(t,0))$ is also continuous. Denote $y(t) = \int\limits_0^t \mathbf{v}(n(u,0))du$. This function is increasing and, therefore, we can introduce inverse function $t(y)$ and the function $n^*(y,a) = n(t(y),a)$. Notice that

$$\begin{aligned} &\frac{\partial n^*}{\partial y}(y,a) - \frac{\partial n^*}{\partial a}(y,a) = \frac{1}{\mathbf{v}(n(t,0))} \frac{\partial n}{\partial t}(t,a) - \frac{\partial n}{\partial a}(t,a) = \\ &= \frac{i(t(y))}{\mathbf{v}(n(t(y),0))}(1-F(a)) \end{aligned}$$

Denote $g(y) = \frac{i(t(y))}{\mathbf{v}(n(t(y),0))}$. Now it can be easily seen that we have a non-homogeneous transport equation:

$$\frac{\partial n^*}{\partial y}(y,a) - \frac{\partial n^*}{\partial a}(y,a) = g(y)(1-F(a))$$

It is well-known, that the solution of such an equation can be written in an integral form. We show here that with our boundary condition we get the integral of interest. First, make a change of variables $n^*(y,a) = n^{**}(\phi_1,\phi_2)$, where

$$\begin{aligned} \phi_1 &= a + y \\ \phi_2 &= y \end{aligned}$$

Second, find the equation on $n^{**}(\phi_1,\phi_2)$ and solve the initial problem:

$$\begin{aligned} \frac{\partial n^*}{\partial y} &= \frac{\partial n^{**}}{\partial \phi_1}\frac{\partial \phi_1}{\partial y} + \frac{\partial n^{**}}{\partial \phi_2}\frac{\partial \phi_2}{\partial y} = \frac{\partial n^{**}}{\partial \phi_1} + \frac{\partial n^{**}}{\partial \phi_2} \\ \frac{\partial n^*}{\partial a} &= \frac{\partial n^{**}}{\partial \phi_1}\frac{\partial \phi_1}{\partial a} + \frac{\partial n^{**}}{\partial \phi_2}\frac{\partial \phi_2}{\partial a} = \frac{\partial n^{**}}{\partial \phi_1} \\ \frac{\partial n^{**}}{\partial \phi_2} &= \frac{\partial n^*}{\partial y} - \frac{\partial n^*}{\partial a} = g(\phi_2)(1 - F(\phi_1 - \phi_2)) \end{aligned}$$

$$\begin{aligned}
&n(t,a) = n^{**}(a+y(t),y(t)) = \\
&= n^{**}(a+y(t),0) + \int\limits_0^y g(\phi_2)(1-F(a+y(t)-\phi_2))d\phi_2 = \\
&= n(0,a+y(t)) + \int\limits_0^t \left(1-F\left(a+y(t)-\int_0^s \mathbf{v}(n(u,0))du\right)\right)i(s)ds = \\
&= \int\limits_{-\infty}^0 \left(1-F\left(a+\int_0^t \mathbf{v}(n(u,0))du-\mathbf{v}(n_0)s\right)\right)i(0)ds+ \\
&+ \int\limits_0^t \left(1-F\left(a+\int_s^t \mathbf{v}(n(u,0))du\right)\right)i(s)ds = \\
&= \int\limits_{-\infty}^t \left(1-F\left(a+\int_s^t \mathbf{v}(n(u,0))du\right)\right)i(s)ds
\end{aligned}$$

□

Proposition 13. *For any $\phi > -1$ the following equalities hold:*

$$\int\limits_0^{+\infty}(1-F(l))l^\phi dl = \frac{1}{\phi+1}\int\limits_0^{+\infty} f(l)l^{\phi+1}dl$$

$$\int\limits_0^{+\infty}\int\limits_0^{+\infty}(1-F(a+l))l^\phi dlda = \frac{1}{(\phi+2)(\phi+1)}\int\limits_0^{+\infty} f(l)l^{\phi+2}dl$$

Proof.

$$\int\limits_0^{+\infty}(1-F(l))l^\phi dl = \int\limits_0^{+\infty} f(x)\int\limits_0^x l^\phi dldx = \frac{1}{\phi+1}\int\limits_0^{+\infty} f(x)x^{\phi+1}dx$$

$$\begin{aligned}
&\int\limits_0^{+\infty}\int\limits_0^{+\infty}(1-F(a+l))l^\phi dlda = \int\limits_0^{+\infty}\int\limits_0^{+\infty} l^\phi \int\limits_{a+l}^{+\infty} f(x)dxdlda = \\
&= \int\limits_0^{+\infty} f(x)\int\limits_0^x l^\phi \int\limits_0^{x-l} dadldx = \int\limits_0^{+\infty} f(x)\int\limits_0^x (x-l)l^\phi dldx = \\
&= \left(\frac{1}{\phi+1}-\frac{1}{\phi+2}\right)\int\limits_0^{+\infty} f(x)x^{\phi+2}dx = \frac{1}{(\phi+2)(\phi+1)}\int\limits_0^{+\infty} f(x)x^{\phi+2}dx
\end{aligned}$$

□

Proposition 14.

$$\dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t)$$

Proof. Using Proposition 9 we can express $n(t,a)$ the following way:

$$\begin{aligned}
n(t,a) &= n(0,a) + \int\limits_0^t (1-F(a))i(\tilde{t})d\tilde{t} - \\
&- \int\limits_0^t \mathbf{v}(n(\tilde{t}))\int\limits_{-\infty}^{\tilde{t}} f\left(a+\int_s^{\tilde{t}} \mathbf{v}(n(u))du\right)i(s)dsd\tilde{t}
\end{aligned}$$

This results in

$$
\begin{aligned}
M(t) = & \int_0^{+\infty} n(t,a)da = \\
= & \int_0^{+\infty} n(0,a)da + \int_0^{+\infty}\int_0^t (1-F(a))i(\tilde{t})d\tilde{t}da - \\
& - \int_0^{+\infty}\int_0^t \mathbf{v}(n(\tilde{t})) \int_{-\infty}^{\tilde{t}} f\left(a + \int_s^{\tilde{t}} \mathbf{v}(n(u))du\right) i(s)dsd\tilde{t}da = \\
= & M(0) + \int_0^t i(\tilde{t}) \int_0^{+\infty} (1-F(a))dad\tilde{t} - \\
& - \int_0^t \mathbf{v}(\tilde{t}) \int_{-\infty}^{\tilde{t}} i(s) \int_0^{+\infty} f\left(a + \int_s^{\tilde{t}} v(u)du\right) dadsd\tilde{t} = \\
= & M(0) + L\int_0^t i(\tilde{t})d\tilde{t} - \int_0^t \mathbf{v}(n(\tilde{t})) \int_{-\infty}^{\tilde{t}} \left(1 - F\left(\int_s^{\tilde{t}} v(u)du\right)\right) i(s)dsd\tilde{t} = \\
= & M(0) + L\int_0^t i(\tilde{t})d\tilde{t} - \int_0^t \mathbf{v}(n(\tilde{t}))n(\tilde{t})d\tilde{t}
\end{aligned}
$$

Note that the function $i(\tilde{t})$ is piecewise continuous and the function $\mathbf{v}(n(\tilde{t}))n(\tilde{t})$ is continuous. By differentiating the last equation one gets

$$\dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t)$$

in the sense that

$$
\begin{cases}
\dot{M}_-(t) = Li_-(t) - \mathbf{v}(n(t))n(t) \\
\dot{M}_+(t) = Li_+(t) - \mathbf{v}(n(t))n(t)
\end{cases}
$$

□

Proposition 15. *If the solution of problem*

$$
\begin{cases}
n(0) = n_0 & \\
M(0) = \frac{L}{\alpha}n_0 & \\
\dot{n}(t) = i(t) - \mathbf{v}(n(t))(\beta_1 n(t) - \beta_2 M(t)) & , t \in [0,T] \\
\dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t) & , t \in [0,T]
\end{cases}
$$

exists, then it is unique.

Proof. If $\mathbf{v}(n)$ is constant then we can rewrite the problem as

$$
\begin{cases}
\begin{bmatrix} n(0) \\ \frac{M(0)}{L} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} n_0 \\
\begin{bmatrix} \dot{n}(t) \\ \frac{\dot{M}(t)}{L} \end{bmatrix} = -\frac{1}{L}A \begin{bmatrix} n(t) \\ \frac{M(t)}{L} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} i(t)
\end{cases}
$$

where $A = \begin{bmatrix} \beta_1 L & -\beta_2 L^2 \\ 1 & 0 \end{bmatrix}$. Obviously, this is a system of linear differential equations with unique solution

$$\begin{bmatrix} n(0) \\ \frac{M(0)}{L} \end{bmatrix} = e^{-At/L} \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} n_0 + \int\limits_0^t e^{-A(t-s)/L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} i(s)ds$$

If $\mathbf{v}(n)$ is conventional then the uniqueness of solution can be guaranteed by showing that functions $\mathbf{v}(n)(\beta_1 n - \beta_2 M)$ and $\mathbf{v}(n)n$ are Lipschitz with respect to n and M. Obviously, Lipschitz constants for M can be taken as $|\beta_2|$ and 0. In the proof of Proposition 1 we already showed that $\mathbf{v}(n)n$ is Lipschitz with constant $1 + Cn_{jam}$. To find Lipschitz constant of $\mathbf{v}(n)(\beta_1 n - \beta_2 M)$ for n we first show that $M(t)$, $t \in (0, T]$ is bounded:

$$M(t) \leqslant \tfrac{L}{\alpha} n_0 + i_{max} LT$$
$$M(t) \geqslant \tfrac{L}{\alpha} n_0 - n_{jam} T$$

If we denote $M_{max} = \max\left(\frac{L}{\alpha} n_0 + i_{max} LT, \left|\frac{L}{\alpha} n_0 - n_{jam} T\right|\right)$ then we can take $|\beta_1|(1 + Cn_{jam}) + |\beta_2| M_{max} C$ as Lipschitz constant for n. □

Proposition 16. *$f(l)$, $l \in (0, +\infty)$ is a PDF satisfying equation of the form $f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$ if and only if it belongs to one of four families:*

F1) $C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l}$
where $\lambda_2 < \lambda_1 < 0$, $C_1 > 0$, $C_2 > 0$, $-\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1$

F2) $C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l}$
where $\lambda_2 < \lambda_1 < 0$, $C_1 > 0$, $C_2 < 0$, $C_1 + C_2 \geqslant 0$, $-\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1$

F3) $C_3 e^{\lambda_3 l}$
where $\lambda_3 < 0$, $C_3 > 0$, $-\frac{C_3}{\lambda_3} = 1$

F4) $(C_1 l + C_2) e^{\lambda_1 l}$
where $\lambda_1 < 0$, $C_1 > 0$, $C_2 \geqslant 0$, $\frac{C_1}{\lambda_1^2} - \frac{C_2}{\lambda_1} = 1$

Proof. First, prove that if $f(l)$, $l \in (0, +\infty)$ is a solution of equation $f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$ and is a PDF, in other words, it satisfies conditions

$$\begin{cases} f(l) \geqslant 0 \\ \int\limits_0^{+\infty} f(l)dl = 1 \end{cases} \tag{3.2}$$

then it belongs to one of four presented families. Consider characteristic equation $\lambda^2 + \beta_1 \lambda + \beta_2 = 0$. Let $D = \beta_1^2 - 4\beta_2$ be its discriminant. There are three possible cases:

1. $D > 0$ - two real roots $\lambda_1 = \frac{-\beta_1+\sqrt{D}}{2}$ and $\lambda_2 = \frac{-\beta_1-\sqrt{D}}{2}$. The general solution has form $f(l) = C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l}$. In the case $C_1 \neq 0$, $C_2 \neq 0$ one can easily prove that $C_1 > 0$. Indeed, if $C_1 < 0$ then $C_2 > 0$ because of the first condition in (3.2), but this condition cannot be fully satisfied since for $l > \frac{1}{\sqrt{D}} \log\left(-\frac{C_2}{C_1}\right)$ we have

$$e^{(\lambda_1-\lambda_2)l} = e^{\sqrt{D}l} > -\frac{C_2}{C_1}$$

and, therefore, $C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l} < 0$. Once we proved that $C_1 > 0$, we can consider cases $C_2 > 0$ and $C_2 < 0$. In the first case it is obvious that $\lambda_1 < 0$, $\lambda_2 < 0$ and $-\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1$, otherwise the second condition in (3.2) cannot be satisfied. In the second case we can prove that $\lambda_1 < 0$, $\lambda_2 < 0$ by contradiction. The case $\lambda_2 \geqslant 0$ is not possible because for $l > \frac{1}{\sqrt{D}} \log\left(1 - \frac{C_2}{C_1}\right)$ we have

$$C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l} > C_1 e^{\lambda_2 l} \geqslant C_1 > 0$$

and the integral $\int\limits_0^{+\infty} f(l)dl$ does not exist. The case $\lambda_2 < 0, \lambda_1 \geqslant 0$ is not possible because the integral $\int\limits_0^{+\infty} C_2 e^{\lambda_2 l} dl$ exists and the integral $\int\limits_0^{+\infty} C_1 e^{\lambda_1 l} dl$ does not exist. Notice also that for $C_1 > 0$, $C_2 < 0$ the necessary condition is $f_+(0) = C_1 + C_2 \geqslant 0$.
In the case when $C_1 = 0$ or $C_2 = 0$ we can say that $f(l) = C_3 e^{\lambda_3 l}$, where $C_3 > 0$ and $\lambda_3 < 0$ is either λ_1 or λ_2. The second condition in (3.2) results in $-\frac{C_3}{\lambda_3} = 1$.
2. $D = 0$ - two equal real roots $\lambda_1 = \lambda_2 = -\frac{\beta_1}{2}$. The general solution has form $f(l) = (C_1 l + C_2)e^{\lambda_1 l}$. If $C_1 = 0$ then we get the previous case. If $C_1 < 0$ then for $l > -\frac{C_2}{C_1}$ we have $f(l) < 0$ which is not possible. Therefore, $C_1 > 0$. Moreover, $C_2 = f_+(0) \geqslant 0$ and $\lambda_1 < 0$. The second condition in (3.2) results in $\frac{C_1}{\lambda_1^2} - \frac{C_2}{\lambda_1} = 1$.
3. $D < 0$ - two complex roots $\lambda_1 = \frac{-\beta_1+\sqrt{D}i}{2}$ and $\lambda_2 = \frac{-\beta_1+\sqrt{D}i}{2}$. The general solution has form

$$\left(C_1 \cos\left(\frac{\sqrt{D}}{2} l\right) + C_2 \sin\left(\frac{\sqrt{D}}{2} l\right)\right) e^{-\beta_1 l/2}$$

Obviously, it cannot be a PDF.

Second, prove that if $f(l)$ belongs to one of four presented families then it is a PDF that satisfies equation of the form $f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$. Indeed, the second condition in (3.2) is satisfied for all the families, the first condition is, obviously, satisfied for the families F1, F3 and F4. For the family F2 it is satisfied because if $C_1 + C_2 \geqslant 0$, $\lambda_1 > \lambda_2$ and $C_1 > 0$ then $C_1 e^{\lambda_1 l} > -C_2 e^{\lambda_2 l}$. Also $f(l)$ satisfies equation of the form $f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$. This can be easily shown. For the families F1, F2 and F4 one should take $\beta_1 = -(\lambda_1 + \lambda_2)$ and $\beta_2 = \lambda_1\lambda_2$. For the family F3 one should take $\beta_1 = -(\lambda_3 + \lambda_4)$ and $\beta_2 = \lambda_3\lambda_4$, where λ_4 is an arbitrary real number. This means that for the families F1, F2 and F4 there exists only one corresponding differential equation and for the family F3 the set of such equations is infinite. □

Proposition 17. *M model with parameters β_1 and β_2 is equivalent to TB model for any input if and only if $f(l)$ is a solution of equation $f''(l)+\beta_1 f'(l)+\beta_2 f(l) = 0$. Moreover, for such a solution the equality $1-\beta_1 L+\frac{\beta_2 L^2}{\alpha} = 0$ always holds.*

Proof. We will prove the equivalence of three statements:

1. $f(l)$, $l \in (0, +\infty)$ satisfies equation

$$f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$$

2. $f(l)$, $l \in (0, +\infty)$ satisfies equation

$$f(l) - \beta_1(1 - F(l)) + \beta_2 \int_0^{+\infty} (1 - F(a + l))da = 0$$

3. M model with parameters β_1 and β_2 is equivalent to TB model with $f(l)$ for any input.

1→2. If Statement 1 holds then $f(l)$ belongs to one of four families presented in Proposition 15. All of them satisfy conditions

$$\begin{cases} \lim\limits_{l\to+\infty} f(l) = 0 \\ \lim\limits_{l\to+\infty} f'(l) = 0 \end{cases}$$

Using them, we can integrate equation $f''(l) + \beta_1 f'(l) + \beta_2 f(l) = 0$ two times:

$$\begin{aligned} 0 = & \int_l^{+\infty} (f''(x) + \beta_1 f'(x) + \beta_2 f(x))\, dx = \\ = & - f'(l) - \beta_1 f(l) + \beta_2(1 - F(l)) \\ 0 = & \int_l^{+\infty} (-f'(x) - \beta_1 f(x) + \beta_2(1 - F(x)))\, dx = \\ = & f(l) - \beta_1(1 - F(l)) + \beta_2 \int_0^{+\infty} (1 - F(a + l))da \end{aligned}$$

2→1. Notice that the function

$$f(l) = \beta_1(1 - F(l)) - \beta_2 \int_0^{+\infty} (1 - F(a + l))da =$$
$$= \beta_1(1 - F(l)) - \beta_2 \int_l^{+\infty} (1 - F(x))dx$$

is differentiable. Therefore,

$$f'(l) = -\beta_1 f(l) + \beta_2(1 - F(l))$$

The function $-\beta_1 f(l) + \beta_2(1 - F(l))$ is also differentiable, because we just showed that $f'(l)$ exists. Therefore,

$$f''(l) = -\beta_1 f'(l) - \beta_2 f(l)$$

2→3. Prove that if

$$f(l) = \beta_1(1 - F(l)) - \beta_2 \int_0^{+\infty} (1 - F(a + l))da$$

then the outflow in TB model with $f(l)$ is

$$o(t) = \mathbf{v}(n(t))(\beta_1 n(t) - \beta_2 M(t))$$

for any input. First notice that

$$M(t) = \int_0^{+\infty} n(t, a)da =$$
$$= \int_0^{+\infty} \int_{-\infty}^{t} \left(1 - F\left(a + \int_s^t \mathbf{v}(n(u))du\right)\right) i(s)dsda =$$
$$= \int_{-\infty}^{t} i(s) \int_0^{+\infty} \left(1 - F\left(a + \int_s^t \mathbf{v}(n(u))du\right)\right) dads$$

This leads to

$$o(t) - \mathbf{v}(n(t))(\beta_1 n(t) - \beta_2 M(t)) =$$
$$= \mathbf{v}(n(t)) \int_{-\infty}^{t} f\left(\int_s^t \mathbf{v}(n(u))du\right) i(s)ds-$$
$$- \mathbf{v}(n(t)) \int_{-\infty}^{t} \beta_1 \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds+$$
$$+ \mathbf{v}(n(t)) \int_{-\infty}^{t} \beta_2 \left(\int_0^{+\infty} \left(1 - F\left(a + \int_s^t \mathbf{v}(n(u))du\right)\right) da\right) i(s)ds = 0$$

Therefore, there exists a solution of M model that is equivalent to solution of TB model. Proposition 15 guarantees the uniqueness of such a solution.
3→2. The proof is similar to the second part of the proof of Proposition 8. Assume

$$f(l) - \beta_1(1 - F(l)) + \beta_2 \int\limits_0^{+\infty} (1 - F(a + l))da \neq 0$$

for some l. This function is piecewise continuous and, therefore, exists an interval (l_1, l_2) such that

$$f(l) - \beta_1(1 - F(l)) + \beta_2 \int\limits_0^{+\infty} (1 - F(a + l))da > 0,\ l \in (l_1, l_2)$$

or

$$f(l) - \beta_1(1 - F(l)) + \beta_2 \int\limits_0^{+\infty} (1 - F(a + l))da < 0,\ l \in (l_1, l_2)$$

Consider the input

$$n_0 = 0$$
$$i(t) = \begin{cases} j,\ t \in (0, h] \\ 0,\ t \in (h, T] \end{cases}$$

where $h < l_2 - l_1$, $j < \frac{n_{jam}}{h}$, $T > \frac{l_2}{\mathbf{v}(jh)}$. These conditions allow to find $t < T$ such that $\int\limits_0^t \mathbf{v}(n(u))du = l_2$. Notice that for all $s \in (0, h]$ we have $\int\limits_s^t \mathbf{v}(n(u))du = \int\limits_0^t \mathbf{v}(n(u))du - \int\limits_0^s \mathbf{v}(n(u))du$, where $\int\limits_0^s \mathbf{v}(n(u))du \in (0, l_2 - l_1)$. Therefore,

$$\int\limits_s^t \mathbf{v}(n(u))du \in (l_1, l_2)$$

and

$$\begin{aligned} &\mathbf{v}(n(t)) \int\limits_{-\infty}^t f\left(\int_s^t \mathbf{v}(n(u))du\right) i(s)ds- \\ &- \mathbf{v}(n(t)) \int\limits_{-\infty}^t \beta_1 \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds+ \\ &+ \mathbf{v}(n(t)) \int\limits_{-\infty}^t \beta_2 \left(\int\limits_0^{+\infty} \left(1 - F\left(a + \int_s^t \mathbf{v}(n(u))du\right)\right) da\right) i(s)ds \end{aligned}$$

has the same sign as function

$$f(l) - \beta_1(1 - F(l)) + \beta_2 \int\limits_0^{+\infty} (1 - F(a + l))da$$

on (l_1, l_2), i.e. $o(t) - \mathbf{v}(n(t))(\beta_1 n(t) - \beta_2 M(t)) \neq 0$.

Now, after proving that presented statements hold simultaneously, we integrate the equation in the second statement using Proposition 13 to show that $1 - \beta_1 L + \frac{\beta_2 L^2}{\alpha} = 0$:

$$0 = \int\limits_0^{+\infty} \left(f(l) - \beta_1(1 - F(l)) + \beta_2 \int\limits_0^{+\infty} (1 - F(a + l))da \right) dl =$$
$$= 1 - \beta_1 L + \beta_2 \frac{L^2 + \sigma^2}{2} = 1 - \beta_1 L + \frac{\beta_2 L^2}{\alpha}$$

□

Proposition 18. *M model with parameter $\beta \neq 0$ is equivalent to PL model for any input if and only if $\alpha = 1$.*

Proof. If $\alpha = 1$ then models are equivalent for any input. Indeed, if $n(t)$ is a solution of problem

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} n(t),\ t \in (0, T], \end{cases}$$

then $n(t)$, $M(t) = Ln(t)$ is a solution of problem

$$\begin{cases} n(0) = n_0 \\ M(0) = Ln_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} \left(n(t) + \beta \left(n(t) - \frac{1}{L} M(t)\right)\right) & , t \in [0, T] \\ \dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t) & , t \in [0, T] \end{cases}$$

The uniqueness of solution follows from Proposition 15. Vice versa, if models are equivalent for any input then their outflows are also equal. Since β is not zero, $n(t) = \frac{\alpha}{L} M(t)$. This results in $\dot{n}(t) = \frac{\alpha}{L} \dot{M}(t)$ and also in $\dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} n(t) = \frac{1}{L} \dot{M}(t)$. Therefore, $\alpha = 1$. □

Proposition 19. *If $\alpha \in \left(0, \frac{4}{3}\right]$ and*

$$\beta \in \begin{cases} (0, +\infty) & , \alpha \in (0, 1) \\ (-\infty, +\infty) \setminus \{0\} & , \alpha = 1 \\ \left[2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}, \frac{1}{\alpha - 1}\right] & , \alpha \in \left(1, \frac{4}{3}\right] \end{cases}$$

then there exists exactly one $f(l)$ with parameters L and α such that M model with parameter $\beta \neq 0$ is equivalent to TB model for any input. Otherwise, there is no such $f(l)$.

Proof. From Proposition 17 we conclude that M model is equivalent to TB model for any input if and only if

$$f''(l) + \beta_1 f'(l) + \beta_2 f(l) = f''(l) + \tfrac{1+\beta}{L} f'(l) + \tfrac{\alpha\beta}{L^2} f(l) = 0$$

From Proposition 16 we conclude that such $f(l)$ comes from one of four families:

F1) $C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l}$
    where $\lambda_2 < \lambda_1 < 0$, $C_1 > 0$, $C_2 > 0$, $-\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1$

F2) $C_1 e^{\lambda_1 l} + C_2 e^{\lambda_2 l}$
    where $\lambda_2 < \lambda_1 < 0$, $C_1 > 0$, $C_2 < 0$, $C_1 + C_2 \geqslant 0$, $-\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1$

F3) $C_3 e^{\lambda_3 l}$
    where $\lambda_3 < 0$, $C_3 > 0$, $-\frac{C_3}{\lambda_3} = 1$

F4) $(C_1 l + C_2) e^{\lambda_1 l}$
    where $\lambda_1 < 0$, $C_1 > 0$, $C_2 \geqslant 0$, $\frac{C_1}{\lambda_1^2} - \frac{C_2}{\lambda_1} = 1$

For each of them we will find all pairs (α, β) that correspond to some PDF with mean L from this family. Notice also that the value of L does not influence this set because the stretching transformation $f(l) \longrightarrow c f(cl)$ does not change the family of $f(l)$ and validity of $f''(l) + \frac{1+\beta}{L} f'(l) + \frac{\alpha\beta}{L^2} f(l) = 0$. The characteristic equation of this ODE has roots

$$\tfrac{1}{L}\left(-\tfrac{1+\beta}{2} + \sqrt{\psi}\right), \ \tfrac{1}{L}\left(-\tfrac{1+\beta}{2} - \sqrt{\psi}\right)$$

where $\psi = \frac{(1+\beta)^2}{4} - \alpha\beta$. For the families F1 and F2 they are equal to λ_1 and λ_2. For the family F3 at least one of them is equal to λ_3. For the family F4 roots are equal to λ_1.

F1) The necessary and sufficient conditions for $f(l)$ to be a PDF with mean L from the family F1 are:

$$\begin{cases} \lambda_2 < \lambda_1 < 0,\ C_1 > 0,\ C_2 > 0 \\ -\frac{C_1}{\lambda_1} - \frac{C_2}{\lambda_2} = 1 \\ \frac{C_1}{\lambda_1^2} + \frac{C_2}{\lambda_2^2} = L \end{cases}$$

Introduce dimensionless parameters

$$\mu_1 = -\lambda_1 L,\ \mu_2 = -\lambda_2 L,\ c_1 = C_1 L,\ c_2 = C_2 L$$

Now conditions can be written as follows:

$$\begin{cases} 0 < \mu_1 = \frac{1+\beta}{2} - \sqrt{\psi} < \mu_2 = \frac{1+\beta}{2} + \sqrt{\psi},\ c_1 > 0,\ c_2 > 0 \\ \frac{c_1}{\mu_1} + \frac{c_2}{\mu_2} = 1 \\ \frac{c_1}{\mu_1^2} + \frac{c_2}{\mu_2^2} = 1 \end{cases}$$

First, express c_1 and c_2 through μ_1 and μ_2. The result is $c_1 = \mu_1^2 \frac{1-\mu_2}{\mu_1-\mu_2}$ and $c_2 = \mu_2^2 \frac{\mu_1-1}{\mu_1-\mu_2}$. Note also that we do not need to include the constraint $\int\limits_0^{+\infty} f(l) l^2 dl = L^2 + \sigma^2$ that takes the form $\frac{c_1}{\mu_1^3} + \frac{c_2}{\mu_2^3} = \frac{1}{\alpha}$ into the system because from Proposition 17 follows that this constraint holds for all pairs (α, β) that produce a PDF. Second, write conditions $0 < \mu_1 < \mu_2$, $c_1 > 0$, $c_2 > 0$ as the system

$$\begin{cases} \mu_1 > 0 \\ \psi > 0 \\ \mu_2 > 1 \\ \mu_1 < 1 \end{cases}$$

The inequality $\mu_1 > 0$ is equivalent to the system

$$\begin{cases} 1 + \beta > 0 \\ (1+\beta)^2 > (1+\beta)^2 - 4\alpha\beta \end{cases}$$

which has solution $\beta > 0$. The system

$$\begin{cases} \mu_2 > 1 \\ \mu_1 < 1 \end{cases}$$

is equivalent to the inequality $\frac{(1-\beta)^2}{4} < \psi = \frac{(1+\beta)^2}{4} - \alpha\beta$ which is equivalent to $\alpha < 1$ if $\beta > 0$. Note that $\psi > 0$ holds automatically. Thus, we showed that any pair of $\alpha \in (0, 1)$ and $\beta \in (0, +\infty)$ corresponds to some PDF from the family F1 and there are no other such pairs.

F2) For the family F2 we also have conditions $c_1 = \mu_1^2 \frac{1-\mu_2}{\mu_1-\mu_2}$, $c_2 = \mu_2^2 \frac{\mu_1-1}{\mu_1-\mu_2}$ and $0 < \mu_1 < \mu_2$. The difference is conditions $c_1 > 0$, $c_2 < 0$, $c_1 + c_2 \geqslant 0$ instead of $c_1 > 0$, $c_2 > 0$. Notice that $c_1 + c_2 = \mu_1 + \mu_2 - \mu_1\mu_2$. Now we need to solve the system

$$\begin{cases} \mu_1 > 0 \\ \psi > 0 \\ \mu_2 > 1 \\ \mu_1 > 1 \\ \mu_1 + \mu_2 - \mu_1\mu_2 \geqslant 0 \end{cases}$$

This system is equivalent to

$$\begin{cases} \psi > 0 \\ \mu_1 > 1 \\ \mu_1 + \mu_2 - \mu_1\mu_2 \geqslant 0 \end{cases}$$

Inequality $\mu_1 > 1$ is equivalent to $\frac{1-\beta}{2} < -\sqrt{\psi}$ which is equivalent to the system

$$\begin{cases} \beta > 1 \\ (1-\beta)^2 > (1+\beta)^2 - 4\alpha\beta \end{cases}$$

The second inequality in this this system is equivalent, given $\beta > 1$, to $\alpha > 1$. Now consider inequality $\psi = \frac{(1+\beta)^2}{4} - \alpha\beta > 0$. It leads to $\beta < 2\alpha - 1 - \sqrt{4\alpha^2 - 4\alpha}$ or $\beta > 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}$. The first case is not possible for $\alpha > 1$, $\beta > 1$ because

$$2\alpha - 1 - \sqrt{4\alpha^2 - 4\alpha} - 1 = 2(\alpha - 1) - \sqrt{4(\alpha-1)^2 + 4(\alpha - 1)} < 0$$

for $\alpha > 1$. In the second case

$$2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha} > 1$$

for $\alpha > 1$. Now consider inequality $\mu_1 + \mu_2 - \mu_1\mu_2 = 1 + \beta - \alpha\beta \geqslant 0$. For $\alpha, \beta > 1$ it is equivalent to $\beta \leqslant \frac{1}{\alpha-1}$. As a last step show that, given $\alpha > 1$, condition $2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha} < \beta \leqslant \frac{1}{\alpha-1}$ holds if and only if $\alpha < \frac{4}{3}$. Indeed, the function $2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha} = 2\alpha - 1 + \sqrt{(2\alpha-1)^2 - 1}$ is increasing and the function $\frac{1}{\alpha-1}$ is decreasing. If $\alpha = \frac{4}{3}$ both of them are equal to 3. Thus, we showed that any pair of $\alpha \in \left(1, \frac{4}{3}\right)$ and $\beta \in \left(2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}, \frac{1}{\alpha-1}\right]$ corresponds to some PDF from the family F2 and there are no other such pairs.

F3) From Proposition 16 we already know that the family F3 consists only of $f(l) = \frac{1}{L}e^{-l/L}$ that has $\alpha = 1$. We know also that this is the case when M model is equivalent to TB model, regardless the value of $\beta \neq 0$. However, it is easy to check that if $\beta \neq 0$ then the solution of the system

$$\begin{cases} \mu_3 > 0,\ c_3 > 0 \\ \frac{c_3}{\mu_3} = 1 \\ \frac{c_3}{\mu_3^2} = 1 \\ \mu_3 \in \left\{\frac{1+\beta}{2} - \sqrt{\psi}, \frac{1+\beta}{2} + \sqrt{\psi}\right\} \end{cases}$$

is $\alpha = 1$, $\beta \in (-\infty, +\infty) \setminus \{0\}$, where $\beta \leqslant 1$ corresponds to $\mu_3 = \frac{1+\beta}{2} + \sqrt{\psi}$

and $\beta \geqslant 1$ corresponds to $\mu_3 = \frac{1+\beta}{2} - \sqrt{\psi}$.
F4) For the family F4 the system is

$$\begin{cases} \mu_1 > 0,\ \psi = 0,\ c_1 > 0,\ c_2 \geqslant 0 \\ \frac{c_1}{\mu_1^2} + \frac{c_2}{\mu_1} = 1 \\ \frac{2c_1}{\mu_1^3} + \frac{c_2}{\mu_1^2} = 1 \end{cases}$$

First, notice that $\frac{c_1}{\mu_1^3} = 1 - \frac{1}{\mu_1}$ and, therefore, $c_1 = \mu_1^3 - \mu_1^2$, $c_2 = 2\mu_1 - \mu_1^2$. As $c_1 > 0$, $c_2 \geqslant 0$, we have condition $\mu_1 \in (1, 2]$. If $\psi = 0$, it is equivalent to $\beta \in (1, 3]$. From $\psi = 0$ also follows

$$\beta = 2\alpha - 1 - \sqrt{4\alpha^2 - 4\alpha}$$

or

$$\beta = 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}$$

Therefore, $\alpha \geqslant 1$ and

$$2\alpha - 1 - \sqrt{4\alpha^2 - 4\alpha} \leqslant 2\alpha - 1 - 2(\alpha - 1) = 1$$

The function $2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}$ is continuous, increasing and takes the value 1 if $\alpha = 1$ and the value 3 if $\alpha = \frac{4}{3}$. Thus, we showed that any pair of $\alpha \in \left(1, \frac{4}{3}\right)$ and $\beta = 2\alpha - 1 - \sqrt{4\alpha^2 - 4\alpha}$ corresponds to some PDF from the family F4.

Note that all the sets of (α, β) that correspond to different families do not intersect. Also all the coefficients of any PDF are expressed through α, β and L. Thus, for any feasible pair (α, β) there exists exactly one corresponding PDF with mean L. $\square$

Proposition 20. *If $f(l)$, $l \in (0, \Lambda L)$ is a PDF with mean L and variance σ^2 then all the possible values of $\frac{\sigma^2}{L^2}$ are $(0, \Lambda - 1)$.*

Proof. We start the proof by relaxing the assumption that distribution is continuous. We will prove first that all the possible values of σ^2 for an arbitrary distribution with domain $[0, \Lambda L]$ are $[0, (\Lambda - 1)L^2]$. Consider the set

$$conv\left((l, l^2),\ l \in [0, \Lambda L]\right)$$

shown in Figure 3.1.

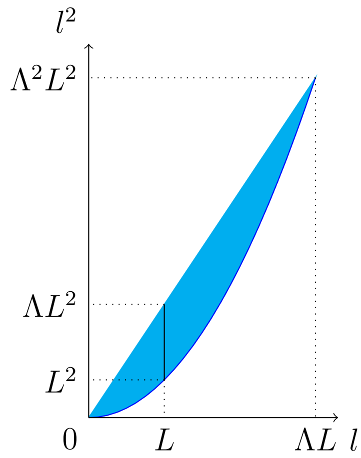


Figure 3.1: Convex hull of the curve (l, l^2), $l \in [0, \Lambda L]$.

Obviously, this set consists of all possible pairs of first and second moments of probability distribution defined on $[0, \Lambda L]$. With the assumption that first moment is equal to L we get the interval $[L^2, \Lambda L^2]$ for the second moment. The values L^2 and ΛL^2 correspond to discrete distributions and cannot be equal to the second moment of some PDF. For all other values we can construct a PDF. Any feasible and not boundary point $(L, L^2 + \sigma^2)$ can be viewed as a convex combination of two points (l_1, l_1^2) and (l_2, l_2^2), where $0 < l_1 < l_2 < \Lambda L$. Denote the weights of convex combination as w_1 and w_2. Consider distribution

$$(1 - \varepsilon)w_1 U[l_1 - \delta, l_1 + \delta] + (1 - \varepsilon)w_2 U[l_2 - \delta, l_2 + \delta] + \varepsilon U[L - \delta, L + \delta]$$

Obviously, the first moment of this distribution is equal to L. The second moment is equal to

$$(1 - \varepsilon)\left(L^2 + \sigma^2 + \tfrac{\delta^2}{3}\right) + \varepsilon\left(L^2 + \tfrac{\delta^2}{3}\right) = L^2 + \sigma^2 - \varepsilon\sigma^2 + \tfrac{\delta^2}{3}$$

If we take $\delta = \sqrt{3\varepsilon}\sigma$ for given ε then the second moment is equal to $L^2 + \sigma^2$. For sufficiently small ε we have $l_1 - \delta > 0$ and $l_2 + \delta < \Lambda L$. These inequalities ensure that the domain of PDF belongs to $(0, \Lambda L)$. □

Proposition 21. *If $f(l)$, $l \in (0, \Lambda L)$ is a PDF with mean L, variance σ^2 and third central moment ρ^3 then all the possible values of $\frac{\rho^3}{L^3}$ are*

$$\left(-\tfrac{\sigma^2}{L^2} + \left(\tfrac{\sigma^2}{L^2}\right)^2, (\Lambda - 1)\tfrac{\sigma^2}{L^2} - \tfrac{1}{\Lambda - 1}\left(\tfrac{\sigma^2}{L^2}\right)^2\right)$$

Proof. As in the proof of Proposition 20, we relax the assumption that distribution is continuous. Consider the set

$$conv\left((l, l^2, l^3),\ l \in [0, \Lambda L]\right)$$

Obviously, this is the set of all possible triplets of first, second and third moments of probability distribution defined on $[0, \Lambda L]$. Each point of the set can be represented as a convex combination of finite number of points (l, l^2, l^3), $l \in [0, \Lambda L]$. We can prove that for the given L and σ^2 the minimum and maximum of ρ^3 are achieved on combinations of one or two points. We prove this by contradiction. Assume the convex combination includes l_1 and l_2 such that $0 < l_1 < l_2$ or $l_1 < l_2 < \Lambda L$. We will show that in the first case ρ^3 can be decreased without changing L and σ^2 and in the second case it can be increased. Therefore, any convex combination of three points cannot give minimum or maximum of ρ^3. Moreover, if combination of two points gives minimum of ρ^3 then it includes 0 and if it gives maximum of ρ^3 then it includes ΛL.

Consider some combination that includes $l_1 < l_2$ with weights $w_1 > 0$ and $w_2 > 0$. We can express the value $c_3 = w_1 l_1^3 + w_2 l_2^3$ through l_1, $c_0 = w_1 + w_2$, $c_1 = w_1 l_1 + w_2 l_2$ and $c_2 = w_1 l_1^2 + w_2 l_2^2$. First, notice that

$$l_2 = \tfrac{c_2 - c_1 l_1}{c_1 - c_0 l_1}$$

Therefore,

$$\begin{aligned}
c_3 &= w_1 l_1^3 + w_2 l_2^3 = \left(w_1 l_1^2 + w_2 l_2^2\right)(l_1 + l_2) - (w_1 l_1 + w_2 l_2)\, l_1 l_2 = \\
&= c_2\left(l_1 + \tfrac{c_2 - c_1 l_1}{c_1 - c_0 l_1}\right) - c_1 l_1 \left(\tfrac{c_2 - c_1 l_1}{c_1 - c_0 l_1}\right) = \\
&= \tfrac{c_2^2 - c_1 c_2 l_1 - \left(c_0 c_2 - c_1^2\right) l_1^2}{c_1 - c_0 l_1} = \\
&= \tfrac{1}{c_1 - c_0 l_1}\left(c_2^2 - c_1 c_2 \left(\tfrac{c_1 - (c_1 - c_0 l_1)}{c_0}\right) - (c_0 c_2 - c_1^2)\left(\tfrac{c_1 - (c_1 - c_0 l_1)}{c_0}\right)^2\right) = \\
&= \tfrac{\left(c_0 c_2 - c_1^2\right)^2}{c_0^2 (c_1 - c_0 l_1)} - \tfrac{2 c_1^3}{c_0^2} + \tfrac{3 c_1 c_2}{c_0} - \tfrac{\left(c_0 c_2 - c_1^2\right)(c_1 - c_0 l_1)}{c_0^2}
\end{aligned}$$

Notice that $c_0 c_2 - c_1^2 = w_1 w_2 (l_2 - l_1)^2 > 0$ and $c_1 - c_0 l_1 = w_2 (l_2 - l_1) > 0$. Therefore, if l_1 increases with fixed c_0, c_1 and c_2 then c_3 also increases. As the difference between ρ^3 and c_3 is constant, $l_1 > 0$ cannot give minimum of ρ^3. Now notice that

$$l_2 = \tfrac{c_2 - c_1 l_1}{c_1 - c_0 l_1} = \tfrac{c_1}{c_0} + \tfrac{c_0 c_2 - c_1^2}{c_0 (c_1 - c_0 l_1)}$$

increases if l_1 increases. Therefore, $l_2 < \Lambda L$ cannot give maximum of ρ^3. Summarizing, the minimum and maximum ρ^3 are achieved on convex combinations of two points. In this case $c_0 = 1$, $c_1 = L$ and $c_2 = L^2 + \sigma^2$. The minimum ρ^3

is achieved on $l_1 = 0$, $l_2 = L + \frac{\sigma^2}{L}$ and takes the value

$$\begin{aligned}\rho^3 &= \tfrac{\sigma^4}{L} - 2L^3 + 3L\left(L^2+\sigma^2\right) - \sigma^2 L - L^3 - 3L\sigma^2 = \\ &= \tfrac{\sigma^4}{L} - \sigma^2 L\end{aligned}$$

The maximum ρ^3 is achieved on $l_2 = \Lambda L$, $l_1 = L - \frac{\sigma^2}{(\Lambda-1)L}$ and takes the value

$$\begin{aligned}\rho^3 &= (\Lambda-1)\sigma^2 L - 2L^3 + 3L\left(L^2+\sigma^2\right) - \tfrac{\sigma^4}{(\Lambda-1)L} - L^3 - 3L\sigma^2 = \\ &= (\Lambda-1)\sigma^2 L - \tfrac{\sigma^4}{(\Lambda-1)L}\end{aligned}$$

Note that for the case of convex combination of two points ρ^3 can take all the feasible values as it is continuous function of l_1. This also means that any not boundary value of ρ^3 corresponds to some case $0 < l_1 < l_2 < \Lambda L$. Now we show how to construct a corresponding PDF for such ρ^3. Notice that for any not boundary value of ρ^3 and sufficiently small ε the value of $\frac{\rho^3}{1-\varepsilon}$ also belongs to the interior of the interval of feasible ρ^3. This means that there exists convex combination of two points $0 < l_1(\varepsilon) < l_2(\varepsilon) < \Lambda L$ such that its third central moment is equal to $\frac{\rho^3}{1-\varepsilon}$. Denote weights of this combination as $w_1(\varepsilon)$ and $w_2(\varepsilon)$. Consider distribution

$$\begin{aligned}&(1-\varepsilon)w_1(\varepsilon)U[l_1(\varepsilon)-\delta, l_1(\varepsilon)+\delta]+ \\ &+(1-\varepsilon)w_2(\varepsilon)U[l_2(\varepsilon)-\delta, l_2(\varepsilon)+\delta]+ \\ &+\varepsilon U[L-\delta, L+\delta]\end{aligned}$$

Obviously, the first moment of this distribution is equal to L. The second moment is equal to

$$(1-\varepsilon)\left(L^2+\sigma^2+\tfrac{\delta^2}{3}\right) + \varepsilon\left(L^2+\tfrac{\delta^2}{3}\right) = L^2+\sigma^2-\varepsilon\sigma^2+\tfrac{\delta^2}{3}$$

The third moment is equal to

$$\begin{aligned}&(1-\varepsilon)\left(L^3+3L\left(\sigma^2+\tfrac{\delta^2}{3}\right)+\tfrac{\rho^3}{1-\varepsilon}\right)+\varepsilon\left(L^3+L\delta^2\right) = \\ &= L^3+3L\sigma^2+\rho^3-3L\varepsilon\sigma^2+L\delta^2\end{aligned}$$

If we take $\delta = \sqrt{3\varepsilon}\sigma$ for given ε then the second moment is equal to $L^2+\sigma^2$ and the third moment is equal to $L^3+3L\sigma^2+\rho^3$. If $\varepsilon \to 0$ then $l_1(\varepsilon)\to l_1(0) > 0$ and $l_2(\varepsilon)\to l_2(0) < \Lambda L$. For sufficiently small ε we have $l_1(\varepsilon)-\delta > 0$ and $l_2(\varepsilon)+\delta < \Lambda L$. These inequalities ensure that the domain of PDF belongs to $(0, \Lambda L)$. □

Proposition 22. *If $\alpha \in (1,2)$ then the solution of M model with $\beta > 0$ exists for any input in the case of constant $\mathbf{v}(n)$ if and only if*

$$\beta \geqslant 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}$$

Proof. Depending on the value of $\psi = \frac{(1+\beta)^2}{4} - \alpha\beta$ we have three different cases:

1. $\psi > 0$

The solution

$$n(t) = e^{-\frac{1+\beta}{2}t/L}\left(cosh(\sqrt{\psi}t/L) + \tfrac{\beta-1}{2\sqrt{\psi}}sinh(\sqrt{\psi}t/L)\right)n_0+$$
$$+\int_0^t e^{-\frac{1+\beta}{2}(t-s)/L}\left(cosh(\sqrt{\psi}(t-s)/L) + \tfrac{(2\alpha-1)\beta-1}{2\sqrt{\psi}}sinh(\sqrt{\psi}(t-s)/L)\right)i(s)ds$$

does not take negative values for any input if and only if

$$\begin{cases} \frac{\beta-1}{2\sqrt{\psi}} \geqslant -1 \\ \frac{(2\alpha-1)\beta-1}{2\sqrt{\psi}} \geqslant -1 \end{cases}$$

As we assume $\alpha \in (1, 2)$ and $\psi > 0$, the first inequality is stronger. Moreover,

$$\psi = \tfrac{(1+\beta)^2}{4} - \alpha\beta < \tfrac{(\beta-1)^2}{4}$$

which means that $\beta \geqslant 1$. As $\psi > 0$ holds if and only if

$$\beta < 2\alpha - 1 - \sqrt{4\alpha^2 - 4\alpha} < 1$$

or

$$\beta > 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha} > 1$$

we conclude that all the feasible values of β are

$$\beta > 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}$$

2. $\psi = 0$

The solution

$$n(t) = e^{-\frac{1+\beta}{2}t/L}\left(1 + \tfrac{\beta-1}{2}\tfrac{t}{L}\right)n_0 + \int_0^t e^{-\frac{1+\beta}{2}(t-s)/L}\left(1 + \tfrac{(2\alpha-1)\beta-1}{2}\tfrac{t-s}{L}\right)i(s)ds$$

does not take negative values for any input if and only if

$$\begin{cases} \beta - 1 \geqslant 0 \\ (2\alpha-1)\beta - 1 \geqslant 0 \end{cases}$$

As $\alpha \in (1,2)$, the first inequality is stronger and, therefore,

$$\beta = 2\alpha - 1 + \sqrt{4\alpha^2 - 4\alpha}$$

3. $\psi < 0$
The solution

$$n(t) = e^{-\frac{1+\beta}{2}t/L}\left(cos(\sqrt{-\psi}t/L) + \frac{\beta-1}{2\sqrt{-\psi}}sin(\sqrt{-\psi}t/L)\right)n_0 +$$
$$+\int_0^t e^{-\frac{1+\beta}{2}(t-s)/L}\left(cos(\sqrt{-\psi}(t-s)/L) + \frac{(2\alpha-1)\beta-1}{2\sqrt{-\psi}}sin(\sqrt{-\psi}(t-s)/L)\right)i(s)ds$$

takes negative values for some inputs, because any function of the form

$$cos(x) + Csin(x) = \sqrt{1+C^2}cos\left(x - arccos\left(\frac{1}{\sqrt{1+C^2}}\right)\right)$$

takes negative values. □

Proposition 23. *The solution of M model exists for any input in the case of constant* $\mathbf{v}(n)$ *if and only if the solution of the same M model exists for any input in the case of conventional* $\mathbf{v}(n)$*.*

Proof. First, we prove that if the solution of M model with constant $\mathbf{v}(n)$ exists for any input then the solution of M model with conventional $\mathbf{v}(n)$ and the same β exists for any input. We prove this by contradiction. Assume that solution of M model with conventional $\mathbf{v}(n)$ does not exist for some input. It means that the solution reaches zero with negative $\dot{n}(t)$ at some $t_0 < T$. Gridlock was not reached on the interval $[0, t_0]$ because otherwise $n(t_0) \geqslant n_{jam}$. Therefore, $\mathbf{v}(n(t)) > 0$, $t \in [0, t_0]$. Consider the change of variable $y = \int_0^t \mathbf{v}(n(u))du$. Define functions $n^*(y) = n(t)$, $M^*(y) = M(t)$ and $g(y) = \frac{i(t)}{\mathbf{v}(n(t))}$. Notice that $\dot{y}(t) = \mathbf{v}(n(t))$. Therefore,

$$\begin{cases} n^*(0) = n_0 \\ M^*(0) = \frac{L}{\alpha}n_0 \\ \frac{dn^*}{dy}(y) = g(y) - \frac{1}{L}\left(n^*(y) + \beta\left(n^*(y) - \frac{\alpha}{L}M^*(y)\right)\right) & , y \in (0, y_0] \\ \frac{dM^*}{dy}(y) = Lg(y) - n^*(y) & , y \in (0, y_0] \end{cases} \tag{3.3}$$

where $y_0 = y(t_0)$. By our assumption, $n^*(y_0) = n(t_0) = 0$, $\frac{dn^*}{dy}(y_0) = \frac{\dot{n}(t_0)}{\mathbf{v}(n(t_0))} = \dot{n}(t_0) < 0$. Thus, if we assume that n_0, y_0 and $g(y)$, $y \in (0, y_0]$ is an input to M model with constant $\mathbf{v}(n)$ then the solution reaches zero at y_0 with negative derivative. If we adjust the input by saying that $g(y) = 0$, $y > y_0$ then the

solution will not exist. Thus, we come to contradiction with an assumption that solution of M model with constant $\mathbf{v}(n)$ exists for any input.
Second, we prove by contradiction that if the solution of M model with conventional $\mathbf{v}(n)$ exists for any input then the solution of M model with constant $\mathbf{v}(n)$ exists for any input. Assume there exists some input n_0, $g(y)$ such that $n^*(y)$ given by (3.3) satisfies $n^*(y_0) = 0$, $\frac{dn^*}{dy}(y_0) < 0$. We can assume that $n^*(y) < n_{jam}$, $y \in (0, y_0]$ because the solution can be always decreased several times by multiplying the input by some coefficient. Thus, $\mathbf{v}(n^*(y)) > 0$. Consider the change of variable $t = \int\limits_0^y \frac{1}{\mathbf{v}(n^*(x))} dx$. Define functions $n(t) = n^*(y)$, $M(t) = M^*(y)$ and $i(t) = g(y)\mathbf{v}(n^*(y)) = g(y)\mathbf{v}(n(t))$. Notice that $\dot{y}(t) = \mathbf{v}(n^*(y)) = \mathbf{v}(n(t))$. Therefore

$$\begin{cases} n(0) = n_0 \\ M(0) = \frac{L}{\alpha} n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L}\left(n(t) + \beta\left(n(t) - \frac{\alpha}{L}M(t)\right)\right) & , t \in (0, t_0] \\ \dot{M}(t) = Li(t) - \mathbf{v}(n(t))n(t) & , t \in (0, t_0] \end{cases}$$

where $t_0 = t(y_0)$. By our assumption,

$$\begin{aligned} n(t_0) &= n^*(y_0) = 0 \\ \dot{n}(t_0) &= \frac{dn^*}{dy}(y_0)\mathbf{v}(n(t_0)) = \frac{dn^*}{dy}(y_0) < 0 \end{aligned}$$

Thus, if we assume that n_0, t_0 and $i(t)$, $t \in (0, t_0]$ is an input to M model with constant $\mathbf{v}(n)$ then the solution reaches zero at t_0 with negative derivative. If we adjust the input by saying that $i(t) = 0$, $t > t_0$ then the solution will not exist. Thus, we come to contradiction with an assumption that solution of M model with conventional $\mathbf{v}(n)$ exists for any input. $\square$

Proposition 24. *The solution of TB model for $i(t) = i_0 + i_1 t + i_2 t^2$ in the case of constant $\mathbf{v}(n)$ is*

$$n^{TB}(t) = L(i_0 + i_1 t + i_2 t^2) - \frac{L^2}{\alpha} i_1 - 2\frac{L^2}{\alpha} i_2 t + 2\left(\frac{1}{\alpha} - \frac{1}{3} + \frac{\rho^3}{6L^3}\right) L^3 i_2,\ t \in [\Lambda L, T]$$

Proof. The general formula for the solution of TB model in the case of constant $\mathbf{v}(n)$ is

$$n(t) = \int\limits_{-\infty}^{t} (1 - F(t-s))i(s)ds = \int\limits_{t-\Lambda L}^{t} (1 - F(t-s))i(s)ds,\ t \in [\Lambda L, T]$$

If $i(t) = i_0 + i_1 t + i_2 t^2$, $t \in (0, T]$ then for $t \in [\Lambda L, T]$ we get

$$
\begin{aligned}
n(t) = & \int_{t-\Lambda L}^{t} (1-F(t-s))\left(i_0 + i_1 s + i_2 s^2\right) ds = \\
= & \int_{t-\Lambda L}^{t} (1-F(t-s))\left(i_0 + i_1(t-(t-s)) + i_2(t-(t-s))^2\right) ds = \\
= & \left(\int_0^{\Lambda L} (1-F(l))dl\right)\left(i_0 + i_1 t + i_2 t^2\right) + \\
& + \left(\int_0^{\Lambda L} (1-F(l))l dl\right)(-i_1 - 2i_2 t) + \\
& + \left(\int_0^{\Lambda L} (1-F(l))l^2 dl\right) i_2
\end{aligned}
$$

From Proposition 13 we can find

$$
\begin{aligned}
& \int_0^{\Lambda L} (1-F(l))dl = L \\
& \int_0^{\Lambda L} (1-F(l))l dl = \tfrac{L^2+\sigma^2}{2} = \tfrac{L^2}{\alpha} \\
& \int_0^{\Lambda L} (1-F(l))l^2 dl = \tfrac{L^3+3L\sigma^2+\rho^3}{3} = \tfrac{2L^3}{\alpha} - \tfrac{2L^3}{3} + \tfrac{\rho^3}{3} = \\
= & \ 2\left(\tfrac{1}{\alpha} - \tfrac{1}{3} + \tfrac{\rho^3}{6L^3}\right) L^3
\end{aligned}
$$

□

Proposition 25. *The solution of PL model for $i(t) = i_0 + i_1 t + i_2 t^2$ in the case of constant $\mathbf{v}(n)$ is*

$$
\begin{gathered}
n^{PL}(t) = L(i_0 + i_1 t + i_2 t^2) - L^2 i_1 - 2L^2 i_2 t + 2L^3 i_2 + \\
+ e^{-t/L}\left(n_0 - L i_0 + L^2 i_1 - 2L^3 i_2\right)
\end{gathered}
$$

Proof. The general formula for the solution of PL model in the case of constant $\mathbf{v}(n)$ is

$$
n(t) = e^{-t/L} n_0 + \int_0^t e^{-(t-s)/L} i(s) ds
$$

If $i(t) = i_0 + i_1 t + i_2 t^2$, $t \in (0, T]$ then

$$\begin{aligned}
n(t) = {} & e^{-t/L} n_0 + \int\limits_0^t e^{-(t-s)/L} \left(i_0 + i_1 s + i_2 s^2\right) ds = \\
= {} & e^{-t/L} n_0 + \int\limits_{-\infty}^t e^{-(t-s)/L} \left(i_0 + i_1(t-(t-s)) + i_2(t-(t-s))^2\right) ds - \\
& - e^{-t/L} \int\limits_{-\infty}^0 e^{s/L} \left(i_0 + i_1 s + i_2 s^2\right) ds = \\
= {} & e^{-t/L} n_0 + \left(\int\limits_0^{+\infty} e^{-l/L} dl \right) \left(i_0 + i_1 t + i_2 t^2\right) + \\
& + \left(\int\limits_0^{+\infty} e^{-l/L} l dl \right) (-i_1 - 2 i_2 t) + \left(\int\limits_0^{+\infty} e^{-l/L} l^2 dl \right) i_2 - \\
& - e^{-t/L} \int\limits_0^{+\infty} e^{-l/L} \left(i_0 - i_1 l + i_2 l^2\right) dl = \\
= {} & e^{-t/L} n_0 + L\left(i_0 + i_1 t + i_2 t^2\right) + L^2(-i_1 - 2 i_2 t) + 2L^3 i_2 - \\
& - e^{-t/L} \left(L i_0 - L^2 i_1 + 2L^3 i_2\right) = \\
= {} & L(i_0 + i_1 t + i_2 t^2) - L^2 i_1 - 2L^2 i_2 t + 2L^3 i_2 + \\
& + e^{-t/L} \left(n_0 - L i_0 + L^2 i_1 - 2L^3 i_2\right)
\end{aligned}$$

□

Proposition 26. *The solution of M model with $\beta > 0$ for $i(t) = i_0 + i_1 t + i_2 t^2$ in the case of constant $\mathbf{v}(n)$ is*

$$\begin{aligned}
\begin{bmatrix} Lo^M(t) \\ n^M(t) \end{bmatrix} = {} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} L(i_0 + i_1 t + i_2 t^2) - \\
- \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} L^2 i_1 - 2 \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} L^2 i_2 t + 2 & \begin{bmatrix} 1 \\ \frac{1}{\alpha^2} + \frac{\frac{1}{\alpha}}{\frac{1}{\alpha^2 \beta}} - \frac{1}{\alpha\beta} \end{bmatrix} L^3 i_2 + \\
+ e^{-At/L} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} (n_0 - L i_0) + \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} L^2 i_1 - 2 \right. & \left. \begin{bmatrix} 1 \\ \frac{1}{\alpha^2} + \frac{\frac{1}{\alpha}}{\frac{1}{\alpha^2 \beta}} - \frac{1}{\alpha\beta} \end{bmatrix} L^3 i_2 \right)
\end{aligned}$$

where $A = \begin{bmatrix} 1+\beta & -\alpha\beta \\ 1 & 0 \end{bmatrix}$.

Proof. The general formula for the solution of M model in the case of constant $\mathbf{v}(n)$ is

$$\begin{aligned}
\begin{bmatrix} Lo(t) \\ n(t) \end{bmatrix} & = e^{-At/L} A \begin{bmatrix} 1 \\ \frac{1}{\alpha} \end{bmatrix} n_0 + \left(\int\limits_0^t e^{-A(t-s)/L} i(s) ds \right) A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \\
= e^{-At/L} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} n_0 + \left(\int\limits_0^t e^{-A(t-s)/L} i(s) ds \right) \begin{bmatrix} 1+\beta-\alpha\beta \\ 1 \end{bmatrix}
\end{aligned}$$

The product of eigenvalues of A is equal to $\alpha\beta$. If $\beta > 0$ and eigenvalues are real then they are positive. If eigenvalues are complex then their real part

$\frac{1+\beta}{2}$ is positive. Therefore, real parts of eigenvalues of A are positive. Consequently, we can claim that

$$\lim_{l\to+\infty} e^{-Al/L} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and also

$$\lim_{l\to+\infty} e^{-Al/L} l^{\phi} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for any $\phi > 0$. Now we prove by induction for $\phi = 0, 1, ...$ that

$$\int\limits_0^{+\infty} e^{-Al/L} l^{\phi} dl = \phi! A^{-\phi-1} L^{\phi+1}$$

Basis. For $\phi = 0$ we get

$$\int\limits_0^{+\infty} e^{-Al/L} dl = L \int\limits_0^{+\infty} e^{-Al/L} d\tfrac{l}{L} = A^{-1}L$$

Induction step.

$$\int\limits_0^{+\infty} e^{-Al/L} l^{\phi+1} dl = A^{-1}L \int\limits_0^{+\infty} e^{-Al/L} \left(l^{\phi+1}\right)' dl =$$
$$= (\phi+1)A^{-1}L \int\limits_0^{+\infty} e^{-Al/L} l^{\phi} dl = (\phi+1)! A^{-\phi-2} L^{\phi+2}$$

Once we proved this formula, we can use it to calculate the solution. First, calculate the integral

$$\begin{aligned}
&\int\limits_0^{t} e^{-A(t-s)/L} i(s) ds = \int\limits_{-\infty}^{t} e^{-A(t-s)/L} i(s) ds - \int\limits_{-\infty}^{0} e^{-A(t-s)/L} i(s) ds = \\
&= \int\limits_{-\infty}^{t} e^{-A(t-s)/L} \left(i_0 + i_1(t-(t-s)) + i_2(t-(t-s))^2\right) ds - \\
&\quad - e^{-At/L} \int\limits_{-\infty}^{0} e^{As/L} \left(i_0 + i_1 s + i_2 s^2\right) ds = \\
&= \int\limits_0^{+\infty} e^{-Al/L} \left(i_0 + i_1(t-l) + i_2(t-l)^2\right) dl - \\
&\quad - e^{-At/L} \int\limits_0^{+\infty} e^{-Al/L} \left(i_0 - i_1 l + i_2 l^2\right) dl = \\
&= A^{-1}L\left(i_0 + i_1 t + i_2 t^2\right) + A^{-2}L^2\left(-i_1 - 2i_2 t\right) + 2A^{-3}L^3 i_2 - \\
&\quad - e^{-At/L}\left(A^{-1}L i_0 - A^{-2}L^2 i_1 + 2A^{-3}L^3 i_2\right)
\end{aligned}$$

Finally, we can find

$$A^{-1}\begin{bmatrix}1+\beta-\alpha\beta\\1\end{bmatrix}=\tfrac{1}{\alpha\beta}\begin{bmatrix}0&\alpha\beta\\-1&1+\beta\end{bmatrix}\begin{bmatrix}1+\beta-\alpha\beta\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$
$$A^{-2}\begin{bmatrix}1+\beta-\alpha\beta\\1\end{bmatrix}=\tfrac{1}{\alpha\beta}\begin{bmatrix}0&\alpha\beta\\-1&1+\beta\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{\alpha}\end{bmatrix}$$
$$A^{-3}\begin{bmatrix}1+\beta-\alpha\beta\\1\end{bmatrix}=\tfrac{1}{\alpha\beta}\begin{bmatrix}0&\alpha\beta\\-1&1+\beta\end{bmatrix}\begin{bmatrix}1\\\frac{1}{\alpha}\end{bmatrix}=\begin{bmatrix}\frac{1}{\alpha}\\\frac{1}{\alpha^2}-\frac{1}{\alpha\beta}+\frac{1}{\alpha^2\beta}\end{bmatrix}$$

Therefore,

$$\begin{bmatrix}Lo^M(t)\\n^M(t)\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}L(i_0+i_1t+i_2t^2)-$$
$$-\begin{bmatrix}1\\\frac{1}{\alpha}\end{bmatrix}L^2i_1-2\begin{bmatrix}1\\\frac{1}{\alpha}\end{bmatrix}L^2i_2t+2\begin{bmatrix}\frac{1}{\alpha}\\\frac{1}{\alpha^2}+\frac{1}{\alpha^2\beta}-\frac{1}{\alpha\beta}\end{bmatrix}L^3i_2+$$
$$+e^{-At/L}\left(\begin{bmatrix}1\\1\end{bmatrix}(n_0-Li_0)+\begin{bmatrix}1\\\frac{1}{\alpha}\end{bmatrix}L^2i_1-2\begin{bmatrix}\frac{1}{\alpha}\\\frac{1}{\alpha^2}+\frac{1}{\alpha^2\beta}-\frac{1}{\alpha\beta}\end{bmatrix}L^3i_2\right)$$

□

Proposition 27. *If $\alpha>1$ and $\beta\geqslant 1$ then the absolute value of matrix exponential $e^{-At/L}$, where $A=\begin{bmatrix}1+\beta&-\alpha\beta\\1&0\end{bmatrix}$, is less than*

$$e^{-(1-e^{-1})t/L}\begin{bmatrix}\frac{3+\beta}{2}&\alpha\beta\\1&\frac{3+\beta}{2}\end{bmatrix}$$

Proof. To calculate matrix exponential $e^{-At/L}$ we introduce matrix

$$Q=A-\tfrac{1}{2}tr(A)I=\begin{bmatrix}1+\beta&-\alpha\beta\\1&0\end{bmatrix}-\tfrac{1+\beta}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}\frac{1+\beta}{2}&-\alpha\beta\\1&-\frac{1+\beta}{2}\end{bmatrix}$$

This gives

$$e^{-At/L}=e^{-\frac{1}{2}tr(A)t/L}e^{-Qt/L}=e^{-\frac{1+\beta}{2}t/L}e^{-Qt/L}$$

From $tr(Q)=0$ follows (for a 2×2 matrix) that $Q^2=-det(Q)I$. This is a trivial consequence of the fact that characteristic polynomial of $n\times n$ matrix P is equal to $\lambda^n-tr(P)\lambda^{n-1}+...+(-1)^n det(P)$. One should just take $n=2$ and apply Cayley-Hamilton theorem. However, manual check looks easier. Denote $\psi=-det(Q)=\frac{(1+\beta)^2}{4}-\alpha\beta$. Thus, depending on the sign of ψ, we get three different cases:

1. $\psi > 0$

$$e^{-At/L} = e^{-\frac{1+\beta}{2}t/L}\left(cosh(\sqrt{\psi}t/L)I - \tfrac{1}{\sqrt{\psi}}sinh(\sqrt{\psi}t/L)Q\right) \tag{3.4}$$

2. $\psi = 0$

$$e^{-At/L} = e^{-\frac{1+\beta}{2}t/L}\left(I - \tfrac{t}{L}Q\right) \tag{3.5}$$

3. $\psi < 0$

$$e^{-At/L} = e^{-\frac{1+\beta}{2}t/L}\left(cos(\sqrt{-\psi}t/L)I - \tfrac{1}{\sqrt{-\psi}}sin(\sqrt{-\psi}t/L)Q\right) \tag{3.6}$$

Obviously, $t/L \leqslant e^{e^{-1}t/L}$ (equality is reached at $t/L = e$). This gives the following upper bounds on the absolute value of $e^{-At/L}$:
1. $\psi > 0$

$$\begin{aligned} \left|e^{-At/L}\right| &\leqslant e^{-\frac{1+\beta}{2}t/L}cosh(\sqrt{\psi}t/L)\left|I - \tfrac{1}{\sqrt{\psi}}tanh(\sqrt{\psi}t/L)Q\right| \leqslant \\ &\leqslant e^{-\frac{1+\beta}{2}t/L}e^{\sqrt{\psi}t/L}\left(I + \tfrac{t}{L}|Q|\right) < \\ &< e^{-\left(\frac{1+\beta}{2} - \sqrt{\psi}\right)t/L}e^{e^{-1}t/L}\left(I + |Q|\right) = \\ &= e^{-\left(\frac{1+\beta}{2} - \sqrt{\psi} - e^{-1}\right)t/L}\begin{bmatrix} \frac{3+\beta}{2} & \alpha\beta \\ 1 & \frac{3+\beta}{2} \end{bmatrix} \end{aligned} \tag{3.7}$$

2 and 3. $\psi \leqslant 0$

$$\begin{aligned} \left|e^{-At/L}\right| &\leqslant e^{-\frac{1+\beta}{2}t/L}\left(I + \tfrac{t}{L}|Q|\right) < e^{-\left(\frac{1+\beta}{2} - e^{-1}\right)t/L}(I + |Q|) = \\ &= e^{-\left(\frac{1+\beta}{2} - e^{-1}\right)t/L}\begin{bmatrix} \frac{3+\beta}{2} & \alpha\beta \\ 1 & \frac{3+\beta}{2} \end{bmatrix} \end{aligned} \tag{3.8}$$

Now notice that if $\alpha > 1$ and $\beta \geqslant 1$ then $\frac{1+\beta}{2} \geqslant 1$ and in the case $\psi > 0$

$$\tfrac{1+\beta}{2} - \sqrt{\psi} = \frac{\alpha\beta}{\frac{1+\beta}{2} + \sqrt{\psi}} > \frac{\alpha\beta}{\frac{1+\beta}{2} + \frac{\beta-1}{2}} = \alpha > 1$$

Thus, for any $\alpha > 1$, $\beta \geqslant 1$ the inequality

$$\left|e^{-At/L}\right| < e^{-(1-e^{-1})t/L}\begin{bmatrix} \frac{3+\beta}{2} & \alpha\beta \\ 1 & \frac{3+\beta}{2} \end{bmatrix}$$

is satisfied. □

Proposition 28. *If $\alpha > 1$ and ρ^3 correspond to some distribution defined on $(0, 3L)$ then*

$$\frac{\frac{1}{3} - \frac{\rho^3}{6L^3}}{1 - \frac{1}{\alpha}} > \frac{1}{\alpha}$$

Proof. From Proposition 21 follows that

$$\begin{aligned} &\tfrac{\rho^3}{L^3} < 2\left(\tfrac{2}{\alpha} - 1\right) - \tfrac{1}{2}\left(\tfrac{2}{\alpha} - 1\right)^2 = -\tfrac{5}{2} + \tfrac{6}{\alpha} - \tfrac{2}{\alpha^2} = \\ &= 2 - 6\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) - 2\left(\tfrac{2}{\alpha} - \tfrac{3}{2}\right)^2 \leqslant 2 - 6\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) \end{aligned}$$

Therefore,

$$\frac{\frac{1}{3} - \frac{\rho^3}{6L^3}}{1 - \frac{1}{\alpha}} > \frac{\frac{1}{3} - \frac{\rho^3}{6L^3}}{1 - \frac{1}{\alpha}} = \frac{\frac{1}{3} - \frac{1}{3} + \left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right)}{1 - \frac{1}{\alpha}} = \frac{1}{\alpha}$$

□

Proposition 29. *If $i_+(0) = \frac{\mathbf{p}(n_0)}{L} + \Delta i$ and $\mathbf{v}(n)$ is differentiable then the outflow in M model has right derivative*

$$\dot{o}_+(0) = \left(\frac{\mathbf{p}'(n_0)}{L} + \beta(1-\alpha)\frac{\mathbf{v}(n_0)}{L}\right)\Delta i$$

Proof.

$$\begin{aligned} &\dot{o}_+(t) = \tfrac{\mathbf{v}'(n(t))}{L}\dot{n}_+(t)\left(n(t) + \beta\left(n(t) - \tfrac{\alpha}{L}M(t)\right)\right) + \\ &+ \tfrac{\mathbf{v}(n(t))}{L}\left(\dot{n}_+(t) + \beta\left(\dot{n}_+(t) - \tfrac{\alpha}{L}\left(Li_+(t) - \mathbf{v}(n(t))n(t)\right)\right)\right) \end{aligned}$$

Obviously, $\dot{n}_+(0) = \Delta i$. This results in

$$\begin{aligned} &\dot{o}_+(0) = \tfrac{\mathbf{v}'(n_0)}{L}n_0\Delta i + \tfrac{\mathbf{v}(n_0)}{L}\Delta i + \beta\tfrac{\mathbf{v}(n_0)}{L}\left(\Delta i - \alpha\Delta i\right) = \\ &= \left(\tfrac{\mathbf{p}'(n_0)}{L} + \beta(1-\alpha)\tfrac{\mathbf{v}(n_0)}{L}\right)\Delta i \end{aligned}$$

□

Proposition 30. *Numerical solution of TB model is equivalent to numerical solution of PL model for any input if and only if f_m, $m = 1, 2, \ldots$ correspond to geometric distribution with mean $\frac{L}{\Delta t}$.*

Proof. The numerical scheme for solving TB model looks as

$$\begin{cases} n_{0,m} = n_0\frac{\Delta t}{L}\sum\limits_{r=m}^{\infty}(1 - F_r) \\ n_{k+1,m} - n_{k,m} = (1 - F_m)i_{k+1}\Delta t - \mathbf{v}(n_{k,0})(n_{k,m} - n_{k,m+1}) \end{cases}$$

To solve PL model we use the scheme

$$n_{k+1} - n_k = i_{k+1}\Delta t - \frac{\mathbf{v}(n_k)}{L} n_k \Delta t$$

First, prove that if $f_m = \frac{\Delta t}{L}\left(1 - \frac{\Delta t}{L}\right)^{m-1}$, $m = 1, 2, ...$ then numerical solution of TB model is equivalent to numerical solution of PL model for any input. It is sufficient to prove that $n_{k+1,0}^{TB} - n_{k,0}^{TB}$ is equal to $n_{k+1}^{PL} - n_k^{PL}$ for $k = 1, 2, ..., K-1$. This condition is equivalent to

$$n_{k,0}^{TB} - n_{k,1}^{TB} = n_{k,0}^{TB}\frac{\Delta t}{L},\ k = 0, 1, ..., K-1$$

We can prove this by induction. However, it is easier to prove

$$n_{k,m}^{TB} - n_{k,m+1}^{TB} = n_{k,m}^{TB}\frac{\Delta t}{L},\ k = 0, 1, ..., K-1,\ m = 0, 1, ...$$

which is more general fact. The CDF that corresponds to f_m is

$$F_m = 1 - \left(1 - \frac{\Delta t}{L}\right)^m,\ m = 0, 1, ...$$

Basis. For $k = 0$ we have initial condition

$$n_{0,m} = n_0 \frac{\Delta t}{L} \sum_{r=m}^{\infty} (1 - F_r),\ m = 0, 1, ...$$

Therefore,

$$n_{0,m} = n_0 \frac{\Delta t}{L} \frac{\left(1 - \frac{\Delta t}{L}\right)^m}{1 - \left(1 - \frac{\Delta t}{L}\right)} = n_0\left(1 - \frac{\Delta t}{L}\right)^m$$

$$n_{0,m} - n_{0,m+1} = n_{0,m}\frac{\Delta t}{L}$$

Induction step.

$$\begin{aligned} n_{k+1,m} - n_{k+1,m+1} &= n_{k,m} - n_{k,m+1} + (F_{m+1} - F_m) i_{k+1}\Delta t - \\ &- \mathbf{v}(n_{k,0})\left(n_{k,m} - n_{k,m+1} - n_{k,m+1} + n_{k,m+2}\right) = \\ = \left(n_{k,m} + i_{k+1}\Delta t - \mathbf{v}(n_{k,0})(n_{k,m} - n_{k,m+1})\right)\frac{\Delta t}{L} &= n_{k+1,m}\frac{\Delta t}{L} \end{aligned}$$

Second, prove that if numerical solution of TB model is equivalent to numerical solution of PL model for any input then $f_m = \frac{\Delta t}{L}\left(1 - \frac{\Delta t}{L}\right)^{m-1}$, $m = 1, 2, ...$ This equality will follow from another statement:
For any $K > 1$ and any $m = 0, 1, ..., K-1$ the equality

$$n_{k,m} - n_{k,m+1} = n_{k,0}\frac{\Delta t}{L}\left(1 - \frac{\Delta t}{L}\right)^m,\ k = 0, 1, ..., K-1-m$$

holds for any input that does not produce gridlock. We prove this equality by

induction.
Basis. If $m = 0$ then

$$n_{k+1,0} - n_{k,1} = i_{k+1}\Delta t - \mathbf{v}(n_{k,0})(n_{k,0} - n_{k,1}) =$$
$$= i_{k+1}\Delta t - \tfrac{\mathbf{v}(n_{k,0})}{L} n_{k,0}\Delta t,\ k = 0, 1, ..., K-1$$

and, therefore,

$$n_{k,0} - n_{k,1} = n_{k,0}\tfrac{\Delta t}{L},\ k = 0, 1, ..., K-1$$

Induction step. By definition, the solution of TB model satisfies

$$n_{k+1,m} - n_{k+1,m+1} = n_{k,m} - n_{k,m+1} + (F_{m+1} - F_m)i_{k+1}\Delta t -$$
$$- \mathbf{v}(n_{k,0})\left(n_{k,m} - n_{k,m+1} - n_{k,m+1} + n_{k,m+2}\right),\ k = 0, 1, ..., K-1$$

for any input. By our assumption,

$$n_{k+1,0}\tfrac{\Delta t}{L}\left(1 - \tfrac{\Delta t}{L}\right)^m = n_{k,0}\tfrac{\Delta t}{L}\left(1 - \tfrac{\Delta t}{L}\right)^m + (F_{m+1} - F_m)i_{k+1}\Delta t -$$
$$- \mathbf{v}(n_{k,0})\left(n_{k,0}\tfrac{\Delta t}{L}\left(1 - \tfrac{\Delta t}{L}\right)^m - n_{k,m+1} + n_{k,m+2}\right),\ k = 0, 1, ..., K-2-m$$

for any TB model which is equivalent to PL model for any input that does not produce gridlock. Any solution of TB model satisfies

$$n_{k+1,0} = n_{k,0} + i_{k+1}\Delta t - \mathbf{v}(n_{k,0})\left(n_{k,0} - n_{k,1}\right),\ k = 0, 1, ..., K-1$$

Therefore, as i_{k+1} is independent from $n_{k,m+1} - n_{k,m+2}$, we have

$$F_{m+1} - F_m = \tfrac{\Delta t}{L}\left(1 - \tfrac{\Delta t}{L}\right)^m$$

and

$$n_{k,m+1} - n_{k,m+2} = n_{k,1}\tfrac{\Delta t}{L}\left(1 - \tfrac{\Delta t}{L}\right)^m =$$
$$= n_{k,0}\tfrac{\Delta t}{L}\left(1 - \tfrac{\Delta t}{L}\right)^{m+1},\ k = 0, 1, ..., K-1-(m+1)$$

The first equality is equivalent to

$$f_{m+1} = \tfrac{\Delta t}{L}\left(1 - \tfrac{\Delta t}{L}\right)^m$$

We proved the statement for all $m = 0, 1, ..., K-2$ and at the same time found all f_{m+1}. Therefore, any f_m, $m = 1, 2, ...$ can be found by taking $K = m+1$. □

α model 4

To derive M model we assumed that the inflow $i(t)$ is piecewise continuous, because we were searching for a simplified model that can be used for the inflow control. However, if we assume that $i(t)$ changes slowly, it is not necessary to approximate the outflow $o(t)$ based on the state variables, like accumulation $n(t)$ or total distance to be traveled $M(t)$. We can try to derive an approximation where $o(t)$ depends also on $i(t)$. In this chapter we present a simple approximation of TB model for slowly changing $i(t)$. It uses only one parameter $\alpha = \frac{2L^2}{L^2+\sigma^2}$ to describe $f(l)$. This model (we refer to it as α model) takes the form of Cauchy problem

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = \alpha \left(i(t) - \frac{\mathbf{v}(n(t))}{L} n(t) \right), \; t \in (0, T] \end{cases} \tag{4.1}$$

The outflow $o(t)$ in α model is equal to

$$o(t) = i(t) - \dot{n}(t) = \alpha \frac{\mathbf{v}(n(t))}{L} n(t) - (\alpha - 1) i(t)$$

We derive α model by showing that accumulation $n^{TB}(t)$ in TB model approximately satisfies

$$\dot{n}(t) \approx \alpha \left(i(t) - \frac{\mathbf{v}(n(t))}{L} n(t) \right)$$

4.1 Derivation of α model

All the steps should be understood as a physical intuition. There will not be rigorous proofs or definitions in this section. The approximation will be derived under the assumptions of slowly changing $i(t)$ and not very small $\mathbf{v}(n(t))$.

If we make a change of variable $y = \int\limits_0^t \mathbf{v}(n(u))du$ and introduce function $g(y) = \frac{i(t(y))}{\mathbf{v}(n(t(y)))}$ then we can also assume that $g(y)$ changes slowly. We do not say that $g(y)$ is necessarily differentiable. For us it is more important to assume that for any y there exists $g_1(y)$ such that

$$g(x) \approx g(y) - g_1(y)(y-x),\ x \in (y - \Lambda L, y)$$

where ΛL is the maximum trip length. Given this approximation and Proposition 13, we conclude that

$$\begin{aligned} n(t) &= \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds = \\ &= \int\limits_{y-\Lambda L}^{y} (1 - F(y-x))g(x)dx \approx \\ &\approx \int\limits_{y-\Lambda L}^{y} (1 - F(y-x))(g(y) - g_1(y)(y-x))dx = \\ &= Lg(y) - \tfrac{L^2+\sigma^2}{2} g_1(y) = Lg(y) - \tfrac{L^2}{\alpha} g_1(y) \end{aligned}$$

The approximate value of $o(t)$ can be calculated as

$$\begin{aligned} o(t) &= \mathbf{v}(n(t)) \int\limits_{-\infty}^{t} f\left(\int_s^t \mathbf{v}(n(u))du\right) i(s)ds = \\ &= \mathbf{v}(n(t)) \int\limits_{y-\Lambda L}^{y} f(y-x)g(x)dx \approx \\ &\approx \mathbf{v}(n(t)) \int\limits_{y-\Lambda L}^{y} f(y-x)(g(y) - g_1(y)(y-x))dx = \\ &= \mathbf{v}(n(t))(g(y) - Lg_1(y)) = i(t) - \mathbf{v}(n(t))L\tfrac{\alpha}{L^2}\left(Lg(y) - n(t)\right) = \\ &= i(t) - \alpha\left(i(t) - \tfrac{\mathbf{v}(n(t))}{L} n(t)\right) = \alpha \tfrac{\mathbf{v}(n(t))}{L} n(t) - (\alpha - 1)i(t) \end{aligned}$$

This leads to

$$\dot{n}(t) \approx \alpha\left(i(t) - \tfrac{\mathbf{v}(n(t))}{L} n(t)\right)$$

4.2 Connection between α and PL models

From the mathematical point of view, PL model is a specific case of α model for $\alpha = 1$. This means that if $\alpha = 1$ then the solutions of PL and TB models are similar for the slowly changing inflow. However, in this work we consider that $f(l)$ with $\alpha = 1$ is not realistic.

Second important property of α model (4.1) is that it can be reformulated as PL model. Consider change of variable $z = \alpha t$ and functions $I(z) = i(z/\alpha)$, $z \in [0, \alpha T]$, $N(z) = n(z/\alpha)$, $z \in [0, \alpha T]$. This leads to

$$\begin{aligned} N'(z) = \dot{n}(t(z))t'(z) = \tfrac{\dot{n}(t)}{\alpha} = \\ = i(t) - \tfrac{\mathbf{v}(n(t))}{L}n(t) = I(z) - \tfrac{\mathbf{v}(N(z))}{L}N(z) \end{aligned}$$

From this equality follows that α model can be solved as PL model for variable z. One needs to stretch the t axis α times to get the inflow $I(z)$, then find for this inflow the solution $N(z)$ of PL model and then contract z axis back to t axis to get the solution $n(t)$ of α model for the inflow $i(t)$. The first important consequence of this reformulation is the existence and uniqueness of solution of α model. The second important consequence is the stability of α model for constant $\mathbf{v}(n)$.

4.3 Connection between α and M models

In this section we show for the case of constant $\mathbf{v}(n)$ that α model can be viewed as a limiting case of M model with parameter β when β tends to $+\infty$.

First, solve α model for the case $\mathbf{v}(n) = 1$. Representation from Section 4.2 leads to

$$n(t) = N(z) = e^{-z/L}n_0 + \int\limits_0^z e^{-(z-q)/L}I(q)dq$$

where $z = \alpha t$ and $I(z) = i(t)$. By putting $q = \alpha s$ we get

$$n(t) = e^{-\alpha t/L}n_0 + \alpha \int\limits_0^t e^{-\alpha(t-s)/L}i(s)ds \tag{4.2}$$

Second, notice that parameter $\psi = \frac{(1+\beta)^2}{4} - \alpha\beta$ tends to $+\infty$. Therefore, starting from rather big β, the solution of M model is

$$\begin{aligned} n^M(t) = e^{-\frac{1+\beta}{2}t/L}\left(cosh(\sqrt{\psi}t/L) + \tfrac{\beta-1}{2\sqrt{\psi}}sinh(\sqrt{\psi}t/L)\right)n_0 + \\ + \int\limits_0^t e^{-\frac{1+\beta}{2}(t-s)/L}\left(cosh(\sqrt{\psi}(t-s)/L) + \tfrac{(2\alpha-1)\beta-1}{2\sqrt{\psi}}sinh(\sqrt{\psi}(t-s)/L)\right)i(s)ds \end{aligned}$$

Now find the limit of this expression. Obviously,

$$\lim_{\beta\to+\infty} \frac{cosh(\sqrt{\psi}t/L)}{e^{\sqrt{\psi}t/L}} = \lim_{\beta\to+\infty} \frac{sinh(\sqrt{\psi}t/L)}{e^{\sqrt{\psi}t/L}} = \frac{1}{2}$$

for positive t. We can also take $t-s$ (where $s<t$) instead of t. Moreover,

$$\lim_{\beta\to+\infty} \frac{\beta-1}{2\sqrt{\psi}} = \lim_{\beta\to+\infty} \frac{1-\frac{1}{\beta}}{\sqrt{\left(1+\frac{1}{\beta}\right)^2-\frac{4\alpha}{\beta}}} = 1$$

and

$$\lim_{\beta\to+\infty} \frac{(2\alpha-1)\beta-1}{2\sqrt{\psi}} = \lim_{\beta\to+\infty} \frac{2\alpha-1-\frac{1}{\beta}}{\sqrt{\left(1+\frac{1}{\beta}\right)^2-\frac{4\alpha}{\beta}}} = 2\alpha-1$$

and

$$\lim_{\beta\to+\infty} \left(\tfrac{1+\beta}{2}-\sqrt{\psi}\right) = \lim_{\beta\to+\infty} \frac{\alpha\beta}{\frac{1+\beta}{2}+\sqrt{\psi}} =$$
$$= \lim_{\beta\to+\infty} \frac{2\alpha}{1+\frac{1}{\beta}+\sqrt{\left(1+\frac{1}{\beta}\right)^2-\frac{4\alpha}{\beta}}} = \alpha$$

Therefore,

$$\lim_{\beta\to+\infty} n^M(t) = e^{-\alpha t/L}n_0 + \alpha\int_0^t e^{-\alpha(t-s)/L}i(s)ds$$

which is the solution of α model.

4.4 Analytical comparison of α and M models

In this section we perform analytical comparison of α and M models as an approximations of TB model. We assume $\mathbf{v}(n)=1$ to simplify the analysis. Similarly to Section 2.4, we consider cases of big and small t. Big t should be understood as $t>\Lambda L$, where Λ is not a very big number such that domain of $f(l)$ belongs to $[0,\Lambda L]$. Small t should be understood as $t<\Lambda L$.

In the case of big t we consider quadratic inflow profile $i(t)=i_0+i_1t+i_2t^2$. Proposition 24 says that the solution of TB model in this case is

$$n^{TB}(t) = L(i_0+i_1t+i_2t^2) - \frac{L^2}{\alpha}i_1 - 2\frac{L^2}{\alpha}i_2t + 2\left(\frac{1}{\alpha}-\frac{1}{3}+\frac{\rho^3}{6L^3}\right)L^3i_2,\ t\in[\Lambda L,T]$$

where ρ^3 is the third central moment of $f(l)$. From the reformulation of α model presented in Section 4.2 and Proposition 25 follows that the solution of

α model is

$$n^\alpha(t) = L\left(i_0 + \tfrac{i_1}{\alpha}(\alpha t) + \tfrac{i_2}{\alpha^2}(\alpha t)^2\right) - L^2\tfrac{i_1}{\alpha} - 2L^2\tfrac{i_2}{\alpha^2}(\alpha t) + 2L^3\tfrac{i_2}{\alpha^2} + \\ + e^{-\alpha t/L}\left(n_0 - Li_0 + L^2\tfrac{i_1}{\alpha} - 2L^3\tfrac{i_2}{\alpha^2}\right)$$

which is equal to

$$n^\alpha(t) = L(i_0 + i_1 t + i_2 t^2) - \tfrac{L^2}{\alpha}i_1 - 2\tfrac{L^2}{\alpha}i_2 t + 2\tfrac{L^3}{\alpha^2}i_2 + \\ + e^{-\alpha t/L}\left(n_0 - Li_0 + \tfrac{L^2}{\alpha}i_1 - 2\tfrac{L^3}{\alpha^2}i_2\right)$$

The difference $n^\alpha(t) - n^{TB}(t)$ converges to the constant

$$2\left(-\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3} - \tfrac{\rho^3}{6L^3}\right)L^3 i_2$$

Thus, if $i_2 = 0$ then α model converges to TB model. Recall that PL model does not converge to TB model for the linear inflow if $\alpha > 1$. In this sense α model is more preferable as an approximation of TB model. To understand the quality of approximation for the quadratic inflow, we look at the value of

$$\delta = -\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3} - \tfrac{\rho^3}{6L^3}$$

for realistic and gamma-like distributions. From Section 2.4 follows that the similar value of δ for M model is equal to

$$\delta = -\left(1 + \tfrac{1}{\beta}\right)\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3} - \tfrac{\rho^3}{6L^3}$$

Obviously, if β tends to $+\infty$ then this value tends to δ that corresponds to α model. Thus, we can use formulas (2.13) and (2.14) to find all the possible values of δ for reasonable and gamma-like distributions in α model. The range of δ for reasonable distributions is

$$\begin{cases} \delta > -\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{12}\left(9 - \tfrac{12}{\alpha} + \tfrac{4}{\alpha^2}\right) = \tfrac{3}{4}\left(\tfrac{4}{3\alpha} - 1\right)^2 \\ \delta < -\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3}\left(\tfrac{3}{\alpha} - \tfrac{2}{\alpha^2}\right) = \tfrac{1}{3\alpha^2} \end{cases}$$

The value of δ for gamma-like distributions is

$$\delta = -\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right) + \tfrac{1}{3} - \tfrac{1}{6}\left(2 - 8\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right)\right) = \tfrac{1}{3}\left(\tfrac{1}{\alpha} - \tfrac{1}{\alpha^2}\right)$$

In Figure 4.1 we compare M and α models in the sense of possible values of δ for reasonable and gamma-like distributions.

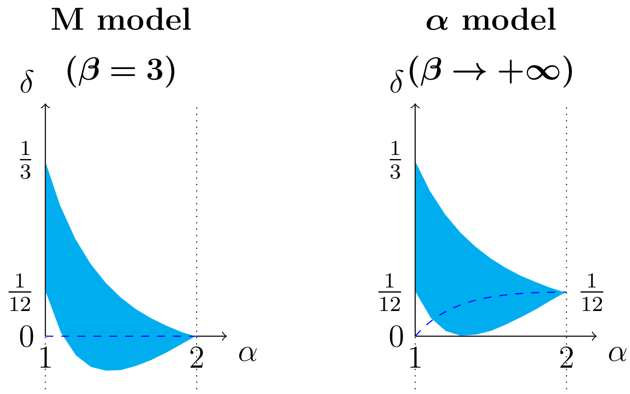


Figure 4.1: Possible values of δ for reasonable (cyan area) and gamma-like (blue, dashed) distributions. Values are given for M model with $\beta = 3$ (left) and α model (right).

We can say that M model works better for realistic distributions which we consider to be almost gamma-like. However, α model is not very bad as it gives rather small values of δ for gamma-like distributions and not very big values for reasonable distributions.

Now consider the case of small t. As in Section 2.4, we assume $i(t) = \frac{n_0}{L} + \Delta i$, $t \in (0, T]$ and take distributions

$$\begin{aligned} \mathrm{D1} &= \tfrac{1}{2}U[0, L] + \tfrac{1}{2}U[0, 3L] \\ \mathrm{D2} &= U\left[\left(1 - \tfrac{\sqrt{3}}{2}\right)L, \left(1 + \tfrac{\sqrt{3}}{2}\right)L\right] \end{aligned}$$

which are reasonable and not gamma-like. D1 has $\alpha = 1.2$ and D2 has $\alpha = 1.6$. We look at the outflow $o(t)$ to see the difference between models more clearly. To solve α model for $i(t) = \frac{n_0}{L} + \Delta i$ we use formula (4.2). It gives

$$n(t) = n_0 + L\Delta i \left(1 - e^{-\alpha t/L}\right)$$

The corresponding outflow can be calculated as $o(t) = i(t) - \dot{n}(t)$. This gives

$$o(t) = \tfrac{n_0}{L} + \Delta i \left(1 - \alpha e^{-\alpha t/L}\right)$$

In Figure 4.2 we compare $o(t)$, $t \in (0, 3L]$ in M and α models.

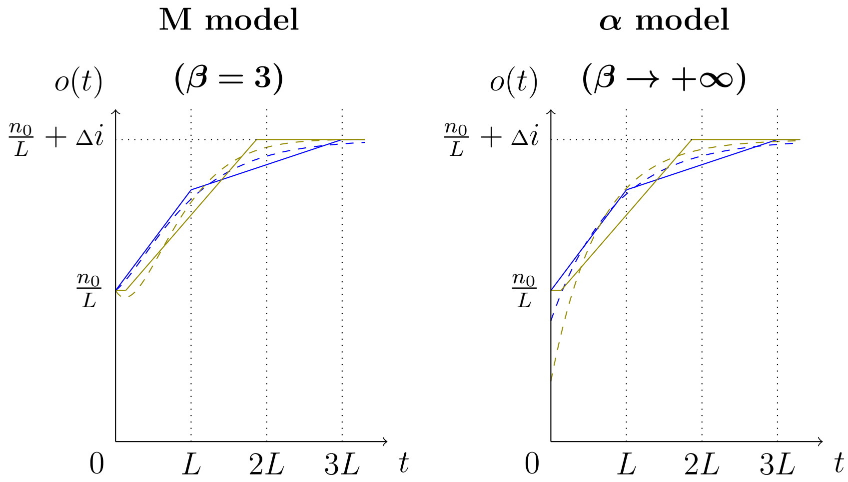


Figure 4.2: Solutions of models for $i(t) = \frac{n_0}{L} + \Delta i$. TB model is given for D1 (blue, solid) and D2 (olive, solid). M model (left) is given for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed). It approximates TB model very well. α model (right) is also given for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed). It approximates TB model not very well.

We can see that M model is more precise in describing the behavior of TB model for the jumping inflow. The biggest problem of α model is in the very beginning of the time period. It cannot approximate TB model well because in the case $\alpha > 1$ the function $o(t)$ makes a jump of the opposite direction to the jump of $i(t)$. The size of the jump is $(\alpha - 1)\Delta i$. This might create also negative values of $o(t)$ for some inputs. However, these inputs are not realistic because in practice we expect the size of Δi to be smaller than $\frac{n_0}{L}$.

4.5 Numerical comparison of α and M models

As we showed in this chapter, α model should be rather good approximation of TB model for the slowly changing inflow. The case of jumping inflow is very unclear. There is a big problem of not accurate approximation just after the jump, but the sign of discrepancy changes very soon. Thus, one can expect descent results if jumps occur not very often. We are mostly interested in inflows that are typical for the transportation systems. To test α model and compare it with M model we take the same setting as in the Section 2.5. The speed-MFD is shown in Figure 4.3:

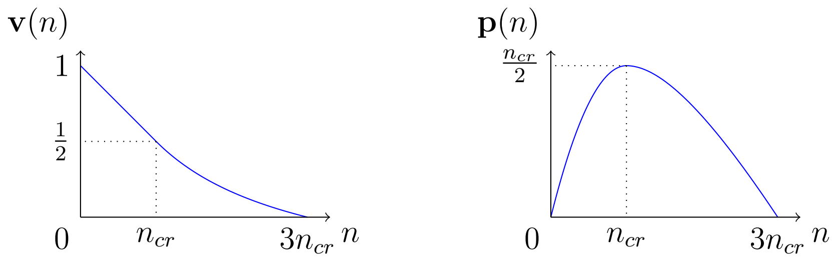


Figure 4.3: Realistic function $\mathbf{v}(n)$ and corresponding $\mathbf{p}(n)$.

The input is $n_0 = 0.3n_{cr}$, $T = 30L$, $i(t) = i_0 + 4\frac{t}{T}\left(1 - \frac{t}{T}\right)\Delta i$, where $i_0 = 0.51c$ and $i_0 + \Delta i \in [0.9c, 1.2c]$. The value of c is the capacity of the zone. It is equal to $\frac{n_{cr}}{2L}$ for the chosen $\mathbf{v}(n)$. We also test jumping inflow $i(t) + (-1)^{[t/L]}0.1c$ to simulate control. The discretization of TB, PL and M models is the same as in Section 2.5. To solve α model we use numerical scheme

$$n_{k+1} - n_k = \alpha\left(i_{k+1}\Delta t - \frac{\mathbf{v}(n_k)}{L}n_k\Delta t\right)$$

In general, α model appeared to be much closer to TB model than PL model. It reproduces hysteresis effect and under some conditions reproduces the effect when the gridlock property of solution depends on $f(l)$. We illustrate these effects in Figure 4.4 for $i_0 + \Delta i = 1.1c$.

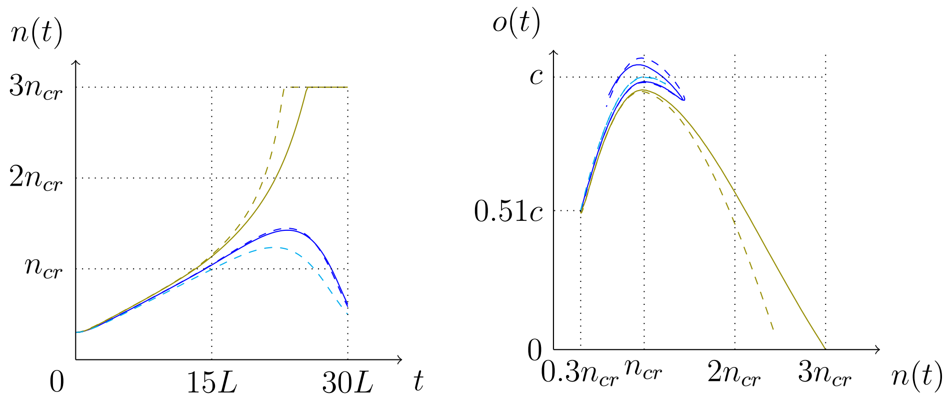


Figure 4.4: Solutions of models for smooth inflow with $i_0 + \Delta i = 1.1c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). α model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) approximates TB model very well. PL model (cyan, dashed) is not a good approximation and cannot predict gridlock for D2.

Similarly to the analysis shown in Section 2.5, we can find the maximum value of accumulation n_{max} depending on $i_0 + \Delta i$. The results for α, TB and PL models are shown in Figure 4.5.

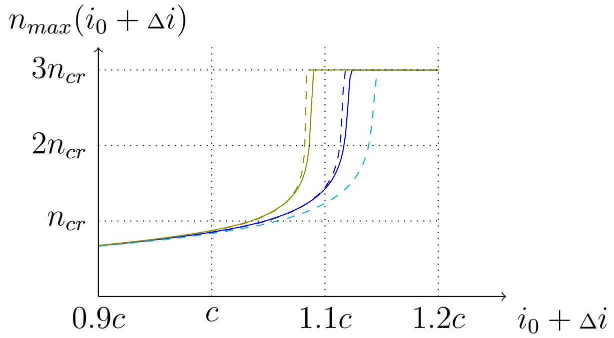


Figure 4.5: Maximum accumulation for smooth inflow depending on $i_0 + \Delta i$. TB model is given for D1 (blue) and D2 (olive). α model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) shows similar values. PL model (cyan, dashed) shows very different values for $i_0 + \Delta i > c$.

One can see from Figures 2.19 and 4.5 that α model is less accurate than M model as an approximation of TB model, however, it is still much more preferable than PL model. We can also plot the average relative error as function of $i_0 + \Delta i$ to visualize the accuracy that each of models gives. The results are shown in Figure 4.6.

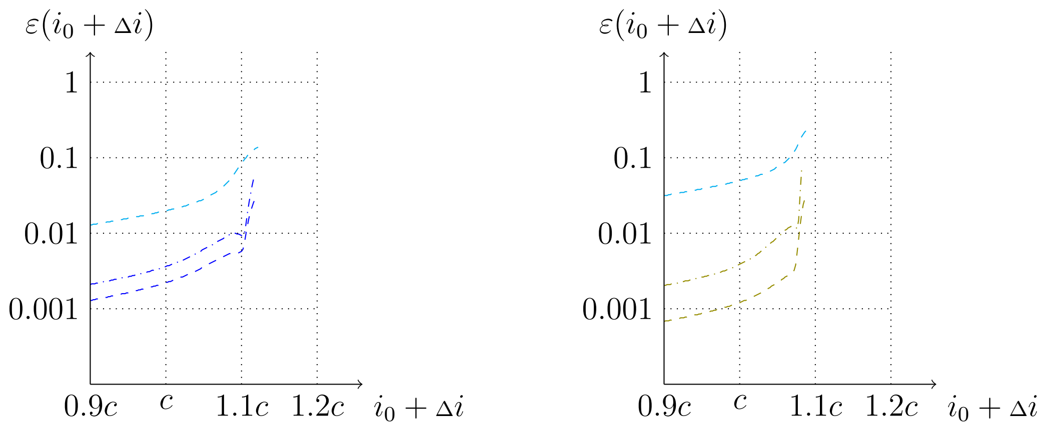


Figure 4.6: Relative error between approximations and TB model for smooth inflow. TB model is given for D1 (left) and D2 (right). The error of M model for $\alpha = 1.2$ (blue, dashed) and M model for $\alpha = 1.6$ (olive, dashed) is much lower than the error of PL model (cyan, dashed). The error of α model for $\alpha = 1.2$ (blue, dash-dotted) and $\alpha = 1.6$ (olive, dash-dotted) lies between the errors of M and PL models.

Interestingly, α model still works well for the jumping inflow. To show this we make similar analysis as in Figures 4.4-4.6. In Figures 4.7-4.9 we take jumping inflow $i(t) + (-1)^{[t/L]}0.1c$ instead of smooth inflow $i(t) = i_0 + 4\frac{t}{T}\left(1 - \frac{t}{T}\right)\Delta i$.

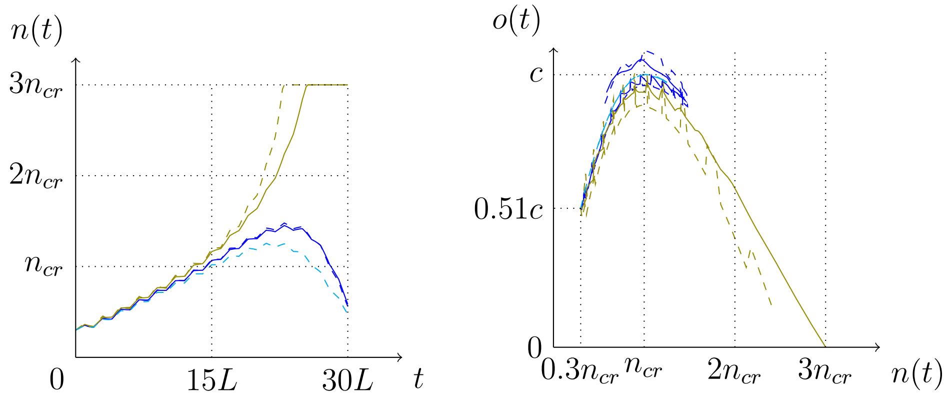


Figure 4.7: Solutions of models for jumping inflow with $i_0 + \Delta i = 1.1c$. TB model is given for D1 (blue, solid) and D2 (olive, solid). α model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) approximates TB model very well. PL model (cyan, dashed) is not a good approximation and cannot predict gridlock for D2.

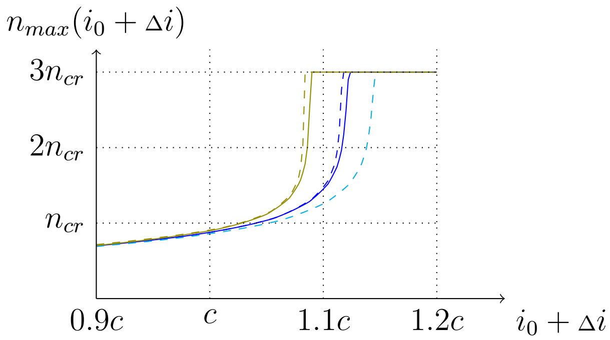


Figure 4.8: Maximum accumulation for jumping inflow depending on $i_0 + \Delta i$. TB model is given for D1 (blue) and D2 (olive). α model for $\alpha = 1.2$ (blue, dashed) and $\alpha = 1.6$ (olive, dashed) shows similar values. PL model (cyan, dashed) shows very different values for $i_0 + \Delta i > c$.

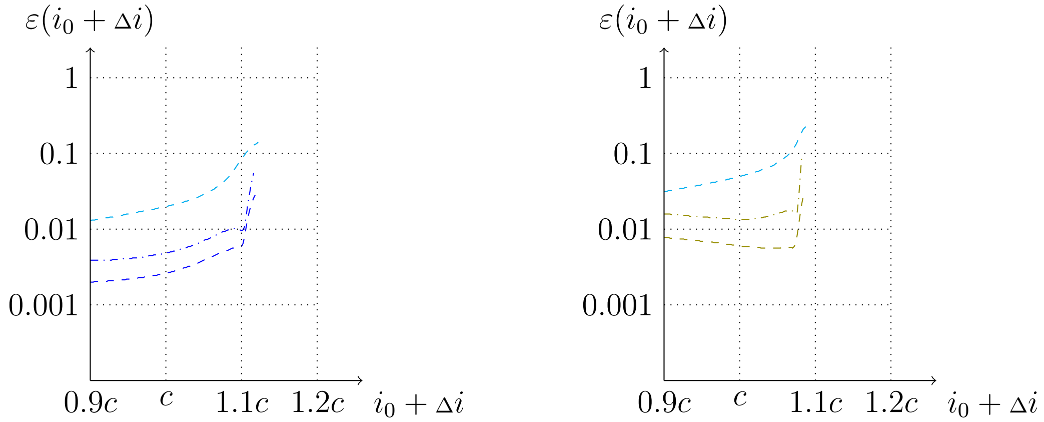


Figure 4.9: Relative error between approximations and TB model for jumping inflow. TB model is given for D1 (left) and D2 (right). The error of M model for $\alpha = 1.2$ (blue, dashed) and M model for $\alpha = 1.6$ (olive, dashed) is much lower than the error of PL model (cyan, dashed). The error of α model for $\alpha = 1.2$ (blue, dash-dotted) and $\alpha = 1.6$ (olive, dash-dotted) lies between the errors of M and PL models.

One can see that the accuracy of α model for the jumping inflow is less than for the smooth inflow. But α model is still much more accurate than PL model and is less accurate than M model.

4.6 Convex formulation of α model

Production-MFD $\mathbf{p}(n) = \mathbf{v}(n)n$ is usually assumed to be concave on the interval $[0, n_{jam}]$. This fact can be used to build a convex formulation of α model. Indeed, if we relax the assumption that the speed is equal to $\mathbf{v}(n)$ and allow all the values from 0 to $\mathbf{v}(n)$ (this can be interpreted as adjustment of speed-MFD by manipulating all the traffic lights inside the zone) then the outflow in α model will satisfy

$$\begin{cases} o(t) \geqslant -(\alpha - 1)i(t) \\ o(t) \leqslant \alpha \frac{\mathbf{v}(n(t))}{L} n(t) - (\alpha - 1)i(t) \end{cases}$$

The derivative of accumulation $\dot{n}(t) = i(t) - o(t)$ will satisfy

$$\begin{cases} \dot{n}(t) \leqslant \alpha i(t) \\ \dot{n}(t) \geqslant \alpha \left(i(t) - \frac{\mathbf{v}(n(t))}{L} n(t) \right) \end{cases}$$

or, equivalently,

$$\begin{cases} \dot{n}(t) - \alpha i(t) \leqslant 0 \\ \alpha \left(i(t) - \frac{\mathbf{v}(n(t))}{L} n(t) \right) - \dot{n}(t) \leqslant 0 \end{cases} \tag{4.3}$$

Now imagine that we want to minimize some convex objective function that depends on $n(t)$, $t \in [0, T]$. Notice that inequality constraints (4.3) are convex. Thus, if we assume that all other constraints are linear equalities or convex inequalities, we get convex optimization problem. Given the optimal solution, we can reconstruct the coefficient of speed adjustment as

$$U(t) = \frac{\alpha i(t) - \dot{n}(t)}{\alpha \frac{\mathbf{v}(n(t))}{L} n(t)}$$

Note that $U(t)$ can vary from 0 to 1. However, we expect that, if the optimization problem makes sense, and the outflow does not produce negative effect (by entering neighboring congested zone), speed adjustment cannot be optimal and $U(t)$ should be equal to 1. In other words, we get the same solution as we were using equality $\dot{n}(t) = \alpha \left(i(t) - \frac{\mathbf{v}(n(t))}{L} n(t) \right)$.

The same relaxation trick can be used for PL model which is α model for $\alpha = 1$ and for any other NEF model (e.g. [2]). If we look at the formulation of M model we can see, that this trick cannot be applied to it directly. Thus, α model might be a better choice than M model in the situations where the accurate formulation of dynamics is more preferable than other techniques like linearization around some point.

5 Conclusion

This thesis investigates properties of different speed-MFD models developed for a single zone. The most popular approach (PL model) says that the dynamics of accumulation $n(t)$, $t \in [0, T]$ can be found as a solution of Cauchy problem

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} n(t),\ t \in [0, T] \end{cases}$$

where $\mathbf{v}(n(t))$ is a speed-MFD and L is a coefficient that can be interpreted for the steady state as a mean trip length. It was shown that PL model cannot accurately describe what happens if $i(t)$ makes jumps. This drawback does not exist in TB model that postulates that incoming vehicles have some trip length distribution and the speed of all the vehicles is the same and equals $\mathbf{v}(n)$. Mathematical formulation of TB model looks as

$$\begin{cases} n(t) = n_0 & ,\ t \in (-\infty, 0] \\ i(t) = \frac{\mathbf{v}(n_0)}{L} n_0 & ,\ t \in (-\infty, 0] \\ n(t) = \int\limits_{-\infty}^{t} \left(1 - F\left(\int_s^t \mathbf{v}(n(u))du\right)\right) i(s)ds & ,\ t \in (-\infty, T] \end{cases}$$

where $F(l)$ is a CDF of trip length distribution. It was shown that the solution of PL model is a solution of TB model for exponential distribution. Thus, TB model seems to have one degree of freedom (trip length distribution) that can lead to more accurate modeling of $n(t)$. However, TB model is computationally expensive.

5.1 M model

The main contribution of the thesis is proposed ODE approximation of TB model. This so-called M model has the form of Cauchy problem

$$\begin{cases} n(0) = n_0 \\ M(0) = \frac{L}{\alpha} n_0 \\ \dot{n}(t) = i(t) - \frac{\mathbf{v}(n(t))}{L} \left(n(t) + 3 \left(n(t) - \frac{\alpha}{L} M(t) \right) \right) & , t \in [0, T] \\ \dot{M}(t) = Li(t) - \mathbf{v}(n(t)) n(t) & , t \in [0, T] \end{cases}$$

where $\alpha = \frac{2L^2}{L^2+\sigma^2} \in (0, 2)$ is dimensionless parameter that depends on the mean L and the standard deviation σ of trip length distribution. M model becomes equivalent to PL model if $\alpha = 1$ ($\sigma = L$). However, the typical distribution that appears in transportation field (for a zone of $1 - 2$ kilometers in size) has $\alpha > 1$ ($\sigma < L$). Analytical and numerical analysis presented in Chapter 2 shows that M model is more accurate than PL model as an approximation of TB model in the case of $\alpha > 1$. Thus, M model becomes an attractive alternative to TB model. Not only because it is computationally cheap, but also because it does not require precise information about the trip length distribution.

5.2 The case of constant speed

It can be expected that M model might be useful in other fields that are not related to transport. Indeed, the formulation of TB model with constant $\mathbf{v}(n) = 1$ (in this case trip length distribution becomes trip time distribution) looks as

$$\begin{cases} n(t) = n_0 & , t \in (-\infty, 0] \\ i(t) = \frac{1}{L} n_0 & , t \in (-\infty, 0] \\ n(t) = \int\limits_{-\infty}^{t} (1 - F(t-s)) i(s) ds & , t \in (-\infty, T] \end{cases}$$

Such a system is called LTI (Linear Time-Invariant) system and can appear in many different applications, from economics to pharmacokinetics. The approximation of this system with M model takes the form

$$\begin{cases} n(0) = n_0 \\ M(0) = \frac{L}{\alpha} n_0 \\ \dot{n}(t) = i(t) - \frac{1}{L} \left(n(t) + 3 \left(n(t) - \frac{\alpha}{L} M(t) \right) \right) & , t \in [0, T] \\ \dot{M}(t) = Li(t) - n(t) & , t \in [0, T] \end{cases}$$

which is constant coefficient non-homogeneous linear differential equation. It can be solved analytically for an arbitrary $i(t)$. The solution is presented in Section 2.4. It should be noted that in this thesis the conclusion that M model is close to TB model is done based on trip length distributions and inflow functions that are typical for transportation systems. However, it was shown that the approximation works very well for distributions that are close to distributions from gamma family (such distributions appear very often in economic and physical models). Moreover, for any $\alpha \leqslant \frac{4}{3}$ there can be found exactly one distribution such that M model is equivalent to TB model. If $\alpha = \frac{4}{3}$ then this will be gamma distribution with PDF $f(l) = \frac{4l}{L^2}e^{-2l/L}$. In the case $\alpha > \frac{4}{3}$ the equivalence is not possible. To show that approximation still works well for distributions that are close to gamma family the quadratic class of $i(t)$ was used. Such $i(t)$ can capture non-linear behavior of any smooth $i(t)$ with high accuracy.

5.3 α model

If the inflow $i(t)$ changes slowly, another ODE approximation of TB model (α model) can be used for the accurate modeling of $n(t)$. Similarly to M model, it requires coefficient α. However, the formulation of α model is much simpler and has the form of Cauchy problem

$$\begin{cases} n(0) = n_0 \\ \dot{n}(t) = \alpha\left(i(t) - \frac{\mathbf{v}(n(t))}{L}n(t)\right),\ t \in (0, T] \end{cases}$$

where $\mathbf{v}(n(t))$ is the speed-MFD function that can be either constant or decreasing. In the case $\alpha = 1$ this model becomes equivalent to PL model, but this case is considered to be not realistic for transportation systems. Numerical analysis performed in this thesis for $\alpha = 1.2$ and $\alpha = 1.6$ showed that, in general, α model is less accurate than M model and more accurate than PL model. The main advantage of α model that can be efficiently utilized in optimization frameworks is its convex relaxation

$$\alpha\left(i(t) - \frac{\mathbf{v}(n(t))}{L}n(t)\right) - \dot{n}(t) \leqslant 0$$

which can be interpreted as a proportional reduction of speed of all the vehicles inside the zone. The relaxation is convex because the production-MFD function $\mathbf{p}(n) = \mathbf{v}(n)n$ is usually assumed to be concave. M model cannot be relaxed to a convex problem this way. To be used inside convex optimization frameworks, it can be linearized around some point, but this reduces the accuracy. The question if α model suits better than M model for the real-time control of transportation systems is not trivial. It can be viewed as potential research direction.

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