Two-Degree-of-Freedom ℓ_2-Optimal Tracking with Preview ⋆

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Abstract

A simple SISO two-degree-of-freedom pole-placement design method is presented that provides ℓ_2 optimal tracking of a given reference signal. The closed-loop pole locations are first chosen by the system designer. The closed-loop zeros are then placed in an optimal fashion by a computationally inexpensive algorithm to achieve asymptotic tracking with an optimal transient response. The preview approach, which has become a common method for dealing with systems which have non-minimum phase behavior, can then optionally be used to further improve the transient behavior for both minimum phase and non-minimum phase systems. Unlike previous results based on the preview approach, the solution presented here takes into consideration the closed-loop pole dynamics, and is ℓ_2 optimal with respect to all other two-degree-of-freedom preview controllers with the same closed-loop poles. A simple solution to the H_2 model matching problem, where the design parameter Q is not rational, but polynomial, is the heart of the solution method.

Key words: Discrete-time control; Non-minimum phase systems; Optimality; Zero assignment; Two-degree-of-freedom controllers; Trajectory tracking; Preview control

1 Introduction

The pole placement approach (sometimes called the *RST approach*) has become a popular method for designing simple controllers [1,6,7]. A reference model $B_m(q)/A_m(q)$ is chosen, and a controller is found so that the closed-loop transfer function is equal to $B_m(q)/A_m(q)$. This simple method is based on classical control theory, and is intuitive for control system designers familiar with the PID approach. Separate tuning of the loop properties and the tracking properties is straightforward, and addition of internal models is intuitive. It is often possible to find an appropriate denominator polynomial $A_m(q)$ by choosing closed-loop poles that lie within a region inside the unit circle with reasonable damping. Simple optimal approaches that permit one to choose the numerator polynomial $B_m(q)$ are lacking, so control system designers often simply select an appropriately scaled polynomial consisting of the plant zeros which are unstable or poorly damped so that the closed-loop system is stable with unit D.C. gain. Although this method works for unit step reference signals, it does not work for more general reference signals. This paper demonstrates in which cases simply setting the D.C. gain is suboptimal, and provides an optimal controller using an easily programmed and conceptually simple solution that is related to H_2 optimization, but which is considerably simpler than the complete H_2 solution.

The method proposed in this paper finds an optimal controller that minimizes a cost function consisting of a weighted sum of terms penalizing the control action and tracking error resulting from a given reference signal. The optimization is performed over all controllers of fixed degree maintaining fixed closed-loop poles and observer polynomial, and eliminating permanent tracking error. The penalty functions are based on the ℓ_2 norm, and various reference signals may be used. The relationship between reference signal complexity and controller order is demonstrated.

The approach incorporates the idea of *preview*, which can optionally be used to further improve performance. For applications in which it is desired to optimally follow a specific reference signal, like motion control, machine-tooling, and robotics, the use of preview can be very beneficial, as well as for non-minimum phase systems with

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undershoot. Although preview has been studied extensively in [11,4,9], these methods which place zeros optimally with respect to a FIR system are difficult to use in practice where it is not realistic to design a controller which cancels all plant poles and stable plant zeros. If one chooses not to cancel all plant poles, these methods lose their optimality. In addition, these methods may result in excessive control action. The method we are proposing takes into consideration closed-loop poles from the beginning, and since the actuator signal is part of the cost function, excessive control action can be avoided.

In [3] it is demonstrated that it is possible to add preview to a controller in order to achieve better tracking without necessarily separating minimum phase and non-minimum phase dynamics and designing two separate controllers. In [3] it is suggested that by choosing the T polynomial that results in the shortest error response for a deadbeat system, a good response for the non deadbeat system should occur. The problem to this approach is that control action may be high, and a short error response for a deadbeat system may decay slowly when system poles are present. Optimizing the true closed-loop error signal seems more appropriate.

This paper is organized as follows: in Section 2 we review the standard two-degree-of-freedom controller. In Section 3 we review the concept of preview, while defining the tracking error signal. We also define the concept of an *admissible* controller and parameterize all admissible controllers. The major results (Theorem 3 and Theorem 4) are derived in Section 4. The theorems provide a method for computing controllers which are H_2 optimal among all admissible controllers. In Section 5, extensions to the basic results are given that allow more flexibility with controller synthesis when preview information is available. Some examples that demonstrate the solution to various control problems are presented in Section 6, followed by a section with some concluding remarks.

As a general rule, capital letters in formulas will represent polynomials in q, the forward shift operator. To enhance readability, this dependence will often not be shown. V and W, the exceptions to the above rule, will be used to represent rational functions in Section 4. The unit impulse signal will be denoted by $\delta(k)$, and the degree of a polynomial P will be written as δP.

2 Two-degree-of-freedom controller design

The problem of designing a two-degree-of-freedom pole-placement controller (Figure 1) for a strictly proper plant $B(q)/A(q)$ is discussed in this section (see also [1,7]). The general pole placement controller is of the form $R(q)u(k) = T(q)y_r(k) - S(q)y(k)$ where $k \in \mathbb{Z}$ is the discrete time instant, $y_r(k)$ is the reference signal, $y(k)$ is the plant output, and $u(k)$ is the actuator signal.

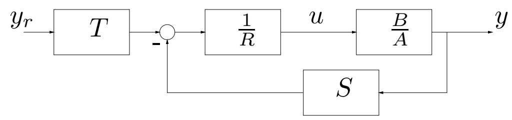


Fig. 1. Two degree of freedom control system

It is simple to derive that $y(k) = \frac{BT}{AR+BS}y_r(k)$. B may contain stable, marginally stable, and unstable zeros. As in [1], factorize $B(q) = B^+(q)B^-(q)$ so that $B^+(q)$ is the highest degree monic polynomial with stable, well-damped zeros. We also define $R = B^+R_fR_1'$ where R_f is a fixed part of R that we may choose, for example, to contain integrators. This results in $y(k) = \frac{B^-T}{AR_fR_1'+B^-S}y_r(k)$. Assume that we would like the closed-loop characteristic polynomial to be equal to A_oA_m, where the closed-loop modes are specified by the stable polynomial A_m, and A_o is an observer polynomial that will be canceled by T, by defining $T = T'A_o$. Then equating denominators we get the following Diophantine equation

$$AR_fR_1' + B^-S = A_oA_m, \tag{1}$$

which is an equation in the unknowns R_1' and S, and the transfer function $y(k) = \frac{B^-T'}{A_m}y_r(k)$. By choosing R_f, A_o and A_m appropriately it is possible to calibrate the sensitivity function to achieve robustness and disturbance rejection [6,1].

A solution set R_1', S to Equation (1) such that $\delta S \leq \delta A + \delta R_f - 1$ exists [3] under the assumption that A_mA_o be divisible by the greatest common divisor of AR_f and B^-. This assumption will almost always be satisfied because in most cases AR_f and B^- are coprime. The above parameterization provides many algebraic solutions to Equation (1). However, all of these solutions may not result in a controller satisfying the causality requirement $\delta R \geq \delta S$. The following lemma indicates under what conditions this will be satisfied. The causality condition $\delta R \geq \delta T$ will be discussed later.

Lemma 1 *If $\delta A_o \geq 2\delta A - \delta A_m - \delta B^+ + \delta R_f - 1$ then $\delta R \geq \delta S$.*

PROOF. From the way that S was chosen, we have

$$\delta S \leq \delta A + \delta R_f - 1. \tag{2}$$

This, combined with the fact that the plant is strictly proper, gives us:

$$\delta(BS) \leq 2\delta A + \delta R_f - 2. \tag{3}$$

[3] In fact a solution for S exists for which equality is achieved. See Theorem 10.3 of [7] for proof.

Equation 1, which implies that $AR + BS = A_o A_m B^+$, and the lemma hypothesis on δA_o lead to:

$$\begin{aligned}\delta(AR + BS) &= \delta B^+ + \delta A_o + \delta A_m \\ &\geq 2\delta A + \delta R_f - 1. \end{aligned} \tag{4}$$

Inequalities (3) and (4) imply the following:

$$\begin{aligned}\delta(AR) &> \delta(BS) \\ \delta(AR + BS) &= \delta(AR) \\ \delta(AR) &\geq 2\delta A + \delta R_f - 1 \\ \delta R &\geq \delta A + \delta R_f - 1.\end{aligned}$$

Comparing this to Inequality (2), we must have $\delta R \geq \delta S$. $\square$

3 Tracking goal

In this section we elaborate on the tracking goal, defining the error signal and discussing preview in the process. A parameterization of controllers that eliminate permanent tracking error is given, and a cost function making clear the tradeoff between tracking performance and control activity is proposed.

Let a *fixed* reference signal $\hat{y}_r(k)$ be given, with $\hat{y}_r(k) = 0$ for $k < 0$ and $\hat{y}_r(k) \neq 0$ for $k = 0$. In the figures which will follow, a unit step signal will be used for simplicity. First, assume that this signal is used directly as the reference signal: $y_r(k) = \hat{y}_r(k)$. If the system has an internal delay of $\gamma_1 = \delta A - \delta B$ samples, the closed-loop system will necessarily have a delay of at least γ_1 samples. Thus, attempting to follow the reference signal immediately doesn't make sense because it is an unachievable goal (see Figure 2).

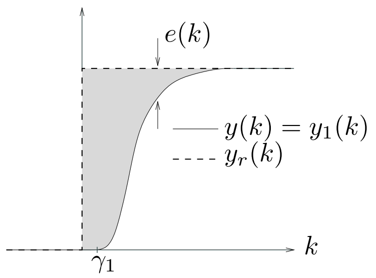
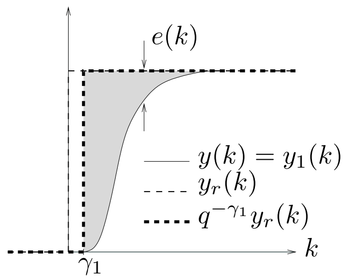


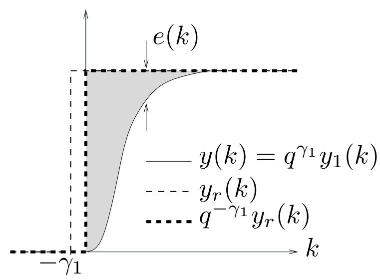
Fig. 2. System with delay, $y_r(k) = \hat{y}_r(k)$, $e(k) = y_r(k) - y(k)$

It is more meaningful to compare a delayed version of the reference signal with the plant output instead (see Figure 3(a)). This figure is the same as Figure 2, except that the error signal is the difference between the reference signal *delayed* by γ_1 samples and the output $y(k)$: $e(k) = q^{-\gamma_1} y_r(k) - y(k)$.

In many tracking applications, the reference signal is available in advance. In this case, it is possible to *cancel* the delay by sending the reference signal exactly γ_1 samples in advance, by using $y_r(k) = q^{\gamma_1} \hat{y}_r(k)$ as seen in Figure 3(b).



(a) $y_r = \hat{y}_r$



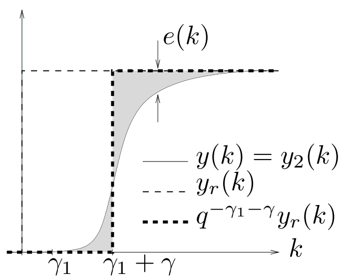
(b) $y_r = q^{\gamma_1} \hat{y}_r$

Fig. 3. System with delay, $e(k) = q^{-\gamma_1} y_r(k) - y(k)$

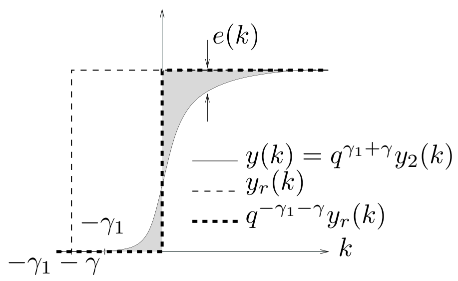
Now, let's return to the case where $y_r(k) = \hat{y}_r(k)$. As was done in Figure 3(a), the design of the error signal takes into consideration the system delay γ_1. However, it may be possible to increase precision (reduce the error), at the cost of the introduction of additional delay γ, by using an error signal which delays $y_r(k)$ by γ additional samples before making the comparison. Through this anticipative behavior, a controller design which minimizes the error signal may result in tracking performance that is unachievable otherwise (see Figure 4(a)). However, this tracking performance comes at the cost of additional delay. This delay, however, is not problematic if the reference signal is sent $\gamma_1 + \gamma$ samples in advance, as is clear in Figure 4(b), where $y_r(k) = q^{\gamma_1+\gamma} \hat{y}_r(k)$. *Preview* is the use of reference signal information (or *preview information*) in advance. Thus we have

$$\begin{aligned} e(k) &= q^{-\gamma-\gamma_1} y_r(k) - y(k) \\ &= \left(\frac{A_m - q^{\gamma+\gamma_1} B^- T'}{q^{\gamma+\gamma_1} A_m} \right) y_r(k). \end{aligned}$$

Now suppose that the reference signal is generated by $y_r(k) = \frac{B_c(q)}{A_c^+(q) A_c^-(q)} \delta(k)$ where A_c^+ is composed of stable poles, and A_c^- is composed of unstable poles. Most useful reference signals can be generated in this way by consulting a table of Z-transforms. A step input can be



(a) $y_r = \hat{y}_r$



(b) $y_r = q^{\gamma_1+\gamma}\hat{y}_r$

Fig. 4. System with delay, $e(k) = q^{-\gamma_1-\gamma} y_r(k) - y(k)$

generated, for example, by defining $B_c = q$, $A_c^+ = 1$, $A_c^- = q - 1$. It is possible to use A_c^+ and B_c to low-pass filter the reference signal. The error signal can now be represented in the following way:

$$e(k) = \left(\frac{A_m - q^{\gamma+\gamma_1} B^- T'}{q^{\gamma+\gamma_1} A_m} \right) \frac{B_c}{A_c^+ A_c^-} \delta(k).$$

If the equation

$$A_c^- P + q^{\gamma+\gamma_1} B^- T' = A_m \tag{5}$$

is solved, then A_c^- will be canceled and we obtain the following equations representing the error and actuator signals:

$$e(k) = \frac{P B_c}{q^{\gamma+\gamma_1} A_m A_c^+} \delta(k), \quad u(k) = \frac{A T'}{B^+ A_m} y_r(k). \tag{6}$$

The H_2 system norm and ℓ_2 norm of the impulse response are equivalent (see [2]), so we can write $\|e(k)\|_2^2 = \left\| \frac{P B_c}{q^{\gamma+\gamma_1} A_m A_c^+} \right\|_{H_2}^2$. Thus, in order to achieve good performance while limiting control action, the cost function J is defined simply as

$$J = \alpha_1^2 \left\| \frac{P B_c}{q^{\gamma+\gamma_1} A_m A_c^+} \right\|_{H_2}^2 + \alpha_2^2 \left\| \frac{A T'}{B^+ A_m} \right\|_{H_2}^2 .$$

A controller is called *admissible* if $\lim_{k\to\infty} e(k) = 0$ and J is finite.

The poles of the transfer functions of Equation (6) are inside the unit circle. The final value theorem shows that $e(k)$ converges to zero. Due to linearity, $e(k)$ converges to zero exponentially and this controller is therefore admissible. Note that we have not chosen to weigh the ℓ_2 norm of $u(k)$ directly, because generally it will not be finite.

Given α_1 and α_2, it is possible to achieve a compromise between maintaining low control excitation and high tracking precision by finding a controller which is admissible and such that J is small.

A solution set P_0, T_0' to Equation (5) such that $\delta T_0' < \delta A_c^-$ exists under the reasonable assumption that $q^{\gamma+\gamma_1} B^-$ and A_c^- have no common zeros.

With no limitations on the degrees of T' and P, it is known that all solutions to Equation (5) can be parameterized with respect to the polynomial Q by

$$T' = T_0' - A_c^- Q, \qquad P = P_0 + q^{\gamma+\gamma_1} B^- Q \tag{7}$$

In our case, however, the degree of T' may be limited to ensure causality. The lemma below shows that it is nevertheless possible to parameterize all solutions that satisfy a degree condition on one of the dependent variables.

Lemma 2 *Let $m \in \mathbb{N}$ and polynomials A, B, and C such that the greatest common divisor of A and B divides C be given. If the equation $AX + BY = C$ possesses a solution set (X, Y) such that the degree condition $\delta Y \leq m$ is satisfied, then all solution sets to this equation that satisfy the degree condition may be parameterized by $(X + BQ, Y - AQ)$, where Q is allowed to vary over the set of all polynomials such that $\delta Q \leq m - \delta A$.*

PROOF.

Assume that $\delta Q \leq m - \delta A$. Then $\delta(Y - AQ) \leq \max(\delta Y, \delta A + \delta Q) \leq m$. So, $(X + BQ, Y - AQ)$ clearly satisfies the equation $AX + BY = C$ and the degree condition. Conversely, let (X_2, Y_2) be another solution set with Y_2 satisfying the degree condition. Then by standard results [10], some polynomial Q exists such that $X_2 = X + BQ$ and $Y_2 = Y - AQ$. Then $\delta AQ = \delta(Y - Y_2) \leq m$, so $\delta Q \leq m - \delta A$. $\square$

We have already determined that the relative degree of S/R will be non-negative. However, certain values of the degree of the parameter Q may result in a negative relative degree of T/R. Application of Lemma 2 shows that if a solution to Equation (5) exists such that $\delta T \leq \delta R$, which is equivalent to $\delta T' \leq \delta R - \delta A_o$, then all solutions satisfying this degree condition may be obtained through the parameterization of Equation 7, where

$$\delta Q \leq \delta R - \delta A_o - \delta A_c^- . \tag{8}$$

If such a solution does not exist, it is possible to increase the order of the polynomial R.

It is now possible to write the cost function J as a function of Q:

$$J(Q) = \alpha_1^2 \left\| \frac{(P_0 + q^{\gamma+\gamma_1} B^- Q) B_c}{q^{\gamma+\gamma_1} A_m A_c^+} \right\|_{H_2}^2 + \alpha_2^2 \left\| \frac{A(T_0' - A_c^- Q)}{B^+ A_m} \right\|_{H_2}^2 . \tag{9}$$

For each Q the controller is admissible. The goal of the controller synthesis problem is to find a polynomial Q^* so that $J(Q^*) = \inf_Q J(Q)$. This problem will be solved in Theorem 4. Note that the preview γ appears in the definition of the cost function $J(Q)$. So if all controller polynomials are fixed to be of a certain degree, it is nevertheless possible to apply the reference signal in advance, and define γ appropriately, to take advantage of preview information.

4 Finding the optimal admissible solution

In the previous section, a controller parameterization and cost function were elaborated. In this section, two theorems are presented, the second of which permits one to find the unique controller minimizing the cost function.

RH_2 denotes the set of strictly proper stable rational transfer functions with real coefficients. Since RH_2 is a Hilbert space, the inner product can be written as a function of the norm. This easily derived formula is called the polarization identity [12]:

$$\langle X, Y \rangle = [\|X + Y\|_{H_2}^2 - \|X - Y\|_{H_2}^2]/4$$

The norm is easily calculated using standard state-space techniques. It is also possible to write the inner product as $\sum_{i=0}^{\infty} x_i y_i$, where $\{x_i\}$ and $\{y_i\}$ are the impulse responses of the $X(q)$ and $Y(q)$. This summation exists since the impulse responses of stable linear systems converge to zero asymptotically. Using this summation to find the approximate inner product appears to be computationally less expensive than the use of the polarization identity and state-space H_2 norm algorithms.

Theorem 3 *Given $n \in \mathbb{Z}$ and stable rational functions $V(q)$, and $W(q)$ such that $V(q)$ and $q^n W(q)$ are strictly proper, and the impulse response of $W(q)$ contains at least one non-null element, define*

$$\begin{aligned} Q &= q_0 + q_1 q + \ldots + q_n q^n, \\ \bar{q} &= [q_0, \ldots, q_n]', \\ \Psi(V, W) &= \begin{bmatrix} \langle V, W \rangle \\ \langle V, qW \rangle \\ \vdots \\ \langle V, q^n W \rangle \end{bmatrix}, \text{ and} \\ \Phi(W) &= \begin{bmatrix} \langle W, W \rangle & \langle qW, W \rangle & \ldots & \langle q^n W, W \rangle \\ \langle W, qW \rangle & \langle qW, qW \rangle & \ldots & \langle q^n W, qW \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle W, q^n W \rangle & \langle qW, q^n W \rangle & \ldots & \langle q^n W, q^n W \rangle \end{bmatrix}. \end{aligned}$$

Then the solution $\bar{q}$ to the regular matrix inversion problem

$$\Psi(V, W) = \Phi(W) \bar{q} \tag{10}$$

exists and gives the unique polynomial solution Q^* of degree n to the minimization problem:

$$\min_Q \|V(q) - W(q)Q(q)\|_{H_2}^2 .$$

PROOF. $V - WQ$ can be written as $V - q_0 W - q_1(qW) - q_2(q^2)W - \ldots - q_n(q^n W)$. The optimal solution is obtained through a simple projection of V onto the linear subspace $\tilde{M} = span\{W, qW, \ldots, q^n W\}$. $\tilde{M}$ is a finite-dimensional subspace of the Hilbert space RH_2, and is therefore closed. By the classical projection theorem [8], an optimal solution Q^* exists, and is unique. The unique minimizing solution Q^* is such that $(V - Q^* W) \perp \tilde{M}$, resulting in $\langle V, q^i W \rangle - q_0 \langle W, q^i W \rangle - q_1 \langle qW, q^i W \rangle - \ldots - q_n \langle q^n W, q^i W \rangle = 0 \ \forall i \in \{0, \ldots, n\}$. Each choice of i yields one row of Equation 10.

In order to verify linear independence of the basis vectors $W, qW, \ldots, q^n W$, define $\tilde{w}_i = \{w_1, w_2, w_3, \ldots\}$ as the impulse response of $q^{n-i} W(q)$. Clearly, $\tilde{w}_i$ is the sequence $\tilde{w}_0$ delayed by i samples, and starting with i zeros. The space of impulse responses of elements of H_2 is isomorphic with H_2 if we use $\langle X, Y \rangle = \sum_{i=1}^{\infty} x_i y_i$ where $\{x_i\}$ and $\{y_i\}$ are the impulse responses of $X(q)$ and $Y(q)$. Assuming that $\tilde{w}_0$ is not identically zero, consider the first non-zero element of $\tilde{w}_0$. It cannot be made zero by any linear combination of $\tilde{w}_i,\ i > 0$. So $\alpha_0 \tilde{w}_0 + \alpha_1 \tilde{w}_1 + \ldots = 0$ implies that $\alpha_0 = 0$. The first non-zero element of $\tilde{w}_1$ cannot be made zero by any linear combination of $\tilde{w}_i,\ i > 1$. So we must also have $\alpha_1 = 0$. This argument can be repeated, showing the linear independence of the finite set $\tilde{w}_i$. So the solution Q^* is represented by the unique vector $\bar{q}$, implying the invertibility of the gram matrix. □

The H_2 norm of a system $G(q)$ is the same as the norm of the system $G(q)q^{-1}$. This is because the energy of the impulse response of a system is backward-shift invariant. If we perform a forward-shift on the impulse response of a system, its norm remains the same until the system becomes non-proper. At this point, it may no longer be possible to calculate the norm using the typical state-space methods. The impulse response of non-causal systems have finite energy which may nevertheless be calculated. Or one may simply calculate the H_2 norm by multiplying by q^{-k} for some sufficiently large k. Thus, Theorem 3 may be used even when $V(q)$ and $q^n W(q)$ are not strictly proper. Simply multiply the denominators of V and W by q^k for some sufficiently large k.

The next theorem provides the solution to a generalization of the problem of Theorem 3. Finding the solution Q^* minimizing Equation (9) will be the main application.

Theorem 4 *Given $n, m \in \mathbb{Z}$, $\alpha_j \in \{\mathbb{R} \setminus 0\}$ and stable rational functions $V^j(q)$, $W^j(q)$, $j \in \{1, \ldots, m\}$ such that $V^j(q)$ and $q^n W^j(q)$ are strictly proper for all j, and the impulse response of $W^j(q)$ contains at least one non-null element for some j, define Q, $\bar{q}$, Ψ, and Φ as in Theorem 3. Then the solution $\bar{q}$ to the regular matrix inversion problem $\sum_{j=1}^m \alpha_j^2 \Psi(V^j, W^j) = (\sum_{j=1}^m \alpha_j^2 \Phi(W^j))\bar{q}$ exists and gives the unique polynomial solution Q^* of degree n to the minimization problem*

$$\min_Q \sum_{j=1}^{m} \alpha_j^2 \left\| V^j(q) - W^j(q)Q(q) \right\|_{H_2}^2. \tag{11}$$

PROOF. Theorem 4, like Theorem 3, is solved using the classical projection theorem. Here, however, the projection theorem is used on the space H which is defined as the m-fold cartesian product of RH_2: $H = RH_2 \times \ldots \times RH_2$. Using the definition of the inner product, induction, and by the completeness of the cartesian product of complete spaces ([5]), it is easily shown that H is a Hilbert space when equipped with the inner product: $\langle (X_1, \ldots, X_m), (Y_1, \ldots, Y_m) \rangle_H = \alpha_1^2 \langle X_1, Y_1 \rangle + \ldots + \alpha_m^2 \langle X_m, Y_m \rangle$. Now, defining $V = (V^1, \ldots, V^m), W = (W^1, \ldots, W^m)$, we have $\|V - q_0 W - \ldots - q_n(q^n W)\|_H^2 = \sum_{j=1}^m \alpha_j^2 \left\| V^j - W^j Q \right\|_{H_2}^2$. Application of the classical projection theorem to minimize the left hand side of this expression results in the solution to 11. The equation $\sum_{j=1}^m \alpha_j^2 \Psi(V^j, W^j) = (\sum_{j=1}^m \alpha_j^2 \Phi(W^j))\bar{q}$ is a simple consequence of the orthogonality condition, and as long as one of the W^j has a non-null impulse response, the unicity of $\bar{q}$ can be shown as in Theorem 3. $\square$

In order to apply this theorem to minimize $J(Q)$ (given by Equation (9)), simply choose $m = 2$, $V^1 = P_0 B_c / (q^{\gamma+\gamma_1} A_m A_c^+)$, $W^1 = -B^- B_c / (A_m A_c^+)$, $V^2 = (A T_0') / (B^+ A_m)$, and $W^2 = (A A_c^-) / (B^+ A_m)$.

5 Increasing design freedom with non-causal controller

Note that the degree of Q is a measure of the amount of design freedom that is available to improve performance. More complicated reference signals are reflected by A_c^- having higher degree, which reduces the degree of Q (see Equation 8). It is possible to increase the degree of Q by increasing the controller order.

Note that the preview factor γ doesn't affect controller structure in any way. γ only results in a time shift of the reference signal that is used in computing the cost function. Choosing a large value of γ tunes the optimization procedure so that it attempts to find an appropriate optimal Q. But this may not be possible without making changes to the degrees of the controller polynomials.

The papers [11,4,9] are based on the principle that by prepending a controller with a FIR filter (which effectively adds zeros to the controller) and using open-loop preview it is possible to improve tracking performance of non-minimum phase systems. These results are encouraging, but do not consider denominator dynamics or actuator behavior. Here we propose making a similar structural change by which we can achieve the similar results, but taking into consideration the pole dynamics. We do this by adding additional zeros to the controller by increasing the order of the T polynomial so that its degree may be larger than the degree of R. Obviously this violates causality, but if additional preview information is available this is not a problem. By slight modification (see Equation 8),

$$\delta T \leq \delta R + \kappa.$$

may be ensured by choosing

$$\delta Q \leq \delta R - \delta A_o - \delta A_c^- + \kappa$$

Thus if an additional κ samples of preview information are available we can increase the degrees of T and Q by κ, resulting in more design freedom.

The choice of κ has no theoretical limitation. As in [4], performance tends to increase as κ increases. However, as κ increases, sensitivity to plant perturbations also increases. This does not result in instability, since only R and S affect loop properties. But it will affect performance, so there are practical limits to the choice of κ. γ generally should be smaller or equal to the order of the controller. This makes sense because a low order controller cannot behave like a long time delay.

6 Examples

Here are a few examples that demonstrate basic controller synthesis. The first two examples show how to apply the results of Section 4. The third example demonstrates how to apply the results of Section 5.

Example 5 With sampling period $h = 0.2$, the zero-hold discretization of the simple second order system $1/(s(s+1)(s+4))$ gives $A = q^3 - 2.2681q^2 + 1.6359q - 0.3679$, $B^+ = q+0.2062$, $B^- = 0.0010q+0.0031$. R and S are chosen to move the discrete-time open-loop poles from $\{1, 0.8187, 0.4493\}$ to $0.8, 0.6+0.1i, 0.6-0.1i$, using a deadbeat observer polynomial $A_o = q$. We choose a step input as the reference signal, and let $\gamma = 0$, assuming that no preview information is available. If we choose not to weigh the actuator signal by choosing $\alpha_1^2 = 1$, $\alpha_2^2 = 0$ and using Theorem 4 to minimize $J(Q)$, we find $Q^* = -63.8$, and get the following controller: $R = q^2 + 0.3989q + 0.0397$, $S = 71.8470q^2 - 86.5798q + 22.9498$, $T = 63.777q^2 - 55.560q$. The signals y, y_r and u are shown in Figure 5. From this figure, we see that the output responds quickly to the change in reference signal, but the actuator signal $u(k)$ might be too aggressive. The behavior of the controller which one obtains with $\alpha_1^2 = 15$ and $\alpha_2^2 = 1$ is shown in Figure 6. R and S are the same as before, but $Q^* = -8$, and $T = 7.9927q^2 + 0.2243q$.

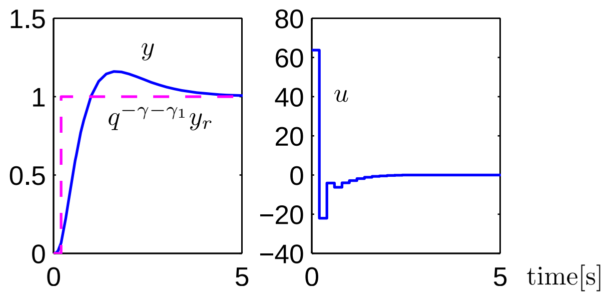


Fig. 5. Optimal step responses and actuator signal $\alpha_1^2 = 1, \alpha_2^2 = 0$

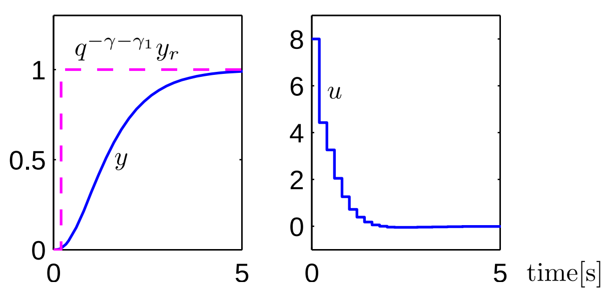


Fig. 6. Optimal step responses and actuator signal; $\alpha_1^2 = 15, \alpha_2^2 = 1$

Example 6 Using the same process as above, assume that it is necessary to track a sinusoidal reference signal. Without preview (Figure 7), the initial tracking error is significant. The use of two samples of preview information results in considerable tracking improvement (Figure 8). Note that the plant output y responds to the sinusoid before the reference signal arrives. This demonstrates that the use preview results in anticipatory action, which is possible since the reference signal actually is known and provided to the controller in advance.

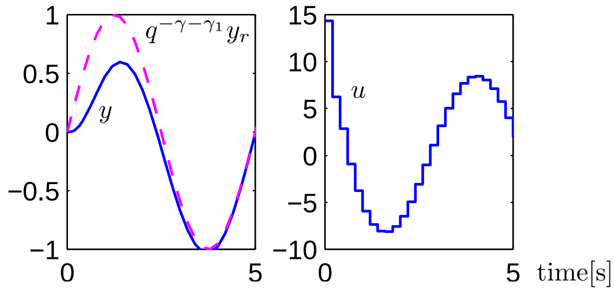


Fig. 7. Optimal sin responses and actuator signal; $\alpha_1^2 = 15, \alpha_2^2 = 10, \gamma = 0$

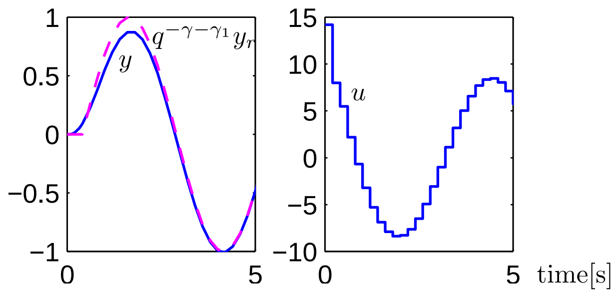


Fig. 8. Optimal sin responses and actuator signal; $\alpha_1^2 = 15, \alpha_2^2 = 10, \gamma = 2$

Example 7 With sampling period $h = 1$, the zero-hold discretization of the simple non-minimum phase system $(s-0.2)/(s+0.4)^2$ gives $B(q) = 0.59q - 0.73$, $A(q) = q^2 - 1.34q + 0.4493$. R and S are chosen to move the discrete-time open-loop poles to $0.6, 0.5+0.1i, 0.5-0.1i$, using observer polynomial $A_o = 1$. We choose a step input as the reference signal, and let $\gamma = 0$, assuming that no preview information is available. Using Theorem 4 with weights $\alpha_1^2 = \alpha_2^2 = 1$, we get $Q^* = 0.635$ and the following controller: $R = q - 0.14$, $S = -0.20q + 0.13$, $T = -0.64q - 0.13$. As we see in Figure 9, there is a considerable amount of undershoot. It is not possible to reduce the undershoot significantly by varying the α_i weights. Assuming that reducing tracking error is important and 12 samples of preview are available, we choose $\gamma = 0$, and $\kappa = 12$ as described above, to get the improved results in Figure 10. Here, Q^* is a polynomial of degree 12, $R = q - 0.14$, $S = -0.20q + 0.13$, and T is a polynomial of degree 13. Since T is of higher degree than R, the system is not causal, but because of the

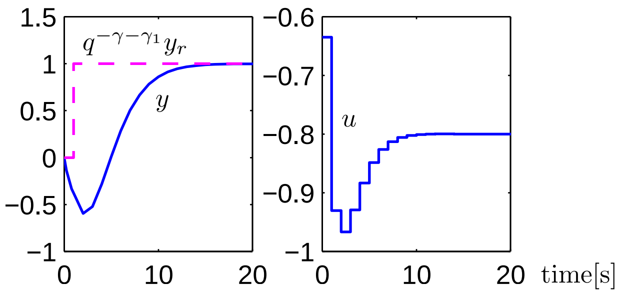


Fig. 9. Non-minimum phase system with no preview; $\alpha_1^2 = 15$,$\alpha_2^2 = 15$,$\gamma = 0$,$\kappa = 0$

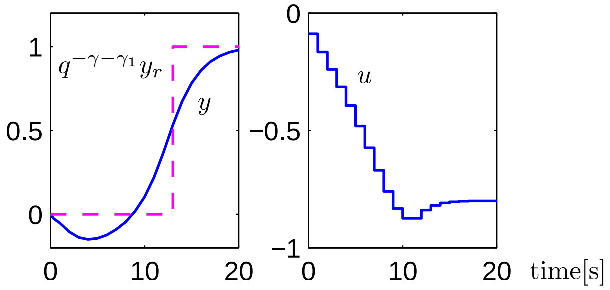


Fig. 10. Non-minimum phase system with preview; $\alpha_1^2 = 15$,$\alpha_2^2 = 50$,$\gamma = 0$,$\kappa = 12$

availability of preview information this causes no problem. In addition, this may seem like high order control, but since T is outside the loop, it behaves like a simple FIR signal prefilter, and doesn't change loop behavior. In addition, as is seen in Figure 10, T actually results in a much smoother actuator signal.

7 Conclusions

We have presented a simple pole-placement based synthesis approach that consists of minimizing a weighted cost function composed of terms penalizing tracking error and control behavior. The optimization algorithm does not place poles, but places the zeros of the closed-loop system, through the solution of a simple projection-type optimization problem which is equivalent to a constrained H_2 model-matching problem (Theorem 3 and Theorem 4). This algorithm is believed to be original.

For non-minimum phase systems which exhibit undershoot, preview may be used to significantly improve tracking response if the reference signal is available in advance. The amount of preview information that is generally necessary for these systems is related to the time constant of the dominant non-minimum phase zero. The amount of preview information that is necessary, therefore, may be significant. The use of preview information may also significantly improve tracking behavior for minimum phase systems. In this case, a few samples of preview information may be sufficient (Example 6).

Equation 8 makes it is clear how reference signal complexity, controller tracking performance, and controller order are related. For fixed controller order, the amount of design freedom available to improve controller performance decreases as reference signal complexity increases. When preview information is available, it is possible to increase tracking performance without modifying the loop dynamics by increasing the order of the T polynomial. This is similar to the approach taken in [11,4,9], except that control amplitude and uncanceled plant poles are taken into consideration when performing the optimization, simplifying practical application of the results. Computational complexity for the proposed method is low.

The optimization algorithm only places the zeros of the closed-loop transfer function in order to improve tracking. The pole locations are chosen by the control designer, so the presented optimization algorithm does not affect stability properties.

Although the general results allow the use of preview, if the reference signal is not available in advance, one can set $\gamma = 0$ and the results apply without change to the case without preview.

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