Condition for Bifurcation of the Region of Attraction in Linear Planar Systems with Saturated Linear Feedback

J-Y. Favez, B. Srinivasan, Ph. Mullhaupt, D. Bonvin
Laboratoire d'Automatique, Ecole Polytechnique Fédérale de Lausanne,
CH-1015 Lausanne, Switzerland
email: jean-yves.favez@epfl.ch; fax: +41 21 693 25 74

Abstract

Bifurcation of the region of attraction for planar systems with one stable and one unstable pole under saturated linear state feedback is considered. The boundary of the region of attraction can either possess an unbounded hyperbolic shape or be a bounded limit cycle. The main contribution of this paper is to provide an analytical condition under which bifurcation occurs. This condition is based on characteristics and position of the stable and unstable manifolds. Furthermore, the exact shape of the region of attraction is provided.

1 Introduction

Linear systems with bounded inputs have been widely studied [6, 4, 2]. This type of study is important since, in most practical situations, the range of inputs is in fact limited.

Two important concepts pertaining to these systems have to be distinguished. First is the *null controllable region*, i.e. the region in state space where there exists an open-loop input that can steer the system to the origin [1, 2, 3, 5]. The second is the *region of attraction with a given controller*, i.e. the region in state space from which the closed-loop system asymptotically reaches the origin [1, 2]. In this paper, only the issues pertaining to the latter, i.e. the region of attraction, will be studied. Also, designing controllers for which the region of attraction is arbitrarily close to the null controllable region [4] will not be studied here.

Single input linear planar systems (systems with 2 states) with saturated linear feedback will be considered. It will be assumed that the feedback makes the origin globally asymptotically stable in the absence of saturation.

The shape of the region of attraction depends on the location of the open-loop poles. With respect to the region of attraction, the poles on the imaginary axis have the same characteristics as the stables ones. If both poles are stable, then the system is globally stabilisable [1, 6]. If both poles are unstable, then the boundary of the region of attraction is a closed trajectory [1]. A method for finding this closed trajectory (limit cycle) is provided in [2, 3]. For systems with one stable and one unstable pole, it has been shown in [1] that topological bifurcation of the region of attraction occurs, i.e. the region of attraction changes between being a hyperbolic type region and a region bounded by a limit cycle. The characteristics of the region of attraction are summarised in Table 1.

Pole configuration	*Region of attraction*
Both stable	$\mathbf{R}^2$
One stable, one unstable	bifurcation
Both unstable	closed by a limit cycle

Table 1: Characteristics of the region of attraction

Since this paper deals with bifurcation, only the case with one stable and one unstable pole will be considered. Although this problem was studied in [1], the bifurcation result presented therein is only existential. Also, no condition for bifurcation is provided. The main contribution of this paper is to derive an analytical condition under which bifurcation occurs. Furthermore, the exact shape of the region of attraction is calculated.

The paper is organised as follows. In Section 2, definitions and terms used in this paper are introduced. Section 3 provides the condition under which the bifurcation of the region of attraction appears. In Section 4, the region of attraction is calculated. Section 5 provides numerical examples, and conclusions are drawn in Section 6.

2 Preliminaries

2.1 System

Consider a single input second-order linear system with a stable and an unstable pole. Upon state transformation, the system can be written as:

$$\dot{x} = Ax + bu = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} u \qquad (1)$$

where, $x \in \mathbf{R}^2$ is the state vector, u the input, A and b appropriate matrices, and λ_1, λ_2 the eigenvalues of the open-loop system. Assume that $\lambda_1 > 0$ and $\lambda_2 < 0$. The symmetric saturation function with unity saturation level will be used:

$$\text{sat}(s) = \begin{cases} -1 & \text{if} \quad s < -1 \\ s & \text{if} \quad -1 \leq s \leq 1 \\ 1 & \text{if} \quad s > 1 \end{cases} \tag{2}$$

With saturated linear state feedback, the closed-loop system is

$$\dot{x} = Ax + b\,\text{sat}(fx), \tag{3}$$

where f is the feedback gain vector. The matrix $(A + bf)$ is assumed to be Hurwitz, i.e. the system is stable without saturation. Let $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ be the eigenvalues of $(A + bf)$. The two conditions that correspond to $(A+bf)$ being Hurwitz are: (i) $\lambda_1(1+f_1)+\lambda_2(1+f_2) < 0$, and (ii) $\lambda_1\lambda_2(1 + f_1 + f_2) > 0$. Since $\lambda_1\lambda_2 < 0$, the second condition gives $(1 + f_1 + f_2) < 0$. Also, it can be verified that $f_1 < 0$, though f_2 can take either sign.

2.2 Equilibrium points and region of attraction

System (3) with one stable and one unstable open-loop pole has three equilibrium points, as opposed to all other open-loop pole configurations (both poles stable or unstable) where the origin is the unique equilibrium point [1].

Theorem 1 [1] *The closed-loop system (3) has three equilibrium points:* $x_{e+} = A^{-1}b = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, $x_{e-} = -A^{-1}b = \begin{bmatrix} -1 & -1 \end{bmatrix}^T$, *and* $x_{e0} = 0$. *Of these,* x_{e0} *is stable, while the other two are saddle points.*

Definition 1 *Let* $\Phi(t, x_0)$ *denote the state of (3) at time* t, *starting with the initial condition* x_0 *at* $t = 0$. *The region of attraction of the stable equilibrium point is defined by:*

$$\mathcal{A} = \left\{ x : \lim_{t\to\infty} \Phi(t, x) = 0 \right\}. \tag{4}$$

The boundary of $\mathcal{A}$ *is denoted by* $\partial\mathcal{A}$.

2.3 Manifolds

Define the following hyperplanes and manifolds (refer to Figures 1-4 for illustration):

- $\partial\mathcal{L}_0 = \{x : fx = 0\}$
- $\partial\mathcal{L}_+ = \{x : fx = 1\}$, $\partial\mathcal{L}_- = \{x : fx = -1\}$
- $\partial\mathcal{C}_+ = \{x : x_1 = 1\}$, $\partial\mathcal{C}_- = \{x : x_1 = -1\}$
- $\partial\mathcal{R}_+ = \{x : x_2 = 1\}$, $\partial\mathcal{R}_- = \{x : x_2 = -1\}$
- $\partial\mathcal{S}_+ = \{x : \lim_{t\to\infty} \Phi(t, x) = x_{e+}\}$
- $\partial\mathcal{S}_- = \{x : \lim_{t\to\infty} \Phi(t, x) = x_{e-}\}$
- $\partial\mathcal{U}_+ = \{x : \lim_{t\to\infty} \Phi(-t, x) = x_{e+}\}$
- $\partial\mathcal{U}_- = \{x : \lim_{t\to\infty} \Phi(-t, x) = x_{e-}\}$

The hyperplanes $\partial\mathcal{L}_+$ and $\partial\mathcal{L}_-$ are the boundaries of the region $\mathcal{L}$ where the control is linear and $\partial\mathcal{L}_0$ is the hyperplane of zero control. The hyperplanes $\partial\mathcal{C}_+$ and $\partial\mathcal{C}_-$ are the boundaries of null controllable region $\mathcal{C}$, while similarly $\mathcal{R}$ is the null reachable region, i.e. the region in state space to which the system can be reached from the origin using a saturated input. $\partial\mathcal{S}$ and $\partial\mathcal{U}$ denote the stable and unstable manifolds, respectively. For the unstable manifolds, evolution in reverse time is considered. All manifolds have two branches, on either side of the saddle points. From the saddle points, the branches of the manifolds $\partial\mathcal{S}$ and $\partial\mathcal{U}$ extend along $\partial\mathcal{R}$ and $\partial\mathcal{C}$, respectively, until they hit the linear region. The two points where the manifolds $\partial\mathcal{S}$ and $\partial\mathcal{U}$ intersect the boundaries of the linear region are given by:

$$c = \partial\mathcal{C}_- \cap \partial\mathcal{L}_+ = \begin{bmatrix} -1 & \frac{(1+f_1)}{f_2} \end{bmatrix}^T \tag{5}$$

$$r = \partial\mathcal{R}_+ \cap \partial\mathcal{L}_- = \begin{bmatrix} -\frac{(1+f_2)}{f_1} & 1 \end{bmatrix}^T \tag{6}$$

2.4 Existence of bifurcation

For a system with one stable and one unstable open-loop pole, an existential result on bifurcation of the region of attraction was given in [1]:

1. If $\partial\mathcal{U}_+ \cap \mathcal{A} \neq \emptyset$ and $\partial\mathcal{U}_- \cap \mathcal{A} \neq \emptyset$, then $\partial\mathcal{A} = \partial\mathcal{S}_+ \cup \partial\mathcal{S}_-$.
2. If $\partial\mathcal{U}_+ \cap \mathcal{A} \neq \emptyset$ and $x_{e-} \neq \partial\mathcal{A}$ then $\partial\mathcal{A} = \partial\mathcal{S}_+$
3. If $\partial\mathcal{U}_- \cap \mathcal{A} \neq \emptyset$ and $x_{e+} \neq \partial\mathcal{A}$ then $\partial\mathcal{A} = \partial\mathcal{S}_-$.
4. If $\partial\mathcal{U}_+ \cap \mathcal{A} = \emptyset$ and $\partial\mathcal{U}_- \cap \mathcal{A} = \emptyset$, then $\partial\mathcal{A}$ is either a closed orbit or a graph of homoclinic/heteroclinic connections.

This result calls for some remarks. Firstly, the result depends on the shape of $\mathcal{A}$ and $\partial\mathcal{A}$ that are unknown. In this paper, an analytical condition for bifurcation will be provided that does not assume *a priori* the shape of $\mathcal{A}$. Secondly, it will be shown that Statements 2 and 3 cannot occur. Also, in Statement 4, homoclinic connections, i.e. manifolds starting from and ending at the same saddle point, do not exist for the system considered. In addition, it is possible to distinguish between the cases when heteroclinic connections occur (manifolds starting from one saddle point and ending at another) and when the boundary is a closed orbit.

3 Condition for bifurcation

For the class of systems considered, the boundary of the region of attraction can be either i) unbounded hyperbolically shaped or ii) a bounded limit cycle, depending on the parameters of the system and the controller. The limiting case between the two types of the region of attraction corresponds to a set bounded by two heteroclinic connections. In this section, a condition depending on the system and controller parameters will be defined, with which it will be possible to distinguish between the two categories and detect the limiting case. However, the link between this condition and the bifurcation is deferred to the next section.

3.1 Intersection of system trajectory with $\partial\mathcal{L}_0$

The condition for bifurcation is based on how the trajectories from points c and r intersect $\partial\mathcal{L}_0$. The first intersection of the trajectory from an arbitrary initial condition is considered. Let T_+ denote the first positive time for which the trajectory intersects $\partial\mathcal{L}_0$, and T_- the first negative time. The analytical expressions for T_+ and T_- depend on the nature of the closed-loop poles $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. Three cases have to be distinguished: (i) distinct and real poles, (ii) double poles, and (iii) complex conjugate poles.

3.1.1 Distinct real poles:

Proposition 1 *Let the eigenvalues of $(A+bf)$ be distinct and real. Given $x_o = \left[\begin{array}{cc} x_1 & x_2 \end{array}\right]^T$ in $\mathcal{L}\setminus 0$, the intersection times T_+ and T_- are given by:*

$$T_+ = \begin{cases} \gamma & \text{if} \quad \gamma > 0 \text{ and } \alpha > 0 \\ \infty & \text{if} \quad \gamma \leq 0 \text{ or } \alpha \leq 0 \end{cases} \tag{7}$$

$$T_- = \begin{cases} \gamma & \text{if} \quad \gamma < 0 \text{ and } \alpha > 0 \\ \text{undefined} & \text{if} \quad \gamma \geq 0 \text{ or } \alpha \leq 0 \end{cases} \tag{8}$$

where

$$\gamma = \frac{\ln(\alpha)}{\tilde{\lambda}_2 - \tilde{\lambda}_1}, \; \alpha = \frac{f_1x_1(\tilde{\lambda}_1 - \lambda_2) + f_2x_2(\tilde{\lambda}_1 - \lambda_1)}{f_1x_1(\tilde{\lambda}_2 - \lambda_2) + f_2x_2(\tilde{\lambda}_2 - \lambda_1)} \tag{9}$$

Proof: The following transformation is used to diagonalise the system (3) without saturation:

$$\tilde{V} = \frac{1}{\tilde{\lambda}_2 - \tilde{\lambda}_1}\left[\begin{array}{cc} f_1(\lambda_2 - \tilde{\lambda}_1) & f_2(\lambda_1 - \tilde{\lambda}_1) \\ f_1(\tilde{\lambda}_2 - \lambda_2) & f_2(\tilde{\lambda}_2 - \lambda_1) \end{array}\right]$$

Then, the evolution of the states is given by

$$x(t) = e^{(A+bf)t}x_o = \tilde{V}^{-1}\left[\begin{array}{cc} e^{\tilde{\lambda}_1 t} & 0 \\ 0 & e^{\tilde{\lambda}_2 t} \end{array}\right]\tilde{V}x_o \tag{10}$$

The intersection time T satisfying $fe^{(A+bf)T}x_o = 0$ is sought. Since $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$, the previous expression reads:

$$\begin{aligned} &e^{\tilde{\lambda}_1 T}(f_1x_1(\lambda_2 - \tilde{\lambda}_1) + f_2x_2(\lambda_1 - \tilde{\lambda}_1)) \\ &\quad + e^{\tilde{\lambda}_2 T}(f_1x_1(\tilde{\lambda}_2 - \lambda_2) + f_2x_2(\tilde{\lambda}_2 - \lambda_1)) = 0 \end{aligned} \tag{11}$$

which always admits the solution $T = \infty$. The other solution is $T = \gamma = \frac{\ln(\alpha)}{\tilde{\lambda}_2 - \tilde{\lambda}_1}$, where

$$\alpha = e^{(\tilde{\lambda_2} - \tilde{\lambda_1})T} = \frac{f_1x_1(\tilde{\lambda}_1 - \lambda_2) + f_2x_2(\tilde{\lambda}_1 - \lambda_1)}{f_1x_1(\tilde{\lambda}_2 - \lambda_2) + f_2x_2(\tilde{\lambda}_2 - \lambda_1)} \tag{12}$$

α is well defined as long as both the numerator and the denominator do not vanish simultaneously. This will not happen due the invertibility of $\tilde{V}$. However, α can be negative, in which case the solution of (12) is imaginary, and so $T = \infty$ is the only solution. Depending upon the sign of γ, the solution is either in forward time or in reverse time. ∎

Note that, when the closed-loop poles are real, in addition to reaching the origin asymptotically, there is at most one intersection of $\partial\mathcal{L}_0$. This intersection can either be in forward (positive) time or in reverse (negative) time. The positive intersection time is always defined since, in the worst case, the system reaches the origin asymptotically. However, there might be no intersection in negative time and T_- may be undefined.

Corollary 1 *T_- is defined from the point c.*

Proof: Assume, without loss of generality, that $(\tilde{\lambda}_1 - \tilde{\lambda}_2) > 0$. Then from (8) and (9), $T_- = \gamma < 0$ exists only if $\alpha > 1$. Substituting $x_o = c$,

$$\alpha = 1 + \frac{(\tilde{\lambda}_1 - \tilde{\lambda}_2)}{f_1(\lambda_2 - \lambda_1) + (\tilde{\lambda}_2 - \lambda_1)}. \tag{13}$$

Since $(\tilde{\lambda}_1 - \tilde{\lambda}_2) > 0$, $\alpha > 1$ when the denominator of (13) is positive. Using the fact that $\tilde{\lambda}_1 + \tilde{\lambda}_2 = (1 + f_1)\lambda_1 + (1 + f_2)\lambda_2$, the denominator becomes:

$$f_1(\lambda_2 - \lambda_1) + (\tilde{\lambda}_2 - \lambda_1) = \lambda_2(1 + f_1 + f_2) - \tilde{\lambda}_1 \tag{14}$$

From $\lambda_2, \tilde{\lambda}_1 < 0$ and the second Hurwitz condition $(1 + f_1 + f_2) < 0$, the denominator is positive, so $\alpha > 1$, $\gamma < 0$, and thus T_- exists. ∎

3.1.2 Double poles:

When $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}$, $\alpha = 1$ and γ is indeterminate. However, the limiting value can be easily found:

$$T_+ = \begin{cases} \gamma & \text{if} \quad \gamma > 0 \\ \infty & \text{if} \quad \gamma \leq 0 \end{cases} \tag{15}$$

$$T_- = \begin{cases} \gamma & \text{if} \quad \gamma < 0 \\ \text{undefined} & \text{if} \quad \gamma \geq 0 \end{cases} \tag{16}$$

where

$$\gamma = \frac{f_1x_1 + f_2x_2}{f_1x_1(\lambda_2 - \tilde{\lambda}) + f_2x_2(\lambda_1 - \tilde{\lambda})} \tag{17}$$

3.1.3 Complex conjugate poles: The expression (9) can also be used when the poles are complex. Note that the numerator and denominator of α are complex conjugates. So, $|\alpha| = 1$, the real part of $\ln(\alpha)$ is zero and so is $\mathrm{Re}(\tilde{\lambda}_2 - \tilde{\lambda}_1)$. However, the important difference is that $\ln(\alpha)$ admits multiple solutions, and there are infinitely many intersections both in positive and negative times. Among the solutions of $\ln(\alpha)$, the first positive solution and the first negative solution are used for the computation of T_+ and T_-. So,

$$T_+ = \text{first positive solution of } \left(\frac{\ln(\alpha)}{\tilde{\lambda}_2 - \tilde{\lambda}_1}\right) \quad (18)$$

$$T_- = \text{first negative solution of } \left(\frac{\ln(\alpha)}{\tilde{\lambda}_2 - \tilde{\lambda}_1}\right) \quad (19)$$

with α given by (9).

3.2 Definition of the Condition on C

Definition 2 *Let $r^* = \Phi(T_+, r) = e^{(A+bf)T_+} r$ be the first intersection of the trajectory starting from r and $\partial\mathcal{L}_0$ and, similarly, $c^* = \Phi(T_-, c) = e^{(A+bf)T_-} c$. Then, C is defined as*

$$C = \|c^*\| - \|r^*\| = \|e^{(A+bf)T_-} c\| - \|e^{(A+bf)T_+} r\|. \quad (20)$$

It will be shown in the next section that the shape of the region of attraction depends on whether $C > 0$, $C < 0$, or $C = 0$. The condition on C can be interpreted as follows: In backward time, $\mathcal{R}$ forms the region to which a trajectory cannot return, once it has left. The sign of C indicates whether or not the trajectory from c leaves $\mathcal{R}$ in backward time. If $C < 0$, the trajectory $\Phi(t, c)$ does not leave $\mathcal{R}$ while when $C > 0$, it leaves $\mathcal{R}$. A similar argument can be made with $\mathcal{C}$ and r.

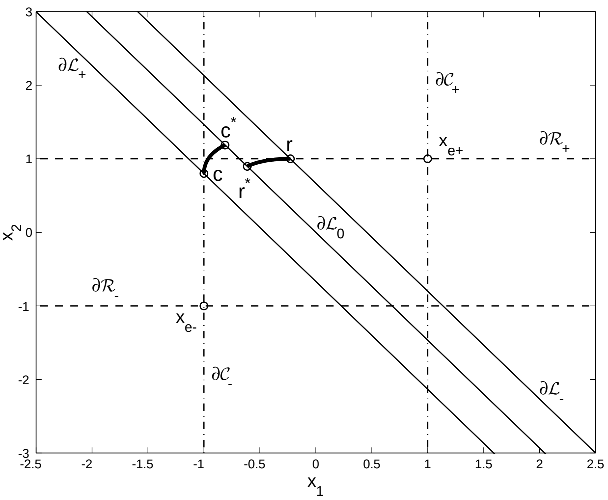

Figure 1: Illustration for the condition on C ($C > 0$).

Figure 1 illustrates (20). If c^* is further from the origin than r^* along the line $\partial\mathcal{L}_0$ then $C > 0$ and vice versa. When $c^* = r^*$, $C = 0$, and it will be shown in the next section that the bifurcation occurs exactly there.

3.3 Simple checks for the Condition on C

Though C has to be computed using (20), there are easier sufficient conditions to check whether $C > 0$ or $C < 0$. From the interpretation of the condition on C, $C > 0$ if $c \notin \mathcal{R}$. Similarly, $C < 0$ if $r \notin \mathcal{C}$. This leads to the following proposition.

Proposition 2 $(1 - f_1 + f_2) < 0 \Rightarrow C < 0$ *and* $(1 + f_1 - f_2) < 0 \Rightarrow C > 0$.

Proof: If $(1 - f_1 + f_2) < 0$, $(1 + f_2) < f_1$. Since $f_1 < 0$, $\frac{1+f_2}{f_1} > 1$. So, the first component of r is smaller than -1, and $r \notin \mathcal{C}$ leads to $(1 - f_1 + f_2) < 0 \Rightarrow C < 0$.

If $(1 + f_1 - f_2) < 0$, then $(1 + f_1) < f_2$. Since f_2 is not sign definite, two cases need to be considered. If $f_2 < 0$, $\frac{1+f_1}{f_2} > 1$. So, the second component of c is larger than 1, and $c \notin \mathcal{R}$. If $f_2 > 0$, the Hurwitz condition $(1 + f_1 + f_2) < 0$ itself indicates that $\frac{1+f_1}{f_2} < -1$. So, in either case, $c \notin \mathcal{R}$, and $(1 + f_1 - f_2) < 0 \Rightarrow C > 0$. ■

Although the conditions are easy to verify, there exists a gap between the two conditions. In this gap, it is necessary to compute C using (20).

Proposition 3 $(\lambda_1 + \lambda_2) \leq 0 \Rightarrow C > 0$.

The proof of this proposition is quite detailed and thus is not provided here. It uses Bendixson's theorem and the result that will be presented in the next section. This proposition implies that, if the unstable open-loop pole is slower than the stable one, then no bifurcation can occur. Also, this shows why that not all systems with one stable and one unstable open-loop pole exhibit bifurcation as a function of the controller parameters.

4 Region of attraction and bifurcation

In this section, the link between the condition on C and the shape of the region of attraction will be established. It will be shown that the bifurcation between a hyperbolic type region of attraction and a region of attraction bounded by a limit cycle occurs at $C = 0$. Due to space limitations, only a sketch of the proof of the main result is provided.

Theorem 1

1. *If $C > 0$, (region bordered by hyperbolae)*
   - *$\partial\mathcal{S}_+$ and $\partial\mathcal{S}_-$ are disjoint and unbounded.*
   - *For both $\partial\mathcal{U}_+$ and $\partial\mathcal{U}_-$, one of the branches of converges to the origin.*
   - *The boundary of the region of attraction is $\partial\mathcal{A} = \partial\mathcal{S}_+ \cup \partial\mathcal{S}_-$.*

2. *If $C < 0$, (region bounded by a limit cycle)*
   - *$\partial\mathcal{U}_+$ and $\partial\mathcal{U}_-$ are disjoint and unbounded.*
   - *For both $\partial\mathcal{S}_+$ and $\partial\mathcal{S}_-$, one of the branches of converges to a limit cycle.*
   - *The boundary of the region of attraction is the unique time-reversed stable limit cycle*

   $$\partial\mathcal{A} = \lim_{t\to\infty} \Phi(-t, x_0)\ \forall\, x_0 \in \mathcal{U}$$

   where the boundary of $\mathcal{U}$ is $\partial\mathcal{U} = \partial\mathcal{U}_+ \cup \partial\mathcal{U}_-$.

3. *If $C = 0$, (region bounded by two heteroclinic connections)*
   - *One of the branches of $\partial\mathcal{U}_+$ is bounded and coincides with that of $\partial\mathcal{S}_-$.*
   - *One of the branches of $\partial\mathcal{U}_-$ is bounded and coincides with that of $\partial\mathcal{S}_+$.*
   - *The boundary of the region of attraction is a double heteroclinic connection, $\partial\mathcal{A} = (\partial\mathcal{U}_+ \cap \partial\mathcal{S}_-) \cup (\partial\mathcal{U}_- \cap \partial\mathcal{S}_+)$.*

The shapes of the regions in the three scenarios are illustrated in Figures 2, 3 and 4.

Sketch of the proof: The condition on C indicates on which side of $\mathcal{C}$ and $\mathcal{R}$ the trajectories lie. If the manifold lies partially outside the respective regions, it goes unbounded. Otherwise, one of its branches is bounded. With this, the characteristics of the stable and unstable manifolds can be deduced.

The unbounded and disjoint boundaries ($\partial\mathcal{S}_+$ and $\partial\mathcal{S}_-$ when $C > 0$ and $\partial\mathcal{U}_+$ and $\partial\mathcal{U}_-$ when $C < 0$) delimit sets inside which the trajectories are trapped both in forward and reverse times. So, they either converge to the origin, converge to the limit cycle, or escape to infinity. The Poincaré and Bendixson theorems are used to detect the existence or non-existence of limit cycles in this region.

It is interesting to note that, in the case of $C < 0$, the theorem not only provides the region of attraction, but also the domain of all initial conditions that converge in reverse time to the limit cycle. The boundaries of this domain are in fact the unstable manifolds.

When $C = 0$, the boundary of the region of attraction is in between a hyperbolic shape and a bounded limit cycle. It consists of two heteroclinic connections, one starting from the saddle point x_{e+} and ending in x_{e-}, and another starting from x_{e-} and ending in x_{e+}.

5 Numerical examples

The three different scenarios for the condition on C will be illustrated on a numerical example in this section. Consider the system (1) with the numerical values: $\lambda_1 = 1$ and $\lambda_2 = -0.5$. Three illustrations with three different linear state feedback controllers (3) are presented:

1. $f_1 = -2.7$ and $f_2 = -1.5$: The computed value of C is $0.35092 > 0$. Thus, the region of attraction is the unbounded set with $\partial\mathcal{A} = \partial\mathcal{S}_+ \cup \partial\mathcal{S}_-$. In Figure 2, the evolution of two trajectories with the following initial conditions is shown:

   $$x_s = [0.5 \quad 1.5]^T \in \mathcal{A},\ x_u = [0.5 \quad 2.5]^T \in \mathbf{R}^2 \setminus \mathcal{A}.$$

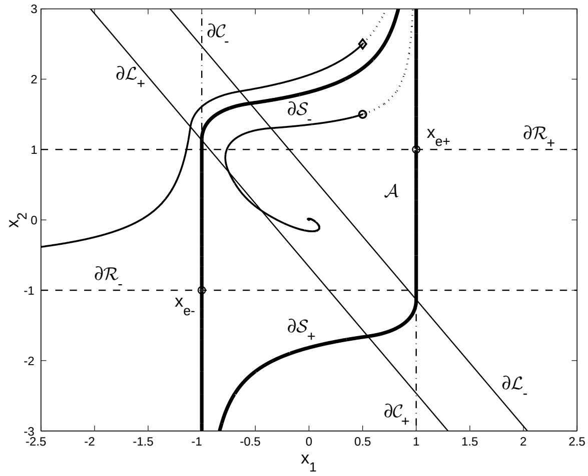

Figure 2: Example of the region of attraction for the condition $C > 0$ ($\circ = x_s$, $\Diamond = x_u$) (solid line - forward time, dotted line - reverse time).

   It is seen that the trajectory from the initial condition inside the region of attraction converges to the origin, while that outside escapes to infinity. For both $\partial\mathcal{U}_+$ and $\partial\mathcal{U}_-$ (not shown in the figure), one of the branches converges to the origin.

2. $f_1 = -1.4$ and $f_2 = -1.5$: The computed value of C is $-0.32136 < 0$. Thus, the region of attraction is bounded by a limit cycle $\partial\mathcal{A}$. In Figure 3, the evolution of three trajectories with the following initial conditions is shown:

   $$x_s = [0.5 \quad 0.4]^T \in \mathcal{A},\ x_u = [0.5 \quad 0.8]^T \in \mathcal{U} \setminus \mathcal{A},$$
   $$x_s = [0.5 \quad 1.5]^T \in \mathbf{R}^2 \setminus \mathcal{U}$$

   In this case, a similar conclusion can be made for the region of attraction. However, what is interesting is that, in reverse time, the trajectories starting from x_s and x_{su} converge to the limit cycle, while that from x_u goes to infinity. For both $\partial\mathcal{S}_+$ and $\partial\mathcal{S}_-$ (not shown in the figure), one of the branches converges, in reverse time, to the limit cycle.

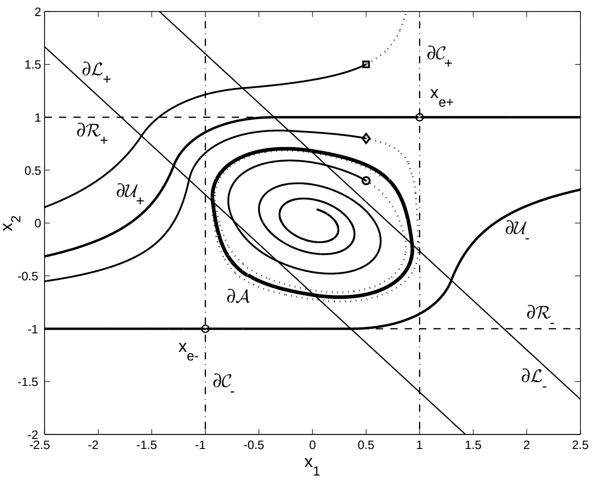

Figure 3: Example of the region of attraction for the condition $C < 0$ ($\circ = x_s$, $\Diamond = x_{su}$, $\Box = x_u$) (solid line - forward time, dotted line - reverse time).

3. $f_1 = -1.7473$ and $f_2 = -1.5$: The computed value of C is $1.49 \times 10^{-5} \approx 0$. Thus, the region of attraction is bounded by two heteroclinic connections $\partial\mathcal{S}_+ \cap \partial\mathcal{U}_-$ and $\partial\mathcal{S}_- \cap \partial\mathcal{U}_+$. In Figure 4, the evolution of two trajectories with the following initial conditions is shown:

$$x_s = [0.5 \quad 0.8]^T \in \mathcal{A},\ x_u = [0.5 \quad 2]^T \in \mathbf{R}^2 \setminus \mathcal{A}.$$

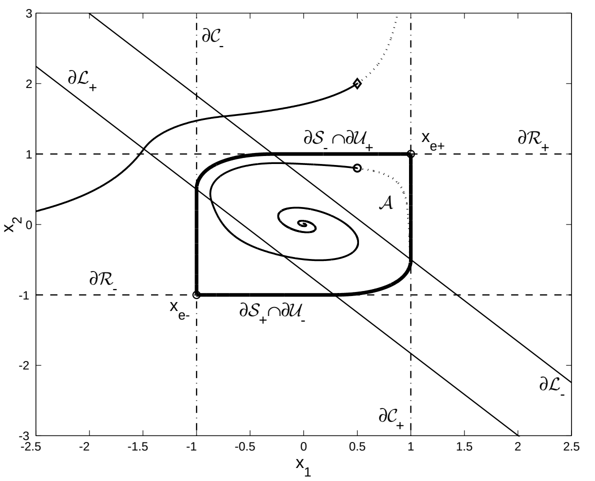

Figure 4: Example of the region of attraction for the condition $C = 0$ ($\circ = x_s$, $\Diamond = x_u$) (solid line - forward time, dotted line - reverse time).

Note that, in reverse time, trajectories with initial conditions within $\mathcal{A}$ converge arbitrary close to the boundary defined by the heteroclinic connections, while those outside $\mathcal{A}$ go to infinity.

In this example, the bifurcation was created by only varying the parameter f_1. Note that the unstable open-loop pole is faster than the stable one and thus the sufficient condition of Proposition 3 cannot be used. The sufficient condition $(1 + f_1 - f_2) < 0$ of Proposition 2 can be used for the first case. However, in the other two cases, the sufficient conditions of Proposition 2 are indecisive.

6 Conclusion

In this paper, the bifurcations of the region of attraction are analysed. It is shown that a planar system with one stable and one unstable pole only exhibits a bifurcation when the unstable pole is faster than the stable one. An analytical condition is provided for which the region of attraction changes from an unbounded hyperbolic region to a bounded limit cycle.

Though this paper dealt only with planar systems with one stable and one unstable pole, it is hoped that the analytical condition presented here can be extended to arbitrary planar systems. Also, the Poincaré and Bendixson theorems provide valuable information regarding the limit cycles, which needs to be explored in the present context.

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