From Euclid to Hilbert (2/2): Bases, Expansions and Matrix Representations

Mathematical Foundations of Signal Processing

Dr. Matthieu Simeoni

October 5, 2020

EPFL

Table of contents

Motivation

Vectors in $\mathbb{R}^N$ are usually represented as 1D arrays of coefficients or coordinates, measuring the *signed distances* of the vector to the canonical axes. In doing so, we implicitly use the canonical basis of $\mathbb{R}^N$:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{e}_2} + \cdots + x_{N-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{e}_{N-1}} + x_N \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{e}_N}$$

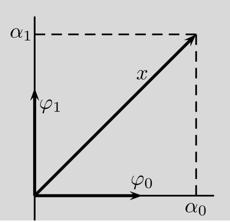


Figure: Coordinates of a vector.

Similarly, finite-dimensional linear operators $A : \mathbb{R}^N \to \mathbb{R}^M$ can be represented as 2D arrays of coefficients called matrices $\boldsymbol{A} \in \mathbb{R}^{M \times N}$, each column being the *image* of an element of the canonical basis: $\boldsymbol{A}_{:,n} = A\boldsymbol{e}_n$.

Such representations are very convenient for practical purposes, since they allow us to perform linear algebra operations via simple *matrix calculations*. We would like similar representations for infinite-dimensional vector spaces!

Objectives and Reading Material

In this lesson we will introduce bases for abstract Hilbert spaces show how they can be used to:

1. Represent/expand vectors as sequences of coefficients,
2. Represent linear operators as (potentially infinite) matrices.

Our aim is to extend linear algebra matrix calculations to abstract Hilbert spaces. This way, we reduce our level of abstraction and get closer to computational tools for signal processing.

Reading Material

- Chapter 2, "From Euclid to Hilbert", of **[?]**, Section 2.5 (subsection 2.5.4 on frames is **optional** and is not part of the exam material).
- **Download link:** `http://www.fourierandwavelets.org/FSP_v1.1_2014.pdf`

Bases

Definition: (Basis)

The set of vectors $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset V$, where $\mathcal{K}$ is countable, is called a basis for a normed vector space V when:

$$\forall x \in V, \text{ there exists a unique sequence } (\alpha_k)_{k\in\mathcal{K}} \in \mathbb{C}^{\mathcal{K}} \text{ such that}^1 \; x = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k.$$

The elements of the sequence $(\alpha_k)_{k\in\mathcal{K}}$ are called the expansion coefficients of x w.r.t the basis Φ.

⚠ *Whether a set is a basis or not depends on the underlying norm (see [**?**, Example 2.30]).*

- Any element of V can be represented uniquely in terms of its expansion coefficients.
- The existence of the expansion coefficients for each $x \in V$ can be re-written as $V = \overline{\mathrm{span}}(\Phi)$.
- The Hilbert spaces that we will work with are separable (they contain a countable dense subset) and have hence countable bases.

[1] We restrict our attention to unconditional bases for which the convergence of the series does not depend on the summation order.

Riesz Bases

When working with bases in practice, one may face numerical stability issues. Riesz bases are special bases with stability constraints, making them particularly suitable for numerical computations.

Definition: (Riesz Basis)

In a Hilbert space $\mathscr{H}$, a basis $\Phi = \{\varphi_k\}_{k\in\mathscr{K}}$ is called a Riesz basis when there exists $0 < \lambda_{min} \leq \lambda_{max} < \infty$ s.t. $\forall x = \sum_{k\in\mathscr{K}} \alpha_k \varphi_k \in \mathscr{H}$,

$$\lambda_{min}\|x\|^2 \leqslant \sum_{k\in\mathscr{K}} |\alpha_k|^2 \leqslant \lambda_{max}\|x\|^2.$$

The constants λ_{max} and λ_{min} are called stability constants. The smallest (respectively largest) λ_{max} and λ_{min} are called the optimal stability constants.

The stability constants bound the ratio $\|\alpha\|_2/\|x\|$ from above and below, hence ensuring that the energy of the expansion coefficients is not arbitrarily large or small w.r.t. the energy of x.

Synthesis Operator Associated to a Riesz Basis

Any vector can be synthesised from its expansion coefficients in a certain Riesz basis using the synthesis operator associated to this basis:

Definition: (Synthesis Operator)

Given a Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for a Hilbert space $\mathcal{H}$, the synthesis operator associated with it is given by

$$\Phi : \begin{cases} \ell_2(\mathcal{K}) \to \mathcal{H}, \\ \alpha \mapsto x = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k. \end{cases}$$

Note that the synthesis operator is bounded. Indeed, we have from the definition of a Riesz basis:

$$\|\Phi\alpha\|^2 = \|x\|^2 \leq \frac{1}{\lambda_{min}} \sum_{k\in\mathcal{K}} |\alpha_k|^2 = \frac{1}{\lambda_{min}} \|\alpha\|_2^2,$$

and hence $\|\Phi\| = \sup_{\|\alpha\|_2=1} \|\Phi\alpha\| \leq 1/\sqrt{\lambda_{min}}$.

Analysis Operator Associated to a Riesz Basis

Let us compute the adjoint of the synthesis operator. For $\alpha \in \ell_2(\mathcal{K})$ and $y \in \mathcal{H}$ we have:

$$\langle \Phi\alpha, y \rangle = \left\langle \sum_{k\in\mathcal{K}} \alpha_k \varphi_k, y \right\rangle = \sum_{k\in\mathcal{K}} \alpha_k \overline{\langle y, \varphi_k \rangle} = \langle \alpha, \Phi^* y \rangle,$$

and hence the adjoint –called the analysis operator– is given by:

Definition: (Analysis Operator)

Given a Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for a Hilbert space $\mathcal{H}$, the analysis operator associated with it is given by

$$\Phi^* : \begin{cases} \mathcal{H} \to \ell_2(\mathcal{K}), \\ y \mapsto (\beta)_k = \langle x, \varphi_k \rangle, \quad k \in \mathcal{K}. \end{cases}$$

Again the analysis operator is bounded: $\|\Phi^*\| = \|\Phi\| \le 1/\sqrt{\lambda_{min}}$. The analysis operator maps a vector $y \in \mathcal{H}$ to a sequence $\beta \in \ell_2(\mathcal{K})$, which in general *differs* from the representing sequence of y. Each coefficient in β measures the linear resemblance of y with an element of the Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$.

Gram Operator

We define now the Gram operator –or Gramian– associated to a Riesz basis:

Definition: (Gram Operator)

Given a Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for a Hilbert space $\mathcal{H}$, the Gram operator associated with it is given by

$$G:\begin{cases}\ell_2(\mathcal{K})\to\ell_2(\mathcal{K}),\\ \alpha\mapsto(\beta)_i=\sum_{k\in\mathcal{K}}\langle\varphi_k,\varphi_i\rangle\alpha_k, \quad i\in\mathcal{K}.\end{cases}$$

Once again, this operator is bounded: $\|\Phi^*\Phi\|=\|\Phi\|^2\leq 1/\lambda_{min}$. The Gramian is the composition $G=\Phi^*\Phi$ between the analysis and synthesis operator. We have indeed, for all $\alpha\in\ell_2(\mathcal{K})$:

$$(G\alpha)_i=\sum_{k\in\mathcal{K}}\langle\varphi_k,\varphi_i\rangle\alpha_k=\left\langle\sum_{k\in\mathcal{K}}\alpha_k\varphi_k,\varphi_i\right\rangle=\langle\Phi\alpha,\varphi_i\rangle=(\Phi^*\Phi x)_i, \qquad i\in\mathcal{K}.$$

Gram Operator and Orthonormal Bases

The Gramian maps sequences on sequences, and can hence be represented as an infinite matrix:

$$G = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \varphi_{-1} \rangle & \langle \varphi_0, \varphi_{-1} \rangle & \langle \varphi_1, \varphi_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \varphi_0 \rangle & \boxed{\langle \varphi_0, \varphi_0 \rangle} & \langle \varphi_1, \varphi_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \varphi_1 \rangle & \langle \varphi_0, \varphi_1 \rangle & \langle \varphi_1, \varphi_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Each entry of this matrix measures the amount of collinearity between two elements of the Riesz basis.

Definition: (Orthonormal Basis)

A basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for a Hilbert space $\mathcal{H}$ is said to be orthonormal if $\langle \varphi_i, \varphi_k \rangle = \delta_{i-k}$, $\forall i, k \in \mathcal{K}$.

From the definition of an orthonormal basis, we have hence: a Riesz basis is orthonormal iff $G = \mathrm{Id}$.

Gramian Eigenvalues & Optimal Stability Constants

$\underline{K} = \{0, \dots, N\}$ $\quad \{\varphi_0, \dots \varphi_N\} \subset V \leftarrow$ Hilbert space $\langle \cdot, \cdot \rangle$

$x \in V : \quad x = \underline{\Phi \alpha} = \sum_{k=0}^{N} \alpha_k \varphi_k$ $\quad \exists !$

$\lambda_{min} = \frac{1}{\mu_{max}}$ $\quad \|x\|^2 = \|\Phi\alpha\|^2 = \langle \Phi\alpha, \Phi\alpha \rangle = \langle \alpha, \Phi^* \Phi \alpha \rangle = \langle \alpha, G\alpha \rangle$

$\lambda_{max} = \frac{1}{\mu_{min}}$ $\quad G \in \mathbb{R}^{(N+1)\times(N+1)} \rightarrow \quad G = \sum_{k=0}^{N} \mu_k \underline{u_k u_k^H} \quad \mu_k > 0$

$\frac{\mu_{max}}{\mu_{min}}$

$\frac{\lambda_{min}}{\lambda_{max}} = \frac{\mu_{min}}{\mu_{max}}$ $\quad \langle \alpha, G\alpha \rangle = \sum_{k=0}^{N} \mu_k \langle \alpha, u_k \underbrace{u_k^H \alpha}_{\langle \alpha, u_k \rangle} \rangle = \sum_{k=0}^{N} \mu_k \langle \alpha, u_k \rangle \langle \alpha, u_k \rangle$

$\|x\|^2 = \sum_{k=0}^{N} \mu_k |\langle \alpha, u_k \rangle|^2 \leq \mu_{max} \sum |\langle \alpha, u_k \rangle|^2$ $\quad |\langle \alpha, u_k \rangle|$

$\mu_{min} \|\alpha\|_2^2 \leq \|x\|^2$

$\frac{1}{\mu_{max}} \|x\|^2 \leq \|\alpha\|_2^2 \leq \frac{1}{\mu_{min}} \|x\|^2$

Computing the Optimal Stability Constants (Example I)

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \qquad G = \underbrace{\Phi^T \Phi}_{} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}$$

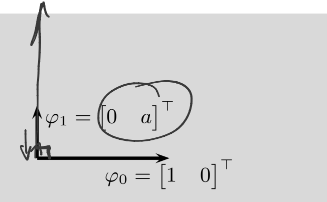
$a > 0$

$$\mu_0 = 1 \qquad \mu_1 = \underbrace{a^2}_{}$$

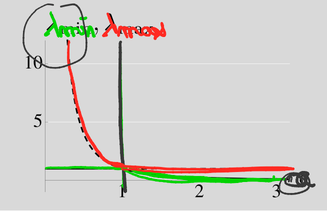
$$\lambda_{min}(a) = \begin{cases} 1 & a \leq 1 \\ 1/a^2 & a > 1 \end{cases}$$

$$\lambda_{max}(a) = \begin{cases} 1/a^2 & a \leq 1 \\ 1 & a > 1 \end{cases}$$

$$= \frac{1}{\mu_{min}}$$



(a) Riesz basis.



(b) Optimal stability constants vs a.

Figure: Example setup.

Computing the Optimal Stability Constants (Example II)

$$\Phi = \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix} \qquad \Phi^T\Phi = \begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$$

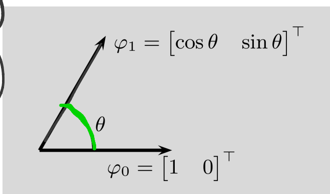
$$\det(\Phi^T\Phi - \lambda I) = \begin{vmatrix} 1-\lambda & \cos\theta \\ \cos\theta & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - \cos^2\theta$$

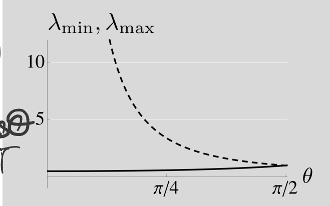
$$= (\underbrace{1-\cos\theta}-\lambda)(\underbrace{1+\cos\theta}-\lambda)$$

$$\mu_0 = 1-\cos\theta \qquad \mu_1 = 1+\cos\theta$$

$$\lambda_{min} = \frac{1}{\mu_{max}}$$



(a) Riesz basis.



(b) Optimal stability constants vs a.

Figure: Example setup.

Gramian & Generalised Parseval's Equality

Proposition: (Generalised Parseval's Equality)

Consider a Hilbert space $\mathscr{H}$ with Riesz basis $\{\varphi_k\}_{k\in\mathscr{K}}$ and associated synthesis operator Φ. Then, for all $x = \Phi\alpha$ and $y = \Phi\beta$ in $\mathscr{H}$, we have

$$\langle x, y\rangle = \langle \Phi\alpha, \Phi\beta\rangle = \langle \alpha, \Phi^*\Phi\beta\rangle = \langle \alpha, G\beta\rangle.$$

In particular we have, for all $x = \Phi\alpha \in \mathscr{H}$:

$$\|x\| = \sqrt{\langle \alpha, G\alpha\rangle} = \|\alpha\|_G. \tag{1}$$

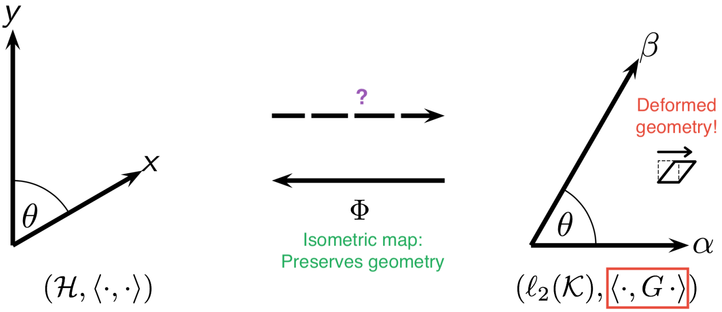
$\|\cdot\|_G$ and $\|\cdot\|_2$ are equivalent norms on $\ell_2(\mathscr{K})$

From (1) and the definition of a Riesz basis, we can show that $\|\cdot\|_G$ and $\|\cdot\|_2$ are equivalent norms:

$$\lambda_{min}\|\alpha\|_G^2 = \lambda_{min}\|x\|^2 \leq \sum_{k\in\mathscr{K}} |\alpha_k|^2 = \|\alpha\|_2^2 \leq \lambda_{max}\|x\|^2 = \lambda_{max}\|\alpha\|_G^2, \qquad \forall \alpha \in \ell_2(\mathscr{K}) \text{ and } x = \Phi\alpha.$$

Gramian & Generalised Parseval's Equality

The inner product in $\mathscr{H}$ becomes a (re-weighted) inner product in $\ell_2(\mathscr{K})$. The synthesis operator Φ is an isometric isomorphism from $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ to $(\ell_2(\mathscr{K}), \langle\cdot, G\cdot\rangle)$ preserving angles and distances.



Useful in practice: geometric manipulations on expansion coefficients instead of abstract vector!
Problem: how to compute the expansion coefficients?

Expansion Coefficients via Φ^{-1}

$\|\Phi^{-1}\| = \sup_{\|x\|=1} \|\Phi^{-1}x\|_2$

From the definition of a Riesz basis, Φ is injective, surjective and hence invertible. Its inverse $\Phi^{-1} : \mathscr{H} \to \ell_2(\mathscr{K})$ is moreover *bounded*. Indeed, we have:

$$\forall x \in \mathscr{H}, \quad \left\|\Phi^{-1}x\right\|_2^2 = \left\|\Phi^{-1}\Phi\alpha\right\|_2^2 = \|\alpha\|_2^2 = \sum_{k\in\mathscr{K}} |\alpha_k|^2 \leq \lambda_{max}\|x\|^2 \quad \Longrightarrow \quad \left\|\Phi^{-1}\right\| \leq \sqrt{\lambda_{max}}.$$

We can hence use Φ^{-1} to obtain the expansion coefficients. Indeed, $\forall x = \Phi\alpha \in \mathscr{H}$, we have $\Phi^{-1}x = \Phi^{-1}\Phi\alpha = \alpha$. But how to compute the inverse of an abstract operator?

Examples:

Sometimes, computing Φ^{-1} can be avoided by directly evaluating the image $\Phi^{-1}x$.

- **Polynomial of degree at most N, $\Pi_{k=1}^N (X-\omega_k) = \sum_{k=0}^N \alpha_k X^k$:**
  $\Phi^{-1}\left(\Pi_{k=1}^N (X-\omega_k)\right) = (\alpha_0, \ldots, \alpha_N)$ can be computed (tediously) via polynomial expansion.
- **Trigonometric polynomial $\cos(Nx) = \sum_{k=0}^N \alpha_k \cos^k(x)$:**
  $\Phi^{-1}(\cos(Nx)) = (\alpha_0, \ldots, \alpha_N)$ can be computed by (very tediously) expanding the degree N Chebyshev polynomial of the first kind $T_N(X)$ (which is such that $\cos(Nx) = T_N(\cos(x))$).

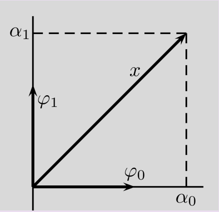
Computing Φ^{-1}: Orthonormal Bases

For orthonormal Riesz bases, Φ^{-1} can be computed very efficiently. In this case, we have indeed $G = \Phi^*\Phi = \mathrm{Id}$ (see Slide 10) and hence[2] $\Phi^{-1} = \Phi^*$ (Φ is unitary).

Theorem: (Expansion with Orthonormal Bases)

Let $\{\varphi_k\}_{k\in\mathcal{K}}$ be an orthonormal Riesz basis for some Hilbert space $\mathcal{H}$, with associated synthesis operator Φ. Then, every $x \in \mathcal{H}$ can be written uniquely as $x = \Phi\alpha$ where the expansion coefficients α are given by

$$\alpha = \Phi^* x = \left(\langle x, \varphi_0\rangle, \langle x, \varphi_1\rangle, \langle x, \varphi_2\rangle, \langle x, \varphi_3\rangle, \cdots\right) \in \ell_2(\mathcal{K}).$$



This can be written in short as: $x = \Phi\Phi^* x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \varphi_k$, $\forall x \in \mathcal{H}$.
The generalised Parseval's equality is moreover given in this case by:

$$\langle x, y\rangle = \langle \Phi^* x, \Phi^* y\rangle = \langle \alpha, \beta\rangle, \quad \text{and in particular} \quad \|x\| = \|\Phi^* x\|_2 = \|\alpha\|_2, \quad \forall x = \Phi\alpha, y = \Phi\beta \in \mathcal{H}.$$

[2] Since Φ is invertible, its left inverse and inverse coincide (check it!).

Computing Φ^{-1}: Non Orthonormal Bases

Consider now a non orthonormal Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$. Then, for all $x=\Phi\alpha\in\mathcal{H}$ we have

$$x=\Phi\alpha \Leftrightarrow \Phi^*x=\Phi^*\Phi\alpha \Leftrightarrow (\Phi^*\Phi)^{-1}\Phi^*x=\alpha \Leftrightarrow \Phi(\Phi^*\Phi)^{-1}\Phi^*x=\Phi\alpha=x.$$

This yields $\Phi^{-1}=(\Phi^*\Phi)^{-1}\Phi^*$. For all $x=\Phi\alpha\in\mathcal{H}$ the expansion coefficients α can hence be computed as

$$\alpha=(\Phi^*\Phi)^{-1}\Phi^*x=G^{-1}\left(\langle x,\varphi_0\rangle,\langle x,\varphi_1\rangle,\langle x,\varphi_2\rangle,\langle x,\varphi_3\rangle,\cdots\right)\in\ell_2(\mathcal{K}). \tag{2}$$

The coefficients are hence obtained by applying the synthesis operator followed by the inverse of the Gramian, correcting for the lack of orthogonality.[3] When $\mathcal{K}$ is finite, the Gramian correction *simply amounts to inverting a matrix*! When $\mathcal{K}$ is infinite, inverting the Gramian can be complex...

Is the Gramian Invertible?

Note that the Gramian $G=\Phi^*\Phi$ is indeed invertible as composition between the synthesis and analysis operators, both invertible. Moreover, its inverse is bounded $\|(\Phi^*\Phi)^{-1}\|=\|\Phi^{-1}\|^2\leq\lambda_{max}$.

[3] Notice that in the orthonormal case, $G=\mathrm{Id}$ and the Gram correction disappears.

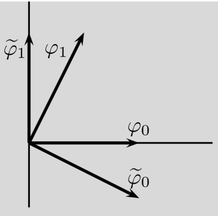
Dual Basis and Biorthogonal Pairs of Bases

Note that the expansion coefficients in (2) can be seen as the image of x through the dual of $\widetilde{\Phi} = \Phi G^{-1} : \ell_2(\mathscr{K}) \to \mathscr{H}$. This is the synthesis operator associated to the Riesz basis of $\mathscr{H}$:

$$\left\{ \widetilde{\varphi}_j = \sum_{k \in \mathscr{K}} \left(G^{-1} \right)_{k,j} \varphi_k \right\}_{j \in \mathscr{K}} \tag{3}$$

with optimal stability constants $1/\lambda_{max}$ and $1/\lambda_{min}$.[4] The basis defined in (3) is called the dual basis of $\{\varphi_k\}_{k \in \mathscr{K}}$. Notice that the two bases are such that

$$\widetilde{\Phi}^* \Phi = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \widetilde{\varphi}_{-1} \rangle & \langle \varphi_0, \widetilde{\varphi}_{-1} \rangle & \langle \varphi_1, \widetilde{\varphi}_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \widetilde{\varphi}_0 \rangle & \boxed{\langle \varphi_0, \widetilde{\varphi}_0 \rangle} & \langle \varphi_1, \widetilde{\varphi}_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \widetilde{\varphi}_1 \rangle & \langle \varphi_0, \widetilde{\varphi}_1 \rangle & \langle \varphi_1, \widetilde{\varphi}_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} = (\Phi^* \Phi)^{-1} \Phi^* \Phi = \mathrm{Id}.$$



We say that the two bases form a biorthogonal pair of bases: $\langle \varphi_k, \widetilde{\varphi}_j \rangle = \delta_{k-j}, \ \forall k, j \in \mathscr{K}$.

[4] See [?, Exercise 2.39] for a proof of this fact.

Expansion with Non Orthonormal Bases

Theorem: (Expansion with Non Orthonormal Bases)

Let $\{\varphi_k\}_{k\in\mathcal{K}}$ be a Riesz basis for some Hilbert space $\mathcal{H}$, with dual basis $\{\tilde{\varphi}_k\}_{k\in\mathcal{K}}$. Then, every $x\in\mathcal{H}$ can be written uniquely as $x=\Phi\alpha$ where

$$\alpha = G^{-1}\Phi^* x = G^{-1}\left(\langle x,\varphi_0\rangle,\langle x,\varphi_1\rangle,\langle x,\varphi_2\rangle,\langle x,\varphi_3\rangle,\cdots\right)$$
$$\Leftrightarrow\ \alpha = \tilde{\Phi}^* x = \left(\langle x,\tilde{\varphi}_0\rangle,\langle x,\tilde{\varphi}_1\rangle,\langle x,\tilde{\varphi}_2\rangle,\langle x,\tilde{\varphi}_3\rangle,\cdots\right).$$

This can be written in short as:

$$x = \Phi G^{-1}\Phi^* x = \Phi\tilde{\Phi}^* x = \sum_{k\in\mathcal{K}}\langle x,\tilde{\varphi}_k\rangle\varphi_k$$
$$= \tilde{\Phi}\Phi^* x = \sum_{k\in\mathcal{K}}\langle x,\varphi_k\rangle\tilde{\varphi}_k, \quad \forall x\in\mathcal{H}.$$

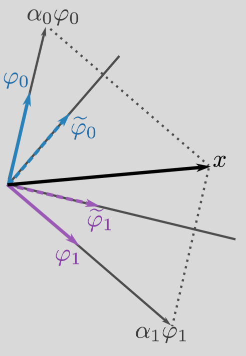


Figure: Expansion in non orthonormal bases.

The generalised Parseval's equality can also be written in this case as:

$$\langle x,y\rangle = \langle\alpha,G\beta\rangle = \langle\tilde{\Phi}^* x, G\tilde{\Phi}^* y\rangle = \langle\tilde{\Phi}^* x,\Phi^* y\rangle = \langle\alpha,\tilde{\beta}\rangle, \quad \forall x=\Phi\alpha=\tilde{\Phi}\tilde{\alpha},\ y=\Phi\beta=\tilde{\Phi}\tilde{\beta}\in\mathcal{H}.$$

Example in $\mathbb{R}^2$

$$x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Phi = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (\varphi_0, \varphi_1)$$

$$G = \phi^T \phi = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$x = \phi \, G^{-1} \phi^T x$$

$$\tilde{\phi} = \phi G^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

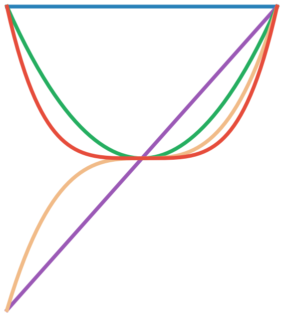
$$G^{-1} = \frac{1}{\det(G)} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$c = G^{-1} \phi^T x$$

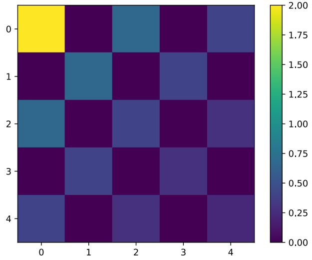
$$c = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha + \beta \end{pmatrix} = \begin{pmatrix} \alpha - \beta \\ \beta \end{pmatrix}$$

$$x = \phi c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha - \beta \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

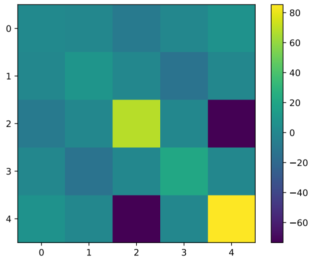
Example: Expansion of Polynomials



(a) Monomials basis: $\varphi_0 = 1$, $\varphi_1 = X$, $\varphi_2 = X^2$, $\varphi_3 = X^3$, $\varphi_4 = X^4$.



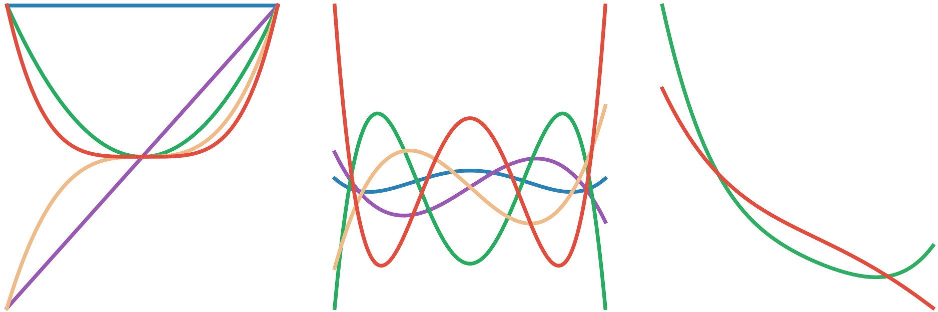
(b) Gram matrix: $G_{ij} = \int_{-1}^{1} \varphi_i(X)\varphi_j(X)\,dX$, $i,j \in \{0,1,2,3,4\}$



(c) Inverse of the Gram matrix: G^{-1}.

Figure: Monomials basis on [-1,1], its Gram matrix and its inverse.

Example: Expansion of Polynomials



(a) Monomials basis: $\varphi_0 = 1$, $\varphi_1 = X$, $\varphi_2 = X^2$, $\varphi_3 = X^3$, $\varphi_4 = X^4$.

(b) Dual basis: $\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_4$.

(c) Effect of Gram correction: $\Phi\Phi^* x$ vs $\Phi G^{-1}\Phi^* x$

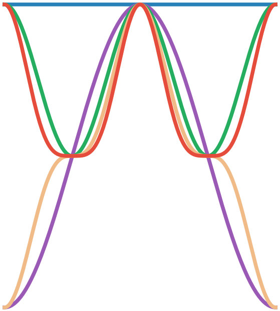
Figure: Expansion of polynomials of degree at most 4 on [-1,1].

Example: Expansion of Polynomials (Python code)

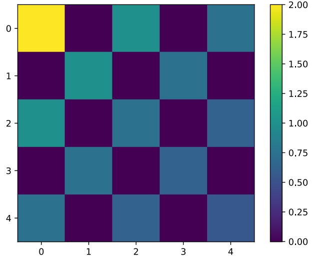
```
import numpy as np

x = np.linspace(-1, 1, 900)
step_size= 2 / x.size
Phi = np.stack([1 + 0 * x, x, x ** 2,x ** 3, x ** 4], axis=-1)
Gram = step_size * (Phi.transpose() @ Phi)
Gram_inv = np.linalg.solve(Gram, np.eye(Phi.shape[1]))
Phi_tilde = Phi @ Gram_inv
```

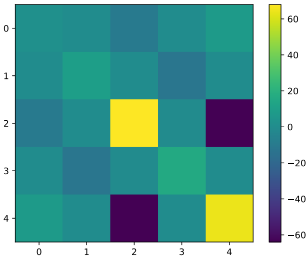
Example: Expansion of Trigonometric Polynomials



(a) Trigonometric monomials basis:
$\varphi_0 = 1$, $\varphi_1 = \cos(t)$, $\varphi_2 = \cos^2(t)$,
$\varphi_3 = \cos^3(t)$, $\varphi_4 = \cos^4(t)$.



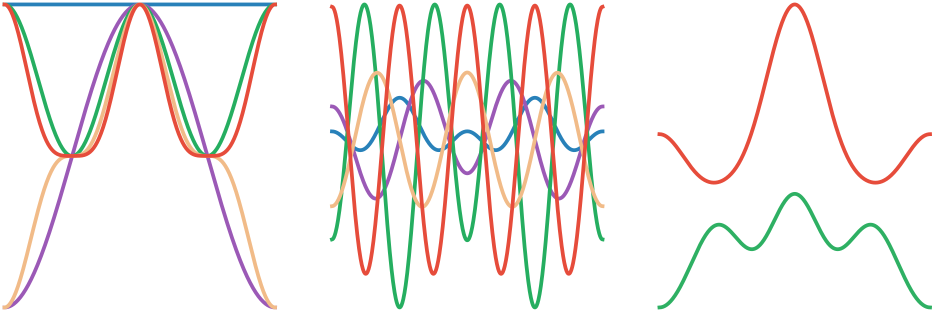
(b) Gram matrix:
$G_{ij} = \int_{-\pi}^{\pi} \varphi_i(X)\varphi_j(X)\,dX$,
$i, j \in \{0, 1, 2, 3, 4\}$



(c) Inverse of the Gram matrix: G^{-1}.

Figure: Trigonometric monomials basis on $[-\pi, \pi]$, its Gram matrix and its inverse.

Example: Expansion of Trigonometric Polynomials



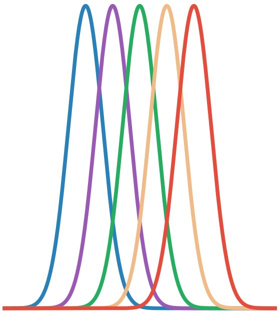
(a) Trigonometric monomials basis: $\varphi_0 = 1$, $\varphi_1 = \cos(t)$, $\varphi_2 = \cos^2(t)$, $\varphi_3 = \cos^3(t)$, $\varphi_4 = \cos^4(t)$.

(b) Dual basis: $\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_4$.

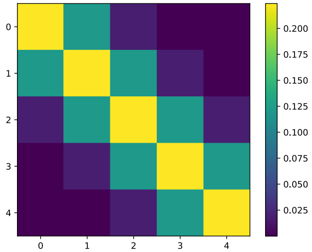
(c) Effect of Gram correction: $\Phi\Phi^* x$ vs $\Phi G^{-1}\Phi^* x$

Figure: Expansion of trigonometric polynomials of degree at most 4 on $[-\pi, \pi]$.

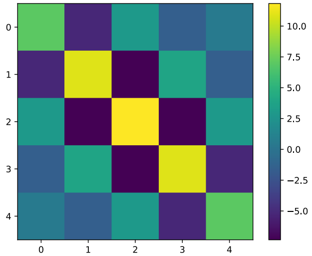
Example: Expansion w.r.t. basis of shifted Gaussians



(a) Basis of shifted Gaussians: φ_0, φ_1, φ_2, φ_3, φ_4.



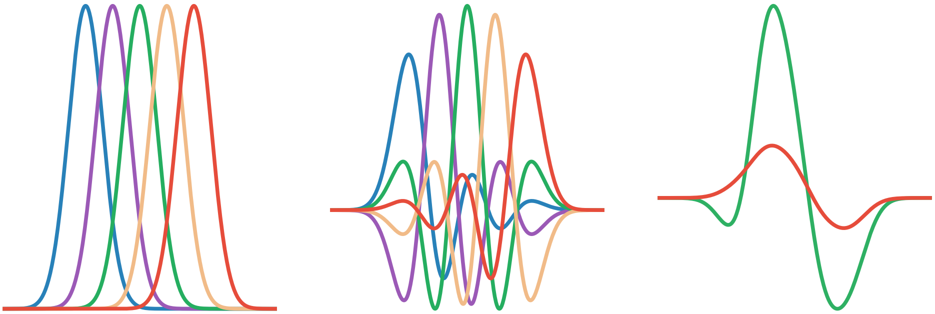
(b) Gram matrix: $G_{ij} = \int \varphi_i(X)\varphi_j(X)\,dX$, $i, j \in \{0, 1, 2, 3, 4\}$



(c) Inverse of the Gram matrix: G^{-1}.

Figure: Basis of shifted Gaussians, its Gram matrix and its inverse.

Example: Expansion w.r.t. basis of shifted Gaussians



(a) Basis of shifted Gaussians: φ_0, φ_1, φ_2, φ_3, φ_4.

(b) Dual basis: $\widetilde{\varphi}_0, \widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\varphi}_3, \widetilde{\varphi}_4$.

(c) Effect of Gram correction: $\Phi\Phi^* x$ vs $\Phi G^{-1}\Phi^* x$

Figure: Expansion w.r.t. basis of shifted Gaussians.

Gramian and Orthogonalisation

Consider factorising the Gramian as $G = G^{1/2}G^{*/2}$, where $G^{1/2}$ is a square root of G. Then, we have $G^{-1} = G^{-1/2}G^{-*/2}$the expansion formula can be rewritten as

$$x = \Phi G^{-1}\Phi^* x = \Phi G^{-1/2}G^{-*/2}\Phi^* x = \left(\Phi G^{-1/2}\right)\left(\Phi G^{-1/2}\right)^* x = \Phi_\perp \Phi_\perp^* x.$$

The operator $\Phi_\perp = \Phi G^{-1/2}$ is unitary:

$$\Phi_\perp^* \Phi_\perp = G^{-*/2}\Phi^*\Phi G^{-1/2} = G^{-*/2}GG^{-1/2} = G^{-*/2}G^* G^{-1/2} = G^{-*/2}G^{*/2}G^{1/2}G^{-1/2} = \mathrm{Id}.$$

The Riesz basis[5] $\left\{\varphi_j^\perp = \sum_{k\in\mathcal{K}} \left(G^{-1/2}\right)_{k,j} \varphi_k\right\}_{j\in\mathcal{K}}$ is hence orthonormal.

The operator $G^{-1/2}$ hence orthogonalises the Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$. Note that the square root of an operator is non unique: the square root obtained via Cholesky factorisation $G^{-1} = LL^*$ is lower-triangular while the square root obtained via the eigenvalue decomposition $G^{-1/2} = U\Lambda^{-1/2}U^*$ is self-adjoint. Each square root yields a different orthogonalised basis.

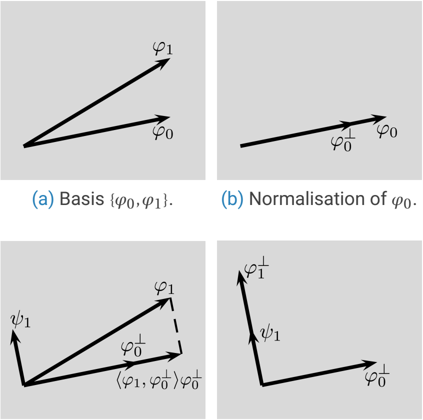
[5]With optimal stability constants $\lambda_{min} = \lambda_{max} = 1$.

Gram-Schmidt Orthogonalisation

Another popular orhtogonalisation process is the Gram-Schmidt algorithm. It transforms a basis $\{\varphi_k\}_{k\in\mathcal{K}}$ into an orthonormal basis $\{\varphi_k^\perp\}_{k\in\mathcal{K}}$ via the iterations:

$$\begin{cases} \varphi_0^\perp = \dfrac{\varphi_0}{\|\varphi_0\|} \\ \psi_k = \varphi_k - \displaystyle\sum_{j=0}^{k-1} \left\langle \varphi_k, \varphi_j^\perp \right\rangle \varphi_j^\perp \\ \varphi_k^\perp = \dfrac{\psi_k}{\|\psi_k\|}, \qquad \text{for } k = 0, 1, 2, 3 \cdots \end{cases}$$

⚠ This algorithm is numerically unstable, and can result in loss of orthogonality due to roundoff errors.



(a) Basis $\{\varphi_0, \varphi_1\}$. (b) Normalisation of φ_0.

(c) Creation of $\psi_1 \perp \varphi_0^\perp$. (d) Normalisation of ψ_1.

Figure: Illustration of the Gram-Schmidt orthogonalisation process.

Gram-Schmidt Orthogonalisation and QR Decomposition

The QR Notice that Gram-Schmidt orthogonalisation performs a QR decomposition of Φ:

$$\Phi = [\varphi_0\, \varphi_1\, \varphi_2 \ldots] = \underbrace{\left[\varphi_0^\perp\, \varphi_1^\perp\, \varphi_2^\perp \ldots\right]}_{Q=\Phi_\perp \text{ unitary}} \underbrace{\begin{bmatrix} \langle \varphi_0, \varphi_0^\perp \rangle & \langle \varphi_1, \varphi_0^\perp \rangle & \langle \varphi_2, \varphi_0^\perp \rangle & \cdots \\ 0 & \langle \varphi_1, \varphi_1^\perp \rangle & \langle \varphi_2, \varphi_1^\perp \rangle & \cdots \\ \vdots & 0 & \langle \varphi_2, \varphi_2^\perp \rangle & \cdots \\ \vdots & \vdots & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{R \text{ upper triangular}}. \tag{4}$$

Indeed, all the quantities involved in (4) are computed during the Gram-Schmidt orthogonalisation process.[6] In numerical linear algebra libraries however, the QR decomposition is computed via Householder reflections or Givens rotations which are more stable than the Gram-Schmidt orthogonalisation process.

[6] Observe indeed that $\left\langle \varphi_k, \varphi_k^\perp \right\rangle = \|\psi_k\|$, for $k \in \mathcal{K}$.

Orthogonalisation in $\mathbb{R}^2$ (Cholesky)

$$\Phi = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(\Phi^T\Phi)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$$

$$\begin{cases} \alpha^2 = 2 \Rightarrow \alpha = \sqrt{2} \\ \alpha\beta = -1 \Rightarrow \beta = -\frac{1}{\sqrt{2}} \\ \beta^2+\gamma^2 = 1 \Rightarrow \gamma^2 = 1 - \frac{1}{2} = 1/2 \Rightarrow \gamma = \frac{1}{\sqrt{2}} \end{cases} = \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2+\gamma^2 \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

ψ

$$\Phi_1 = \Phi L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Orthogonalisation in $\mathbb{R}^2$ (Gram-Schmidt)

$$\Phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_0^\perp = \frac{\varphi_0}{\|\varphi_0\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

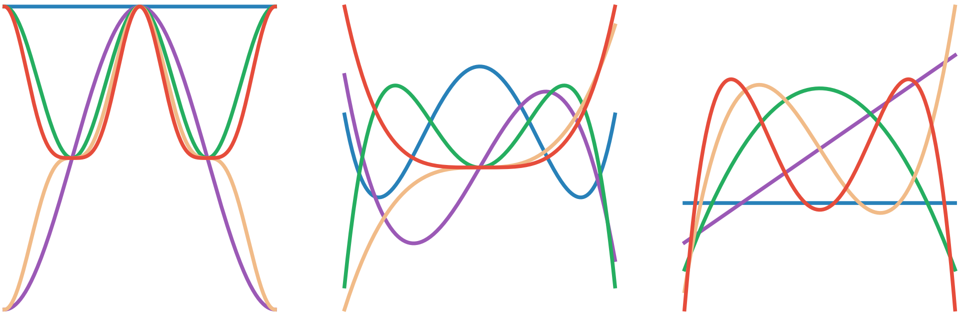
$$\varphi_1 = \varphi_1 - \langle \varphi_1, \varphi_0^\perp \rangle \varphi_0^\perp$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\varphi_1^\perp = \varphi_1 \qquad \|\varphi_1\| = 1$$

$$\varphi_0^\perp = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \varphi_1^\perp = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example: Orthogonal Bases for Polynomials



(a) Monomials basis: $\varphi_0 = 1$, $\varphi_1 = X$, $\varphi_2 = X^2$, $\varphi_3 = X^3$, $\varphi_4 = X^4$.

(b) Orthogonalised basis (via Cholesky): $\varphi_0^\perp, \varphi_1^\perp, \varphi_2^\perp, \varphi_3^\perp, \varphi_4^\perp$.

(c) Legendre basis (Gram-Schmidt): $\varphi_0^\perp$, $\varphi_1^\perp, \varphi_2^\perp, \varphi_3^\perp, \varphi_4^\perp$.

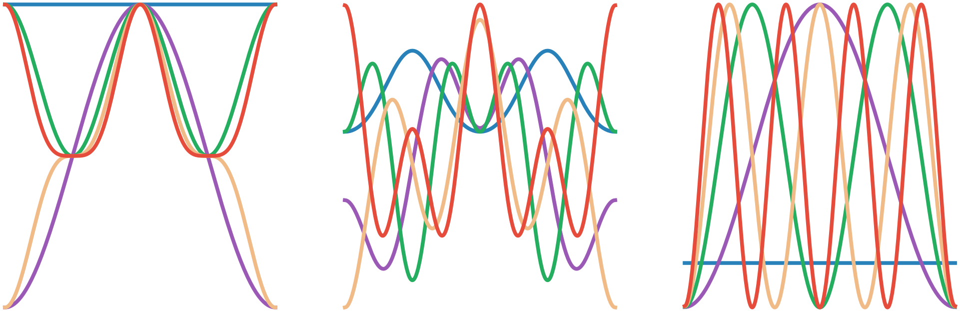
Figure: Orthogonalisation of the monomials basis on [-1,1].

Example: Orthogonal Bases for Polynomials (Python code)

```
import numpy as np

x = np.linspace(-1, 1, 900)
step_size= 2 / x.size
Phi = np.stack([1 + 0 * x, x, x ** 2,x ** 3, x ** 4], axis=-1)
Gram = step_size * (Phi.transpose() @ Phi)
Gram_root1 = np.linalg.cholesky(Gram_inv)
Phi_perp1 = Phi @ Gram_root1
w,v = np.linalg.eig(Gram)
Gram_root2 = (v * 1/np.sqrt(w[None,:])) @ v.transpose()
Phi_perp2 = Phi @ Gram_root2
Phi_legendre, r=np.linalg.qr(Phi, mode='reduced')
```

Example: Orthogonal Bases for Trigonometric Polynomials



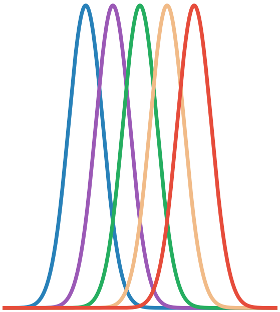
(a) Trigonometric monomials basis: $\varphi_0 = 1$, $\varphi_1 = \cos(t)$, $\varphi_2 = \cos^2(t)$, $\varphi_3 = \cos^3(t)$, $\varphi_4 = \cos^4(t)$.

(b) Orthogonalised basis (via EVD): $\varphi_0^\perp$, $\varphi_1^\perp$, $\varphi_2^\perp$, $\varphi_3^\perp$, $\varphi_4^\perp$.

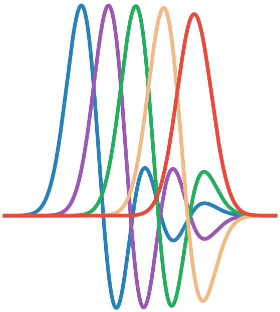
(c) Fourier basis (Gram-Schmidt): $\varphi_0^\perp \propto 1$, $\varphi_1^\perp \propto \cos(t)$, $\varphi_2^\perp \propto \cos(2t)$, $\varphi_3^\perp \propto \cos(3t)$, $\varphi_4^\perp \propto \cos(4t)$.

Figure: Orthogonalisation of the monomials basis on $[-\pi, \pi]$.

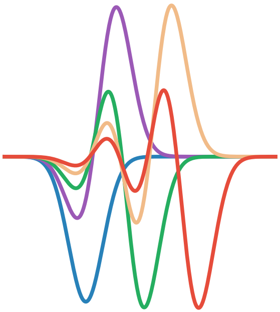
Example: Orthogonalisation of shifted Gaussians



(a) Basis of shifted Gaussians: $\varphi_0 = 1$, φ_1, φ_2, φ_3, φ_4.



(b) Orthogonalised basis (via Cholesky): $\varphi_0^\perp$, $\varphi_1^\perp$, $\varphi_2^\perp$, $\varphi_3^\perp$, $\varphi_4^\perp$.



(c) Orthogonalised basis (via Gram-Schmidt): $\varphi_0^\perp$, $\varphi_1^\perp$, $\varphi_2^\perp$, $\varphi_3^\perp$, $\varphi_4^\perp$.

Figure: Expansion w.r.t. basis of shifted Gaussians.

Orthogonal Projection Onto a Subspace

Theorem: (Orthogonal Projection Onto a Subspace)

Let $\{\varphi_k\}_{k\in\mathscr{I}}$ be a Riesz basis for a closed subspace $S_\mathscr{I} = \overline{\text{span}}(\{\varphi_k\}_{k\in\mathscr{I}})$ in a Hilbert space $\mathscr{H}$, with synthesis operator $\Phi_\mathscr{I} : \ell_2(\mathscr{I}) \to \mathscr{H}$. Then, for every $x \in \mathscr{H}$,

$$P_\mathscr{I} x = \Phi_\mathscr{I} \left(\Phi^*_\mathscr{I}\Phi_\mathscr{I}\right)^{-1} \Phi^*_\mathscr{I} x \tag{5}$$

is the orthogonal projection of x onto $S_\mathscr{I}$.

- When $\{\varphi_k\}_{k\in\mathscr{I}}$ is orthonormal, (5) simplifies to $P_\mathscr{I} x = \Phi_\mathscr{I}\Phi^*_\mathscr{I} x$.
- We can rewrite (5) as $P_\mathscr{I} x = \widetilde{\Phi}_\mathscr{I}\Phi^*_\mathscr{I} x$, where $\widetilde{\Phi}_\mathscr{I} = \Phi_\mathscr{I}\left(\Phi^*_\mathscr{I}\Phi_\mathscr{I}\right)^{-1}$ is the synthesis operator of the dual basis $\{\widetilde{\varphi}_k\}_{k\in\mathscr{I}}$ of $\{\varphi_k\}_{k\in\mathscr{I}}$. Note that $\widetilde{\Phi}_\mathscr{I}$ is a right inverse of $\Phi^*_\mathscr{I}$:
$$\Phi^*_\mathscr{I}\widetilde{\Phi}_\mathscr{I} = \Phi^*_\mathscr{I}\Phi_\mathscr{I}\left(\Phi^*_\mathscr{I}\Phi_\mathscr{I}\right)^{-1} = \mathrm{Id}.$$
- The projection residual $x - P_\mathscr{I} x$ is consequently orthogonal to $S_\mathscr{I}$:
$$\Phi^*_\mathscr{I}\left(x - P_\mathscr{I} x\right) = \Phi^*_\mathscr{I} x - \underbrace{\Phi^*_\mathscr{I}\widetilde{\Phi}_\mathscr{I}}_{=\mathrm{Id}}\Phi^*_\mathscr{I} x = 0 \quad \Longrightarrow \quad x - P_\mathscr{I} x \perp S_\mathscr{I}.$$

Bessel's Inequality

Since the projection $P_{\mathscr{I}}x$ and the residual $x - P_{\mathscr{I}}x$ are orthogonal, we can use Pythagorean theorem to deduce the so-called Bessel's inequality:

Bessel's inequality

Let $P_{\mathscr{I}}$ be an orthogonal projection operator as in (5). Then,

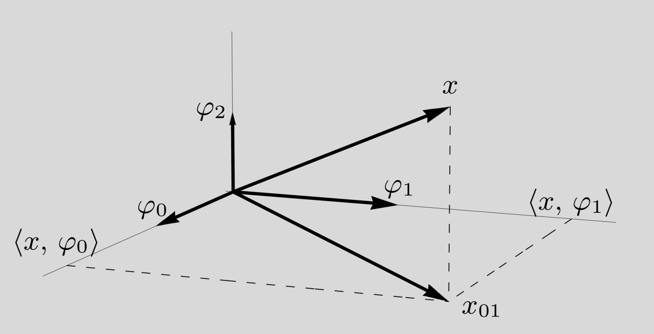
$$\|x\|^2 = \|P_{\mathscr{I}}x\|^2 + \|x - P_{\mathscr{I}}x\|^2 \geq \|P_{\mathscr{I}}x\|^2 = \langle \Phi^*_{\mathscr{I}}x, \widetilde{\Phi}^*_{\mathscr{I}}x \rangle = \sum_{k\in\mathscr{I}} \langle x, \varphi_k \rangle \overline{\langle x, \widetilde{\varphi}_k \rangle}, \qquad \forall x \in \mathscr{H}.$$

Bessel's inequality becomes an equality when $S_{\mathscr{I}} = \overline{\text{span}}(\{\varphi_k\}_{k\in\mathscr{I}}) = \mathscr{H}$. When $\{\varphi_k\}_{k\in\mathscr{I}}$ is orthonormal moreover, it simplifies into:

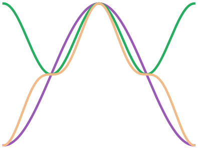
$$\|x\|^2 \geq \|P_{\mathscr{I}}x\|^2 = \sum_{k\in\mathscr{I}} |\langle x, \varphi_k \rangle|^2.$$

Geometrically speaking, Bessel's inequality tells us that orthogonal projections shrink the norm of their input. This is not true in general for oblique projections!

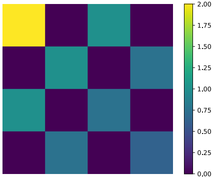
Illustration of Bessel's Inequality in $\mathbb{R}^3$



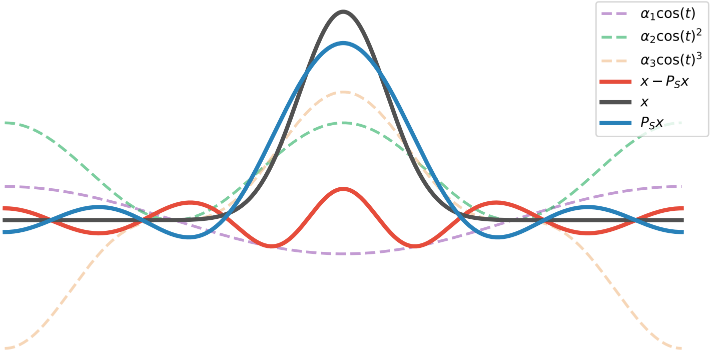
Example of Orthogonal Projection



(a) Basis functions $\{\cos(t), \cos(t)^2, \cos(t)^3\}$.



(b) Gramian.



(c) Orthogonal projection of a Gaussian $x(t) = \exp(-3t^2)$ onto $S = \text{span}\left(\{\cos(t), \cos(t)^2, \cos(t)^3\}\right)$. The residual $x - P_S x$ is orthogonal to S.

Figure: Example of orthogonal projection.

Oblique Projection with Biorthogonality

Theorem: (Oblique Projection with Biorthogonality)

Let $\{\varphi_k\}_{k\in\mathcal{I}}$ and $\{\tilde{\varphi}_k\}_{k\in\mathcal{I}}$ be two Riesz bases for some closed subspaces $S_{\mathcal{I}} = \overline{\text{span}}(\{\varphi_k\}_{k\in\mathcal{I}})$ and $\tilde{S}_{\mathcal{I}} = \overline{\text{span}}(\{\tilde{\varphi}_k\}_{k\in\mathcal{I}})$ of a Hilbert space $\mathcal{H}$. Assume further the biorthogonality condition:

$$\langle \varphi_i, \tilde{\varphi}_k \rangle = \delta_{i-k}, \qquad \forall i, k \in \mathcal{I}. \tag{6}$$

Then, for every $x \in \mathcal{H}$,

$$P_{\mathcal{I}} x = \Phi_{\mathcal{I}} \tilde{\Phi}^*_{\mathcal{I}} x \tag{7}$$

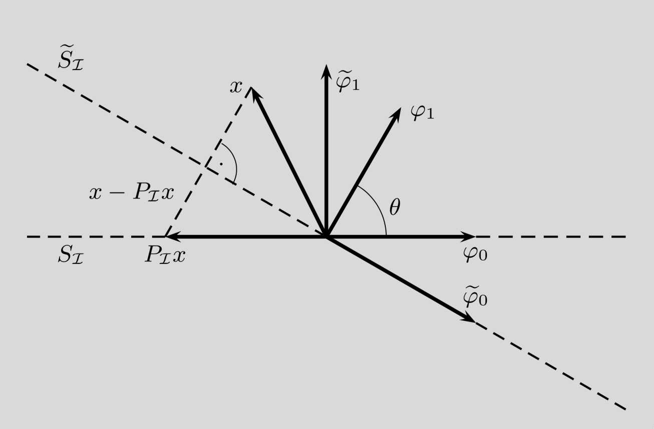
is an oblique projection of x onto $S_{\mathcal{I}}$ and

$$\tilde{P}_{\mathcal{I}} x = \tilde{\Phi}_{\mathcal{I}} \Phi^*_{\mathcal{I}} x \tag{8}$$

is is an oblique projection of x onto $\tilde{S}_{\mathcal{I}}$. Moreover, we have: $x - P_{\mathcal{I}} x \in \tilde{S}^{\perp}_{\mathcal{I}}$ and $x - \tilde{P}_{\mathcal{I}} x \in S^{\perp}_{\mathcal{I}}$.

The biorthogonality condition (6) implies that $\tilde{\Phi}^*_{\mathcal{I}} \Phi_{\mathcal{I}} = \Phi^*_{\mathcal{I}} \tilde{\Phi}_{\mathcal{I}} = \mathrm{Id}$, which shows that (7) and (8) are indeed projections.

Illustration of Oblique Projection with Biorthogonality $\mathbb{R}^2$



General Oblique Projections

Theorem: (General Oblique Projections)

Let $\{\varphi_k\}_{k\in\mathcal{I}}$ and $\{\psi_k\}_{k\in\mathcal{E}}$ be two Riesz bases for some closed subspaces $S_{\mathcal{I}} = \overline{\text{span}}(\{\varphi_k\}_{k\in\mathcal{I}})$ and $S_{\mathcal{E}} = \overline{\text{span}}(\{\psi_k\}_{k\in\mathcal{E}})$ of a Hilbert space $\mathcal{H}$. Then, for every $x \in \mathcal{H}$,

$$P_{\mathcal{E}} x = \Psi_{\mathcal{E}} \left(\Phi^*_{\mathcal{I}} \Psi_{\mathcal{E}}\right)^{\dagger} \Phi^*_{\mathcal{I}} x \tag{9}$$

is an oblique projection of x onto $\mathcal{R}\left(\Psi_{\mathcal{E}}(\Phi^*_{\mathcal{I}}\Psi_{\mathcal{E}})^{\dagger}\right) \subset S_{\mathcal{E}}$ and

$$P_{\mathcal{I}} x = \Phi_{\mathcal{I}} \left(\Psi^*_{\mathcal{E}} \Phi_{\mathcal{I}}\right)^{\dagger} \Psi^*_{\mathcal{E}} x \tag{10}$$

is an oblique projection of x onto $\mathcal{R}\left(\Phi_{\mathcal{I}}(\Psi^*_{\mathcal{E}}\Phi_{\mathcal{I}})^{\dagger}\right) \subset S_{\mathcal{I}}$.

When $\Phi^*_{\mathcal{I}}\Psi_{\mathcal{E}}$ is invertible (requires $\mathcal{I} = \mathcal{E}$) then the ranges of the oblique projections (9) and (10) coincide with $S_{\mathcal{E}}$ and $S_{\mathcal{I}}$ respectively.

Proof

$$A^+ A A^+ = A^+$$

$$P_\varepsilon = \Psi_\varepsilon \left(\Phi_\varepsilon^* \Psi_\varepsilon \right)^+ \Phi_\varepsilon^*$$

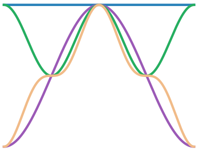
$$\left(\Phi_\varepsilon^* \Psi_\varepsilon \right)^+$$

$$P_\varepsilon^2 = \underbrace{\Psi_\varepsilon \left(\Phi_\varepsilon^* \Psi_\varepsilon \right)^+ \Phi_\varepsilon^*}_{P_\varepsilon} \underbrace{\Psi_\varepsilon \left(\Phi_\varepsilon^* \Psi_\varepsilon \right)^+ \Phi_\varepsilon^*}_{P_\varepsilon}$$

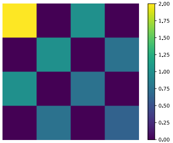
$$= \Psi_\varepsilon \left(\Phi_\varepsilon^* \Psi_\varepsilon \right)^+ \Phi_\varepsilon^* = P_\varepsilon$$

$$P_\varepsilon^* = \Phi_\varepsilon \left(\Phi_\varepsilon^* \Phi_\varepsilon \right)^+ \Psi_\varepsilon^* \neq P_\varepsilon$$

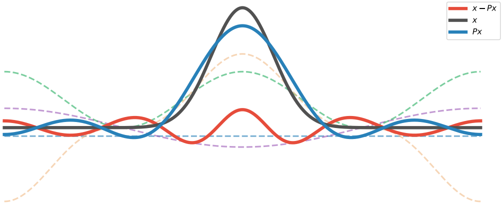
Oblique vs. Orthogonal Projection



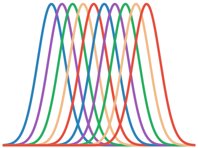
(a) Sampling functions $\Phi^* : \mathscr{H} \to \mathbb{R}^3$.



(c) Gramian $\Phi^*\Phi \in \mathbb{R}^{3\times 3}$.

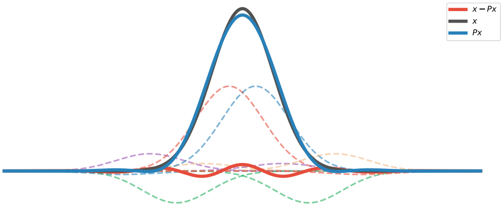


(e) Orthogonal projection $\Phi(\Phi^*\Phi)^{-1}\Phi^* x$ of $x(t) = \exp(-3t^2)$.



(b) Interpolating functions $\Psi : \mathbb{R}^{10} \to \mathscr{H}$.

(d) Gramian $\Phi^*\Psi \in \mathbb{R}^{3\times 10}$.



(f) Oblique projection $\Psi(\Phi^*\Psi)^\dagger\Phi^* x$ of $x(t) = \exp(-3t^2)$.

Matrix Representations of Linear Operators

We want to represent a bounded linear operator $A : \mathscr{H}_0 \to \mathscr{H}_1$ between two Hilbert spaces $\mathscr{H}_0$ and $\mathscr{H}_1$ as a matrix. To this end, we consider two Riesz bases $\{\varphi_k\}_{k\in\mathcal{K}_0}$ and $\{\psi_k\}_{k\in\mathcal{K}_1}$ for $\mathscr{H}_0$ and $\mathscr{H}_1$ respectively. Then, for any $y = \Psi\beta \in \mathscr{H}_1$ and $x = \Phi\alpha \in \mathscr{H}_0$ such that $y = Ax$ we have:

$$y = Ax \quad \Leftrightarrow \quad \Psi\beta = A\Phi\alpha \quad \Leftrightarrow \quad \underbrace{\tilde{\Psi}^*\Psi}_{=\mathrm{Id}}\beta = \tilde{\Psi}^* A\Phi\alpha \quad \Leftrightarrow \quad \beta = \Gamma\alpha,$$

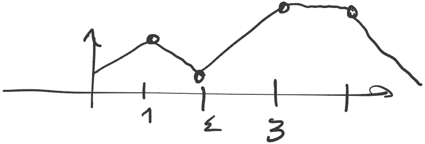
where $\tilde{\Psi}^*$ denotes the analysis operator associated to the dual basis $\{\tilde{\psi}_k\}_{k\in\mathcal{K}_1}$ of $\{\psi_k\}_{k\in\mathcal{K}_1}$.
The operator A can hence be *represented* as a (potentially infinite) matrix $\Gamma : \ell_2(\mathcal{K}_0) \to \ell_2(\mathcal{K}_1)$ given by:

$$\Gamma = \tilde{\Psi}^* A\Phi = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle A\varphi_{-1}, \tilde{\psi}_{-1}\rangle & \langle A\varphi_0, \tilde{\psi}_{-1}\rangle & \langle A\varphi_1, \tilde{\psi}_{-1}\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \tilde{\psi}_0\rangle & \boxed{\langle A\varphi_0, \tilde{\psi}_0\rangle} & \langle A\varphi_1, \tilde{\psi}_0\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \tilde{\psi}_1\rangle & \langle A\varphi_0, \tilde{\psi}_1\rangle & \langle A\varphi_1, \tilde{\psi}_1\rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} = (\Psi^*\Psi)^{-1}(\Psi^* A\Phi).$$

Matrix Representations of Linear Operators

$$\begin{tikzcd}
H_0 & H_1 \\
x \arrow[r, "A"] \arrow[d, bend right, "\Phi^{-1}"'] & y \arrow[d, bend right, "\Psi^{-1}"'] \\
\alpha \arrow[r, "\Gamma"'] \arrow[u, bend right, "\Phi"'] & \beta \arrow[u, bend right, "\Psi"'] \\
\ell^2(\mathcal{K}_0) & \ell^2(\mathcal{K}_1)
\end{tikzcd}$$

Example: Derivative Operator I



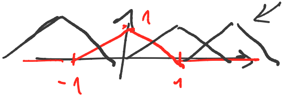
Let $A : \mathscr{H}_0 \to \mathscr{H}_1$ be the derivative operator from

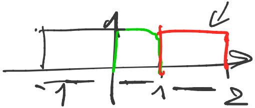
$\mathscr{H}_0$: space of piecewise-linear, continuous, finite-energy functions with breakpoints at integers

to

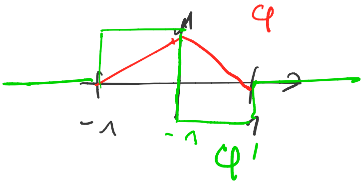
$\mathscr{H}_1$: space of piecewise-constant, finite-energy functions with breakpoints at integers.

- **Basis for $\mathscr{H}_0$:** $\{\varphi_k(t)\}_{k\in\mathbb{Z}} = \{\varphi(t-k)\}_{k\in\mathbb{Z}}, \quad \varphi(t) = \begin{cases} 1-|t|, & |t|<1; \\ 0, & \text{otherwise} \end{cases}$
- **Basis for $\mathscr{H}_1$:** $\{\psi_i(t)\}_{i\in\mathbb{Z}} = \{\chi_{[i,i+1)}(t)\}_{i\in\mathbb{Z}}$. This basis is orthogonal so $\widetilde{\Psi} = \Psi$.





Example: Derivative Operator II



We evaluate $\underbrace{\langle A\varphi_k, \psi_i \rangle}_{\Gamma}$ for all k and i.

$$A\varphi(t) = \varphi'(t) = \begin{cases} 1, & \text{for } -1 < t < 0; \\ -1, & \text{for } 0 < t < 1; \\ 0, & \text{for } |t| > 1, \end{cases}$$

Then $\langle A\varphi_0, \psi_i \rangle = \begin{cases} 1, & \text{for } i = -1; \\ -1, & \text{for } i = 0; \\ 0, & \text{otherwise.} \end{cases}$ and $\langle A\varphi_k, \psi_i \rangle = \begin{cases} 1, & \text{for } i = k-1; \\ -1, & \text{for } i = k; \\ 0, & \text{otherwise.} \end{cases}$

This yields

$$\Gamma = \begin{bmatrix} \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & -1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{-1} & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & -1 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Example: Derivative Operator III

Consider

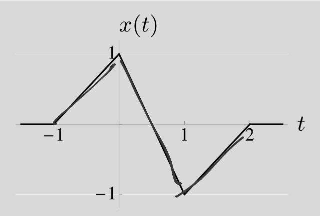
$$x(t) = \varphi(t) - \varphi(t-1)$$
$$= \Phi\underbrace{\left(\cdots, 0, \boxed{1}, -1, 0, \cdots\right)}_{=\alpha}.$$

Its derivative is given by

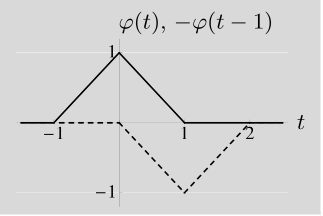
$$x'(t) = \psi(t+1) - 2\psi(t) + \psi(t-1)$$
$$= \Psi\underbrace{\left(\cdots, 0, 1, \boxed{-2}, 1, 0, \cdots\right)}_{=\beta}.$$

We have indeed (check it!)

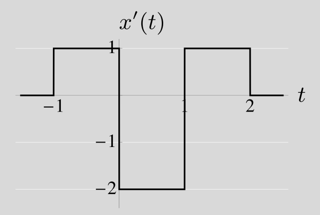
$$\beta = \Gamma\alpha.$$



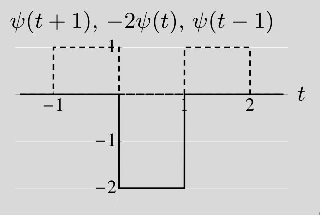
(g) Original function $x(t)$.



(h) Its decomposition in the $\mathscr{H}_0$ basis.



(i) Derivative function $x'(t)$.



(j) Its decomposition in the $\mathscr{H}_1$ basis.

References I