# Predictive Control of Fast Unstable and Nonminimum-phase Nonlinear Systems

K. Guemghar, B. Srinivasan, Ph. Mullhaupt, D. Bonvin
Laboratoire d'Automatique, École Polytechnique Fédérale de Lausanne,
CH-1015 Lausanne, Switzerland.
e-mail: kahina.guemghar@epfl.ch

**Abstract**

The problem of controlling fast, unstable, and nonminimum-phase systems is considered. With standard predictive control, the time required for optimization is typically larger than the sampling interval that is needed for stabilization of the fast dynamics. On the other hand, due to the nonminimum-phase behavior, control based on input-output feedback linearisation leads to unstable internal dynamics. In this paper, a cascade structure is proposed, with control based on input-output feedback linearisation forming the inner loop and predictive control the outer loop. Assuming high-gain feedback for the inner loop, a stability analysis of the global scheme is provided based on singular perturbation theory. The approach is illustrated *via* the simulation of an inverted pendulum system.

**Keywords:** Predictive control, Feedback linearization, Unstable systems, Nonminimum-phase systems, Nonlinear systems, Singular perturbation.

## 1 Introduction

Predictive control is an effective approach for tackling problems with constraints and nonlinear dynamics, especially when the analytical computation of the control law is difficult [8, 12]. This methodology is widely used in the process industry, where system dynamics are sufficiently slow to permit its implementation [11]. In contrast, applications to fast systems are rather limited since it is often not possible to complete the optimization within one sampling interval, the duration of which is limited by Nyquist's sampling theorem. When the fast dynamics are stable, the violation of the sampling limit leads to degradation in performance. However, when the fast dynamics are unstable, the system cannot even be stabilized without respecting this limit. The way predictive control can be used to stabilize fast unstable systems is considered in this paper.

The idea often used in the literature for the predictive control of fast unstable systems is to first design a feedback that pre-stabilizes the system [8, 13]. Predictive control is then applied to the pre-stabilized system. In this paper, input-output feedback linearization is proposed as a systematic way of designing a pre-stabilizer. This idea has been used in [10] with a special emphasis on input constraints. However therein, the issue of internal dynamics (that can be unstable) is not addressed.

To highlight the issue of internal dynamics, the control of nonminimum-phase systems is considered. For such systems, input-output feedback linearization leads to internal dynamics that are unstable [3]. Thus, in the control of fast unstable and nonminimum-phase systems, the two aspects of fast unstable dynamics and nonminimum-phase behavior prevent the application of standard predictive control and input-output feedback linearization, respectively. It will be shown in this paper that, though each of the methods cannot independently handle the class of systems considered, a combination of the two can be used satisfactorily.

A cascade structure involving feedback linearization and the stabilization of internal dynamics has been considered in [2]. However, no systematic procedure for stabilization is provided therein. In this paper, predictive control is used for that purpose.

Another contribution is the stability proof based on singular perturbation theory, for the application of which a time-scale separation is artificially introduced through the use of high-gain controllers for the linearized part.

The paper is organized as follows: Section 2 introduces basic concepts and notations in the fields of predictive control, singular perturbation, and feedback linearization. Section 3 develops the proposed cascade control scheme, while the stability analysis is in Section 4. Section 5 uses the example of an inverted pendulum on a cart to illustrate the proposed method, and Section 6 concludes the paper.

## 2 Preliminaries

### 2.1 Predictive Control

The attention in this paper is restricted to single-input single-output systems. Consider the nonlinear affine-in-input system represented by:

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0 \tag{1}$$

where $x \in R^n$ are the states, $u \in R$ the input, $x_0$ the initial conditions, and $f$ and $g$ the functions describing the system dynamics. Assume that $x = 0$ is an equilibrium point with $f(0) = 0$.

In predictive control, the current control action is obtained by solving, at each time instant $t$, a finite horizon open-loop optimal control problem:

$$\begin{aligned} \min_{u([t,t+T])} \; J = & \frac{1}{2} x(t+T)^T P\, x(t+T) \qquad (2)\\ & + \frac{1}{2} \int_t^{t+T} \left( x(\tau)^T Q\, x(\tau) + R\, u^2(\tau) \right) d\tau \\ s.t. \qquad & \dot{x} = f(x) + g(x)u, \qquad x(t) = x_t \\ & u(\cdot) \in \mathcal{U},\; x(\cdot) \in \mathcal{X},\; x(t+T) \in \mathcal{X}_f \end{aligned}$$

where $P$, $Q$, and $R$ are positive-definite weighting matrices of appropriate dimensions, $T$ the prediction horizon, $x_t$ the states measured at time instant $t$, $\mathcal{X}$ and $\mathcal{U}$ the sets of admissible states and inputs, and $\mathcal{X}_f \subset \mathcal{X}$ a closed set such that $0 \in \mathcal{X}_f$. Numerical optimization yields the control sequence, $u^*([t, t+T])$, of which only the first part, $u^*([t, t+\delta])$, is applied open-loop to the plant, where $\delta$ is the sampling period. Then, numerical optimization is repeated at every sampling instant.

### 2.2 Singularly-perturbed Systems

Consider a system that exhibits two-time-scale behavior, i.e., slow and fast dynamics, as given in:

$$\dot{\eta} = f_\eta(\eta, \xi, \epsilon) + g_\eta(\eta, \xi, \epsilon)u \tag{3}$$

$$\epsilon\dot{\xi} = f_\xi(\eta, \xi, \epsilon) + g_\xi(\eta, \xi, \epsilon)u \tag{4}$$

where $\epsilon > 0$ is a small parameter. As $\epsilon \to 0$, the dynamics of $\xi$ act quickly, thus leading to a time-scale separation, with $\eta$ and $\xi$ representing the slow and fast states, respectively. Such a separation can either represent the physics of the system or can be artificially created by the use of high-gain controllers. As $\epsilon \to 0$, $\xi$ can be approximated by its quasi-steady state $\bar{\xi} = \phi(\eta, u)$ obtained by solving $f_\xi(\eta, \xi, 0) + g_\xi(\eta, \xi, 0)u = 0$. So, the reduced (slow) system is given by:

$$\dot{\eta} = f_\eta(\eta, \phi(\eta, u), 0) + g_\eta(\eta, \phi(\eta, u), 0)u = \bar{f}_\eta(\eta, u) \tag{5}$$

Note that the reduced system (5) is not necessarily affine-in-input. One of the main results from the singular perturbation theory, which will be used in this paper, is presented next:

**Theorem 1 (Theorem 9.3 of [5])** *Assume that the following conditions are satisfied:*

- *The origin is an equilibrium point for (3)-(4) .*
- *$\phi(\eta, u)$ has a unique solution.*
- *The functions $f_\eta$, $f_\xi$, $g_\eta$, $g_\xi$, $\phi$ and their partial derivatives up to order 2 are bounded for $\xi$ in the neighborhood of $\bar{\xi}$.*
- *The origin of the boundary-layer system (4) is exponentially stable for all $\eta$.*
- *The origin of the reduced system (5) is exponentially stable.*

*Then, there exists $\epsilon^* > 0$ such that, for all $\epsilon < \epsilon^*$, the origin of (3)-(4) is exponentially stable.*

Next, the test for exponential stability is presented:

**Theorem 2 (Corollary 3.4 of [5])** *Given system (1), if there exists a Lyapunov function $V(x)$ and positive constants $c_1$, $c_2$, and $c_3$ such that $c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$ and $\dot{V}(x) \le -c_3\|x\|^2$, then the origin is exponentially stable.*

### 2.3 Input-output Feedback Linearisation

Consider the output $y = h(x)$, $y \in R$, $h(0) = 0$, for system (1). The relative degree, $r$, is the number of times the output has to be differentiated with respect to time before the input appears. If $r < n$, then system (1) can be feedback linearized into Byrnes-Isidori normal form [3] using the following steps:

1. Applying a feedback law that compensates the nonlinearities in the input-output behavior:

$$u = \frac{v - L_f^r h(x)}{L_f^{r-1} L_g h(x)} \tag{6}$$

   where $L_f N(x) = \frac{\partial N}{\partial x} f(x)$ is the Lie derivative of $N$. Since $r$ is the relative degree, $L_f^{r-1} L_g h(x) \neq 0$, $\forall x$ and $L_f^i L_g h(x) = 0, \forall i < r-1$.

2. Using the nonlinear transformation $z = T(x)$, $z = [y\ \dot{y} \cdots y^{(r-1)}\ \eta^T]^T$, with $\eta \in R^{n-r}$, the system can be transformed to:

$$y^{(r)} = v, \qquad \dot{\eta} = \mathcal{Q}(\eta, y, \dot{y}, \cdots, y^{(r-1)}, v) \tag{7}$$

   Here, the notation $\eta$ is used for the states corresponding to the internal dynamics. In (3), $\eta$ was used for the slow states. The reason for using the same notation is that it will be shown later that the internal dynamics are in fact slow. Note that the above transformation decouples the internal dynamics from the input-output behavior, i.e., $\eta$ has no effect on $y$.

## 3 Combining Feedback Linearisation and Predictive Control

### 3.1 Cascade Control Scheme

For fast unstable systems, since the time taken for optimization (2) is typically larger than the sampling interval, the goal is to transform the problem so that predictive control can be done at lower rate. Towards this end, the following procedure is proposed (Figure 1):

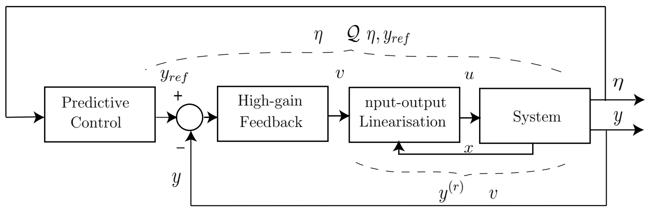


**Figure 1:** Cascade control scheme

1. *Input-output Linearization:* Using the input (6), the system is input-output feedback linearized to $y^{(r)} = v$, $\dot{\eta} = \mathcal{Q}(\eta, y, \dot{y}, \cdots, y^{(r-1)}, v)$ as in (7).

2. *High-gain Feedback:* High-gain state feedback is applied to the linearized input-output system:

$$v = \frac{K_1}{\epsilon^r}(y_{ref} - y) - \sum_{i=1}^{r-1} \frac{K_{i+1}}{\epsilon^{(r-i)}} y^{(i)} \qquad (8)$$

where $y_{ref}$ is the reference for the output, $\epsilon \to 0$ a small parameter, and $K_1, \cdots, K_r$ are coefficients of a Hurwitz polynomial. The gains are chosen this way since, for any choice of $\epsilon > 0$, the closed loop is stable and $\epsilon$ can be used as a single tuning parameter. The dynamics of the inner loop read:

$$\begin{aligned} y^{(r)} &= \frac{K_1}{\epsilon^r}(y_{ref} - y) - \sum_{i=1}^{r-1} \frac{K_{i+1}}{\epsilon^{(r-i)}} y^{(i)} && (9) \\ \dot{\eta} &= \mathcal{Q}(\eta, y, \dot{y}, \cdots, y^{(r-1)}, v) && (10) \end{aligned}$$

This system can be written in the form (3)-(4) by defining the fast variables $\xi_i = \epsilon^{(i-1)} y^{(i-1)}$, $i = 1, \ldots, r$:

$$\begin{aligned} \dot{\eta} &= \mathcal{Q}(\eta, \xi_1, \frac{\xi_2}{\epsilon}, \cdots, \frac{\xi_r}{\epsilon^{(r-1)}}, && (11) \\ & \quad \frac{1}{\epsilon^r}[K_1(y_{ref} - \xi_1) - \sum_{i=1}^{r-1} K_{i+1}\xi_{i+1}]) \\ \epsilon \dot{\xi}_i &= \xi_{i+1}, \quad i = 1, \ldots, r-1 && (12) \\ \epsilon \dot{\xi}_r &= K_1(y_{ref} - \xi_1) - \sum_{i=1}^{r-1} K_{i+1}\xi_{i+1} && (13) \end{aligned}$$

As $\epsilon \to 0$, the quasi-steady state assumption leads to $\bar{\xi} = [y_{ref} \ 0 \cdots 0]^T$. Then, the reduced system becomes:

$$\dot{\eta} = \mathcal{Q}(\eta, y_{ref}, 0, \cdots, 0) = \bar{\mathcal{Q}}(\eta, y_{ref}) \qquad (14)$$

3. *Predictive Control:* If the internal dynamics $\dot{\eta} = \bar{\mathcal{Q}}(\eta, y_{ref})$ is stabilizable using $y_{ref}$, then the predictive control problem that determines the stabilizing $y_{ref}$ is given by:

$$\begin{aligned} y^*_{ref} = \arg \min_{y_{ref}([t,t+T])} & \Big\{ \frac{1}{2}\eta(t+T)^T P \, \eta(t+T) && (15) \\ & + \frac{1}{2}\int_t^{t+T} \left(\eta(\tau)^T Q \, \eta(\tau) + R \, y^2_{ref}(\tau)\right) d\tau \Big\} \\ s.t. \quad & \dot{\eta} = \bar{\mathcal{Q}}(\eta, y_{ref}), \qquad \eta(t) = \eta_t \\ & y_{ref}(\cdot) \in \mathcal{Y}, \ \eta(\cdot) \in \mathcal{N}, \ \eta(t+T) \in \mathcal{N}_f \end{aligned}$$

where $\mathcal{Y}$ and $\mathcal{N}$ are the sets of admissible outputs and internal states, respectively, and $\mathcal{N}_f \subset \mathcal{N}$ is a closed set that contains the origin.

Note that, by choosing a sufficiently large gain for the input-output part, a two-time-scale behavior has been artificially created. The mathematical time-scale separation that is introduced, normally follows the physics of the system. If the physics of the system are not respected, the assumption on the stabilizability of the internal dynamics using $y_{ref}$ is not verified.

### 3.2 Dealing with Fast Zeros

The aforementioned procedure is useful only when (15) can be solved at a lower rate compared to (2). This means that the poles of the linearized internal dynamics should be slower than those of the linearized original dynamics. Since the context is a nonlinear one, the discussion of zeros and poles in the sequel is always for the linearized approximation. In fact, the poles of the internal dynamics correspond to the zeros of the original dynamics. Thus, this procedure is beneficial only when the original system has slow zeros.

Fast zeros can often be neglected without affecting the performance [1, 9]. Thus, it is proposed to eliminate the fast zeros and thereby deal only with a system that has slow zeros. As shown in [9], elimination of fast zeros imposes a lower bound on the choice of $\epsilon$.

One of the methods to eliminate fast zeros is to use the pseudo-relative degree $r^* = n - n^*_z$ in (6), instead of the relative degree $r$, where $n^*_z$ is the number of slow zeros that have to be retained. Thus, $\eta$ is of dimension $n^*_z$, and the internal dynamics contain only the slow system zeros. Then, the proposed methodology can be applied effectively.

## 4 Stability Analysis

The stability of the cascade scheme is discussed in this section. The key idea is the same as with singularly perturbed systems. If both the fast and the slow subsystems are exponentially stable, then the combination is also exponentially stable. The reason for insisting on exponential stability is as follows: If a system is exponentially stable, then there exists a margin in the reduction of the Lyapunov function, $\dot{V} \leq -c_3\|x\|^2 \leq 0$ (converse of Theorem 2). This margin, in turn, can be used to accommodate the perturbations.

The following logic will be used in this section: (i) First, exponential stability of the linearized system with high-gain controller is established. (ii) Next, exponential stability of the predictive control scheme is addressed. (iii) Then, these two pieces are combined together to provide a proof of stability for the cascade system.

**Proposition 1** *The system (12)-(13), with $K_1, \cdots, K_r$ chosen as coefficients of a Hurwitz polynomial, is exponentially stable. The following Lyapunov function can be proposed for the system:*

$$V = \frac{1}{2}\xi^T Q \xi, \quad Q = \int_0^\infty e^{A^T t} M e^{At} dt \qquad (16)$$

*where $M \in R^{r \times r}$ is a positive definite matrix and*

$$A = \frac{1}{\epsilon}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -K_1 & -K_2 & \cdots & \cdots & -K_r \end{bmatrix} \qquad (17)$$

**Proof:** System (12)-(13) can be written as: $\dot{\xi} = A\xi$, with the matrix $A$ given in (17). Considering the Lyapunov function candidate $V$ given in (16), its derivative reads:

$$\dot{V} = \frac{1}{2}\xi^T(A^T Q + QA)\xi \qquad (18)$$

For $Q$ defined in (16), it can be worked out as in [14] that $A^T Q + QA = -M$. Thus,

$$\dot{V} = -\frac{1}{2}\xi^T M \xi \qquad (19)$$

Since $M$ is positive definite, from Theorem 2, (12)-(13) is exponentially stable. ∎

The conditions that guarantee stability for the predictive control approach are provided in the following theorem [4, 6]:

**Proposition 2** *If (i) the system (14) is stabilizable via the choice of $y_{ref}$, (ii) $\eta = 0$, $y_{ref} = 0$ correspond to an equilibrium point, (iii) $P$, $Q$, and $R$ used in (15) are positive definite, and (iv) $\eta^T P \eta$ is a local control Lyapunov function, i.e., there exists a reachable set $\mathcal{N}_f$ such that $\forall \eta \in \mathcal{N}_f$, the optimal input, $y^*_{ref}$, computed by solving (15) satisfies:*

$$\eta^T P \bar{\mathcal{Q}}(\eta, y^*_{ref}) + \eta^T Q \eta + R\, {y^*_{ref}}^2 \leq 0 \qquad (20)$$

*then, the controller (15) stabilizes the system (14) exponentially.*

**Proof:** Consider the Lyapunov function candidate

$$W = \min_{y_{ref}} \frac{1}{2}\eta(t+T)^T P\, \eta(t+T) + \frac{1}{2}\int_t^{t+T} \left(\eta(\tau)^T Q\, \eta(\tau) + R\, y_{ref}^2(\tau)\right) d\tau \qquad (21)$$

Then, the time derivative of $W$ is given by:

$$\begin{aligned} \dot{W} \;=\; & -\eta(t)^T Q \eta(t) - R\, y^*_{ref}(t)^2 \\ & +\eta(t+T)^T Q \eta(t+T) + R\, y^*_{ref}(t+T)^2 \\ & +\eta(t+T)^T P \bar{\mathcal{Q}}(\eta(t+T), y^*_{ref}(t+T)) \end{aligned} \qquad (22)$$

Under the hypothesis that (20) is satisfied, the sum of the last three terms of (22) is negative. So, $\dot{W} \leq -\eta^T Q \eta$, with $Q$ being positive definite. Thus, the assumptions of Theorem 2 are satisfied and, (14) with controller (15) is exponentially stable. ∎

**Theorem 3** *For system (1), consider a controller where $y_{ref}$ is obtained by solving the optimization problem (15) and the input is computed using (6) and (8). If $P$, $Q$, and $R$ of (15) are positive definite and satisfies (20), then there exists an $\epsilon > 0$ that would exponentially stabilize (1).*

**Proof:** The following observations can be made:

- The origin ($\xi = 0$, $\eta = 0$, $y_{ref} = 0$) is an equilibrium point for the subsystems (12)-(13) and (14).
- The set of equations resulting from the quasi-steady state assumption ($\epsilon = 0$ in equations (12)-(13)) has a unique solution $\bar{\xi} = [y_{ref}\ 0 \cdots 0]^T$. Furthermore, as a result of the predictive control, $y_{ref}$ is a function of $\eta$.
- $\mathcal{Q}$ and its partial derivatives up to order 2 are bounded for $\xi$ in the neighborhood of $\bar{\xi}$.
- From Proposition 1, the origin of the boundary-layer system (12)-(13) is exponentially stable $\forall\, \eta$.
- From Proposition 2, the origin of the reduced system (14) is exponentially stable.

Since all hypotheses of Theorem 1 are satisfied, it can be concluded that there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the origin of (1) is exponentially stable. ∎

## 5 Example: Inverted Pendulum on a Cart

### 5.1 The Model

The model of the inverted pendulum on a cart can be found in many references (e.g. [7]) and is given by:

$$m\ddot{p} + \mu\ddot{\theta}\cos\theta - \mu\dot{\theta}^2\sin\theta = u \qquad (23)$$
$$\mu\ddot{p}\cos\theta + J\ddot{\theta} - \mu g\sin\theta = 0 \qquad (24)$$

where $p$ is the position of the cart, $\theta$ the angle between the vertical upright position and the pendulum (positive clockwise), $m$ the total mass of the system, $\mu = m_p l_p/2$ with $m_p$ and $l_p$ being the mass and length of the pendulum, respectively, $J = J_p + m_p l_p^2/4$ with $J_p$ the inertia of the pendulum, $g$ the gravity, and $u$ the force applied to the cart. Note that the friction is neglected in this model. The equations (23)-(24) can be rearranged to read:

$$\ddot{\theta} = \frac{m\mu g\sin\theta - \mu\cos\theta(u + \mu\dot{\theta}^2\sin\theta)}{(mJ - \mu^2 cos^2\theta)} \qquad (25)$$
$$\ddot{p} = \frac{J(u + \mu\dot{\theta}^2\sin\theta) - \mu^2 g\sin\theta\cos\theta}{(mJ - \mu^2 cos^2\theta)} \qquad (26)$$

The initial conditions are $p(0) = \dot{p}(0) = \dot{\theta}(0) = 0$ and $\theta(0) = -\pi$, the downward position for the pendulum. The system parameters are given in Table 1.

| | | |
|---|---|---|
| $m$ | 0.3235 | kg |
| $\mu$ | $1.3625 \times 10^{-3}$ | kg m |
| $J$ | $1.5265 \times 10^{-4}$ | kg m$^2$ |
| $g$ | 9.81 | m/s$^2$ |

**Table 1:** System parameters

### 5.2 Cascade Control

Considering $\theta$ as the output, the three steps mentioned in Section 3 are followed:

- The relative degree of system (23)-(24) is $r = 2 < 4$. The input to be applied for input-output linearization is:

$$u = \frac{m\mu g\sin\theta - (mJ - \mu^2 cos^2\theta)v}{\mu\cos\theta} - \mu\dot{\theta}^2\sin\theta \qquad (27)$$

  with which the dynamic system becomes:

$$\ddot{\theta} = v \qquad (28)$$
$$\ddot{p} = \frac{1}{\cos\theta}(g\sin\theta - \frac{J}{\mu}v) \qquad (29)$$

- The high-gain feedback is given by:

$$v = \frac{K_1}{\epsilon^2}(\theta^*_{ref} - \theta) - \frac{K_2}{\epsilon}\dot{\theta} \qquad (30)$$

  The quasi-steady state assumption leads $\theta = \theta_{ref}$, $\dot{\theta} = \ddot{\theta} = v = 0$. So, the reduced system is :

$$\ddot{p} = g\tan(\theta_{ref}) \qquad (31)$$

- $\theta_{ref}(p, \dot{p})$ that stabilizes the internal dynamics (31) is computed by solving the optimization problem:

$$\theta^\star_{ref} = \arg\min_{\theta_{ref}([t,t+T])} \left\{ \frac{1}{2} \begin{bmatrix} p & \dot{p} \end{bmatrix} P \begin{bmatrix} p \\ \dot{p} \end{bmatrix} (t+T) + \frac{1}{2}\int_t^{t+T} \begin{bmatrix} p & \dot{p} \end{bmatrix} Q \begin{bmatrix} p \\ \dot{p} \end{bmatrix} (\tau) + R\,\theta^2_{ref}(\tau)d\tau \right\} \qquad (32)$$

Instead of $\theta$, if $p$ had been considered as the output, the reduced internal dynamics would be: $\ddot{\theta} = \frac{\mu g}{J}\sin\theta$. Due to the absence of $p_{ref}$ in the internal dynamics, $p_{ref}$ cannot be used for its stabilization and the cascade scheme cannot be applied. Also, it can be be reasoned with the physics of the system that the dynamics of the pendulum are much faster than those of the cart, while the cascade scheme with $p$ as the output tries to impose the contrary. Thus, the proposed methodology cannot be applied with $p$ as the output.

### 5.3 Simulation Results

In simulation, the standard predictive control scheme is compared with the cascade scheme. For the standard predictive control (2), the parameters $R = 100$, $Q = \begin{bmatrix} 50 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$, $P = \begin{bmatrix} 265 & 27 & 7.5 & 20 \\ 27 & 2.8 & 0.8 & 2.1 \\ 7.5 & 0.8 & 2.8 & 3.9 \\ 20 & 2.1 & 3.9 & 10 \end{bmatrix}$, and $T = 0.05\,\text{s}$ are used. $P$ is computed by solving the algebraic Ricatti equation of the linear quadratic regulation problem of the linearized system [8]. The choice of the time between two optimizations ($\delta$) is dictated by the Nyquist's sampling theorem, $\delta = 0.01\,\text{s}$.

For the cascade scheme, the parameters chosen for the inner loop are: $K_1 = 1$, $K_2 = 2$, and $\epsilon = 0.1$. For the outer loop, the following choice is made: $R = 1000$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 2.91 & 3.22 \\ 3.22 & 9.37 \end{bmatrix}$, and $T = 3\,\text{s}$. As before, $P$ is computed by solving the algebraic Ricatti equation. Here, $\delta$ is chosen from an implementation perspective, $\delta = 1\,\text{s}$.

The simulation results for the standard and cascade schemes are presented in Figure 2. The system response with either of the two schemes is quite similar. In the cascade scheme, the reference angle is adjusted every second so as to bring the cart back to the origin. The pendulum dynamics are much faster and follows the changes in its reference as can be seen in Figure 3.

An important point to note is that the reoptimization frequency of the cascade scheme (once every 1s) is much

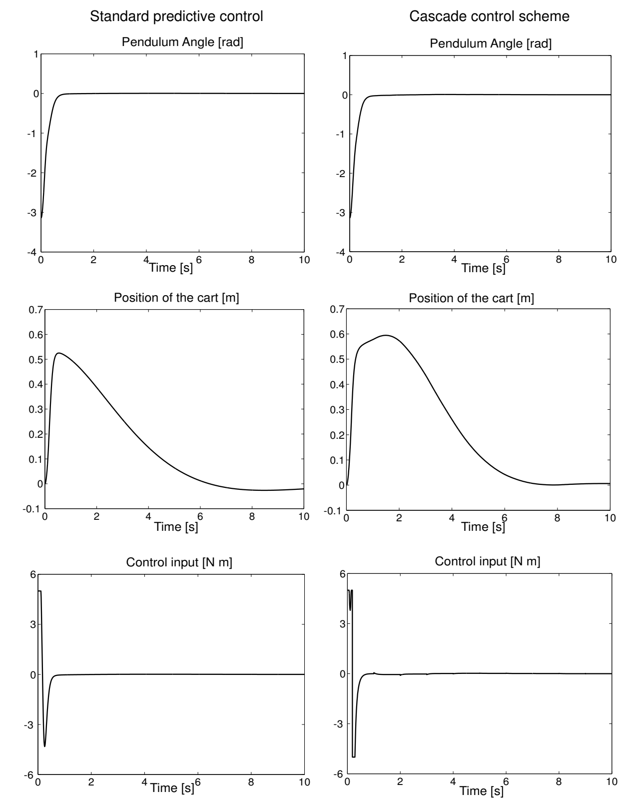


**Figure 2:** Standard predictive control of an inverted pendulum on a cart

smaller than that of the standard scheme (100 times per second). Despite such a large factor in the frequency of reoptimization, similar performance can be obtained due to the feedback provided by the inner loop that consists of input-output feedback linearization and high-gain feedback. The reoptimization frequency can be lowered even further, if a larger excursion and slower cart response is acceptable. Thus, the main advantage of the proposed approach is the reduction in reoptimization frequency and thereby, the computational requirements.

In addition, it was much easier to tune the parameters of the cascade scheme than those of the standard predictive control, especially the matrix associated with the terminal cost $P$. The cascade scheme works well for a wide range of $P$, whereas solving a Ricatti equation was necessary in the case of standard predictive control, especially when the prediction horizon is small.

## 6 Conclusion

This paper has presented a cascade scheme that combines input-output feedback linearization and predictive control. From a feedback linearization point of view, this scheme proposes an elegant way of handling the issue of unstable internal dynamics. From a predictive control point of view, the reoptimization frequency

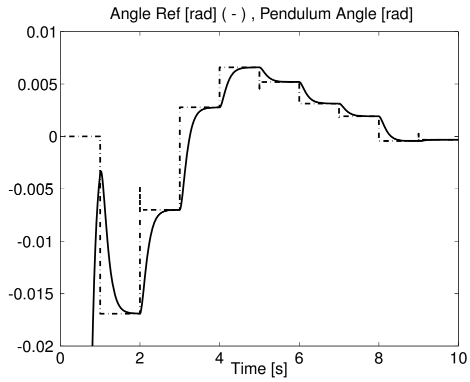


**Figure 3:** $\theta_{ref}$ and $\theta$ in the cascade control scheme

can be considerably reduced, thereby making its implementation much easier. A stability analysis of the cascade scheme has been provided based on singular perturbation theory. The results obtained in the simulation of an inverted pendulum on a cart are indeed very promising.

## References

[1] S. Gopalswamy and J. Hedrick. Tracking nonlinear non-minimum phase systems using sliding control. *International Journal of Control*, 57(5):1141–1158, 1993.

[2] R. Gurumoorthy. *Output Tracking of Nonminimum-Phase Nonlinear Systems.* PhD thesis, University of California at Berkeley, 1992.

[3] A. Isidori. *Nonlinear Control Systems.* Springer-Verlag, 1989.

[4] A. Jadbabaie, J. Yu, and J. Hauser. Unconstrained receding horizon control of nonlinear systems. In $38^{th}$ *Conference on Decision and Control*, pages 3376–3381, Phoenix-Arizona, 1999.

[5] H. K. Khalil. *Nonlinear Systems.* Prentice Hall, New Jersey, 1996.

[6] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

[7] F. Mazenc and L. Praly. Adding integrations, saturated controls, and stabilisation of feedforward systems. *IEEE Transactions on Automatic Control*, 41(11):1559–1578, 1996.

[8] M. Morari and J. H. Lee. Model predictive control: Past, present, and future. *Comp. Chem. Eng.*, 23:667–682, 1999.

[9] P. Mullhaupt. *Analysis and Control of Underactuated Mechanical Nonminimum-Phase Systems.* PhD thesis, $N^o$ 2024 , École Polytechnique Fédérale de Lausanne-Switzerland, 1999.

[10] V. Nevistić and M. Morari. Constrained control of feedback-linearizable systems. In *European Control Conference*, pages 1726–1731, Rome, 1995.

[11] S. J. Qin and T. A. Badgwell. An overview of industrial model predictive technology. In *Chemical Process Control V*, pages 232–256, Tahoe City, CA, 1997.

[12] J. B. Rawlings, E. S. Meadows, and K. R. Muske. Nonlinear model predictive control: A tutorial and survey. In *IFAC ADCHEM'94*, pages 185–197, Kyoto, Japan, 1994.

[13] E. Ronco, B. Srinivasan, J. Y. Favez, and D. Bonvin. Predictive control with added feedback for fast nonlinear systems. In *European Control Conference*, pages 3167–3172, Porto, 2001.

[14] W. J. Rugh. *Linear System Theory*. Prentice Hall, New Jersey, 1989.