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On the Nonminimum-phase Characteristics of Two-link Underactuated Mechanical Systems

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Abstract

Underactuated mechanical systems are those which possess fewer actuators than the number of degrees of freedom. It is shown that, except for some pathological cases, exclusively kinetic. two-link underactuated mechanisms are of nonminimum phase when the natural outputs are considered.
Keywords: Underactuated mechanical system, Nonminimum-phase system.

1 Introduction

Underactuated mechanical systems have recently gained research attention due to the variety of new problems they have generated [5]. In most cases, these systems are not feedback linearizable, and sophisticated control methodologies need to be employed.

In this paper, the difficulty encountered in controlling these systems is attributed to the nonminimum phase characteristics. The nonminimum-phase property is an input-output property well understood in linear systems and associated with right-half-plane transmission zeros. However, in the nonlinear scenario, since the notion of transmission zeros does not hold, nonminimum-phase systems are defined based on the stability of the zero (internal) dynamics [1].

As mentioned above, the nonminimum-phase nature is an input-output characteristic and hence the outputs have to be properly defined. In this study, we will consider only natural outputs, which are the generalized coordinates used in the standard robotics literature [4]. On the other hand, it is possible that there exists outputs for which the system is minimum phase. This issue, however, is not addressed here.

The link between underactuated systems and nonminimum-phase systems has not been rigorously established. The advantage of such a link is that the techniques used in either fields [2] can be exchanged to shed more light on the subject. Along that direction, the conjecture which forms the basis of this work is as follows: "a wide class of non-pathological underactuated systems possess nonminimum phase characteristics".

As a step towards the goal, the analysis in this paper is confined to two-link underactuated systems evolving in the absence of gravity and friction (exclusively kinetic system). It is proven that these systems with the natural outputs are nonminimum phase except for some rare situations.

Section 2 introduces certain preliminaries, while instability of a generic system is shown in Section 3. In Section 4, it is shown that the internal dynamics of underactuated two-link manipulators have the generic form described in Section 3 and, hence, is unstable. A few examples are presented in Section 5 which is followed by conclusions.

2 Preliminaries

In this section we will recall definitions of stability and instability in the sense of Lyapunov and a general theorem on instability. The definition of nonlinear nonminimum-phase systems will be given, and we will finish this section by presenting the dynamics of exclusively kinetic Lagrangian systems.

2.1 Stability

In the following, the stability definitions will be given with respect to the origin. This does not impart on generality since any equilibrium point can be handled using an appropriate translation of coordinates. Consider the system

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x \in R^n \tag{1}$$

where $f(0) = 0$ (i.e., $x = 0$ is an equilibrium point). Let the solution to the above differential equation starting from the initial condition x_0 and evaluated at time t be denoted by $x(t, x_0)$.

Definition 1 *(Stability about the origin [7]) The system (1) is Lyapunov stable if,* $\forall \epsilon > 0$, $\exists \delta > 0$ *such* *that,* $\|x_0\| < \delta \Longrightarrow \|x(t, x_0)\| < \epsilon, \forall\, t > 0.$

If the system does not satisfy definition 1, it is unstable. The result presented next generalizes Lyapunov's instability theorem.

Theorem 1 *(Chetaev [7]) The system (1) has an unstable equilibrium at $x = 0$, if there exist is a C^1 function $V : R^n \to R$, a ball $\mathcal{B}_r$, an open set $\Omega \subseteq \mathcal{B}_r$ and a function γ of class $\mathcal{K}$ such that:*

- $0 < V(x) < \infty,\ \forall\, x \in \Omega,$
- $0 \in \delta\Omega$
- $\dot{V}(x) \geq \gamma(\|x\|),\ \forall\, x \in \Omega,$
- $V(x) = 0,\ \forall\, x \in \delta\Omega \cap \mathcal{B}_r$

This theorem states that if the initial condition is in Ω, then the system can only escape Ω through the exterior of the ball $\mathcal{B}_r$. Since this happens for any initial condition arbitrarily close to the origin, the system is unstable.

2.2 Nonminimum Phase Systems

Since we will be concerned with two-degree of freedom underactuated mechanical systems, it is sufficient to state the definitions of zero dynamics and nonlinear nonminimum-phase systems for single-input single-output systems. Note that this definition extends easily to the multi-input multi-output setting.

Definition 2 *(Zero Dynamics [1]) Consider the control-affine nonlinear system:*

$$\dot{x} = f(x) + g(x)\, u, \quad y = h(x), \quad x(0) = x_0 \tag{2}$$

where $x \in R^n$ and $u, y \in R$. Now suppose there exist a maximal manifold Z^* having the following properties:

1. *$h(x) = 0, \forall\, x \in Z^*$;*
2. *$\exists$ a unique $u^*(x)$ $\forall x \in Z^*$, such that $f(x) + g(x)\, u^*(x)$ is tangent to Z^*;*

Then the zero dynamics of the system is given by

$$\dot{x} = f^*(x) = f(x) + g(x)\, u^*(x), \qquad x \in Z^* \tag{3}$$

Definition 3 *(Minimum-phase Systems [1]) The system (2) is said to be minimum phase at x_e, if x_e is a stable equilibrium point of $f^*(x)$. Otherwise, the system is nonminimum phase at x^e.*

2.3 Exclusively kinetic rigid body dynamics

Definition 4 *(Degrees of Freedom) [3] Generalized coordinates are a set of scalars which unambiguously describes the mechanical configuration. The number of degree of freedom of the system is the number of generalized coordinates minus the number of independent equations of constraint.*

In the following, we will be concerned exclusively with unconstrained systems. Thus the number of degrees of freedom is equal to the number of generalized coordinates.

Definition 5 *(Underactuated mechanical system). A system is underactuated if the system possesses fewer independent actuators than its number of degrees of freedom.*

Definition 6 *(Two-link underactuated mechanical system) A two-link mechanical system is a set of two rigid bodies connected to each other by an articulation. A two-link underactuated mechanical system is a two-link system which is underactuated.*

In a two-link underactuated mechanical system, there is only one generalized force that is associated with the first coordinate, q_1, or with the second coordinate, q_2. Thereby, one of the coordinates is left unactuated.

Definition 7 *(Natural Outputs) The natural outputs of a two-link mechanical system are: (i) the coordinate defining the position of the first link with respect to the base, and (ii) the second link with respect to the first.*

Note that the natural outputs are also the generalized coordinates. The systems that we will consider in the following are two-link underactuated mechanical systems without gravity or friction (termed the exclusively kinetic case). Then, the rigid body dynamics are given by

$$\ddot{q} = -\begin{bmatrix} \dot{q}^T \Gamma^1 \dot{q} \\ \dot{q}^T \Gamma^2 \dot{q} \end{bmatrix} + D^{-1}\tau \tag{4}$$

where $q = [q_1, q_2]^T$ and Γ^m, $m = 1, 2$ are the Christoffel symbols of the second kind:

$$\Gamma^m_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial d_{ki}}{\partial q_j} + \frac{\partial d_{kj}}{\partial q_i} - \frac{\partial d_{ij}}{\partial q_k}\right) n_{mk} \tag{5}$$

with n_{mk} denoting elements of the inverse of the inertia matrix $N = D^{-1}$. Notice that the Christoffel symbols are symmetric $\Gamma^m_{ij} = \Gamma^m_{ji}$ from the symmetry of the inertia matrix $d_{ij} = d_{ji}$. In [6], certain properties of the inertia matrix are explored. The results that will be used here are listed below.

Property 1 *(Properties of inertia matrix)*

1. *no element of the inertia matrix depends on* q_1
2. d_{22} *does not depend on* q_2.

3 A Result on Instability

It will be shown in the next section that internal dynamics of two-link underactuated systems have a specific structure due to the centrifugal terms. The instability of such a generic system is studied here.

Lemma 1 *Let the system be described by*

$$\ddot{\xi} = k(\xi)\dot{\xi}^2 \tag{6}$$

where $k(\xi)$ *is analytic outside a finite set in the neighborhood of the equilibrium point.* $[\xi, \dot{\xi}]^T = 0$. *Then (6) is Lyapunov unstable around* $[\xi, \dot{\xi}]^T = 0$.

Proof : When $k(\xi) = 0$ over an open interval around the origin, the system reads $\ddot{\xi} = 0$, which is trivially unstable. Considering the case where $k(\xi)$ is non-zero and analytic outside a finite set, there exists an $\epsilon > 0$ such that $k(\xi)$ is bounded and positive (without loss of generality), in the interval $\xi \in (0, \epsilon)$. From the openness of this interval, there exists an arbitrarily small $\epsilon_1 > 0$ such that $k(\xi)$ is positive and bounded for ξ such that $\epsilon_1 \leq \xi < \epsilon$. Define the Lyapunov function candidate:

$$V(\xi, \dot{\xi}) = \dot{\xi}^2 \int_{\epsilon_1}^{\xi} k(\eta) d\eta \tag{7}$$

Define the open set $\Omega = \{(\xi, \dot{\xi}) \mid \dot{\xi} > 0, \xi > \epsilon_1, \|[\xi, \dot{\xi}]^T\| < \epsilon\}$. Notice that on the boundaries $\delta\Omega_1 = \{(\xi, \dot{\xi}) \mid \dot{\xi} = 0\}$ and $\delta\Omega_2 = \{(\xi, \dot{\xi}) \mid \xi = \epsilon_1\}$, $V(\xi, \dot{\xi}) = 0$. Furthermore, $V(\xi, \dot{\xi}) > 0$, $\forall [\xi, \dot{\xi}]^T \in \Omega$. Now,

$$\begin{aligned} \dot{V}(\xi, \dot{\xi}) &= k(\xi)\dot{\xi}^3 + 2\dot{\xi}\ddot{\xi} \int_{\epsilon_1}^{\xi} k(\eta) d\eta \\ &= k(\xi) \left(1 + 2\int_{\epsilon_1}^{\xi} k(\eta) d\eta\right) \dot{\xi}^3 \end{aligned} \tag{8}$$

This clearly shows that $\dot{V}(\xi, \dot{\xi}) > 0$, $\forall [\xi, \dot{\xi}]^T \in \Omega$. Thus, the hypotheses of Chetaev's theorem are satisfied, and the system (6) is unstable.

4 Two-link underactuated mechanical systems

The dynamics of the two-link mechanical underactuated system are given by:

$$\ddot{q}_1 = -\Gamma_{11}^1 \dot{q}_1^2 - 2\Gamma_{12}^1 \dot{q}_1 \dot{q}_2 - \Gamma_{22}^1 \dot{q}_2^2 + n_{11}\tau_1 + n_{12}\tau_2 \tag{9}$$

$$\ddot{q}_2 = -\Gamma_{11}^2 \dot{q}_1^2 - 2\Gamma_{12}^2 \dot{q}_1 \dot{q}_2 - \Gamma_{22}^2 \dot{q}_2^2 + n_{21}\tau_1 + n_{22}\tau_2 \tag{10}$$

	$\tau_1 \neq 0$	$\tau_2 \neq 0$
q_1	$\ddot{q}_2 = 0$	$\ddot{q}_2 = -\frac{1}{d_{12}}\left(\frac{\partial d_{12}}{\partial q_2}\right)\dot{q}_2^2$
q_2	$\ddot{q}_1 = \frac{1}{d_{12}}\left(\frac{1}{2}\frac{\partial d_{11}}{\partial q_2}\right)\dot{q}_1^2$	$\ddot{q}_1 = 0$

Table 1: Equations of internal dynamics

where either $\tau_1 \neq 0$ or $\tau_2 \neq 0$. In this section, we will detail the four possibilities for a two-link underactuated mechanical system with natural outputs. The actuator can either be on the first or second coordinate. For each case, the output can either be chosen as the first or second coordinate (q_1 or q_2). We are then confronted with four cases. In all four cases, the internal dynamics will be derived when the controlled output is maintained constant (rather than equal to zero as the definition of the zero dynamics imposes).

Case 1 ($\tau_1 \neq 0$, $y = q_1$): In this case, the input τ_1 which guarantees evolution on the manifold $\dot{q}_1 = 0$ is given by $\tau_1 = \frac{\Gamma_{22}^1}{n_{11}}\dot{q}_2^2$. This induces the following dynamics on the other coordinate

$$\ddot{q}_2 = \left(-\Gamma_{22}^2 + \Gamma_{22}^1 \frac{n_{21}}{n_{11}}\right)\dot{q}_2^2 = -\frac{1}{2}\left(\frac{1}{d_{22}}\frac{\partial d_{22}}{\partial q_2}\right)\dot{q}_2^2 \tag{11}$$

Case 2 ($\tau_1 \neq 0$, $y = q_2$): Considering the second coordinate as the controlled output, $\tau_1 = \frac{\Gamma_{11}^2}{n_{21}}\dot{q}_1^2$ and the internal dynamics on the coordinate q_1 are given by:

$$\ddot{q}_1 = \left(-\Gamma_{11}^1 + \Gamma_{11}^2 \frac{n_{11}}{n_{21}}\right)\dot{q}_1^2 = -\frac{1}{d_{12}}\left(\frac{\partial d_{12}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2}\right)\dot{q}_1^2 \tag{12}$$

The other two cases ($\tau_2 \neq 0$) follow in a similar manner and yield $\ddot{q}_2 = -\frac{1}{d_{12}}\left(\frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1}\right)\dot{q}_2^2$ when the output is q_1 and $\ddot{q}_1 = -\frac{1}{2}\frac{1}{d_{11}}\frac{\partial d_{11}}{\partial q_1}\dot{q}_1^2$ when the output is q_2.

Exploiting the structure of the inertia matrix described in Property 1, the dynamics can be simplified. The four cases obtained are summarized in Table 1.

It can be seen that the internal dynamics has the general structure $\ddot{\xi} = k(\xi)\dot{\xi}^2$, where $\xi = q_1$ or $\xi = q_2$ as appropriate. In most cases, $k(\xi)$ is defined everywhere and the analysis is straightforward. However, when $k(\xi)$ is undefined, so is the internal dynamics. There, two cases of singularity have to be distingushed: (i) when $k(\xi)$ is undefined only at isolated points in the interval $(-\epsilon, \epsilon)$ and (ii) when $k(\xi)$ is independent of ξ and undefined everywhere in $(-\epsilon, \epsilon)$. In the non-singular scenario, the following result can be stated:

Theorem 2 *If the internal dynamics are well defined outside a finite set in the neighborhood of the equilibrium point, then the two-link underactuated mechanical* *systems are non minimum-phase systems with respect to the natural outputs.*

Proof: The equilibrium point of the internal dynamics is $[\xi, \dot{\xi}]^T = 0$. If the internal dynamics are undefined only at isolated points, it can be seen from Table 1 that the hypotheses of Lemma 1 are satisfied. Hence, the result follows.

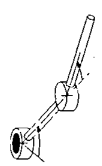
The singular situations require a more in-depth study. We will show with examples in the next section that the mechanism is uncontrollable under these situations.

5 Examples

In this section, we will discuss a few examples of two-link underactuated mechanical systems. The examples will be divided into two categories corresponding to whether or not singularity ($d_{12}(q_2) = 0$) can occur. In every case, the corresponding inertia matrix will be given and only one case of internal dynamics will be discussed. Table 1 can be used to calculate the remaining internal dynamics.

5.1 Systems without singularity

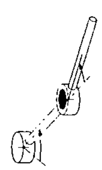
• The pendobot (Planar 2R Robot). ($\tau_1 \neq 0$, $y = q_2$)



$$D = \begin{bmatrix} J_1 + 2J_3 \cos q_2 & J_2 + J_3 \cos q_2 \\ J_2 + J_3 \cos q_2 & J_2 \end{bmatrix}$$

The internal dynamics are $\ddot{q}_1 = -\frac{J_3 \sin q_2}{J_2 + J_3 \cos q_2} \dot{q}_1^2$. Notice that these dynamics are always well defined and unstable according to Theorem 2.

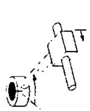
• The acrobot. ($\tau_2 \neq 0$, $y = q_1$) This system has the same inertia matrix as the pendobot. However, the actuation and the ouput are changed.



$$D = \begin{bmatrix} J_1 + 2J_3 \cos q_2 & J_2 + J_3 \cos q_2 \\ J_2 + J_3 \cos q_2 & J_2 \end{bmatrix}$$

The internal dynamics are $\ddot{q}_2 = \frac{J_3 \sin q_2}{J_2 + J_3 \cos q_2} \dot{q}_2^2$ which are well defined and unstable.

• The rotary prismatic system. ($\tau_1 \neq 0$, $y = q_2$)



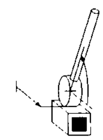
$$D = \begin{bmatrix} J_1 + J_3 q_2^2 & -J_4 \\ -J_4 & J_2 \end{bmatrix}$$

The internal dynamics are $\ddot{q}_1 = -\frac{J_3}{J_4} q_2 \dot{q}_1^2$ which show no singularity and are unstable.

5.2 Systems with singularity

In the first two examples, though there are a few values of q_2 which lead to singularity, the system is nonsingular at the other points where Theorem 2 can be applied. However, in the third example, $d_{12} = 0$ for all coordinate values, thereby depicting an extreme singular situation.

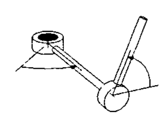
• The inverted pendulum. ($\tau_1 \neq 0$, $y = q_2$)



$$D = \begin{bmatrix} J_1 & -J_3 \sin q_2 \\ -J_3 \sin q_2 & J_2 \end{bmatrix}$$

When $\sin q_2 \neq 0$, the internal dynamics are $\ddot{q}_1 = 0$ which are trivially unstable. The singularity occurs when $\sin q_2 = 0$, which means the pendulum lies in the position $q_2 = 0$ or $q_2 = \pi$. Also, by the definition of the internal dynamics, $\dot{q}_2 = 0$. If the system happens to be in a position where $q_2 = 0$ or $q_2 = \pi$ with zero velocity, it can be seen from (10) that the system cannot leave the manifold $q_2 = 0$. Hence, the system is uncontrollable from the equilibrium point. Note that gravity is assumed to be absent.

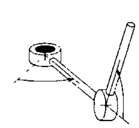
• The rotational inverted pendulum. ($\tau_1 \neq 0$, $y = q_2$)



$$D = \begin{bmatrix} J_1 + J_3 \cos(2q_2) & -J_4 \sin q_2 \\ -J_4 \sin q_2 & J_2 \end{bmatrix}$$

The singularity is similar to the inverted pendulum case. When $\sin q_2 \neq 0$, the internal dynamics are $\ddot{q}_1 = 2 \cos q_2 \dot{q}_1^2$. If the system happens to be in a position where $q_2 = 0$ or $q_2 = \pi$ with zero velocity, singularity occurs and the system is uncontrollable as well.

• Perpendicular rotational inverted pendulum. ($\tau_1 \neq 0$, $y = q_2$)



$$D = \begin{bmatrix} J_1 + J_3 \cos q_2 + J_4 \cos(2q_2) & 0 \\ 0 & J_2 \end{bmatrix}$$

This system exhibits the "worst" possible singularity in the sense that $d_{12} = 0$ independently of

q_1 and q_2. Moreover, the system is uncontrollable everywhere. Intuitively, it can be seen from the fact that, since the motion along q_2 is only through the centrifugal terms, the force can act only in one direction.

6 Conclusions

The analysis of internal dynamics of two-link exclusively kinetic underactuated mechanical systems with the natural outputs was undertaken. It has been shown that, in the nonsingular scenario the internal dynamics are unstable making the system nonminimum phase. Also, in the few examples studied, a link between singularity and uncontrollability was seen.

Generalization of the result on nonminimum phase to more than two links is an open problem. Future work will also be in the direction of rigourously establishing the relationship between uncontrollability and singularity.

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