

A Two-time-scale Controller for a Differentially Cross-coupled System

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Abstract

Control of underactuated mechanical systems often leads to unstable internal dynamics, which can be handled by resorting to prediction when the system bandwidth is small. The present paper considers systems with a high bandwidth and proposes a two-time-scale controller for decoupling the system while ensuring internal stability. A toy helicopter in which the speed of the propellers is manipulated to vary the aerodynamic force is taken up as a case study.

Keywords: Nonminimum-phase system, Underactuated system, Two-time-scale control, Decoupling.

1. Introduction

Control of underactuated systems is quite challenging since they are multi-input, nonlinear, nonminimum-phase systems. Although control techniques for linear nonminimum-phase systems are readily available, the nonlinear counterparts are difficult with regard to meeting desired specifications while ensuring internal stability [1]. A compromise solution is normally resorted to. One possibility is to redefine the tracking output so as to stabilize the internal dynamics. The original output is then only tracked asymptotically [2].

When dealing with multi-input systems, noninteracting (or decoupling) control is rather appealing. In [5], techniques using dynamic state feedback with guaranteed internal stability have been developed for linear systems. The approach has been extended to certain classes of nonlinear systems in [4].

The problem addressed in this paper can be formulated as follows: Given a nonlinear underactuated system, find a feedback law such that the outputs are decoupled and internal stability is guaranteed.

The noninteraction problem in underactuated systems has been addressed in [3] where decoupling of the outputs at the end of a prespecified interval is envisaged. The time lag introduced allows for the stabilization of the internal dynamics, thereby achieving the compromise mentioned above. This technique has the disadvantage that the time lag can in certain cases be quite large and, hence, is restricted to slow systems.

This paper refines the approach proposed in [3] to deal with systems having a large bandwidth. The key idea lies in the use of a cascaded scheme and a two-time-scale structure consisting of an inner loop working at a sufficiently-high rate to reduce the bandwidth of the original system and an outer feedback achieving stable noninteraction. The scheme is illustrated in simulation on an underactuated toy helicopter.

2. The Toy Helicopter

The system under study (Figure 1) is a helicopter setup in which the propeller speed is varied to manipulate the aerodynamic force. The model of such a system is given by

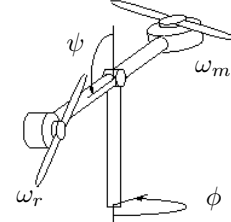


Figure 1: Helicopter setup

$$\begin{aligned}
 I_\psi \ddot{\psi} + I_r \dot{\omega}_r &= C_m \omega_m + I_m \dot{\phi} \omega_m \cos \psi - G_s \sin \psi \\
 &\quad - G_c \cos \psi - C_\psi \dot{\psi} + \frac{1}{2} I_c \dot{\phi}^2 \sin(2\psi) \\
 [I_\phi + I_c \sin^2(\psi)] \ddot{\phi} + I_m \sin \psi \dot{\omega}_m &= C_r \omega_r \sin \psi \\
 &\quad - I_m \dot{\psi} \omega_m \cos \psi - C_\phi \dot{\phi} - I_c \dot{\phi} \dot{\psi} \sin(2\psi) \\
 I_m \dot{\omega}_m &= u_m \\
 I_r \dot{\omega}_r &= u_r
 \end{aligned} \tag{1}$$

First, a simplified linear system that contains all the coupling elements of the nonlinear model will be investigated in an attempt to find an appropriate control law. The idea developed will then be applied to the nonlinear plant. Note that the simplified linear system given below is not the linearization of (1).

$$\begin{aligned}
 I_\psi \ddot{\psi} + I_r \dot{\omega}_r &= C_m \omega_m - G\psi \\
 I_\phi \ddot{\phi} + I_m \dot{\omega}_m &= C_r \omega_r \\
 I_m \dot{\omega}_m &= u_m \\
 I_r \dot{\omega}_r &= u_r
 \end{aligned} \tag{2}$$

3. Predictive Decoupling and Feasibility

This section outlines the predictive control scheme described in [3] that can be used to achieve decoupling. It also sets the stage to indicate under which circumstances such a scheme can work. For the ease of development, the simplified linear model will be used with $x = [\psi \ \dot{\psi} \ \phi \ \dot{\phi} \ \omega_m \ \dot{\omega}_m \ \omega_r \ \dot{\omega}_r]^T$ designating the state vector.

3.1. Predictive scheme

Let the system be discretized under the usual assumptions of zero-order hold on the inputs. The decoupling scheme consists in finding new discrete-time inputs $v_m(k)$ and $v_r(k)$ and a feedback $[u_m(k) \ u_r(k)]^T = f(v_m(k), v_r(k), x(k), x(k+1))$ such that $\dot{\psi}(k+1) = v_m(k)$ and $\dot{\phi}(k+1) = v_r(k)$. The decoupling feedback is obtained as:

$$\begin{bmatrix} u_m(k) \\ u_r(k) \end{bmatrix} = \mathcal{B}^{-1} \begin{bmatrix} I_\psi v_m(k) - C_m \omega_m(k) + G\psi(k+1) \\ I_\phi v_r(k) - C_r \omega_r(k) \end{bmatrix},$$

$$\text{where } \mathcal{B} = \begin{bmatrix} \frac{C_m T}{I_m} & -1 \\ -1 & \frac{C_r T}{I_r} \end{bmatrix} \quad (3)$$

Two crucial aspects of this scheme are: (i) the feedback depends on the future states $x(k+1)$ requiring prediction, and (ii) an additional delay is introduced in order to stabilize the internal dynamics. It is important to note that, due to the discrete-time nature of the scheme, decoupling of the angular acceleration $\ddot{\psi}$ and $\ddot{\phi}$ is achieved on a grid defined by the sampling instants and not for all times. Hence, an intersample coupling ripple of ψ and ϕ will remain (see Section 6). Furthermore, the choice of the sampling period is crucial in ensuring internal stability, which will be discussed next.

3.2. Feasibility

Theorem 1 (from [3]) : System (2) can be decoupled with internal stability using the discrete-time controller (3) if the sampling period satisfies $2\tau < T \leq \frac{1}{2}T_b$ where $\tau = \sqrt{\frac{I_m I_r}{C_r C_m}}$ is the dominant unstable eigenvalue of the internal dynamics, and $T_b = \frac{1}{f_b}$ with f_b [Hz] being the bandwidth of the system.

Definition 1 The decoupling bound 2τ for a differentially cross-coupled system is the lower bound on the sampling period such that the internal dynamics are stable.

Definition 2 A differentially cross-coupled system is feasible with respect to its decoupling bound if $T_b > 4\tau$.

The last definition stems from the fact that, for a feasible system, the sampling period can be chosen such that both Shannon's upper bound and the decoupling lower bound are respected and, hence, the intersample behavior does not affect stability.

The bandwidth of system (2) is obtained by computing the eigenvalues of the system matrix. The corresponding eigenvalues are $\lambda(\mathcal{A}) = \{\sqrt{G/I_\psi}, -\sqrt{G/I_\psi}, 0, 0, 0, 0\}$. Then, $T_b = 2\pi\sqrt{I_\psi/G}$ and, hence, (2) is feasible only when $G < \frac{\pi^2 I_\psi C_m C_r}{4 I_m I_r}$. Clearly, for large values of G , which corresponds to the gravitational torque in this setup, the system becomes infeasible.

4. Two-time-scale Control of the Simplified Linear Model

If the system is infeasible with respect to its decoupling bound, then the discrete decoupling scheme described in the previous section cannot be used. To get around this difficulty, a two-time-scale controller as illustrated in Figure 2 is proposed. The inner feedback operates in continuous time to render the system feasible. The decoupling feedback operates in discrete time with the sampling period $T > 2\tau$ to achieve decoupling. Finally, two outer controllers g_ψ and g_ϕ control the resulting chain of integrators with delay.

The present scheme can be viewed from a different perspective. Due to infeasibility with $T > 2\tau$, the intersample behavior does not necessarily have a loop gain less than unity. The role of the inner feedback is to decrease the loop gain of intersample ripple so that stability can be achieved.

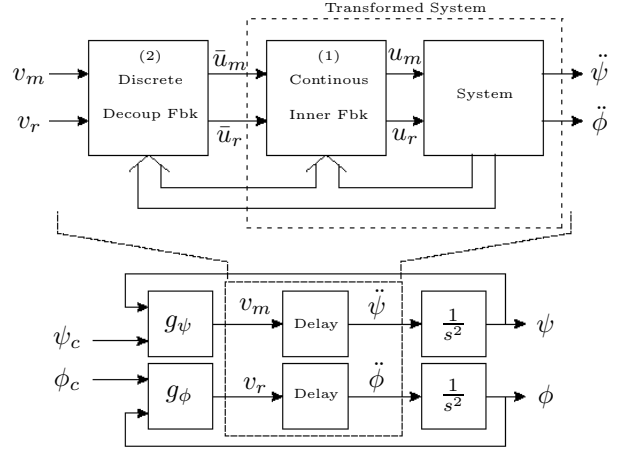


Figure 2: Two-time-scale controller

4.1. Continuous Inner Feedback

For the inner feedback, since the large bandwidth is due to the term $G\psi$ that creates a torque along the ψ axis, it is natural to compensate it using the main aerodynamic force.

Proposition 1 Under the feedback $u_m = \bar{u}_m + G \frac{I_m}{C_m} \dot{\psi}$, $u_r = \bar{u}_r$ and the state transformation $\bar{\omega}_m = \omega_m - \frac{G}{C_m} \psi$, $\bar{\omega}_r = \omega_r$, the initial system (2) reads

$$\begin{aligned} I_\psi \ddot{\psi} + I_r \dot{\omega}_r &= C_m \bar{\omega}_m \\ I_\phi \ddot{\phi} + I_m \dot{\omega}_m &= C_r \bar{\omega}_r - G \frac{I_m}{C_m} \dot{\psi} \\ I_m \dot{\omega}_m &= \bar{u}_m \\ I_r \dot{\omega}_r &= \bar{u}_r \end{aligned} \quad (4)$$

which is feasible with respect to its decoupling bound.

The proof of the proposition is straightforward and not provided here for the sake of brevity. The eigenvalues of the transformed system matrix are $\lambda(\mathcal{A}) = \{0, 0, 0, 0, 0, 0\}$. Hence, $T_b = \infty$ making the transformed system (4) feasible.

4.2. Discrete Decoupling Feedback

The decoupling controller with sampling period $T > 2\tau$ is obtained following the lines of Section 3.1. This results in :

$$\begin{bmatrix} \bar{u}_m(k) \\ \bar{u}_r(k) \end{bmatrix} = \mathcal{B}^{-1} \begin{bmatrix} I_\psi v_m(k) - C_m \bar{\omega}_m(k) \\ I_\phi v_r(k) - C_r \bar{\omega}_r(k) + G \frac{I_m}{C_m} \dot{\psi}(k+1) \end{bmatrix} \quad (5)$$

From this expression, it is seen that the value of $\dot{\psi}$ at time $k+1$ must be known at time k . This prediction can be accomplished in two different ways:

- (i) Use an analytical prediction for $\dot{\psi}(k+1)$.
 - (ii) Use another feedback to eliminate the term $\dot{\psi}(k+1)$.
- The $\dot{\psi}$ term in (4) can be compensated by the rear propeller. The state transformation is $\bar{\omega}_r = \omega_r - G \frac{I_m}{C_m C_r} \dot{\psi}$ and the feedback reads:

$$u_r = \frac{1}{1 + \frac{G I_m I_r}{I_\psi C_m C_r}} \left[\bar{u}_r + \frac{G I_m I_r}{I_\psi C_m C_r} (C_m \omega_m - G \dot{\psi}) \right] \quad (6)$$

5. Two-time-scale Control of the Nonlinear Model

The control of the nonlinear model will be an extension of the idea used for the simplified linear model. The following feedback law is proposed for the continuous inner feedback:

$$\begin{aligned}
 u_m &= f_1 \left\{ \bar{u}_m - \frac{I_m^2}{C_m [I_\phi + I_c \sin^2(\psi)]} \cos(\psi) \omega_m \right. \\
 &\quad [C_r \omega_r \sin(\psi) - I_m \dot{\psi} \cos(\psi) - C_\phi \dot{\phi} - I_c \dot{\psi} \sin(2\psi)] \\
 &\quad \left. + \frac{I_m}{C_m} [I_m \dot{\psi} \dot{\phi} \cos(\psi) \omega_m + (G_s \cos(\psi) - G_c \sin(\psi)) \dot{\psi}] \right\} \\
 u_r &= \bar{u}_r \\
 f_1 &= \frac{1}{1 + \frac{I_m}{C_m} \dot{\phi} \cos(\psi) - \frac{I_m^2}{C_m [I_\phi + I_c \sin^2(\psi)]} \cos(\psi) \omega_m \sin(\psi)}
 \end{aligned} \quad (7)$$

The feedback (7) along with the state transformation $\bar{\omega}_m = \omega_m + \frac{1}{C_m} (I_m \dot{\phi} \omega_m \cos(\psi) - G_s \sin(\psi) - G_c \cos(\psi))$ and $\bar{\omega}_r = \omega_r$ renders the system feasible. The terms depending on the centrifugal and coriolis forces are assumed not to contribute to the infeasibility of the system.

Since the gravity and coriolis terms compensated by (7), the scheme of Section 3.1 can be applied giving the following discrete-time decoupling feedback:

$$\begin{bmatrix} \bar{u}_m(k) \\ \bar{u}_r(k) \end{bmatrix} = \tilde{B}^{-1} \begin{bmatrix} I_\psi v_m(k) - C_m \bar{\omega}_m(k) - \tilde{A}_\psi \\ I_\phi v_r(k) - \tilde{C}_m \bar{\omega}_m(k) - \tilde{C}_r \bar{\omega}_r(k) - \tilde{A}_\phi \end{bmatrix} \quad (8)$$

where $\tilde{B} = \begin{bmatrix} \tilde{C}_m \frac{T}{I_m} & -1 \\ f_m & \tilde{C}_r \frac{T}{I_r} \end{bmatrix}$

with \tilde{C}_m , \tilde{C}_r , \tilde{A}_ψ , \tilde{A}_ϕ and f_m being appropriate nonlinear functions of the states at time $(k+1)$. Because of the nonlinearities, it is not possible to use an analytical prediction for the states at time $(k+1)$. On the other hand, using another feedback for the rear propeller does not help in removing terms at time $(k+1)$. Nevertheless, a nonlinear programming algorithm can be used to compute the inputs $\bar{u}_m(k)$ and $\bar{u}_r(k)$ so that the coordinate accelerations $\dot{\phi}(k+1) = v_m(k)$ and $\dot{\psi}(k+1) = v_r(k)$.

6. Simulation Study

Simulation of the nonlinear system with the proposed two-time-scale controller is performed. The system is infeasible with respect to its decoupling bound. Had the decoupling scheme been directly applied to the nonlinear system without the continuous controller, the internal stability would be lost as depicted in Figure 3. In contrast, the simulation with the two-time-scale controller shows excellent decoupling of the two axes ψ and ϕ (Figure 4). Furthermore, the fast input u_m differs greatly from the slow one \bar{u}_m (which is nearly zero). This is due to the large effort of the fast controller to compensate for the gravity terms $G_s \sin(\psi)$ and $G_c \cos(\psi)$.

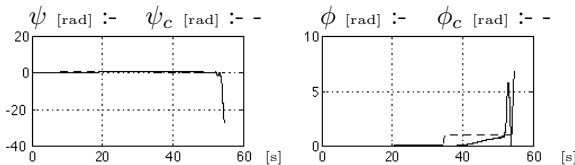


Figure 3: Simulation with no inner feedback

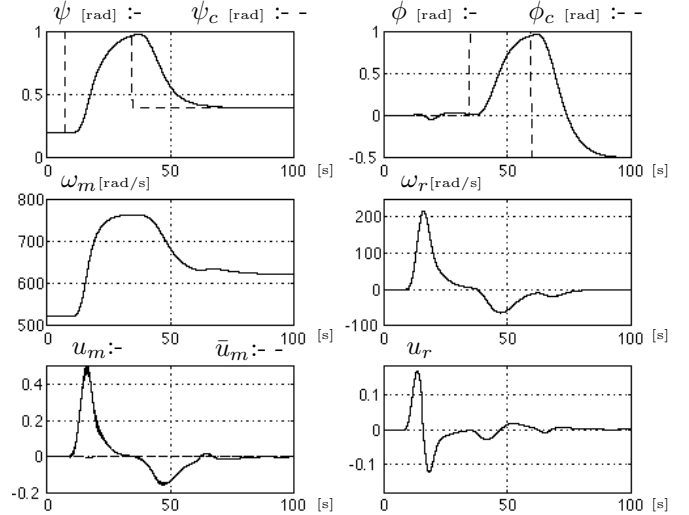


Figure 4: Simulation with two-time-scale controller

7. Conclusion

A two-time-scale controller was proposed for decoupling systems for which Shannon's upper limit and the lower bound for decoupling led to an infeasible region of the sampling period. The proposed scheme was applied in simulation to an underactuated mechanical system. It showed excellent decoupling properties as illustrated in the simulation section.

This paper has nevertheless left the following problems open: (i) How can the properties of the transformed nonlinear system upon application of the fast stabilizing feedback be proven analytically? (ii) Can this scheme be generalized to other nonminimum-phase systems? (iii) How robust is the proposed control scheme?

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