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DESIGN OF A FLEXURE BASED LOW FREQUENCY FOUCAULT PENDULUM

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ABSTRACT

The Foucault pendulum is a well-known mechanism used to demonstrate the rotation of the Earth. It consists in a pendulum launched on linear orbits and, following Mach's Principle, this line of oscillation will remain fixed with respect to absolute space but appear to slowly precess for a terrestrial observer due to the turning of the Earth. The theoretical proof of this phenomenon uses the fact that, to first approximation, the Foucault pendulum is a harmonic isotropic two degree of freedom (2-DOF) oscillator. Our interest in this mechanism follows from our research on flexure-based implementations of 2-DOF oscillators for their application as time bases for mechanical timekeeping.

The concept of the Foucault pendulum therefore applies directly to 2-DOF flexure based harmonic oscillators. In the Foucault pendulum experiment, the rotation of the Earth is not the only source of precession. The unavoidable defects in the isotropy of the pendulum along with its well-known intrinsic isochronism defect induce additional precession which can easily mask the precession due to Earth rotation. These effects become more prominent as the frequency increases, that is, when the length of the pendulum decreases. For this reason, short Foucault pendulums are difficult to implement, museum Foucault pendulum are typically at least 7 meters long. These effects are also present in our flexure based oscillators and reducing these parasitic effects, requires decreasing their frequency.

This paper discusses the design and dimensioning of a new flexure based 2-DOF oscillator which can reach low frequencies of the order of 0.1[Hz]. The motion of this oscillator is approximately planar, like the classical Foucault pendulum, and will have the same Foucault precession rate. The construction of a low frequency demonstrator is underway and will be followed by quantitative measurements which will examine both the Foucault effect as well as parasitic precession.

1 INTRODUCTION

In previous work [9], we showed that two degree of freedom (2-DOF) harmonic oscillators can be used advantageously as time bases for mechanical timekeepers since they bypass escapement mechanisms. Several oscillator designs were implemented as time bases for clocks. The planar 2-DOF harmonic oscillators constructed should exhibit Foucault pendulum behavior when in a rotating frame such as the Earth. This means that the direction of purely linear oscillations remains fixed for an inertial frame. In fact, the proof of the Foucault phenomenon for the classical spherical pendulum assumes that the spherical pendulum is a planar 2-DOF oscillator, i.e., that the vertical displacement of the spherical pendulum is negligible. When first confronted with the Foucault pendulum, G.B. Airy showed that if pendulum trajectories are not perfectly linear but ellipses, then a parasitic precession will occur, possibly masking the Foucault effect [1]. The rate of this precession is proportional to the frequency, so it turns out that it becomes increasingly difficult to observe the Foucault effect for pendulums shorter than 7 meters due to increasing frequency [5]. The goal of this paper is to design 2-DOF harmonic oscillators whose frequency can be lowered arbitrarily while keeping the dimensions of the oscillator mechanism small, in particular, placed on a table. This would enable Foucault's demonstration of the Earth's rotation to be easily accessible since no longer requiring 7 meter tall ceilings to place the demonstrator.

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The design consists of a 2-DOF harmonic oscillator consisting of a mass supported by three cylindrical beams placed symmetrically. The frequency of oscillation can be decreased arbitrarily by increasing the mass which decreases beam stiffness by axial compression. The Lagrangian of this oscillator will be computed using standard methods for compliant mechanisms (flexures). Using an analogue of Airy's formula, we can estimate the parasitic precession and the dimensions required to observe the actual precession due to the rotation of the Earth.

2 THEORETICAL BACKGROUND

2.1 The Foucault Pendulum

The Foucault pendulum is a physics experiment that enables the rotation of the Earth to be visible to the naked eye and was first imagined and constructed in 1851 by the French physicist Léon Foucault. This simple device consists of a spherical pendulum launched in a straight line towards the center of attraction, which defines a line of oscillations. Following Mach's Principle, the mass will follow a straight line with respect to absolute space. Due to the Earth's rotation, for a terrestrial observer, the plane of oscillations will seem to turn. This precession depends on the sine of latitude, the plane of oscillation follows the horizontal motion of a star at the horizon. For an observer at the North Pole, the plane of oscillations will have a precession at a rate of one full turn per twenty-four hours, whereas for an observer at the equator, there will be no precession. See [3] [10] for historical summaries.

2.2 Foucault behaviour of planar oscillators

Planar oscillators In his Principia Mathematica [6, Book I, Proposition X], Isaac Newton showed that if gravitation were to have a linear restoring force (Hooke's Law), then the orbits of a planet around the Sun would be ellipses with the Sun in the center, as opposed to a focus of the ellipse for the inverse square law. Moreover, Newton showed that all orbits of the planets would have the same period. This means that isochronism holds, this property being the basic requirement for an oscillator to be a precise time base for a timekeeper. In previous work, we used this fact to design and construct time bases for mechanical clocks [9] [11], the point being that this eliminates the traditional but complex escapement mechanism. A schematic representation of such an oscillator is shown in Fig. 1.

There are numerous designs of 2-DOF harmonic oscillators. A first design consisting of two flexure based translation stages (Fig.2) was used as a timebase for a clock [9]. In this paper, we study another design consisting of three vertical flexible beams anchored at their lower end and upon which rests the oscillating mass (Fig. 5). The advantage of this design is that the mass's weight compresses the beams and reduces the overall stiffness, therefore reducing the oscillation frequency. This aspect is effective for detecting slowly rotating frames.

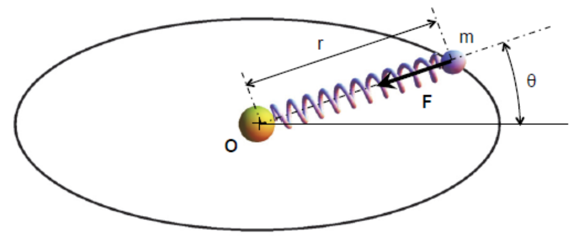

FIGURE 1: Ideal 2-DOF isotropic harmonic oscillator of Newton's modified solar system.

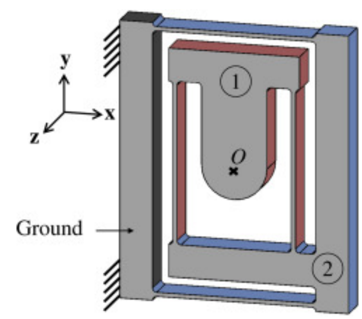

FIGURE 2: Flexure based 2-DOF harmonic isotropic oscillator consisting of two serially connected stages.

2.2.1 Foucault effect in planar oscillators

As is well-known, the circular pendulum has an isochronism defect. This is due to the fact that its restoring force is non-linear, as it is proportional to the sine of the displacement angle. The classical modelization of the Foucault pendulum uses the small angle approximation, assuming the restoring force is linear and restricting the movement of the mass to a plane. With these assumptions, one understands that the approximated Foucault pendulum is nothing else than the ideal planar oscillator previously mentioned in Newton's model, see [2, Chapter 4]. It follows that linear oscillators of 2-DOF harmonic oscillator will exhibit the classical Foucault effect.

2.3 Detecting rotation with a low frequency planar oscillator

Newton's model of a 2-DOF harmonic oscillator assumes that the restoring force is isotropic, i.e., the same in all directions. However, any physical implementation will have an isotropy defect and this will cause linear orbits to degenerate into Lissajous figures. Even for a carefully constructed prototype, there will still be precession due to the non-linearity in the restoring force. Moreover, orbits are never perfectly linear, but slightly elliptic

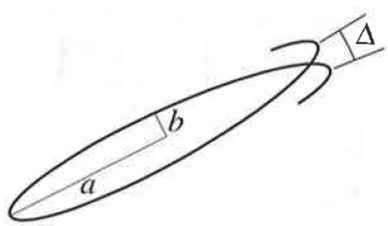

FIGURE 3: Airy precession of an elliptic orbit with major axis a and minor axis b. [10].

causing a spurious precession. In particular, elliptic orbits can be seen as two separate oscillations along the major and minor axes of the ellipse, and in the case of the pendulum, its isochronism defect means that these two oscillations do not have the same period, so elliptic orbits precess.

This effect was first discovered by Airy in 1851 and he developed the formula for the precession rate Ω_{Airy} of a pendulum on an elliptic orbit differing little from a straight line [1] [5] [8]:

$$\Omega_{Airy} = \frac{3}{8}\frac{ab}{l^2}\sqrt{\frac{g}{\ell}}\,, \tag{1}$$

with a, b the major resp. minor axes of the elliptic orbit, ℓ the pendulum length and g the gravitational acceleration. This is also known as the *area formula,* since πab is the area of the ellipse. One can easily notice that this parasitic precession becomes more prominent in short pendulums. Short Foucault pendulums are indeed difficult to obtain. The first functional short Foucault pendulum was only created thirthy years after Foucault's invention, by Kamerlingh Onnes in his master's thesis, he is most famous for later having discovered liquid helium and superconductivity, see [2, Chapter 4], [7].

Alternative 2-DOF oscillators having isochronism defects will also exhibit Airy precession, though the formula will not be exactly the same if their Lagrangrian is different. But, as before the Airy precession will be proportional to frequency and therefore reduced by lowering the frequency. The advantage of a flexure based mechanism over a pendulum is that we can design a mechanism whose frequency can be significantly reduced without significantly changing size. In the case of the pendulum, frequency is determined only by pendulum length. The frequency in the flexure based approach can be reduced by reducing the stiffness of the flexible elements by various methods, as shown below.

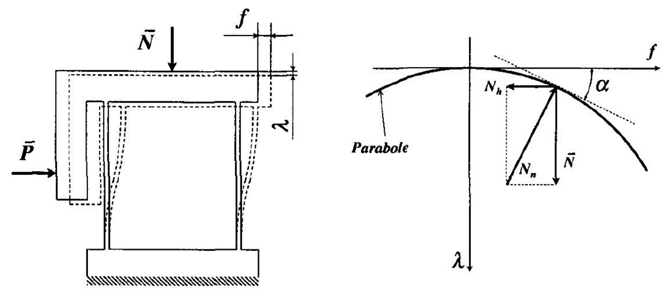

FIGURE 4: Left: parallel leaf spring translational stage. Bending force $\bar{P}$ and compressive force $\bar{N}$ are applied to the mobile block. Right: parabolic trajectory of the parallel spring stages [4].

3 DESIGN OF A LOW FREQUENCY 2DOF PLANAR OSCILLATOR

The proposed design is based on a parallel leaf spring stage (Fig. 4). It is modified by replacing the two leaf springs with circular beams, enabling translation along both the x and y directions. An extra beam needs to be added to block rotation around the x axis. We limit ourselves to three beams, to avoid any hyperstatism. It can be noticed that this design actually has three degrees of freedom. It has the two desired translations along x and y, but also the unwanted rotation along z. This issue can be solved by ensuring that the natural frequency in rotation is much higher than the frequency in translation.

As we wish to limit the Airy precession, the oscillations will be assumed to have small amplitude of the order of a degree so small deformation approximations can be used for the beam bending. Under these conditions, the stiffness of the beams as well as the position of the mass are well known [4]. The stiffness for a single beam $k_{1beam}$is given by:

$$k_{1beam} = \frac{12EI}{\ell^3}\,, \tag{2}$$

with ℓ the beam length, I the beam's moment of inertia and E its Young modulus. The position of the oscillating mass does not lie on a plane, but on a paraboloid, in the same way as the trajectory of a (1-DOF) parallel leaf spring stage is parabolic [4](Fig. 4). This is in fact the main advantage of this design. As in the parallel leaf spring stage, by loading the beams in compression (either with the mass's weight or with an external force), the overall stiffness is reduced. Indeed, the effect of gravity can be seen as a spring with a constant negative stiffness k_g [4]. This effect can be understood using an energetic analysis. For a horizontal displacement x of the parallel leaf spring stage, there is a vertical displacement λ along $\mathbf{z}$:

$$\lambda = \frac{-3}{5\ell}x^2\,, \tag{3}$$

with ℓ the beam length, hence the parabolic trajectory. If we consider the potential energy of gravity $E_g = mgz$, with m the mass and g the gravitational acceleration and knowing that its resulting force F_g along $\mathbf{x}$ is $-dE_g/dx$, one obtains:

$$F_g = \frac{-dE_g}{dx} = \frac{d}{dx}\frac{3mgx^2}{5\ell^2} = \frac{6mg}{5\ell^2}x \,. \tag{4}$$

This force being linear with respect to x and of the same sign as x, the effect of gravity can be seen as a spring with a constant negative stiffness k_g:

$$k_g = \frac{-6mg}{5\ell} \,. \tag{5}$$

This negative stiffness counteracts the restoring force of the bending beams. By considering three beams and adding their stiffness to the effect of loading, one obtains the overall stifness k_{tot}:

$$k_{tot} = \frac{36EI}{\ell^3} - \frac{6mg}{5l} \tag{6}$$

By increasing the mass m, the overall stiffness is reduced and will eventually become negative. With this formula, it is in theory possible to obtain zero stiffness. Experimentally, on a preexisting parallel leaf spring stage of similar dimensions as the foreseen three beam prototype, frequencies as low as 0.1[Hz] were reached.

For Eqn. 2-6 to be valid, the bending force (within the line of oscillation) must be applied at mid-height of the beams. As this force is due to the oscillating mass, one must place the center of gravity of the oscillating mass mid-height of the beams. Without this condition, the bending force will induce tension and compression on the beams and the mobile mass will not remain horizontal (tilting). Moreover, the loading force of gravity must act identically on each beam, again to avoid tilting. For this reason, the mass's weight must act at the center of gravity of the three beams' anchoring.

Regarding the unwanted degree of freedom in rotation along the z axis, its frequency can be increased by reducing the mass's moment of inertia along z. This means that the oscillating mass has to be as close as possible to a thin rod, as shown in Fig.5.

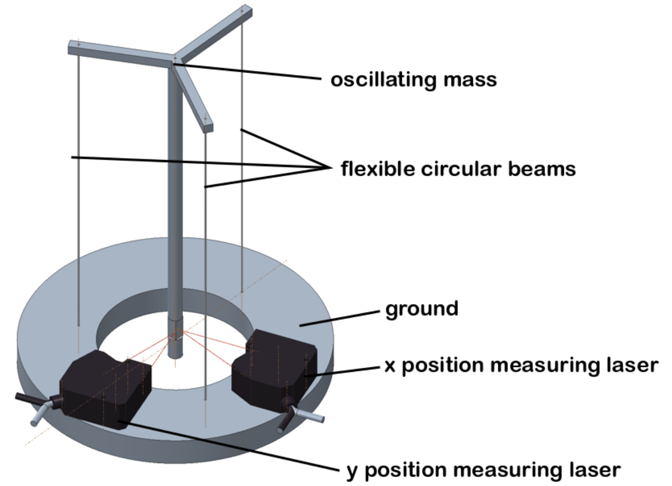

FIGURE 5: Abstract design of the three beam oscillator. The beams have circular cross-section. Two distance measuring lasers are present to determine the x-y position of the mobile mass.

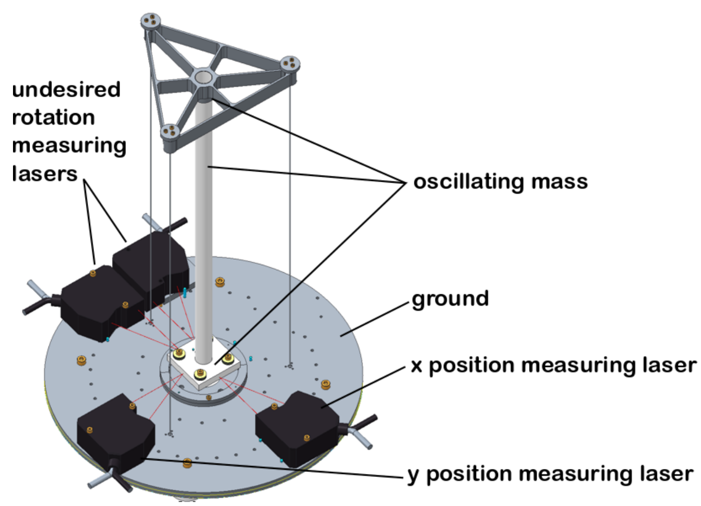

FIGURE 6: Detailed CAD view of the three beam oscillator. An extra set of lasers is present to verify that the third degree of freedom (rotation around the vertical axis) is not excited. The oscillating mass has a square section at its end for the lasers to target during motion.

3.1 Balancing of the oscillator

We refer to *balancing* as the adjustments used to reduce the isotropy defect of the oscillator. As the isotropy defect is mainly due to manufacturing and assembly errors, the balancing is performed after assembly in an iterative manner. The solution found is to adjust the stiffness by filing the beams in the direction with the highest frequency, reducing the stiffness and therefore the frequency. This method is inspired from Kamerlingh Onnes' work, see [7], by filing the bob in his short Foucault pendulum. This technique also has a built-in gross and fine adjustment method: as the beams bend mainly at the anchored extremities, filing them there would have a large effect on their stiffness. On the other hand, filing at mid-height will bring about small changes in the stiffness.

3.2 Lagrangian of the three beam oscillator

To study the Airy precession of this oscillator, a standard method is to evaluate its Lagrangian. Airy's original calculations did not use Lagrangian mechanics, but instead a fourth-order differential equation which he solved at each order [8]. A method provided by Rousseaux et al. [8] revisits Airy's result for the pendulum and we have followed this method. It involves starting from the one dimensional (non-linear) differential equation of position x of the oscillator (as if the orbit were indeed linear). The problem then becomes two-dimensional by substituting x with a complex variable Z. Solving this differential equation assumes an orbit differing little from a straight line, thereby providing several simplifications. An analytic solution for the Airy precession was found and will be described in later work following its experimental verification.

The calculation of this precession requires its Lagrangian. The proposed calculation of the Lagrangian for the three beam oscillator makes the following assumptions:

1. The mass m of the mobile part is concentrated to a point at midheight of the beams
2. The position at rest of m is at the cylindrical coordinate system (r, φ, z) origin, z being vertical and in the direction opposite to gravity.
3. The beams have a length l and the position of the mass m is therefore restricted to a paraboloid given by the equation:

$$z = -ar^2, \text{ with } a = \frac{3}{5\ell} \tag{7}$$

The kinetic energy K in cylindrical coordinates is given by:

$$\begin{aligned} K &= \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\varphi})^2 + \dot{z}^2\right] &(8)\\ &= \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\varphi})^2 + (-2ar\dot{r})^2\right] &(9)\\ &= \frac{1}{2}m\left[(1+4a^2r^2)\dot{r}^2 + (r\dot{\varphi})^2\right] &(10)\\ &= \frac{1}{2}m\left[(1+\frac{36}{25\ell^2}r^2)\dot{r}^2 + (r\dot{\varphi})^2\right] &(11) \end{aligned}$$

The potential V consists in the potential of the beams and the potential of gravity:

$$V = \frac{1}{2}k_{tot}r^2 \tag{12}$$

The Lagrangian $\mathscr{L} = K - V$ is then:

$$\mathscr{L} = \frac{1}{2}m\left[(1+\frac{36}{25l^2}r^2)\dot{r}^2 + (r\dot{\varphi})^2\right] - \frac{1}{2}k_{tot}r^2 \tag{13}$$

If we compare this results with the Lagrangian of Newton's ideal 2-DOF oscillator (for which there is no Airy precession) $\mathscr{L}_N$:

$$\mathscr{L}_N = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\varphi})^2\right] - \frac{1}{2}k_{tot}r^2 \tag{14}$$

one notices that the isochronism defect is reduced by having longer beams (the same applies to the pendulum by increasing its length).

3.3 Proposed dimensions

The Foucault effect at our latitude at Neuchatel, Switzerland of 47° is a period of $24/\sin 47^\circ \approx 32.8$ hours, so the Foucault precession has frequency $1/(32.8 \times 3600) = 8.4 \times 10^{-6}$[Hz]. This is quite small, but since a 7 meter spherical pendulum is typically accepted for museum exhibits of the Foucault phenomenon, the dimensions for our mechanisms were chosen to be within the same range. A 7 meter pendulum has frequency $1/2 \times 1/\sqrt{7} \approx 0.19$[Hz], so we want our mechanism to have frequency of the order of 0.1[Hz]. The critical aspects in the construction is to guarantee a reasonable isotropy defect and to be able to obtain low frequencies of order of 0.1[Hz]. We also aim for a height of maximum 500[mm], since the main goal was to have an easily transportable Foucault demonstrator. However, longer beams are advantageous as they reduce the Airy precession. The total mass of the prototype also has to be within a range making the demonstrator easily transportable.

The isotropy defect is affected by factors such as beam cross-section circularity and beam straightness. A solution proposed is to manufacture our beams from injection mould ejector pins, which are manufactured with high tolerances and straightness. Out of the available beams, the longest that satisfy the specification has diameter 2[mm] and length 380[mm]. The mass required to reach a frequency of 0.1[Hz] is 3.3[kg]. The range of motion can go up to ± 4.5[cm], but we limit ourselves to ± 2[cm] to reduce the Airy precession.

Concerning the construction, particular care is taken to ensure that the beams' anchoring is as isotropic as possible as well as that all beams are parallel and have the same active length . The beams' anchoring will be performed by gluing the beams into a tighly toleranced blind hole. To guarantee an identical spacing between beams for them to be parallel, the holes of both the base and top arms shall be machined on the same fixture.

The final CAD model is underway and fabrication will start shortly.

4 CONCLUSION AND FUTURE WORK

The subject of this paper is to design a transportable Foucault demonstrator. In order to achieve this, we use flexure mechanisms to construct a reasonable size oscillator whose frequency is comparable to that of a 7 meter Foucault pendulum. We propose a design capable of frequencies as low as 0.1[Hz]. It involves three vertical flexible beams of circular cross section which are anchored to the ground at their lower end and connected to the mobile part at their upper end. The mobile part's movement is restricted to a paraboloid, which enables the overall stiffness to be reduced by increasing the mass of the mobile part. Effects that limit the ability to detect the Foucault effect include isotropy defects and Airy precession. The isotropy defect is dealt by methods developed by Kamerlingh-Onnes and the Airy precession is minimized by reducing the frequency.

Future work includes a verification of the analytical solution for the Airy precession of the new three beam oscillator by numerical simulation of its Euler-Lagrange equations. A final CAD model is underway and fabrication will start shortly. This prototype will also include position measuring lasers for quantitative results.

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