

Effective equations of arbitrary order for wave propagation in periodic media

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Abstract

While the standard homogenized wave equation describes the effective behavior of the wave at short times, it fails to capture the macroscopic dispersion that appears at long times. To describe the dispersion, the effective model must include additional operators of higher order. In this work, we present a practical way to construct effective equations of arbitrary order in periodic media, with a focus on their numerical approximation. In particular, we exhibit an important structure hidden in the definition of the high order effective tensors which allows a significant reduction of the computational cost for their approximation.

Keywords: homogenization, long time behavior, dispersion

1 Introduction

Let $a(y)$ be a $[0, 1]^d$ -periodic tensor, $\Omega \subset \mathbb{R}^d$ be a hypercube and for $\varepsilon > 0$ let $u^\varepsilon : [0, T] \times \Omega \rightarrow \mathbb{R}$ be the solution of the wave equation

$$\partial_t^2 u^\varepsilon(t, x) - \nabla_x \cdot \left(a\left(\frac{x}{\varepsilon}\right) \nabla_x u^\varepsilon(t, x) \right) = f(t, x), \quad (1)$$

for $(t, x) \in (0, T] \times \Omega$, where we impose Ω -periodic boundary conditions, the initial conditions and the source f are assumed to have $\mathcal{O}(1)$ frequencies and $\mathcal{O}(1)$ support, and Ω is arbitrarily large. To accurately approximate u^ε , standard numerical methods require a grid resolution of order $\mathcal{O}(\varepsilon)$ in the whole domain, which leads to a prohibitive computational cost as $\varepsilon \rightarrow 0$. In the regime $\varepsilon \ll 1$, homogenization theory provides a way to approximate u^ε at a cost that is independent of ε : the result states that $\lim_{\varepsilon \rightarrow 0} u^\varepsilon = u^0$ in $L^\infty(0, T; L^2(\Omega))$, where u^0 solves the *homogenized equation*

$$\partial_t^2 u^0(t, x) - a_{ij}^0 \partial_{ij}^2 u^0(t, x) = f(t, x), \quad (2)$$

equipped with the same initial and boundary conditions as (1). The homogenized tensor a^0 is constant and can be computed by means of (first

order) correctors, solutions of (first order) *cell problems*, i.e., periodic elliptic PDEs in $[0, 1]^d$ involving $a(y)$. In practice, we observe that for long times $t = \mathcal{O}(\varepsilon^{-\alpha})$ $\alpha \geq 2$, dispersion effects that appear in the L^2 behavior of $u^\varepsilon(t, \cdot)$ are not captured by $u^0(t, \cdot)$. *High order effective equations* are effective models that describe the dispersion (with an accuracy that should increase with the order). Several definitions of high order effective equations were recently proposed [1–3]. Although the form of the equations are not the same, they all involve the same high order effective quantities.

2 Family of effective equations of arbitrary order

We present the high order models introduced in [1]. For $q \in \text{Sym}^n(\mathbb{R}^d)$, a symmetric tensor of order n , we denote the operator $q \nabla_x^n = \sum q_{i_1 \dots i_n} \partial_{i_1 \dots i_n}^n$. For a timescale $\mathcal{O}(\varepsilon^{-\alpha})$, the effective equations have the form

$$\partial_t^2 \tilde{u} - a^0 \nabla_x^2 \tilde{u} - \sum_{r=1}^{\lfloor \alpha/2 \rfloor} (-1)^r \varepsilon^{2r} L^{2r} \tilde{u} = Qf, \quad (3)$$

where the operators L^{2r} and Q are defined as

$$L^{2r} = a^{2r} \nabla_x^{2r+2} - b^{2r} \nabla_x^{2r} \partial_t^2, \\ Q = 1 + \sum_{r=1}^{\lfloor \alpha/2 \rfloor} (-1)^r \varepsilon^{2r} b^{2r} \nabla_x^{2r},$$

and $a^{2r} \in \text{Sym}^{2r+2}(\mathbb{R}^d)$, $b^{2r} \in \text{Sym}^{2r}(\mathbb{R}^d)$. Note that if a^{2r}, b^{2r} are non-negative, (3) is well-posed.

To derive the values of the effective tensors a^{2r}, b^{2r} , we use asymptotic expansion. We look for an adaptation $\mathcal{B}^\varepsilon \tilde{u}$ that approximates u^ε . An error estimate tells us that for \tilde{u} to be close to u^ε up to $\mathcal{O}(\varepsilon^{-\alpha})$ timescales, $\mathcal{B}^\varepsilon \tilde{u} - u^\varepsilon$ must satisfy the wave equation with a right hand side of order $\mathcal{O}(\varepsilon^{\alpha+1})$ in the $L^\infty(0, \varepsilon^{-\alpha}T; L^2(\Omega))$ -norm. We then combine (i) the ansatz

$$\mathcal{B}^\varepsilon \tilde{u}(t, x) = \tilde{u}(t, x) + \sum_{k=1}^{\alpha+2} \chi^k(t, x, y) \nabla_x^k u(t, x),$$

where the k -th order corrector $\chi^k = \{\chi_{i_1 \dots i_k}^k\}$ has value in $\text{Sym}^k(\mathbb{R}^d)$, and (ii) inductive Boussinesq tricks (use (3) to replace time derivatives with space derivatives) and obtain the *cell problems*, which have the cascade form:

$$\begin{aligned} \mathcal{A}\chi_{i_1}^1 &= \mathcal{F}_{i_1}^1(a), \\ \mathcal{A}\chi_{i_1 i_2}^2 &= \mathcal{F}_{i_1 i_2}^2(a, \chi^1, a^0), \\ \mathcal{A}\chi_{i_1 \dots i_{2r+1}}^{2r+1} &= \mathcal{F}_{i_1 \dots i_{2r+1}}^{2r+1}(a, \chi^1, \dots, \chi^{2r}), \\ \mathcal{A}\chi_{i_1 \dots i_{2r+2}}^{2r+2} &= \mathcal{F}_{i_1 \dots i_{2r+2}}^{2r+2}(a, \chi^1, \dots, \chi^{2r+1}, a^{2r} - a^0 \otimes b^{2r}), \end{aligned} \quad (4)$$

where $\mathcal{A} = -\nabla_y \cdot (a \nabla_y \cdot)$ and $\mathcal{F}_{i_1 \dots i_k}^k$ are specified in [1]. While the odd order cell problems are well-posed unconditionally, the solvability of the even order cell problems provides constraints on the tensors a^{2r}, b^{2r} :

$$a^{2r} - a^0 \otimes b^{2r} =_S \tilde{q}^r(\chi^1, \dots, \chi^{2r+1}), \quad (5)$$

where $\tilde{q}^r(\chi^1, \dots, \chi^{2r+1})$ is a constant tensor of order $2r+2$ computed by means of the correctors χ^1 to χ^{2r+1} and $=_S$ means that the equality holds up to symmetry.

Theorem 1 *Assume sufficient regularity of the data and let $\{a^{2r}, b^{2r}\}_{r=1}^{\lfloor \alpha/2 \rfloor}$ be symmetric, non-negative tensors satisfying (5). Then it holds*

$$\|u^\varepsilon - \tilde{u}\|_{L^\infty(0, \varepsilon^{-\alpha} T; W)} \leq C\varepsilon,$$

where the constant C is independent of ε and Ω and the norm $\|\cdot\|_W$ is equivalent to the $L^2(\Omega)$ -norm up to the Poincaré constant.

Theorem 1 ensures that any set $\{a^{2r}, b^{2r}\}_{r=1}^{\lfloor \alpha/2 \rfloor}$ satisfying the requirements gives an effective equation. Hence, this result implicitly defines a family of effective equations over timescales $\mathcal{O}(\varepsilon^{-\alpha})$.

3 Substantial cost reduction to compute the high order effective tensors

In [1], we provide an explicit procedure to compute the effective tensors $\{a^{2r}, b^{2r}\}$ in practice. As \tilde{q}^r may happen to be negative, the main challenge is to build non-negative a^{2r} that satisfy (5). The preminent computational cost of the procedure is the calculation of \tilde{q}^r . The natural—but naive—formula for \tilde{q}^r requires to solve the cell problems for χ^1 to χ^{2r+1} . However, exploiting a hidden structure of the cell problems, we prove the following result:

Theorem 2 *The tensor \tilde{q}^r involved in (5) can in fact be computed from $\chi^1, \dots, \chi^{r+1}$.*

Thanks to this result, the computational cost to compute the effective tensors $\{a^{2r}, b^{2r}\}_{r=1}^{\lfloor \alpha/2 \rfloor}$ is significantly reduced. Specifically, it allows to avoid solving

$$N(\alpha, d) = \binom{2\lfloor \alpha/2 \rfloor + 1 + d}{d} - \binom{\lfloor \alpha/2 \rfloor + 1 + d}{d}$$

cell problems (e.g., $N(6, 2) = 21$, $N(6, 3) = 85$).

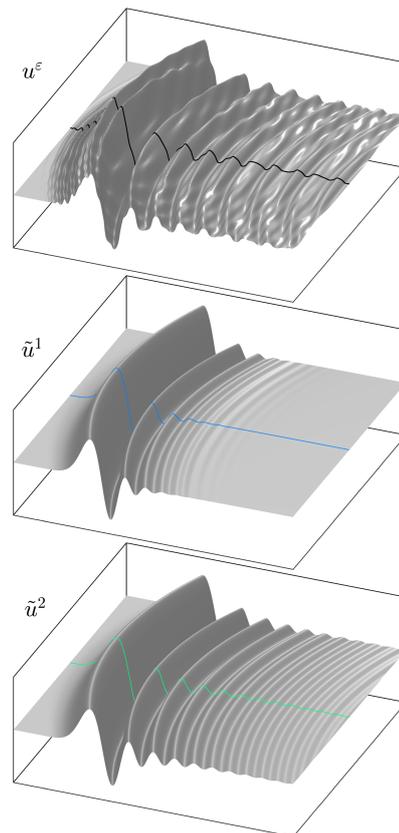


Figure 1: Comparison between u^ε (top) and high order effective solutions for $\alpha=2$ (\tilde{u}^1 , middle) and $\alpha=4$ (\tilde{u}^2 , bottom). See [1] for details.

References

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