integrability. As a practical example for the problem might serve the Darcy flow as a model of ground water flow. Here, $\boldsymbol{\mathcal{A}}$ is a random diffusion process modeling the unknown diffusion coefficient of the ground material. The right-hand side $\boldsymbol{f}$ would be the random (unknown) injection of pollutants into the ground water. Ultimately, the user would be interested in the average distribution of pollutants in the ground. Obviously, computing the right-hand side of (1) requires the user to sample Ω and Ξ successively, whereas computing the left-hand side of (1) forces the user to sample the much larger product space $\Omega \times \Xi$. Therefore, an accurate discretization of $\mathcal{A}$ can help saving significant computational cost.

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Approximation of high order homogenized wave equations for long time wave propagation

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(joint work with Assyr Abdulle)

Abstract

While the standard homogenized wave equation describes the effective behavior of the wave at short times, it fails to capture the macroscopic dispersion that appears at long times. To describe the dispersion, the effective model must include additional operators of higher order. In this work, we present a practical way to construct effective equations of arbitrary order in periodic media, with a focus on their numerical approximation. In particular, we exhibit an important structure hidden in the definition of the high order effective tensors which allows a significant reduction of the computational cost for their approximation.

1. Introduction

Let $a(y)$ be a $[0,1)^d$-periodic tensor, $\Omega \subset \mathbb{R}^d$ be a hypercube and for $\varepsilon > 0$ let $u^\varepsilon : [0,T] \times \Omega \to \mathbb{R}$ be the solution of the wave equation

$$\partial_t^2 u^\varepsilon(t,x) - \nabla_x \cdot \Big(a\big(\tfrac{x}{\varepsilon}\big) \nabla_x u^\varepsilon(t,x) \Big) = f(t,x), \tag{1}$$

for $(t,x) \in (0,T] \times \Omega$, where we impose Ω-periodic boundary conditions, the initial conditions and the source f are assumed to have $\mathcal{O}(1)$ frequencies and $\mathcal{O}(1)$ support. The hypercube Ω can be arbitrarily large but its length in every directions must be an integer multiple of ε. To accurately approximate u^ε, standard numerical methods require a grid resolution of order $\mathcal{O}(\varepsilon)$ in the whole domain, which leads to a prohibitive computational cost as $\varepsilon \to 0$. In the regime $\varepsilon \ll 1$, homogenization theory provides a way to approximate u^ε at a cost that is independent of ε: the result states that $\lim_{\varepsilon\to 0} u^\varepsilon = u^0$ in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$, where u^0 solves the *homogenized equation*

$$\partial_t^2 u^0(t,x) - a^0_{ij}\partial^2_{ij} u^0(t,x) = f(t,x), \tag{2}$$

equipped with the same initial and boundary conditions as (1). The homogenized tensor a^0 is constant and can be computed by means of (first order) correctors, solutions of (first order) *cell problems*, i.e., periodic elliptic PDEs in $[0,1)^d$ involving $a(y)$. In practice, we observe that for long times $t = \mathcal{O}(\varepsilon^{-\alpha})$ $\alpha \geq 2$, dispersion effects that appear in the L^2 behavior of $u^\varepsilon(t,\cdot)$ are not captured by $u^0(t,\cdot)$. *High order effective equations* are effective models that describe the dispersion (with an accuracy that should increase with the order). Several definitions of high order effective equations were recently proposed [5, 4, 3]. Although the form of the equations are not the same, they all involve the same high order effective quantities.

2. Family of effective equations of arbitrary order

We present the high order models introduced in [3]. For $q \in \mathrm{Sym}^n(\mathbb{R}^d)$, a symmetric tensor of order n, we denote the operator $q\nabla_x^n = \sum q_{i_1\cdots i_n}\partial^n_{i_1\cdots i_n}$. For a timescale $\mathcal{O}(\varepsilon^{-\alpha})$, the effective equations have the form

$$\partial_t^2 \tilde u - a^0\nabla_x^2 \tilde u - \sum_{r=1}^{\lfloor \alpha/2\rfloor} (-1)^r \varepsilon^{2r} L^{2r}\tilde u = Qf, \tag{3}$$

where the operators L^{2r} and Q are defined as

$$L^{2r} = a^{2r}\nabla_x^{2r+2} - b^{2r}\nabla_x^{2r}\partial_t^2, \qquad Q = 1 + \sum_{r=1}^{\lfloor \alpha/2\rfloor} (-1)^r \varepsilon^{2r} b^{2r}\nabla_x^{2r},$$

and $a^{2r} \in \mathrm{Sym}^{2r+2}(\mathbb{R}^d)$, $b^{2r} \in \mathrm{Sym}^{2r}(\mathbb{R}^d)$. Note that if a^{2r}, b^{2r} are non-negative, (3) is well-posed.

The effective tensors a^{2r}, b^{2r} are derived by generalizing the technique introduced in [2, 1] for $\mathcal{O}(\varepsilon^{-2})$ timescales. Using asymptotic expansion we construct an adaptation $\mathcal{B}^\varepsilon\tilde u$ that approximates u^ε. An energy estimate tells us that for $\tilde u$ to be close to u^ε up to $\mathcal{O}(\varepsilon^{-\alpha})$ timescales, $\mathcal{B}^\varepsilon\tilde u - u^\varepsilon$ must satisfy the wave equation with a right hand side of order $\mathcal{O}(\varepsilon^{\alpha+1})$ in the $\mathrm{L}^\infty(0,\varepsilon^{-\alpha}T;\mathrm{L}^2(\Omega))$-norm. We then combine (i) the ansatz

$$\mathcal{B}^\varepsilon\tilde u(t,x) = \tilde u(t,x) + \sum_{k=1}^{\alpha+2} \chi^k(t,x,y)\nabla_x^k u(t,x),$$

where the *k-th order corrector* $\chi^k = \left\{\chi^k_{i_1\cdots i_k}\right\}$ has value in $\mathrm{Sym}^k(\mathbb{R}^d)$, and (ii) inductive Boussinesq tricks (we use (3) to replace time derivatives with space derivatives) and obtain the *cell problems*, which have the cascade form:

$$
\begin{aligned}
\mathcal{A}\chi^1_{i_1} &= \mathcal{F}^1_{i_1}(a),\\
\mathcal{A}\chi^2_{i_1 i_2} &= \mathcal{F}^2_{i_1}(a,\chi^1,a^0),\\
\mathcal{A}\chi^{2r+1}_{i_1\cdots i_{2r+1}} &= \mathcal{F}^{2r+1}_{i_1\cdots i_{2r+1}}\big(a,\chi^1,\ldots,\chi^{2r}\big),\\
\mathcal{A}\chi^{2r+2}_{i_1\cdots i_{2r+2}} &= \mathcal{F}^{2r+2}_{i_1\cdots i_{2r+2}}\big(a,\chi^1,\ldots,\chi^{2r+1},a^{2r}-a^0\otimes b^{2r}\big),
\end{aligned}
\tag{4}
$$

where $\mathcal{A} = -\nabla_y \cdot \big(a\nabla_y \,\cdot\big)$ and $\mathcal{F}^k_{i_1\cdots i_k}$ are explicitly defined in [3]. While the odd order cell problems are well-posed unconditionally, the solvability of the even order cell problems provides constraints on the tensors a^{2r}, b^{2r}:

$$
a^{2r} - a^0 \otimes b^{2r} =_S \check{q}^r\big(\chi^1,\ldots,\chi^{2r+1}\big),
\tag{5}
$$

where $\check{q}^r\big(\chi^1,\ldots,\chi^{2r+1}\big)$ is a constant tensor of order $2r+2$ computed by means of the correctors χ^1 to χ^{2r+1} and $=_S$ indicates that the equality is relaxed up to symmetry.

Under sufficient regularity of the data, we prove that if the tensors $\{a^{2r},b^{2r}\}_{r=1}^{\lfloor\alpha/2\rfloor}$ are non-negative and verify (5), then (3) is well-posed and its solution satisfies

$$
\|u^\varepsilon - \tilde{u}\|_{\mathrm{L}^\infty(0,\varepsilon^{-\alpha}T;W)} \le C\varepsilon,
$$

where the constant C is independent of ε and Ω and the norm $\|\cdot\|_W$ is equivalent to the $\mathrm{L}^2(\Omega)$-norm up to the Poincaré constant. This result ensures that any set $\{a^{2r},b^{2r}\}_{r=1}^{\lfloor\alpha/2\rfloor}$ satisfying the requirements gives an effective equation. Hence, this result implicitly defines a family of effective equations over timescales $\mathcal{O}(\varepsilon^{-\alpha})$.

3. Cost reduction for the computation of the effective tensors

In [3], we provide an explicit procedure to compute the effective tensors $\{a^{2r},b^{2r}\}$ in practice. As $\check{q}^r$ may happen to be negative, the main challenge is to build non-negative a^{2r} that satisfy (5). The preeminent computational cost of the procedure is the calculation of $\check{q}^r$. The natural—but naive—formula for $\check{q}^r$ requires to solve the cell problems for all the distinct entries of χ^1 to χ^{2r+1}. However, exploiting a hidden structure of the cell problems, we prove that the tensor $\check{q}^r$ involved in (5) can in fact be computed from $\chi^1,\ldots,\chi^{r+1}$. Thanks to this result, the computational cost to compute the effective tensors $\{a^{2r},b^{2r}\}_{r=1}^{\lfloor\alpha/2\rfloor}$ is significantly reduced. Specifically, it allows to avoid solving

$$
N(\alpha,d) = \binom{2\lfloor\alpha/2\rfloor+1+d}{d} - \binom{\lfloor\alpha/2\rfloor+1+d}{d}
$$

cell problems (e.g., $N(6,2)=21$, $N(6,3)=85$).

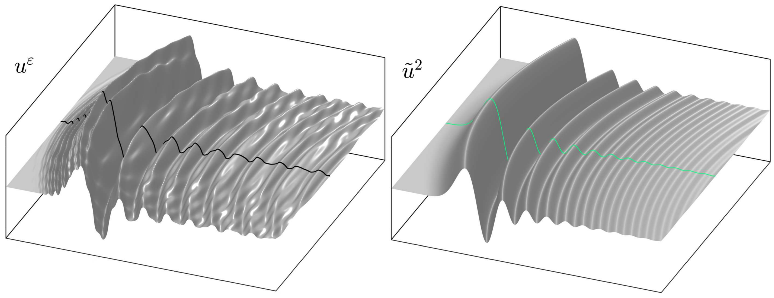

FIGURE 1. Comparison of u^ε and $\tilde{u}$ for $\alpha = 4$ ($\tilde{u}^2$). See [3] for details.

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Numerical Homogenization of Multiscale Fault Networks

Ralf Kornhuber

(joint work with Martin Heida, Joscha Podlesny, Harry Yserentant)

Viscoelastic contact problems involving rate- and state-dependent (RSD) friction conditions on multiscale fault networks play a crucial role in understanding the scaling properties of deformation accumulation.

After a short revision of recent results concerning analysis and numerical analysis of rigid contact problems with RSD friction [6, 7], we concentrate on a scalar elliptic model problem with jump conditions on a hierarchy of networks

$$\Gamma^{(K)} = \bigcup_{k=1}^{K} \Gamma_k \subset \boldsymbol{Q} \subset \mathbb{R}^d$$

of interfaces Γ_k, $k = 1, \dots,$ with fractal limit $\Gamma = \Gamma^{(\infty)}$. We derive an associated 'fractal' function space $\mathcal{H}$ which then is characterized in terms of generalized jumps and gradients, and we prove continuous embeddings of $\mathcal{H}$ into $L^2(\boldsymbol{Q})$ and $H^s(\boldsymbol{Q})$ with $s < 1/2$ [1].