Semiclassical $\mathcal{S}$-matrix and black hole entropy in dilaton gravity

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ABSTRACT: We use complex semiclassical method to compute scattering amplitudes of a point particle in dilaton gravity with a boundary. This model has nonzero minimal black hole mass M_{cr}. We find that at energies below M_{cr} the particle trivially scatters off the boundary with unit probability. At higher energies the scattering amplitude is exponentially suppressed. The corresponding semiclassical solution is interpreted as formation of an intermediate black hole decaying into the final-state particle. Relating the suppression of the scattering probability to the number of the intermediate black hole states, we find an expression for the black hole entropy consistent with thermodynamics. In addition, we fix the constant part of the entropy which is left free by the thermodynamic arguments. We rederive this result by modifying the standard Euclidean entropy calculation.

Contents

1   Introduction

Black hole (BH) information paradox [1, 2] has long history ever since the discovery of
BH evaporation [3]. Recently there has been a remarkable progress towards its resolution.
Within the framework of the AdS/CFT correspondence, refs. [4, 5] performed semiclassical
calculations of the entanglement entropy of an evaporating BH and demonstrated that it
follows the Page curve [6, 7], consistent with unitarity. To derive the expression for the
entanglement entropy these calculations use complex saddle points of the gravitational path
integral — replica wormholes [8–12]. It has been suggested that this approach applies also
beyond the holographic setting leading to the "island rule" for the entropy of the Hawking

radiation [11–15]. It still remains to be understood, however, how quantum correlations are encoded in the state of the emitted quanta. Only then the information paradox will be completely resolved [16, 17].

A direct approach to study unitarity of BH evaporation is to compute the related elements of the gravitational $\mathcal{S}$-matrix [18]. In this case one treats BH formation and its subsequent decay as a scattering process [18, 19] mediated by a metastable bound state. On general grounds, consideration of this complete process appears more adequate than its splitting into separate stages of collapse and evaporation. It was argued in [20] that when both initial and final states of the scattering process are semiclassical, the related amplitudes can be evaluated using complex saddle points of the path integral with appropriate boundary conditions, cf. [21, 22].

In this paper we further develop complex semiclassical method for gravitational $\mathcal{S}$-matrix. Using this method, we compute the scattering amplitudes and probe the entropy of black holes in $(1+1)$-dimensional dilaton gravity.

We start with an outline of the method. Consider complex quantum transition including collapse of matter in pure initial state Ψ_i into a black hole and evaporation of the latter into the state Ψ_f. This process interpolates between the free flat-space states Ψ_i and Ψ_f and therefore defines a gravitational $\mathcal{S}$-matrix [18]. Schematically, one can write a path integral for the transition amplitude as

$$\mathcal{A}_{fi} \equiv \langle \Psi_f | \hat{\mathcal{S}} | \Psi_i \rangle = \int \mathcal{D}\Phi \, e^{iS'[\Phi]} \, \Psi_f^*[\Phi] \, \Psi_i[\Phi] \,, \tag{1.1}$$

where $S'[\Phi]$ is the classical action and Φ includes all fields of the model — matter fields, metric, and Faddeev-Popov ghosts. Precise definition of the gravitational path integral (1.1) is a formidable task. One can assume, however, that the initial and final states of the process are semiclassical. In field theory this means that they contain many quanta at high occupation numbers. Then the integral can be evaluated in the saddle-point approximation, giving $\mathcal{A}_{fi} \simeq e^{iS'[\Phi_{cl}]} \, \Psi_f^*[\Phi_{cl}] \, \Psi_i[\Phi_{cl}]$, where the semiclassical configuration Φ_{cl} extremizes the integrand in eq. (1.1) i.e. solves the classical field equations.

Importantly, Φ_{cl} does not coincide with the classical collapsing solution: like all configurations in the path integral (1.1) it starts from the flat space in the past and arrives to it in the future. Since real solutions with these properties do not exist, Φ_{cl} is a complex saddle point describing an exponentially suppressed process. This is to be expected: the intermediate black hole mainly emits Hawking radiation with low occupancies, and the probability of producing a semiclassical state Ψ_f is exponentially small.

Generically, there may exist many complex saddle points for eq. (1.1), and one has to select the physical one giving the main contribution into the path integral. To this end, we use the method suggested in [20] (see [23, 24] for quantum mechanical applications). The main idea is to enforce the scattering boundary conditions in the path integral (1.1) with a special variant of a constrained instanton method. After that the physical complex solutions are obtained by smooth deformation of the real solutions that describe classical low-energy scattering without black hole production.
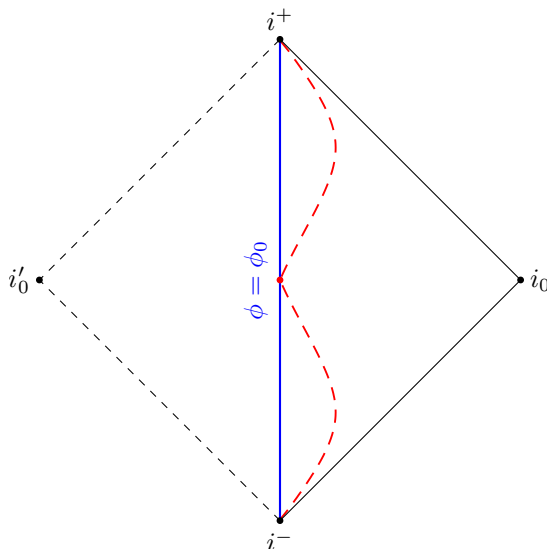
Figure 1. Penrose diagram for the vacuum solution in the CGHS model. The boundary $\phi = \phi_0$ cuts off the strongly coupled region to the left making the model semiclassically tractable. Dashed line shows the trajectory of a particle reflecting off the boundary.

Our method reduces construction of the semiclassical gravitational $\mathcal{S}$-matrix to solution of the classical field equations in the complex domain. Though this is in principle managable, applications to four-dimensional field theories with dynamical gravity are challenging. So far this method has been applied only in spherically reduced models with simplified matter content [20].

Below we consider another simplified model based on the two-dimensional Callan-Giddings-Harvey-Strominger (CGHS) [25, 26] dilaton gravity. The model describes interaction of a non-dynamical metric $g_{\mu\nu}(x)$ and dilaton $\phi(x)$ with matter. The action of this model is qualitatively similar to that of spherically-reduced multidimensional gravity, where $g_{\mu\nu}$ includes the time and radial metric components and $\mathrm{e}^{-2\phi}$ is related to the areas of the extra spheres [26]. The vacuum solution in this model has flat $g_{\mu\nu}$ and linear dilaton field ϕ changing from $-\infty$ to $+\infty$. For positive values of ϕ, gravity becomes strongly coupled precluding the semiclassical analysis. To make the model tractable, we cut off the strongly coupled region by introducing a reflective boundary along the line of constant dilaton $\phi(x) = \phi_0$, where ϕ_0 is negative and large [27–31]. All fields in the path integral are then restricted to the submanifold $\phi < \phi_0$ (the rightmost region in figure 1). This model was shown to be equivalent to the flat limit of the Jackiw-Teitelboim gravity [32, 33] with a boundary both at the classical [34] and quantum level [35].

We also make the second radical simplification. Instead of a full-fledged field theory, we represent the matter sector with a point particle of mass m moving along the trajectory $x^\mu = x_*^\mu(\tau)$. One can interpret it as a toy model for the narrow wavepacket in field theory. We find complex semiclassical solutions $\Phi_{cl} = \{g_{\mu\nu}(x), \phi(x), x_*^\mu(\tau)\}$ and compute the transition amplitudes of the particle. At low energies M the particle trivially scatters off the boundary with unit probability, see the dashed line in figure 1. However, once the

energy exceeds a certain critical value M_{cr} the semiclassical solutions become complex. Initial and final parts of these solutions describe formation of an intermediate BH with mass M from the particle and, after complex evolution, a particle in the final state. The transition probability equals

$$\mathcal{P}_{fi} = |\mathcal{A}_{fi}|^2 \simeq e^{-2\pi(M - M_{cr})/\lambda}, \qquad\qquad M > M_{cr}, \qquad\qquad (1.2)$$

independently of the particle mass m. Here λ is the CGHS energy scale. Notably, M_{cr} coincides with the minimal mass of black holes in the model. It is worth stressing that our semiclassical method provides the phase of the amplitude, in addition to its absolute value.

One can interpret the probability (1.2) as follows [22]. The intermediate BH has entropy $\Sigma_{\mathrm{BH}}(M)$ and an exponentially large number of states $\exp(\Sigma_{\mathrm{BH}})$. Then it is expected to decay into the single-particle final state with probability $\mathcal{P} \propto \exp(-\Sigma_{\mathrm{BH}})$. Comparing to eq. (1.2), we find the entropy of the CGHS black holes,

$$\Sigma_{\mathrm{BH}} = 2\pi(M - M_{cr})/\lambda. \qquad\qquad (1.3)$$

This expression is consistent with the results for BH entropy in similar models [36–39].

Our result, however, raises a puzzle. A naive extrapolation to our model of the Gibbons-Hawking Euclidean calculation [40] of the BH entropy gives,

$$\Sigma_{\mathrm{BH}}^{naive} = 2\pi M/\lambda, \qquad\qquad (1.4)$$

independently of the boundary parameter ϕ_0. The expression (1.4) would imply that the entropy of the critical black hole with mass M_{cr} is non-zero. If this were the case, one would see an unphysical jump of the scattering probability $\mathcal{P}_{fi}$ at $M = M_{cr}$. Our result in eq. (1.2), quite consistently, has no jump.

Note that the constant term in BH entropy is not fixed by the laws of BH thermodynamics. In previous Euclidean calculations of BH entropy in dilaton gravity, this constant was added somewhat ad hoc. We show that eq. (1.3) can be recovered naturally by a suitable modification of the Euclidean procedure once the presence of the boundary at $\phi = \phi_0$ is taken into account.

It is worth stressing that the arguments leading to eq. (1.3) do not apply to multidimensional gravity, where critical BHs are known to have nonzero entropy [40, 41]. The masses of the latter are minimal only among the black holes with given charges and/or angular momenta, whereas the absolute minimum is reached by the neutral BH with the Planckian mass. In this case collision of charged particles may lead to formation of a neutral BH, with charge and angular momentum carried away by bremsstrahlung. Then the corresponding scattering probability is a continuous function of energy [20].

The present paper is organized as follows. In section 2 we introduce our setup. The scattering amplitude is calculated in section 3. In section 4 we discuss the entropic interpretation of the scattering probability and the Euclidean calculation of BH entropy. Section 5 is devoted to conclusions. Several appendices contain details of the calculations.

2 The setup

2.1 Dilaton gravity

We consider non-perturbative scattering in two-dimensional dilaton gravity with a boundary [31], see also [25, 27–30, 35]. The gravitational action[1]

$$S_{\text{gr}} = \int\limits_{\phi < \phi_0} d^2x \sqrt{-g}\, \mathrm{e}^{-2\phi} \left[R + 4(\nabla\phi)^2 + 4\lambda^2 \right] + 2 \int\limits_{\phi = \phi_0} d\tau_0\, \mathrm{e}^{-2\phi} \left(K + 2\lambda \right) \tag{2.1}$$

describes the CGHS model [25] with non-dynamical metric $g_{\mu\nu}(x)$ and dilaton $\phi(x)$. Besides, it includes the timelike boundary at $\phi = \phi_0$ which cuts off the region of strong coupling. Importantly, a regulating boundary should be present in all configurations in the path integral (1.1), otherwise the CGHS fields would become singular at the quantum level [35, 42]. In eq. (2.1) we included the Gibbons-Hawking term [40] at $\phi = \phi_0$ with proper time τ_0, extrinsic curvature $K = \nabla_\mu n_0^\mu$ and outer normal[2] n_0^μ. Parameter λ sets the energy scale of the model.

The semiclassical expansion is controlled by the combination $\mathrm{e}^{2\phi_0}$. Indeed, a shift $\phi \mapsto \phi + \phi_0$ brings this parameter in front of the classical action, at the place of the Planck constant in the path integral. In what follows we consider the case

$$\mathrm{e}^{2\phi_0} \ll 1\,, \tag{2.2}$$

and work to the leading order in this parameter.

Without matter, the general solution in the bulk is,

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)}\,, \qquad \phi = -\lambda r\,, \qquad f(r) = 1 - \frac{M}{2\lambda}\, \mathrm{e}^{-2\lambda r}\,, \tag{2.3}$$

where M is the Arnowitt-Deser-Misner (ADM) mass. This constitutes the two-dimensional analog on the Birkhoff theorem [43], which we derive in appendix A.1 for completeness. For $M = 0$ the spacetime is flat, while for $M > 0$ it describes a black hole. In eqs. (2.3) we use Schwarzschild coordinates with the "radius" $r = -\phi/\lambda$ and the orthogonal time t. The light-like line $r = r_h$,

$$r_h = \frac{1}{2\lambda} \log\left(\frac{M}{2\lambda}\right)\,, \tag{2.4}$$

with $f(r_h) = 0$ is a black hole horizon. Penrose diagrams of the solutions with $M = 0$ and $M > 0$ are shown in figures 1 and 2, respectively.

It is not enough, however, to solve the bulk field equations: one should also add the boundary. This amounts to cutting off the spacetime at $\phi = \phi_0$ and imposing the boundary condition

$$n_0^\mu \nabla_\mu \phi = \lambda \qquad \text{at} \quad \phi = \phi_0\,, \tag{2.5}$$

which follows from variation of the action (2.1) with respect to the boundary metric, see appendix A.3. The spacetime (2.3) satisfies eq. (2.5) only for $M = 0$ when it is flat. The

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[1]We use the metric signature $(-, +)$ and Greek indices $\mu, \nu, \ldots = 0, 1$.

[2]The direction of the normal is fixed by the condition $n_0^\mu \nabla_\mu \phi > 0$.
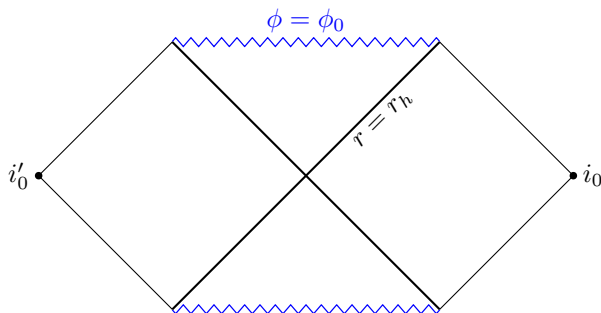
Figure 2. Black hole in the CGHS model with a boundary. The field equations break down at the line $\phi = \phi_0$, indicating a singularity.

breakdown of the equations of motion at the line $\phi = \phi_0$ for $M \neq 0$ implies that it should be interpreted as a singularity. This line is spacelike and hidden under the black hole horizon if $r_h > -\phi_0/\lambda$ or $M > M_{cr}$, where

$$M_{cr} = 2\lambda \mathrm{e}^{-2\phi_0} \tag{2.6}$$

is the critical mass, see figure 2. At $M < M_{cr}$, $M \neq 0$ the solution (2.3) is a spacetime with timelike naked singularity. The latter does not form in the collapse of a regular matter [31].

2.2 Classical scattering

Now we want to consider scattering of a point particle with mass m and action

$$S_m = -m \int d\tau \tag{2.7}$$

off the boundary. Here the parameter τ is a proper time of the particle. One can find the particle trajectories using the well-known techniques developed for thin shells in multidimensional gravity [44]. We describe the particle trajectory with radius $r = -\phi/\lambda$ as a function of the proper time $r = r_*(\tau)$. The two-dimensional Birkhoff theorem guarantees that the empty spacetime regions to the left and to the right of the particle are either Schwarzschild or Minkowski.

Then, if the particle starts evolution in Minkowski spacetime, the solution in the "inner" region $r < r_*$ remains flat,

$$ds^2 = -dT^2 + dr^2, \qquad \phi = -\lambda r. \tag{2.8}$$

Note that we introduced the notation T for the time coordinate in the inner region to emphasize its difference from the time t of the distant observer. Similarly, the "outer" region $r > r_*$ is described by the Schwarzschild metric (2.3) with conserved gravitational mass M.

Since the particle energy-momentum tensor is concentrated at the worldline, the derivatives of the metric and dilaton change discontinuously across it. In appendix A.2 we derive

the Israel junction conditions for the jumps of the extrinsic curvature and normal derivative of the dilaton,

$$[n^\mu \nabla_\mu \phi] = \frac{m}{4} e^{2\phi(r_*)}, \qquad [K] = 2 [n^\mu \nabla_\mu \phi]. \qquad (2.9)$$

Here the square brackets represent difference of the values at $r_* + 0$ and $r_* - 0$; the worldline normal n^μ points towards large r. Substituting the inner and outer spacetimes (2.8), (2.3) into eq. (2.9) one finds equation of motion for the particle,

$$\dot{r}_*^2 + V_{\rm eff}(r_*) = 0, \qquad V_{\rm eff}(r) = 1 - \left(\frac{M}{m} + \frac{m}{8\lambda} e^{-2\lambda r} \right)^2, \qquad (2.10)$$

where dot is a derivative with respect to the proper time τ. Recall that $M > m$ is the particle total energy measured at infinity, cf. eq. (A.10). This equation has an intuitive form of non-relativistic "energy conservation law" with effective potential $V_{\rm eff}$. The latter is negative everywhere, it monotonically increases from a finite value at the boundary $r = r_0$,

$$r_0 = -\phi_0/\lambda, \qquad (2.11)$$

to $1 - M^2/m^2 < 0$ as $r \to +\infty$. The details of the derivation are given in appendix A.2.

Now it is clear that the left-moving particle with energy $M < M_{cr}$ always reaches the boundary $r = r_0$ at some moment of time $\tau = \tau_\times$. Then it reflects back. In appendix A.3 we demonstrate that reflection of the particle from the boundary simply flips the sign of its radial velocity, $\dot{r}_*(\tau_\times + 0) = -\dot{r}_*(\tau_\times - 0)$. At late times the particle goes to $r \to +\infty$.

The classical story changes completely if the particle energy M exceeds M_{cr}. In this case it first crosses the horizon $r_h > r_0$ of the outer metric (2.3) and thus forms a black hole. Whence the particle can be retrieved only quantum mechanically with exponentially small probability.

3 Semiclassical scattering amplitude

3.1 Semiclassical method

Quantum $\mathcal{S}$-matrix is an operator connecting initial and final Fock states of the process. It is formally defined as

$$\hat{\mathcal{S}} = \hat{U}_0(0, t_f) \, \hat{U}(t_f, t_i) \, \hat{U}_0(t_i, 0), \qquad (3.1)$$

where $\hat{U}$ and $\hat{U}_0$ are the interacting and free evolution operators, and the limits $t_i \to -\infty$, $t_f \to +\infty$ are assumed. In the path integral representation eq. (3.1) reads,

$$\mathcal{A}_{fi} \equiv \langle \Psi_f | \hat{\mathcal{S}} | \Psi_i \rangle = \int \mathcal{D}\Phi \, e^{iS_0(0_+, t_f) + iS(t_f, t_i) + iS_0(t_i, 0_-)} \, \Psi_f^*[\Phi] \, \Psi_i[\Phi], \qquad (3.2)$$

where Φ denotes all fields of the model on the time contour in figure 3, while S and S_0 are the interacting and free classical actions[3] on the respective parts of the contour. Note that the fields at the endpoints of the contour $t = 0_-$ and $t = 0_+$ do not coincide. We also introduced the wave functions Ψ_i, Ψ_f of the free initial and final states.

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[3]We shortly denoted $S'[\Phi] \equiv S_0(0_+, t_f) + S(t_f, t_i) + S_0(t_i, 0_-)$ in eq. (1.1).
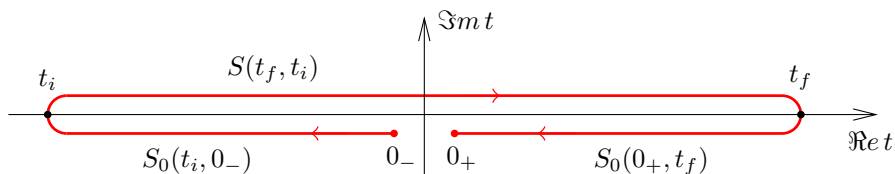
Figure 3. Time contour in the path integral for the scattering amplitude.

We generalize eq. (3.2) to gravity in a straightforward way. In this case Φ includes the particle trajectory $r_*(\tau)$, metric $g_{\mu\nu}$, dilaton ϕ and Faddeev-Popov ghosts. The interacting action

$$S(t_f, t_i) = S_{\text{gr}} + S_m + S_{\text{GH}}, \qquad (3.3)$$

involves gravitational and matter contributions, as well as the standard Gibbons-Hawking term S_{GH} at infinity; see its definition in appendix C.

Importantly, we assume that configurations $\Phi = \{g_{\mu\nu}(x), \phi(x), r_*(\tau)\}$ in the path integral (3.2) have trivial topology expected from the scattering processes. First, they should contain the boundary $\phi = \phi_0$. Second, they should start from flat spacetime at $t = t_i$ and arrive to it in the future. This gives a preferred choice of the asymptotic time t changing from t_i to t_f by the clock of the distant observer. The free actions $S_0 = S_m$ describe a particle in flat spacetime and $\Psi_{i,f} \propto e^{\mp ipr}$ are the momentum eigenstates of this particle with $p = \sqrt{M^2 - m^2}$.

In the semiclassical limit $e^{2\phi_0} \ll 1$ the classical action S becomes large and the integral (3.2) can be evaluated in the saddle-point approximation,

$$\mathcal{A}_{fi} \simeq e^{iS_{\text{tot}}[\Phi_{cl}]}, \qquad (3.4)$$

where

$$S_{\text{tot}}[\Phi] = S_0(0_+, t_f) + S(t_f, t_i) + S_0(t_i, 0_-) - i\ln \Psi_f^* - i\ln \Psi_i \qquad (3.5)$$

is the total action and Φ_{cl} is a complex classical solution extremizing S_{tot}. The Faddeev-Popov ghosts can be neglected at this point as they don't contribute to the leading exponential term.

3.2 From low to high energies

It is straightforward to compute the amplitude at $M < M_{cr}$ substituting the real classical solution into eq. (3.4), see appendix C. In the overcritical case, however, the task of finding the relevant saddle-point configuration becomes non-trivial. The ordinary collapsing solutions are of no use here, since they describe formation of black holes and therefore violate the requirement of flat spacetime in the asymptotic future.

To enforce this requirement, we introduce a positive-definite and diffeomorphism-invariant functional $\mathcal{T}_{\text{int}}[\Phi]$ estimating the duration of the scattering process from the viewpoint of a distant observer. Namely, $\mathcal{T}_{\text{int}}[\Phi]$ should be finite on any scattering configuration Φ interpolating between flat spacetimes at $t \to \pm\infty$, and infinite otherwise. Then

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we constrain the path integral (3.2) to run only over configurations with finite values of $\mathcal{T}_{\text{int}}[\Phi]$. Technically, this is implemented by inserting the unity

$$
1 = \int\limits_0^{+\infty} d\mathcal{T}_0 \, \delta(\mathcal{T}_{\text{int}}[\Phi] - \mathcal{T}_0) = \int\limits_0^{+\infty} d\mathcal{T}_0 \int\limits_{-i\infty}^{+i\infty} \frac{d\varepsilon}{2\pi i} \, e^{-\varepsilon(\mathcal{T}_{\text{int}} - \mathcal{T}_0)}
\tag{3.6}
$$

into the integrand of eq. (3.2) and interchanging the order of integration over $\mathcal{D}\Phi$ and $d\mathcal{T}_0 d\varepsilon$.

To have a specific example, consider the choice

$$
\mathcal{T}_{\text{int}}[\Phi] = \int d^2x \, \sqrt{-g} \, L(\phi) \left[\lambda^2 - (\nabla\phi)^2\right]^2 , \qquad L(\phi) = e^{-4\phi} \delta(\phi - \phi_\varepsilon)/\lambda^2 ,
\tag{3.7}
$$

where the integration is concentrated on the line $\phi = \phi_\varepsilon$ which is far away from the boundary, $|\phi_\varepsilon| \gg |\phi_0|$. Clearly, $\mathcal{T}_{\text{int}}$ in eq. (3.7) is positive-definite for real $g_{\mu\nu}$ and ϕ. Besides, in the asymptotically Schwarzschild spacetime with mass M one finds $\mathcal{T}_{\text{int}} = \int dt \, M^2/4\lambda$. Thus, this functional estimates the asymptotic time spent by the ADM mass M in the "interaction region" to the left of $\phi = \phi_\varepsilon$. We stress that our method is not specific to the choice (3.7) and can exploit any appropriate positive-definite $\mathcal{T}_{\text{int}}$.

Inserting the unity (3.6) into eq. (3.2), one finds the path integral with the "regularized" interacting action

$$
S_\varepsilon[\Phi] = S[\Phi] + i\varepsilon \, \mathcal{T}_{\text{int}}[\Phi] - i\varepsilon \, \mathcal{T}_0
\tag{3.8}
$$

and the additional integrations over ε and $\mathcal{T}_0$. The δ-function (3.6) ensures that the configurations Φ leave the "interaction region" in a finite "time" $\mathcal{T}_0$. Besides, we can use the positive definiteness of $\mathcal{T}_{\text{int}}$, to improve convergence of the path integral. To this end, we deform the contour of ε-integration into the region $\Re e \, \varepsilon \geq 0$.

At fixed $\mathcal{T}_0$ and ε the semiclassical solutions extremize $S_\varepsilon[\Phi]$. The additional saddle-point integrals with respect to ε and $\mathcal{T}_0$ give $\varepsilon = 0$. We therefore perform calculations at $\varepsilon > 0$ and send $\varepsilon \to +0$ in the end, restoring the original saddle-point equations. The "regularized" semiclassical solutions at $\varepsilon > 0$ have three important properties [20, 23, 24]. First, they leave the "interaction region" $\phi > \phi_\varepsilon$ in finite time. Second, the corresponding fields are generically complex-valued. Third, they can be obtained by smooth deformation of the classical reflecting solutions.

To demonstrate these properties, we consider the "shell-like" term (3.7) concentrated at $\phi = \phi_\varepsilon$. Junction at this shell changes the metric to the left of the shell, at $r < -\phi_\varepsilon/\lambda$. By Birkhoff theorem, the form of this metric is still Schwarzschild, eq. (2.3), but with the complex mass

$$
M \mapsto M_\varepsilon = M + i \, \varepsilon' .
\tag{3.9}
$$

In appendix B we show that ε' is positive and proportional to ε. After this replacement the regularized saddle-point configurations change continuously with energy. At $M < M_{cr}$ they are close to the real classical solutions: the particle trajectory $r_*(\tau)$ reaches the boundary at $r = r_0$ and reflects from it, see figure 4a. The outer time t changes almost along the real axis (figure 4b). Importantly, the horizon of the outer metric now acquires a positive imaginary part, $\Im m \, r_h > 0$, see eq. (2.4). Thus, even at $M > M_{cr}$ the particle continues to
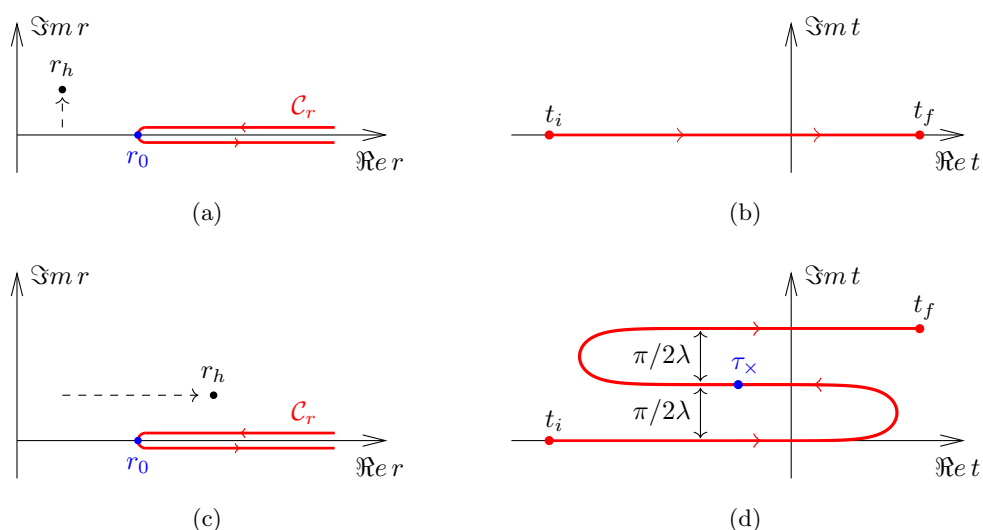
Figure 4. Trajectory of the particle in complex planes of the radial and temporal Schwarzschild coordinates for the regularized solutions at $M < M_{cr}$ (top) and $M > M_{cr}$ (bottom). The radial coordinate $r_*(\tau)$ varies along the almost real contour $\mathcal{C}_r$ as the particle's proper time τ changes from $-\infty$ to $+\infty$. The particle bypasses the event horizon which is shifted upwards in the complex plane.

evolve along the contour $\mathcal{C}_r$ in figure 4c. It bypasses the horizon in complex r-plane, both on the way in and on the way out. But now the outer Schwarzschild time of the particle is essentially complex. Equations (2.3) and (2.10) imply,

$$t(r_*) = \int_{r_i}^{r_*} dr \, \frac{\sqrt{f(r) - V_{\text{eff}}(r)}}{f(r) \, \dot{r}_*(r)}, \qquad \dot{r}_*(r) = \mp\sqrt{-V_{\text{eff}}(r)}, \tag{3.10}$$

where the integral runs along the contour $\mathcal{C}_r$ in figure 4c and the minus (plus) sign of $\dot{r}_*$ correspond to motion prior to (after) reflection at r_0. The integrand in eq. (3.10) has a pole at the horizon giving an imaginary time change

$$\Im m \, (t_f - t_i) = 2\pi \operatorname*{Res}_{r=r_h} f^{-1}(r) = \frac{\pi}{\lambda}. \tag{3.11}$$

Notably, the time contour in eq. (3.10) is smooth at finite $\varepsilon > 0$, see figure 4d. Since the regularized solutions are now connected to the classical ones, we assume that they also represent the physical saddle points of the path integral (3.2).[4] Once the amplitude (3.4) is computed, we send $\varepsilon \to +0$.

3.3 The result

By construction, the regularized saddle-point configurations have trivial topology, just like the reflective classical solutions at low energies. Their action S_{tot} is computed in a straightforward way, given eq. (2.10) for the particle trajectory $r_*(\tau)$ and the inner and

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[4]This assumption has been confirmed in quantum-mechanical systems by direct comparison with the solutions of the Schrödinger equation [23, 24, 45, 46].
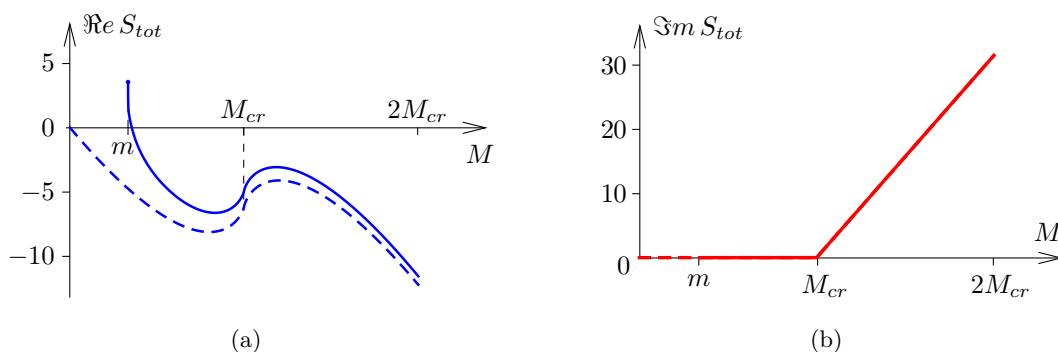
Figure 5. Real (a) and imaginary (b) parts of the total action (3.12) for $m = M_{\rm cr}/3$ (solid) and $m = 0$ (dashed) as functions of the particle energy M. The critical black hole mass is $M_{\rm cr} = 10\lambda$. The interval $M < m$ is kinematically forbidden.

outer metrics (2.3), (2.8). We perform this computation in appendix C. Here is the result,

$$
\begin{aligned}
S_{\rm tot} = {} & -\frac{M - M_{cr}}{\lambda} \log\left(1 - \frac{M + i\varepsilon'}{M_{cr}}\right) + \frac{p}{\lambda}(1 + 2\phi_0) \\
& - \frac{p}{\lambda} \log\left(\frac{1}{2} + \frac{m^2 M}{8 M_{cr} p^2} + \frac{p_\times}{2p}\right) + \frac{2M_{cr}}{\lambda} \log\left(\frac{4M_{cr}(p_\times + M) + m^2}{4M_{cr}(p_\times + M) - m^2}\right) \\
& + \frac{M}{\lambda} \log\left[\frac{4M^3 - 3m^2 M + (4M^2 - m^2)p_\times}{(p + M)^3} + \frac{m^2(4M^2 + m^2)}{4M_{cr}(p + M)^3}\right],
\end{aligned}
\tag{3.12}
$$

where

$$
p_\times = \sqrt{(M + m^2/4M_{cr})^2 - m^2}
\tag{3.13}
$$

is the radial momentum of the particle immediately after the collision with the boundary. This result is finite and valid at all energies. It provides the absolute value and the phase of the amplitude (3.4). In the massless case $m = 0$ the expression (3.12) simplifies,

$$
S_{\rm tot} = -\frac{M - M_{\rm cr}}{\lambda} \log\left(1 - \frac{M + i\varepsilon'}{M_{\rm cr}}\right) + \frac{M}{\lambda}\left(1 - \log\frac{M_{\rm cr}}{2\lambda}\right).
\tag{3.14}
$$

The infinitesimal mass shift $i\varepsilon'$ in eqs. (3.12), (3.14) fixes the branch of the first logarithm at $M > M_{cr}$ leading to the imaginary part,

$$
\Im m\, S_{\rm tot} = \frac{\pi}{\lambda}(M - M_{cr})\,\theta(M - M_{cr}),
\tag{3.15}
$$

which is independent of the particle mass m. This gives the probability of overcritical scattering $\mathcal{P}_{fi} = |\mathcal{A}_{fi}|^2$ in eq. (1.2). The real and imaginary parts of the expression (3.12) are shown in figure 5.

Let us outline where the imaginary part of $S_{\rm tot}$ comes from. We consider[5] the space-time contour in figures 4c,d with almost real $r_*(\tau)$. Then the particle action (2.7) is almost real as well by eq. (2.10). In appendix C we show that the bulk CGHS Lagrangian

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[5]The analytic integral $S_{\rm tot}$ is independent of the choice of the complex contour. However, separate contributions to it depend on this choice.
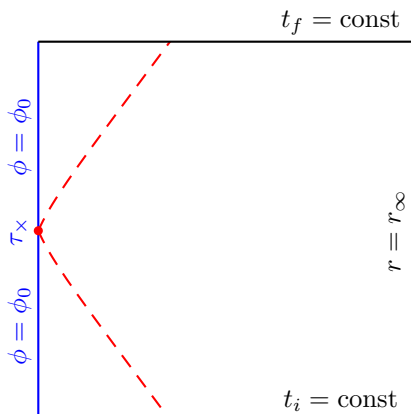
Figure 6. Schematic representation of the regularized scattering solution. Red dashed line shows the particle trajectory.

in eq. (2.1) is a total derivative. This means that the interacting action S is a sum of integrals along the boundary $\phi = \phi_0$, initial and final Cauchy surfaces $t = t_i, t_f$, and the timelike surface at the spatial infinity $r = r_\infty$, see figure 6. We find that the latter term vanishes. The integrals at $t = t_i, t_f$ combined with the free action S_0 and the wavefunctions Ψ_i, Ψ_f give complex contribution into S_{tot}. Its imaginary part comes from the residue of the Schwarzschild time at the horizon,

$$\Im m\, S_{\text{tot}}^{(1)} = M \Im m (t_f - t_i) = \frac{\pi}{\lambda} M\, \theta(M - M_{cr}). \tag{3.16}$$

This is similar to the results of the previous studies [20, 22]. Remarkably, the contribution of the boundary is also complex. One may notice from figure 6 that before and after the collision the boundary lives in flat spacetime. These parts do not contribute into S_{tot}. We find, however, that the collision point $\tau_\times$ corresponds to a non-analytic break of the boundary with the extrinsic curvature proportional to a δ-function,

$$K_{\phi = \phi_0} = 2\delta(\tau_0 - \tau_{0,\times}) \left[\text{arcsh} \sqrt{-V_{\text{eff}}} - \text{arcsh} \sqrt{-V_{\text{eff}}/f} \right] \Big|_{r = r_0}, \tag{3.17}$$

where τ_0 is the boundary proper time and $\tau_{0,\times}$ is its value at the collision point. The expression (3.17) is complex because $f < 0$ at $r = r_0 < r_h$. Substituting it into eq. (2.1), one finds an imaginary term

$$\Im m\, S_{\text{tot}}^{(2)} = \frac{M_{cr}}{\lambda} \Im m \log \left(1 - \frac{M + i\varepsilon'}{M_{cr}} \right) = -\frac{\pi}{\lambda} M_{cr}\, \theta(M - M_{cr}). \tag{3.18}$$

There are no imaginary contributions in addition to eqs. (3.16) and (3.18). Summing up these terms, we arrive to the expression (3.15).

4 Relation to black entropy

4.1 Euclidean calculation of entropy: a puzzle

Our semiclassical result (3.15) is natural from the quantum-mechanical viewpoint: the probability $\mathcal{P}_{fi}$ of particle reflection is a continuous function of energy M, as it should
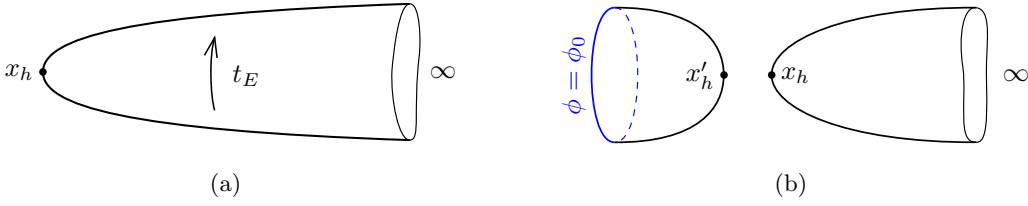
Figure 7. Geometries for Euclidean calculations of the black hole entropy.

be. At $M > M_{cr}$ i.e. above the threshold for classical BH production, this probability is exponentially suppressed. The respective transitions are interpreted as two-stage processes. First, the left-moving particle creates the BH of mass M classically. Second, the intermediate BH decays into the final-state particle with exponentially small probability. One expects [22, 47] that the probability of the latter stage is suppressed by the number of BH states $\exp(\Sigma_{\mathrm{BH}})$. This implies the expression (1.3) for the black hole entropy Σ_{BH}.

A following puzzle arises. There is an alternative method for calculating BH entropy based on Euclidean path integral [40]. When applied to our model, this method apparently gives a different result (1.4). Let us briefly review the relevant calculation [38, 39]. One computes the thermal partition function

$$\mathcal{Z}(\beta) = \int\limits_{\mathrm{periodic}} \mathcal{D}\Phi \, \mathrm{e}^{-S_E[\Phi]} \,, \tag{4.1}$$

where S_E is the Euclidean CGHS action, see appendix D for the precise definition. The integral is taken over configurations with period β in Euclidean time $t_E = it$. In the semiclassical limit the integral is saturated by the saddle point. The instanton corresponds to Euclidean continuation of the BH exterior with the metric

$$ds^2 = f(r) \, dt_E^2 + \frac{dr^2}{f(r)} \,, \qquad t_E \in [0; \beta] \,. \tag{4.2}$$

This spacetime has topology of a half-tube, where t_E serves as a periodic coordinate, see figure 7a. The black hole horizon corresponds to the tip of the tube. Notably, this tip is a conical singularity if the period β is not equal to the inverse Hawking temperature $T_H^{-1} = 2\pi/\lambda$. As a consequence, the curvature has a δ-function contribution at the tip of the cone,

$$R = 4\pi(1 - \beta T_H) \, \frac{\delta^{(2)}(x - x_h)}{\sqrt{g}} + 2\lambda M \mathrm{e}^{-2\lambda r} \,. \tag{4.3}$$

The singular contribution vanishes for $\beta = T_H^{-1}$.

Now one evaluates the Euclidean action on this solution,

$$S_E = M(\beta - T_H^{-1}) \,, \tag{4.4}$$

and the free energy,

$$F(\beta) \equiv -\beta^{-1} \log \mathcal{Z}(\beta) \simeq \beta^{-1} S_E(\beta) \,. \tag{4.5}$$

Note that the only non-vanishing contribution into the action comes from the δ-function in eq. (4.3). Then the thermodynamical formula

$$\Sigma_{\text{BH}} = \beta^2 \frac{\partial F}{\partial \beta} = \beta \frac{\partial S_E}{\partial \beta} - S_E \qquad (4.6)$$

yields the "naive" entropy (1.4).

Thus, we have two different expressions for the BH entropy — eqs. (1.3) and (1.4) — and we have to decide which one is correct.[6]

4.2 Experiments with the thermal gas

We now present several physical arguments against eq. (1.4). To this end, we couple the dilaton gravity to the gas of massless particles — quanta of some massless scalar field. Notice that the BHs cannot form classically from arbitrary configuration of this field, even if its total mass is higher than M_{cr}. Indeed, the gravitational Lagrangian (2.1) is explicitly proportional to the factor $e^{-2\phi} \equiv e^{2\lambda r}$. This means that the gravitational interaction decreases exponentially at coordinate distance $\Delta r \sim \lambda^{-1}$ from the boundary. Then formation of BHs requires the energy M_{cr} to be concentrated within the interval $\Delta r \sim \lambda^{-1}$. A configuration satisfying this condition, however, cannot carry large coarse-grained entropy. Indeed, the entropy reaches maximum in a thermal state providing the bound

$$\Sigma_{\text{gas}} \leq \frac{2M_{cr}}{T_{\text{gas}}} \sim e^{-\phi_0} \, . \qquad (4.7)$$

Here we related the gas temperature to its energy density $T_{\text{gas}} = \sqrt{6\rho_{\text{gas}}/\pi}$ and substituted $\rho_{\text{gas}} \sim M_{cr}/\Delta r$. On the other hand, the entropy (1.4) is parametrically larger than eq. (4.7): $\Sigma_{\text{BH}}^{naive} = 4\pi e^{-2\phi_0}$ at $M = M_{cr}$. If this expression were correct, it would be puzzling why the critical BH cannot be formed from states with large entropy.

Further, the results on classical subcritical scattering [31] suggest that the entropy of the critical BH is even smaller than (4.7). Namely, near the threshold of critical BH formation reflection of the classical field from the boundary proceeds as follows. A part of the incoming wavepacket reflects immediately, whereas the remaining part forms a long-lived state with mass $M \approx M_{cr}$. The latter state decays into a narrow wavepacket carrying a few highly blue-shifted particles. This may be interpreted as formation of a slightly subcritical black hole decaying classically into a low-entropy state. Extrapolating this picture to the critical black hole, we conclude that it should have an order-one entropy. Our semiclassical formula (1.3) is consistent with this picture.

One may also try to form the black hole in an essentially quantum way. Namely, suppose the spacetime is filled with a massless gas of temperature $T \sim \lambda$. Eventually, the black hole of mass M may appear, eating a part of the gas and providing the first order phase transition. According to eq. (1.3), a part of gas entropy $\Delta\Sigma_{\text{gas}} = 2M_{cr}/\lambda \sim e^{-2\phi_0}$ disappears in this process even if the black hole is critical. We have argued, however, that the black hole cannot form in classical collapse of the low-temperature gas. Thus,

---

[6]Note that both expressions agree with the first law of BH thermodynamics $T_H \Delta\Sigma_{\text{BH}} = \Delta M$.

the probability of this process is exponentially suppressed by the CGHS action $S_{\mathrm{gr}} \propto \mathrm{e}^{-2\phi_0} \sim \Delta\Sigma_{\mathrm{gas}}$. Now, we recall that the entropy of a thermal ensemble can decrease with exponentially small probability due to large fluctuations. The above process appears to be one of them.

It could appear that the problem with entropy might be fixed by adding to the Euclidean action a topological term

$$\Delta S_E^{(\chi)} = 4\pi Y \chi = Y \int\limits_{\phi < \phi_0} d^2 x_E \sqrt{g}\, R + 2Y \int\limits_{\text{boundary}} d\tau_0\, \kappa K \tag{4.8}$$

with $\kappa = +1$. Here $\chi = 2 - 2\mathrm{g} - b$ is the Euler characteristic of spacetime with g handles and b boundaries. Being a topological invariant, it does not affect the semiclassical dynamics. Also, the action of the flat vacuum with $\chi = 0$ remains unchanged. At the same time, the new term adds a constant $\Delta S_E^{(\chi)} = 4\pi Y$ to the action of the instanton in figure 7a and therefore shifts the entropy in eq. (4.6) by $-4\pi Y$. This reproduces eq. (1.3) if $Y = \mathrm{e}^{-2\phi_0}$. Note, however, that the term (4.8) with positive $Y \gg 1$ leads to severe divergence[7] of the path integral (4.1) due to exponentially enhanced contributions of multihandle geometries. Thus, it introduces strong coupling and does not cure the problem.

To see this more explicitly, let us focus on the case with massless matter. Then, at the classical level, the term (4.8) can be completely absorbed by the field redefinition

$$g_{\mu\nu} = \frac{g'_{\mu\nu}}{1 + Y\mathrm{e}^{2\phi'}}\,, \qquad \mathrm{e}^{-2\phi} = \mathrm{e}^{-2\phi'} + Y\,. \tag{4.9}$$

This gives the CGHS action (2.1) for $g'_{\mu\nu}$ and ϕ' with two different parameters: new semiclassical constant $\mathrm{e}^{2\phi'_0} = [\mathrm{e}^{-2\phi_0} - Y]^{-1}$ in place of $\mathrm{e}^{2\phi_0} \ll 1$ and new "mass" parameter $\lambda' = \lambda \mathrm{e}^{\phi'_0 - \phi_0}$ in the boundary term.[8] The choice $Y = \mathrm{e}^{-2\phi_0}$ corresponds to a strongly coupled model with $\phi'_0 = +\infty$.

4.3 Correcting the Euclidean calculation

We now suggest a modification of the Euclidean calculation that reproduces the result (1.3) for the entropy. The approach of section 4.1 misses an important property of our model, namely, the presence of the boundary at $\phi = \phi_0$. This boundary is necessary because it shields the singularities of the CGHS fields in the original Lorentzian path integral [27, 35, 42, 48–50]. Our complex scattering solutions satisfy this property, whereas the Euclidean instanton in figure 7a does not.

We cure this problem by adding to the Euclidean spacetime a disjoint cap-like portion with a closed boundary $\phi = \phi_0$, see figure 7b. By Birkhoff theorem, the geometry of the cap is given by the black hole metric (4.2), possibly with a different mass parameter M'. The latter must be larger than M_{cr} for the cap to be compact and satisfy the inequality $\phi < \phi_0$. The radial coordinate on the cap runs in the interval $r_0 < r < r_h(M')$. Importantly, the signature of the metric on the cap is $(-,-)$ instead of $(+,+)$ in the exterior region.

---

[7]Note that $g_s \sim \mathrm{e}^{4\pi Y}$ is a direct analog of the coupling constant in string theory.

[8]The bulk parameter λ remains unchanged. Recall that we related the boundary "mass" to λ by requiring existence of a flat vacuum. This condition is not Weyl invariant and therefore broken by eq. (4.9).

This configuration does not satisfy the boundary condition (2.5) at $\phi = \phi_0$ and thus it is not an exact saddle point of the path integral (4.1). Rather, as shown in appendix D, it should be interpreted as a constrained instanton extremizing the Euclidean action within a subset of geometries with the boundary. Instead of solving the boundary conditions, one minimizes the action with respect to the free parameter M'.

The action of the additional Euclidean cap equals (see appendix D)

$$\Delta S_E = M'/T_H \,, \tag{4.10}$$

where the terms proportional to β have cancelled. The only remaining contribution comes from the δ-function of the curvature at the horizon r'_h. The latter has an opposite sign to that in eq. (4.3) due to the metric signature $(-, -)$. The minimum of ΔS_E is reached at the boundary $M' \to M_{cr}$ of the parameter region where the solution in figure 7b exists.

Adding up eqs. (4.4) and (4.10) at $M' = M_{cr}$ one reproduces eq. (1.3). This restores agreement between the semiclassical entropy and the scattering probability.

In the generalized model with topological term (4.8) one still obtains correct entropy (1.3). Indeed, the additional cap in figure 7b has the same topology as the original Gibbons-Hawking instanton, but its contribution[9] into eq. (4.8) has opposite sign due to $(-, -)$ signature. Thus, $\Delta S_E^{(\chi)} = 0$ for any Y.

5 Conclusions

In this paper we further developed complex semiclassical method for calculating $\mathcal{S}$-matrix elements in gravity. We considered a simplified setup where the point-like quantum particle scatters off the boundary in two-dimensional Callan-Giddings-Harvey-Strominger (CGHS) model. The semiclassical method provided the amplitude of a complete transition between the initial particle moving with energy M towards the boundary and an outgoing final particle with the same energy. At low energies this reflection proceeds classically and the transition probability is of order one. However, once the particle energy exceeds the minimal mass of black holes (BHs) in the model, the amplitude becomes exponentially suppressed. Then the respective transition can be interpreted as production of an intermediate BH and its subsequent decay into an outgoing particle. The probability of such transition is naturally identified with $\exp(-\Sigma_{\rm BH})$, where $\Sigma_{\rm BH}$ is the BH entropy. It is important to stress that our analysis provides not only the absolute value of the amplitude, but also its phase.

Our result implies that the entropy of the minimal-mass BH vanishes. This is consistent with the expressions for BH entropy in similar two-dimensional models [36–39]. We noticed, however, an apparent conflict between this result and the calculation of the BH entropy using the Gibbons-Hawking Euclidean approach. We suggested a natural modification of the Euclidean calculation that takes into account the presence of the boundary and recovers the correct entropy obtained from the scattering probability.

Our results demonstrate that the semiclassical $\mathcal{S}$-matrix provides important insights about black holes, even if the simplified matter content is considered. It is straightforward

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[9]In this case $\kappa = n_0^\mu n_{0\,\mu} = -1$, or the term is not a topological invariant.

to apply our approach to spherically-symmetric sectors of multidimensional gravities. In particular, the case of 4 dimensions was considered in [23]. A simplified matter content in this case is provided by thin spherical shells with dynamical radius $R = R(\tau)$.

Let us outline several directions for future research.

The phase of the amplitude is known to contain information about temporal properties of the scattering process [51]. It will be interesting to extract this information from our results and compare it to the characteristic time scales of the BH evaporation, e.g. the scrambling time [7].

Further development of the semiclassical $\mathcal{S}$-matrix approach will be inclusion of full-fledged matter fields. In field theory, the semiclassical amplitudes can be used for studying quantum correlations in the Hawking radiation and for direct tests of unitarity. As an example, consider the identity satisfied in any $(d + 1)$-dimensional unitary theory,

$$\mathrm{e}^{\int d^d \boldsymbol{k}\, a_{\boldsymbol{k}}^* b_{\boldsymbol{k}}} = \langle a|b\rangle = \langle a|\hat{\mathcal{S}}^\dagger \hat{\mathcal{S}}|b\rangle = \int \mathcal{D}c^* \mathcal{D}c\, \mathrm{e}^{-\int d^d \boldsymbol{k}\, c_{\boldsymbol{k}}^* c_{\boldsymbol{k}}} \left[\langle c|\hat{\mathcal{S}}|a\rangle\right]^* \langle c|\hat{\mathcal{S}}|b\rangle, \qquad (5.1)$$

where $|a\rangle$, $|b\rangle$ and $|c\rangle$ are the flat-space coherent states in the beginning and end of the scattering process. One can write the r.h.s. in eq. (5.1) as a path integral using eq. (1.1). At large $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}$ the initial states are semiclassical. If they are different, their overlap is exponentially suppressed. Then the integral on the r.h.s. can be evaluated in the saddle-point approximation. Importantly, the relevant saddle-point solutions should interpolate between flat spacetimes in the initial and final asymptotic regions. Comparing the saddle-point result to the l.h.s. of eq. (5.1), one will perform a nontrivial check of unitarity.

In the context of 2-dimensional dilaton gravity one can add one-loop corrections by including the Polyakov effective action [25] and, optionally, Russo-Susskind-Thorlacius (RST) counterterm [27, 35]. This modification may clarify relation between our semiclassical $\mathcal{S}$-matrix and the conventional black hole evaporation due to one-loop quantum corrections. Note, however, that the Polyakov term is nonlocal and therefore introduces an additional effective field into the model. Solving the semiclassical equations in this case will require full field-theoretical treatment and goes beyond the scope of this paper.

Another interesting direction of research would be to relate the semiclassical $\mathcal{S}$-matrix to the new "island" method for calculating the entanglement entropy of the Hawking radiation [4, 5, 11–15]. The latter method indicates purification of the Hawking radiation in the final state. Technically, it makes use of "replica wormholes" [8–12], saddle points of the gravitational path integral for the trace $\mathrm{Tr}\,\hat{\rho}^n$, where $\hat{\rho}$ is the density matrix of the radiation and n is an arbitrary power. If BH is formed from a pure state $|\Psi_i\rangle$, the final density matrix equals $\hat{\rho} = \hat{\mathcal{S}}|\Psi_i\rangle\langle\Psi_i|\hat{\mathcal{S}}^\dagger$. Thus, $\mathrm{Tr}\,\hat{\rho}^n$ can be formally written using the path integral (1.1) for the $\mathcal{S}$-matrix. This suggests that the relevant saddle-point solutions in the two methods may be related to each other by some kind of analytic continuation.

Acknowledgments

A Classical solutions

In this appendix we summarize the field equations and discuss the relevant solutions.

A.1 Birkhoff theorem

Varying the action (2.1), (2.7) with respect to $g_{\mu\nu}$ and ϕ, we find,

$$\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\left[(\nabla\phi)^2 - \Box\phi - \lambda^2\right] = \mathrm{e}^{2\phi}\, T_{\mu\nu}/4\,, \tag{A.1}$$

$$(\nabla\phi)^2 - \Box\phi - \lambda^2 = R/4\,, \tag{A.2}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$ is the covariant d'Alembertian and

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{A.3}$$

is the matter energy-momentum tensor. It will be discussed later.

In an empty spacetime region one sets $T_{\mu\nu} = 0$ and arrives to the system

$$2\nabla_\mu\nabla_\nu\phi = g_{\mu\nu}\Box\phi\,, \qquad (\nabla\phi)^2 - \frac{1}{2}\Box\phi = \lambda^2\,, \qquad \Box\phi = -R/2\,, \tag{A.4}$$

where the second equation is a trace of eq. (A.1). It will be convenient to use the Schwarzschild gauge,

$$ds^2 = -h(r,t)\,dt^2 + \frac{dr^2}{f(r,t)}\,, \qquad \phi = -\lambda r\,, \tag{A.5}$$

where the spatial coordinate r tracks the dilaton field ϕ and the time t is orthogonal to r. The first of eqs. (A.4) gives,

$$\partial_t f = \partial_r(h/f) = 0\,,$$

implying that the metric component $f = f(r)$ is time-independent and $h = c(t)f(r)$ with arbitrary $c(t)$. One can fix $h = f(r)$ using the residual time reparametrization invariance in eq. (A.5).

With these simplifications the second of eqs. (A.4) reduces to

$$\partial_r f = 2\lambda(1 - f)\,, \tag{A.6}$$

with general solution (2.3). The Schwarzschild mass M is an arbitrary integration constant in this solution. It is straightforward to check that the third of eqs. (A.4) is now automatically satisfied.

To summarize, we demonstrated that the static black hole spacetime (2.3) with arbitrary mass M is the only solution in an empty patch of spacetime. This is the analog of the Birkhoff theorem in the present context [43].

A.2 Junction conditions and equation of motion for the particle

It will be convenient to introduce Gaussian normal coordinates (τ, n) near the particle trajectory $x_*^\mu(\tau)$. Here n measures the geodesic distance to the trajectory and τ is orthogonal to n,

$$ds^2 = -a(\tau, n)\, d\tau^2 + dn^2\,. \tag{A.7}$$

In these coordinates the particle trajectory is $n = 0$. We choose τ to coincide with the proper time along the trajectory: $a(\tau, 0) = 1$. By construction, $a(\tau, n)$ is continuous at $n = 0$, as opposed to the metric components in the Schwarzschild gauge.

Variation of the action (2.7) with respect to $g^{\mu\nu}$ gives the particle energy-momentum tensor (A.3),

$$T^{\mu\nu} = m\, \dot{x}_*^\mu \dot{x}_*^\nu\, \delta(n)\,,$$

where $(\dot{x}_*^\tau, \dot{x}_*^n) = (1, 0)$ is the particle velocity. Since eq. (A.1) has a δ-function in the r.h.s., the normal derivatives of a and ϕ are discontinuous at $n = 0$. Equations (A.1), (A.2) take the from

$$\partial_n^2 \phi = m\, \mathrm{e}^{2\phi} \delta(n)/4 + (\text{regular terms})\,,$$
$$\partial_n(\partial_n a/a) = 4\partial_n^2 \phi + (\text{regular terms})\,, \tag{A.8}$$

where we have kept only the "singular" terms with the second n-derivatives, which are proportional to $\delta(n)$. Now, we integrate eqs. (A.8) from $n = -0$ to $n = +0$ and rewrite them in the covariant form using $\partial_n \phi = n^\mu \nabla_\mu \phi$ and $\partial_n a/(2a) = K$. We arrive to the junction conditions (2.9).

Since $n^\mu \nabla_\mu \phi$ and K are frame-independent, one can compute them in different coordinate systems (T, r) and (t, r) at the two sides of the particle trajectory. The outer trajectory normal in these regions is

$$(n^T, n^r) = (\dot{r}_*,\ \dot{T}_*)\,, \qquad (n^t, n^r) = \left(\frac{\dot{r}_*}{f},\ \dot{t}_* f\right)\,, \tag{A.9}$$

where $\dot{T}_*$ and $\dot{t}_*$ can be expressed from eqs. (2.8) and (2.3),

$$\dot{T}_* = \sqrt{1 + \dot{r}_*^2}\,, \qquad \dot{t}_* = \frac{\sqrt{f(r_*) + \dot{r}_*^2}}{f(r_*)}\,.$$

Substituting the normal into the first of eqs. (2.9) one obtains the energy conservation law

$$M = m\sqrt{1 + \dot{r}_*^2} - \frac{m^2}{8\lambda} \mathrm{e}^{-2\lambda r_*}\,. \tag{A.10}$$

The second junction condition in eqs. (2.9) is a time derivative of eq. (A.10). It is trivially satisfied once eq. (A.10) is solved. Equation of motion (2.10) from the main text is obtained by squaring eq. (A.10).

A.3 Boundary condition and reflection law

The saddle-point configurations $\Phi_{cl}(x)$ should extremize the action (2.1) with respect to all variables, in particular, the metrics $g_{\mu\nu}$ at the boundary $\phi = \phi_0$. Due to the reparametrization invariance, it is enough to consider only the variations preserving the coordinate position of this boundary. Then $\delta\phi = 0$ and $\delta n_{0\,\mu} \propto n_{0\,\mu}$ at the line $\phi = \phi_0$. We vary eq. (2.1) with respect to $g_{\mu\nu}$ and leave only the boundary terms,

$$\delta S_{\mathrm{gr}} = 2\mathrm{e}^{-2\phi_0} \int\limits_{\phi=\phi_0} d\tau_0 \; (n_0^\kappa \nabla_\kappa \phi - \lambda) \; \tau^\mu \tau^\nu \, \delta g_{\mu\nu} \,,$$

where $\tau^\mu = dx^\mu/d\tau_0$ is the unit vector along the line $\phi = \phi_0$. Requiring the variation to vanish, we obtain eq. (2.5).

To derive reflection law for the particle from the boundary, we notice that the collision point $\tau_\times$ divides the particle trajectory $x_*^\mu(\tau)$ into two smooth parts, see figure 6. Thus,

$$S_m = -m \int\limits_{\tau_i}^{\tau_\times} d\tau - m \int\limits_{\tau_\times}^{\tau_f} d\tau \,, \tag{A.11}$$

where τ_i and τ_f are the initial and final times of the process. We vary eq. (A.11) with respect to $x_*^\mu(\tau)$, again keeping the position of the boundary intact: $n_{0\,\mu}\delta x_*^\mu(\tau_\times) = 0$. We obtain,

$$\delta S_m = m\tau_\nu \delta x_*^\nu \left[\tau_\mu \dot{x}_*^\mu(\tau_\times + 0) - \tau_\mu \dot{x}_*^\mu(\tau_\times - 0)\right] \,,$$

where again only the boundary terms are shown. We obtain two equations,

$$\tau_\mu \dot{x}_*^\mu(\tau_\times - 0) = \tau_\mu \dot{x}_*^\mu(\tau_\times + 0)\,, \qquad n_{0\,\mu} \dot{x}_*^\mu(\tau_\times - 0) = -n_{0\,\mu} \dot{x}_*^\mu(\tau_\times + 0)\,, \tag{A.12}$$

where the second one follows from the normalization $\dot{x}_*^\mu \dot{x}_{*\,\mu} = -1$.

Now, we rewrite eqs. (A.12) in the coordinates (T, r) of flat spacetime patches immediately prior to the collision and after it, see figure 6. This gives the reflection law

$$\dot{T}_*(\tau_\times - 0) = \dot{T}_*(\tau_\times + 0)\,, \qquad \dot{r}_*(\tau_\times - 0) = -\dot{r}_*(\tau_\times + 0)\,, \tag{A.13}$$

which is used in the main text.

B Regularization method

Let us demonstrate that the regularization (3.7), (3.8) of the classical action is equivalent to the imaginary shift of the Schwarzschild mass M inside the regulating "shell" $r_* < r < r_\varepsilon$, where $r_\varepsilon = -\phi_\varepsilon/\lambda$. To this end, we solve the field equations at $r > r_*$. Additional term in the regularized action (3.8) produces imaginary energy-momentum tensor,

$$T_{\mu\nu,\,\varepsilon} = i\varepsilon\, L(\phi)\, (\lambda^2 - (\nabla\phi)^2) \left[4\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}(\lambda^2 - (\nabla\phi)^2)\right] \,,$$

in the right-hand side of eq. (A.1). In the Schwarzschild gauge (A.5) the (rt) and (tt) components of eq. (A.1) give

$$\partial_t f = 0\,, \qquad \partial_r \tilde{M}^{-1}(r) = \frac{i\varepsilon}{4}\lambda^2 L(-\lambda r)\,\mathrm{e}^{-4\lambda r}\,, \qquad (B.1)$$

where $\tilde{M}(r)$ is the coordinate-dependent mass entering the metric as

$$f(r) = 1 - \frac{\tilde{M}(r)}{2\lambda}\,\mathrm{e}^{-2\lambda r}\,.$$

Integrating the second of eqs. (B.1) one arrives to the matching condition

$$\frac{1}{M} - \frac{1}{M_\varepsilon} = \frac{i\varepsilon\lambda^2}{4}\int dr\,L(-\lambda r)\,\mathrm{e}^{-4\lambda r} = \frac{i\varepsilon}{4\lambda}\,. \qquad (B.2)$$

Here $M = \tilde{M}(+\infty)$ is the real conserved energy of the particle and $M_\varepsilon = \tilde{M}(r < r_\varepsilon)$ is the Schwarzschild mass parameter inside the regulating "shell" at $\phi = \phi_\varepsilon$. Expressing M_ε from eq. (B.2), one obtains eq. (3.9) from the main text.

C Computing the action

Let us calculate the total action (3.5) on the semiclassical solution. We work in the limit $\varepsilon \to +0$. To start, we note that the bulk Lagrangian in eq. (2.1) is a total derivative on the field equation (A.2),

$$\mathrm{e}^{-2\phi}\left[R + 4(\nabla\phi)^2 + 4\lambda^2\right] = 2\,\Box\mathrm{e}^{-2\phi}\,.$$

Thus, the interacting action (3.3) is a sum of one-dimensional contour integrals over the spatial infinity $r = r_\infty \to +\infty$, the boundary $\phi = \phi_0$, Cauchy surfaces $t = t_i, t_f$, and the particle worldline $r = r_*(\tau)$, see figure 6,

$$S(t_f, t_i) = S_{r_\infty} + S_{\phi=\phi_0} + S_{t_i} + S_{t_f} + S_m\,. \qquad (C.1)$$

This expression includes the Gibbons-Hawking term,

$$S_{\mathrm{GH}} = 2\int_\infty d\sigma\,\mathrm{e}^{-2\phi}\kappa\,(K - K_0)\,, \qquad (C.2)$$

over the spatial infinity $r = r_\infty$ and surfaces $t = t_i, t_f \to \mp\infty$. Here σ is the proper time or the proper distance, K is the outer-normal extrinsic curvature, and $\kappa = n_\mu n^\mu$ equals $+1$ (-1) at the timelike line $r = r_\infty$ (spacelike curves $t = t_i, t_f$). The parameter K_0 is introduced in eq. (C.2) to subtract the vacuum contribution; it equals 2λ at $r = r_\infty$ and zero at $t = t_i, t_f$. Recall that in addition to the terms in eq. (C.1), the total action S_{tot} includes the free actions S_0 and the wave functionals $\Psi_{i,f}$. We now evaluate all the listed contributions one by one.
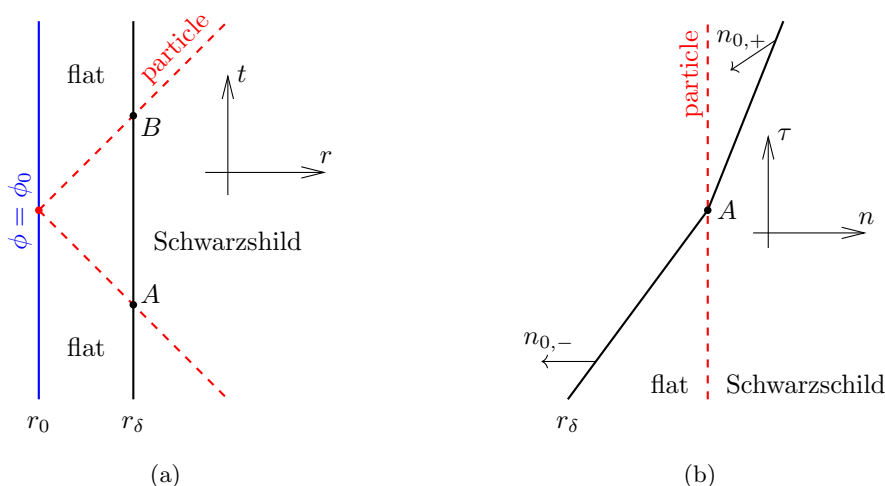
Figure 8. Regularized boundary $r = r_\delta$ in (a) the original coordinates and (b) Gaussian normal frame attached to the particle.

Spatial infinity. It is straightforward to check that the term at $r = r_\infty$ vanishes as $e^{-2\lambda r_\infty}$,

$$S_{r_\infty} = 2 \int\limits_{r=r_\infty} d\sigma \, e^{-2\phi} \left(K - 2\lambda - 2n^\mu \nabla_\mu \phi \right) \to 0 \,, \tag{C.3}$$

where we evaluated $n^\mu \nabla_\mu \phi = -\lambda \sqrt{f}$ and $K = f'/2\sqrt{f}$ using the metric (2.3), then sent $r_\infty \to +\infty$.

The boundary. Adding the boundary term in eq. (2.1) to the contribution from the bulk action, one obtains,

$$S_{\phi=\phi_0} = 2 \, e^{-2\phi_0} \int\limits_{\phi=\phi_0} d\tau_0 \, K \,, \tag{C.4}$$

where the term proportional to λ has cancelled due to the boundary condition (2.5). Note that $\phi = \phi_0$ is a straight line in flat spacetime prior to and after the collision, see figure 6. In these regions $K = 0$. Thus, the only non-zero contribution into $S_{\phi=\phi_0}$ comes from the singularity of the extrinsic curvature at the collision point.

To evaluate it, we use several technical steps. We regulate the calculation by slightly shifting the line of integration to $r_\delta \equiv r_0 + \delta r$, see figure 8a. In contrast to the boundary, the regulating line intersects the particle trajectory twice, going from the flat geometry to Schwarzschild and back. We will see that each of these intersection points gives a δ-functional contribution into the integral (C.4).

Let us focus on the first intersection point A. In its vicinity we introduce the Gaussian normal coordinates (τ, n) which are continuous at the particle worldline. In these coordinates the line $r = r_\delta$ has a break at A, see figure 8b. This is because the normal n_0^μ to this line has a discontinuity, as we now demonstrate. In the Schwarzschild and flat patches it has the components

$$\left(n_{0,+}^t, n_{0,+}^r \right) = \left(0, -\sqrt{f(r_\delta)} \right) \,, \qquad \left(n_{0,-}^T, n_{0,-}^r \right) = (0, -1) \,.$$

At the intersection point A we can decompose $n^\mu_{0,\pm}$ in the basis of the tangential and normal vectors to the particle trajectory. In the two patches the former equals to

$$(\tau^t_+, \tau^r_+) = (\dot{t}_*, \dot{r}_*), \qquad\qquad (\tau^T_-, \tau^r_-) = (\dot{T}_*, \dot{r}_*),$$

whereas the latter is given by eq. (A.9). In this way we find the components of n^μ_0 in the Gaussian normal frame which are different on the two sides of the intersection point A,

$$(n^\tau_{0,\pm}, n^n_{0,\pm}) = (-\operatorname{sh}\psi_\pm, -\operatorname{ch}\psi_\pm), \qquad \text{where} \quad \operatorname{sh}\psi_+ = -\dot{r}_*/\sqrt{f(r_*)}, \qquad \operatorname{sh}\psi_- = -\dot{r}_*.$$

Here all the quantities are evaluated at A.

Now we regularize the break approximating the line $r = r_\delta$ with a smooth curve. Its normal is

$$(n^\tau_0, n^n_0) = (-\operatorname{sh}\psi(\tau), -\operatorname{ch}\psi(\tau)), \tag{C.5}$$

where $\psi(\tau)$ interpolates between ψ_- and ψ_+. The proper time and extrinsic curvature of the curve are readily computed: $d\tau_0 = d\tau/\operatorname{ch}\psi$, $K = -\operatorname{ch}\psi\,\partial_\tau\psi$. Integrating K in the vicinity of the point A, we obtain,

$$\int_A d\tau_0\, K = \psi_- - \psi_+. \tag{C.6}$$

We see that K contains a δ-function at A. Another δ-function with the same coefficient comes from the second intersection point B. Taking the limit $r_\delta \to r_0$, we conclude that the boundary extrinsic curvature is proportional to $\delta(\tau_0 - \tau_{0,\times})$. Then eqs. (C.6) and (2.10) yield eq. (3.17) from the main text.

Finally, using the formula for V_{eff} and expressing r_0 in terms of M_{cr}, we arrive to the boundary action,

$$S_{\phi=\phi_0} = \frac{M_{cr}}{\lambda} \log\left(1 - \frac{M + i\varepsilon'}{M_{cr}}\right) + \frac{2M_{cr}}{\lambda} \log\left(\frac{4M_{cr}(p_\times + M) + m^2}{4M_{cr}(p_\times + M) - m^2}\right), \tag{C.7}$$

where $p_\times$ is defined in eq. (3.13) and the imaginary part of the logarithm is fixed by the regularization procedure from appendix B.

Initial and final Cauchy surfaces. We define them as the lines of constant Schwarzschild time $t = t_{i,f}$ to the right of the initial and final particle positions $r_{i,f}$ continued as $T = \text{const}$ to the left, see figure 9. The interacting action at the final surface equals
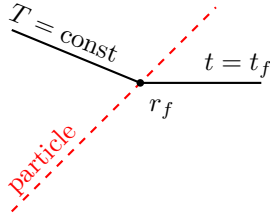
$$S_{t_f} = -2 \int_{t=t_f} d\sigma\, e^{-2\phi} K. \tag{C.8}$$

One can check that $K = 0$ on the outer and inner parts of this surface. Thus, the only non-zero contribution comes from the jump of the normal at the particle position $r = r_f$ where the two parts of the surface join. The same calculation as before gives,

$$S_{t_f} = -2\, e^{2\lambda r_f} \left[\operatorname{arcsh}\dot{r}_* - \operatorname{arcsh}\frac{\dot{r}_*}{\sqrt{f}}\right]_{r_f} = \frac{p}{2\lambda}, \tag{C.9}$$

where $p = \sqrt{M^2 - m^2}$ and in the second equality we have sent $r_f \to +\infty$.

Figure 9. The final Cauchy surface.

Computing the initial contribution at $t = t_i$ in a similar way, one obtains,

$$S_{t_i} = -2 \int\limits_{t=t_i} d\sigma \ e^{-2\phi} K = \frac{p}{2\lambda} , \qquad (\text{C}.10)$$

which doubles the contribution (C.9).

Particle worldline. The particle action (2.7) is already expressed as a contour integral. We divide it into two parts, prior to the collision with the boundary and after it,

$$S_m = -m \int_{r_0}^{r_i} \frac{dr}{\sqrt{-V_{\text{eff}}(r)}} - m \int_{r_0}^{r_f} \frac{dr}{\sqrt{-V_{\text{eff}}(r)}} , \qquad (\text{C}.11)$$

where we also changed the integration variable to $r \in \mathcal{C}_r$ using eq. (2.10), see figures 4a,c, and recalled that reflection flips the sign of $\dot{r}_*$. As before, $r_{i,f}$ are the particle positions at $t = t_{i,f}$. Explicitly calculating the integral (C.11) we find,

$$S_m = \frac{m^2}{\lambda p} \log \left(\frac{1}{2} + \frac{Mm^2}{8M_{cr}p^2} + \frac{p_\times}{2p} \right) - \frac{m^2}{p}(r_i + r_f - 2r_0) , \qquad (\text{C}.12)$$

where we extracted the asymptotics at $r_{i,f} \to +\infty$. Note that this contribution diverges linearly. The divergence will cancel, however, when we add the initial and final terms.

Initial and final terms. The expression (3.5) for S_{tot} includes contributions from the initial and final wavefunctions $\Psi_{i,f}(r_\mp) = \exp(\mp ipr_\mp)$, as well as the free actions S_0. The latter describe freely moving particle with momenta $\mp p$,

$$S_0(t_i, 0_-) = p(r_- - r_i) - Mt_i , \qquad\qquad S_0(0_+, t_f) = p(r_+ - r_f) + Mt_f ,$$

where $r_\mp$ are the positions of the free particle at $t = 0_\mp$. Combining the terms, one obtains

$$S_0(t_i, 0_-) + S_0(0_+, t_f) - i \log \Psi_f^* - i \log \Psi_i = -p(r_i + r_f) + M(t_f - t_i) . \qquad (\text{C}.13)$$

The change of the Schwarzschild time appearing here is given by the integral (3.10). Taking it explicitly, we obtain,

$$M(t_f - t_i) = -\frac{M}{\lambda} \log \left(1 - \frac{M + i\varepsilon'}{M_{cr}} \right) - \frac{M^2}{\lambda p} \log \left(\frac{1}{2} + \frac{Mm^2}{8M_{cr}p^2} + \frac{p_\times}{2p} \right) \qquad (\text{C}.14)$$

$$+ \frac{M}{\lambda} \log \left[\frac{4M^3 - 3m^2M + (4M^2 - m^2)p_\times}{(M+p)^3} + \frac{m^2(4M^2 + m^2)}{4M_{cr}(M+p)^3} \right] + \frac{M^2(r_f + r_i - 2r_0)}{p} .$$

Note that the contribution (C.13), (C.14) also diverges as $r_{i,f} \to +\infty$.

Collecting the terms (C.3), (C.7), (C.9), (C.10), (C.12), (C.13), and (C.14), one finally arrives to the total action (3.12). Note that the divergences at $r_{i,f} \to +\infty$ cancel between eqs. (C.12) and (C.13), (C.14).

D Constrained instantons for the entropy

In this appendix we give details of the Euclidean derivation of BH entropy. Performing the Wick rotation $t = -it_E$ in eq. (2.1), one obtains the Euclidean action,

$$
S_{\mathrm{gr},E} = -iS_{\mathrm{gr}} = -\int d^2x_E \sqrt{g}\, e^{-2\phi} \left[R + 4(\nabla\phi)^2 + 4\lambda^2 \right] - 2 \int\limits_{\phi=\phi_0} d\tau_0\, e^{-2\phi} (\kappa\, K + 2\lambda)
$$
$$
- 2 \int\limits_{r=r_\infty} d\sigma\, e^{-2\phi} (\kappa\, K - 2\lambda)\,, \tag{D.1}
$$

where we explicitly added the Gibbons-Hawking term at infinity. The parameter $\kappa = n^\mu n_\mu = \pm 1$ discriminates between the signatures $(+,+)$ and $(-,-)$ of the Euclidean space-time. Note that this parameter is implicitly present[10] in the original Minkowski action, or the latter would be inconsistent.

To warm up, consider the standard Gibbons-Hawking instanton in figure 7a. Since the solution (4.2) is stationary, one may naively expect that its Euclidean action is proportional to $\int dt_E = \beta$. This would give zero entropy in eq. (4.6). However, in the vicinity of the horizon $r - r_h \ll r_h$ the metric (4.2) takes the form

$$
ds^2 = d\rho^2 + \rho^2 d\theta^2\,, \tag{D.2}
$$

where $\rho = \sqrt{2(r - r_h)/\lambda}$ and $\theta = \lambda t_E$ are the radial and angular coordinates. Since θ changes between 0 and $\lambda\beta$, this metric describes a cone with angle deficit $2\pi - \lambda\beta$. The respective δ-contribution in curvature, eq. (4.3), is proportional to the angle deficit, not to β. That is why the standard calculation gives non-zero black hole entropy.

Now, let us ensure that every single configuration in the Euclidean path integral (4.1) includes a boundary $x^\mu = x_b^\mu(\tau_0)$ and ϕ equals ϕ_0 at this boundary. The latter condition is enforced by a δ-function in the integration measure,

$$
\prod_{\tau_0} \delta\big(\phi(x_b(\tau_0)) - \phi_0\big) = \int \mathcal{D}\Lambda\, e^{-\int d\tau_0\, \Lambda(\tau_0)\,[\phi(x_b) - \phi_0]}\,, \tag{D.3}
$$

with the boundary function $\Lambda(\tau_0)$ playing the role of a Lagrange multiplier. The product on the l.h.s. is taken over all points on the boundary. This adds an extra term to the Euclidean action,

$$
S_E = S_{\mathrm{gr},E} + \int d\tau_0\, \Lambda(\tau_0)\,(\phi_b - \phi_0)\,, \tag{D.4}
$$

where $\phi_b \equiv \phi(x_b(\tau_0))$.

---

[10]We fixed $\kappa = +1$ in eq. (2.1) because the scattering solutions included timelike boundary.

Importantly, the term (D.4) changes the boundary conditions at $x = x_b$. Indeed, variations with respect to $g_{\mu\nu}$ and ϕ now give[11]

$$n_0^\mu \nabla_\mu \phi = \kappa\lambda - \frac{\kappa}{4} e^{2\phi} \Lambda (\phi_b - \phi_0) , \qquad -2n_0^\mu \nabla_\mu \phi + K + 2\kappa\lambda = -\frac{\kappa}{4} e^{2\phi} \Lambda \qquad (D.5)$$

at $x = x_b(\tau_0)$. Besides, variation with respect to $\Lambda(\tau_0)$ gives equation $\phi_b = \phi_0$. In what follows we find solutions at a fixed Λ and then take the limit $\phi_b \to \phi_0$.

It is clear that the Gibbons-Hawking instanton in figure 7a does not satisfy the condition $\phi_b = \phi_0$, as it has $\phi < \phi_h < \phi_0$. Thus, we have to suggest an alternative. Let us assume that the true saddle-point configuration exists and it is real. Besides, we will consider only the solutions with τ_0-independent $\phi_b = -\lambda r_b$. This is reasonable because we will eventually send $\phi_b \to \phi_0$.

With the above assumptions, the only candidate for correct instanton is the disconnected configuration in figure 7b. The Birkhoff theorem guarantees that the additional cap-like part in this configuration is described by the Schwarzschild metric (4.2) with mass parameter M' in place of M. Besides, the instanton should include precisely one infinity with fixed ADM mass M. This specifies the patch $r_b < r < r_h'$ of the cap. Note that the metric (4.2) is this case has signature $(-, -)$ and $\kappa = -1$ in eqs. (D.5).

Substituting eq. (4.2) and constant ϕ_b into eqs. (D.5), one finds boundary conditions,

$$\Lambda(\phi_b - \phi_0) = 2M_b \left(1 + \sqrt{M'/M_b - 1}\right) , \qquad (D.6a)$$

$$(\Lambda/2 + 2M_b) \sqrt{M'/M_b - 1} = 2M_b - M' , \qquad (D.6b)$$

where $M_b = 2\lambda e^{-2\phi_b}$. The limit $\phi_b \to \phi_0$ corresponds to $\Lambda \to \infty$ and $M' \to M_{cr}$, with the combinations on the l.h.s. of eqs. (D.6) held fixed. At this point we obtained a unique solution.

The only additional contribution into the Euclidean action which is not proportional to β comes from the conical singularity at its second horizon r_h'. This time, however, the metric in the vicinity of r_h' is negative-definite: $ds^2 = -d\rho^2 - \rho^2 d\theta^2$, where $\rho = \sqrt{2(r_h' - r)/\lambda}$ and $\theta = \lambda t_E$. Thus, the respective δ-term of the curvature has opposite sign,

$$R = -4\pi(1 - \beta T_H) \frac{\delta^{(2)}(x - x_h')}{\sqrt{g}} + 2\lambda M' e^{-2\lambda r} , \qquad (D.7)$$

cf. eq. (4.3). Substituting eqs. (D.6), (D.7) into eq. (D.4) one finds the action (4.10) of the additional cap. The latter is β-independent. Note that M' in this expression is related to ϕ_b via eqs. (D.6). The limit $\phi_b \to \phi_0$ implies $M' \to M_{cr}$. Then eq. (4.6) reproduces the entropy (1.3).

Note that alternatively one can ignore some saddle-point equations and directly minimize the Euclidean action over the free parameters of the solutions. In particular, the discussion of section 4.3 corresponds to neglecting eq. (D.6b) and minimizing with respect to the single remaining parameter M'.

---

[11]In general, the second of eqs. (D.5) is obtained from the first by taking a derivative along the boundary and dividing the equation by $d\phi_b/d\tau_0$. The two equations are independent, however, if ϕ_b is constant at $x = x_b(\tau_0)$.

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