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To my father…

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Lausanne, July 20, 2020 Çağıl Koçyiğit

Abstract

Mechanism design theory examines the design of allocation mechanisms or incentive systems involving multiple rational but self-interested agents and plays a central role in many societally important problems in economics (auctions, contract theory, pricing, market design, voting, taxation, *etc.*). In mechanism design problems, agents typically hold private information that is unknown to the others. Traditionally, the mechanism design literature models this information asymmetry through random variables that are governed by a probability distribution, which is known precisely by all agents. Although this assumption usually facilitates the analysis, the knowledge of such a distribution may be difficult to justify, especially when the available data is scarce. In this thesis, we address this concern by assuming that the underlying probability distribution is unknown but belongs to an ambiguity set, which is commonly known. An ambiguity set is a set of distributions and typically contains all distributions consistent with the information available. We leverage techniques from distributionally robust optimization to investigate how decisions of a mechanism designer are affected by distributional ambiguity. In the first part of the thesis, we study a distributionally robust single-item auction design problem, where the seller aims to design a revenue maximizing auction that is not only immunized against the ambiguity of the bidder values but also against the uncertainty about the bidders' attitude towards ambiguity. For ambiguity sets that contain all distributions supported on a hypercube or that contain only distributions under which the bidders' values are independent, we show that the classical second price auctions are essentially optimal. For more realistic ambiguity sets under which the bidders' values are characterized through moment bounds, we identify a new mechanism, the optimal highest-bidder-lottery, whose revenues cannot be matched by any second price auction with a constant number of additional bidders. We also show that the optimal highest-bidder-lottery provides a 2-approximation of the (unknown) optimal mechanism, whereas the best second price auction fails to provide any constant-factor approximation guarantee. In the second part of the thesis, we study a robust monopoly pricing problem with a minimax regret objective, where a seller endeavors to sell multiple goods to a single buyer and has no knowledge of the underlying distribution apart from its support. We interpret this pricing problem as a zero-sum game between the seller, who chooses a selling mechanism, and a fictitious adversary or 'nature', who chooses the buyer's values from within an uncertainty set. Using duality techniques rooted in robust optimization, we prove that this game admits a Nash equilibrium in mixed strategies that can be computed in closed form. The Nash strategy of the seller is a randomized posted price mechanism under which the goods are sold separately, while the Nash strategy of nature is a distribution on the uncertainty set under which the buyer's values are comonotonic. We further show that the restriction of the pricing problem to deterministic mechanisms is solved by a deteministic posted price mechanism under which the goods are sold separately. In the last part of the thesis, we consider the multi-bidder variant of the mechanism design problem studied in the second part. We show that the separation result persists and that the optimal auction is separable across items, that is, at optimality the seller uses an individual auction separately for each item. To our best knowledge, this is the first separation result for mechanism design problems involving multiple bidders.

Résumé

La théorie de la conception des mécanismes examine des mécanismes d'allocation ou des systèmes d'incitation impliquant multiples agents rationnels à la recherche de leurs avantages personnels. Elle joue un rôle central dans des nombreux problèmes économiques importants pour nos sociétés (enchères, théorie des contrats, tarification, conception du marché, vote, fiscalité, etc.). Dans les problèmes de conception de mécanisme, les agents détiennent généralement des informations privées inconnues des autres. Traditionnellement, la littérature modélise cette asymétrie d'information par le biais de variables aléatoires gouvernées par une distribution de probabilité connue avec précision par tous les agents. Bien que cette hypothèse facilite généralement l'analyse, la connaissance d'une telle distribution peut être difficile à justifier, surtout lorsque peu de données sont disponibles. Dans cette thèse, nous adressons cette problématique en supposant que la distribution de probabilité sous-jacente est inconnue mais appartient à un ensemble d'ambiguïté connu par tous les agents. Un ensemble d'ambiguïté est un ensemble de distributions et contient typiquement toutes les distributions cohérentes avec les informations disponibles. Nous utilisons des techniques de l'optimisation distributionnellement robuste pour étudier comment les décisions d'un concepteur de mécanisme sont affectées par l'ambiguïté distributionnelle. Dans la première partie de la thèse, nous étudions un problème de conception d'enchères mono-unitaire distributionnellement robuste, où le vendeur vise à concevoir une enchère maximisant les revenus qui est non seulement immune contre l'ambiguïté des valeurs des soumissionnaires mais également contre l'incertitude concernant l'attitude envers l'ambiguïté des soumissionnaires. Pour les ensembles d'ambiguïté contentant toutes les distributions dont le support est un hypercube ou ne contenant que des distributions dans lesquelles les valeurs des soumissionnaires sont indépendantes, nous montrons que les enchères classiques au deuxième prix sont essentiellement optimales. Pour des ambiguïtés plus réalistes sous lesquelles les valeurs des soumissionnaires sont caractérisées par des limites de moments, nous identifions un nouveau mécanisme, la loterie optimale à la plus haute offre, dont les revenus ne peuvent être égalés par aucune enchère au deuxième prix avec un nombre fini de soumissionnaires supplémentaires. Nous montrons également que la loterie optimale à la plus haute offre fournit une approximation de facteur 2 du mécanisme optimal (inconnu), tandis que la meilleure enchère au deuxième prix ne fournit aucune garantie d'approximation de facteur constant. Dans la deuxième partie de la thèse, nous étudions un problème de tarification monopolistique robuste avec un objectif de regret minimax, où un vendeur s'efforce de vendre plusieurs biens à un seul acheteur en n'ayant aucune connaissance de la distribution sous-jacente autre que son support. Nous interprétons ce problème de tarification comme un jeu à somme nulle entre le vendeur, qui choisit un mécanisme de vente, et un adversaire fictif ou ≪ la nature ≫, qui choisit les valeurs de l'acheteur dans un ensemble d'incertitude. En utilisant des techniques de dualité ancrées dans l'optimisation robuste, nous prouvons que ce jeu admet un équilibre de Nash en stratégies mixtes qui peut être calculé analytiquement. La stratégie de Nash du vendeur est un mécanisme de prix affiché aléatoire selon lequel les biens sont vendues séparément, tandis que la stratégie de Nash de la nature est une distribution sur le set d'incertitude sous laquelle les valeurs de l'acheteur sont comonotoniques. Nous montrons en outre que la restriction du problème de tarification aux mécanismes déterministes admet comme solution un mécanisme de prix affiché déterministe selon lequel les biens sont vendues séparément. Dans la dernière partie de la thèse, nous considérons la variante multi-soumissionnaire du problème de conception de mécanisme étudié dans la deuxième partie. Nous montrons que le résultat de séparation persiste et que l'enchère optimale est séparable entre les biens, c'est-à-dire que le vendeur utilise une enchère individuelle pour chaque article à l'optimalité. À notre connaissance, il s'agit du premier résultat de séparation pour les problèmes de conception de mécanismes à plusieurs soumissionnaires.

Contents

Introduction

I choose questions to work on according to how much they excite me.

–Eric Maskin

Mechanism design theory plays a central role in many societally important problems in economics (auctions, contract theory, pricing, market design, voting, taxation, *etc.*) and has attracted tremendous interest. In fact, Leonid Hurwicz, Eric Maskin and Roger Myerson were awarded the Nobel Prize in economics (2007) "for having laid the foundations of mechanism design theory." In this thesis, we investigate mechanism design through the lens of optimization under uncertainty, which offers powerful modern methods that can open the door to solving practical mechanism design problems with complicating features.

Mechanism design theory examines the design of allocation mechanisms or incentive systems involving multiple rational but self-interested agents. Mechanism design problems are affected by uncertain parameters due to private information held by different agents. For example, in auctions, each bidder's willingness to pay (value) is unknown to the seller and to the other bidders. Traditionally, the mechanism design literature models this information asymmetry through random variables that are governed by a probability distribution, which is known precisely by all agents. Although this assumption usually facilitates the analysis, the knowledge of such a distribution may be difficult to justify, especially when the available data is scarce. For example, if an auction house is auctioning a tableau for the first time, there is simply no past sales data that could be used to estimate a reliable distribution of the bidders' values. It is thus natural to investigate the design of mechanisms when only partial information about the distribution is available. There is a large body of research trying to address this question across different communities. This thesis aims to contribute to this stream of research. Particularly, we address this question by assuming that the underlying probability distribution is unknown but belongs to an ambiguity set, which is commonly known. An ambiguity set is a set of distributions and typically contains all distributions consistent with the information available such as the support, mean value or standard deviation of the distribution. We leverage techniques from distributionally robust optimization to investigate how the decisions of a mechanism designer are affected by distributional ambiguity. In doing so, we mainly focus on auction design and pricing.

Distributionally robust optimization seeks decisions that have minimum risk under the most adverse distribution in the ambiguity set. The solutions to distributionally robust optimization problems are guaranteed to display an out-of-sample risk that falls below the worst-case optimal risk whenever the ambiguity set contains the unknown true distribution. Thus, the decisions taken with respect to an empirical distribution over-promise and under-deliver, while distributionally robust decisions under-promise and over-deliver. For a general introduction to distributionally robust optimization we refer to Delage and Ye (2010), Goh and Sim (2010) and Wiesemann et al. (2014). In the context of mechanism design, the respective decisions constitute mechanisms. Intuitively, a distributionally robust mechanism design problem can be viewed as a zero-sum game between the mechanism designer, who chooses a mechanism, and a fictitious adversary, who chooses a distribution with the goal to inflict maximum damage to the designer.

Contributions and Structure of the Thesis

The main contributions of this thesis are divided into three self-contained chapters.

In Chapter 1 we study a distributionally robust variant of the single-item auction design problem, where a seller aims to design a revenue maximizing auction that is not only immunized against the ambiguity of the bidder values but also against the uncertainty about the bidders' attitude towards ambiguity. In particular, we consider three popular classes of ambiguity sets: *(i)* support-only ambiguity sets containing all distributions supported on a hypercube, *(ii)* independence ambiguity sets comprising symmetric and regular distributions supported on a hypercube under which the bidder values are independent, and *(iii)* Markov ambiguity sets containing all distributions that are supported on a hypercube and satisfy a first-order moment constraint. For support-only ambiguity sets, we show that the Vickrey auction is not only optimal but also Pareto robustly optimal over all mechanisms in the sense that there exists no other feasible mechanism that generates higher revenues to the seller under some realization of the bidders' values while generating at least the same revenues under every other realization. For independence ambiguity sets, the added value of the (to date unknown) optimal mechanism over the Vickrey auction is offset by attracting just one additional bidder. For Markov ambiguity sets, we specify the best second price auction with reserve price. We show that while this auction asymptotically maximizes the worst-case expected seller revenues as the number of bidders grows, it may extract an arbitrarily small fraction of the optimal mechanism's revenues for a finite number of bidders. We also propose a new class of auctions—the *highest-bidder-lotteries*—in which the seller offers the highest bidder a lottery that determines the allocation of the good as well as the payment. We analytically determine the best highest-bidder-lottery, and we show that it can generate significantly higher revenues than the best second price auction with reserve price. Specifically, we show that the proposed mechanism is a 2-approximation of the (unknown) optimal one. We also prove that the revenues of the optimal highest-bidder-lottery cannot be matched by any second price auction with a constant number of additional bidders.

The contents of Chapter 1 are published in the following paper.

- Koçyiğit, Ç., Iyengar, G., Kuhn, D., and Wiesemann, W. (2020). Distributionally robust mechanism design. *Management Science*, 66(1), 159-189.

In Chapter 2 we study a robust monopoly pricing problem with minimax regret objective, where a seller endeavors to sell multiple goods to a single buyer, only knowing that the buyer's values for the goods range over a rectangular uncertainty set. We first show that the pricing problem at hand admits an explicit analytical solution by leveraging duality techniques rooted in robust optimization. The solution obtained represents a randomized mechanism under which the goods are sold *separately*. We then interpret the robust pricing problem as a zero-sum game between the seller and a fictitious adversary or 'nature', who chooses the buyer's value profile in the uncertainty set with the aim to inflict maximum damage. We demonstrate that this game admits a Nash equilibrium in mixed strategies, which can be computed in closed form. The Nash strategy of the seller coincides with the optimal randomized mechanism, while the Nash strategy of nature is a (non-discrete) distribution on the uncertainty set under which the buyer's values for the items are comonotonic. We also study a restriction of the robust pricing problem that optimizes only over deterministic mechanisms, which is essentially equivalent to searching over all posted price mechanisms. We solve this problem analytically and show that the different goods are again sold *separately* at optimality.

The contents of Chapter 2 can be found in the following paper.

- Koçyiğit, Ç., Rujeerapaiboon, N., and Kuhn, D. (2020). Robust multidimensional pricing: Separation without regret. *Under Review for Mathematical Programming*.

In Chapter 3 we study a multi-bidder variant of the mechanism design problem studied in Chapter 2. Specifically, we study a robust auction design problem with minimax regret objective, where a seller seeks an auction mechanism to sell multiple items to multiple anonymous bidders. The seller knows that the bidders' values range over a box uncertainty set but has no other information about their probability distribution. We interpret this auction design problem as a zero sum game between the seller and a fictitious adversary or 'nature', who chooses the bidders' value profiles from within a box uncertainty set with the aim to inflict maximum damage to the seller. We characterize the Nash equilibrium of this game analytically and prove that the seller's Nash strategy is a mechanism under which each item is auctioned *separately*. The separate mechanisms for the individual items can be interpreted as second price auctions with random reserve prices. We show that nature's Nash strategy is mixed and represents a probability distribution under which each bidder's values for the items are comonotic. We also demonstrate that the seller's worst-case regret under her Nash strategy is independent of the number of bidders.

The contents of Chapter 3 are based on the following technical note.

- Koçyiğit, Ç., Rujeerapaiboon, N., and Kuhn, D. (2020). Regret Minimization and Separation in Multi-Bidder Multi-Item Auctions. *Under Review.*

We discuss the contributions and positioning relative to the existing literature in more detail in each chapter.

Statement of Originality

I hereby certify that this thesis is the result of my own work, where some parts are the result of collaborations with my supervisor Dr. Daniel Kuhn and my co-authors Dr. Garud Iyengar, Dr. Napat Rujeerapaiboon and Dr. Wolfram Wiesemann. No other person's work has been used without due acknowledgment.

1 Distributionally Robust Mechanism Design

We study a mechanism design problem where an indivisible good is auctioned to multiple bidders, for each of whom it has a private value that is unknown to the seller and the other bidders. The agents perceive the ensemble of all bidder values as a random vector governed by an ambiguous probability distribution, which belongs to a commonly known ambiguity set. The seller aims to design a revenue maximizing mechanism that is not only immunized against the ambiguity of the bidder values but also against the uncertainty about the bidders' attitude towards ambiguity. We argue that the seller achieves this goal by maximizing the worst-case expected revenue across all value distributions in the ambiguity set and by positing that the bidders have Knightian preferences. For ambiguity sets containing all distributions supported on a hypercube, we show that the Vickrey auction is the unique mechanism that is optimal, efficient and Pareto robustly optimal. If the bidders' values are additionally known to be independent, then the revenue of the (unknown) optimal mechanism does not exceed that of a second price auction with only one additional bidder. For ambiguity sets under which the bidders' values are dependent and characterized through moment bounds, on the other hand, we provide a new class of randomized mechanisms, the highest-bidder-lotteries, whose revenues cannot be matched by any second price auction with a constant number of additional bidders. Moreover, we show that the optimal highest-bidder-lottery is a 2-approximation of the (unknown) optimal mechanism, whereas the best second price auction fails to provide any constant-factor approximation guarantee.

1.1 Introduction

When traders from the Ottoman Empire first brought tulip bulbs to Holland in the seventeenth century, the combination of a limited supply and a rapidly increasing popularity led to highly non-stationary and volatile prices. Faced with the challenge of selling scarce items with a largely unknown demand, the flower exchange invented the Dutch auction, in which an artificially high asking price is gradually decreased until the first participant is willing to accept the trade. Nowadays, auctions are routinely used in economic transactions that are characterized by demand uncertainty, ranging from the sale of financial instruments (*e.g.*, U.S. Treasury bills), antiques, collectibles and commodities (*e.g.*, radio spectra, electricity and carbon emissions) to livestock and holidays.

Despite their long history, the scientific study of auctions only started in the sixties of the last century when the then emerging discipline of mechanism design began to model auctions as incomplete information games between rational but self-interested agents. In the most basic such game, a seller wishes to auction a single product to multiple bidders. Each bidder is fully aware of the value that he attaches to the good, whereas the other bidders and the seller only know the probability distribution from which this value has been drawn. This information structure is referred to as the *private value* setting. The seller aims to design a mechanism that allocates the good and charges the bidders based on a single-shot or iterative bidding process so as to maximize her expected revenues (*optimal* mechanism design), sometimes under the additional constraint that the resulting allocation should maximize the overall welfare (*efficient* mechanism design). The bidders, in turn, seek to submit bids that maximize their expected utility arising from the difference of the value obtained from receiving the good (if they do so) and the charges incurred.

In the private value setting outlined above, the bidders' values for the good are typically modeled as independent random variables. Under this assumption, Vickrey (1961) argues that the second price auction without reserve price, which allocates the good to the highest bidder and charges him the value of the second highest bid, generates maximum revenues among all efficient mechanisms. Myerson (1981) proves that in the same setting, the second price auction maximizes the seller's revenues if it is augmented with a suitable reserve price. In this case, however, efficiency is typically lost since the good resides with the seller whenever the highest bid falls short of the reserve price. Cremer and McLean (1988) show that if the bidders' values are described by *correlated* random variables, then second price auctions no longer maximize the seller's revenues, and the seller can extract all surplus by combining an auction with a menu of side bets with the bidders. For a review of the mechanism design literature, we refer to Klemperer (1999) and Krishna (2009).

Traditionally, the mechanism design literature models the bidders' values as a random vector that is governed by a probability distribution which is known precisely by all participants. Although this assumption greatly facilitates the analysis, the existence and common knowledge of such a distribution may be difficult to justify in settings where the demand is poorly understood, which arguably form the auctions' *raison d'être*. The literature on robust mechanism design addresses this concern by assuming that the bidders' willingness to pay is only known to be governed by some probability distribution from within an *ambiguity set*. In this setting, the agents take decisions that maximize their expected utility under the most adverse value distribution in the ambiguity set.

The early literature on robust mechanism design has studied the impact of ambiguity on traditional auction schemes. Salo and Weber (1995) show that the experimentally observed deviations from the theoretically optimal bidding strategy in a first price auction can be explained by the presence of ambiguity as well as ambiguity averse decision-making on behalf of the agents. In a similar study, Chen et al. (2007) show that the presence of ambiguity leads to lower bids in first price auctions. Lo (1998) and Ozdenoren (2002) derive the optimal bidding strategy for an ambiguity averse bidder in a first price auction, and they show that in contrast to the traditional theory, first price and second price auctions yield different revenues in the presence of ambiguity. Chiesa et al. (2015) study a variant of the private value setting where the bidders are unsure about both their own value and the other bidders' values for the auctioned good, and they show that the Vickrey mechanism maximizes the worst-case social welfare in this setting.

More recently, the robust mechanism design literature has focused on characterizing revenue maximizing auctions for different variants of the mechanism design problem under ambiguity. Bose et al. (2006) show that full insurance mechanisms, which either make the seller or the bidders indifferent between the possible bids of the (other) bidders, maximize the seller's worst-case revenues in several variants of the optimal auction design problem under ambiguity. Bodoh-Creed (2012) generalizes a well-known payoff equivalence result to ambiguous auctions, and he uses it to provide further intuition about the optimality of full insurance mechanisms. Bose and Daripa (2009) show that for certain classes of ϵ-contamination ambiguity sets, the seller can extract almost all surplus by a variant of the Dutch auction.

Bose et al. (2006), Bose and Daripa (2009) and Bodoh-Creed (2012) all model the bidders' values as independent random variables, and they assume that the agents exhibit maxmin preferences, that is, the agents judge actions in view of their expected utility under the worst probability distribution in the ambiguity set. In contrast, Lopomo et al. (2014) consider agents that exhibit Knightian preferences, that is, an action A is preferred over an action B only if A yields a weakly higher expected utility than B under every probability distribution in the ambiguity set. They derive necessary and sufficient conditions for full surplus extraction under ambiguity in a mechanism design problem where a principal interacts with a single agent. In a similar spirit, Koçyiğit et al. (2018a) study a single-item auction where the bidders' values are private and ambiguous, and the agents exhibit Knightian preferences. The authors show that second price auctions are no longer optimal in this setting, and they develop a numerical solution scheme for determining the optimal mechanism under the premise that the ambiguity set consists of two distributions.

Bandi and Bertsimas (2014) point out that the mechanism design problem is amenable to a formulation as a robust optimization problem (Ben-Tal et al., 2009; Bertsimas et al., 2011). To this end, they model the bidders' values as a deterministic vector that is chosen adversely from an uncertainty set. They show that in this setting, the second price auction with item and bidder dependent reserve prices is optimal for multi-item auctions with budget constrained buyers. They also show that the optimal reserve prices can be calculated through an optimization problem.

Apart from the robust mechanism design literature, relaxations of the assumptions underlying the information structure of auction problems have been studied in the related fields of prior-free and prior-independent mechanism design. While the informational assumptions of prior-free mechanisms are akin to those of robust mechanisms, their goal is to minimize the worst-case regret (in terms of the actual revenues) over all scenarios relative to a judiciously chosen benchmark, rather than to maximize the worst-case revenues. Notable examples are discussed by Goldberg et al. (2006), who provide a 4-approximation to the revenues generated by the optimal posted price in a digital goods environment, as well as Hartline and Roughgarden (2008), who derive an O(1)-approximation to the maximum revenue achieved by any Bayesian optimal mechanism in a general symmetric auction framework. For further details, we refer to Nisan et al. (2007).

In a similar spirit, prior-independent mechanisms aim to minimize the worst-case regret (now in terms of the expected revenues) over all value distributions contained in an ambiguity set and relative to the respective Bayesian optimal mechanism. Dhangwatnotai et al. (2015) propose the Single Sample mechanism, which is a second price auction with a random reserve price, and show that this mechanism provides constant-factor approximations in different auction settings. This mechanism has subsequently been generalized by Roughgarden and Talgam-Cohen (2013) to an interdependent values setting, where the bidders' values are determined through private signals.

Finally, we note that there is a distinct branch of the mechanism design literature that is also referred to as robust mechanism design, see, *e.g.*, Bergemann and Morris (2005), Chung and Ely (2007) and Bergemann et al. (2016). Contrary to the robust mechanism design literature reviewed above, which assumes that the value distribution is ambiguous but the ambiguity set is common knowledge, this literature stream exclusively works with non-ambiguous value distributions, but relaxes the common knowledge assumption in the sense that the agents may be unsure about the (higher-order) beliefs of the other agents. As a consequence, the findings in the two literature streams are complementary due to the different informational assumptions made. For a review of this literature stream, we refer to Bergemann and Morris

(2013).

In this chapter, we study the single-item auction design problem under ambiguity, where we follow the approach of Lopomo et al. (2014) and assume that the agents exhibit Knightian preferences. We show that this assumption not only protects the seller against the ambiguity of the bidders' values, but it also immunizes her against the bidders' attitude towards this ambiguity. We then argue that the resulting mechanism design problem under ambiguity is amenable to a formulation as a *distributionally* robust optimization problem (Delage and Ye, 2010; Wiesemann et al., 2014). We use this insight to study three popular classes of ambiguity sets: *(i)* support-only ambiguity sets containing all distributions supported on a hypercube, *(ii)* independence ambiguity sets comprising symmetric and regular distributions supported on a hypercube under which the bidder values are independent, and *(iii)* Markov ambiguity sets containing all distributions that are supported on a hypercube and satisfy a first-order moment constraint.

The contributions of this chapter to the three classes of ambiguity sets are summarized below.

1. For support-only ambiguity sets, we show that the Vickrey auction is not only optimal but also Pareto robustly optimal over *all* (efficient and inefficient) mechanisms in the sense that there exists no other feasible mechanism that generates higher revenues to the seller under some realization of the bidders' values while generating at least the same revenues under every other realization. Moreover, we show that the Vickrey auction generates the highest revenues among all efficient mechanisms under every possible realization of the bidders' values.

2. For independence ambiguity sets, we prove that the Vickrey auction generates the highest worst-case expected revenue among all *efficient* (but not necessarily inefficient) mechanisms. We also show that the added value of the (to date unknown) optimal mechanism over the Vickrey auction is offset by attracting just one additional bidder.

3. For Markov ambiguity sets, we specify the best second price auction with reserve price. We show that while this auction asymptotically maximizes the worst-case expected seller revenues as the number of bidders grows, it may extract an arbitrarily small fraction of the optimal mechanism's revenues for a finite number of bidders. We also propose a new class of auctions—the *highest-bidder-lotteries*—in which the seller offers the highest bidder a lottery that determines the allocation of the good as well as the payment. We analytically determine the best highest-bidder-lottery, and we show that it can generate significantly higher revenues than the best second price auction with reserve price. Specifically, we show that the proposed mechanism is a 2-approximation of the (unknown) optimal one. We also prove that the revenues of the optimal highest-

bidder-lottery cannot be matched by any second price auction with a constant number of additional bidders.

Notation.

For any $\boldsymbol{v} \in \mathbb{R}^I$ we denote by v_i its i^{th} component and by $\boldsymbol{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_I)$ its subvector excluding v_i. The vector of ones is denoted by $\mathbf{e}$. Random variables are designated by tilde signs (*e.g.*, $\tilde{v}$) and their realizations by the same symbols without tildes (*e.g.*, v). For any Borel set $\mathscr{A} \in \mathscr{B}(\mathbb{R}^I)$ we use $\mathscr{P}_0(\mathscr{A})$ to represent the set of all probability distributions on $\mathscr{A}$. The family of all bounded Borel-measurable functions from $\mathscr{A} \in \mathscr{B}(\mathbb{R}^n)$ to $\mathscr{C} \in \mathscr{B}(\mathbb{R}^m)$ is denoted by $\mathscr{L}_\infty(\mathscr{A}, \mathscr{C})$. For $\mathscr{A} \in \mathscr{B}(\mathbb{R}^I)$, $A_i \in \mathscr{B}(\mathbb{R})$, $f, g \in \mathscr{L}_\infty(\mathscr{A}, \mathbb{R})$ and $\mathscr{P} \subseteq \mathscr{P}_0(\mathscr{A})$, statements of the form $\inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\boldsymbol{v}}) | \tilde{v}_i = v_i] \geq \inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}}[g(\tilde{\boldsymbol{v}}) | \tilde{v}_i = v_i] \; \forall v_i \in \mathscr{A}_i$, which are not well-defined because conditional expectations under $\mathbb{P}$ are only defined up to sets of $\mathbb{P}$-measure zero, should be interpreted as $\inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\boldsymbol{v}}) h(\tilde{v}_i)] \geq \inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}}[g(\tilde{\boldsymbol{v}}) h(\tilde{v}_i)] \; \forall h \in \mathscr{L}_\infty(\mathscr{A}_i, \mathbb{R}_+)$. The latter statement is well-defined but cumbersome.

1.2 Problem Formulation and Preliminaries

We consider the following mechanism design problem. A seller aims to sell an indivisible good which is of zero value to her. There are $I \geq 2$ potential buyers (or bidders) indexed by $i \in \mathscr{I} = \{1, \ldots, I\}$. The buyers' values for the good are modeled as a random vector $\tilde{\boldsymbol{v}}$ that follows a probability distribution $\mathbb{P}^0$ in some ambiguity set $\mathscr{P} \subseteq \mathscr{P}_0(\mathbb{R}_+^I)$. We denote the realizations of $\tilde{\boldsymbol{v}}$ by $\boldsymbol{v}$ and refer to them as scenarios. The probability distribution $\mathbb{P}^0$ is unknown to the agents, but the ambiguity set $\mathscr{P}$ is common knowledge. We assume that the smallest closed set that has probability 1 under every distribution $\mathbb{P} \in \mathscr{P}$ is of the form $\mathscr{V}^I = \mathscr{V} \times \cdots \times \mathscr{V}$ with marginal projections $\mathscr{V} \subseteq \mathbb{R}_+$; this is a standard assumption in the mechanism design literature (McAfee and McMillan, 1987).

The seller aims to determine a mechanism for selling the good. A mechanism $(\mathscr{B}_1, \ldots, \mathscr{B}_I, \boldsymbol{q}, \boldsymbol{m})$ consists of a set $\mathscr{B}_i$ of messages (or bids) available to each buyer i, an allocation rule $\boldsymbol{q}$: $\mathscr{B}_1 \times \cdots \times \mathscr{B}_I \to \mathbb{R}_+^I$ and a payment rule $\boldsymbol{m} : \mathscr{B}_1 \times \cdots \times \mathscr{B}_I \mapsto \mathbb{R}^I$. Depending on his value v_i, each buyer i reports a message $b_i \in \mathscr{B}_i$ to the seller. Once all messages are collected, the seller allocates the good to buyer i with probability $q_i(\boldsymbol{b})$ and charges this buyer an amount $m_i(\boldsymbol{b})$, where $\boldsymbol{b} = (b_1, \ldots, b_I)$.

Example 1.2.1 (First Price Sealed Bid Auction)**.** *The first price sealed bid auction is a widely used mechanism, where bidders simultaneously report their bids* $b_i \in \mathscr{B}_i = \mathbb{R}_+$, $i \in \mathscr{I}$. *The highest bidder wins the good with probability* 1 *and pays an amount equal to his bid, whereas all other bidders win the good with probability* 0 *and do not make a payment. If there is a tie* *(the highest bidder is not unique), then the winner is determined at random (or by some other tie-breaking rule).*

We assume that all agents are risk-neutral with respect to the uncertainty of the allocation.

Definition 1.2.1 (Ex-post Utility)**.** *The* ex-post utility *of bidder i with value v_i and reporting message b_i is defined as*

$$u_i(b_i; v_i, \boldsymbol{b}_{-i}) \;=\; q_i(b_i, \boldsymbol{b}_{-i})\, v_i - m_i(b_i, \boldsymbol{b}_{-i}),$$

where $\boldsymbol{b}_{-i}$ denotes the vector of messages reported by the other bidders.

The ex-post utility of a bidder quantifies his expected payoff after all messages are revealed. Note that the ex-post utility depends critically on the allocation and payment rules of the mechanism at hand. We will suppress this dependence notationally, however, in order to avoid clutter.

We assume that the buyers have incomplete preferences as in Knightian decision theory, see, *e.g.*, Knight (1921) and Bewley (2002). In this setting, a buyer prefers an action to another one if it results in a higher expected utility to him under every distribution $\mathbb{P} \in \mathscr{P}$.

Given a mechanism, the buyers play a game of incomplete information and select their bids strategically to induce the most desirable outcome in view of their individual preferences. Recall that buyer i selects a message depending on his value v_i. Thus, his strategy must be modeled as a function $\beta_i : \mathcal{V} \to \mathscr{B}_i$ that maps each of his possible values to a message. An I-tuple of strategies $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_I)$ constitutes an equilibrium for a given mechanism if no agent i has an incentive to unilaterally change his strategy β_i.

Definition 1.2.2 (Knightian Nash Equilibrium)**.** *An I-tuple of strategies $\beta_i : \mathcal{V} \to \mathscr{B}_i$, $i \in \mathscr{I}$, constitutes a* Knightian Nash equilibrium *for a mechanism $(\mathscr{B}_1, \ldots, \mathscr{B}_I, \boldsymbol{q}, \boldsymbol{m})$ if*

$$\inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(\beta_i(v_i); v_i, \boldsymbol{\beta}_{-i}(\tilde{\boldsymbol{v}}_{-i})) - u_i(b_i; v_i, \boldsymbol{\beta}_{-i}(\tilde{\boldsymbol{v}}_{-i})) \mid \tilde{v}_i = v_i \right] \geq 0 \quad \forall i \in \mathscr{I}, \forall v_i \in \mathcal{V}, \forall b_i \in \mathscr{B}_i.$$

In the absence of ambiguity, that is, for $\mathscr{P} = \{\mathbb{P}^0\}$, a Knightian Nash equilibrium collapses to a Bayesian Nash equilibrium as introduced by Harsanyi (1967). If $\mathscr{P} = \mathscr{P}_0(\mathcal{V}^I)$, on the other hand, then the Knightian Nash equilibrium reduces to an ex-post Nash equilibrium (Fudenberg and Tirole, 1991, Section 1.2). Note also that every ex-post Nash equilibrium is automatically a Knightian Nash equilibrium, but the converse implication is generally wrong.

The mechanism design problem is the decision problem of the seller. We assume that the seller is ambiguity averse in the sense that she aims to maximize the worst-case expected revenue in view of all distributions $\mathbb{P} \in \mathscr{P}$. However, she may not know how ambiguity is perceived by the bidders and may wish to hedge against uncertainty in the buyers' preferences. We will argue later that this is achieved by adopting the view that the buyers have Knightian preferences, which in a sense represent the worst-case buyer preferences from the seller's perspective. Hence, the seller is interested in selecting allocation and payment rules that maximize her worst-case expected revenue, anticipating that the buyers' strategies will be in a Knightian Nash equilibrium. Note that a mechanism is of interest only if it has a Knightian Nash equilibrium because, otherwise, its outcome is unpredictable.

We assume that bidder i with value $v_i \in \mathcal{V}$ will walk away from a mechanism if his expected utility under a Knightian Nash equilibrium is negative for some $\mathbb{P} \in \mathscr{P}$. Nevertheless, the seller only needs to consider mechanisms that attract all buyers. Indeed, imagine that bidder i with value v_i prefers to walk away under the mechanism $(\boldsymbol{q}, \boldsymbol{m})$. The same outcome is achieved by setting $q_i(\beta_i(v_i), \boldsymbol{\beta}_{-i}(\boldsymbol{v}_{-i})) = 0$ and $m_i(\beta_i(v_i), \boldsymbol{\beta}_{-i}(\boldsymbol{v}_{-i})) = 0$ for all $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, which results in an ex-post utility of zero to him so that participating remains weakly dominant.

The set of all mechanisms is extremely large. An important subset is the family of *direct* mechanisms in which the set of messages available to buyer i is equal to the set of his values, that is, $\mathscr{B}_i = \mathcal{V}$ for all $i \in \mathscr{I}$. Yet a smaller subset is the family of *truthful* direct mechanisms, in which it is optimal for each buyer to report his true value. In fact, due to the celebrated revelation principle by Myerson (1981), we can restrict attention to truthful direct mechanisms without loss of generality.

Theorem 1.2.1 (The Revelation Principle)**.** *Given any mechanism $(\mathscr{B}_1, \ldots, \mathscr{B}_I, \boldsymbol{q}, \boldsymbol{m})$ with a corresponding Knightian Nash equilibrium $\beta_i : \mathcal{V} \to \mathscr{B}_i$, $i \in \mathscr{I}$, there exists a truthful direct mechanism resulting in the same ex-post utilities for the bidders and the same ex-post revenue for the seller for every $\boldsymbol{v} \in \mathcal{V}^I$.*

The proof is a straightforward adaptation of the proof of Proposition 5.1 in Krishna (2009). The intuition is as follows. Consider any mechanism $(\mathscr{B}_1, \ldots, \mathscr{B}_I, \boldsymbol{q}, \boldsymbol{m})$ as well as an equilibrium $\boldsymbol{\beta}$ for this mechanism. Then, the seller can construct an equivalent truthful direct mechanism $(\mathcal{V}, \ldots, \mathcal{V}, \boldsymbol{q}', \boldsymbol{m}')$ by asking the bidders to report their true values, allocating the good according to the rule $\boldsymbol{q}'(\boldsymbol{v}) = \boldsymbol{q}(\boldsymbol{\beta}(\boldsymbol{v}))$ and charging payments $\boldsymbol{m}'(\boldsymbol{v}) = \boldsymbol{m}(\boldsymbol{\beta}(\boldsymbol{v}))$ as if the bidders had implemented their equilibrium strategies for the original mechanism. In this case, the bidders have no incentive to misreport their true values because truthful bidding is the equilibrium strategy for the new mechanism by construction. Also, the ex-post revenue of the seller and the ex-post utilities of the bidders do not change.

From now on, we focus exclusively on truthful direct mechanisms and use the shorthand $(\boldsymbol{q}, \boldsymbol{m})$ to denote $(\mathcal{V}, \ldots, \mathcal{V}, \boldsymbol{q}, \boldsymbol{m})$ because the set of messages available to each buyer is always equal to the interval of his possible values. A direct mechanism is truthful under Knightian preferences if and only if it is distributionally robust incentive compatible.

Definition 1.2.3 (Distributionally Robust Incentive Compatibility)**.**
A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ *is called* distributionally robust incentive compatible *if*

$$\inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) - u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq 0 \quad \forall i \in \mathscr{I}, \forall v_i, w_i \in \mathscr{V}. \tag{IC-D}$$

Distributionally robust incentive compatibility ensures that reporting the true value $v_i \in \mathscr{V}$ is a dominant strategy for bidder i under Knightian preferences.

Recall that the seller only needs to consider mechanisms that attract all bidders. If the bidders are willing to participate in a given mechanism, the corresponding truthful direct mechanism will be distributionally robust individually rational.

Definition 1.2.4 (Distributionally Robust Individual Rationality)**.**
A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ *is called* distributionally robust individually rational *if*

$$\inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq 0 \quad \forall i \in \mathscr{I}, \forall v_i \in \mathscr{V}. \tag{IR-D}$$

Distributionally robust individual rationality ensures that the expected utility of bidder i conditional on his own value v_i is non-negative under truthful bidding for any possible value $v_i \in \mathscr{V}$ and any possible probability distribution $\mathbb{P} \in \mathscr{P}$.

We can now formalize the seller's problem of finding the best truthful direct mechanism as

$$\begin{array}{ll} \sup\limits_{\boldsymbol{q}\in\mathscr{Q},\boldsymbol{m}\in\mathscr{M}} & \inf\limits_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[\sum\limits_{i\in\mathscr{I}} m_i(\tilde{\boldsymbol{v}})\right] \\ \text{s.t.} & \text{(IC-D), (IR-D),} \end{array} \tag{$\mathscr{MDP}$}$$

where

$$\mathscr{Q} = \{\boldsymbol{q} \in \mathscr{L}_\infty(\mathscr{V}^I, \mathbb{R}^I_+) \mid \sum_{i\in\mathscr{I}} q_i(\boldsymbol{v}) \leq 1 \ \ \forall \boldsymbol{v} \in \mathscr{V}^I\}$$

is the set of all possible allocation rules of direct mechanisms. The definition of $\mathscr{Q}$ captures the idea that the seller can sell the good at most once. Similarly, $\mathscr{M} = \mathscr{L}_\infty(\mathscr{V}^I, \mathbb{R}^I)$ denotes the set of all possible payment rules of direct mechanisms. By the revelation principle, solving ($\mathscr{MDP}$) is equivalent to finding the best mechanism among all direct *and* indirect mechanisms.

Sometimes we will further restrict problem ($\mathscr{MDP}$) to optimize only over efficient mechanisms.

Definition 1.2.5 (Efficiency)**.** *A mechanism* $(\boldsymbol{q}, \boldsymbol{m})$ *is called* efficient *if* $\boldsymbol{q} \in \mathcal{Q}^{\text{eff}}$, *where*

$$\mathcal{Q}^{\text{eff}} = \left\{ \boldsymbol{q} \in \mathcal{Q} \mid q_i(\boldsymbol{v}) > 0 \implies v_i = \max_{j \in \mathcal{I}} v_j \quad \forall i \in \mathcal{I}, \quad \sum_{i \in \mathcal{I}} q_i(\boldsymbol{v}) = 1 \quad \forall \boldsymbol{v} \in \mathcal{V}^I \right\}.$$

An efficient mechanism allocates the good with probability 1 to a bidder who values it most. Hence, it maximizes the ex-post total social welfare across all agents (seller and bidders), which coincides with the highest bidder value because the payments of the bidders and the revenue of the seller cancel each other. Recall that the good has zero value to the seller. Allocative efficiency plays a crucial role in the sale of public goods such as railway lines, plots of public land or specific bands of the electromagnetic spectrum. Efficient allocations do not normally emerge from mechanisms with inefficient allocation rules, even if we allow for the existence of an aftermarket with zero transaction costs (Krishna, 2009, Section 1.4).

We will now demonstrate that distributionally robust incentive compatible mechanisms protect the seller against uncertainty in the bidders' attitude towards ambiguity. Indeed, depending on the bidders' preferences, one can envisage other types of incentive compatibility.

Definition 1.2.6. *A mechanism* $(\boldsymbol{q}, \boldsymbol{m})$ *is called*

(i) ex-post incentive compatible *if for all* $i \in \mathcal{I}$, $\boldsymbol{v} \in \mathcal{V}^I$, $w_i \in \mathcal{V}$,

$$u_i(v_i; v_i, \boldsymbol{v}_{-i}) \geq u_i(w_i; v_i, \boldsymbol{v}_{-i}),$$

(ii) maxmin incentive compatible *if for all* $i \in \mathcal{I}$, $v_i, w_i \in \mathcal{V}$,

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right],$$

(iii) Hurwicz incentive compatible *with respect to* $\alpha \in (0,1)$ *if for all* $i \in \mathcal{I}$, $v_i, w_i \in \mathcal{V}$,

$$\begin{aligned} &\alpha \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] + (1-\alpha) \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] \\ &\quad \geq \alpha \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] + (1-\alpha) \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right], \end{aligned}$$

(iv) Bayesian incentive compatible *with respect to a Borel distribution* $\mathbb{Q}$ *on* $\mathcal{P}$ *(where* $\mathcal{P}$ *is equipped with its weak topology) if for all* $i \in \mathcal{I}$, $v_i, w_i \in \mathcal{V}$,

$$\mathbb{E}_{\mathbb{Q}} \Big[\mathbb{E}_{\tilde{\mathbb{P}}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] \Big] \geq \mathbb{E}_{\mathbb{Q}} \Big[\mathbb{E}_{\tilde{\mathbb{P}}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] \Big],$$

where $\tilde{\mathbb{P}} \sim \mathbb{Q}$ *is a random value distribution.*

One can define individual rationality with respect to other preferences analogously.

Proposition 1.2.1. *Ex-post incentive compatibility implies distributionally robust incentive compatibility, whereas distributionally robust incentive compatibility implies maxmin incentive compatibility, Hurwicz incentive compatibility and Bayesian incentive compatibility.*

Using similar arguments as in Proposition 1.2.1, one can verify that ex-post individual rationality implies distributionally robust individual rationality, and that distributionally robust individual rationality implies maxmin, Hurwicz and Bayesian individual rationality. Thus, agents have no incentive to walk away from a distributionally robust individually rational mechanism or to misreport their values in a distributionally robust incentive compatible mechanism even if they have maxmin, Bayesian or Hurwicz preferences. Hence, adopting a distributionally robust perspective allows the seller to hedge against uncertainty about the bidders' attitude towards ambiguity. Use of ex-post individual rationality and incentive compatibility would provide even stronger protection against uncertainty in the bidders' preferences, but it would also lead to more conservative mechanisms that do not benefit from any distributional information that might be available.

So far, the literature on mechanism design has used almost exclusively the maxmin criterion to model the ambiguity aversion of the bidders. However, while being less conservative, the resulting mechanism design problems may fail to protect against the uncertainty about the bidders' attitude towards ambiguity.

Example 1.2.2. *Consider an auction with two bidders whose values are governed by a probability distribution from the ambiguity set* $\mathscr{P} = \{\mathbb{P}_1, \mathbb{P}_2\}$ *over* $\mathcal{V}^2 = \{0,4\} \times \{0,4\}$. *The probabilities of the four scenarios under* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *are given in the following table.*

$\boldsymbol{v}$	$(0,0)$	$(4,0)$	$(0,4)$	$(4,4)$
$\mathbb{P}_1$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$
$\mathbb{P}_2$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

We consider an all-pay mechanism where the highest bidder wins, and every bidder pays half of his bid (irrespective of whether the bid was successful or not). Ties are broken lexicographically, i.e., the first bidder wins if there is a tie. One can verify that this mechanism is maxmin incentive compatible over $\mathscr{P}$. *Below we list the expected utilities of bidder* 1 *with true value* 4 *with respect to* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *when he reports the values* 4 *and* 0, *respectively.*

$$\mathbb{E}_{\mathbb{P}_1}\left[u_1(4;\tilde{v}_1,\tilde{v}_2) \mid \tilde{v}_1 = 4\right] \;=\; \mathbb{E}_{\mathbb{P}_2}\left[u_1(4;\tilde{v}_1,\tilde{v}_2) \mid \tilde{v}_1 = 4\right] \;=\; 2$$
$$\mathbb{E}_{\mathbb{P}_1}\left[u_1(0;\tilde{v}_1,\tilde{v}_2) \mid \tilde{v}_1 = 4\right] \;=\; \frac{10}{3}, \quad \mathbb{E}_{\mathbb{P}_2}\left[u_1(0;\tilde{v}_1,\tilde{v}_2) \mid \tilde{v}_1 = 4\right] \;=\; 2$$

Hence, we have

$$\mathbb{E}_{\mathbb{P}_1}\left[u_1(4;\tilde{v}_1,\tilde{v}_2)\mid \tilde{v}_1=4\right] \;=\; 2 \;<\; \frac{10}{3} \;=\; \mathbb{E}_{\mathbb{P}_1}\left[u_1(0;\tilde{v}_1,\tilde{v}_2)\mid \tilde{v}_1=4\right],$$

that is, the all-pay mechanism is not distributionally robust incentive compatible.

For $\alpha = 1/2$, Hurwicz incentive compatibility is violated because

$$\begin{aligned}
&\alpha \inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(4;\tilde{v}_1,\tilde{v}_2)\mid \tilde{v}_1=4\right] + (1-\alpha)\sup_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(4;\tilde{v}_1,\tilde{v}_2)\mid \tilde{v}_1=4\right] \;=\; 2\\
&<\; \frac{8}{3} \;=\; \alpha \inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(0;\tilde{v}_1,\tilde{v}_2)\mid \tilde{v}_1=4\right] + (1-\alpha)\sup_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(0;\tilde{v}_1,\tilde{v}_2)\mid \tilde{v}_1=4\right].
\end{aligned}$$

One similarly verifies that Bayesian incentive compatibility is violated if $\mathbb{Q}(\mathbb{P}_1) = \mathbb{Q}(\mathbb{P}_2) = 1/2$.

In the following sections, we will investigate the *optimal mechanisms*, which maximize the worst-case expected revenues without any restrictions on the allocation rule, and the *best efficient mechanisms*, which maximize only over efficient allocation rules, for different classes of ambiguity sets $\mathscr{P}$. Before that, we review and extend some important results from the literature that will be used throughout the chapter.

1.2.1 The Revenue Equivalence

We first review a cornerstone result from the mechanism design literature stating that the payment rule of an ex-post incentive compatible mechanism is uniquely determined by the allocation rule up to an additive constant for each bidder. In addition, we derive a related result for distributionally robust incentive compatible mechanisms. When applicable, these results will help us to simplify problem ($\mathscr{MDP}$) by substituting out the payment rule. For ease of exposition, we will henceforth assume that $\mathscr{V} = [\underline{v}, \overline{v}]$.

We first define monotonicity of allocation rules.

Definition 1.2.7 (Monotone Allocation Rule)**.** *An allocation rule is called*

(i) ex-post monotone *if it belongs to the set*

$$\begin{aligned}
\mathscr{Q}^{\text{m-p}} \;=\; \{\boldsymbol{q}\in\mathscr{Q} \mid\ & q_i(v_i,\boldsymbol{v}_{-i}) - q_i(w_i,\boldsymbol{v}_{-i}) \geq 0\\
& \forall i\in\mathscr{I}, \forall v_i, w_i \in \mathscr{V} : v_i \geq w_i, \forall \boldsymbol{v}_{-i}\in\mathscr{V}^{I-1}\},
\end{aligned}$$

(ii) distributionally robust monotone *if it belongs to the set*

$$\mathscr{Q}^{\text{m-d}} \;=\; \{\boldsymbol{q}\in\mathscr{Q} \mid \inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}[q_i(v_i,\tilde{\boldsymbol{v}}_{-i}) - q_i(w_i,\tilde{\boldsymbol{v}}_{-i})] \geq 0 \quad \forall i\in\mathscr{I}, \forall v_i, w_i\in\mathscr{V} : v_i \geq w_i\}.$$

Ex-post monotonicity implies that bidder i's probability to win the good is non-decreasing in his value v_i if $\boldsymbol{v}_{-i}$ is kept constant. On the other hand, distributionally robust monotonicity ensures that the expected allocation to bidder i is non-decreasing in v_i under all distributions $\mathbb{P} \in \mathscr{P}$. Note that $\mathscr{Q}^{\text{m-p}} \subseteq \mathscr{Q}^{\text{m-d}}$ by construction.

Some results below will rely on the assumption that the bidders' values are independent.

Definition 1.2.8 (Independence)**.** *We say that the bidders' values are* independent *if the random variables $\tilde{v}_i$, $i \in \mathscr{I}$, are mutually independent under every $\mathbb{P} \in \mathscr{P}$.*

From now on, we use the shorthand $u_i(v_i, \boldsymbol{v}_{-i})$ to denote the ex-post utility $u_i(v_i; v_i, \boldsymbol{v}_{-i})$ under truthful bidding. The next proposition shows that the actual (expected) payment of each bidder under an ex-post (distributionally robust) incentive compatible mechanism is completely determined by the allocation rule and the ex-post (expected) utility of the bidder under his lowest value.

Proposition 1.2.2. *We have the following equivalent characterizations of incentive compatibility.*

(i) A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is ex-post incentive compatible if and only if $\boldsymbol{q} \in \mathscr{Q}^{\text{m-p}}$ and

$$m_i(v_i, \boldsymbol{v}_{-i}) = q_i(v_i, \boldsymbol{v}_{-i}) v_i - u_i(\underline{v}, \boldsymbol{v}_{-i}) - \int_{\underline{v}}^{v_i} q_i(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \quad \forall i \in \mathscr{I}, \forall \boldsymbol{v} \in \mathscr{V}^I. \tag{1.1}$$

(ii) If the bidders' values are independent, then a mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is distributionally robust incentive compatible if and only if $\boldsymbol{q} \in \mathscr{Q}^{\text{m-d}}$ and

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[m_i(v_i, \tilde{\boldsymbol{v}}_{-i})] = \mathbb{E}_{\mathbb{P}}\Big[q_i(v_i, \tilde{\boldsymbol{v}}_{-i}) v_i - u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) - \int_{\underline{v}}^{v_i} q_i(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x \Big] \\ \forall i \in \mathscr{I}, \forall v_i \in \mathscr{V}, \forall \mathbb{P} \in \mathscr{P}. \end{aligned} \tag{1.2}$$

Proof. Assertion (ii) follows directly from Krishna (2009, Section 5.1.2), and thus its proof is omitted. Assertion (i), on the other hand, is an immediate consequence of assertion (ii). This can be seen by defining $\mathscr{P}$ as the set of all distributions supported on $\mathscr{V}^I$ under which the bidders' values are independent. Since $\mathscr{P}$ contains all Dirac distributions supported on $\mathscr{V}^I$, equation (1.2) implies equation (1.1). ■

Proposition 1.2.2 is the main ingredient for the following generalized revenue equivalence theorem, which is an extension of the revenue equivalence theorem by Myerson (1981) and Riley and Samuelson (1981).

Theorem 1.2.2 (The Revenue Equivalence)**.** *If the bidders' values are independent, then all distributionally robust individually rational and incentive compatible mechanisms with the* *same allocation rule* $\boldsymbol{q}$*, for which the ex-post utility of each bidder under his lowest value is* 0*, result in the same worst-case expected revenue for the seller.*

The revenue equivalence theorem naturally extends to all *indirect* mechanisms by virtue of the revelation principle (see Theorem 1.2.1). Note that the assertion (ii) of Proposition 1.2.2 ceases to hold if the bidders' values are dependent, even if the ambiguity set is a singleton, which implies that the revenue equivalence breaks down (Milgrom and Weber, 1982).

1.2.2 Second Price Auctions with Reserve Prices

The most widely used incentive compatible mechanisms are the second price auctions with reserve prices. From now on, we let $\mathcal{W}^i$, $i \in \mathcal{I}$, be any partition of $\mathcal{V}^I$ (*i.e.*, $\cup_{i\in\mathcal{I}}\mathcal{W}^i = \mathcal{V}^I$ and $\mathcal{W}^i \cap \mathcal{W}^{i'} = \emptyset$ for all $i, i' \in \mathcal{I}$ with $i \neq i'$) such that $\mathcal{W}^i$ contains only scenarios $\boldsymbol{v}$ in which i is among the highest bidders. Note that these requirements almost uniquely determine $\mathcal{W}^i$. In scenarios with multiple highest bidders, however, an arbitrary tie-breaking rule must be used to ensure that each $\boldsymbol{v} \in \mathcal{V}^I$ is assigned to exactly one $\mathcal{W}^j$ (*e.g.*, the lexicographic tie-breaker assigns $\boldsymbol{v}$ to $\mathcal{W}^i$ if and only if $i = \min \arg\max_{j\in\mathcal{I}} v_j$).

Definition 1.2.9 (Second Price Auction with Reserve Price)**.** *A mechanism* $(\boldsymbol{q}^{sp}, \boldsymbol{m}^{sp})$ *is called a* second price auction with a reserve price r *if* $\forall i \in \mathcal{I}$, $\forall \boldsymbol{v} \in \mathcal{V}^I$,

$$q_i^{sp}(v_i, \boldsymbol{v}_{-i}) = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathcal{W}^i \text{ and } v_i \geq r, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$m_i^{sp}(v_i, \boldsymbol{v}_{-i}) = \begin{cases} \max\left\{\max\limits_{j\neq i} v_j,\ r\right\} & \text{if } q_i^{sp}(v_i, \boldsymbol{v}_{-i}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The allocation rule $\boldsymbol{q}^{\text{sp}}$ depends on the tie-breaker that was used in the definition of the sets $\mathcal{W}^i$, $i \in \mathcal{I}$. Our subsequent results will not depend on the particular choice of the tie-breaker. Intuitively, in a second price auction with a reserve price, the good is allocated to the highest bidder provided that his value exceeds the reserve price r, and the winner pays an amount equal to the maximum of the second highest bid and r. Second price auctions with reserve prices are known to be incentive compatible and individually rational at the ex-post stage (Krishna, 2009, Section 2.2). Since ex-post individual rationality and incentive compatibility imply distributionally robust individual rationality and incentive compatibility (see Proposition 1.2.1 and the subsequent discussion), second price auctions with reserve prices are feasible in ($\mathcal{MDP}$).

A second price auction with a non-zero reserve price is not necessarily efficient. Indeed, the seller may keep the object for herself if the highest bid falls short of the reserve price. The Vickrey mechanism is an instance of an *efficient* second price auction.

Definition 1.2.10 (Vickrey Mechanism)**.** *The* Vickrey mechanism *is the second price auction* $(\boldsymbol{q}^v, \boldsymbol{m}^v)$ *with reserve price* $r = 0$.

Under the Vickrey mechanism, the highest bidder always receives the good (using any tie-breaking rule). As the Vickrey mechanism is a special instance of a second price auction with reserve price, it is ex-post individually rational and incentive compatible.

Corollary 1.2.1. *The Vickrey mechanism is ex-post individually rational and incentive compatible.*

Proof. The claim follows from Section 2.2 in Krishna (2009). ■

1.3 Robust Mechanism Design

Assume that the seller and the bidders only know the support $\mathcal{V}^I$ of the values that are possible but have no information about their probabilities. In this case, the ambiguity set reduces to $\mathcal{P} = \mathcal{P}_0(\mathcal{V}^I)$.

Proposition 1.3.1. *If* $\mathcal{P} = \mathcal{P}_0(\mathcal{V}^I)$*, then the optimal mechanism design problem* $(\mathcal{MDP})$ *reduces to the robust optimization problem*

$$\begin{array}{lll} \sup\limits_{\boldsymbol{q}\in\mathcal{Q},\boldsymbol{m}\in\mathcal{M}} & \inf\limits_{\boldsymbol{v}\in\mathcal{V}^I} \sum\limits_{i\in\mathcal{I}} m_i(v_i, \boldsymbol{v}_{-i}) & \\ \text{s.t.} & u_i(v_i; v_i, \boldsymbol{v}_{-i}) - u_i(w_i; v_i, \boldsymbol{v}_{-i}) \geq 0 & \forall i \in \mathcal{I},\ \forall \boldsymbol{v} \in \mathcal{V}^I,\ \forall w_i \in \mathcal{V} \\ & u_i(v_i; v_i, \boldsymbol{v}_{-i}) \geq 0 & \forall i \in \mathcal{I},\ \forall \boldsymbol{v} \in \mathcal{V}^I. \end{array} \tag{$\mathcal{RMDP}$}$$

The proof of Proposition 1.3.1 is elementary and therefore omitted. Note that under a support-only ambiguity set, distributionally robust individual rationality and incentive compatibility reduce to ex-post individual rationality and incentive compatibility, respectively.

Bandi and Bertsimas (2014, Section 3) show that the Vickrey mechanism solves $(\mathcal{RMDP})$. Thus, $(\mathcal{RMDP})$ admits an optimal mechanism that is efficient even though efficiency was not imposed. This is unusual because requiring efficiency generically reduces revenues (see, *e.g.*, Krishna 2009, Section 2.5). We formalize this result in the following theorem.

Theorem 1.3.1 (Bandi and Bertsimas (2014))**.** *The Vickrey mechanism is optimal in* $(\mathcal{RMDP})$.

Bandi and Bertsimas (2014) show that Theorem 1.3.1 generalizes to any non-rectangular bounded support sets. However, the rectangularity assumption is essential for Propositions

1.3.2 and 1.3.3 below, which establish that the Vickrey mechanism is not only efficient but also happens to display two other useful properties that were not imposed in ($\mathscr{RMDP}$). First, we demonstrate that the Vickrey mechanism is Pareto robustly optimal with respect to the theory of Pareto optimality in robust optimization due to Iancu and Trichakis (2013).

Definition 1.3.1 (Pareto Robust Optimality)**.** *An ex-post individually rational and incentive compatible mechanism* $(\boldsymbol{q}, \boldsymbol{m})$ *is called* Pareto robustly optimal *if there exists no ex-post individually rational and incentive compatible mechanism* $(\boldsymbol{q}', \boldsymbol{m}')$ *such that*

$$\sum_{i \in \mathscr{I}} m_i'(\boldsymbol{v}) \geq \sum_{i \in \mathscr{I}} m_i(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{V}^I$$

and the above inequality is strict for at least one $\boldsymbol{v} \in \mathcal{V}^I$*.*

Proposition 1.3.2. *The Vickrey mechanism is Pareto robustly optimal.*

The idea of the proof can be summarized as follows. If the Vickrey mechanism was not Pareto robustly optimal, then there would exist a mechanism $(\boldsymbol{q}, \boldsymbol{m})$ that generates higher revenues to the seller under some scenario $\boldsymbol{v}'$ while generating at least the same revenues under every other scenario. Note that the revenues generated by the Vickrey mechanism equal the second highest value under every scenario. At the same time, $(\boldsymbol{q}, \boldsymbol{m})$ cannot charge any bidder more than his value due to individual rationality. Thus, in order to generate higher revenues than the Vickrey auction in scenario $\boldsymbol{v}'$, the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ must allocate the item with strictly positive probability to the highest bidder i^1 and charge him more than the probability of him winning times the second highest value.

Given that scenario $\boldsymbol{v}'$ exists, we can construct another scenario $\boldsymbol{w}$ in which we reduce the value of i^1 (while ensuring that his bid remains the highest one) but keep all other values unchanged. In scenario $\boldsymbol{w}$, the probability of i^1 winning cannot have increased due to the assumed incentive compatibility of $(\boldsymbol{q}, \boldsymbol{m})$, see Proposition 1.2.2$(i)$. Due to individual rationality, on the other hand, i^1's payment has to decrease in scenario $\boldsymbol{w}$. Thus, to match the revenues generated by the Vickrey auction, the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ has to assign the good to the second highest bidder i^2 with a strictly positive probability.

Finally, we construct a third scenario $\boldsymbol{w}'$ in which we increase the value of i^2 to the value of i^1. In scenario $\boldsymbol{w}'$, the probability of i^2 winning cannot have decreased due to Proposition 1.2.2(i). To match the revenues of the Vickrey auction, the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ has to charge bidder i^2 his probability of winning the good times his value. Thus, the utility of bidder i^2 is 0 in scenario $\boldsymbol{w}'$. Since i^2 would have also won the good with a positive probability when reporting w_{i^2} (while being charged less due to individual rationality), however, he receives a strictly positive utility when he misreports his value. Since this contradicts incentive compatibility,

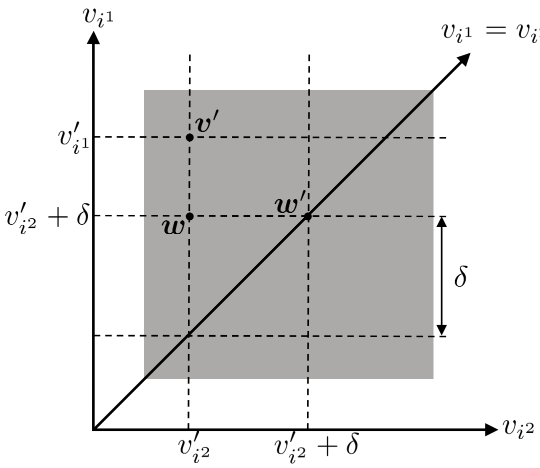


Figure 1.1 – Visualization of $\boldsymbol{v}'$, $\boldsymbol{w}$ and $\boldsymbol{w}'$. The gray rectangle represents $\mathcal{V}^2$ whereas v'_{i^1} and v'_{i^2} are the values of the highest bidder i^1 and the second highest bidder i^2 in scenario $\boldsymbol{v}'$, respectively.

we conclude that no such mechanism $(\boldsymbol{q}, \boldsymbol{m})$ can exist. Figure 1 visualizes the scenarios $\boldsymbol{v}'$, $\boldsymbol{w}$ and $\boldsymbol{w}'$.

Pareto optimality is important in classical robust optimization because there are typically multiple optimal solutions. To see this, consider the mechanism that allocates the good to the first bidder with probability 1 and that charges this bidder $\underline{v}$. One can show that this mechanism is ex-post individually rational and incentive compatible. Moreover, it generates the same worst-case revenue $\underline{v}$ for the seller as the Vickrey mechanism and is thus optimal in ($\mathcal{RMDP}$). However, the Vickrey mechanism generates weakly higher revenues in *every* fixed scenario $\boldsymbol{v} \in \mathcal{V}^I$, and strictly higher revenues in every scenario in which the second highest bid exceeds $\underline{v}$.

The Vickrey mechanism also displays a powerful Pareto dominance property among all *efficient* ex-post individually rational and incentive compatible mechanisms.

Proposition 1.3.3. *Among all efficient ex-post individually rational and incentive compatible mechanisms, the Vickrey mechanism generates the highest revenues in every fixed scenario* $\boldsymbol{v} \in \mathcal{V}^I$.

Proposition 1.3.3 provides a stronger result than Pareto robust optimality. It shows that among all efficient solutions to ($\mathcal{RMDP}$) the Vickrey mechanism offers the highest revenues in *every* fixed scenario. One can show that the Pareto dominance property of Proposition 1.3.3 ceases to hold if we compare the Vickrey mechanism against every (not necessarily efficient) ex-post individually rational and incentive compatible mechanism.

Theorem 1.3.1, Proposition 1.3.2 and Proposition 1.3.3 imply the following corollary.

Corollary 1.3.1. *The Vickrey mechanism is the unique mechanism that is optimal, efficient and Pareto robustly optimal.*

1.4 Mechanism Design under Independent Values

Throughout this section, we assume that the bidders' values are independent in the sense of Definition 1.2.8. Some results of this section will further require that each $\mathbb{P} \in \mathscr{P}$ is symmetric and regular.

Definition 1.4.1 (Symmetry)**.** *A distribution $\mathbb{P} \in \mathscr{P}_0(\mathscr{V}^I)$ is called* symmetric *if the random variables $\tilde{v}_i$, $i \in \mathscr{I}$, share the same marginal distribution under $\mathbb{P}$.*

Definition 1.4.2 (Regularity)**.** *A distribution $\mathbb{P} \in \mathscr{P}_0(\mathscr{V}^I)$ is called* regular *if the marginal density $\rho_i^{\mathbb{P}}(v_i)$ of $\tilde{v}_i$ under $\mathbb{P}$ exists and is strictly positive for all $v_i \in \mathscr{V}$, while the* virtual valuation

$$\psi_i^{\mathbb{P}}(v_i) = v_i - \frac{1 - \int_{\underline{v}}^{v_i} \rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}{\rho_i^{\mathbb{P}}(v_i)}$$

is non-decreasing in v_i for all $i \in \mathscr{I}$.

Independence, symmetry and regularity are standard assumptions of the benchmark model for auctions as defined by McAfee and McMillan (1987). The virtual valuations can be interpreted as marginal revenues (see Krishna, 2009, Section 5.2.3). They were first introduced by Myerson (1981) in order to solve the optimal mechanism design problem in the absence of ambiguity. Note that a sufficient condition for regularity is that the hazard function $\frac{\rho_i^{\mathbb{P}}(v_i)}{1-\int_{\underline{v}}^{v_i} \rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}$ is non-decreasing in v_i.

Proposition 1.4.1. *If the bidders' values are independent, then the optimal mechanism design problem $(\mathscr{MDP})$ reduces to*

$$\begin{array}{lll} \sup\limits_{\boldsymbol{q} \in \mathscr{Q}^{\text{m-d}}, \boldsymbol{m} \in \mathscr{M}} & \inf\limits_{\mathbb{P} \in \mathscr{P}} \sum\limits_{i \in \mathscr{I}} \mathbb{E}_{\mathbb{P}}\Big[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \Big] & \\ \text{s.t.} & \mathbb{E}_{\mathbb{P}}\left[u_i(v_i, \tilde{\boldsymbol{v}}_{-i})\right] & (\mathscr{IMDP}) \\ & \quad = \mathbb{E}_{\mathbb{P}}\Big[u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \Big] \quad \forall i \in \mathscr{I}, \forall v_i \in \mathscr{V}, \forall \mathbb{P} \in \mathscr{P} & \\ & \mathbb{E}_{\mathbb{P}}\left[u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\right] \ \geq\ 0 \quad \forall i \in \mathscr{I}, \forall \mathbb{P} \in \mathscr{P}. & \end{array}$$

Proof. By Proposition 1.2.2(ii), distributionally robust incentive compatibility is equivalent to the first constraint of $(\mathscr{IMDP})$ and the requirement that $\boldsymbol{q} \in \mathscr{Q}^{\text{m-d}}$. The first constraint in

($\mathscr{IMDP}$) then implies that distributionally robust individual rationality simplifies to

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\left[u_i(v_i,\tilde{\boldsymbol{v}}_{-i})\right] \;=\; \mathbb{E}_{\mathbb{P}}\left[u_i(\underline{v},\tilde{\boldsymbol{v}}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x,\tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\right] \;\geq\; 0 \quad \forall i\in\mathscr{I}, \forall v_i\in\mathscr{V}, \forall\mathbb{P}\in\mathscr{P} \\ \iff\;& \mathbb{E}_{\mathbb{P}}\left[u_i(\underline{v},\tilde{\boldsymbol{v}}_{-i})\right] \;\geq\; 0 \qquad \forall i\in\mathscr{I}, \forall\mathbb{P}\in\mathscr{P}, \end{aligned}$$

where the equivalence holds because the integral in the first line is always non-negative.

To see that the objective function of ($\mathscr{MDP}$) reduces to the objective function of ($\mathscr{IMDP}$), we proceed as in the proof of Theorem 1.2.2. Details are omitted for brevity. ■

We now show that the Vickrey mechanism is the best *efficient* mechanism in ($\mathscr{IMDP}$).

Theorem 1.4.1. *The Vickrey mechanism generates the highest worst-case expected revenue in* ($\mathscr{IMDP}$) *among all efficient mechanisms.*

Proof. By Corollary 1.2.1, the Vickrey mechanism is ex-post individually rational and incentive compatible. Hence, by Proposition 1.2.1, it is also distributionally robust individually rational and incentive compatible. We conclude that the Vickrey mechanism is feasible in ($\mathscr{MDP}$) and thus, by Proposition 1.4.1, in ($\mathscr{IMDP}$).

We now show that the ex-post utility of each bidder i with value $\underline{v}$ vanishes under the Vickrey mechanism. If bidder i with value $\underline{v}$ does not win the good, then he does not have to make a payment, and his ex-post utility is zero. Otherwise, if he wins the good, then we have

$$u_i(\underline{v},\boldsymbol{v}_{-i}) \;=\; q_i(\underline{v},\boldsymbol{v}_{-i})\underline{v} - m_i(\underline{v},\boldsymbol{v}_{-i}) \;=\; \underline{v} - \max_{j\neq i} v_j \;=\; \underline{v}-\underline{v} \;=\; 0,$$

where the third equality holds because $\underline{v} \leq \max_{j\neq i} v_j \leq v_i = \underline{v}$. Thus, the ex-post utility of each bidder i with value $\underline{v}$ is always zero.

Next, we show that the Vickrey mechanism generates a weakly higher worst-case expected revenue than any other efficient mechanism $(\boldsymbol{q}',\boldsymbol{m}') \in \mathscr{Q}^{\mathrm{eff}}\times\mathscr{M}$ that is feasible in ($\mathscr{IMDP}$). Indeed, as the ex-post utility of each bidder i with value $\underline{v}$ vanishes under the Vickrey mechanism, the objective value of the Vickrey mechanism in ($\mathscr{IMDP}$) satisfies

$$\begin{aligned} &\inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathscr{I}} q_i(\tilde{v}_i,\tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i(x,\tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\Big] \;=\; \inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathscr{I}} q_i'(\tilde{v}_i,\tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i'(x,\tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\Big] \\ \geq\; &\inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathscr{I}} q_i'(\tilde{v}_i,\tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i'(x,\tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\Big] - \mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathscr{I}} q_i'(\underline{v},\tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i'(\underline{v},\tilde{\boldsymbol{v}}_{-i})\Big]. \end{aligned}$$

Here, the first equality follows from efficiency, which implies that $\sum_{i\in\mathscr{I}} q_i(\boldsymbol{v}) = \sum_{i\in\mathscr{I}} q_i'(\boldsymbol{v}) = 1$ and that $\sum_{i\in\mathscr{I}} q_i(\boldsymbol{v})v_i = \sum_{i\in\mathscr{I}} q_i'(\boldsymbol{v})v_i = \max_{i\in\mathscr{I}} v_i$, while the inequality is due to the second constraint of ($\mathscr{IMDP}$). The claim then follows because the last line of the above expression represents the objective function value of $(\boldsymbol{q}', \boldsymbol{m}')$ in $(\mathscr{IMDP})$. ■

In order to examine the properties of the optimal (not necessarily efficient) mechanism, we reformulate problem $(\mathscr{IMDP})$ in terms of the virtual valuations introduced in Definition 1.4.2. The following proposition extends Lemma 3 by Myerson (1981) to ambiguous value distributions. Its proof is relegated to the appendix.

Proposition 1.4.2. *If the bidders' values are independent and each $\mathbb{P} \in \mathscr{P}$ is regular, then problem $(\mathscr{IMDP})$ is equivalent to*

$$\sup_{\boldsymbol{q} \in \mathscr{Q}^{\text{m-d}}} \inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}} \Big[\sum_{i \in \mathscr{I}} \psi_i^{\mathbb{P}}(\tilde{v}_i) q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i}) \Big]. \tag{1.3}$$

We now review a celebrated result by Myerson (1981), which asserts that in the absence of ambiguity ($\mathscr{P} = \{\mathbb{P}\}$), a second price auction with a reserve price is optimal if the bidders' values are independent and the distribution $\mathbb{P}$ is symmetric and regular.

Theorem 1.4.2 (Myerson (1981))**.** *If $\mathscr{P} = \{\mathbb{P}\}$, the bidders' values are independent, and the distribution $\mathbb{P}$ is symmetric and regular, then the allocation rule*

$$q_i^\star(\boldsymbol{v}) = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathscr{W}^i \text{ and } \psi_1^{\mathbb{P}}(v_i) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in \mathscr{I}$ and $\boldsymbol{v} \in \mathscr{V}^I$, is optimal in (1.3) *and generates expected revenues of*

$$\mathbb{E}_{\mathbb{P}} \Big[\max \Big\{ \max_{i \in \mathscr{I}} \psi_1^{\mathbb{P}}(\tilde{v}_i), 0 \Big\} \Big]. \tag{1.4}$$

The allocation rule $\boldsymbol{q}^\star$ can be used to construct a payment rule $\boldsymbol{m}^\star$ defined through

$$m_i^\star(v_i, \boldsymbol{v}_{-i}) = q_i^\star(v_i, \boldsymbol{v}_{-i}) v_i - \int_{\underline{v}}^{v_i} q_i^\star(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \quad \forall i \in \mathscr{I}, \forall \boldsymbol{v} \in \mathscr{V}^I.$$

One can show that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is optimal in $(\mathscr{IMDP})$ when $\mathscr{P}$ is a singleton.

Note that if the virtual valuation $\psi_1^{\mathbb{P}}$ is continuous, then the optimal mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is the second price auction with reserve price $r = \inf\{v_1 \in \mathscr{V} : \psi_1^{\mathbb{P}}(v_1) \geq 0\}$. Note also that this mechanism can be inefficient because $\psi_1^{\mathbb{P}}(v_1)$ can be negative.

Koçyiğit et al. (2018a) show that second price auctions with reserve prices are generally suboptimal as soon as $\mathscr{P}$ contains two distributions, even if the bidders' values are independent and each $\mathbb{P} \in \mathscr{P}$ is symmetric and regular. Unfortunately, we are unable to solve problem (1.3) analytically unless $\mathscr{P}$ is a singleton. Even though the optimal mechanism remains elusive, Dhangwatnotai et al. (2015) have identified a mechanism, called the modified Single Sample mechanism, which is guaranteed to generate at least half of the expected revenue of the optimal mechanism under every distribution $\mathbb{P} \in \mathscr{P}$. This mechanism can be viewed as a second price auction with a random reserve price, and the corresponding constant-factor approximation guarantee critically relies on the independence of the bidder values. As we will argue below, a simple second price auction without reserve price also offers compelling optimality guarantees, which suggest that the added value of the unknown optimal mechanism is negligible.

In the absence of ambiguity, Bulow and Klemperer (1996) demonstrate that the second price auction without reserve price for $I+1$ bidders yields higher expected revenues than the optimal auction for I bidders. In the non-ambiguous case, the optimal mechanism for I bidders is known to be a second price auction with a reserve price (see Theorem 1.4.2). The following theorem generalizes the result by Bulow and Klemperer (1996) to mechanism design problems under ambiguity even though the optimal mechanism remains unknown in this setting.

Theorem 1.4.3. *Assume that the bidders' values are independent and that each distribution in the ambiguity set is symmetric and regular. Then, a second price auction without reserve price for $I+1$ bidders yields a weakly higher worst-case expected revenue than an optimal auction for I bidders.*

Theorem 1.4.3 shows that the added value of the optimal mechanism over a simple second price auction without reserve price is offset by just attracting one additional bidder. Theorem 1.4.3 critically relies on the independence of the bidders' values, which facilitates the reformulation (1.3). In the next section, we will investigate the mechanism design problem under moment ambiguity sets where the bidders' values may be correlated. In this case, the added value of the optimal mechanism over even the best second price auction can be significant.

1.5 Mechanism Design under Moment Information

While commonly employed in the mechanism design literature, the assumption of independent bidder values can be restrictive in practice, where bidders may interact with one another or share common information sources. This motivates us to investigate settings where the bidders' values can be dependent. Specifically, we assume that the agents have information about some (generalized) moments of the value distribution. We thus consider moment ambiguity sets of the form

$$\mathscr{P} = \left\{\mathbb{P} \in \mathscr{P}_0(\mathbb{R}^I_+) : \mathbb{P}(\tilde{\boldsymbol{v}} \in \mathcal{V}^I) = 1, \mathbb{E}_{\mathbb{P}}\left[\boldsymbol{h}(\tilde{\boldsymbol{v}})\right] \geq \boldsymbol{\mu}\right\}, \tag{1.5}$$

where $\boldsymbol{h} = (h_1, \ldots, h_J)$ represents a vector of generalized moment functions $h_j : \mathcal{V}^I \to \mathbb{R}$, and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_J)$ denotes a vector of given moment bounds $\mu_j \in \mathbb{R}$. The following non-restrictive technical condition will be assumed to hold throughout this section.

Assumption 1.5.1 (Slater Condition). *There exists a Slater point* $\mathbb{P}_s \in \mathscr{P}$ *with* $\mathbb{E}_{\mathbb{P}_s}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] > \boldsymbol{\mu}$.

The following proposition shows that if $\mathscr{P}$ is of the form (1.5), the bidders will require ex-post individual rationality and incentive compatibility.

Proposition 1.5.1. *If* $\mathscr{P}$ *is a moment ambiguity set of the form* (1.5) *and Assumption 1.5.1 holds, then distributionally robust individual rationality and incentive compatibility simplify to ex-post individual rationality and incentive compatibility, respectively.*

Proof. Select an arbitrary bidder $i \in \mathscr{I}$ with value $v_i \in \mathcal{V}$, and note that the inequality

$$\inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}}[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i] \geq \inf_{\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}} u_i(v_i, \boldsymbol{v}_{-i})$$

is trivially satisfied. To establish the converse inequality, we use Assumption 1.5.1, whereby there exists $\mathbb{P}_{\mathrm{s}} \in \mathscr{P}$ with $\mathbb{E}_{\mathbb{P}_{\mathrm{s}}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] > \boldsymbol{\mu}$. By the Richter-Rogosinski theorem, we can assume without loss of generality that $\mathbb{P}_{\mathrm{s}}$ is discrete and representable as

$$\mathbb{P}_s = \sum_{j=1}^{J+1} p_j \delta_{\boldsymbol{v}^{(j)}} \text{ with } \sum_{j=1}^{J+1} p_j = 1,\ p_j \geq 0 \text{ and } \boldsymbol{v}^{(j)} \in \mathcal{V}^I \quad \forall j = 1, \ldots, J+1,$$

where $\delta_{\boldsymbol{v}^{(j)}}$ denotes the Dirac point mass at $\boldsymbol{v}^{(j)}$, see Theorem 7.23 in Shapiro et al. (2014). Moreover, by a standard perturbation argument, we can assume without loss of generality that $v_i^{(j)} \neq v_i$ for all $j = 1, \ldots, J+1$.

Select now an arbitrary $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$ and set $\boldsymbol{v} = (v_i, \boldsymbol{v}_{-i})$ as usual. As $\mathbb{E}_{\mathbb{P}_{\mathrm{s}}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] > \boldsymbol{\mu}$, there exists $\lambda \in (0,1)$ small enough such that the distribution $\mathbb{P}_{\boldsymbol{v}_{-i}} = \lambda \delta_{\boldsymbol{v}} + (1-\lambda)\mathbb{P}_{\mathrm{s}}$ satisfies

$$\mathbb{E}_{\mathbb{P}_{\boldsymbol{v}_{-i}}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] = \lambda \boldsymbol{h}(\boldsymbol{v}) + (1-\lambda)\mathbb{E}_{\mathbb{P}_{\mathrm{s}}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] \geq \boldsymbol{\mu}.$$

Hence, $\mathbb{P}_{\boldsymbol{v}_{-i}} \in \mathscr{P}$. This implies that

$$\inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}}[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i] \leq \mathbb{E}_{\mathbb{P}_{\boldsymbol{v}_{-i}}}[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i] = u_i(v_i, \boldsymbol{v}_{-i}),$$

where the inequality holds because $\mathbb{P}_{\boldsymbol{v}_{-i}} \in \mathscr{P}$ and the equality holds due to construction of $\mathbb{P}_{\boldsymbol{v}_{-i}}$. Since $\boldsymbol{v}_{-i}$ was chosen arbitrarily, the above inequality holds for all $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, that is, $\inf_{\mathbb{P} \in \mathscr{P}} \mathbb{E}_{\mathbb{P}}[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i] \leq \inf_{\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}} u_i(v_i, \boldsymbol{v}_{-i})$. Thus, distributionally robust individual rationality simplifies to ex-post individual rationality.

Using similar arguments, one can also prove the assertion about incentive compatibility. Details are omitted for brevity. ■

We now use the above proposition to simplify problem ($\mathscr{MDP}$).

Proposition 1.5.2. *If $\mathscr{P}$ is a moment ambiguity set of the form* (1.5) *and Assumption 1.5.1 holds, then the optimal mechanism design problem* ($\mathscr{MDP}$) *reduces to*

$$\begin{array}{lll}
\sup\limits_{\boldsymbol{q}\in\mathscr{Q}^{\text{m-p}},\boldsymbol{m}\in\mathscr{M}} & \inf\limits_{\mathbb{P}\in\mathscr{P}} \sum\limits_{i\in\mathscr{I}} \mathbb{E}_{\mathbb{P}}\left[q_i(\tilde{v}_i,\tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - u_i(\underline{v},\tilde{\boldsymbol{v}}_{-i}) - \int_{\underline{v}}^{\tilde{v}_i} q_i(x,\tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\right] & \\
\text{s.t.} & u_i(v_i,\boldsymbol{v}_{-i}) = u_i(\underline{v},\boldsymbol{v}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x,\boldsymbol{v}_{-i})\,\mathrm{d}x \qquad \forall i\in\mathscr{I}, \forall \boldsymbol{v}\in\mathscr{V}^I & (\mathscr{MMDP})\\
& u_i(\underline{v},\boldsymbol{v}_{-i}) \geq 0 \qquad \forall i\in\mathscr{I}, \forall \boldsymbol{v}_{-i}\in\mathscr{V}^{I-1}. &
\end{array}$$

Proof. By Proposition 1.5.1, distributionally robust individual rationality and incentive compatibility simplify to ex-post individual rationality and incentive compatibility, respectively. By Proposition 1.2.2 (i), ex-post incentive compatibility is equivalent to the first constraint of ($\mathscr{MMDP}$) and the requirement that $\boldsymbol{q}\in\mathscr{Q}^{\text{m-p}}$. The reformulation of the objective function immediately follows from the first constraint in ($\mathscr{MMDP}$). This constraint also implies that ex-post individual rationality simplifies to

$$\begin{aligned}
& u_i(v_i,\boldsymbol{v}_{-i}) = u_i(\underline{v},\boldsymbol{v}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x,\boldsymbol{v}_{-i}) \geq 0 \quad && \forall i\in\mathscr{I}, \forall \boldsymbol{v}\in\mathscr{V}^I \\
\iff\ & u_i(\underline{v},\boldsymbol{v}_{-i}) \geq 0 && \forall i\in\mathscr{I}, \forall \boldsymbol{v}_{-i}\in\mathscr{V}^{I-1},
\end{aligned}$$

where the equivalence holds because the integral in the first line is always non-negative. ■

Theorem 1.5.1. *The Vickrey mechanism generates the highest worst-case expected revenue in* ($\mathscr{MMDP}$) *among all efficient mechanisms.*

Proof. The proof is immediate from Proposition 1.3.3 and Proposition 1.5.1. ■

1.5.1 Markov Ambiguity Sets

If the efficiency condition is relaxed, then the Vickrey mechanism is suboptimal for generic moment ambiguity sets. To show this, we will henceforth focus on Markov ambiguity sets with first-order moment information of the form

$$\mathscr{P} = \left\{\mathbb{P}\in\mathscr{P}_0(\mathbb{R}_+^I) : \mathbb{P}(\tilde{\boldsymbol{v}}\in[0,1]^I) = 1, \mathbb{E}_{\mathbb{P}}\left[\tilde{v}_i\right] \geq \mu, \forall i\in\mathscr{I}\right\}, \tag{1.6}$$

where $\mu \in [0,1]$. The Markov ambiguity set stipulates that the bidder values range over the unit interval $[0,1]$ and are not smaller than μ in expectation. Markov ambiguity sets are intuitively appealing as they only require the specification of the smallest, the highest and the most likely bidder value for the good. As the seller's revenue is non-decreasing in the bidder values, we could actually require $\mathbb{E}_{\mathbb{P}}[\tilde{v}_i] = \mu$, $i \in \mathcal{I}$, without affecting the objective function of problem ($\mathcal{MMDP}$). We prefer to work with inequality constraints, however, to ensure that $\mathcal{P}$ admits a Slater point. Note also that, although the description of the Markov ambiguity set (1.6) is permutation symmetric, it contains distributions that are not symmetric.

It is instructive to investigate what would happen if the seller knew the bidder values from the outset. In this case, the seller's optimal strategy would be to give the good to the highest bidder and to charge him an amount equal to his value. In this manner, the seller could both maximize and appropriate the total social welfare. In other words, the seller could extract full surplus.

Definition 1.5.1 (Worst-Case Expected Full Surplus)**.** *The* worst-case expected full surplus *corresponding to an ambiguity set* $\mathcal{P}$ *is* $\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\max_{i\in\mathcal{I}} \tilde{v}_i]$.

The worst-case expected full surplus clearly provides an upper bound on the worst-case expected revenue the seller can obtain by implementing any ex-post individually rational and incentive compatible mechanism. This is because, by ex-post individual rationality, the seller cannot charge the winner more than his value.

Proposition 1.5.3. *If* $\mathcal{P}$ *is a Markov ambiguity set of the form* (1.6)*, then the worst-case expected full surplus is equal to* μ*.*

Proof. Let $\delta_{\mu\mathbf{e}}$ be the Dirac point mass at $\mu\mathbf{e}$. Since $\delta_{\mu\mathbf{e}} \in \mathcal{P}$, we have

$$\mu \;=\; \mathbb{E}_{\delta_{\mu\mathbf{e}}}[\max_{i\in\mathcal{I}} \tilde{v}_i] \;\geq\; \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\max_{i\in\mathcal{I}} \tilde{v}_i] \;\geq\; \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\tilde{v}_1] \;=\; \mu,$$

and thus the claim follows. ■

If $\mu = 1$, then the Markov ambiguity set is a singleton that contains only the Dirac distribution at $\boldsymbol{v} = \mathbf{e}$. In this case, the seller can implement a second price auction without reserve price to extract the worst-case full surplus μ. On the other hand, if $\mu = 0$, then the Dirac distribution at $\boldsymbol{v} = \mathbf{0}$ is contained in the Markov ambiguity set. In this case, the worst-case expected revenue is 0 independent of the mechanism implemented. To exclude these trivial special cases, we will henceforth assume that $\mu \in (0,1)$.

We now offer two equivalent reformulations of the worst-case expected revenues in ($\mathcal{MMDP}$) when $\mathcal{P}$ is a Markov ambiguity set.

Proposition 1.5.4. *If $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *and* $\mu \in (0,1)$, *then the objective function value of a fixed allocation rule* $\boldsymbol{q} \in \mathscr{Q}^{\text{m-p}}$ *in* $(\mathscr{M}\mathscr{M}\mathscr{D}\mathscr{P})$ *coincides with the (equal) optimal values of the primal and dual semi-infinite linear programs*

$$\inf_{\mathbb{P}\in\mathscr{P}} \sum_{i\in\mathscr{I}} \int_{[0,1]^I} \left[q_i(v_i, \boldsymbol{v}_{-i})v_i - \int_0^{v_i} q_i(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \right] \mathrm{d}\mathbb{P}(\boldsymbol{v}) \tag{1.7}$$

and

$$\begin{array}{lll} \sup\limits_{\boldsymbol{\sigma}\in\mathbb{R}^I_+,\lambda\in\mathbb{R}} & \lambda + \sum\limits_{i\in\mathscr{I}} \sigma_i \mu & \\ \text{s.t.} & \sum\limits_{i\in\mathscr{I}} \left[q_i(v_i, \boldsymbol{v}_{-i})v_i - \int_0^{v_i} q_i(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \right] \geq \lambda + \sum\limits_{i\in\mathscr{I}} \sigma_i v_i & \forall \boldsymbol{v} \in [0,1]^I. \end{array} \tag{1.8}$$

Proof. Note that $u_i(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathscr{I}$, $\boldsymbol{v}_{-i} \in [0,1]^{I-1}$ because $\underline{v} = 0$. Hence, the objective value of $\boldsymbol{q} \in \mathscr{Q}^{\text{m-p}}$ in $(\mathscr{M}\mathscr{M}\mathscr{D}\mathscr{P})$ is equal to (1.7).

By the definition of the Markov ambiguity set in (1.6), problem (1.7) can be represented as the generalized moment problem

$$\begin{array}{lll} \inf\limits_{\mathbb{P}\in\mathscr{P}_0(\mathbb{R}^I_+)} & \sum\limits_{i\in\mathscr{I}} \int_{[0,1]^I} \left[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_0^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \right] \mathrm{d}\mathbb{P}(\boldsymbol{v}) & \\ \text{s.t.} & \int_{[0,1]^I} \mathrm{d}\mathbb{P}(\boldsymbol{v}) = 1 & \\ & \int_{[0,1]^I} v_i \,\mathrm{d}\mathbb{P}(\boldsymbol{v}) \geq \mu & \forall i \in \mathscr{I}. \end{array}$$

The Lagrangian dual of this moment problem is given by the semi-infinite linear program (1.8). Strong duality holds due to Proposition 3.4 in Shapiro (2001), which is applicable because $\mu \in (0,1)$. Hence, problems (1.7) and (1.8) share the same optimal value. ■

Note that, by Proposition 1.5.4, two equivalent reformulations of problem $(\mathscr{M}\mathscr{M}\mathscr{D}\mathscr{P})$ are obtained by maximizing (1.7) or (1.8) over $\boldsymbol{q} \in \mathscr{Q}^{\text{m-p}}$. Solving either of these problems yields an optimal allocation rule. The corresponding optimal payment rule can then be recovered from the first constraint in $(\mathscr{M}\mathscr{M}\mathscr{D}\mathscr{P})$.

1.5.2 The Optimal Second Price Auction with Reserve Price

Consider problem $(\mathscr{M}\mathscr{M}\mathscr{D}\mathscr{P})$ with a Markov ambiguity set of the form (1.6) with $\mu \in (0,1)$, and assume for now that the seller aims to find the best second price auction $(\boldsymbol{q}^{\text{sp}}, \boldsymbol{m}^{\text{sp}})$ with reserve price $r \in [0,1]$. Recall that all second price auctions with reserve prices are indeed ex-post individually rational and incentive compatible and thus feasible in $(\mathscr{M}\mathscr{M}\mathscr{D}\mathscr{P})$, see

Section 2.2 in Krishna (2009). As $\underline{v} = 0$ for all $i \in \mathscr{I}$, it follows from Proposition 1.2.2(i) that

$$m_i^{\rm sp}(v_i, \boldsymbol{v}_{-i}) \;=\; q_i^{\rm sp}(v_i, \boldsymbol{v}_{-i})v_i - \int_0^{v_i} q_i^{\rm sp}(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \quad \forall i \in \mathscr{I},\ \forall \boldsymbol{v} \in [0,1]^I. \tag{1.9}$$

The correctness of (1.9) can also be checked directly. Imagine that bidder i wins the good in scenario $\boldsymbol{v}$. Thus, the first term on the right-hand side of (1.9) reduces to v_i. As $q_i^{\rm sp}(x, \boldsymbol{v}_{-i}) = 1$ only if $x \geq \max\{\max_{j\neq i} v_j, r\}$ and $q_i(x, \boldsymbol{v}_{-i}) = 0$ whenever $x < \max\{\max_{j\neq i} v_j, r\}$, the integral in (1.9) evaluates to the difference between v_i and $\max\{\max_{j\neq i} v_j, r\}$. As expected, the payment of bidder i is therefore equal to the maximum of the second highest value and the reserve price.

We now calculate the worst-case expected revenue generated by a fixed second price auction $(\boldsymbol{q}^{\rm sp}, \boldsymbol{m}^{\rm sp})$ with reserve price $r \in [0,1]$, which coincides with the (equal) optimal values of the problems (1.7) and (1.8) for $\boldsymbol{q} = \boldsymbol{q}^{\rm sp}$ (see Proposition 1.5.4).

Assume first that $r > \mu$. In this case, the worst-case expected revenue is 0, which is attained by the Dirac distribution at $\boldsymbol{v} = \mu\mathbf{e}$. Therefore, the seller will only consider reserve prices $r \leq \mu$. The subsequent discussion is based on the following partition of the interval $[0,\mu]$ of all reasonable candidate reserve prices.

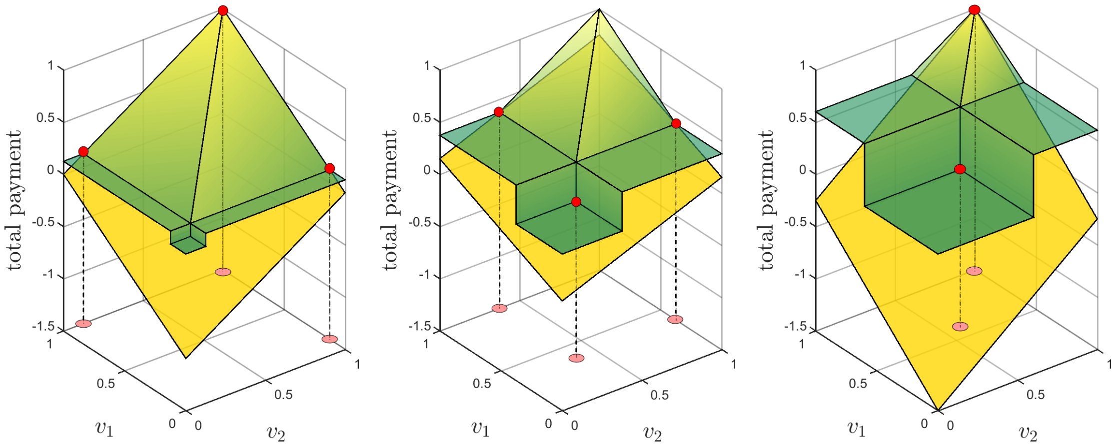
$$\begin{aligned}
\mathscr{R}_1 \;&=\; \left\{ r \in \mathbb{R}_+ : \min\left\{\frac{1}{I}, \frac{I\mu - 1}{I-1}\right\} \;\geq\; r \right\}\\
\mathscr{R}_2 \;&=\; \left\{ r \in \mathbb{R}_+ : \min\left\{\mu, \frac{1}{I}\right\} \;\geq\; r \;>\; \frac{I\mu - 1}{I-1} \right\}\\
\mathscr{R}_3 \;&=\; \left\{ r \in \mathbb{R}_+ : \mu \;\geq\; r \;>\; \frac{1}{I} \right\}
\end{aligned}$$

One can verify that

$$\mu \;\geq\; \frac{I\mu - 1}{I-1} \quad \forall \mu \in (0,1),\ I \in \mathbb{N}, \tag{1.10}$$

which ensures that $\mathscr{R}_1 \subseteq [0,\mu]$ as desired. Later in this section, we will show that $\mathscr{R}_1, \mathscr{R}_2$ and $\mathscr{R}_3$ indeed form a partition of the interval $[0,\mu]$.

We now show that the structure of the worst-case distribution in problem (1.7) depends on whether the reserve price belongs to $\mathscr{R}_1$, $\mathscr{R}_2$ or $\mathscr{R}_3$. To this end, consider the semi-infinite constraint in the dual problem (1.8). By equation (1.9), the left-hand side of this constraint quantifies the total revenue in scenario $\boldsymbol{v}$. On the other hand, the right-hand side represents a linear function of $\boldsymbol{v}$. The objective function of (1.8) tries to push this linear function upwards. At optimality, the linear function touches the total revenue function at a finite number of points in $\mathcal{V}^I$. By complementary slackness, the support of the worst-case distribution that solves (1.7), if it exists, is confined to these discrete points. The following propositions provide explicit formulas for these extremal distributions and the corresponding worst-case expected



(a) $r \in \mathscr{R}_1$ (b) $r \in \mathscr{R}_2$ (c) $r \in \mathscr{R}_3$

Figure 1.2 – Complementary slackness between the distribution $\mathbb{P}$ in (1.7) and the semi-infinite constraint in (1.8) for two bidders and $\mu = \frac{5}{8}$. The green (dark shaded) area and the yellow (light shaded) area represent the left-hand side and right-hand side values of the semi-infinite constraint in (1.8), respectively. The atoms of the distribution $\mathbb{P}$ in (1.7) are visualized by the red dots.

revenues.

Proposition 1.5.5. *If $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *and* $\mu \in (0,1)$, *then the worst-case expected revenue of a second price auction with reserve price $r \in \mathscr{R}_1$ amounts to*

$$\inf_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathscr{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\Big] = \frac{I\mu - 1}{I-1}$$

and is attained by the extremal distribution

$$\mathbb{Q}^{(1)} = \left(1 - \frac{I(\mu-1)}{(I-1)(r-1)}\right)\delta_{\mathbf{e}} + \sum_{i\in\mathscr{I}} \frac{\mu-1}{(I-1)(r-1)}\,\delta_{\mathbf{e}_i + r\mathbf{e}_{-i}}.$$

The atoms of the distribution $\mathbb{Q}^{(1)}$ are visualized by the red dots in Figure 1.2a. Note that the worst-case expected revenue is independent of the reserve price r as long as $r \in \mathscr{R}_1$. This independence emerges because two opposite effects offset each other: When r increases, the probability of the scenario $\boldsymbol{v} = \mathbf{e}$, in which the seller earns the highest payments, decreases so that the expected value of $\tilde{v}_i$ is preserved at μ. At the same time, the payments in all other scenarios increase due to the change in r.

Proposition 1.5.6. *If $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *and* $\mu \in (0,1)$, *then the* *worst-case expected revenue of a second price auction with reserve price* $r \in \mathcal{R}_2$ *is equal to*

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\Big] = Ir\,\frac{\mu-r}{1-r},$$

which is attained asymptotically by the sequence of distributions

$$\mathbb{Q}_\epsilon^{(2)} = \left(1-\frac{I(\mu-(r-\epsilon))}{1-(r-\epsilon)}\right)\delta_{(r-\epsilon)\mathbf{e}} + \sum_{i\in\mathcal{I}}\frac{\mu-(r-\epsilon)}{1-(r-\epsilon)}\,\delta_{\mathbf{e}_i+(r-\epsilon)\mathbf{e}_{-i}}$$

for $\epsilon \downarrow 0$.

The atoms of the distribution $\mathbb{Q}_\epsilon^{(2)}$ (for ϵ close to 0) are visualized by the red dots in Figure 1.2b. Note that the probabilities assigned to the scenarios $\mathbf{e}_i + (r-\epsilon)\mathbf{e}_{-i}$, $i \in \mathcal{I}$, are independent of the number of bidders. These scenarios each contribute an ex-post revenue of r. This explains why the worst-case expected revenue increases linearly in the number of bidders.

Proposition 1.5.7. *If* $\mathcal{P}$ *is a Markov ambiguity set of the form* (1.6) *and* $\mu \in (0,1)$*, then the worst-case expected revenue of a second price auction with reserve price* $r \in \mathcal{R}_3$ *amounts to*

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\Big] = \frac{\mu-r}{1-r},$$

which is attained asymptotically by the sequence of distributions

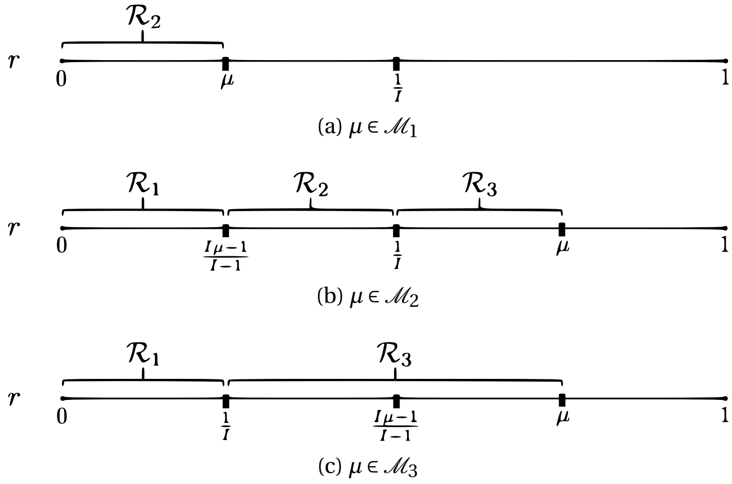
$$\mathbb{Q}_\epsilon^{(3)} = \left(1-\frac{1-\mu}{1-(r-\epsilon)}\right)\delta_{\mathbf{e}} + \frac{1-\mu}{1-(r-\epsilon)}\,\delta_{(r-\epsilon)\mathbf{e}}$$

for $\epsilon \downarrow 0$.

The atoms of the distribution $\mathbb{Q}_\epsilon^{(3)}$ (for ϵ close to 0) are indicated by the red dots in Figure 1.2c. In this case, the worst-case expected revenue does not depend on the number of bidders because $\mathbb{Q}_\epsilon^{(3)}$ is itself independent of the number of bidders.

We are now ready to determine the optimal reserve price as a function of μ and the number of bidders I. Recall from (1.10) that μ is always larger than or equal to $\frac{I\mu-1}{I-1}$. However, $\frac{1}{I}$ can be larger than μ, between $\frac{I\mu-1}{I-1}$ and μ or smaller than $\frac{I\mu-1}{I-1}$. As $\frac{1}{I}$ is greater (smaller) than or equal to $\frac{I\mu-1}{I-1}$ if and only if $\frac{2I-1}{I^2}$ is greater (smaller) than or equal to μ, the interval $(0,1)$ of possible mean values μ can be partitioned into the following disjoint subsets.

$$\begin{aligned}
\mathcal{M}_1 &= \left\{\mu\in\mathbb{R}_+ : 0 < \mu \le \frac{1}{I}\right\}\\
\mathcal{M}_2 &= \left\{\mu\in\mathbb{R}_+ : \frac{1}{I} < \mu \le \frac{2I-1}{I^2}\right\}\\
\mathcal{M}_3 &= \left\{\mu\in\mathbb{R}_+ : \frac{2I-1}{I^2} < \mu < 1\right\}
\end{aligned}$$



(a) $\mu \in \mathcal{M}_1$

(b) $\mu \in \mathcal{M}_2$

(c) $\mu \in \mathcal{M}_3$

Figure 1.3 – Relation between $\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3$ and $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$.

The intervals $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, and their relations to $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$ are visualized in Figure 1.3. If $\mu \in \mathcal{M}_1$, then $\frac{I\mu-1}{I-1}$ is non-positive. Hence, $\mathscr{R}_1$ is empty unless $\mu = \frac{1}{I}$, in which case we have $\mathscr{R}_1 = \{0\}$. Moreover, $\mathscr{R}_2$ is nonempty and $\mathscr{R}_3$ is empty. If $\mu \in \mathcal{M}_2$, then $\frac{I\mu-1}{I-1} \leq \frac{1}{I}$, which implies that $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$ are all nonempty. Finally, if $\mu \in \mathcal{M}_3$, then $\frac{I\mu-1}{I-1} > \frac{1}{I}$, in which case $\mathscr{R}_2$ is empty while $\mathscr{R}_1$ and $\mathscr{R}_3$ are nonempty. In particular, Figure 3 illustrates that $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$ form a partition of $[0, \mu]$ for any value of μ.

Theorem 1.5.2. *Assume that $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *with $\mu \in (0,1)$, and let $r^\star$ and $z^\star$ denote the optimal reserve price and the corresponding worst-case expected revenue, respectively.*

(i) If $\mu \in \mathcal{M}_1 \cup \mathcal{M}_2$, then $r^\star = 1 - \sqrt{1-\mu}$ and $z^\star = I(1-\sqrt{1-\mu})^2$.

(ii) If $\mu \in \mathcal{M}_3$, then any reserve price $r^\star \in \mathscr{R}_1$ is optimal and $z^\star = \frac{I\mu-1}{I-1}$.

Recall from Theorem 1.4.2 that a second price auction with reserve price is optimal in ($\mathcal{MDP}$) if $\mathscr{P} = \{\mathbb{P}\}$ is a singleton, the bidders' values are independent and $\mathbb{P}$ is symmetric and regular. Moreover, in this case, the optimal reserve price is independent of the number of bidders. In contrast, Theorem 1.5.2 asserts that, for a Markov ambiguity set, the optimal reserve price depends on the number of bidders through the sets $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$. Specifically, for $\mu \in \mathcal{M}_1 \cup \mathcal{M}_2$, the optimal reserve price depends on μ but not on the number of bidders I. However, as I increases, $\mathcal{M}_3$ will eventually cover μ, which results in a decrease of the optimal reserve price. In this case, an interval of the reserve prices becomes optimal. We note that while each reserve price in this interval maximizes the worst-case revenues, the choice $r^\star = 0$ is preferable in practice as it additionally ensures efficiency.

We close this section by proving that the second price auction without reserve price is asymptotically optimal in ($\mathcal{MMDP}$) as the number of bidders tends to infinity.

Proposition 1.5.8. *If $\mathcal{P}$ is a Markov ambiguity set of the form* (1.6) *with* $\mu \in (0,1)$, *then for every* $\epsilon > 0$ *there exists* $I_\epsilon \in \mathbb{N}$ *such that the second price auction without reserve price is* ϵ*-suboptimal in* ($\mathcal{MMDP}$) *for all* $I \geq I_\epsilon$.

Proof. Fix any $\epsilon > 0$ and select $I_\epsilon \in \mathbb{N}$ such that $\frac{I_\epsilon \mu - 1}{I_\epsilon - 1} \geq \max\{\mu - \epsilon, 0\}$. Thus, $0 \in \mathcal{R}_1$ whenever $I \geq I_\epsilon$. This implies via Proposition 1.5.5 that the objective value of the second price auction without reserve price in ($\mathcal{MMDP}$) is at least $\mu - \epsilon$ for all $I \geq I_\epsilon$. The claim then follows because the optimal value of ($\mathcal{MMDP}$) is at most μ by Proposition 1.5.3. ■

Proposition 1.5.8 does not imply that second price auctions are optimal when the number of bidders is finite. Indeed, this is not the case as we will see in the next section.

1.5.3 The Optimal Highest-Bidder-Lottery

Consider again problem ($\mathcal{MMDP}$) with a Markov ambiguity set of the form (1.6) and $\mu \in (0,1)$. Assume that the seller aims to optimize over all mechanisms in which only the highest bidder has a chance to win the good. Note that these mechanisms are not necessarily efficient because the seller can keep the good or assign the good to the highest bidder with some probability smaller than 1.

By Proposition 1.5.4, the mechanism design problem ($\mathcal{MMDP}$) can thus be reformulated as

$$\begin{aligned}
\sup_{\boldsymbol{q} \in \mathcal{Q}^{\text{m-p}}, \boldsymbol{\sigma} \in \mathbb{R}^I_+, \lambda \in \mathbb{R}} \quad & \lambda + \sum_{i \in \mathcal{I}} \sigma_i \mu && \text{(1.11a)}\\
\text{s.t.} \quad & \sum_{i \in \mathcal{I}} \left[q_i(v_i, \boldsymbol{v}_{-i}) v_i - \int_0^{v_i} q_i(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \right] \geq \lambda + \sum_{i \in \mathcal{I}} \sigma_i v_i \quad \forall \boldsymbol{v} \in [0,1]^I && \text{(1.11b)}\\
& q_i(\boldsymbol{v}) = 0 \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in [0,1]^I : \boldsymbol{v} \notin \mathcal{W}^i, && \text{(1.11c)}
\end{aligned}$$

where the last constraint ensures that only the highest bidder (with respect to some prescribed tie-breaker) has a chance to win the good. Thus, we refer to the mechanisms feasible in (1.11) as *highest-bidder-lotteries.*

Theorem 1.5.3. *Assume that $\mathcal{P}$ is a Markov ambiguity set of the form* (1.6) *with* $\mu \in (0,1)$, *and set* $\sigma^\star = -(W_{-1}(-\mu I e^{-I}) + 1)^{-1}$, *where* W_{-1} *denotes the lower branch of the Lambert-W* *function (Corless et al., 1996). Moreover, set* $r = e^{(I-1-\frac{1}{\sigma^\star})}$, $\lambda^\star = -\sigma^\star r$ *and*

$$q_i^\star(\boldsymbol{v}) = \begin{cases} \sigma^\star \log\Big(\dfrac{v_i}{\max\limits_{j\neq i} v_j}\Big) + I\sigma^\star - \dfrac{\sigma^\star r}{\max\limits_{j\neq i} v_j} & \text{if } \boldsymbol{v}\in\mathcal{W}^i \text{ and } v_i \geq \max\limits_{j\neq i} v_j \geq r, \quad (1.12a)\\ \sigma^\star \log(v_i) + 1 & \text{if } v_i \geq r > \max\limits_{j\neq i} v_j, \quad (1.12b)\\ (I-1)\sigma^\star & \text{if } \boldsymbol{v}\in\mathcal{W}^i \text{ and } r > v_i \geq \max\limits_{j\neq i} v_j, \quad (1.12c)\\ 0 & \text{if } \boldsymbol{v}\notin\mathcal{W}^i. \quad (1.12d) \end{cases}$$

Then, $(\boldsymbol{q}^\star, \sigma^\star \mathbf{e}, \lambda^\star)$ *is optimal in* (1.11) *with corresponding objective value* $r = e^{(I-1-\frac{1}{\sigma^\star})}$.

By Proposition 1.2.2(i), we can construct a payment rule $\boldsymbol{m}^\star$ from the allocation rule $\boldsymbol{q}^\star$ by

$$m_i^\star(\boldsymbol{v}) = q_i^\star(\boldsymbol{v}) v_i - \int_0^{v_i} q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in [0,1]^I. \tag{1.13}$$

Theorem 1.5.3 implies that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is an optimal highest-bidder-lottery in ($\mathcal{MMDP}$). The optimal allocation rule $\boldsymbol{q}^\star$ is randomized and can be interpreted as follows. The highest bidder i earns the right to participate in a lottery, which allows him to win the object with probability $q_i^\star(\boldsymbol{v})$. The probability $q_i^\star(\boldsymbol{v})$ is increasing in v_i if $v_i \geq r$ and constant otherwise. Moreover, $q_i^\star(\boldsymbol{v})$ is constant in $\boldsymbol{v}_{-i}$ if $\max_{j\neq i} v_j \leq r$. Finally, $q_i^\star(\boldsymbol{v})$ is decreasing in the second highest bid as long as both exceed r. It is perhaps surprising that the optimal allocation rule is randomized. As shown by Delage et al. (2019), however, agents with maxmin preferences can derive substantial benefits from randomization when facing a discrete choice (such as choosing a buyer out of I bidders).

Proposition 1.5.9. *As the number of bidders tends to infinity, the optimal highest-bidder-lottery* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *converges uniformly to the second price auction without reserve price.*

Figure 1.4 compares the optimal highest-bidder-lottery against the best second price auction. Figure 1.4a shows the worst-case expected revenues generated by the optimal highest-bidder-lottery and the optimal second price auction with reserve price as a function of the number of bidders for $\mu = 0.5$. The gap between them relative to the worst-case expected revenue of the optimal highest-bidder-lottery is visualized in Figure 1.4b. We observe that the optimal highest-bidder-lottery generates substantially higher revenues when μ or I are small.

Even though the optimal highest-bidder-lottery was derived under the restriction that the good can be allocated only to the highest bidder, Proposition 1.5.8 implies that it is asymptotically optimal in ($\mathcal{MMDP}$). Next, we show that the optimal highest-bidder-lottery also offers a constant-factor approximation guarantee, which holds for any number of bidders.

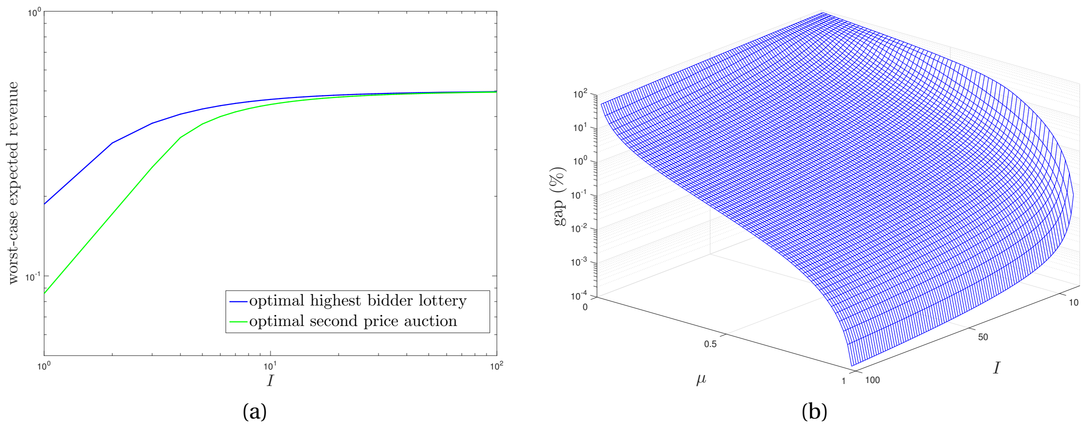


Figure 1.4 – Optimal highest-bidder-lottery versus optimal second price auction.

Theorem 1.5.4. *The optimal highest-bidder-lottery extracts at least* 50% *of the worst-case expected revenue of the unknown optimal mechanism.*

Theorem 1.5.4 shows that the disadvantage of the optimal highest-bidder-lottery relative to the unknown optimal mechanism is bounded. On the other hand, the seller can be arbitrarily worse off by using the best second price auction instead of the optimal highest-bidder-lottery.

Theorem 1.5.5. *For every* $\epsilon > 0$ *and every fixed* $I \in \mathbb{N}$*, there exists* $\mu \in (0,1)$ *such that the optimal second price auction extracts less than* $\epsilon \cdot 100\%$ *of the worst-case expected revenue of the optimal highest-bidder-lottery.*

Since the worst-case expected revenue of the optimal highest-bidder-lottery provides a lower bound on the worst-case expected revenue of the optimal mechanism, the following corollary holds.

Corollary 1.5.1. *For every* $\epsilon > 0$ *and every fixed* $I \in \mathbb{N}$*, there exists* $\mu \in (0,1)$ *such that the optimal second price auction extracts less than* $\epsilon \cdot 100\%$ *of the worst-case expected revenue of the unknown optimal mechanism.*

In Section 1.4 we have seen that if the bidders' values are independent and each distribution in the ambiguity set is symmetric and regular, then the added value of the optimal mechanism over a simple second price auction without reserve price is offset by just attracting one additional bidder. We now demonstrate that this result ceases to hold if the bidders' values can be dependent, as is the case under some distributions in a Markov ambiguity set. To this end, we denote by $\Delta(I) \in \mathbb{N}$ the least number of additional bidders needed by the best second price auction (which generates higher revenues than the second price auction without reserve price) to outperform the optimal highest-bidder-lottery with I bidders (which may be suboptimal in ($\mathscr{MMDP}$)). Figure 1.5 shows that $\Delta(I)$ can be much larger than 1. In fact, we can even prove that there does not exist any finite upper bound on $\Delta(I)$ that holds uniformly across all $I \in \mathbb{N}$.

Proposition 1.5.10. *If $\mu \in (0,1)$, then the set $\{\Delta(I) : I \in \mathbb{N}\}$ does not have a finite upper bound.*

Proposition 1.5.10 suggests that a Bulow and Klemperer (1996) type result does not hold in the setting of this section even if it is relaxed to any finite additional number of bidders.

Unlike $\boldsymbol{q}^\star$, the optimal payment rule $\boldsymbol{m}^\star$ is deterministic, implying that the highest bidder has to make a payment even if he is unlucky in the lottery. That is, he is charged a fee for the right to participate in the lottery. In the following, we construct a new mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ equivalent to $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, where both the allocation rule and the payment rule are randomized, and which charges the highest bidder only if he actually receives the good. To this end, we assume that the seller has access to a randomization device which generates a uniformly distributed sample $\tilde{u}$ from the interval $[0,1]$ that is independent of $\tilde{\boldsymbol{v}}$. Then, we define the new mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ through

$$q_i'(\boldsymbol{v}, u) = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathscr{W}^i \text{ and } u \leq q_i^\star(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_i'(\boldsymbol{v}, u) = \begin{cases} v_i - \displaystyle\int_0^{v_i} \frac{q_i^\star(x, \boldsymbol{v}_{-i})}{q_i^\star(v_i, \boldsymbol{v}_{-i})} \, \mathrm{d}x & \text{if } \boldsymbol{v} \in \mathscr{W}^i \text{ and } u \leq q_i^\star(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$

for every $i \in \mathscr{I}$, $\boldsymbol{v} \in [0,1]^I$, and $u \in [0,1]$. Note that both the allocation rule $\boldsymbol{q}'$ and the payment rule $\boldsymbol{m}'$ depend on the outcome u of the randomization device and are thus randomized. By construction, however, the winner is not required to pay unless he receives the good in the lottery.

It is easy to verify that $(\boldsymbol{q}', \boldsymbol{m}')$ is equivalent to $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$. Indeed, we have

$$\mathbb{E}[q_i'(\boldsymbol{v}, \tilde{u})] = q_i^\star(\boldsymbol{v}), \quad \mathbb{E}[m_i'(\boldsymbol{v}, \tilde{u})] = q_i^\star(\boldsymbol{v}) v_i - \int_0^{v_i} q_i^\star(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x = m_i^\star(\boldsymbol{v}) \;\; \forall i \in \mathscr{I}, \forall \boldsymbol{v} \in [0,1]^I,$$

which implies that the expected revenues of the seller and the expected utilities of the bidders are identical under $(\boldsymbol{q}', \boldsymbol{m}')$ and $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, irrespective of the value distribution $\mathbb{P} \in \mathscr{P}$.

In this section, we introduced the optimal highest-bidder-lottery, a randomized mechanism that offers to the seller significantly higher revenues than the best second price auction. Based on numerical experiments, we conjecture that the best second price auction from Section 1.5.2 is also the best deterministic mechanism for problem ($\mathscr{MMDP}$).We emphasize that

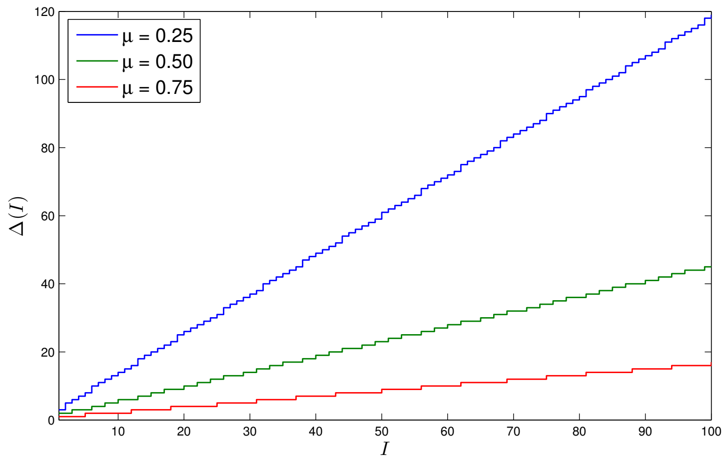


Figure 1.5 – The minimum number of additional bidders necessary to outperform the optimal highest-bidder-lottery with the best second price auction.

Theorem 1.5.5, Corollary 1.5.1 and Proposition 1.5.10 remain valid even if the bidders display maxmin preferences instead of Knightian preferences. Indeed, relaxing the Knightian incentive compatibility to maxmin incentive compatibility constraints increases the set of admissible mechanisms in ($\mathcal{MMDP}$); see Proposition 1.2.1. As all second price auctions are ex-post incentive compatible, however, this relaxation does not alter the best second price auction. Thus, the highest-bidder-lottery $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ continues to outperform the best second price auction if the bidders have maxmin preferences.

1.6 Conclusion

The standard assumption in the classical mechanism design literature whereby the seller has full knowledge of the bidders' value distribution is not tenable in practice. This prompts us to question whether the popular second price auctions remain optimal in the presence of distributional ambiguity. If not, we aim to identify the optimal mechanism or, if that is not possible, a near-optimal mechanism that offers strong performance guarantees. If only the range of all possible bidder values is known, we prove that the Vickrey auction is the unique optimal, efficient and Pareto robustly optimal mechanism. Thus, sellers who have no information about the bidders' values or have little trust in their information, as is typically the case for one-off auctions, might settle for a simple second price auction without reserve price. If the seller lacks information about the distribution but knows that the bidders' values are independent, the globally optimal auction is unknown, but second price auctions remain near-optimal: there exists a second price auction with a random reserve price that offers a 2-approximation for the mechanism design problem, and even the naïve second price auction without reserve price generates higher revenues than the unknown optimal mechanism if the former attracts only one additional bidder. However, independence can rarely be ascertained in reality as the bidders might interact with each other or might share common information sources. This concern motivates us to study Markov ambiguity sets, which allow the bidders' values to be correlated. Markov ambiguity sets capture the knowledge of more informed sellers who not only know the range of possible bidder values but are also aware of the expected or 'most likely' values. In this setting, which makes honest assumptions about the information that is realistically available to the seller, the performance of second price auctions drops significantly. Indeed, we prove that even the best second price auction, which we can characterize analytically, fails to offer any constant-factor approximation guarantee. While the globally optimal mechanism remains unknown, we can explicitly construct a randomized near-optimal mechanism, the optimal highest-bidder-lottery, which offers a 2-approximation for the mechanism design problem. Moreover, we demonstrate that the number of additional bidders needed by the best second price auction to match the revenues of the optimal highest-bidder-lottery (and, a fortiori, of the unknown globally optimal mechanism) is unbounded. Under realistic assumptions about the available distributional information, second price auction can therefore be severely suboptimal.

1.7 Appendix

Proof of Proposition 1.2.1. It is easy to verify that ex-post incentive compatibility implies distributionally robust incentive compatibility.

For any fixed $i \in \mathcal{I}$ and $v_i, w_i \in \mathcal{V}$, distributionally robust incentive compatibility requires that

$$\mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \mathbb{E}_{\mathbb{P}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \quad \forall \mathbb{P} \in \mathcal{P}. \tag{1.14}$$

Since (1.14) holds for all $\mathbb{P} \in \mathcal{P}$, we have

$$\mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \quad \forall \mathbb{P} \in \mathcal{P}.$$

Now, taking the infimum over $\mathbb{P} \in \mathcal{P}$ on the left-hand side yields

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right]. \tag{1.15}$$

This establishes maxmin incentive compatibility.

Note that (1.14) implies

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \mathbb{E}_{\mathbb{P}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \quad \forall \mathbb{P} \in \mathcal{P}.$$

Since this condition holds for all $\mathbb{P} \in \mathscr{P}$, we can take the supremum over $\mathbb{P} \in \mathscr{P}$ on the right-hand side to obtain

$$\sup_{\mathbb{Q}\in\mathscr{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \sup_{\mathbb{P}\in\mathscr{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right]. \tag{1.16}$$

Summing α times (1.15) and $1-\alpha$ times (1.16) for any $\alpha \in (0,1)$ yields Hurwicz incentive compatibility.

By taking expectations on both sides of (1.14) with respect to the distribution $\mathbb{Q}$ on $\mathscr{P}$, finally, one concludes that Bayesian incentive compatibility holds as well. ■

Proof of Theorem 1.2.2. By Proposition 1.2.2(ii) and by the assumption that $u_i(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathscr{I}$ and $\boldsymbol{v}_{-i} \in \mathcal{V}$, we have

$$\mathbb{E}_{\mathbb{P}}\left[m_i(v_i, \tilde{\boldsymbol{v}}_{-i})\right] = \mathbb{E}_{\mathbb{P}}\left[q_i(v_i, \tilde{\boldsymbol{v}}_{-i})v_i - \int_{\underline{v}}^{v_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\right] \quad \forall i \in \mathscr{I}, \forall v_i \in \mathcal{V}, \forall \mathbb{P} \in \mathscr{P}. \tag{1.17}$$

Hence, the expected revenue of the seller with respect to some $\mathbb{P} \in \mathscr{P}$ is equal to

$$\mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathscr{I}} m_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\Big] = \sum_{i\in\mathscr{I}} \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}[m_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})|\tilde{v}_i]\Big] = \sum_{i\in\mathscr{I}} \mathbb{E}_{\mathbb{P}}\Big[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\Big],$$

where the second equality follows from (1.17). This indicates that the seller's expected revenue under any $\mathbb{P}$ is determined solely by the allocation rule $\boldsymbol{q}$. Hence, the seller earns the same worst-case expected revenue from all mechanisms with identical allocation rules. ■

Proof of Proposition 1.3.2. We will prove the claim by contradiction. Assume that the Vickrey mechanism is not Pareto robustly optimal. Thus, there exists a mechanism $(\boldsymbol{q}, \boldsymbol{m})$ feasible in ($\mathscr{RMDP}$) with $\sum_{i\in\mathscr{I}} m_i(\boldsymbol{v}) \geq \sum_{i\in\mathscr{I}} m_i^{\mathrm{v}}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathcal{V}^I$, and there exists $\boldsymbol{v}' \in \mathcal{V}^I$ with $\sum_{i\in\mathscr{I}} m_i(\boldsymbol{v}') > \sum_{i\in\mathscr{I}} m_i^{\mathrm{v}}(\boldsymbol{v}')$. Let $i^1 \in \mathscr{I}$ be the winning bidder in the Vickrey mechanism under scenario $\boldsymbol{v}'$, which implies that $i^1 \in \arg\max_{j\in\mathscr{I}} v'_j$, and let $i^2 \in \arg\max_{j\neq i^1} v'_j$ be any second highest bidder.

If $v'_{i^1} = v'_{i^2}$, then we have

$$\sum_{j\in\mathscr{I}} m_j(\boldsymbol{v}') > \sum_{j\in\mathscr{I}} m_j^{\mathrm{v}}(\boldsymbol{v}') = v'_{i^2} = v'_{i^1} \geq \sum_{j\in\mathscr{I}} q_j(\boldsymbol{v}')v'_j,$$

where the last inequality holds because $\sum_{j\in\mathscr{I}} q_j(\boldsymbol{v}') \leq 1$. This contradicts ex-post individual rationality, which is imposed in ($\mathscr{RMDP}$) and which implies that $\sum_{j\in\mathscr{I}} m_j(\boldsymbol{v}') \leq \sum_{j\in\mathscr{I}} q_j(\boldsymbol{v}')v'_j$.

Assume from now on that $v'_{i^1} > v'_{i^2}$. Our analysis is divided into three steps. In the first step, we construct a scenario $\boldsymbol{w} \in \mathcal{V}^I$ such that $w_{i^1} = v'_{i^2} + \delta$ for a judiciously chosen $\delta > 0$ and

$w_j = v'_j$ for all $j \neq i^1$. In the second step, we show that the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ has to assign the good to a second highest bidder i^2 with strictly positive probability under scenario $\boldsymbol{w}$ in order to satisfy individual rationality and incentive compatibility at the ex-post stage. In the third step, we construct a scenario $\boldsymbol{w}' \in \mathcal{V}^I$ such that $w'_{i^1} = w'_{i^2} = v'_{i^2} + \delta$ and $w'_j = v'_j$ for all $j \notin \{i^1, i^2\}$. Leveraging the results from Steps 1 and 2, we then show that if $(\boldsymbol{q}, \boldsymbol{m})$ satisfies ex-post individual rationality in scenario $\boldsymbol{w}'$, then it has to violate ex-post incentive compatibility because bidder i^2 has an incentive to misreport his value as w_{i^2} instead of w'_{i^2}. We will conclude that the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ cannot simultaneously satisfy individual rationality and incentive compatibility at the ex-post stage, which contradicts our assumption that $(\boldsymbol{q}, \boldsymbol{m})$ is feasible in ($\mathscr{RMDP}$). Figure 1 visualizes the scenarios $\boldsymbol{v}', \boldsymbol{w}$ and $\boldsymbol{w}'$ constructed below.

Step 1: Let $\epsilon = \frac{m_{i^1}(\boldsymbol{v}')}{q_{i^1}(\boldsymbol{v}')} - v'_{i^2}$ and $\gamma = \epsilon/2v'_{i^1}$. Note that ϵ is well-defined since $q_{i^1}(\boldsymbol{v}') > 0$ must hold for $(\boldsymbol{q}, \boldsymbol{m})$ to generate higher revenues than the Vickrey mechanism in scenario $\boldsymbol{v}'$. Note also that γ is well-defined due to assumption $v'_{i^1} > v'_{i^2}$ and because the bidders' values are non-negative. Define $\boldsymbol{w}$ such that $w_{i^1} = v'_{i^2} + \delta$ and $w_j = v'_j$ for all $j \neq i^1$, where

$$\delta = \min\left\{\frac{\epsilon}{2}, \frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{v}')(1-\gamma)} - (v'_{i^2} - v'_{i^3})\right\} \tag{1.18}$$

and $i^3 \in \arg\max_{j \in \mathscr{I}}\{v_j : v_j < v'_{i^2}\}$. If $\arg\max_{j \in \mathscr{I}}\{v_j : v_j < v'_{i^2}\} = \emptyset$, in which $\boldsymbol{v}'$ accounts only for two different bids, we may set $v'_{i^3} = 0$. In this step, we show that $\delta \in (0, v'_{i^1} - v'_{i^2})$, which ensures that $\boldsymbol{w} \in \mathcal{V}^I$. Specifically, we show that both terms inside the minimum in (1.18) are positive and that the first term is smaller than $v'_{i^1} - v'_{i^2}$.

We start by showing that $\frac{\epsilon}{2} \in (0, v'_{i^1} - v'_{i^2})$. By assumption, we have $\sum_{j \in \mathscr{I}} m_j(\boldsymbol{v}') > \sum_{j \in \mathscr{I}} m_j^{\mathrm{v}}(\boldsymbol{v}') = v'_{i^2}$. Moreover, by individual rationality, for all $j \neq i^1$, we have

$$m_j(\boldsymbol{v}') \leq q_j(\boldsymbol{v}')v'_j \leq q_j(\boldsymbol{v}')v'_{i^2},$$

where the second inequality holds because $v'_j \leq v'_{i^2}$. As $\sum_{j \in \mathscr{I}} q_j(\boldsymbol{v}') \leq 1$, our initial assumption that $\sum_{j \in \mathscr{I}} m_j(\boldsymbol{v}') > v'_{i^2}$ can hold only if $m_{i^1}(\boldsymbol{v}') > q_{i^1}(\boldsymbol{v}')v'_{i^2}$. Thus $\epsilon = \frac{m_{i^1}(\boldsymbol{v}')}{q_{i^1}(\boldsymbol{v}')} - v'_{i^2} > 0$. We also have that $\frac{\epsilon}{2} < v'_{i^1} - v'_{i^2}$ because $\frac{\epsilon}{2} < \epsilon = \frac{m_{i^1}(\boldsymbol{v}')}{q_{i^1}(\boldsymbol{v}')} - v'_{i^2} \leq v'_{i^1} - v'_{i^2}$ as individual rationality requires that $m_{i^1}(\boldsymbol{v}') \leq q_{i^1}(\boldsymbol{v}')v'_{i^1}$. Finally, the second term inside the minimum in (1.18) is greater than 0 since

$$\frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{v}')(1-\gamma)} > \frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{v}')} \geq v'_{i^2} - v'_{i^3},$$

where the first inequality holds because $0 < \gamma < 1$ as $0 < \frac{\epsilon}{2} < v'_{i^1} - v'_{i^2} < v'_{i^1}$, and the second inequality follows from $0 < q_{i^1}(\boldsymbol{v}') \leq 1$. As both terms inside the minimum in (1.18) are positive and the first term is smaller than $v'_{i^1} - v'_{i^2}$, we conclude that $\delta \in (0, v'_{i^1} - v'_{i^2})$.

Step 2: Assume without loss of generality that amongst all second highest bidders, i^2 has the highest probability to win under scenario $\boldsymbol{w}$. In this step, we show that $q_{i^2}(\boldsymbol{w}) > 0$. Assume to the contrary that $q_{i^2}(\boldsymbol{w}) = 0$, which implies that the probability to win for any second highest bidder equals 0. By ex-post individual rationality and the assumption that the Vickrey mechanism is not Pareto robustly optimal, we have

$$\begin{aligned} q_{i^1}(\boldsymbol{w})w_{i^1} + \sum_{j\neq i^1} q_j(\boldsymbol{w})w_j \; &= \; q_{i^1}(\boldsymbol{w})(v'_{i^2}+\delta) + \sum_{j\neq i^1} q_j(\boldsymbol{w})w_j \\ &\geq \; \sum_{j\in\mathcal{I}} m_j(\boldsymbol{w}) \;\geq\; \sum_{j\in\mathcal{I}} m^{\mathrm{v}}_j(\boldsymbol{w}) \;=\; v'_{i^2}. \end{aligned} \tag{1.19}$$

Since $q_{i^2}(\boldsymbol{w}) = 0$, equation (1.19) can only hold if $q_{i^1}(\boldsymbol{w})(v'_{i^2}+\delta) + (1-q_{i^1}(\boldsymbol{w}))v'_{i^3} \;\geq\; v'_{i^2}$. Reordering term, this implies that

$$\delta \;\geq\; \frac{v'_{i^2}-v'_{i^3}}{q_{i^1}(\boldsymbol{w})} - (v'_{i^2}-v'_{i^3}). \tag{1.20}$$

We next prove that (1.20) contradicts the definition of δ if $(\boldsymbol{q},\boldsymbol{m})$ is ex-post individually rational and incentive compatible. Note that we have $m_{i^1}(\boldsymbol{w}) \leq q_{i^1}(\boldsymbol{w})(v'_{i^2}+\delta) < q_{i^1}(\boldsymbol{v}')(v'_{i^2}+\epsilon) = m_{i^1}(\boldsymbol{v}')$, where the first inequality follows from individual rationality, and the second inequality holds because $\epsilon > \delta$ by definition and $q_{i^1}(\boldsymbol{v}') \geq q_{i^1}(\boldsymbol{w})$ by Proposition 1.2.2(i). Combining the inequality $m_{i^1}(\boldsymbol{w}) < m_{i^1}(\boldsymbol{v}')$ with the incentive compatibility of $(\boldsymbol{q},\boldsymbol{m})$, which requires that $q_{i^1}(\boldsymbol{v}')v'_{i^1} - m_{i^1}(\boldsymbol{v}') \;\geq\; q_{i^1}(\boldsymbol{w})v'_{i^1} - m_{i^1}(\boldsymbol{w})$, implies that $1 \geq q_{i^1}(\boldsymbol{v}') > q_{i^1}(\boldsymbol{w})$ and

$$q_{i^1}(\boldsymbol{v}') - q_{i^1}(\boldsymbol{w}) \;\geq\; \frac{m_{i^1}(\boldsymbol{v}') - m_{i^1}(\boldsymbol{w})}{v'_{i^1}} \;>\; \frac{q_{i^1}(\boldsymbol{v}')\epsilon}{2v'_{i^1}} \;=\; q_{i^1}(\boldsymbol{v}')\gamma, \tag{1.21}$$

where the second inequality holds because $m_{i^1}(\boldsymbol{v}') = q_{i^1}(\boldsymbol{v}')(v'_{i^2}+\epsilon)$ by definition of ϵ and $m_{i^1}(\boldsymbol{w}) \leq q_{i^1}(\boldsymbol{w})(v'_{i^2}+\delta) < q_{i^1}(\boldsymbol{v}')(v'_{i^2}+\frac{\epsilon}{2})$ due to $q_{i^1}(\boldsymbol{v}') > q_{i^1}(\boldsymbol{w})$ and $\delta \leq \frac{\epsilon}{2}$. The inequalities (1.20) and (1.21) imply that

$$\delta \;\geq\; \frac{v'_{i^2}-v'_{i^3}}{q_{i^1}(\boldsymbol{w})} - (v'_{i^2}-v'_{i^3}) > \frac{v'_{i^2}-v'_{i^3}}{q_{i^1}(\boldsymbol{v}')(1-\gamma)} - (v'_{i^2}-v'_{i^3}),$$

which contradicts the definition of δ. Thus $q_{i^2}(\boldsymbol{w}) > 0$, which concludes Step 2.

Step 3: Define $\boldsymbol{w}'$ such that $w'_{i^1} = w'_{i^2} = v'_{i^2}+\delta$ and $w'_j = v'_j$ for all $j \not\in \{i^1, i^2\}$. Note that $\boldsymbol{w}' \in \mathcal{V}^I$ because $\mathcal{V} = [\underline{v}, \bar{v}]$ and $v'_{i^2}+\delta \leq v'_{i^1} \leq \bar{v}$ by construction. Note that we have

$$q_{i^2}(\boldsymbol{w})(v'_{i^2}+\delta) - m_{i^2}(\boldsymbol{w}) \;>\; q_{i^2}(\boldsymbol{w})v'_{i^2} - m_{i^2}(\boldsymbol{w}) \;\geq\; 0, \tag{1.22}$$

where the first inequality holds because $\delta > 0$ due to Step 1 and $q_{i^2}(\boldsymbol{w}) > 0$ due to Step 2, and the second inequality holds due to ex-post individual rationality at scenario $\boldsymbol{w}$. Next, we prove that $q_{i^2}(\boldsymbol{w}')(v'_{i^2}+\delta) - m_{i^2}(\boldsymbol{w}') = 0$ if $(\boldsymbol{q},\boldsymbol{m})$ is ex-post individually rational at scenario $\boldsymbol{w}'$. By Proposition 1.2.2(i), we have that $q_{i^2}(\boldsymbol{w}') \geq q_{i^2}(\boldsymbol{w}) > 0$. Moreover, by ex-post individual rationality, we have that $\sum_{j\in\mathscr{I}} m_j(\boldsymbol{w}') \leq \sum_{j\in\mathscr{I}} q_j(\boldsymbol{w}')w'_j \leq \sum_{j\in\mathscr{I}} q_j(\boldsymbol{w}')(v'_{i^2}+\delta)$. Since $\sum_{j\in\mathscr{I}} q_j(\boldsymbol{w}') \leq 1$, the assumption $\sum_{j\in\mathscr{I}} m_j(\boldsymbol{w}') \geq \sum_{j\in\mathscr{I}} m^{\mathrm{v}}_j(\boldsymbol{w}') = v'_{i^2}+\delta$ thus can hold only if $m_{i^2}(\boldsymbol{w}') = q_{i^2}(\boldsymbol{w}')(v'_{i^2}+\delta)$, which contradicts the incentive compatibility of $(\boldsymbol{q},\boldsymbol{m})$ as we have

$$q_{i^2}(\boldsymbol{w}')(v'_{i^2}+\delta) - m_{i^2}(\boldsymbol{w}') \;=\; 0 \;<\; q_{i^2}(\boldsymbol{w})(v'_{i^2}+\delta) - m_{i^2}(\boldsymbol{w}),$$

where the inequality follows from (1.22). The claim thus follows. ■

Proof of Proposition 1.3.3. Select any efficient ex-post individually rational and incentive compatible mechanism $(\boldsymbol{q},\boldsymbol{m})$. Suppose that $\sum_{j\in\mathscr{I}} m_j(\boldsymbol{v}) > \sum_{j\in\mathscr{I}} m^{\mathrm{v}}_j(\boldsymbol{v})$ for some fixed $\boldsymbol{v}\in\mathcal{V}^I$, and note that $\boldsymbol{v}\in\mathcal{W}^i$ for some $i\in\mathscr{I}$. Then, we have

$$\sum_{j\in\mathscr{I}} m_j(\boldsymbol{v}) \;>\; \sum_{j\in\mathscr{I}} m^{\mathrm{v}}_j(\boldsymbol{v}) \;=\; m^{\mathrm{v}}_i(\boldsymbol{v}) \;=\; \max_{j\neq i} v_j.$$

We will show that if the above strict inequality holds, then $(\boldsymbol{q},\boldsymbol{m})$ cannot simultaneously satisfy ex-post individual rationality, ex-post incentive compatibility and efficiency.

If the second highest bid equals the highest bid v_i, then we have

$$\sum_{j\in\mathscr{I}} m_j(\boldsymbol{v}) \;>\; \max_{j\neq i} v_j \;=\; v_i \;=\; \sum_{j\in\mathscr{I}} q_j(\boldsymbol{v})v_i \;\geq\; \sum_{j\in\mathscr{I}} q_j(\boldsymbol{v})v_j,$$

where the second inequality holds because $\sum_{j\in\mathscr{I}} q_j(\boldsymbol{v}) = 1$. This contradicts ex-post individual rationality, which implies that $\sum_{j\in\mathscr{I}} m_j(\boldsymbol{v}) \leq \sum_{j\in\mathscr{I}} q_j(\boldsymbol{v})v_j$.

If there is no tie such that $v_i > \max_{j\neq i} v_j$, then $m_i(\boldsymbol{v}) = \sum_{j\in\mathscr{I}} m_j(\boldsymbol{v}) > \sum_{j\in\mathscr{I}} m^{\mathrm{v}}_j(\boldsymbol{v}) = m^{\mathrm{v}}_i(\boldsymbol{v})$, where the first equality holds because the mechanism $(\boldsymbol{q},\boldsymbol{m})$ is efficient and ex-post individually rational. Select $\epsilon > 0$ small enough such that $\epsilon < v_i - \max_{j\neq i} v_j$ and $\epsilon < m_i(\boldsymbol{v}) - m^{\mathrm{v}}_i(\boldsymbol{v})$. Moreover, set $v'_i = \max_{j\neq i} v_j + \epsilon$ and note that $v'_i < v_i$ and that $(v'_i, \boldsymbol{v}_{-i}) \in \mathcal{V}^I$ because $\mathcal{V} = [\underline{v}, \bar{v}]$. Then, the mechanism $(\boldsymbol{q},\boldsymbol{m})$ violates ex-post incentive compatibility because

$$q_i(\boldsymbol{v})v_i - m_i(\boldsymbol{v}) \;<\; v_i - v'_i \;\leq\; q_i(v'_i,\boldsymbol{v}_{-i})v_i - m_i(v'_i,\boldsymbol{v}_{-i}),$$

where the first inequality holds because $m_i(\boldsymbol{v}) - m^{\mathrm{v}}_i(\boldsymbol{v}) > \epsilon$, which implies that $m_i(\boldsymbol{v}) > v'_i$. The second inequality holds because $v'_i > \max_{j\neq i} v_j$, which implies that $q_i(v'_i,\boldsymbol{v}_{-i}) = 1$ due to efficiency, and because $m_i(v'_i,\boldsymbol{v}_{-i}) \leq q_i(v'_i,\boldsymbol{v}_{-i})v'_i = v'_i$ due to ex-post individual rationality. Thus $(\boldsymbol{q},\boldsymbol{m})$ violates ex-post incentive compatibility if it is ex-post individually rational and efficient. ■

We need the following auxiliary result to prove Proposition 1.4.2.

Lemma 1.7.1. *For each mechanism* $(\boldsymbol{q}, \boldsymbol{m})$ *feasible in* $(\mathcal{IMDP})$*, there exists a mechanism* $(\boldsymbol{q}', \boldsymbol{m}')$ *with*

$$\mathbb{E}_{\mathbb{P}}[q_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})] = 0 \quad \forall i \in \mathcal{I}, \ \forall \mathbb{P} \in \mathcal{P} \tag{1.23}$$

that is also feasible in $(\mathcal{IMDP})$ *and results in a weakly higher worst-case expected revenue to the seller.*

Proof. Construct $(\boldsymbol{q}', \boldsymbol{m}')$ by setting $\boldsymbol{q}' = \boldsymbol{q}$ and

$$m_i'(v_i, \boldsymbol{v}_{-i}) = q_i'(v_i, \boldsymbol{v}_{-i})v_i - \int_{\underline{v}}^{v_i} q_i'(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \quad \forall i \in \mathcal{I}, \ \forall \boldsymbol{v} \in \mathcal{V}^I. \tag{1.24}$$

Note that the ex-post utility under mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ satisfies $q_i'(\underline{v}, \boldsymbol{v}_{-i})\underline{v} - m_i'(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathcal{I}$, $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, which implies that $\mathbb{E}_{\mathbb{P}}[q_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})] = 0$ for all $i \in \mathcal{I}$, $\mathbb{P} \in \mathcal{P}$. Thus, $(\boldsymbol{q}', \boldsymbol{m}')$ satisfies the second constraint in $(\mathcal{IMDP})$. Moreover, we have

$$q_i'(v_i, \boldsymbol{v}_{-i})v_i - m_i'(v_i, \boldsymbol{v}_{-i}) = \int_{\underline{v}}^{v_i} q_i'(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \quad \forall i \in \mathcal{I}, \ \forall \boldsymbol{v} \in \mathcal{V}^I,$$

which implies that $(\boldsymbol{q}', \boldsymbol{m}')$ satisfies also the first constraint in $(\mathcal{IMDP})$.

We next show that $(\boldsymbol{q}', \boldsymbol{m}')$ results in a weakly higher worst-case expected revenue to the seller than $(\boldsymbol{q}, \boldsymbol{m})$. As the ex-post utility satisfies $q_i'(\underline{v}, \boldsymbol{v}_{-i})\underline{v} - m_i'(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathcal{I}$, $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, the objective function of $(\boldsymbol{q}', \boldsymbol{m}')$ in $(\mathcal{IMDP})$ reduces to

$$\begin{aligned}
&\inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\left[q_i'(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i'(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\right] \\
&\geq \inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\left[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\right] - \mathbb{E}_{\mathbb{P}}[q_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})],
\end{aligned}$$

where the inequality holds because $\boldsymbol{q}' = \boldsymbol{q}$ and $\mathbb{E}_{\mathbb{P}}[q_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})] \geq 0$ for all $i \in \mathcal{I}$ and $\mathbb{P} \in \mathcal{P}$ due to the second constraint in $(\mathcal{IMDP})$. The statement now follows from the fact that right-hand side of this equation coincides with the objective function value of $(\boldsymbol{q}, \boldsymbol{m})$ in $(\mathcal{IMDP})$. ■

Proof of Proposition 1.4.2. Denote by $\rho^{\mathbb{P}}$ the density function of $\tilde{\boldsymbol{v}}$ and by $\rho_i^{\mathbb{P}}$ the marginal density function of $\tilde{v}_i$, $i \in \mathcal{I}$, under $\mathbb{P} \in \mathcal{P}$. By Lemma 1.7.1, without loss of generality, we can restrict the feasible set of $(\mathcal{IMDP})$ to mechanisms that satisfy (1.23). Fix now an arbitrary mechanism $(\boldsymbol{q}, \boldsymbol{m})$ feasible in this restriction of $(\mathcal{IMDP})$.

Proposition 1.2.2(ii) implies that the expected payment of bidder i under $\mathbb{P}$ can be expressed as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[q_i(\tilde{v}_i,\tilde{\boldsymbol{v}}_{-i})\,\tilde{v}_i-\int_{\underline{v}}^{\tilde{v}_i} q_i(x,\tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\right]\\
&=\int_{\mathcal{V}^{I-1}}\int_{\mathcal{V}} q_i(v_i,\boldsymbol{v}_{-i})\,v_i\,\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}-\int_{\mathcal{V}^{I-1}}\int_{\mathcal{V}}\int_{\underline{v}}^{v_i} q_i(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\,\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}.
\end{aligned}$$

Using Fubini's theorem, we can re-write the second term as

$$\int_{\mathcal{V}^{I-1}}\int_{\mathcal{V}}\int_{\underline{v}}^{v_i} q_i(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\,\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i} = \int_{\mathcal{V}^{I-1}}\int_{\underline{v}}^{\bar{v}} q_i(x,\boldsymbol{v}_{-i})\left(\int_{v_i}^{\bar{v}}\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}x\right)\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}.$$

Thus, the expected payment of bidder i under $\mathbb{P}$ simplifies to

$$\begin{aligned}
&\int_{\mathcal{V}^{I-1}}\int_{\mathcal{V}}\left(v_i-\frac{\int_{v_i}^{\bar{v}}\rho^{\mathbb{P}}(x,\boldsymbol{v}_{-i})\,\mathrm{d}x}{\rho^{\mathbb{P}}(\boldsymbol{v})}\right)q_i(v_i,\boldsymbol{v}_{-i})\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}\\
&=\int_{\mathcal{V}^{I-1}}\int_{\mathcal{V}}\left(v_i-\frac{1-\int_{\underline{v}}^{v_i}\rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}{\rho_i^{\mathbb{P}}(v_i)}\right)q_i(v_i,\boldsymbol{v}_{-i})\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i},
\end{aligned}$$

where the equality holds because the bidders' values are independent.

Recalling the definition of the virtual valuation

$$\psi_i^{\mathbb{P}}(v_i)=v_i-\frac{1-\int_{\underline{v}}^{v_i}\rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}{\rho_i^{\mathbb{P}}(v_i)},$$

we can now rewrite the objective function of $(\boldsymbol{q},\boldsymbol{m})$ in $(\mathscr{I}\mathscr{M}\mathscr{D}\mathscr{P})$ as

$$\inf_{\mathbb{P}\in\mathscr{P}}\sum_{i\in\mathscr{I}}\left[\int_{\mathcal{V}^{I-1}}\int_{\mathcal{V}}\psi_i^{\mathbb{P}}(v_i)q_i(v_i,\boldsymbol{v}_{-i})\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}\right] = \inf_{\mathbb{P}\in\mathscr{P}}\mathbb{E}_{\mathbb{P}}\Big[\sum_{i\in\mathscr{I}}\psi_i^{\mathbb{P}}(\tilde{v}_i)q_i(\tilde{v}_i,\tilde{\boldsymbol{v}}_{-i})\Big].$$

Thus, the claim follows. ■

Proof of Theorem 1.4.3. Throughout this proof, we write $\mathscr{Q}_I^{\text{m-d}}$ instead of $\mathscr{Q}^{\text{m-d}}$ and $\mathscr{P}_I$ instead of $\mathscr{P}$ in order to highlight the dependence on the number of bidders. Moreover, we denote by $f_I(\boldsymbol{q},\mathbb{P})$ the objective function value of an allocation rule $\boldsymbol{q}\in\mathscr{Q}_I^{\text{m-d}}$ and distribution $\mathbb{P}\in\mathscr{P}_I$ in problem (1.3).

Select an arbitrary $\epsilon>0$. Assume first that the seller attracts $I+1$ bidders, and denote by $\boldsymbol{q}^{\text{sp}}\in\mathscr{Q}_{I+1}^{\text{m-d}}$ the allocation rule of the second price auction without reserve price for $I+1$ bidders. Then, there exists an ϵ-worst-case distribution $\mathbb{P}_\epsilon\in\mathscr{P}_{I+1}$ such that

$$f_{I+1}(\boldsymbol{q}^{\text{sp}},\mathbb{P}_\epsilon) < \inf_{\mathbb{P}\in\mathscr{P}_{I+1}} f_{I+1}(\boldsymbol{q}^{\text{sp}},\mathbb{P})+\epsilon. \tag{1.25}$$

Denote by $\rho_1^{\mathbb{P}_\epsilon}$ the common marginal density function of the values $\tilde{v}_i$ under the distribution $\mathbb{P}_\epsilon$, $i \in \mathscr{I}$. Note that the virtual valuation $\psi_1^{\mathbb{P}_\epsilon}(v_i)$ is non-decreasing in v_i because $\mathbb{P}_\epsilon \in \mathscr{P}_{I+1}$ is regular. As second price auctions allocate the good to the highest bidder, Proposition 1.4.2 implies that

$$f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon) = \mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1})\right\}\right].$$

Next, we derive a lower bound on $f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon)$ by conditioning the above expectation separately on the events $\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0$ and $\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0$. First, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1})\right\} \middle| \max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0\right] \\
&\geq \mathbb{E}_{\mathbb{P}_\epsilon}\left[\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \middle| \max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0\right] \\
&= \mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), 0\right\} \middle| \max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0\right].
\end{aligned} \tag{1.26}$$

Similarly, we find

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1})\right\} \middle| \max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0\right] \\
&\geq \max\left\{\mathbb{E}_{\mathbb{P}_\epsilon}\left[\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \middle| \max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0\right], \mathbb{E}_{\mathbb{P}_\epsilon}\left[\psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1})\right]\right\} \\
&= \max\left\{\mathbb{E}_{\mathbb{P}_\epsilon}\left[\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \middle| \max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0\right], 0\right\} = 0 \\
&= \mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), 0\right\} \middle| \max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0\right],
\end{aligned} \tag{1.27}$$

where the inequality follows from Jensen's inequality and the independence of the bidders' values. In the third line, we use the fact that $\mathbb{E}_{\mathbb{P}_\epsilon}[\psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1})] = 0$, which can be verified through a direct calculation using integration by parts. By combining (1.26) and (1.27), we then obtain

$$f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon) = \mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1})\right\}\right] \geq \mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i\in\mathscr{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), 0\right\}\right]. \tag{1.28}$$

Consider now the mechanism design problem with I bidders. There exists an ϵ-suboptimal allocation rule $\boldsymbol{q}_\epsilon \in \mathscr{Q}_I^{\text{m-d}}$ with

$$\inf_{\mathbb{P}\in\mathscr{P}_I} f_I(\boldsymbol{q}_\epsilon, \mathbb{P}) > \sup_{\boldsymbol{q}\in\mathscr{Q}_I^{\text{m-d}}} \inf_{\mathbb{P}\in\mathscr{P}_I} f_I(\boldsymbol{q}, \mathbb{P}) - \epsilon. \tag{1.29}$$

Denote by $\mathbb{P}_\epsilon^-$ the marginal distribution of $(\tilde{v}_1, \ldots, \tilde{v}_I)$ under $\mathbb{P}_\epsilon$, and observe that $\mathbb{P}_\epsilon^- \in \mathscr{P}_I$ because the bidders' values are independent and $\mathbb{P}_\epsilon$ is symmetric. Let $\boldsymbol{q}^{\mathbb{P}_\epsilon^-} \in \mathscr{Q}_I^{\text{m-d}}$ be the allocation rule that maximizes the expected revenues in problem (1.3) under the distribution

$\mathbb{P}_\epsilon^-$. Note that this allocation rule exists due to Theorem 1.4.2. Thus, we have

$$\sup_{\boldsymbol{q}\in\mathcal{Q}_I^{\text{m-d}}} \inf_{\mathbb{P}\in\mathcal{P}_I} f_I(\boldsymbol{q},\mathbb{P}) - \epsilon \;<\; \inf_{\mathbb{P}\in\mathcal{P}_I} f_I(\boldsymbol{q}_\epsilon,\mathbb{P}) \;\le\; f_I(\boldsymbol{q}_\epsilon,\mathbb{P}_\epsilon^-) \;\le\; \sup_{\boldsymbol{q}\in\mathcal{Q}_I^{\text{m-d}}} f_I(\boldsymbol{q},\mathbb{P}_\epsilon^-) \;=\; f_I(\boldsymbol{q}^{\mathbb{P}_\epsilon^-},\mathbb{P}_\epsilon^-).$$

Here, the first inequality holds by the construction of $\boldsymbol{q}_\epsilon$, and the equality follows from the optimality of $\boldsymbol{q}^{\mathbb{P}_\epsilon^-}$ for the given distribution $\mathbb{P}_\epsilon^-$. Hence,

$$\begin{aligned}
f_I(\boldsymbol{q}^{\mathbb{P}_\epsilon^-},\mathbb{P}_\epsilon^-) \;&=\; \mathbb{E}_{\mathbb{P}_\epsilon^-}\Big[\max\Big\{\max_{i\in\mathcal{I}} \psi_1^{\mathbb{P}_\epsilon^-}(\tilde{v}_i),0\Big\}\Big] \;=\; \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{\max_{i\in\mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i),0\Big\}\Big] \\
&\le\; f_{I+1}(\boldsymbol{q}^{\text{sp}},\mathbb{P}_\epsilon) \;<\; \inf_{\mathbb{P}\in\mathcal{P}_{I+1}} f_{I+1}(\boldsymbol{q}^{\text{sp}},\mathbb{P}) + \epsilon,
\end{aligned}$$

where the first equality holds due to Theorem 1.4.2, the second equality follows from the definition of $\mathbb{P}_\epsilon^-$, and the inequalities follow from (1.28) and (1.25), respectively.

Since ϵ was chosen arbitrarily, the above implies that

$$\sup_{\boldsymbol{q}\in\mathcal{Q}_I^{\text{m-d}}} \inf_{\mathbb{P}\in\mathcal{P}_I} f_I(\boldsymbol{q},\mathbb{P}) \;\le\; \inf_{\mathbb{P}\in\mathcal{P}_{I+1}} f_{I+1}(\boldsymbol{q}^{\text{sp}},\mathbb{P}),$$

and thus the claim follows. ■

We need the following auxiliary results to prove Proposition 1.5.5.

Lemma 1.7.2. *If $r\in\mathcal{R}_1$ and $\mathcal{P}$ is a Markov ambiguity set of the form* (1.6) *with $\mu\in(0,1)$, then $\mathbb{Q}^{(1)}\in\mathcal{P}$ and*

$$\mathbb{E}_{\mathbb{Q}^{(1)}}\Big[\sum_{i\in\mathcal{I}} m_i^{\text{sp}}(\tilde{\boldsymbol{v}})\Big] \;=\; \frac{I\mu-1}{I-1}.$$

Proof. Using the inequalities $0\le r\le \frac{I\mu-1}{I-1}<1$, which hold because $r\in\mathcal{R}_1$, one can show that the atoms of $\mathbb{Q}^{(1)}$ have non-negative probabilities that add up to 1. Moreover, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{(1)}}[\tilde{v}_i] \;&=\; 1\Big(1-\frac{I(\mu-1)}{(I-1)(r-1)}\Big) + 1\frac{(\mu-1)}{(I-1)(r-1)} + r(I-1)\frac{(\mu-1)}{(I-1)(r-1)} \\
&= 1-\frac{(I-1)(\mu-1)}{(I-1)(r-1)} + \frac{r(I-1)(\mu-1)}{(I-1)(r-1)} \;=\; 1+\frac{(r-1)(\mu-1)}{(r-1)} \;=\; \mu \quad \forall i\in\mathcal{I}.
\end{aligned}$$

This confirms that $\mathbb{Q}^{(1)}\in\mathcal{P}$. A direct calculation further yields

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{(1)}}\Big[\sum_{i\in\mathcal{I}} m_i^{\text{sp}}(\tilde{\boldsymbol{v}})\Big] \;&=\; 1-\frac{I(\mu-1)}{(I-1)(r-1)} + I\frac{(\mu-1)}{(I-1)(r-1)}r \;=\; 1+\frac{I(r-1)(\mu-1)}{(I-1)(r-1)} \\
&=\; \frac{(I-1)+I(\mu-1)}{(I-1)} \;=\; \frac{I\mu-1}{I-1},
\end{aligned}$$

and thus the claim follows. ■

Lemma 1.7.3. *If $r \in \mathscr{R}_1$ and $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *with $\mu \in (0,1)$, then $\boldsymbol{\sigma}^{(1)} = \frac{1}{I-1}\mathbf{e}$ and $\lambda^{(1)} = 1 - \frac{I}{I-1}$ are feasible in* (1.8) *with objective value $\frac{I\mu-1}{I-1}$.*

Proof. Select an arbitrary $\boldsymbol{v} \in \mathscr{V}^I$ and assume without loss of generality that $\boldsymbol{v} \in \mathscr{W}^i$. Due to (1.9) and the convention that in a second price auction only the winner makes a payment, the left-hand side of the semi-infinite constraint in (1.8) reduces to $m_i^{\mathrm{sp}}(\boldsymbol{v})$. Moreover, by construction of $\boldsymbol{\sigma}^{(1)}$ and $\lambda^{(1)}$, the right-hand side of the semi-infinite constraint reduces to

$$\lambda^{(1)} + \sum_{j\in\mathscr{I}} \sigma_j^{(1)} v_j \;=\; 1 - \frac{I}{I-1} + \sum_{j\in\mathscr{I}} \left(\frac{1}{I-1}\right) v_j.$$

If $v_i < r$, we then have

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) \;=\; 0 \;\geq\; \frac{Ir-1}{I-1} \;=\; 1 - \frac{I}{I-1} + I\Big(\frac{1}{I-1}\Big) r \;\geq\; 1 - \frac{I}{I-1} + \sum_{j\in\mathscr{I}} \left(\frac{1}{I-1}\right) v_j,$$

where the first inequality holds because $r \leq \frac{1}{I}$. If $v_i \geq r$, on the other hand, we have

$$\begin{aligned} m_i^{\mathrm{sp}}(\boldsymbol{v}) \;&=\; \max\{\max_{j\neq i} v_j, r\} \;\geq\; 1 - \frac{I}{I-1} + \left(\frac{1}{I-1}\right) v_i + (I-1)\Big(\frac{1}{I-1}\Big)\max\{\max_{j\neq i} v_j, r\} \\ &\geq\; 1 - \frac{I}{I-1} + \sum_{j\in\mathscr{I}} \left(\frac{1}{I-1}\right) v_j, \end{aligned}$$

where the first inequality exploits that $v_i \leq 1$. Finally, the objective value of $(\boldsymbol{\sigma}^{(1)}, \lambda^{(1)})$ in (1.8) amounts to

$$\lambda^{(1)} + \sum_{i\in\mathscr{I}} \sigma_i^{(1)} \mu \;=\; 1 - \frac{I}{I-1} + I\Big(\frac{1}{I-1}\Big)\mu \;=\; \frac{I\mu-1}{I-1},$$

and thus the claim follows. ■

Proof of Proposition 1.5.5. The distribution $\mathbb{Q}^{(1)}$ is feasible in (1.7) due to Lemma 1.7.2, and $(\boldsymbol{\sigma}^{(1)}, \lambda^{(1)})$ is feasible in (1.8) due to Lemma 1.7.3. Since the objective value of $\mathbb{Q}^{(1)}$ in (1.7) is equal to the objective value of $(\boldsymbol{\sigma}^{(1)}, \lambda^{(1)})$ in the dual problem (1.8) (see Lemmas 1.7.2 and 1.7.3), $\mathbb{Q}^{(1)}$ is optimal in (1.7) by weak duality, implying that the worst-case expected revenue amounts to $\frac{I\mu-1}{I-1}$. ■

The proof of Proposition 1.5.6 requires the following auxiliary results.

Lemma 1.7.4. *If $r \in \mathscr{R}_2$ and $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *with $\mu \in (0,1)$, then $\mathbb{Q}_\epsilon^{(2)} \in \mathscr{P}$ for every sufficiently small $\epsilon > 0$, and we have*

$$\lim_{\epsilon\downarrow 0} \mathbb{E}_{\mathbb{Q}_\epsilon^{(2)}}\Big[\sum_{i\in\mathscr{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\Big] \;=\; Ir\,\frac{\mu-r}{1-r}.$$

Proof. One can show that the atoms of $\mathbb{Q}_\epsilon^{(2)}$ have non-negative probabilities that add up to 1 for every $\epsilon \le r - \frac{I\mu-1}{I-1}$. Note that this upper bound on ϵ is strictly positive because $r \in \mathscr{R}_2$. Moreover, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_\epsilon^{(2)}}[\tilde{v}_i] &= (r-\epsilon)\Big(1-\frac{I(\mu-(r-\epsilon))}{1-(r-\epsilon)}\Big)+1\,\frac{\mu-(r-\epsilon)}{1-(r-\epsilon)}+(r-\epsilon)(I-1)\frac{\mu-(r-\epsilon)}{1-(r-\epsilon)}\\ &= (r-\epsilon)-(r-\epsilon)\frac{(\mu-(r-\epsilon))}{1-(r-\epsilon)}+\frac{\mu-(r-\epsilon)}{1-(r-\epsilon)} = (r-\epsilon)+(1-(r-\epsilon))\frac{(\mu-(r-\epsilon))}{1-(r-\epsilon)} = \mu \quad \forall i \in \mathscr{I}.\end{aligned}$$

This confirms that $\mathbb{Q}_\epsilon^{(2)} \in \mathscr{P}$ for every sufficiently small $\epsilon > 0$.

Note that the i^{th} bidder receives the good only in scenario $\boldsymbol{v} = \mathrm{e}_i + (r-\epsilon)\mathbf{e}_{-i}$, in which case he has to pay the reserve price r. Note also that all other bids are below r in this scenario. Thus, we find

$$\lim_{\epsilon\downarrow 0} \mathbb{E}_{\mathbb{Q}_\epsilon^{(2)}}\Big[\sum_{i\in\mathscr{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\Big] = \lim_{\epsilon\downarrow 0} Ir\,\frac{\mu-(r-\epsilon)}{1-(r-\epsilon)} = Ir\,\frac{\mu-r}{1-r}.$$

This observation completes the proof. ■

Lemma 1.7.5. *If $r \in \mathscr{R}_2$ and $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *with $\mu \in (0,1)$, then $\boldsymbol{\sigma}^{(2)} = \frac{r}{1-r}\mathbf{e}$ and $\lambda^{(2)} = \frac{Ir^2}{r-1}$ are feasible in problem* (1.8) *with objective value $Ir\,\frac{\mu-r}{1-r}$.*

Proof. Select an arbitrary $\boldsymbol{v} \in \mathcal{V}^I$ and assume without loss of generality that $\boldsymbol{v} \in \mathscr{W}^i$. Recall from the proof of Lemma 1.7.3 that the left-hand side of the semi-infinite constraint in (1.8) reduces to $m_i^{\mathrm{sp}}(\boldsymbol{v})$. Using the definitions of $\boldsymbol{\sigma}^{(2)}$ and $\lambda^{(2)}$, we can further rewrite the right-hand side of the semi-infinite constraint as

$$\lambda^{(2)} + \sum_{j\in\mathscr{I}} \sigma_j^{(2)} v_j = \frac{Ir^2}{r-1} + \sum_{j\in\mathscr{I}} \Big(\frac{r}{1-r}\Big) v_j.$$

If $v_i < r$, then we have

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) = 0 = \frac{Ir^2}{r-1} + I\Big(\frac{r}{1-r}\Big)r \ge \frac{Ir^2}{r-1} + \sum_{j\in\mathscr{I}} \Big(\frac{r}{1-r}\Big) v_j.$$

If $v_i \ge r$, on the other hand, note that

$$\begin{aligned} & r \le \max\{\max_{j\neq i} v_j, r\} = m_i^{\mathrm{sp}}(\boldsymbol{v})\\ \iff & (1-Ir)r \le (1-Ir)m_i^{\mathrm{sp}}(\boldsymbol{v})\\ \iff & \frac{r}{1-r}[(1-Ir)+(I-1)m_i^{\mathrm{sp}}(\boldsymbol{v})] \le m_i^{\mathrm{sp}}(\boldsymbol{v}),\end{aligned}$$

where the first equivalence holds because $r \le \frac{1}{I}$. Hence, we obtain

$$\begin{aligned} m_i^{\text{sp}}(\boldsymbol{v}) &\ge \frac{r}{1-r}[(1-Ir)+(I-1)m_i^{\text{sp}}(\boldsymbol{v})] \\ &\ge \frac{Ir^2}{r-1}+\frac{r}{1-r}v_i+(I-1)\Big(\frac{r}{1-r}\Big)\max\{\max_{j\neq i} v_j, r\} \;\ge\; \frac{Ir^2}{r-1}+\sum_{j\in\mathscr{I}}\Big(\frac{r}{1-r}\Big)v_j, \end{aligned}$$

where the second inequality holds because $v_i \le 1$ and $m_i(\boldsymbol{v}) = \max\{\max_{j\neq i} v_j, r\}$.

Finally, the objective value of $(\boldsymbol{\sigma}^{(2)}, \lambda^{(2)})$ in problem (1.8) amounts to

$$\lambda^{(2)} + \sum_{i\in\mathscr{I}} \sigma_i^{(2)}\mu = \frac{Ir^2}{r-1} + I\mu\frac{r}{1-r} = Ir\frac{\mu-r}{1-r},$$

and thus the claim follows. ■

Proof of Proposition 1.5.6. For every $\epsilon > 0$ small enough, the discrete distribution $\mathbb{Q}_\epsilon^{(2)}$ is feasible in (1.7) by Lemma 1.7.4, and $(\boldsymbol{\sigma}^{(2)}, \lambda^{(2)})$ is feasible in (1.8) by Lemma 1.7.5. Since the limiting objective value of the distributions $\mathbb{Q}_\epsilon^{(2)}$ in (1.7) for $\epsilon \downarrow 0$ coincides with the objective value of $(\boldsymbol{\sigma}^{(2)}, \lambda^{(2)})$ in the dual problem (1.8) (see Lemmas 1.7.4 and 1.7.5), we conclude via weak duality that the distributions $\mathbb{Q}_\epsilon^{(2)}$, $\epsilon \downarrow 0$, are asymptotically optimal in (1.7), implying that the worst-case expected revenue amounts to $Ir\frac{\mu-r}{1-r}$. ■

To prove Proposition 1.5.7, we will need the following results.

Lemma 1.7.6. *If $r \in \mathscr{R}_3$ and $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *with $\mu \in (0,1)$, then $\mathbb{Q}_\epsilon^{(3)} \in \mathscr{P}$ for every $\epsilon > 0$, and we have*

$$\lim_{\epsilon\downarrow 0} \mathbb{E}_{\mathbb{Q}_\epsilon^{(3)}}\Big[\sum_{i\in\mathscr{I}} m_i^{\text{sp}}(\tilde{\boldsymbol{v}})\Big] = \frac{\mu-r}{1-r}.$$

Proof. One can show that the atoms of $\mathbb{Q}_\epsilon^{(3)}$ have non-negative probabilities that add up to 1 because $r \in \mathscr{R}_3$ implies that $r \le \mu < 1$. Moreover, we have

$$\mathbb{E}_{\mathbb{Q}_\epsilon^{(3)}}[\tilde{v}_i] = 1\Big(1-\frac{1-\mu}{1-(r-\epsilon)}\Big)+(r-\epsilon)\Big(\frac{1-\mu}{1-(r-\epsilon)}\Big) = 1-\frac{(1-(r-\epsilon))(1-\mu)}{1-(r-\epsilon)} = \mu \quad \forall i \in \mathscr{I}.$$

This confirms that $\mathbb{Q}_\epsilon^{(3)} \in \mathscr{P}$ for every $\epsilon > 0$.

Note that the good is allocated only if $\boldsymbol{v} = \mathbf{e}$, in which case the winner pays an amount equal to 1, that is, the second highest bid. Therefore, we find

$$\lim_{\epsilon\downarrow 0} \mathbb{E}_{\mathbb{Q}_\epsilon^{(3)}}\Big[\sum_{i\in\mathscr{I}} m_i^{\text{sp}}(\tilde{\boldsymbol{v}})\Big] = \lim_{\epsilon\downarrow 0} 1-\frac{1-\mu}{1-(r-\epsilon)} = \lim_{\epsilon\downarrow 0}\frac{\mu-(r-\epsilon)}{1-(r-\epsilon)} = \frac{\mu-r}{1-r}.$$

This observation completes the proof. ■

Lemma 1.7.7. *If $r \in \mathscr{R}_3$ and $\mathscr{P}$ is a Markov ambiguity set of the form* (1.6) *with $\mu \in (0,1)$, then $\boldsymbol{\sigma}^{(3)} = \frac{1}{I(1-r)}\mathbf{e}$ and $\lambda^{(3)} = \frac{r}{r-1}$ are feasible in problem* (1.8) *with objective value $\frac{\mu-r}{1-r}$.*

Proof. Select an arbitrary $\boldsymbol{v} \in \mathcal{V}^I$ and assume without loss of generality that $\boldsymbol{v} \in \mathscr{W}^i$. Recall from the proof of Lemma 1.7.3 that the left-hand side of the semi-infinite constraint in (1.8) reduces to $m_i^{\text{sp}}(\boldsymbol{v})$. Using the definitions of $\boldsymbol{\sigma}^{(3)}$ and $\lambda^{(3)}$, we can rewrite the right-hand side of the semi-infinite constraint as

$$\lambda^{(3)} + \sum_{j\in\mathscr{I}} \sigma_j^{(3)} v_j \;=\; \frac{r}{r-1} + \sum_{j\in\mathscr{I}} \frac{1}{I(1-r)} v_j.$$

If $v_i < r$, then we find

$$m_i^{\text{sp}}(\boldsymbol{v}) \;=\; 0 \;=\; \frac{r}{r-1} + I\frac{1}{I(1-r)}r \;\geq\; \frac{r}{r-1} + \sum_{j\in\mathscr{I}} \frac{1}{I(1-r)} v_j.$$

If $v_i \geq r$, on the other hand, we have

$$\begin{aligned}
1 \;\geq\; \max\{\max_{j\neq i} v_j, r\} \;&=\; m_i^{\text{sp}}(\boldsymbol{v}) \\
\iff 1 - Ir \;&\leq\; (1-Ir)m_i^{\text{sp}}(\boldsymbol{v}) \\
\iff \frac{1}{I(1-r)}[1 - Ir + (I-1)m_i^{\text{sp}}(\boldsymbol{v})] \;&\leq\; m_i^{\text{sp}}(\boldsymbol{v}),
\end{aligned}$$

where the first equivalence holds because $r > \frac{1}{I}$, which implies that $(1 - Ir) < 0$. Hence, we have

$$\begin{aligned}
&m_i^{\text{sp}}(\boldsymbol{v}) \;\geq\; \frac{1}{I(1-r)}[1 - Ir + (I-1)m_i^{\text{sp}}(\boldsymbol{v})] \\
&\geq\; \frac{r}{r-1} + \frac{1}{I(1-r)}v_i + (I-1)\frac{1}{I(1-r)}\max\{\max_{j\neq i} v_j, r\} \;\geq\; \frac{r}{r-1} + \sum_{j\in\mathscr{I}} \frac{1}{I(1-r)} v_j.
\end{aligned}$$

Finally, the objective value of $(\boldsymbol{\sigma}^{(3)}, \lambda^{(3)})$ in problem (1.8) amounts to

$$\lambda^{(3)} + \sum_{i\in\mathscr{I}} \sigma_i^{(3)}\mu \;=\; \frac{r}{r-1} + \frac{I\mu}{I(1-r)} \;=\; \frac{\mu - r}{1-r},$$

and thus the claim follows. ■

Proof of Proposition 1.5.7. For every $\epsilon > 0$, the discrete distribution $\mathbb{Q}_\epsilon^{(3)}$ is feasible in (1.7) by Lemma 1.7.6, and $(\boldsymbol{\sigma}^{(3)}, \lambda^{(3)})$ is feasible in (1.8) by Lemma 1.7.7. Since the limiting objective value of the distributions $\mathbb{Q}_\epsilon^{(3)}$, $\epsilon \downarrow 0$, in (1.7) coincides with the objective value of $(\boldsymbol{\sigma}^{(3)}, \lambda^{(3)})$ in the dual problem (1.8) (see Lemmas 1.7.6 and 1.7.7), we conclude via weak duality that the distributions $\mathbb{Q}_\epsilon^{(3)}$, $\epsilon \downarrow 0$, are asymptotically optimal in (1.7), implying that the worst-case expected revenue amounts to $\frac{\mu - r}{1-r}$. ■

Proof of Theorem 1.5.2. As for (i), assume first that $\mu \in \mathscr{M}_1$. In this case $\mathscr{R}_3$ is empty. Moreover, the interval $\mathscr{R}_1$ is nonempty only if $\mu = \frac{1}{I}$, which implies that $\frac{I\mu - 1}{I-1} = 0$ and leads to a worst-case expected revenue of 0. By the definition of second price auctions, their worst-case expected revenue is at least 0 since $\boldsymbol{m}^{\mathrm{sp}}$ is non-negative. Hence, the optimal reserve price must reside within $\mathscr{R}_2$.

We know from Proposition 1.5.6 that, for $r \in \mathscr{R}_2$, the worst-case expected revenue amounts to $Ir\,\frac{\mu-r}{1-r}$. Elementary calculus shows that

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(Ir\,\frac{\mu-r}{1-r}\Big) = I - \frac{I(1-\mu)}{(1-r)^2} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big(Ir\,\frac{\mu-r}{1-r}\Big) = \frac{2(r-1)I(1-\mu)}{(1-r)^4}.$$

Thus, the worst-case expected revenue is strictly concave and maximized by $r^\star = 1 - \sqrt{1-\mu}$. Note that $r^\star$ is indeed an element of $\mathscr{R}_2$ and results in a worst-case expected revenue of $I(1-\sqrt{1-\mu})^2$.

Assume next that $\mu \in \mathscr{M}_2$. In this case, the sets $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$ are all nonempty. If $r \in \mathscr{R}_1$, by Proposition 1.5.5, the seller's worst-case expected revenue amounts to $\frac{I\mu-1}{I-1}$ irrespective of r. If $r \in \mathscr{R}_3$, on the other hand, by Proposition 1.5.7 the worst-case expected revenue is given by $\frac{\mu-r}{1-r}$, which is a decreasing function of r because $\mu < 1$. Hence, the highest possible worst-case expected revenue corresponding to any reserve price $r \in \mathscr{R}_3$ is given by $\frac{I\mu-1}{I-1}$, which is attained asymptotically as r tends to $\frac{1}{I}$, the left boundary of $\mathscr{R}_3$. If $r \in \mathscr{R}_2$, finally, by Proposition 1.5.6 the worst-case expected revenue amounts to $Ir\,\frac{\mu-r}{1-r}$. At both boundary points $r = \frac{I\mu-1}{I-1}$ and $r = \frac{1}{I}$ of $\mathscr{R}_2$, this function evaluates to $\frac{I\mu-1}{I-1}$. Inside interval $\mathscr{R}_2$ this function is concave and attains its maximum at $r^\star = 1-\sqrt{1-\mu}$, resulting in a worst-case expected revenue of $I(1-\sqrt{1-\mu})^2$. Hence, the seller obtains a worst-case expected revenue of $I(1-\sqrt{1-\mu})^2$ by imposing the optimal reserve price $r^\star = 1-\sqrt{1-\mu}$ whenever $\mu \in \mathscr{M}_1 \cup \mathscr{M}_2$.

As for (ii), recall that $\mathscr{R}_2$ is empty if $\mu \in \mathscr{M}_3$. We already know that for $r \in \mathscr{R}_3$, the worst-case expected revenue amounts to $\frac{\mu-r}{1-r}$ which is decreasing in r and attains its maximum $\frac{I\mu-1}{I-1}$ as r tends to $\frac{1}{I}$. For $r \in \mathscr{R}_1$ the worst-case expected revenue $\frac{I\mu-1}{I-1}$ does not depend on the reserve price. Hence, if $\mu \in \mathscr{M}_3$, the seller earns a worst-case expected revenue of $\frac{I\mu-1}{I-1}$ by imposing any reserve price $r \in \mathscr{R}_1$. ■

Before proving Theorem 1.5.3, we first show that $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ is feasible in problem (1.11). To this end, we need the following auxiliary result.

Lemma 1.7.8. *For any fixed* $\mu \in (0,1)$, $\sigma^\star = -(W_{-1}(-\mu I e^{-I}) + 1)^{-1}$ *is the unique solution of the* *equation*

$$\left(\frac{1+\sigma}{I\sigma}\right)e^{(I-1-\frac{1}{\sigma})} = \mu, \tag{1.30}$$

in the interval $(0, \frac{1}{I-1})$*, where* W_{-1} *denotes the lower branch of the Lambert-W function.*

Proof. Set $f(\sigma) = \frac{1+\sigma}{\sigma} e^{(I-1-\frac{1}{\sigma})}$, and note that $\lim_{\sigma\downarrow 0} f(\sigma) = 0$, which follows from L'Hôpital's rule, and that $f(\frac{1}{I-1}) = 1$. Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} f(\sigma) = \frac{1}{I\sigma^3} e^{(I-1-\frac{1}{\sigma})} > 0 \quad \forall \sigma \in \left(0, \frac{1}{I-1}\right).$$

Thus, for any $\mu \in (0,1)$ the equation $f(\sigma) = \mu$ has a unique solution in the interval $(0, \frac{1}{I-1})$.

Equation (1.30) is equivalent to

$$-\left(\frac{1+\sigma}{\sigma}\right)e^{-\frac{1+\sigma}{\sigma}} = -\mu I e^{-I} \iff \frac{1+\sigma}{\sigma} = W(-\mu I e^{-I}) \iff \sigma = -\frac{1}{W(-\mu I e^{-I})+1},$$

where the first equivalence follows from the definition of the Lambert-W function (Corless et al., 1996). As we are interested in finding a solution of (1.30) in the interval $(0, \frac{1}{I-1})$ and as the lower branch of the Lambert-W function is at most -1, we thus have that $\sigma^\star = -(W_{-1}(-\mu I e^{-I}) + 1)^{-1}$. ■

Lemma 1.7.9. *The solution* $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ *is feasible in problem* (1.11).

Proof. Note that $r \in [0,1]$ because $\sigma^\star \in [0, \frac{1}{I-1}]$. Note also that constraint (1.11c) trivially holds by the construction of $\boldsymbol{q}^\star$. Similarly, it is easy to see that $q_i^\star(\boldsymbol{v})$ is non-decreasing in v_i for every $i \in \mathscr{I}$. Thus, we only have to show that the proposed solution satisfies constraint (1.11b) and that the elements of $\boldsymbol{q}^\star(\boldsymbol{v})$ are non-negative and sum up to at most 1.

Select an arbitrary $\boldsymbol{v} \in \mathscr{V}^I$ and assume without loss of generality that $\boldsymbol{v} \in \mathscr{W}^i$ so that bidder i is the winner. We denote the second highest bid by $v_{j^\star} = \max_{j\neq i} v_j$, where $j^\star$ represents an arbitrary second highest bidder, *i.e.*, $j^\star \in \arg\max_{j\neq i} v_j$. Using the definitions of $\lambda^\star$ and the highest-bidder-lottery allocation rule $\boldsymbol{q}^\star$, we can rewrite (1.11b) in scenario $\boldsymbol{v}$ as

$$q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - \int_{v_{j^\star}}^{v_i} q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \geq \sigma^\star\Big(\sum_{j\in\mathscr{I}} v_j\Big) - \sigma^\star r. \tag{1.31}$$

In the remainder of the proof, we show that $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ satisfies (1.31), $\boldsymbol{q}^\star(\boldsymbol{v}) \geq 0$ and $\mathbf{e}^\top \boldsymbol{q}^\star(\boldsymbol{v}) \leq 1$ when scenario $\boldsymbol{v}$ satisfies the conditions in (1.12a), (1.12b) and (1.12c), respectively.

Case 1 ($v_{j^\star} \geq r$): In this case, $q_i^\star(\boldsymbol{v})$ is given by (1.12a). Using integration by parts, we can rewrite (1.31) as

$$
\begin{aligned}
& q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - xq_i^\star(x, \boldsymbol{v}_{-i})\Big|_{v_{j^\star}}^{v_i} + \int_{v_{j^\star}}^{v_i} x\partial_x q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \\
&= v_{j^\star}\left[\sigma^\star \log(1) + I\sigma^\star - \frac{\sigma^\star r}{v_{j^\star}}\right] + \sigma^\star(v_i - v_{j^\star}) \\
&= \sigma^\star(v_i + (I-1)v_{j^\star}) - \sigma^\star r \;\geq\; \sigma^\star\Big(\sum_{j\in\mathscr{I}} v_j\Big) - \sigma^\star r.
\end{aligned}
$$

The first equality holds because the allocation $q_i^\star(x, \boldsymbol{v}_{-i})$ is of the form (1.12a) for all $x \in [v_{j^\star}, v_i]$. The last inequality holds as $v_{j^\star} \geq v_j$ for all $j \neq i$.

Next, we prove that $q_j^\star(\boldsymbol{v}) \geq 0$ for all $j \in \mathscr{I}$. By construction, we have $q_j^\star(\boldsymbol{v}) = 0$ for all $j \neq i$. To prove that $q_i^\star(\boldsymbol{v}) \geq 0$, we observe that

$$
q_i^\star(\boldsymbol{v}) \;=\; \sigma^\star \log\Big(\frac{v_i}{v_{j^\star}}\Big) + I\sigma^\star - \frac{\sigma^\star r}{v_{j^\star}} \;=\; \sigma^\star \log\Big(\frac{v_i}{v_{j^\star}}\Big) + \sigma^\star\Big(\frac{Iv_{j^\star} - r}{v_{j^\star}}\Big) \;\geq\; 0,
$$

where the inequality holds because $\sigma^\star \geq 0$, $v_i \geq v_{j^\star}$, and $v_{j^\star} \geq r$.

To prove that the sum of the allocation probabilities is at most 1, we note that

$$
\begin{aligned}
\sum_{j\in\mathscr{I}} q_j^\star(\boldsymbol{v}) \;=\; q_i^\star(\boldsymbol{v}) \;&=\; \sigma^\star[\log(v_i) - \log(v_{j^\star})] + I\sigma^\star - \frac{\sigma^\star r}{v_{j^\star}} \\
&=\; \sigma^\star \log(v_i) - \sigma^\star\Big(\frac{v_{j^\star}\log(v_{j^\star}) + r}{v_{j^\star}}\Big) + I\sigma^\star \\
&\leq\; I\sigma^\star - \sigma^\star\Big(\frac{v_{j^\star}\log(v_{j^\star}) + r}{v_{j^\star}}\Big) \;\leq\; I\sigma^\star - \sigma^\star\Big(I - \frac{1}{\sigma^\star}\Big) \;=\; 1,
\end{aligned}
$$

where the first equality follows from the definition of the highest-bidder-lottery allocation rule $\boldsymbol{q}^\star$, the first inequality holds because $v_i \leq 1$, and the second inequality holds because the expression in the third line is non-increasing in $v_{j^\star} \in [r, v_i]$, while $r = e^{(I-1-\frac{1}{\sigma^\star})}$.

Case 2 ($v_i \geq r > v_{j^\star}$): In this case, $q_i^\star(\boldsymbol{v})$ is of the form (1.12b), whereby (1.31) reduces to

$$
q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - \int_{v_{j^\star}}^{r} q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x - \int_{r}^{v_i} q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \;\geq\; \sigma^\star\Big(\sum_{j\in\mathscr{I}} v_j\Big) - \sigma^\star r. \tag{1.32}
$$

Note that $q_i^\star(x, \boldsymbol{v}_{-i})$ is of the form (1.12c) for all $x \in [v_{j^\star}, r)$ and of the form (1.12b) for all $x \in [r, v_i]$. Using integration by parts and recalling that $r = e^{(I-1-\frac{1}{\sigma^\star})}$, we thus obtain

$$
\int_{v_{j^\star}}^{r} q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \;=\; xq_i^\star(x, \boldsymbol{v}_{-i})\Big|_{v_{j^\star}}^{r} - \int_{v_{j^\star}}^{r} x\partial_x q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \;=\; (I-1)\sigma^\star(r - v_{j^\star})
$$

and

$$
\begin{aligned}
q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - \int_r^{v_i} q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x &= q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - xq_i^\star(x, \boldsymbol{v}_{-i})\Big|_r^{v_i} + \int_r^{v_i} x\partial_x q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \\
&= r\Big[\sigma^\star\Big(I-1-\frac{1}{\sigma^\star}\Big)+1\Big] + \sigma^\star(v_i - r) \;=\; \sigma^\star v_i + (I-2)\sigma^\star r.
\end{aligned}
$$

Hence, the left-hand side of (1.32) is equal to

$$
\sigma^\star v_i + (I-2)\sigma^\star r - (I-1)\sigma^\star(r - v_{j^\star}) \;=\; \sigma^\star v_i + (I-1)\sigma^\star v_{j^\star} - \sigma^\star r.
$$

The inequality (1.32) then follows because $v_{j^\star} \geq v_j$ for all $j \neq i$.

To show that the allocation probabilities are non-negative and that their sum is at most 1, we first note that $q_j^\star(\boldsymbol{v}) = 0$ for all $j \neq i$. Moreover, we have

$$
q_i^\star(\boldsymbol{v}) \;=\; \sigma^\star \log(v_i) + 1 \;\geq\; \sigma^\star(I-1) \;\geq\; 0,
$$

where the first inequality holds because $v_i \geq r$ and $r = e^{(I-1-\frac{1}{\sigma^\star})}$, while the last inequality follows from Lemma 1.7.8. Finally, since $v_i \leq 1$, we obtain

$$
\sum_{j\in\mathscr{I}} q_j^\star(\boldsymbol{v}) \;=\; q_i^\star(\boldsymbol{v}) \;=\; \sigma^\star \log(v_i) + 1 \;\leq\; 1.
$$

Case 3 ($r > v_i \geq v_{j^\star}$): In this case, $q_i^\star(\boldsymbol{v})$ is given by (1.12c). Moreover, $q_i^\star(x, \boldsymbol{v}_{-i})$ is of the form (1.12c) for all $x \in [v_{j^\star}, v_i]$. Using integration by parts, we can thus rewrite the left-hand side of (1.31) as

$$
q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - xq_i^\star(x, \boldsymbol{v}_{-i})\Big|_{v_{j^\star}}^{v_i} + \int_{v_{j^\star}}^{v_i} x\partial_x q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \;=\; q_i^\star(v_{j^\star}, \boldsymbol{v}_{-i})v_{j^\star} \;=\; (I-1)\sigma^\star v_{j^\star}.
$$

We conclude that the inequality (1.31) is equivalent to

$$
(I-1)\sigma^\star v_{j^\star} \;\geq\; \sigma^\star\Big(\sum_{j\in\mathscr{I}} v_j\Big) - \sigma^\star r \iff 0 \;\geq\; \sigma^\star\Big(v_i - r - (I-1)v_{j^\star} + \sum_{j\neq i} v_j\Big),
$$

which is manifestly satisfied because $\sigma^\star \geq 0$, $v_i < r$ and $v_{j^\star} \geq v_j$ for all $j \neq i$.

As $\sigma^\star \in [0, \frac{1}{I-1}]$ by Lemma 1.7.8, it is easy to see that the allocation probabilities are non-negative and their sum is at most 1. ■

Lemma 1.7.10. *The objective value of* $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ *in problem* (1.11) *amounts to* r.

Proof. By using (1.30) and the definition of r, we find

$$\mu = \Big(\frac{1+\sigma^\star}{I\sigma^\star}\Big) e^{(I-1-\frac{1}{\sigma^\star})} = \Big(\frac{1+\sigma^\star}{I\sigma^\star}\Big) r.$$

Recalling that $\lambda^\star = -\sigma^\star r$, the objective value of $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ in (1.11) can then be expressed as

$$\lambda^\star + \sum_{j\in\mathcal{I}} \sigma^\star \mu \;=\; I\sigma^\star \Big(\frac{1+\sigma^\star}{I\sigma^\star}\Big) r - \sigma^\star r \;=\; r,$$

and thus the claim follows. ■

To prove Theorem 1.5.3, we first ignore the monotonicity condition on the allocation rule and show that $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ is an optimal solution to the relaxed problem (1.11) where $\mathcal{Q}^{\text{m-p}}$ is replaced with $\mathcal{Q}$. As $\boldsymbol{q}^\star$ happens to be ex-post monotone, we may then conclude that this solution is also optimal in (1.11).

Lemma 1.7.11. *The Lagrangian dual of problem* (1.11) *with $\mathcal{Q}$ in lieu of $\mathcal{Q}^{\text{m-p}}$ is equal to*

$$\begin{array}{lll} \displaystyle\inf_{\alpha\in\mathcal{L}_\infty(\mathcal{V},\mathbb{R}_+)} & \displaystyle\int_{[0,1]^I} \max\Big\{0, \sum_{i\in\mathcal{I}} \mathbb{1}_{\mathcal{W}^i}(\boldsymbol{v})\Big(\alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\Big)\Big\}\,\mathrm{d}\boldsymbol{v} & \\ \text{s.t.} & \displaystyle\int_{[0,1]^I} \alpha(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \;=\; 1 & \\ & \displaystyle\int_{[0,1]^I} \alpha(\boldsymbol{v})v_i\,\mathrm{d}\boldsymbol{v} \;=\; \mu & \forall i\in\mathcal{I}. \end{array} \tag{1.33}$$

Proof. The Lagrangian dual of problem (1.11) with $\mathcal{Q}$ in lieu of $\mathcal{Q}^{\text{m-p}}$ is given by

$$\begin{array}{lll} \displaystyle\inf_{\alpha,\beta\in\mathcal{L}_\infty(\mathcal{V},\mathbb{R}_+)} & \displaystyle\int_{[0,1]^I} \beta(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} & \\ \text{s.t.} & \displaystyle\alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x,\boldsymbol{v}_{-i})\,\mathrm{d}x \;\le\; \beta(\boldsymbol{v}) & \forall \boldsymbol{v}\in\mathcal{W}^i,\ \forall i\in\mathcal{I} \\ & \displaystyle\int_{[0,1]^I} \alpha(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \;=\; 1 & \\ & \displaystyle\int_{[0,1]^I} \alpha(\boldsymbol{v})v_i\,\mathrm{d}\boldsymbol{v} \;=\; \mu & \forall i\in\mathcal{I}. \end{array}$$

It is clear that β plays the role of an epigraphical variable. Indeed, for any $\boldsymbol{v}\in\mathcal{W}^i$, $\beta(\boldsymbol{v})$ will be equal to the maximum of 0 and $\alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x,\boldsymbol{v}_{-i})\,\mathrm{d}x$ at optimality. We can thus eliminate β and rewrite the above dual problem as (1.33). ■

Note that α can be viewed as the density function of some probability distribution on $[0,1]^I$ with mean μ.

Theorem 1.7.1. *The optimal objective value of problem* (1.33) *is asymptotically attained by the*

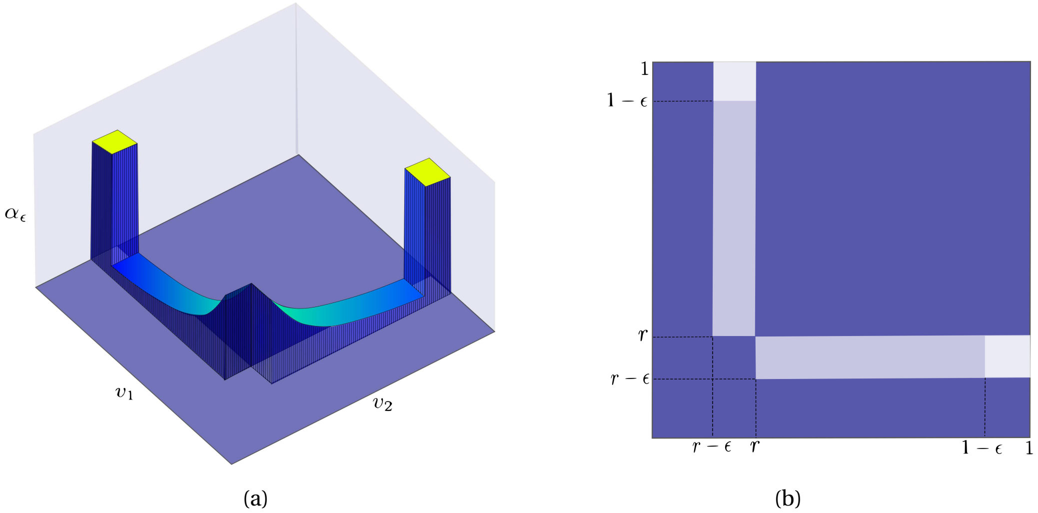


Figure 1.6 – Visualization of α_ϵ.

sequence of density functions

$$\alpha_\epsilon(\boldsymbol{v}) = \begin{cases} \rho_\epsilon \dfrac{r}{I v_i^2 \epsilon^{(I-1)}} + \delta_\epsilon \dfrac{r}{I\epsilon^I} & \text{if } \exists i \in \mathcal{I} \text{ with } \boldsymbol{v} \in \mathcal{W}^i, 1 \geq v_i \geq 1-\epsilon \text{ and } r \geq v_j \geq r-\epsilon \; \forall j \neq i, \\ \rho_\epsilon \dfrac{r}{I v_i^2 \epsilon^{(I-1)}} & \text{if } \exists i \in \mathcal{I} \text{ with } \boldsymbol{v} \in \mathcal{W}^i, 1-\epsilon > v_i \geq r \text{ and } r \geq v_j \geq r-\epsilon \; \forall j \neq i, \\ 0 & \text{otherwise,} \end{cases} \tag{1.34}$$

where

$$\delta_\epsilon = \frac{1 - r\left(\frac{1}{\sigma^\star} - (I-2)\right) - \frac{(I-1)}{2}(1-r^{-1})\epsilon}{1 - \frac{\epsilon}{2} - r\left(\frac{1}{\sigma^\star} - \frac{\epsilon}{2} - (I-2)\right)} \quad \text{and} \quad \rho_\epsilon = \frac{1-\delta_\epsilon r}{1-r}$$

for $\epsilon \downarrow 0$.

One can verify that $\delta_\epsilon > 1$ and $0 < \rho_\epsilon < 1$ for small enough $\epsilon > 0$. Figure 1.6 visualizes α_ϵ.

We prove Theorem 1.7.1 together with Theorem 1.5.3. The proof relies on the following auxiliary results.

Lemma 1.7.12. *The function α_ϵ defined in* (1.34) *is feasible in* (1.33) *for every $\epsilon > 0$ small enough.*

Proof. Note first that δ_ϵ and ρ_ϵ are positive for $\epsilon > 0$ small enough because $r \in [0,1]$. Thus, we have $\alpha_\epsilon(\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in [0,1]^I$. It remains to be shown that α_ϵ satisfies the normalization and mean constraints in (1.33).

As for the normalization constraint, we have

$$\int_{[0,1]^I} \alpha_\epsilon(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \;=\; \sum_{i\in\mathscr{I}} \int_{\mathscr{W}^i} \alpha_\epsilon(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \;=\; \sum_{i\in\mathscr{I}} \int_r^1 \int_{[r-\epsilon,r]^{I-1}} \alpha_\epsilon(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}_{-i}\,\mathrm{d}v_i \;=\; \sum_{i\in\mathscr{I}} \int_r^1 \epsilon^{(I-1)} \alpha_\epsilon(\boldsymbol{v})\,\mathrm{d}v_i$$
$$= \sum_{i\in\mathscr{I}} \Big[\int_r^1 \rho_\epsilon \frac{r}{I v_i^2}\,\mathrm{d}v_i + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I\epsilon}\,\mathrm{d}v_i \Big] \;=\; \int_r^1 \rho_\epsilon \frac{r}{v_1^2}\,\mathrm{d}v_1 + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{\epsilon}\,\mathrm{d}v_1. \tag{1.35}$$

The second equality holds because, for $\boldsymbol{v} \in \mathscr{W}^i$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_i \in [r,1]$ and $\boldsymbol{v}_{-i} \in [r-\epsilon, r]^{(I-1)}$, while the third equality holds because $\alpha(\boldsymbol{v})$ is constant in $\boldsymbol{v}_{-i}$ as long as $\boldsymbol{v} \in \mathscr{W}^i$. The last equality exploits the permutation symmetry of α_ϵ. By explicitly calculating the integrals, (1.35) simplifies to

$$I\Big(\rho_\epsilon \frac{(1-r)}{I} + \delta_\epsilon \frac{r}{I}\Big) \;=\; \frac{1-\delta_\epsilon r}{1-r}(1-r) + \delta_\epsilon r \;=\; 1,$$

where the first equality follows from the definition of ρ_ϵ.

Next, we verify that α_ϵ satisfies the mean constraint. For an arbitrary $i \in \mathscr{I}$, we have

$$\begin{aligned}
\int_{[0,1]^I} \alpha_\epsilon(\boldsymbol{v})\, v_i\,\mathrm{d}\boldsymbol{v} \;&=\; \sum_{j\in\mathscr{I}} \int_{\mathscr{W}^j} \alpha_\epsilon(\boldsymbol{v})\, v_i\,\mathrm{d}\boldsymbol{v} \\
&= \int_r^1 \int_{[r-\epsilon,r]^{I-1}} \alpha_\epsilon(\boldsymbol{v})\, v_i\,\mathrm{d}\boldsymbol{v}_{-i}\,\mathrm{d}v_i + \sum_{j\neq i} \int_r^1 \int_{[r-\epsilon,r]^{I-1}} \alpha_\epsilon(\boldsymbol{v})\, v_i\,\mathrm{d}\boldsymbol{v}_{-j}\,\mathrm{d}v_j \\
&= \int_r^1 \rho_\epsilon \frac{r}{I v_i}\,\mathrm{d}v_i + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I\epsilon} v_i\,\mathrm{d}v_i \\
&\quad + \sum_{j\neq i} \left[\int_r^1 \int_{(r-\epsilon)}^r \rho_\epsilon \frac{r}{I v_j^2 \epsilon} v_i\,\mathrm{d}v_i\,\mathrm{d}v_j + \int_{1-\epsilon}^1 \int_{(r-\epsilon)}^r \delta_\epsilon \frac{r}{I\epsilon^2} v_i\,\mathrm{d}v_i\,\mathrm{d}v_j \right].
\end{aligned} \tag{1.36}$$

The second equality holds because, for $\boldsymbol{v} \in \mathscr{W}^j$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_j \in [r,1]$ and $\boldsymbol{v}_{-j} \in [r-\epsilon, r]^{(I-1)}$, while the last equality holds because $\alpha_\epsilon(\boldsymbol{v})$ is constant in $\boldsymbol{v}_{-j}$ as long as $\boldsymbol{v} \in \mathscr{W}^j$.

An explicit calculation yields

$$\int_r^1 \rho_\epsilon \frac{r}{I v_i}\,\mathrm{d}v_i + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I\epsilon} v_i\,\mathrm{d}v_i \;=\; \rho_\epsilon \frac{r}{I}\Big(\frac{1}{\sigma^\star} - I + 1\Big) + \delta_\epsilon \frac{r}{I}\Big(1 - \frac{\epsilon}{2}\Big), \tag{1.37a}$$

where we use the relation $\log(r) = I - 1 - \frac{1}{\sigma^\star}$, which follows from the definition of r. Similarly, for an arbitrary $j \neq i$, we find

$$\int_r^1 \int_{(r-\epsilon)}^r \rho_\epsilon \frac{r}{I v_j^2 \epsilon} v_i\,\mathrm{d}v_i\,\mathrm{d}v_j \;=\; \int_r^1 \rho_\epsilon \frac{r}{I v_j^2}\Big(r - \frac{\epsilon}{2}\Big)\mathrm{d}v_j \;=\; \rho_\epsilon \frac{(1-r)}{I}\Big(r - \frac{\epsilon}{2}\Big) \tag{1.37b}$$

and

$$\int_{1-\epsilon}^1 \int_{(r-\epsilon)}^r \delta_\epsilon \frac{r}{I\epsilon^2} v_i\,\mathrm{d}v_i\,\mathrm{d}v_j \;=\; \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I\epsilon}\Big(r - \frac{\epsilon}{2}\Big)\mathrm{d}v_j \;=\; \delta_\epsilon \frac{r}{I}\Big(r - \frac{\epsilon}{2}\Big). \tag{1.37c}$$

Substituting (1.37) into (1.36) and using the permutation symmetry of α_ϵ, we obtain

$$
\begin{aligned}
&\rho_\epsilon \frac{r}{I}\Big(\frac{1}{\sigma^\star} - I + 1\Big) + \delta_\epsilon \frac{r}{2I}(2-\epsilon) + (I-1)\left[\rho_\epsilon \frac{(1-r)}{I}\Big(r - \frac{\epsilon}{2}\Big) + \delta_\epsilon \frac{r}{I}\Big(r - \frac{\epsilon}{2}\Big)\right] \\
&= \frac{1}{I}\rho_\epsilon\Big(r\Big(\frac{1}{\sigma^\star} - (I-1)r + (I-1)\frac{\epsilon}{2}\Big) - (I-1)\frac{\epsilon}{2}\Big) + \frac{1}{I}\delta_\epsilon r\Big(1 - \frac{I\epsilon}{2} + (I-1)r\Big) \\
&= \frac{1}{I}\left(\frac{1}{1-r}\right)\Big(r\Big(\frac{1}{\sigma^\star} - (I-1)r + (I-1)\frac{\epsilon}{2}\Big) - (I-1)\frac{\epsilon}{2}\Big) + \frac{1}{I}\delta_\epsilon r\left(\frac{1}{1-r}\right)\Big(1 - \frac{\epsilon}{2} - r\Big(2 - I - \frac{\epsilon}{2} + \frac{1}{\sigma^\star}\Big)\Big) \\
&= \frac{1}{I}\left(\frac{1}{1-r}\right)\Big(r\Big(\frac{1}{\sigma^\star} - (I-1)r + (I-1)\frac{\epsilon}{2}\Big) - (I-1)\frac{\epsilon}{2}\Big) \\
&\quad + \frac{1}{I}\Big(1 - r\Big(\frac{1}{\sigma^\star} - (I-2)\Big) - \frac{(I-1)}{2}(1 - r^{-1})\epsilon\Big) r\left(\frac{1}{1-r}\right) \\
&= \frac{1}{I}\left(\frac{1}{1-r}\right)\Big[r\Big(\frac{1}{\sigma^\star} + 1 - r\Big(\frac{1}{\sigma^\star} + 1\Big)\Big)\Big] \\
&= \frac{1}{I} r\left(\frac{1}{\sigma^\star} + 1\right) = \mu.
\end{aligned}
$$

Here, the first equality follows from grouping terms that involve ρ_ϵ and terms that involve δ_ϵ. The second equality follows from replacing ρ_ϵ with its definition and rearranging terms. Similarly, the fourth equality follows from replacing δ_ϵ with its definition. The remaining reformulations are based on elementary algebra. ■

Lemma 1.7.13. *As ϵ tends to zero, the objective value of α_ϵ in* (1.33) *converges to r.*

Proof. Throughout the proof we assume that $\epsilon < \frac{1}{2}$. Substituting α_ϵ into the objective function of (1.33) yields

$$
\begin{aligned}
&\int_{[0,1]^I} \max\Big\{0, \sum_{i\in\mathscr{I}} \mathbb{1}_{\mathscr{W}^i}(\boldsymbol{v})\Big(\alpha_\epsilon(\boldsymbol{v})v_i - \int_{\boldsymbol{v}_i}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\Big)\Big\}\,\mathrm{d}\boldsymbol{v} \\
&= \sum_{i\in\mathscr{I}} \int_{\mathscr{W}^i} \max\Big\{0, \alpha_\epsilon(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v} \\
&= I\int_r^1 \int_{[r-\epsilon,r]^{I-1}} \max\Big\{0, \alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1.
\end{aligned}
\tag{1.38}
$$

Here, the first equality is obtained by partitioning the integration domain into the subsets $\mathscr{W}^i$, $i \in \mathscr{I}$. The second equality follows from symmetry and because, for $\boldsymbol{v} \in \mathscr{W}^i$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_i \in [r,1]$ and $\boldsymbol{v}_{-i} \in [r-\epsilon, r]^{(I-1)}$. We can decompose the integral in (1.38) into the two terms

$$
\begin{aligned}
&\int_r^{1-\epsilon} \int_{[r-\epsilon,r]^{I-1}} \max\Big\{0, \alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&+ \int_{1-\epsilon}^1 \int_{[r-\epsilon,r]^{I-1}} \max\Big\{0, \alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1,
\end{aligned}
$$

which we investigate separately below. The first integral reduces to

$$
\begin{aligned}
&\int_r^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \alpha_\epsilon(\boldsymbol{v})\, v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\right\} \mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&= \int_r^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \rho_\epsilon \frac{r}{I\, v_1\, \epsilon^{(I-1)}} - \int_{v_1}^1 \rho_\epsilon \frac{r}{I\, x^2\, \epsilon^{(I-1)}}\,\mathrm{d}x - \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I\epsilon^I}\,\mathrm{d}x\right\} \mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&= \int_r^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \rho_\epsilon \frac{r}{I\epsilon^{(I-1)}} - \delta_\epsilon \frac{r}{I\epsilon^{(I-1)}}\right\} \mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \;=\; 0,
\end{aligned}
$$

where the last equality holds because $\rho_\epsilon \le 1 \le \delta_\epsilon$. Similarly, the second integral can be rewritten as

$$
\begin{aligned}
&\int_{1-\epsilon}^1\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \alpha_\epsilon(\boldsymbol{v})\, v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\right\} \mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&\quad= \int_{1-\epsilon}^1\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \rho_\epsilon \frac{r}{I\, v_1\, \epsilon^{(I-1)}} + \delta_\epsilon \frac{r}{I\epsilon^I}\, v_1 \right.\\
&\quad\left. - \int_{v_1}^1 \left(\rho_\epsilon \frac{r}{I\, x^2\, \epsilon^{(I-1)}} + \delta_\epsilon \frac{r}{I\epsilon^I}\right)\mathrm{d}x\right\} \mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1.
\end{aligned} \tag{1.39}
$$

The second argument of the max function in (1.39) is equal to

$$
\begin{aligned}
&\rho_\epsilon \frac{r}{I\, v_1\, \epsilon^{(I-1)}} + \delta_\epsilon \frac{r}{I\epsilon^I}\, v_1 + \rho_\epsilon \frac{r}{I\, x \epsilon^{(I-1)}}\Big|_{v_1}^1 - \delta_\epsilon \frac{r}{I\epsilon^I}\, x\,\Big|_{v_1}^1 \\
&= 2\delta_\epsilon \frac{r}{I\epsilon^I}\, v_1 + \rho_\epsilon \frac{r}{I\epsilon^{(I-1)}} - \delta_\epsilon \frac{r}{I\epsilon^I} \;=\; \frac{r}{I\epsilon^I}\left(\delta_\epsilon(2v_1-1) + \rho_\epsilon \epsilon\right).
\end{aligned}
$$

Note that the last expression is non-negative for all $v_1 \in [1-\epsilon, 1]$ because ρ_ϵ and δ_ϵ are non-negative and because $\epsilon < \frac{1}{2}$. In summary, (1.39) thus reduces to

$$
\begin{aligned}
&\int_{1-\epsilon}^1\int_{[r-\epsilon,r]^{I-1}} \frac{r}{I\epsilon^I}\left(\delta_\epsilon(2v_1-1) + \rho_\epsilon\epsilon\right)\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \;=\; \int_{1-\epsilon}^1 \frac{r}{I\epsilon}\left(\delta_\epsilon(2v_1-1) + \rho_\epsilon\epsilon\right)\mathrm{d}v_1 \\
&= \frac{r}{I\epsilon}\delta_\epsilon(v_1^2 - v_1)\Big|_{1-\epsilon}^1 + \frac{r}{I}\rho_\epsilon v_1\Big|_{1-\epsilon}^1 \;=\; \frac{r}{I}\delta_\epsilon(1-\epsilon) + \frac{r}{I}\rho_\epsilon\epsilon.
\end{aligned}
$$

Therefore, the asymptotic objective value of α_ϵ for small ϵ is given by

$$
\lim_{\epsilon\downarrow 0} r\,\delta_\epsilon(1-\epsilon) + r\,\rho_\epsilon\epsilon \;=\; r
$$

because both δ_ϵ and ρ_ϵ converge to 1 as ϵ tends to 0. ■

We are now ready to prove Theorems 1.5.3 and 1.7.1.

Proof of Theorems 1.5.3 and 1.7.1. By Lemmas 1.7.9 and 1.7.10, $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ is feasible in (1.11) with the objective value r. By Lemmas 1.7.12 and 1.7.13, on the other hand, α_ϵ is feasible in problem (1.33) and asymptotically attains the objective value r for $\epsilon \downarrow 0$. As (1.33) is the dual of a relaxation of (1.11) (obtained by replacing $\mathcal{Q}^{\text{m-p}}$ with $\mathcal{Q}$), it is a restriction of the dual of

(1.11). Thus, α_ϵ is a feasible solution in the dual of (1.11) that certifies via weak duality that $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ is optimal in (1.11). The corresponding worst-case expected revenue amounts to $r = e^{(I-1-\frac{1}{\sigma^\star})}$. ■

Proof of Proposition 1.5.9. For the purpose of this proof we let $\sigma^\star(I) = -\frac{1}{W_{-1}(-\mu I e^{-I})+1}$ denote the value of $\sigma^\star$ from Theorem 1.5.3 for a fixed number of bidders I. We first show that $I\sigma^\star(I)$ converges to 1 as I tends to infinity. Since $-\mu I e^{-I}$ drops to 0 as I grows, we obtain

$$\begin{aligned} W_{-1}(-\mu I e^{-I}) &= \log(\mu I e^{-I}) - \log(-\log(\mu I e^{-I})) + o(1) \\ &= -I + \log(\mu I) - \log(-\log(\mu I) + I) + o(1), \end{aligned} \tag{1.40}$$

where the first equality follows from a well-known asymptotic expansion of the Lambert-W function (see Corless et al. 1996). Thus, we have

$$\lim_{I\to\infty} I\sigma^\star(I) = \lim_{I\to\infty} \frac{I}{I - \log(\mu I) + \log(-\log(\mu I) + I) + o(1)} = 1,$$

which implies that $\sigma^\star(I)$ converges to 0 as I tends to infinity. By (1.12a)-(1.12d) and (1.13), it is immediate that the optimal highest-bidder-lottery $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ converges uniformly to the second price auction without reserve price. ■

To prove Theorem 1.5.4, we first derive an upper bound on the supremum of problem ($\mathcal{MMDP}$). Note that ($\mathcal{MMDP}$) is equivalent to problem (1.11) without the constraint (1.11c) that restricts attention to highest-bidder-lotteries. An upper bound on ($\mathcal{MMDP}$) is thus obtained by solving problem (1.11) without the constraint (1.11c) and by ignoring the monotonicity condition on $\boldsymbol{q}$, which is tantamount to relaxing $\mathcal{Q}^{\text{m-p}}$ to $\mathcal{Q}$. By weak duality, the dual of this relaxation also provides an upper bound on ($\mathcal{MMDP}$).

Lemma 1.7.14. *The Lagrangian dual of problem* (1.11) *without the constraint* (1.11c) *and with* $\mathcal{Q}$ *in lieu of* $\mathcal{Q}^{\text{m-p}}$ *is equivalent to*

$$\begin{aligned} \inf_{\alpha\in\mathcal{L}_\infty(\mathcal{V},\mathbb{R}_+)} \quad & \int_{[0,1]^I} \max\left\{0, \max_{i\in\mathcal{I}}\left(\alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v} \\ \text{s.t.} \quad & \int_{[0,1]^I} \alpha(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = 1 \\ & \int_{[0,1]^I} \alpha(\boldsymbol{v})v_i\,\mathrm{d}\boldsymbol{v} = \mu \qquad \forall i\in\mathcal{I}. \end{aligned} \tag{1.41}$$

Proof. The proof widely parallels that of Lemma 1.7.11. Details are omitted for brevity. ■

The feasible set of (1.41) is equivalent to that of (1.33). Thus, by Lemma 1.7.12, α_ϵ defined in (1.34) is feasible in (1.41). We can now use the objective value of α_ϵ in (1.41) as an upper bound on the supremum of ($\mathcal{MMDP}$).

Lemma 1.7.15. *The objective value of problem ($\mathcal{MMDP}$) is bounded above by $r(2-r)$, where r denotes the worst-case expected revenue of the optimal highest-bidder-lottery as defined in Theorem 1.5.3.*

Proof. By Lemma 1.7.12 and Lemma 1.7.14, α_ϵ defined in (1.34) is feasible in (1.41) and its objective value in (1.41) provides an upper bound on the supremum of problem ($\mathcal{MMDP}$). Throughout the proof we assume that $\epsilon \leq \frac{1-\sqrt{r}}{2}$. Note that $r \leq \mu$ by Proposition 1.5.3 and because the worst-case expected full surplus provides an upper bound on the optimal objective value of ($\mathcal{MMDP}$). Recall also that $\mu < 1$ by assumption. This implies that $\frac{1-\sqrt{r}}{2} > 0$. Substituting α_ϵ into the objective function of (1.41) yields

$$\begin{aligned}
&\int_{[0,1]^I} \max\left\{0, \max_{j\in\mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{\boldsymbol{v}_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}\\
&= \sum_{i\in\mathcal{I}}\int_{\mathcal{W}^i} \max\left\{0, \max_{j\in\mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}\\
&= I\int_r^1\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \max_{j\in\mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1.
\end{aligned} \tag{1.42}$$

Here, the first equality is obtained by partitioning the integration domain into the subsets $\mathcal{W}^i$, $i\in\mathcal{I}$. The second equality follows from symmetry and because, for $\boldsymbol{v}\in\mathcal{W}^i$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_i\in[r,1]$ and $\boldsymbol{v}_{-i}\in[r-\epsilon,r]^{(I-1)}$. We can decompose the integral in (1.42) into the two terms

$$\begin{aligned}
&\int_r^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \max_{j\in\mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1\\
&+\int_{1-\epsilon}^1\int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \max_{j\in\mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1,
\end{aligned} \tag{1.43}$$

which we will investigate separately. We first consider the first integral. To this end, select an arbitrary $\boldsymbol{v}\in\mathcal{W}^1$ such that $v_1\in[r,1-\epsilon)$ and $v_j\in[r-\epsilon,r]$ for all $j\neq 1$. We have

$$\begin{aligned}
\alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x &= \rho_\epsilon\frac{r}{Iv_1\epsilon^{(I-1)}} - \int_{v_1}^1 \rho_\epsilon\frac{r}{Ix^2\epsilon^{(I-1)}}\,\mathrm{d}x - \int_{1-\epsilon}^1 \delta_\epsilon\frac{r}{I\epsilon^I}\,\mathrm{d}x\\
&= \rho_\epsilon\frac{r}{I\epsilon^{(I-1)}} - \delta_\epsilon\frac{r}{I\epsilon^{(I-1)}} \;\leq\; 0,
\end{aligned}$$

where the last equality holds because $\rho_\epsilon\leq\delta_\epsilon$. For any $j\neq 1$, we have

$$\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x \;\leq\; \alpha_\epsilon(\boldsymbol{v})v_j \;\leq\; \alpha_\epsilon(\boldsymbol{v})r \;=\; \rho_\epsilon\frac{r^2}{Iv_1^2\epsilon^{(I-1)}}.$$

In summary, for the first integral in (1.43), we have

$$\begin{aligned}
&\int_r^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}} \max\Big\{0, \max_{j\in\mathcal{I}}\Big(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\Big)\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&\leq \int_r^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}} \max\Big\{0, \rho_\epsilon\frac{r^2}{Iv_1^2\epsilon^{(I-1)}}\Big)\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&= \int_r^{1-\epsilon}\rho_\epsilon\frac{r^2}{Iv_1^2}\,\mathrm{d}v_1 \;=\; -\rho_\epsilon\frac{r^2}{Iv_1}\Big|_r^{1-\epsilon} \;=\; \rho_\epsilon\frac{r(1-\epsilon-r)}{I(1-\epsilon)}.
\end{aligned}$$

We now investigate the second integral in (1.43). To this end, select an arbitrary $\boldsymbol{v}\in\mathcal{W}^1$ such that $v_1 \in [1-\epsilon,1]$ and $v_j\in[r-\epsilon,r]$ for all $j\neq 1$. We have

$$\begin{aligned}
\alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x \quad &= \rho_\epsilon\frac{r}{I\,v_1\,\epsilon^{(I-1)}} + \delta_\epsilon\frac{r}{I\epsilon^I}v_1 - \int_{v_1}^1\Big(\rho_\epsilon\frac{r}{I\,x^2\,\epsilon^{(I-1)}} + \delta_\epsilon\frac{r}{I\epsilon^I}\Big)\,\mathrm{d}x \\
&= \rho_\epsilon\frac{r}{I\,v_1\,\epsilon^{(I-1)}} + \delta_\epsilon\frac{r}{I\epsilon^I}v_1 + \rho_\epsilon\frac{r}{I\,x\epsilon^{(I-1)}}\Big|_{v_1}^1 - \delta_\epsilon\frac{r}{I\epsilon^I}x\Big|_{v_1}^1 \\
&= 2\delta_\epsilon\frac{r}{I\epsilon^I}v_1 + \rho_\epsilon\frac{r}{I\epsilon^{(I-1)}} - \delta_\epsilon\frac{r}{I\epsilon^I} \;=\; \frac{r}{I\epsilon^I}\big(\delta_\epsilon(2v_1-1)+\rho_\epsilon\epsilon\big).
\end{aligned}$$

Similarly, for any $j\neq 1$, we have

$$\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1\alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x \;\leq\; \alpha_\epsilon(\boldsymbol{v})v_j \;\leq\; \alpha_\epsilon(\boldsymbol{v})r \;=\; \frac{r^2}{I\epsilon^{(I-1)}}\Big(\rho_\epsilon\frac{1}{v_1^2}+\delta_\epsilon\frac{1}{\epsilon}\Big).$$

It is easy to show that $\frac{r}{I\epsilon^I}\big(\delta_\epsilon(2v_1-1)+\rho_\epsilon\epsilon\big) \geq \frac{r^2}{I\epsilon^{(I-1)}}\Big(\rho_\epsilon\frac{1}{v_1^2}+\delta_\epsilon\frac{1}{\epsilon}\Big)$ as the coefficients of δ_ϵ and ρ_ϵ on the left-hand side are greater than or equal to the respective coefficients on the right-hand side, which follows from $\epsilon\leq\frac{1-\sqrt{r}}{2}$. Thus, the second integral in (1.43) simplifies to

$$\begin{aligned}
&\int_{1-\epsilon}^1\int_{[r-\epsilon,r]^{I-1}} \max\Big\{0, \max_{j\in\mathcal{I}}\Big(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\Big)\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&= \int_{1-\epsilon}^1\int_{[r-\epsilon,r]^{I-1}} \max\Big\{0, \alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \\
&= \int_{1-\epsilon}^1\int_{[r-\epsilon,r]^{I-1}} \frac{r}{I\epsilon^I}\big(\delta_\epsilon(2v_1-1)+\rho_\epsilon\epsilon\big)\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \;=\; \int_{1-\epsilon}^1\frac{r}{I\epsilon}\big(\delta_\epsilon(2v_1-1)+\rho_\epsilon\epsilon\big)\,\mathrm{d}v_1 \\
&= \frac{r}{I\epsilon}\delta_\epsilon(v_1^2-v_1)\Big|_{1-\epsilon}^1 + \frac{r}{I}\rho_\epsilon v_1\Big|_{1-\epsilon}^1 = \frac{r}{I}\delta_\epsilon(1-\epsilon)+\frac{r}{I}\rho_\epsilon\epsilon.
\end{aligned}$$

As both δ_ϵ and ρ_ϵ converge to 1 when ϵ tends to 0, the asymptotic objective value of α_ϵ for small ϵ is

$$\lim_{\epsilon\downarrow 0}\rho_\epsilon\frac{r(1-\epsilon-r)}{(1-\epsilon)} + r\,\delta_\epsilon(1-\epsilon)+r\rho_\epsilon\epsilon \;=\; r(2-r).$$

Moreover, as (1.41) is the dual of a relaxation of ($\mathcal{MMDP}$), we conclude via weak duality that the optimal value of ($\mathcal{MMDP}$) is bounded above by $r(2-r)$. ■

We are now ready to prove Theorem 1.5.4.

Proof of Theorem 1.5.4. By Lemma 1.7.15, the supremum of problem ($\mathcal{MMDP}$) is bounded above by $r(2-r)$, where r denotes the worst-case expected revenue of the optimal highest-bidder-lottery. Thus, the ratio of worst-case expected revenues of the optimal highest-bidder-lottery and the unknown optimal mechanism is at least $\frac{r}{r(2-r)} = \frac{1}{2-r} \geq \frac{1}{2}$. ■

Proof of Theorem 1.5.5. Fix a number of bidders $I \in \mathbb{N}$. By Theorem 1.5.2, the worst-case expected revenue of the best second price auction amounts to $I(1-\sqrt{1-\mu})^2$ for any $\mu \in \mathcal{M}_1 = (0, \frac{1}{I}]$. By Theorem 1.5.3, we also know that the worst-case expected revenue of the optimal highest-bidder-lottery equals $e^{(I-1-\frac{1}{\sigma^\star})}$, where $\sigma^\star = -(W_{-1}(-\mu I e^{-I})+1)^{-1}$.

We will show that, as μ approaches zero, the ratio of worst-case expected revenues of the best second price auction and the optimal highest-bidder-lottery becomes arbitrarily small. Indeed, we have

$$\lim_{\mu\downarrow 0} \frac{I(1-\sqrt{1-\mu})^2}{e^{(I-1-\frac{1}{\sigma^\star})}} = \lim_{\mu\downarrow 0} \frac{I(1-\sqrt{1-\mu})^2}{e^{(\log(\mu I)-\log(-\log(\mu I)+I)+o(1))}} = \lim_{\mu\downarrow 0} \frac{(1-\sqrt{1-\mu})^2(I-\log(\mu I))}{\mu e^{o(1)}},$$

where the first equality follows from the definition of $\sigma^\star$ and the asymptotic expansion (1.40) of the Lambert-W function. As $\lim_{\mu\downarrow 0} e^{o(1)} = 1$, the last expression equals

$$\lim_{\mu\downarrow 0} \frac{(1-\sqrt{1-\mu})^2}{\frac{\mu}{(I-\log(\mu I))}} = \lim_{\mu\downarrow 0} \frac{\frac{1-\sqrt{1-\mu}}{\sqrt{1-\mu}}}{\frac{I+1-\log(\mu I)}{(I-\log(\mu I))^2}} = \lim_{\mu\downarrow 0} \frac{1}{2(1-\mu)^{(3/2)}} \frac{\mu(I-\log(\mu I))^3}{(I+2-\log(\mu I))} = 0,$$

where the first and the second equalities both follow from L'Hôpital's rule. The last equality holds because $\lim_{\mu\downarrow 0} \mu(I-\log(\mu I))^3 = 0$, which can also be derived using L'Hôpital's rule. Thus the claim follows. ■

Proof of Proposition 1.5.10. By Theorem 1.5.3, the worst-case expected revenue of the optimal highest-bidder-lottery amounts to

$$e^{(I-1-\frac{1}{\sigma^\star(I)})} = e^{(I+W_{-1}(-\mu I e^{-I}))}.$$

Moreover, as $\mu \in (0,1)$, there exists I_μ such that $\mu \in \mathcal{M}_3$ for all $I \geq I_\mu$. By Theorem 1.5.2, the worst-case expected revenue generated by the best second price auction thus amounts to $\frac{I\mu-1}{I-1}$

for all $I \geq I_\mu$. This implies that

$$\begin{aligned}\Delta(I) &= \min\left\{\Delta \in \mathbb{N} : \frac{(I+\Delta)\mu - 1}{(I+\Delta)-1} \geq e^{(I+W_{-1}(-\mu I e^{-I}))}\right\} \\ &= \left\lceil \frac{1-e^{(I+W_{-1}(-\mu I e^{-I}))} - I(\mu - e^{(I+W_{-1}(-\mu I e^{-I}))})}{\mu - e^{(I+W_{-1}(-\mu I e^{-I}))}} \right\rceil\end{aligned}$$

for all $I \geq I_\mu$. Assume now that there exists an upper bound $\overline{\Delta} \in \mathbb{N}$ on $\Delta(I)$ for all $I \in \mathbb{N}$. Any such $\overline{\Delta}$ must satisfy

$$\overline{\Delta} \geq \lim_{I\to\infty} \frac{1-e^{(I+W_{-1}(-\mu I e^{-I}))} - I(\mu - e^{(I+W_{-1}(-\mu I e^{-I}))})}{\mu - e^{(I+W_{-1}(-\mu I e^{-I}))}}. \tag{1.44}$$

As the best second price auction is an instance of a highest-bidder-lottery, we have

$$\frac{I\mu - 1}{I-1} \leq e^{(I+W_{-1}(-\mu I e^{-I}))} \leq \mu \tag{1.45}$$

for all $I \geq I_\mu$, which implies that $e^{(I+W_{-1}(-\mu I e^{-I}))}$ converges from below to μ as I grows. Next, we show that $\lim_{I\to\infty} I(\mu - e^{(I+W_{-1}(-\mu I e^{-I}))}) < 1-\mu$, which implies that the limit in (1.44) evaluates to infinity and that there cannot exist any uniform upper bound $\overline{\Delta}$ on $\Delta(I)$. Specifically, we have

$$\begin{aligned}\lim_{I\to\infty} I(\mu - e^{(I+W_{-1}(-\mu I e^{-I}))}) &= \lim_{\epsilon\downarrow 0} \frac{\mu - e^{(\frac{1}{\epsilon}+W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}))}}{\epsilon} \\ &= \lim_{\epsilon\downarrow 0}\left(\frac{1}{\epsilon^2} + \frac{(\epsilon-1)W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})}{\epsilon^2(W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})+1)}\right) e^{(\epsilon^{-1}+W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}))} \\ &\leq \lim_{\epsilon\downarrow 0}\left(\frac{1}{\epsilon^2} + \frac{(\epsilon-1)W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})}{\epsilon^2(W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})+1)}\right)\mu,\end{aligned}$$

where the second equality holds by L'Hôpital's rule and the analytical formula for the derivative of the Lambert-W function (see Corless et al. 1996), while the inequality follows from (1.45). As $-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}$ drops to 0 when ϵ tends to 0, we have

$$\begin{aligned}W_{-1}(-\tfrac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}) &= \log(\tfrac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}) - \log(-\log(\tfrac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})) + o(1) \\ &= -\frac{1}{\epsilon} + \log(\tfrac{\mu}{\epsilon}) - \log(-\log(\tfrac{\mu}{\epsilon}) + \tfrac{1}{\epsilon}) + o(1),\end{aligned} \tag{1.46}$$

where the first equality follows from a well-known asymptotic expansion of the Lambert-W

function (see Corless et al. 1996). We thus have

$$\lim_{\epsilon\downarrow 0}\left(\frac{1}{\epsilon^2}+\frac{(\epsilon-1)W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})}{\epsilon^2(W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})+1)}\right)\mu = \lim_{\epsilon\downarrow 0}(\log(-\log(\tfrac{\mu}{\epsilon})+\tfrac{1}{\epsilon}))-\log(\tfrac{\mu}{\epsilon}))\mu$$
$$= \lim_{\epsilon\downarrow 0}\log\left(\frac{(\frac{1}{\epsilon}-\log(\frac{\mu}{\epsilon}))\epsilon}{\mu}\right)\mu = \lim_{I\to\infty}\log\left(\frac{1}{\mu}-\epsilon\frac{\log(\frac{\mu}{\epsilon})}{\mu}\right)\mu = \log\left(\frac{1}{\mu}\right)\mu < 1-\mu,$$

where the first equality follows from (1.46) and elementary rearrangements, while the last inequality holds because $\mu\in(0,1)$. This implies that $\lim_{I\to\infty} I(\mu-e^{(I+W_{-1}(-\mu I e^{-I}))})<1-\mu$, and thus the claim follows. ■

2 Robust Multidimensional Pricing: Separation without Regret

We study a robust monopoly pricing problem with a minimax regret objective, where a seller endeavors to sell multiple goods to a single buyer, only knowing that the buyer's values for the goods range over a rectangular uncertainty set. We interpret this pricing problem as a zero-sum game between the seller, who chooses a selling mechanism, and a fictitious adversary or 'nature', who chooses the buyer's values from within the uncertainty set. Using duality techniques rooted in robust optimization, we prove that this game admits a Nash equilibrium in mixed strategies that can be computed in closed form. The Nash strategy of the seller is a *randomized posted price mechanism* under which the goods are sold *separately*, while the Nash strategy of nature is a distribution on the uncertainty set under which the buyer's values are *comonotonic*. We further show that the restriction of the pricing problem to deterministic mechanisms is solved by a *deteministic posted price mechanism* under which the goods are sold *separately*.

2.1 Introduction

We address the fundamental question of how much money one should charge for new products when there is only minimal information about the buyers' willingness to pay. More precisely, we study a robust monopoly pricing problem, where a seller ("she") endeavors to sell multiple indivisible goods to a single buyer ("he"). The buyer assigns each good a private value, which reflects the maximum amount of money he would be willing to pay for this good. The value assigned to a bundle (*i.e.,* a set of multiple goods) equals the sum of the included goods' values. The seller perceives the overall value profile (*i.e.*, the list of values for *all* goods) as an uncertain parameter that is only known to range over a rectangular uncertainty set spanned by the origin and a vector of non-negative upper bounds. This set-based uncertainty model is appropriate in the absence of any trustworthy distributional information or when the acquisition of such information—*e.g.*, via market research or by observations of buyer behavior in prior sales—

would be overly expensive or time-consuming.

We assume that the seller aims to design a mechanism for liquidating the goods with the goal to minimize her worst-case regret. The regret of a mechanism is defined as the difference between the hypothetical revenues that could have been realized under full knowledge of the buyer's value profile and the actual revenues generated by the mechanism. The worst-case regret is obtained by maximizing the realized regret across all possible value profiles in the uncertainty set. The minimax regret criterion was introduced by Savage (1951) and captures the idea that decision makers have a low tolerance for missing out on opportunities to earn revenue. It is less pessimistic than the ordinary maxmin criterion commonly used in robust optimization, which seeks mechanisms that generate maximum revenues under the worst possible value profile in the uncertainty set.

The family of possible selling mechanism is vast. For example, the seller could set individual posted prices for different bundles and ask the buyer to self-select his preferred price-bundle-pair. More generally, the seller could offer the buyer a menu of lotteries for winning the goods with different probabilities, set an individual price (or participation fee) for each lottery, and ask the buyer to self-select his preferred price-lottery-pair.

The classical mechanism design literature models the buyer's value profile as a random vector that is governed by a known probability distribution. If there is only one good, it is well known that setting a deterministic posted price (a take-it-or-leave-it offer) maximizes the seller's expected revenues (Myerson (1981), Riley and Zeckhauser (1983)). Moreover, the optimal posted price can be calculated analytically. In the presence of multiple goods, on the other hand, the expected revenue maximizing mechanism is notoriously difficult to characterize and compute. Even if the buyer's values are independent across the goods and his utility function is quasilinear and additively separable, offering discounts on bundles and using randomized allocation rules can yield strictly higher expected revenues than selling the goods separately (see, *e.g.*, Manelli and Vincent (2006) or Thanassoulis (2004)). Daskalakis et al. (2014) show that, under standard complexity theoretic assumptions, the multidimensional mechanism design problem with expected revenue objective admits no expected polynomial-time solution algorithm even in unrealistically simple settings where the buyer's values are independently distributed on two rational numbers with rational probabilities. Thus, the generic multidimensional mechanism design problem is severely intractable. Nevertheless, closed-form solutions are available for special probability distributions and/or for small numbers of goods (see, *e.g.*, Bhargava (2013), Daskalakis et al. (2013), Giannakopoulos and Koutsoupias (2014) and Daskalakis et al. (2017)). Moreover, under the restrictive assumption that the buyer's values are independent, simple mechanisms (such as selling the goods separately or as a single bundle) provide constant-factor approximations to the expected revenue of the unknown optimal mechanism (see, *e.g.*, Hart and Nisan (2017) or Li and Yao (2013)). If the buyer's values are correlated, the optimal mechanism becomes even hard to approximate. Indeed, Hart and Nisan (2019) show that the optimal mechanism for selling more than one good may involve a menu of infinitely many price-lottery-pairs and that no deterministic mechanism can guarantee to extract any positive fraction of the optimal expected revenue. This implies that the seller can be significantly worse off by setting deterministic posted prices for the bundles instead of implementing an optimal mechanism. Note that this inapproximability result holds in spite of the quasilinearity and additive separability of the buyer's utility function.

Modeling the uncertainty in the buyer's value profile through a crisp distribution not only compromises the problem's computational tractability but is also difficult to justify in situations when the demand is poorly understood. A recent stream of literature thus investigates the impact of distributional uncertainty or *ambiguity* on pricing problems. Most existing studies focus on the single-item case and assume that the seller aims to maximize her worst-case expected revenues in view of all distributions in some *ambiguity set* (see, *e.g.*, Bergemann and Schlag (2011), Carrasco et al. (2018) and Pınar and Kızılkale (2017)). By definition, the ambiguity set contains all distributions that are consistent with the seller's information about the buyer's value profile such as its support, its mean or certain higher-order moments. The maxmin expected revenue criterion results in non-trivial mechanisms only if the seller knows more about the value distribution than just its support. Otherwise, the worst-case expectation reduces to the worst-case realization of the revenue, in which case the underlying pricing problem becomes too conservative to be practically useful. In fact, if the lowest possible value the buyer assigns to any good is zero, then it would be optimal for the seller to keep all goods for herself. This observation prompts Bergemann and Schlag (2008) to study a single-item pricing problem with minimax regret objective. Assuming that there is only support information, they show that the seller's worst-case regret is minimized by setting a *randomized* posted price, whose distribution can be calculated in closed form. In addition, they also identify the best *deterministic* posted price under the minimax regret criterion. The multi-item pricing problem under ambiguity is perceived as challenging and has therefore received only limited attention in the literature. As a notable exception, Carroll (2017) explicitly characterizes the optimal mechanism of a screening problem with maxmin expected revenue objective, where the marginal distributions of the agent's multidimensional type are precisely known to the principal, while their dependence structure (or copula) remains uncertain. In the special case of monopoly pricing when only the marginal distributions of the buyer's values are known, Carroll (2017) shows that the seller does not benefit from bundling and that it is optimal to post a deterministic price for each good separately. Gravin and Lu (2018) show that this separation result continues to hold even if the buyer has a budget for his total payment.

This chapter contributes to the rapidly expanding literature on mechanism design from the perspective of mathematical optimization (see, *e.g.*, Vohra (2012), Bichler (2017), Lopomo et al. (2011) or Fanzeres et al. (2019)) and endeavors to further our understanding of multi-item pricing under extreme ambiguity. Specifically, we postulate that the buyer's value profile may follow any distribution on a given rectangular uncertainty set. This assumption leads to pricing problems that can be analyzed with methods from modern robust optimization (see, *e.g.*, Ben-Tal et al. (2009) or Bertsimas et al. (2011)). If the seller aims to maximize her worst-case revenue, for example, then the robust pricing problem can be interpreted as a robust auction design problem with a single bidder. Bandi and Bertsimas (2014) describe an efficient numerical solution to this problem for any number of bidders. For a single bidder, however, the optimal mechanism collapses to the trivial mechanism under which the seller keeps all items for herself. To mitigate the conservatism of robust pricing, we assume here that the seller minimizes her worst-case regret. In contrast to the single-item pricing model by Bergemann and Schlag (2008), which optimizes over all randomized posted prices, we formulate the robust multi-item pricing problem as an explicit mechanism design problem that searches over all incentive compatible and individually rational allocation and payment rules. While one can show that the two formulations are essentially equivalent in the single-item case (all single-item mechanisms with a right-continuous allocation rule give rise to a randomized posted price), only the explicit formulation advocated here easily generalizes to multiple items. Indeed, randomized posted price mechanisms for multiple goods would already be cumbersome to characterize; they would require a specification of separate (possibly correlated) posted prices for all possible bundles. Moreover, in order to evaluate the seller's worst-case regret, one would have to anticipate the buyer's preferred bundle for each realization of the posted prices and compute the expectation of an implicitly defined piecewise linear function with exponentially many pieces, which seems excruciating.

We highlight the following main contributions of this chapter.

1. We formulate the multi-item pricing problem with minimax regret objective as an adaptive robust optimization problem. While such problems are generically NP-hard (Guslitser, 2002, Theorem 3.5) and typically only solved approximately using linear decision rules, we show that the pricing problem at hand admits an explicit analytical solution in piecewise logarithmic decision rules. This solution is obtained by leveraging duality techniques rooted in robust optimization, and it represents a randomized mechanism under which the goods are sold *separately*.

2. The robust pricing problem can be interpreted as a zero-sum game between the seller and a fictitious adversary or 'nature', who chooses the buyer's value profile in the uncertainty set with the aim to inflict maximum damage. We demonstrate that this game admits a Nash equilibrium in mixed strategies, which can be computed in closed form.

The Nash strategy of the seller coincides with the optimal randomized mechanism, while the Nash strategy of nature is a (non-discrete) distribution on the uncertainty set under which the components of the value profile are comonotonic.

3. We study a restriction of the robust pricing problem that optimizes only over deterministic mechanisms, which is essentially equivalent to searching over all posted price mechanisms. We solve this problem analytically and show that the different goods are again sold *separately* at optimality.

4. We demonstrate that the single-item pricing theory by Bergemann and Schlag (2008) emerges as a special case of the proposed multi-item pricing theory.

The remainder of this chapter is structured as follows. Section 2.2 reviews key microeconomic concepts and formulates the robust multi-item pricing problem as an abstract mechanism design problem. By using duality techniques from robust optimization, Section 2.3 solves the general pricing problem in closed form and shows that a separable randomized posted price mechanism is optimal. Section 2.4 solves a restriction of the pricing problem that optimizes only over deterministic mechanisms and shows that a separable deterministic posted price mechanism is optimal.

Notation.

For any $\mathscr{S} \subseteq \mathscr{J} = \{1,\ldots,J\}$, the vector $\mathbf{1}_{\mathscr{S}} \in \mathbb{R}^J$ is defined through $(\mathbf{1}_{\mathscr{S}})_j = 1$ if $j \in \mathscr{S}$; $= 0$ if $j \in \mathscr{J}\backslash\mathscr{S}$. We use $\mathbf{1}$ as a shorthand for $\mathbf{1}_{\mathscr{J}}$. Similarly, for any $\boldsymbol{v} \in \mathbb{R}^J$, the vector $\boldsymbol{v}^+ \in \mathbb{R}^J$ is defined through $(\boldsymbol{v}^+)_j = \max\{v_j, 0\}$, $j \in \mathscr{J}$. For a logical expression $\mathscr{E}$, we define $\mathbb{1}_{\mathscr{E}} = 1$ if $\mathscr{E}$ is true; $= 0$ otherwise. For any Borel set $\mathscr{A} \subseteq \mathbb{R}^I$, $\Delta(\mathscr{A})$ represents the family of all probability distributions on $\mathscr{A}$. The set of all bounded Borel-measurable functions from a Borel set $\mathscr{D} \subseteq \mathbb{R}^J$ to a Borel set $\mathscr{R} \subseteq \mathbb{R}^J$ is denoted by $\mathscr{L}(\mathscr{D},\mathscr{R})$. Random variables are designated by tildes (*e.g.*, $\tilde{\boldsymbol{v}}$), and their realizations are denoted by the same symbols without tildes (*e.g.*, $\boldsymbol{v}$).

2.2 Problem Formulation and Preliminaries

We consider the problem of designing a mechanism for selling $J \in \mathbb{N}$ different items to a single buyer. The set of items is denoted by $\mathscr{J} = \{1,2,\ldots,J\}$. The buyer assigns each item $j \in \mathscr{J}$ a value v_j that reflects his willingness to pay. While the buyer has full knowledge of his value profile $\boldsymbol{v} = (v_1,\ldots,v_J)^\top$, the seller perceives $\boldsymbol{v}$ as an uncertain parameter. Specifically, we assume that the seller only knows an upper bound $\overline{v}_j$ on the buyer's value v_j for each $j \in \mathscr{J}$. However, she has no information about the distribution of $\boldsymbol{v}$ or suspects that any available information is not trustworthy. In the following, we denote by $\mathscr{V} = \times_{j\in\mathscr{J}} \left[0,\overline{v}_j\right]$ the uncertainty set of the buyer's value profiles. The seller incurs a cost $c_j \in \mathbb{R}_+$ for supplying item j to the buyer. This cost may capture the expenses for producing and/or delivering the item. We denote by $\boldsymbol{c} = (c_1, \ldots, c_J)^\top$ the seller's cost vector. Without loss of generality, we assume that $\boldsymbol{c} < \overline{\boldsymbol{v}} = (\overline{v}_1, \ldots, \overline{v}_J)^\top$. This assumption is justified more rigorously in Remark 2.3.1 of Section 2.3.

The sale proceeds as follows. First, the seller announces a mechanism $(\mathscr{B}, \boldsymbol{q}, m)$ that consists of a set $\mathscr{B}$ of *messages* available to the buyer, an *allocation rule* $\boldsymbol{q} = (q_1, \ldots, q_J)^\top : \mathscr{B} \to [0,1]^J$ and a *payment rule* $m : \mathscr{B} \to \mathbb{R}$. Next, the buyer transmits a message $\boldsymbol{b} \in \mathscr{B}$ to the seller. Depending on this message, the seller then allocates item j to the buyer with probability $q_j(\boldsymbol{b})$ for each $j \in \mathscr{J}$ in return for a total payment equal to $m(\boldsymbol{b})$. If the goods are divisible, we can alternatively interpret $q_j(\boldsymbol{v})$ as the proportion of item j acquired by the buyer.

We assume that the buyer is risk-neutral with respect to the randomness of the allocation rule. Thus, the buyer's expected utility coincides with the expected profit $\boldsymbol{q}(\boldsymbol{b})^\top \boldsymbol{v} - m(\boldsymbol{b})$ and is additively separable with respect to the items. Given a mechanism, the buyer selects his message strategically depending on his value profile $\boldsymbol{v}$ so as to maximize his utility, *i.e.*, he reports $\boldsymbol{b}^\star(\boldsymbol{v}) \in \arg\max_{\boldsymbol{b} \in \mathscr{B}} \boldsymbol{q}(\boldsymbol{b})^\top \boldsymbol{v} - m(\boldsymbol{b})$.

A mechanism $(\mathscr{B}, \boldsymbol{q}, m)$ is called *direct* if the set $\mathscr{B}$ of messages coincides with the set $\mathscr{V}$ of value profiles. We henceforth use the shorthand $(\boldsymbol{q}, m)$ to denote any direct mechanism $(\mathscr{V}, \boldsymbol{q}, m)$ because there is no freedom in specifying the set of messages. A direct mechanism $(\boldsymbol{q}, m)$ is called *incentive compatible* if the buyer's optimal strategy is to truthfully report his value profile $\boldsymbol{v}$.

Definition 2.2.1 (Incentive Compatibility)**.** *A direct mechanism $(\boldsymbol{q}, m)$ is incentive compatible if*

$$\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \;\geq\; \boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w}) \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}. \tag{IC}$$

The incentive compatibility constraint (IC) formalizes the requirement that reporting the true values $\boldsymbol{v}$ should result in the highest expected utility to the buyer. By virtue of the celebrated Revelation Principle due to Myerson (1981), the seller can restrict attention to incentive compatible direct mechanisms without loss of generality. The intuition behind the Revelation Principle is as follows. Given any mechanism $(\mathscr{B}, \boldsymbol{q}, m)$, the seller can construct an equivalent incentive compatible direct mechanism $(\boldsymbol{q}', m')$ by asking the buyer to report his true value profile, allocating the items according to the rule $\boldsymbol{q}'(\boldsymbol{v}) = \boldsymbol{q}(\boldsymbol{b}^\star(\boldsymbol{v}))$ and charging a payment $m'(\boldsymbol{v}) = m(\boldsymbol{b}^\star(\boldsymbol{v}))$ as if the buyer had reported his optimal message $\boldsymbol{b}^\star(\boldsymbol{v})$ for the original mechanism $(\mathscr{B}, \boldsymbol{q}, m)$. Thus, the buyer has no incentive to misreport his value profile. Moreover, the ex-post outcomes are identical under the mechanisms $(\boldsymbol{q}', m')$ and $(\mathscr{B}, \boldsymbol{q}, m)$.

The buyer will participate in the sale only if his utility is non-negative. In order to prevent the buyer from walking away, the seller should thus focus on *individually rational* mechanisms.

Definition 2.2.2 (Individual Rationality)**.** *A direct mechanism* $(\boldsymbol{q}, m)$ *is individually rational if*

$$\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0 \quad \forall \boldsymbol{v} \in \mathcal{V}. \tag{IR}$$

The individual rationality constraint (IR) guarantees that the buyer's utility under truthful reporting is non-negative irrespective of his value profile $\boldsymbol{v}$. The seller can restrict attention to individually rational mechanisms without loss of generality. Indeed, assume that $(\boldsymbol{q}, m)$ is an incentive compatible mechanism that results in a negative utility for the buyer and thus to cancellation of the sale under some value profiles. In this case, the seller can construct an equivalent individually rational mechanism $(\boldsymbol{q}', m')$ defined through $\boldsymbol{q}' = \boldsymbol{q}\,\mathbb{1}_{\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0}$ and $m'(\boldsymbol{v}) = m(\boldsymbol{v})\,\mathbb{1}_{\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0}$ for all $\boldsymbol{v} \in \mathcal{V}$, which sets the allocation probabilities and the payment to zero whenever the original mechanism would result in a cancellation of the sale. This mechanism is still incentive compatible because

$$\begin{aligned}
\sup_{\boldsymbol{w} \in \mathcal{V}} \boldsymbol{q}'(\boldsymbol{w})^\top \boldsymbol{v} - m'(\boldsymbol{w}) &= \sup_{\boldsymbol{w} \in \mathcal{V}} \left[\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w})\right] \mathbb{1}_{\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{w} - m(\boldsymbol{w}) \geq 0} \\
&\leq \sup_{\boldsymbol{w} \in \mathcal{V}} \left[\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w})\right]^+ \mathbb{1}_{\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{w} - m(\boldsymbol{w}) \geq 0} \\
&\leq \sup_{\boldsymbol{w} \in \mathcal{V}} \left[\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w})\right]^+ \\
&= \left[\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v})\right]^+ = \boldsymbol{q}'(\boldsymbol{v})^\top \boldsymbol{v} - m'(\boldsymbol{v}).
\end{aligned}$$

Moreover, the ex-post outcomes are identical under the mechanisms $(\boldsymbol{q}', m')$ and $(\boldsymbol{q}, m)$ when correctly accounting for the walk-away option.

Throughout the rest of the chapter, without loss of generality, we focus only on direct mechanisms that are both incentive compatible and individually rational.

The seller's ex-post regret is defined as the difference between the maximum profit that could have been realized under complete information about $\boldsymbol{v}$ and the expected profit $m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c}$ earned with the mechanism $(\boldsymbol{q}, m)$. If the seller was fully aware of the buyer's willingness to pay, she would sell item j at price v_j whenever $v_j \geq c_j$ and would keep the item otherwise. The maximum profit under complete information can thus be expressed as $\mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+$, while the ex-post regret equals $\mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - \big(m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c}\big)$. The worst-case regret is obtained by maximizing the ex-post regret over all value profiles $\boldsymbol{v} \in \mathcal{V}$.

Throughout this chapter we assume that the seller aims to design an incentive compatible and individually rational mechanism that minimizes the worst-case regret. This mechanism design problem can be formalized as follows.

$$\begin{aligned} z^\star \;=\; \inf_{\boldsymbol{q},m} \quad & \sup_{\boldsymbol{v}\in\mathcal{V}} \; \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - \left(m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c}\right) \\ \text{s.t.} \quad & \boldsymbol{q}\in\mathcal{L}(\mathcal{V},\mathbb{R}^J_+),\; m\in\mathcal{L}(\mathcal{V},\mathbb{R}) \\ & \boldsymbol{q}(\boldsymbol{v})^\top\boldsymbol{v} - m(\boldsymbol{v}) \;\geq\; \boldsymbol{q}(\boldsymbol{w})^\top\boldsymbol{v} - m(\boldsymbol{w}) \quad \forall \boldsymbol{v},\boldsymbol{w}\in\mathcal{V} \\ & \boldsymbol{q}(\boldsymbol{v})^\top\boldsymbol{v} - m(\boldsymbol{v}) \;\geq\; 0 \quad \forall \boldsymbol{v}\in\mathcal{V} \\ & \boldsymbol{q}(\boldsymbol{v}) \;\leq\; \mathbf{1} \quad \forall \boldsymbol{v}\in\mathcal{V} \end{aligned} \tag{2.1}$$

The last constraint in (2.1) ensures that each item is sold at most once. In the following, we use will the shorthand $\mathcal{X}$ to denote the set of all mechanisms feasible in (2.1).

Remark 2.2.1. *Problem* (2.1) *can be interpreted as a zero-sum game between the seller, who chooses the mechanism* $(\boldsymbol{q}, m)$*, and some fictitious adversary or nature, who chooses the buyer's value profile* $\boldsymbol{v}$ *with the goal to inflict maximum damage to the seller. As we allow for randomized allocation rules, the seller plays a mixed strategy and thus solves a convex minimization problem. Nature, on the other hand, chooses a pure strategy from within the uncertainty set* $\mathcal{V}$ *but solves a* non-*convex maximization problem. This problem can be convexified by allowing nature to play a mixed strategy* $\mathbb{P}\in\Delta(\mathcal{V})$*, thereby replacing the non-convex inner maximization problem in* (2.1) *with an infinite-dimensional linear program.*

$$z^\star \;=\; \inf_{(\boldsymbol{q},m)\in\mathcal{X}} \quad \sup_{\mathbb{P}\in\Delta(\mathcal{V})} \; \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}^\top(\tilde{\boldsymbol{v}}-\boldsymbol{c})^+ - \left(m(\tilde{\boldsymbol{v}}) - \boldsymbol{q}(\tilde{\boldsymbol{v}})^\top\boldsymbol{c}\right)\right] \tag{2.2}$$

Problem (2.2) *is clearly equivalent to* (2.1) *because* $\Delta(\mathcal{V})$ *contains all Dirac point measures supported on* $\mathcal{V}$*. Using this formulation, we will show below that the game between the seller and nature admits a Nash equilibrium in mixed strategies.*

The set of mechanisms feasible in (2.1) is vast. Posted price mechanisms sell different bundles of the items at fixed posted prices. They range among the most popular selling mechanisms.

Definition 2.2.3 (Posted Price Mechanism)**.** *A mechanism* $(\boldsymbol{q}, m)$ *is called a* posted price mechanism *if there exists a vector of posted prices* $\boldsymbol{p}\in\mathbb{R}^{2^J}$ *such that* $\boldsymbol{q}(\boldsymbol{v}) = \mathbf{1}_{s(\boldsymbol{v})}$ *and* $m(\boldsymbol{v}) = p_{s(\boldsymbol{v})}$*, where* $s\in\mathcal{L}(\mathcal{V},2^{\mathcal{J}})$ *represents a* bundle allocation rule *that satisfies* $s(\boldsymbol{v})\in\arg\max_{\mathcal{S}\subseteq\mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top\boldsymbol{v} - p_{\mathcal{S}}$ *for all* $\boldsymbol{v}\in\mathcal{V}$*.*

The set-valued bundle allocation rule s maps any value profile $\boldsymbol{v}\in\mathcal{V}$ to a bundle $\mathcal{S}\subseteq\mathcal{J}$ that maximizes the utility of the buyer of type $\boldsymbol{v}$. It is almost uniquely determined by the vector of posted prices $\boldsymbol{p}$, but ties are broken at the discretion of the seller in cases when multiple bundles are optimal. One can prove that any posted price mechanism induced by $\boldsymbol{p}$ is individually rational if and only if $p_{\mathcal{S}}\leq 0$ for at least one bundle $\mathcal{S}\subseteq\mathcal{J}$. Any posted price

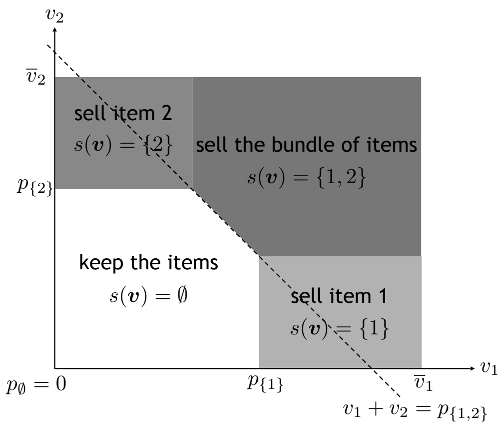


Figure 2.1 – Visualization of the bundle allocation rule corresponding to a posted price mechanism with $\boldsymbol{p} = [p_\emptyset, p_{\{1\}}, p_{\{2\}}, p_{\{1,2\}}]^\top$ and $p_\emptyset = 0$. If the buyer's value v_1 for item 1 exceeds $p_{\{1\}}$ but his value v_2 for item 2 falls short of $p_{\{1,2\}} - p_{\{1\}}$, then he receives only item 1 at price $p_{\{1\}}$. Similarly, if $v_1 + v_2 \geq p_{\{1,2\}}$, $v_1 \geq p_{\{1,2\}} - p_{\{2\}}$ and $v_2 \geq p_{\{1,2\}} - p_{\{1\}}$, then the buyer acquires the bundle $\{1,2\}$ at price $p_{\{1,2\}}$, *etc.*

mechanism satisfying this condition is feasible in (2.1). Intuitively, the seller may implement a posted price mechanism by setting an individual price for each bundle and let the buyer choose the bundle that maximizes his utility. Figure 2.1 visualizes the allocation rule of a posted price mechanism for two items. A more in-depth discussion of posted price mechanisms is relegated to Section 2.4.

In the following sections, we will derive optimal randomized and deterministic mechanisms in closed form. Many of the subsequent results will depend on a well-know equivalent characterization of incentive compatible mechanisms for selling a single item due to Myerson and Satterthwaite (1983). Note that, for $J = 1$, the allocation rule reduces to a scalar function denoted by q.

Proposition 2.2.1. *For $J = 1$, a mechanism (q, m) is incentive compatible if and only if*

(i) $q(v)$ *is non-decreasing in* $v \in \mathcal{V}$,

(ii) $m(v) \;=\; m(0) + q(v)v - \int_0^v q(x)\,\mathrm{d}x \quad \forall v \in \mathcal{V}.$

2.3 Optimal Mechanism

One particularly simple policy for the seller would be to sell each item individually via a separable mechanism that ignores the possibility of bundling.

Definition 2.3.1 (Separability)**.** *A mechanism $(\boldsymbol{q}, m)$ is called* separable *if there exists* $\hat{q}_j \in$

$\mathscr{L}([0,\overline{v}_j],[0,1])$ and $\hat{m}_j \in \mathscr{L}([0,\overline{v}_j],\mathbb{R})$ for all $j \in \mathscr{J}$ such that $\boldsymbol{q}(\boldsymbol{v}) = [\hat{q}_1(v_1),\ldots,\hat{q}_J(v_J)]^\top$ and $m(\boldsymbol{v}) = \sum_{j\in\mathscr{J}} \hat{m}_j(v_j)$ for all $\boldsymbol{v} \in \mathcal{V}$.

Under a separable mechanism, the sale of each item j is processed according to a separate single-item mechanism $(\hat{q}_j, \hat{m}_j)$ that depends exclusively on the value v_j.

In the remainder of this section we will investigate the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ with corresponding single-item mechanisms $(\hat{q}_j, \hat{m}_j)$, $j \in \mathscr{J}$, defined through

$$\big(\hat{q}_j(v_j), \hat{m}_j(v_j)\big) = \begin{cases} \left(1+\log\left(\frac{v_j-c_j}{\overline{v}_j-c_j}\right), v_j - \frac{(\overline{v}_j-c_j)}{e} + c_j \log\left(\frac{v_j-c_j}{\overline{v}_j-c_j}\right)\right) & \text{if } \frac{v_j-c_j}{\overline{v}_j-c_j} \geq \frac{1}{e}, \\ (0,0) & \text{otherwise,} \end{cases} \tag{2.3}$$

for all $v_j \in [0, \overline{v}_j]$.

Lemma 2.3.1. *The separable mechanism $(\boldsymbol{q}^\star, m^\star)$ defined via* (2.3) *is feasible in* (2.1) *and attains an objective value of $\frac{1}{e} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})$.*

Proof. The proof is divided into two parts. We first show that the mechanism $(\boldsymbol{q}^\star, m^\star)$ is feasible in (2.1) (Step 1), and then we calculate its objective value (Step 2).

Step 1: By the construction of $(\boldsymbol{q}^\star, m^\star)$ in (2.3), each component of $\boldsymbol{q}^\star(\boldsymbol{v})$ is non-negative and bounded above by 1 for all $\boldsymbol{v} \in \mathcal{V}$ because $-1 \leq \log\left(\frac{v_j-c_j}{\overline{v}_j-c_j}\right) \leq 0$ for all $v_j \leq \overline{v}_j$ with $\frac{v_j-c_j}{\overline{v}_j-c_j} \geq \frac{1}{e}$. Thus, we have $\mathbf{0} \leq \boldsymbol{q}^\star(\boldsymbol{v}) \leq \mathbf{1}$ for all $\boldsymbol{v} \in \mathcal{V}$.

Next, it is easy to see that $(\boldsymbol{q}^\star, m^\star)$ inherits incentive compatibility and individual rationality from the single-item mechanisms $(\hat{q}_j, \hat{m}_j)$, $j \in \mathscr{J}$. By Proposition 2.2.1, $(\hat{q}_j, \hat{m}_j)$ is incentive compatible if and only if the allocation rule $\hat{q}_j(v_j)$ is non-decreasing in v_j, which follows immediately from (2.3), while the payment rule satisfies

$$\hat{m}_j(v_j) = \hat{m}_j(0) + \hat{q}_j(v_j)v_j - \int_0^{v_j} \hat{q}_j(x)\,\mathrm{d}x \quad \forall v_j \in [0, \overline{v}_j].$$

This equality trivially holds when $\frac{v_j-c_j}{\overline{v}_j-c_j} < \frac{1}{e}$, in which case both sides reduce to 0. Moreover, for $\frac{v_j-c_j}{\overline{v}_j-c_j} \geq \frac{1}{e}$, the right-hand side of the above equation simplifies to

$$\begin{aligned} \hat{q}_j(v_j)v_j - \int_0^{v_j} \hat{q}_j(x)\,\mathrm{d}x &= \left(1+\log\left(\frac{v_j-c_j}{\overline{v}_j-c_j}\right)\right)v_j - \int_{\frac{\overline{v}_j-c_j}{e}+c_j}^{v_j} 1+\log\left(\frac{x-c_j}{\overline{v}_j-c_j}\right)\mathrm{d}x \\ &= \left(1+\log\left(\frac{v_j-c_j}{\overline{v}_j-c_j}\right)\right)v_j - \left[(x-c_j)\log\left(\frac{x-c_j}{\overline{v}_j-c_j}\right)\right]_{\frac{\overline{v}_j-c_j}{e}+c_j}^{v_j} \\ &= v_j + c_j\log\left(\frac{v_j-c_j}{\overline{v}_j-c_j}\right) + \left(\frac{\overline{v}_j-c_j}{e}\right)\log\left(\frac{1}{e}\right), \end{aligned}$$

which manifestly equals $\hat{m}_j(v_j)$. Hence, the mechanism $(\hat{q}_j, \hat{m}_j)$ is incentive compatible as it satisfies both conditions of Proposition 2.2.1.

Finally, the mechanism $(\hat{q}_j, \hat{m}_j)$ is also individually rational because

$$\hat{q}_j(v_j)v_j - \hat{m}_j(v_j) = \int_0^{v_j} \hat{q}_j(x)\,\mathrm{d}x - \hat{m}_j(0) \geq 0,$$

where the equality follows again from Proposition 2.2.1, while the inequality holds because $\hat{m}_j(0) = 0$ and $\hat{q}_j$ is non-negative. This concludes Step 1.

Step 2: Thanks to its separability, the objective function value of $(\boldsymbol{q}^\star, m^\star)$ can be expressed as

$$\sup_{\boldsymbol{v}\in\mathcal{V}} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - \left(m^\star(\boldsymbol{v}) - \boldsymbol{q}^\star(\boldsymbol{v})^\top \boldsymbol{c}\right) = \sum_{j\in\mathcal{J}} \sup_{0\leq v_j\leq \overline{v}_j} r_j(v_j), \tag{2.4}$$

where $r_j(v_j) = (v_j - c_j)^+ - \hat{m}_j(v_j) + c_j\hat{q}_j(v_j)$. For each $j \in \mathcal{J}$, the supremum of $r_j(v_j)$ can be determined by distinguishing two cases as in (2.3). If $\frac{v_j-c_j}{\overline{v}_j-c_j} < \frac{1}{e}$, then both $\hat{q}_j(v_j)$ and $\hat{m}_j(v_j)$ vanish, and v_j belongs to the interval $[0, c_j + \frac{\overline{v}_j-c_j}{e})$. The supremum of $r_j(v_j) = (v_j - c_j)^+$ over this interval is attained at the interval's right boundary and amounts to $\frac{1}{e}\cdot(\overline{v}_j - c_j)$. If $\frac{v_j-c_j}{\overline{v}_j-c_j} \geq \frac{1}{e}$, on the other hand, we find

$$\begin{aligned} r_j(v_j) &= (v_j - c_j) - \left(v_j - \frac{\overline{v}_j - c_j}{e} + c_j \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right) + c_j\left(1 + \log\left(\frac{v_j-c_j}{\overline{v}_j-c_j}\right)\right) \\ &= \frac{1}{e}\cdot(\overline{v}_j - c_j). \end{aligned}$$

In summary, we have that $r_j(v_j) \leq \frac{1}{e}\cdot(\overline{v}_j - c_j)$ for all $v_j \in [0, \overline{v}_j]$ and that this inequality is tight for all $v_j \in [c_j + \frac{\overline{v}_j-c_j}{e}, \overline{v}_j]$. Thus the objective value (2.4) of the mechanism $(\boldsymbol{q}^\star, m^\star)$ simplifies to $\frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})$. This observation completes the proof. ■

Next, we will show that the mechanism $(\boldsymbol{q}^\star, m^\star)$ is not only feasible but also optimal in (2.1). To this end, consider the following discrete approximation of (2.1).

$$\begin{aligned} z_n^\star \;=\; \inf \quad & \sup_{\boldsymbol{v}\in\mathcal{V}_n} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - \left(m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top\boldsymbol{c}\right) \\ \text{s.t.} \quad & \boldsymbol{q}\in\mathcal{L}(\mathcal{V}_n, \mathbb{R}_+^J),\ m\in\mathcal{L}(\mathcal{V}_n,\mathbb{R}) \\ & \boldsymbol{q}(\boldsymbol{v})^\top\boldsymbol{v} - m(\boldsymbol{v}) \;\geq\; 0 && \forall \boldsymbol{v}\in\mathcal{V}_n \\ & \boldsymbol{q}(\boldsymbol{v})^\top\boldsymbol{v} - m(\boldsymbol{v}) \;\geq\; \boldsymbol{q}(\boldsymbol{w})^\top\boldsymbol{v} - m(\boldsymbol{w}) && \forall \boldsymbol{v},\boldsymbol{w}\in\mathcal{V}_n \\ & \boldsymbol{q}(\boldsymbol{v}) \;\leq\; \mathbf{1} && \forall \boldsymbol{v}\in\mathcal{V}_n \end{aligned} \tag{2.5}$$

Problem (2.5) differs from (2.1) only in that it involves a discrete uncertainty set $\mathcal{V}_n = \times_{j\in\mathcal{J}}\mathcal{V}_{n,j}$, where $\mathcal{V}_{n,j} = \left\{c_j, \frac{1}{n}(\overline{v}_j - c_j)+c_j, \frac{2}{n}(\overline{v}_j-c_j)+c_j, \ldots, \overline{v}_j\right\}$ represents a uniform one-dimensional grid with $n+1$ discretization points for some $n\in\mathbb{N}$. Note also that any (scalar) function defined on $\mathcal{V}_n$ corresponds to an $(n+1)^J$-dimensional vector.

Lemma 2.3.2. *For any $n \in \mathbb{N}$, we have $z_n^\star \leq z^\star$.*

Proof. By construction it is clear that $\mathcal{V}_n \subseteq \mathcal{V}$. Thus, the objective function of (2.5) is majorized by that of (2.1) uniformly across all $\boldsymbol{q}$ and m, and the feasible set of (2.5) contains that of (2.1) as it relaxes all constraints associated with value profiles $\boldsymbol{v} \in \mathcal{V} \setminus \mathcal{V}_n$. The optimal value of (2.5) is therefore non-inferior to that of (2.1). ∎

As its objective function is convex and piecewise linear, problem (2.5) can be reformulated as an equivalent finite linear program of the form

$$
\begin{array}{rll}
z_n^\star = \inf & r & \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+^J),\ m \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}),\ r \in \mathbb{R} & \\
& r \geq \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - \left(m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c}\right) & \forall \boldsymbol{v} \in \mathcal{V}_n \\
& \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0 & \forall \boldsymbol{v} \in \mathcal{V}_n \\
& \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq \boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w}) & \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}_n \\
& \boldsymbol{q}(\boldsymbol{v}) \leq \mathbf{1} & \forall \boldsymbol{v} \in \mathcal{V}_n,
\end{array}
\tag{2.6}
$$

where r represents an auxiliary epigraphical variable.

The linear program dual to (2.6) is given by

$$
\begin{array}{rll}
z_n^\star = \sup & \displaystyle\sum_{j \in \mathcal{J}} \sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha(\boldsymbol{v})(v_j - c_j)^+ - \sum_{j \in \mathcal{J}} \sum_{\boldsymbol{v} \in \mathcal{V}_n} \lambda_j(\boldsymbol{v}) & \\
\text{s.t.} & \alpha, \beta \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+),\ \gamma \in \mathcal{L}(\mathcal{V}_n \times \mathcal{V}_n, \mathbb{R}_+),\ \boldsymbol{\lambda} \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+^J) & \\
& \displaystyle\sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha(\boldsymbol{v}) = 1 & \\
& \displaystyle\beta(\boldsymbol{v}) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{v}, \boldsymbol{w}) - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v}) = \alpha(\boldsymbol{v}) & \forall \boldsymbol{v} \in \mathcal{V}_n \\
& \displaystyle\lambda_j(\boldsymbol{v}) + \alpha(\boldsymbol{v}) c_j \geq \beta(\boldsymbol{v}) v_j + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{v}, \boldsymbol{w}) v_j - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v}) w_j & \forall j \in \mathcal{J},\ \forall \boldsymbol{v} \in \mathcal{V}_n,
\end{array}
\tag{2.7}
$$

where α represents the dual variable of the epigraphical constraint, β and γ are the dual variables of the individual rationality and incentive compatibility constraints, respectively, and $\boldsymbol{\lambda}$ collects the dual variables of the upper probability bounds in (2.6). Strong duality holds because the trivial mechanism that sets $\boldsymbol{q}(\boldsymbol{v}) = \mathbf{0}$ and $m(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \mathcal{V}$ is feasible in (2.6) for every $r \geq \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Since the linear program (2.7) seeks to make the decision variables $\lambda_j(\boldsymbol{v})$ as small as possible while ensuring that they remain non-negative and satisfy the last constraint of (2.7), it is clear that

$$\begin{aligned}\lambda_j(\boldsymbol{v}) &= \left(\beta(\boldsymbol{v})v_j + \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma(\boldsymbol{v},\boldsymbol{w})v_j - \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma(\boldsymbol{w},\boldsymbol{v})w_j - \alpha(\boldsymbol{v})c_j\right)^+ \\ &= \left(\alpha(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma(\boldsymbol{w},\boldsymbol{v})(v_j - w_j)\right)^+ \qquad \forall j\in\mathcal{J}, \forall \boldsymbol{v}\in\mathcal{V}_n\end{aligned}$$

at optimality, where the second equality exploits the second equality constraint in (2.7) to eliminate $\beta(\boldsymbol{v})$. By substituting the above expression for $\lambda_j(\boldsymbol{v})$ into the objective function of problem (2.7), we then obtain the following equivalent non-linear program in the decision variables α and γ.

$$\begin{aligned} z_n^\star \; = \; \sup \quad & \sum_{j\in\mathcal{J}}\sum_{\boldsymbol{v}\in\mathcal{V}_n}\alpha(\boldsymbol{v})(v_j - c_j)^+ - \left(\alpha(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma(\boldsymbol{w},\boldsymbol{v})(v_j - w_j)\right)^+ \\ \text{s.t.} \quad & \alpha\in\mathcal{L}(\mathcal{V}_n,\mathbb{R}_+),\ \gamma\in\mathcal{L}(\mathcal{V}_n\times\mathcal{V}_n,\mathbb{R}_+) \\ & \sum_{\boldsymbol{v}\in\mathcal{V}_n}\alpha(\boldsymbol{v}) \; = \; 1 \\ & \alpha(\boldsymbol{v}) + \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma(\boldsymbol{w},\boldsymbol{v}) - \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma(\boldsymbol{v},\boldsymbol{w}) \; \geq \; 0 \quad \forall \boldsymbol{v}\in\mathcal{V}_n \end{aligned} \tag{2.8}$$

Note that, by construction, the optimal objective value of (2.8) is still equal to $z_n^\star$.

Lemma 2.3.3 below constructs a feasible solution for problem (2.8) that asymptotically attains the objective value $\frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$ as n tends to infinity. This will allow us later to conclude that the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ defined via (2.3) is indeed optimal in (2.1).

Lemma 2.3.3. *We have* $\liminf_{n\to\infty} z_n^\star \;\geq\; \frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$.

Proof. For any $n\in\mathbb{N}$ satisfying $n > e(1+e)$, we define

$$\alpha_n(\boldsymbol{v}) = \begin{cases} \frac{n}{ek(k+1)} & \text{if } \exists k\in\left\{\lfloor\frac{n}{e}\rfloor,\ldots,n-1\right\} \text{ with } \boldsymbol{v}-\boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}}-\boldsymbol{c}), \\ 1-\sum_{k=\lfloor\frac{n}{e}\rfloor}^{n-1}\dfrac{n}{ek(k+1)} & \text{if } \boldsymbol{v} \;=\; \overline{\boldsymbol{v}}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\gamma_n(\boldsymbol{w},\boldsymbol{v}) \;=\; \begin{cases} \frac{(n-e-e^2)n}{e(n-e)(k+1)} & \text{if } \exists k\in\left\{\lfloor\frac{n}{e}\rfloor,\ldots,n-1\right\} \text{ with } \boldsymbol{v}-\boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}}-\boldsymbol{c}) \text{ and } \boldsymbol{w} = \boldsymbol{v}+\frac{1}{n}(\overline{\boldsymbol{v}}-\boldsymbol{c}), \\ 0 & \text{otherwise.} \end{cases}$$

We will show that (α_n,γ_n) is feasible in (2.8) (Step 1) and yields a lower bound on $z_n^\star$ (Step 2). The claim then follows by showing that this lower bound converges to $\frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$ as n tends to infinity (Step 3).

Step 1: From the definitions of α_n and γ_n, it is easy to verify that $\sum_{\boldsymbol{v}\in\mathcal{V}_n}\alpha_n(\boldsymbol{v}) = 1$, while $\alpha_n(\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in \mathcal{V}_n \setminus \{\overline{\boldsymbol{v}}\}$ and $\gamma_n(\boldsymbol{w},\boldsymbol{v}) \geq 0$ for all $\boldsymbol{w},\boldsymbol{v} \in \mathcal{V}$. In addition, we observe that

$$\begin{aligned}\alpha_n(\overline{\boldsymbol{v}}) = 1 - \sum_{k=\lfloor\frac{n}{e}\rfloor}^{n-1}\frac{n}{ek(k+1)} &= 1 - \frac{n}{e}\left(\sum_{k=\lfloor\frac{n}{e}\rfloor}^{n-1}\frac{1}{k} - \frac{1}{k+1}\right)\\ = 1 - \frac{n}{e}\left(\frac{1}{\lfloor\frac{n}{e}\rfloor} - \frac{1}{n}\right) &> 1 - \frac{n}{n-e} + \frac{1}{e},\end{aligned} \tag{2.9}$$

where the third equality follows from the cancellation of all intermediate terms within the telescoping series, and the inequality holds because $\lfloor\frac{n}{e}\rfloor > \frac{n}{e} - 1$. The assumption $n > e(1+e)$ further ensures that

$$1 - \frac{n}{n-e} + \frac{1}{e} = -\frac{e}{n-e} + \frac{1}{e} > 0.$$

Hence, we may conclude that $\alpha_n(\overline{\boldsymbol{v}}) > 0$. To show that (α_n, γ_n) is feasible in (2.8), it thus remains to show that $\alpha_n(\boldsymbol{v}) + \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma_n(\boldsymbol{w},\boldsymbol{v}) - \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma_n(\boldsymbol{v},\boldsymbol{w}) \geq 0$ for all $\boldsymbol{v} \in \mathcal{V}_n$. By the definitions of α_n and γ_n, it suffices to show that this inequality holds when there exists an integer $k \in \{\lfloor\frac{n}{e}\rfloor, \ldots, n\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$. Otherwise, the left-hand side of the inequality trivially evaluates to 0. When $k = \lfloor\frac{n}{e}\rfloor$, we have $\gamma_n(\boldsymbol{v},\boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \mathcal{V}_n$, and the postulated inequality indeed holds because $\alpha_n(\boldsymbol{v}) \geq 0$ and $\gamma_n(\boldsymbol{w},\boldsymbol{v}) \geq 0$ for all $\boldsymbol{w} \in \mathcal{V}_n$. On the other hand, when $k = n$ or, equivalently, when $\boldsymbol{v} = \overline{\boldsymbol{v}}$, we have $\gamma_n(\boldsymbol{w},\overline{\boldsymbol{v}}) = 0$ for all $\boldsymbol{w} \in \mathcal{V}_n$ and $\gamma_n(\overline{\boldsymbol{v}},\boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \mathcal{V}_n \setminus \{\overline{\boldsymbol{v}} - (\overline{\boldsymbol{v}} - \boldsymbol{c})/n\}$. Hence

$$\begin{aligned}\alpha_n(\overline{\boldsymbol{v}}) + \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma_n(\boldsymbol{w},\overline{\boldsymbol{v}}) - \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma_n(\overline{\boldsymbol{v}},\boldsymbol{w}) &= \alpha_n(\overline{\boldsymbol{v}}) - \gamma_n(\overline{\boldsymbol{v}}, \overline{\boldsymbol{v}} - (\overline{\boldsymbol{v}} - \boldsymbol{c})/n)\\ &> \left(1 - \frac{n}{n-e} + \frac{1}{e}\right) - \frac{n-e-e^2}{e(n-e)} = 0,\end{aligned}$$

where the inequality follows from (2.9). Finally, when $\lfloor\frac{n}{e}\rfloor + 1 \leq k \leq n-1$, we have

$$\begin{aligned}\alpha_n(\boldsymbol{v}) + \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma_n(\boldsymbol{w},\boldsymbol{v}) - \sum_{\boldsymbol{w}\in\mathcal{V}_n}\gamma_n(\boldsymbol{v},\boldsymbol{w}) &= \alpha_n(\boldsymbol{v}) + \gamma_n(\boldsymbol{v} + (\overline{\boldsymbol{v}} - \boldsymbol{c})/n, \boldsymbol{v}) - \gamma_n(\boldsymbol{v}, \boldsymbol{v} - (\overline{\boldsymbol{v}} - \boldsymbol{c})/n)\\ &= \frac{n}{ek(k+1)} + \frac{(n-e-e^2)n}{e(n-e)}\left(\frac{1}{k+1} - \frac{1}{k}\right)\\ &= \frac{n}{ek(k+1)} - \frac{(n-e-e^2)n}{ek(k+1)(n-e)} = \frac{ne}{k(k+1)(n-e)} > 0.\end{aligned}$$

We may therefore conclude that (α_n, γ_n) is feasible in (2.8) as postulated.

Step 2: The objective function value of (α_n, γ_n) in the non-linear program (2.8) can be expressed as $z_n^+ - z_n^-$, where

$$
\begin{aligned}
z_n^+ &= \sum_{j\in\mathcal{J}} \sum_{\boldsymbol{v}\in\mathcal{V}_n} \alpha_n(\boldsymbol{v})(v_j - c_j)^+, \\
z_n^- &= \left(\sum_{j\in\mathcal{J}} \sum_{\boldsymbol{v}\in\mathcal{V}_n} \alpha_n(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w}\in\mathcal{V}_n} (v_j - w_j)\gamma_n(\boldsymbol{w},\boldsymbol{v}) \right)^+ .
\end{aligned}
$$

By construction, $\alpha_n(\boldsymbol{v})$ is non-zero if and only if there exists an integer $k \in \left\{\lfloor \frac{n}{e} \rfloor, \dots, n\right\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$. Therefore, z_n^+ can be reformulated as

$$
\begin{aligned}
z_n^+ &= \sum_{j\in\mathcal{J}} \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{n}{ek(k+1)} \cdot \frac{k}{n} (\overline{v}_j - c_j)^+ + \sum_{j\in\mathcal{J}} \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j)^+ \\
&= \sum_{j\in\mathcal{J}} \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{1}{e(k+1)} \cdot (\overline{v}_j - c_j) + \sum_{j\in\mathcal{J}} \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j),
\end{aligned}
$$

where the second equality follows from our standing assumption that $\boldsymbol{c} < \overline{\boldsymbol{v}}$.

In order to reformulate z_n^-, we first observe that

$$
\alpha_n(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w}\in\mathcal{V}_n} (v_j - w_j)\gamma_n(\boldsymbol{w},\boldsymbol{v}) = \begin{cases} \alpha_n(\boldsymbol{v})(v_j - c_j) - \frac{1}{n}(\overline{v}_j - c_j)\gamma_n\left(\boldsymbol{v} + \frac{1}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c}), \boldsymbol{v}\right) \\ \qquad\qquad \text{if } \boldsymbol{v} \in \mathcal{V}_n \setminus \{\overline{\boldsymbol{v}}\}, \\ \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j) \qquad \text{if } \boldsymbol{v} = \overline{\boldsymbol{v}}, \end{cases}
$$

where the equality follows from the definition of γ_n, which implies that $\gamma_n(\boldsymbol{w},\boldsymbol{v}) = 0$ unless $\boldsymbol{w} = \boldsymbol{v} + \frac{1}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c}) \in \mathcal{V}_n$. By the definitions of α_n and γ_n, the right-hand side of the above equality trivially vanishes if $\boldsymbol{v} \neq \overline{\boldsymbol{v}}$ and there is no $k \in \left\{\lfloor \frac{n}{e} \rfloor, \dots, n-1\right\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$. Moreover, if $\boldsymbol{v} \neq \overline{\boldsymbol{v}}$ and there exists $k \in \left\{\lfloor \frac{n}{e} \rfloor, \dots, n-1\right\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$, then the right-hand side evaluates to

$$
\begin{aligned}
\frac{n}{ek(k+1)} \cdot \frac{k}{n}(\overline{v}_j - c_j) - \frac{1}{n}(\overline{v}_j - c_j) \cdot \frac{(n-e-e^2)n}{e(n-e)(k+1)} &= \frac{1}{e(k+1)}(\overline{v}_j - c_j)\left(1 - \frac{n-e-e^2}{n-e}\right) \\
&= \frac{e}{(n-e)(k+1)}(\overline{v}_j - c_j).
\end{aligned}
$$

This observation together with the definition of z_n^- implies that

$$
\begin{aligned}
z_n^- &= \left(\frac{e}{(n-e)} \sum_{j\in\mathcal{J}} \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{1}{k+1}(\overline{v}_j - c_j) + \sum_{j\in\mathcal{J}} \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j) \right)^+ \\
&= \frac{e}{(n-e)} \sum_{j\in\mathcal{J}} \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{1}{k+1}(\overline{v}_j - c_j) + \sum_{j\in\mathcal{J}} \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j),
\end{aligned}
$$

where the second equality holds because $\boldsymbol{c} < \overline{\boldsymbol{v}}$ by assumption.

Step 3: Our insights from Steps 1 and 2 imply that

$$z_n^\star \geq z_n^+ - z_n^- = \left(\frac{1}{e} - \frac{e}{n-e}\right) \sum_{j \in \mathscr{J}} \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{1}{k+1} \cdot (\overline{v}_j - c_j) = \left(\frac{1}{e} - \frac{e}{n-e}\right) \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \sum_{k=\lfloor \frac{n}{e} \rfloor + 1}^{n} \frac{1}{k},$$

where the inequality follows from the feasibility of (α_n, γ_n) in (2.8) established in Step 1, the first equality exploits the explicit formulas for z_n^+ and z_n^- derived in Step 2, and the last equality is due to the index shift $k \leftarrow k+1$. We may thus conclude that

$$\liminf_{n\to\infty} z_n^\star \geq \frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \lim_{n\to\infty} \sum_{k=\lfloor \frac{n}{e} \rfloor + 1}^{n} \frac{1}{k} = \frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}).$$

Here, the last equality follows from the observation that $\sum_{k=\lfloor \frac{n}{e} \rfloor + 1}^{n} \frac{1}{k}$ constitutes a difference of two Harmonic series. Using Theorem 2.2.1 by Lagarias (2013), its limit can thus be calculated in closed form as

$$\lim_{n\to\infty} \sum_{k=\lfloor \frac{n}{e} \rfloor + 1}^{n} \frac{1}{k} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{\lfloor \frac{n}{e} \rfloor} \frac{1}{k} \right) = \lim_{n\to\infty} \log(n) - \log\left(\left\lfloor \frac{n}{e} \right\rfloor\right) = 1.$$

Thus, the claim follows. ■

We are now ready to prove that the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ is indeed optimal.

Theorem 2.3.1. *The separable mechanism* $(\boldsymbol{q}^\star, m^\star)$ *defined through* (2.3) *is optimal in* (2.1). *The optimal value of* (2.1) *is given by* $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Proof. By Lemma 2.3.1, $(\boldsymbol{q}^\star, m^\star)$ is feasible in (2.1). Moreover, its objective value satisfies

$$\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \geq z^\star \geq \liminf_{n\to\infty} z_n^\star \geq \frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}),$$

where the three inequalities follow from Lemma 2.3.1, Lemma 2.3.2 and Lemma 2.3.3, respectively. This implies that $z^\star = \frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$, and thus $(\boldsymbol{q}^\star, m^\star)$ is optimal in (2.1). ■

Remark 2.3.1. *All results of this section remain valid if the assumption that* $\mathbf{c} < \overline{\boldsymbol{v}}$ *is relaxed. To see this, denote by* $\mathscr{J}^-$ *the set of items with* $c_j \geq \overline{v}_j > 0$, *and construct a separable mechanism* $(\boldsymbol{q}^\star, m^\star)$ *where the underlying single-item mechanism* $(\hat{q}_j, \hat{m}_j)$ *is given by* (2.3) *for all items* $j \in \mathscr{J} \setminus \mathscr{J}^-$ *and equals the trivial mechanism* $(0,0)$ *for all items* $j \in \mathscr{J}^-$. *One can then show that* $(\boldsymbol{q}^\star, m^\star)$ *is optimal in* (2.1) *and attains a worst-case regret of* $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})^+$. *The proof of this result critically relies on the rectangularity of the uncertainty set* $\mathcal{V} = \times_{j \in \mathscr{J}} [0, \overline{v}_j]$. *Details are omitted for brevity.*

The discretized linear program (2.5) and its dual (2.7) not only enable us to solve the mechanism design problem (2.1) but also allow us to construct a Nash equilibrium for the game between the seller and nature described in Remark 2.2.1. To see this, for any $n > e(1+e)$ we define $\mathbb{P}_n \in \Delta(\mathcal{V})$ as the discrete distribution that assigns probability $\alpha_n(\boldsymbol{v})$ to any $\boldsymbol{v} \in \mathcal{V}_n$, where $\alpha_n(\boldsymbol{v})$ is defined as in the proof of Lemma 2.3.3. Note that $\mathbb{P}_n$ is normalized because $\sum_{\boldsymbol{v}\in\mathcal{V}_n} \alpha_n(\boldsymbol{v}) = 1$. Moreover, we define $\mathbb{P}^\star \in \Delta(\mathcal{V})$ via the relations

$$\mathbb{P}^\star(\tilde{\boldsymbol{v}} \le \boldsymbol{v}) = \begin{cases} \min_{j\in\mathcal{J}} \left(1 - \frac{1}{e}\left(\frac{\overline{v}_j - c_j}{v_j - c_j}\right)\right)^+ & \text{if } \boldsymbol{v} \in \mathcal{V} \setminus \{\overline{\boldsymbol{v}}\}, \\ 1 & \text{if } \boldsymbol{v} = \overline{\boldsymbol{v}}, \end{cases} \tag{2.10}$$

$\boldsymbol{v} \in \mathcal{V}$, which fully characterize the cumulative distribution function of $\mathbb{P}^\star$. If we define the marginal distributions $\mathbb{P}^\star_j \in \Delta([0,\overline{v}_j])$, $j \in \mathcal{J}$, through $\mathbb{P}^\star_j(\tilde{v}_j \le v_j) = \left(1 - \frac{1}{e}\left(\frac{\overline{v}_j - c_j}{v_j - c_j}\right)\right)^+$ if $v_j \in [0,\overline{v}_j)$; $= 1$ if $v_j = \overline{v}_j$, then (2.10) simplifies to $\mathbb{P}^\star(\tilde{\boldsymbol{v}} \le \boldsymbol{v}) = \min_{j\in\mathcal{J}} \mathbb{P}^\star_j(\tilde{v}_j \le v_j)$. This reveals that $\mathbb{P}^\star$ is the unique comonotone distribution with marginals $\mathbb{P}^\star_j$, $j \in \mathcal{J}$. Moreover, it is easy to verify that the support of $\mathbb{P}^\star$ is confined to the line segment $\{\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}) : s \in [\frac{1}{e}, 1]\}$. We are now ready to prove that $\mathbb{P}^\star$ can be viewed as the limit of the discrete distributions $\mathbb{P}_n$.

Lemma 2.3.4. *The discrete distributions $\mathbb{P}_n$ converge weakly to $\mathbb{P}^\star$.*

Proof. For any $n > e(1+e)$, by construction of $\mathbb{P}_n$, we have

$$\mathbb{P}_n(\tilde{\boldsymbol{v}} \le \boldsymbol{v}) = \begin{cases} \displaystyle\sum_{k=\lfloor n/e\rfloor}^{\left\lfloor n\left(\min_{j\in\mathcal{J}} \frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right\rfloor} \frac{n}{ek(k+1)} & \text{if } \boldsymbol{v} \in \mathcal{V} \setminus \{\overline{\boldsymbol{v}}\}, \\ 1 & \text{if } \boldsymbol{v} = \overline{\boldsymbol{v}}. \end{cases}$$

As $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, the sum in the above expression can be viewed as a telescoping series. For any $\boldsymbol{v} \in \mathcal{V} \setminus \{\overline{\boldsymbol{v}}\}$, we thus have

$$\mathbb{P}_n(\tilde{\boldsymbol{v}} \le \boldsymbol{v}) = \frac{n}{e}\left(\frac{1}{\lfloor n/e\rfloor} - \frac{1}{\left\lfloor n\left(\min_{j\in\mathcal{J}} \frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right\rfloor + 1}\right)^+,$$

which in turn implies that

$$\lim_{n\to\infty} \mathbb{P}_n(\tilde{\boldsymbol{v}} \le \boldsymbol{v}) = \frac{1}{e}\left(e - \frac{1}{\left(\min_{j\in\mathcal{J}} \frac{v_j - c_j}{\overline{v}_j - c_j}\right)}\right)^+ = \min_{j\in\mathcal{J}}\left(1 - \frac{1}{e}\left(\frac{\overline{v}_j - c_j}{v_j - c_j}\right)\right)^+ = \mathbb{P}^\star(\tilde{\boldsymbol{v}} \le \boldsymbol{v}).$$

Finally, we note that $\lim_{n\to\infty} \mathbb{P}_n\left(\tilde{\boldsymbol{v}} \le \overline{\boldsymbol{v}}\right) = 1 = \mathbb{P}^\star\left(\tilde{\boldsymbol{v}} \le \overline{\boldsymbol{v}}\right)$. Thus, the claim follows. ∎

In Section 2.2 we have argued that problem (2.1) can be interpreted as a zero-sum game between the seller, who chooses a mechanism $(\boldsymbol{q}, m) \in \mathcal{X}$, and nature, who chooses a probability distribution $\mathbb{P} \in \Delta(\mathcal{V})$ over the buyer's value profiles; see Remark 2.2.1. We can now show that the distribution $\mathbb{P}^\star$, which was extracted from the (discretized) dual mechanism design problem (2.7), actually represents nature's Nash strategy. To simplify the subsequent discussion, we denote by

$$z(\boldsymbol{q}, m; \mathbb{P}) \;=\; \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}^\top(\tilde{\boldsymbol{v}} - \boldsymbol{c})^+ - (m(\tilde{\boldsymbol{v}}) - \boldsymbol{q}(\tilde{\boldsymbol{v}})^\top \boldsymbol{c})\right]$$

the expected regret of the mechanism $(\boldsymbol{q}, m)$ under the probability distribution $\mathbb{P}$.

Theorem 2.3.2. *The separable mechanism* $(\boldsymbol{q}^\star, m^\star)$ *defined in* (2.3) *and the comonotone probability distribution* $\mathbb{P}^\star$ *defined in* (2.10) *satisfy the saddle point condition*

$$\max_{\mathbb{P}\in\Delta(\mathcal{V})} z(\boldsymbol{q}^\star, m^\star; \mathbb{P}) \;\le\; z(\boldsymbol{q}^\star, m^\star; \mathbb{P}^\star) \;\le\; \min_{(\boldsymbol{q},m)\in\mathcal{X}} z(\boldsymbol{q}, m; \mathbb{P}^\star). \tag{2.11}$$

Theorem 2.3.2 implies that $(\boldsymbol{q}^\star, m^\star)$ and $\mathbb{P}^\star$ form a Nash equilibrium of the game between the seller and nature. Indeed, the first inequality in (2.11) implies that $\mathbb{P}^\star$ is a best response to the mechanism $(\boldsymbol{q}^\star, m^\star)$, while the second inequality in (2.11) implies that $(\boldsymbol{q}^\star, m^\star)$ is a best response to the probability distribution $\mathbb{P}^\star$.

Proof of Theorem 2.3.2. We first show that $\mathbb{P}^\star$ solves the maximization problem on the left-hand side of (2.11) (Step 1), and then we prove that $(\boldsymbol{q}^\star, m^\star)$ solves the minimization problem on the right-hand side of (2.11) (Step 2).

Step 1: Fix the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ and an arbitrary distribution $\mathbb{P} \in \Delta(\mathcal{V})$. Then, the expected regret $z(\boldsymbol{q}^\star, m^\star; \mathbb{P})$ admits the upper bound

$$\begin{aligned}
z(\boldsymbol{q}^\star, m^\star; \mathbb{P}) \;&=\; \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}} (\tilde{v}_j - c_j)^+ - \hat{m}_j(\tilde{v}_j) + \hat{q}_j(\tilde{v}_j)c_j\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}} (\tilde{v}_j - c_j)^+ + \mathbb{1}_{\tilde{v}_j \ge c_j + \frac{1}{e}(\overline{v}_j - c_j)}\left(-\tilde{v}_j + \frac{\overline{v}_j - c_j}{e} - c_j \log\left(\frac{\tilde{v}_j - c_j}{\overline{v}_j - c_j}\right) + c_j + c_j \log\left(\frac{\tilde{v}_j - c_j}{\overline{v}_j - c_j}\right)\right)\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}} (\tilde{v}_j - c_j)^+ + \mathbb{1}_{\tilde{v}_j \ge c_j + \frac{1}{e}(\overline{v}_j - c_j)}\left(-\tilde{v}_j + \frac{\overline{v}_j - c_j}{e} + c_j\right)\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}} (\tilde{v}_j - c_j)^+ - \left(\tilde{v}_j - c_j - \frac{1}{e}(\overline{v}_j - c_j)\right)^+\right] \\
&\le \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}} \frac{1}{e}(\overline{v}_j - c_j)\right] \;=\; \frac{1}{e}\cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}),
\end{aligned}$$

where the first and the second equalities follow from definitions of $(\boldsymbol{q}^\star, m^\star)$ and $(\hat{q}_j, \hat{m}_j)$, $j \in \mathcal{J}$, respectively, while the inequality holds due to the sub-additivity of the maximum operator and the assumption that $\overline{\boldsymbol{v}} > \boldsymbol{c}$. As the support of $\mathbb{P}^\star$ is a subset of the rectangle

$\times_{j\in\mathscr{J}}[c_j + \frac{1}{e}(\overline{v}_j - c_j), \overline{v}_j]$, the inequality is tight for $\mathbb{P}^\star$. Thus, $\mathbb{P}^\star$ solves nature's maximization problem in (2.11).

Step 2: Fix now the distribution $\mathbb{P}^\star$, and consider a relaxation of the minimization problem on the right-hand side of (2.11) that enforces the incentive compatibility and individual rationality constraints only on the line segment $\mathscr{V}^s = \{\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}) : s \in [0,1]\}$ containing the support of $\mathbb{P}^\star$. Defining $\mathbb{P}^s \in \Delta([0,1])$ through $\mathbb{P}^s(\tilde{s} \le s) = \left(1 - \frac{1}{es}\right)^+$ if $s \in [0,1)$; $=1$ if $s = 1$, this relaxation can be reformulated as

$$\begin{aligned}
\inf \quad & \mathbb{E}_{\mathbb{P}^s}\left[\tilde{s}\cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - m^s(\tilde{s}) + \boldsymbol{q}^s(\tilde{s})^\top \boldsymbol{c}\right] \\
\text{s.t.} \quad & \boldsymbol{q}^s \in \mathscr{L}([0,1],[0,1]^J),\ m^s \in \mathscr{L}([0,1],\mathbb{R}) \\
& \boldsymbol{q}^s(s)^\top(\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c})) - m^s(s) \ge \boldsymbol{q}^s(t)^\top(\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c})) - m^s(t) \quad \forall s,t \in [0,1] \\
& \boldsymbol{q}^s(s)^\top(\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c})) - m^s(s) \ge 0 \quad \forall s \in [0,1].
\end{aligned} \tag{2.12a}$$

Indeed, note that any mechanism $(\boldsymbol{q}, m)$ feasible on the right-hand side of (2.11) induces a pair of functions $(\boldsymbol{q}^s, m^s)$ feasible in (2.12a) with the same objective value, where $\boldsymbol{q}^s(s) = \boldsymbol{q}(\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}))$ and $m^s(s) = m(\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}))$ for all $s \in [0,1]$. Note also that the two constraints in (2.12a) are easily recognized as the incentive compatibility and individual rationality constraints restricted to the line segment $\mathscr{V}^s$, respectively. Using the variable substitution $f(s) \leftarrow \boldsymbol{q}^s(s)^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})$ and $g(s) \leftarrow m^s(s) - \boldsymbol{q}^s(s)^\top \boldsymbol{c}$, problem (2.12a) can be reformulated as

$$\begin{aligned}
\inf \quad & \mathbb{E}_{\mathbb{P}^s}\left[\tilde{s}\cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - g(\tilde{s})\right] \\
\text{s.t.} \quad & f \in \mathscr{L}([0,1],[0,\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})]),\ g \in \mathscr{L}([0,1],\mathbb{R}) \\
& sf(s) - g(s) \ge sf(t) - g(t) \quad \forall s,t \in [0,1] \\
& sf(s) - g(s) \ge 0 \quad \forall s \in [0,1].
\end{aligned} \tag{2.12b}$$

Normalizing f and g by the positive constant $\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})$ further simplifies problem (2.12b) to

$$\begin{aligned}
\inf \quad & \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) \cdot \mathbb{E}_{\mathbb{P}^s}\left[\tilde{s} - g(\tilde{s})\right] \\
\text{s.t.} \quad & f \in \mathscr{L}([0,1],[0,1]),\ g \in \mathscr{L}([0,1],\mathbb{R}) \\
& sf(s) - g(s) \ge sf(t) - g(t) \quad \forall s,t \in [0,1] \\
& sf(s) - g(s) \ge 0 \quad \forall s \in [0,1].
\end{aligned} \tag{2.12c}$$

Note that minimizing $\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) \cdot \mathbb{E}_{\mathbb{P}^s}\left[\tilde{s} - g(\tilde{s})\right]$ is tantamount to maximizing $\mathbb{E}_{\mathbb{P}^s}\left[g(\tilde{s})\right]$. This reveals that problem (2.12c) is equivalent to a single-item pricing problem with the objective of maximizing expected revenues under the probability distribution $\mathbb{P}^s$, where the cost of procuring the item vanishes. Note that the auxiliary variables f and g in (2.12c) are naturally interpreted as univariate allocation and payment rules, respectively.

It is well-known that the maximum expected revenue under $\mathbb{P}^s$ is given by $\max_{p\in\mathbb{R}} p(1-\mathbb{P}^s(\tilde{s}\le p))$; see Myerson (1981) or Riley and Zeckhauser (1983). By the definition of $\mathbb{P}^s$, we have

$$p(1-\mathbb{P}^s(\tilde{s}\le p)) = \begin{cases} p & \text{if } p < \frac{1}{e}, \\ \frac{1}{e} & \text{if } p \in [\frac{1}{e}, 1), \\ 0 & \text{otherwise,} \end{cases}$$

and thus $\max_{p\in\mathbb{R}} p(1-\mathbb{P}^s(\tilde{s}\le p)) = \frac{1}{e}$. Moreover, a direct calculation shows that $\mathbb{E}_{\mathbb{P}^s}[\tilde{s}] = \frac{2}{e}$. The optimal value of (2.12c) therefore amounts to $\frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$. As (2.12c) was obtained by relaxing the mechanism design problem on the right-hand side of (2.11) and as $z(\boldsymbol{q}^\star, m^\star; \mathbb{P}^\star) = \frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$ by our reasoning in Step 1, we may thus conclude that $(\boldsymbol{q}^\star, m^\star)$ solves the seller's minimization problem in (2.11). ■

We are now ready to elucidate how our results relate to those by Bergemann and Schlag (2008), who investigate a *single-item* pricing problem with worst-case regret objective that minimizes over all randomized posted price mechanisms encoded by univariate distributions $\mathbb{Q}\in\Delta([0,\overline{v}])$. Under any such mechanism, the seller draws a random price $\tilde{p}$ from $\mathbb{Q}$ and sells the good to the buyer at price $\tilde{p}$ whenever $\tilde{p}$ is smaller or equal to the buyer's value $\tilde{v}$. The best randomized posted price mechanism can thus be found by solving the worst-case regret minimization problem

$$\inf_{\mathbb{Q}\in\Delta([0,\overline{v}])}\ \sup_{\mathbb{P}\in\Delta([0,\overline{v}])} \int_0^{\overline{v}}\int_0^{\overline{v}} (v-c)^+ - \mathbb{1}_{p\le v}(p-c)\,\mathrm{d}\mathbb{Q}(p)\,\mathrm{d}\mathbb{P}(v), \tag{2.13}$$

which can again be viewed as a zero-sum game akin to (2.2). In the following, denote by $(q^\star, m^\star)$ the mechanism defined via (2.3) for $J=1$, and define the univariate distribution $\mathbb{Q}^\star\in\Delta([0,\overline{v}])$ through $\mathbb{Q}^\star(\tilde{p}\le p) = q^\star(p)$ for all $p\in[0,\overline{v}]$. Moreover, denote by $\mathbb{P}^\star\in\Delta([0,\overline{v}])$ the distribution defined in (2.10) for $J=1$. Bergemann and Schlag (2008) show that $\mathbb{Q}^\star$ and $\mathbb{P}^\star$ form a Nash equilibrium for problem (2.13) and that the optimal value of (2.13) evaluates to $\frac{1}{e}\cdot(\overline{v}-c)$. Our Theorems 2.3.1 and 2.3.2 thus encompass the single-item pricing theory by Bergemann and Schlag (2008) as a special case. More specifically, one can show that there is a one-to-one correspondence between the single-item mechanisms (q,m) feasible in (2.1) that involve a *right-continuous* allocation rule and the randomized posted price mechanisms $\mathbb{Q}$ feasible in (2.13) that satisfy $\mathbb{Q}(\tilde{p}\le p) = q(p)$ for all $p\in[0,\overline{v}]$. The analysis of multi-item pricing problems portrayed in this section critically relies on our representation of the selling mechanisms in terms of generic allocation and payment rules that are subject to explicit incentive compatibility and individual rationality constraints. In contrast, randomized posted price mechanisms for multiple items are difficult to characterize because they require a separate posted price for each of the exponentially many bundles $\mathscr{S}\in 2^{\mathscr{J}}$. Moreover, each realization of the posted prices leads to a different tessellation of the uncertainty set $\mathcal{V}$ into 2^J polytopes (see Figure 2.1 for a visualization when $J = 2$), and the revenue of the seller depends on the particular polytope that accommodates the uncertain value profile $\tilde{\boldsymbol{v}}$. In order to evaluate the seller's expected revenue, one would thus have to compute the probabilities of exponentially many (random) polytopes with respect to $\mathbb{P}$ and integrate a (random) weighted sum of these probabilities with respect to $\mathbb{Q}$, which seems excruciating.

When $J = 1$, the optimal randomized posted price mechanism $\mathbb{Q}^\star$ offers distinct implementational advantages over the optimal single-item mechanism $(q^\star, m^\star)$ even though the agents' expected utilities are identical under both mechanisms irrespective of the value distribution $\mathbb{P}$. Specifically, under the randomized posted price mechanism the buyer only needs to make a payment if he actually receives the good. In contrast, under the optimal single-item mechanism the seller offers the buyer a lottery to win the good with probability $q^\star(v)$, and the payment $m^\star(v)$ can be interpreted as a participation fee that is due upfront. It could thus happen that the buyer ends up making a payment without obtaining the good. On the other hand, buyers who are lucky to win the good under the optimal single-item mechanism incur a lower cost than under the randomized posted price mechanism. We conclude that the randomized posted price mechanism is more likely to be accepted in practice because the prospect of making a payment without any reward is likely to disconcert potential buyers.

When $J > 1$, we have shown here that the optimal multi-item mechanism $(\boldsymbol{q}^\star, m^\star)$ is separable. Therefore, it can still easily be implemented as a randomized posted price mechanism without the need to specify separate posted prices for all possible bundles. Instead, one only needs one randomized posted price per item, and the buyer only has to compare his value for a particular item with the respective posted price.

2.4 Optimal Deterministic Mechanism

Some agents may feel uncomfortable about randomized selling mechanisms. Thus, we study now a *deterministic* variant of problem (2.1), where the allocation rule $\boldsymbol{q}$ must be chosen from $\mathcal{L}(\mathcal{V}, \{0,1\}^J)$. This means that the items are assigned to the buyer either with probability 0 or 1.

Every deterministic allocation rule $\boldsymbol{q} \in \mathcal{L}(\mathcal{V}, \{0,1\}^J)$ induces a unique *bundle allocation rule* $s \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}})$ defined through $s(\boldsymbol{v}) = \{j \in \mathcal{J} : q_j(\boldsymbol{v}) = 1\}$ for all $\boldsymbol{v} \in \mathcal{V}$. Thus, $s(\boldsymbol{v}) \subseteq \mathcal{J}$ represents the bundle acquired by a buyer with value profile $\boldsymbol{v}$. Moreover, the incentive compatibility constraint implies that $m(\boldsymbol{v}) = m(\boldsymbol{w})$ whenever $s(\boldsymbol{v}) = s(\boldsymbol{w})$. This means that the seller receives the same payment from all buyers who acquire the same bundle. Thus, for every $\mathcal{S} \subseteq \mathcal{J}$, there exists a bundle price $p_{\mathcal{S}}$, and the payment rule must satisfy $m(\boldsymbol{v}) = p_{s(\boldsymbol{v})}$. This observation allows us to reformulate the deterministic mechanism design problem over $\boldsymbol{q} \in \mathscr{L}(\mathcal{V}, \{0,1\}^J)$ and $m \in \mathscr{L}(\mathcal{V}, \mathbb{R})$ as an equivalent problem over all bundle allocation rules $s \in \mathscr{L}(\mathcal{V}, 2^{\mathscr{I}})$ and vectors of posted prices $\boldsymbol{p} \in \mathbb{R}^{2^J}$. The deterministic pricing problem thus simplifies to

$$
\begin{aligned}
z_{\mathrm{d}}^\star \;=\; \inf \quad & \sup_{\boldsymbol{v}\in\mathcal{V}} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - (p_{s(\boldsymbol{v})} - \mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{c}) \\
\text{s.t.} \quad & s \in \mathscr{L}(\mathcal{V}, 2^{\mathscr{I}}),\ \boldsymbol{p} \in \mathbb{R}^{2^J} \\
& \mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{v} - p_{s(\boldsymbol{v})} \geq \mathbf{1}_{s(\boldsymbol{w})}^\top \boldsymbol{v} - p_{s(\boldsymbol{w})} \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V} \\
& \mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{v} - p_{s(\boldsymbol{v})} \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V},
\end{aligned}
\tag{2.14}
$$

where the constraints ensure incentive compatibility and individual rationality, respectively.

We can now prove that optimizing over all deterministic mechanisms is equivalent to optimizing over all (deterministic) posted price mechanisms in the sense of Definition 2.2.3 and that the price for the empty bundle must vanish at optimality. This equivalence is actually a consequence of the so called 'taxation principle' from the standard mechanism design literature (Rochet, 1985).

Lemma 2.4.1. *For any* $(s, \boldsymbol{p}) \in \mathscr{L}(\mathcal{V}, 2^{\mathscr{I}}) \times \mathbb{R}^{2^J}$ *feasible in* (2.14)*, there exists* $\hat{\boldsymbol{p}} \in \mathbb{R}^{2^J}$ *such that* $(s, \hat{\boldsymbol{p}})$ *is also feasible in* (2.14)*, attains a weakly lower objective value than* $(s, \boldsymbol{p})$*, and satisfies*

(i) $s(v) \in \arg\max_{\mathscr{S} \subseteq \mathscr{I}} \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - \hat{p}_{\mathscr{S}}$ *for all* $\boldsymbol{v} \in \mathcal{V}$*,*

(ii) $\hat{p}_\emptyset = 0$.

Condition (i) ensures that the buyer acquires a bundle that maximizes his utility, which in turn implies that the mechanism induced by $\hat{\boldsymbol{p}}$ represents a posted price mechanism in the sense of Definition 2.2.3. Condition (ii) eliminates the arbitrage opportunity that would allow the buyer to earn free money when acquiring no item.

Proof of Lemma 2.4.1. We first show that for every deterministic mechanism there exists an equally desirable one that satisfies condition (i) (Step 1). Next, we prove that for every mechanism satisfying condition (i) there exists a weakly preferable one that satisfies both conditions (i) and (ii) (Step 2).

Step 1: For a fixed bundle allocation rule $s \in \mathscr{L}(\mathcal{V} \times 2^{\mathscr{I}})$, we define range$(s) = \{s(\boldsymbol{v}) : \boldsymbol{v} \in \mathcal{V}\}$ as the collection of all purchasable bundles. Since the posted prices $p_{\mathscr{S}}$, $\mathscr{S} \notin$ range(s), do not enter the deterministic pricing problem (2.14) at all, we can assign arbitrary values to them. In particular, we may introduce a new posted price vector $\hat{\boldsymbol{p}}$ with $\hat{p}_{\mathscr{S}} = p_{\mathscr{S}}$ for all $\mathscr{S} \in$ range(s) and $\hat{p}_{\mathscr{S}} = \mathbf{1}_{\mathscr{S}}^\top \overline{\boldsymbol{v}}$ for all $\mathscr{S} \notin$ range(s). By construction, $(s, \hat{p})$ is feasible in (2.14) and attains the same objective value as $(s, \boldsymbol{p})$. Moreover, we have

$$s(\boldsymbol{v}) \in \arg\max_{\mathscr{S}\in\mathrm{range}(s)} \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - p_{\mathscr{S}} \subseteq \arg\max_{\mathscr{S}\subseteq\mathscr{J}} \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - \hat{p}_{\mathscr{S}} \quad \forall \boldsymbol{v}\in\mathcal{V},$$

where the first inclusion follows from the incentive compatibility of $(s, \boldsymbol{p})$ as implied by the first constraint in (2.14), while the second inclusion follows from the individual rationality of $(s, \boldsymbol{p})$ and the definition of $\hat{\boldsymbol{p}}$, which ensure that $\max_{\mathscr{S}\in\mathrm{range}(s)} \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - p_{\mathscr{S}} \geq 0$ and that $\mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - \hat{p}_{\mathscr{S}} = \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - \mathbf{1}_{\mathscr{S}}^\top \overline{\boldsymbol{v}} \leq 0$ for all $\mathscr{S} \notin \mathrm{range}(s)$, respectively. Hence, $(s, \hat{p})$ satisfies condition (i).

Step 2: By the insights gained in Step 1, we may assume without loss of generality that $(s, \boldsymbol{p})$ satisfies condition (i). Next, set $\delta = \inf_{\boldsymbol{v}\in\mathcal{V}} \mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{v} - p_{s(\boldsymbol{v})}$ and note that $\delta \geq 0$ because of individual rationality. We may now introduce a new posted price vector $\hat{\boldsymbol{p}} = \boldsymbol{p} + \delta\mathbf{1}$, which increases the price of each bundle by δ. It is easy to verify that $(s, \hat{\boldsymbol{p}})$ remains incentive compatible and individually rational and still satisfies condition (i). Moreover, $(s, \hat{\boldsymbol{p}})$ incurs a weakly lower regret than $(s, \boldsymbol{p})$. It remains to be shown that $\hat{p}_\emptyset = 0$. To this end, denote by $\{\boldsymbol{v}_k\}_{k\in\mathbb{N}}$ a sequence of value profiles that asymptotically attain the infimal buyer utility δ under the mechanism $(s, \boldsymbol{p})$. Thus, we have

$$\lim_{k\to\infty} \mathbf{1}_{s(\boldsymbol{v}_k)}^\top \boldsymbol{v}_k - p_{s(\boldsymbol{v}_k)} = \delta,$$

which in turn implies

$$0 = \lim_{k\to\infty} \mathbf{1}_{s(\boldsymbol{v}_k)}^\top \boldsymbol{v}_k - p_{s(\boldsymbol{v}_k)} - \delta = \lim_{k\to\infty} \mathbf{1}_{s(\boldsymbol{v}_k)}^\top \boldsymbol{v}_k - \hat{p}_{s(\boldsymbol{v}_k)} \geq \lim_{k\to\infty} \mathbf{1}_{\emptyset}^\top \boldsymbol{v}_k - \hat{p}_{\emptyset}. = -\hat{p}_\emptyset$$

Hence, $\hat{p}_\emptyset \geq 0$. Finally, if $\emptyset \in \mathrm{range}(s)$, the individual rationality constraint implies $-\hat{p}_\emptyset \geq 0$, and if $\emptyset \notin \mathrm{range}(s)$, then we are free to set $\hat{p}_\emptyset = \mathbf{1}_\emptyset^\top \overline{\boldsymbol{v}} = 0$ without compromising condition (i) (see Step 1). In both cases, we have $\hat{p}_\emptyset = 0$. ■

Lemma 2.4.2. *Any posted price mechanism* $(s, \boldsymbol{p}) \in \mathscr{L}(\mathcal{V}, 2^{\mathscr{J}}) \times \mathbb{R}^{2^J}$ *that satisfies conditions (i) and (ii) from Lemma 2.4.1 is both incentive compatible and individually rational, and thus it is feasible in* (2.14).

Proof. Condition (i) implies that

$$\mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{v} - p_{s(\boldsymbol{v})} = \max_{\mathscr{S}\subseteq\mathscr{J}} \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - p_{\mathscr{S}} \geq \mathbf{1}_{s(\boldsymbol{w})}^\top \boldsymbol{v} - p_{s(\boldsymbol{w})} \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$$

and is thus a sufficient condition for the incentive compatibility constraint in (2.14). By condition (i), we further have

$$\mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{v} - p_{s(\boldsymbol{v})} \geq \mathbf{1}_{\emptyset}^\top \boldsymbol{v} - p_{\emptyset} = 0,$$

where the equality follows from condition (ii). Thus, condition (ii) can be viewed as a sufficient condition for the individual rationality constraint in (2.14). ■

Lemma 2.4.1 implies that conditions (i) and (ii) may be appended as constraints to problem (2.14) without increasing its optimal value, while Lemma 2.4.2 shows that the incentive compatibility and individual rationality constraints are redundant in the resulting optimization problem and may thus be eliminated. Thus, the deterministic mechanism design problem (2.14) is equivalent to

$$\begin{aligned} z_{\mathrm{d}}^{\star} \;=\; \inf \quad & \sup_{\boldsymbol{v}\in\mathcal{V}} \; \mathbf{1}^{\top}(\boldsymbol{v}-\boldsymbol{c})^{+} - (p_{s(\boldsymbol{v})} - \mathbf{1}_{s(\boldsymbol{v})}^{\top}\boldsymbol{c}) \\ \text{s.t.} \quad & s \in \mathscr{L}(\mathcal{V}, 2^{\mathscr{J}}),\; \boldsymbol{p}\in\mathscr{P} \\ & s(\boldsymbol{v}) \in \arg\max_{\mathscr{S}\subseteq\mathscr{J}} \; \mathbf{1}_{\mathscr{S}}^{\top}\boldsymbol{v} - p_{\mathscr{S}} \quad \forall \boldsymbol{v}\in\mathcal{V}, \end{aligned} \tag{2.15}$$

where $\mathscr{P} = \{\boldsymbol{p}\in\mathbb{R}^{2^J} : p_{\emptyset} = 0\}$ will henceforth be referred to as the set of admissible posted prices.

Problem (2.15) is easily recognized as the worst-case regret minimization problem over all posted price mechanisms that set the price of the empty bundle to zero; see also Definition 2.2.3.

Remark 2.4.1. *Problem* (2.15) *can be interpreted as a two-stage robust bilevel program, where the seller acts as the leader, and the buyer acts as the follower. Indeed, the leader chooses a posted price vector* $\boldsymbol{p}$ *before* $\boldsymbol{v}$ *is revealed with the aim to minimize her worst-case regret, while the follower chooses the bundle* $s(\boldsymbol{v})$ *after* $\boldsymbol{v}$ *is revealed with the aim to maximize his utility. More precisely, problem* (2.15) *constitutes an* optimistic *bilevel program because ties are broken at the discretion of the leader whenever the follower's problem admits multiple optimal solutions. This optimistic bilevel program essentially models an* indirect *implementation of a posted price mechanism, whereby the buyer first reports his value profile* $\boldsymbol{v}$ *to the seller, and the seller then picks a utility-maximizing bundle on behalf of the buyer.*

In order to solve the deterministic mechanism design problem (2.15), we introduce the sets

$$\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},\delta) \;=\; \left\{\boldsymbol{v}\in\mathcal{V} : \mathbf{1}_{\mathscr{S}}^{\top}\boldsymbol{v} - p_{\mathscr{S}} \geq \mathbf{1}_{\mathscr{S}'}^{\top}\boldsymbol{v} - p_{\mathscr{S}'} + \delta \quad \forall \mathscr{S}'\subseteq\mathscr{J} : \mathscr{S}'\neq\mathscr{S}\right\}$$

parameterized by $\boldsymbol{p}\in\mathscr{P}$, $\mathscr{S}\subseteq\mathscr{J}$ and $\delta\geq 0$. By definition, $\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},0)$ is the set of all value profiles under which the buyer weakly prefers bundle $\mathscr{S}$ to any other bundle $\mathscr{S}'$, and the interior of $\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},0)$ contains those value profiles under which the buyer strictly prefers $\mathscr{S}$. Note that the polytopes $\{\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},0)\}_{\mathscr{S}\subseteq\mathscr{J}}$ have disjoint interiors but may have overlapping boundaries. Thus, for any fixed $\boldsymbol{p}\in\mathscr{P}$, the bundle allocation rule $s(\boldsymbol{v})$ is uniquely determined *almost* everywhere.

We now define an auxiliary problem parameterized by $\delta \geq 0$.

$$z'_{\mathrm{d}}(\delta) \;=\; \inf_{\boldsymbol{p}\in\mathscr{P}} \max_{\mathscr{S}\subseteq\mathscr{J}} \max_{\boldsymbol{v}\in\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},\delta)} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - (p_{\mathscr{S}} - \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{c}) \tag{2.16}$$

Remark 2.4.2. *Problem* (2.16) *with* $\delta = 0$ *admits again an intuitive interpretation as a two-stage robust bilevel program, where the leader chooses a posted price vector* $\boldsymbol{p}$ *that minimizes her worst-case regret, anticipating that the follower will choose a bundle* $\mathscr{S}$ *that maximizes his utility. Indeed, note that the regret of a particular value profile* $\boldsymbol{v}$ *in* (2.16) *is always evaluated under the bundle* $\mathscr{S}$ *that maximizes the utility of the buyer of type* $\boldsymbol{v}$*. More precisely, problem* (2.16) *with* $\delta = 0$ *constitutes a* pessimistic *bilevel program because ties are broken at the discretion of the follower, while the leader hedges against the most adverse of the follower's optimal solutions. This pessimistic bilevel program essentially models a* direct *implementation of a posted price mechanism, whereby the buyer picks a utility-maximizing bundle on his own.*

In the remainder of this section we will demonstrate that the optimistic bilevel program (2.15) is in fact equivalent to the pessimistic bilevel program (2.16) with $\delta = 0$, which implies that it is immaterial whether a posted price mechanism is implemented in a direct or an indirect fashion. To prove this equivalence, we will also study the auxiliary problem (2.16) with $\delta > 0$, which lacks an intuitive physical interpretation.

Lemma 2.4.3. *We have* $z'_{\mathrm{d}}(\delta) \leq z^\star_{\mathrm{d}} \leq z'_{\mathrm{d}}(0)$ *for all* $\delta > 0$.

Proof. Problem (2.15) is equivalent to

$$\begin{aligned} z^\star_{\mathrm{d}} \;=\; \inf \quad & \sup_{\mathscr{S}\subseteq\mathscr{J}} \; \sup_{\boldsymbol{v}:s(\boldsymbol{v})=\mathscr{S}} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - (p_{\mathscr{S}} - \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{c}) \\ \text{s.t.} \quad & s \in \mathscr{L}(\mathcal{V}, 2^{\mathscr{J}}),\ \boldsymbol{p}\in\mathscr{P} \\ & s(\boldsymbol{v}) \in \arg\max_{\mathscr{S}\subseteq\mathscr{J}} \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{v} - p_{\mathscr{S}} \quad \forall \boldsymbol{v}\in\mathcal{V}, \end{aligned}$$

where the maximization over all value profiles $\boldsymbol{v}$ is decomposed into a maximization over all bundles $\mathscr{S}$ and a subsequent maximization over all value profiles $\boldsymbol{v}$ under which bundle $\mathscr{S}$ is acquired. Assume first that $\delta > 0$. As $\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},\delta)$ only contains value profiles under which the buyer strictly prefers bundle $\mathscr{S}$ to any other bundle, we have $\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},\delta) \subseteq \{v : s(\boldsymbol{v}) = \mathscr{S}\}$. This inclusion holds for all admissible $(\boldsymbol{p}, s)$, and thus we find $z^\star_d \geq z'_d(\delta)$. Similarly, for $\delta = 0$, the inequality $z^\star_d \leq z'_d(0)$ follows from the observation that $\mathcal{V}_{\mathscr{S}}(\boldsymbol{p},0) \supseteq \{v : s(\boldsymbol{v}) = \mathscr{S}\}$ for all admissible $(\boldsymbol{p}, s)$. ■

In the following we prove that the posted price vector $\boldsymbol{p}^\star \in \mathbb{R}^{2^J}$ defined through

$$p^\star_{\mathscr{S}} = \frac{1}{2}\cdot \mathbf{1}_{\mathscr{S}}^\top(\overline{\boldsymbol{v}} + \boldsymbol{c}) \quad \forall \mathscr{S}\subseteq\mathscr{J} \tag{2.17}$$

is optimal in (2.15). Note that under this pricing scheme, each item $j \in \mathscr{J}$ is assigned a price $\frac{1}{2}(\overline{v}_j + c_j)$, while the price of each bundle $\mathscr{S} \subseteq \mathscr{J}$ is obtained by summing up the prices of all items in the bundle. Thus, the items are priced separately, and there are no discounts for bundles. Note also that $\boldsymbol{p}^\star$ is feasible in (2.15) as $p^\star_\emptyset = 0$.

Lemma 2.4.4. *We have $z'_{\mathrm{d}}(0) \leq \frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})$.*

Proof. Since $\boldsymbol{p}^\star \in \mathscr{P}$, it suffices to show that

$$\max_{\boldsymbol{v} \in \mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)} \mathbf{1}^\top(\boldsymbol{v} - \boldsymbol{c})^+ - (p^\star_{\mathscr{S}} - \mathbf{1}^\top_{\mathscr{S}}\boldsymbol{c}) \;\leq\; \frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) \tag{2.18}$$

for all $\mathscr{S} \subseteq \mathscr{J}$. Consider a fixed bundle $\mathscr{S} \subseteq \mathscr{J}$. Note that (2.18) trivially holds if $\mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)$ is empty, in which case the left-hand side evaluates to $-\infty$. Suppose now that $\mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0) \neq \emptyset$ and recall that, for any $\boldsymbol{v} \in \mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)$, the buyer weakly prefers bundle $\mathscr{S}$ over any larger bundle $\mathscr{S}' \supseteq \mathscr{S}$. Formally, for any $\boldsymbol{v} \in \mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)$ and $\mathscr{S}' \supseteq \mathscr{S}$, we have

$$\begin{aligned}
\mathbf{1}^\top_{\mathscr{S}}\boldsymbol{v} - p^\star_{\mathscr{S}} \geq \mathbf{1}^\top_{\mathscr{S}'}\boldsymbol{v} - p^\star_{\mathscr{S}'} \;&\Longleftrightarrow\; \mathbf{1}^\top_{\mathscr{S}}\boldsymbol{v} - \frac{1}{2} \cdot \mathbf{1}^\top_{\mathscr{S}}(\overline{\boldsymbol{v}} + \boldsymbol{c}) \geq \mathbf{1}^\top_{\mathscr{S}'}\boldsymbol{v} - \frac{1}{2} \cdot \mathbf{1}^\top_{\mathscr{S}'}(\overline{\boldsymbol{v}} + \boldsymbol{c}) \\
&\Longleftrightarrow\; \frac{1}{2} \cdot \mathbf{1}^\top_{\mathscr{S}' \setminus \mathscr{S}}(\overline{\boldsymbol{v}} + \boldsymbol{c}) \geq \mathbf{1}^\top_{\mathscr{S}' \setminus \mathscr{S}}\boldsymbol{v}.
\end{aligned}$$

For any $j \notin \mathscr{S}$, evaluating the last inequality at $\mathscr{S}' = \mathscr{S} \cup \{j\}$ shows that $v_j \leq \frac{1}{2}(\overline{v}_j + c_j)$ for all $\boldsymbol{v} \in \mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)$. For any $j \in \mathscr{S}$, on the other hand, we have the trivial upper bound $v_j \leq \overline{v}_j$ for all $\boldsymbol{v} \in \mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)$. These uniform upper bounds on the components of $\boldsymbol{v}$ imply that

$$\begin{aligned}
\mathbf{1}^\top(\boldsymbol{v} - \boldsymbol{c})^+ = \sum_{j \in \mathscr{S}} (v_j - c_j)^+ + \sum_{j \in \mathscr{J} \setminus \mathscr{S}} (v_j - c_j)^+ &\leq \sum_{j \in \mathscr{S}} (\overline{v}_j - c_j)^+ + \sum_{j \in \mathscr{J} \setminus \mathscr{S}} \left(\frac{1}{2}(\overline{v}_j + c_j) - c_j\right)^+ \\
&= \sum_{j \in \mathscr{S}} (\overline{v}_j - c_j)^+ + \frac{1}{2} \sum_{j \in \mathscr{J} \setminus \mathscr{S}} (\overline{v}_j - c_j)^+ \\
&= \frac{1}{2}(2 \cdot \mathbf{1}_{\mathscr{S}} + \mathbf{1}_{\mathscr{J} \setminus \mathscr{S}})^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) \\
&= \frac{1}{2}(\mathbf{1} + \mathbf{1}_{\mathscr{S}})^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}),
\end{aligned}$$

for all $\boldsymbol{v} \in \mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)$, where the second equality follows from the standing assumption that $\boldsymbol{c} < \overline{\boldsymbol{v}}$. Thus, we find

$$\max_{\boldsymbol{v} \in \mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star, 0)} \mathbf{1}^\top(\boldsymbol{v} - \boldsymbol{c})^+ - (p^\star_{\mathscr{S}} - \mathbf{1}^\top_{\mathscr{S}}\boldsymbol{c}) \;\leq\; \frac{1}{2}(\mathbf{1} + \mathbf{1}_{\mathscr{S}})^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - (p^\star_{\mathscr{S}} - \mathbf{1}^\top_{\mathscr{S}}\boldsymbol{c}) \;=\; \frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}),$$

where the equality follows from the definition of $\boldsymbol{p}^\star$. Thus, (2.18) holds for all $\mathscr{S} \subseteq \mathscr{J}$. This observation completes the proof. ■

Remark 2.4.3. *It is possible to prove that the inequality* (2.18) *holds in fact as an equality.*

Specifically, one can show that $\boldsymbol{v}^\star \in \mathbb{R}^J$ defined through $v_j^\star = \overline{v}_j$ if $j \in \mathscr{S}$; $= \frac{1}{2}(\overline{v}_j + c_j)$ otherwise, solves the maximization problem on the left-hand side of (2.18). *However, this stronger statement does not help to prove that $\boldsymbol{p}^\star$ is optimal in* (2.15).

Lemma 2.4.5. *We have $z_{\mathrm{d}}'(\delta) \geq \frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - J\delta$ for all $\delta \in (0, \frac{1}{2}\min_{j\in\mathscr{J}} \overline{v}_j - c_j]$.*

Note that the interval $(0, \frac{1}{2}\min_{j\in\mathscr{J}} \overline{v}_j - c_j]$ is non-empty because of the assumption that $\overline{\boldsymbol{v}} > \boldsymbol{c}$.

Proof of Lemma 2.4.5. Fix an arbitrary $\boldsymbol{p} \in \mathscr{P}$ and $\delta \in (0, \frac{1}{2}\min_{j\in\mathscr{J}} \overline{v}_j - c_j]$. The claim will follow if we can show that

$$\max_{\mathscr{S}\subseteq\mathscr{J}} \max_{\boldsymbol{v}\in\mathscr{V}_{\mathscr{S}}(\boldsymbol{p},\delta)} \mathbf{1}^\top(\boldsymbol{v} - \boldsymbol{c})^+ - (p_{\mathscr{S}} - \mathbf{1}_{\mathscr{S}}^\top \boldsymbol{c}) \;\geq\; \frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - J\delta. \tag{2.19}$$

To this end, we define the perturbation vector $\boldsymbol{\epsilon} = \boldsymbol{p} - \boldsymbol{p}^\star$ and prove (2.19) separately for the cases $\boldsymbol{\epsilon} \ngeq \mathbf{0}$ (Step 1) and $\boldsymbol{\epsilon} \geq \mathbf{0}$ (Step 2).

Step 1 ($\boldsymbol{\epsilon} \ngeq \mathbf{0}$): Denote by $\mathscr{S}'$ the bundle with the smallest perturbation, *i.e.*, $\epsilon_{\mathscr{S}'} \leq \epsilon_{\mathscr{S}}$ for all $\mathscr{S} \subseteq \mathscr{J}$. As $\boldsymbol{\epsilon} \ngeq \mathbf{0}$, this implies that $\epsilon_{\mathscr{S}'} < 0$. Next, we define an auxiliary value profile $\hat{\boldsymbol{v}} \in \mathbb{R}^J$ through $\hat{v}_j = \overline{v}_j$ if $j \in \mathscr{S}'$; $= \frac{1}{2}(\overline{v}_j + c_j) - \delta$ otherwise. In order to establish (2.19), we will prove that $\hat{\boldsymbol{v}} \in \mathscr{V}_{\mathscr{S}'}(\boldsymbol{p},\delta)$ and that the seller's regret under the value profile $\hat{\boldsymbol{v}}$ strictly exceeds $\frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - J\delta$. By construction, $\mathbf{0} \leq \hat{\boldsymbol{v}} \leq \overline{\boldsymbol{v}}$ and thus $\hat{\boldsymbol{v}} \in \mathscr{V}$. In order to prove the stronger statement that $\hat{\boldsymbol{v}} \in \mathscr{V}_{\mathscr{S}'}(\boldsymbol{p},\delta) \subseteq \mathscr{V}$, we first reformulate the buyer's utility from choosing bundle $\mathscr{S}'$ as

$$\mathbf{1}_{\mathscr{S}'}^\top \hat{\boldsymbol{v}} - p_{\mathscr{S}'} = \mathbf{1}_{\mathscr{S}'}^\top \overline{\boldsymbol{v}} - \left(\frac{1}{2} \cdot \mathbf{1}_{\mathscr{S}'}^\top(\overline{\boldsymbol{v}} + \boldsymbol{c}) + \epsilon_{\mathscr{S}'}\right) = \frac{1}{2} \cdot \mathbf{1}_{\mathscr{S}'}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - \epsilon_{\mathscr{S}'},$$

where the first equality exploits the relation $p_{\mathscr{S}'} = p_{\mathscr{S}'}^\star + \epsilon_{\mathscr{S}'}$, and compare it against his utility from choosing any other bundle $\mathscr{S} \subseteq \mathscr{J}$, which is given by

$$\begin{aligned}
\mathbf{1}_{\mathscr{S}}^\top \hat{\boldsymbol{v}} - p_{\mathscr{S}} &= \left(\mathbf{1}_{\mathscr{S}\cap\mathscr{S}'}^\top \overline{\boldsymbol{v}} + \frac{1}{2} \cdot \mathbf{1}_{\mathscr{S}\setminus\mathscr{S}'}^\top(\overline{\boldsymbol{v}} + \boldsymbol{c}) - |\mathscr{S} \setminus \mathscr{S}'|\delta\right) - \left(\frac{1}{2} \cdot \mathbf{1}_{\mathscr{S}}^\top(\overline{\boldsymbol{v}} + \boldsymbol{c}) + \epsilon_{\mathscr{S}}\right) \\
&= \frac{1}{2} \cdot \mathbf{1}_{\mathscr{S}\cap\mathscr{S}'}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - |\mathscr{S} \setminus \mathscr{S}'|\delta - \epsilon_{\mathscr{S}}.
\end{aligned}$$

As $\boldsymbol{c} < \overline{\boldsymbol{v}}$, $\mathscr{S} \cap \mathscr{S}' \subseteq \mathscr{S}'$ and $\epsilon_{\mathscr{S}'} \leq \epsilon_{\mathscr{S}}$, it becomes evident that the buyer's utility arising from bundle $\mathscr{S}'$ exceeds that from bundle $\mathscr{S}$ by at least $\frac{1}{2} \cdot \mathbf{1}_{\mathscr{S}'\setminus\mathscr{S}}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) + |\mathscr{S} \setminus \mathscr{S}'|\delta \geq \delta$. Thus, we have $\hat{\boldsymbol{v}} \in \mathcal{V}_{\mathcal{S}'}(\boldsymbol{p},\delta)$. Next, the seller's regret under the value profile $\hat{\boldsymbol{v}}$ can be expressed as

$$\begin{aligned}
\mathbf{1}^\top(\hat{\boldsymbol{v}}-\boldsymbol{c})^+ - (p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}) &= \mathbf{1}^\top(\hat{\boldsymbol{v}}-\boldsymbol{c}) - (p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}) \\
&= \mathbf{1}_{\mathcal{S}'}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) + \mathbf{1}_{\mathcal{J}\setminus\mathcal{S}'}^\top\left(\frac{1}{2}(\overline{\boldsymbol{v}}+\boldsymbol{c}) - \boldsymbol{c} - \delta\cdot\mathbf{1}\right) - \left(p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}\right) \\
&= \frac{1}{2}\left(2\cdot\mathbf{1}_{\mathcal{S}'} + \mathbf{1}_{\mathcal{J}\setminus\mathcal{S}'}\right)^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - \left(p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}\right) - (J-|\mathcal{S}'|)\delta \\
&= \frac{1}{2}\left(\mathbf{1} + \mathbf{1}_{\mathcal{S}'}\right)^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - \left(p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}\right) - (J-|\mathcal{S}'|)\delta \\
&= \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - \epsilon_{\mathcal{S}'} - (J-|\mathcal{S}'|)\delta \\
&> \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - J\delta,
\end{aligned}$$

where the first equality holds because $\delta \le \frac{1}{2}\min_{j\in\mathcal{J}}\overline{v}_j - c_j$ while the second and the last equalities follow from the definitions of $\hat{\boldsymbol{v}}$ and $\boldsymbol{p}$, respectively. This concludes Step 1 because

$$\max_{\mathcal{S}\subseteq\mathcal{J}}\max_{\boldsymbol{v}\in\mathcal{V}_{\mathcal{S}}(\boldsymbol{p},\delta)} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - (p_{\mathcal{S}} - \mathbf{1}_{\mathcal{S}}^\top\boldsymbol{c}) \ \ge\ \mathbf{1}^\top(\hat{\boldsymbol{v}}-\boldsymbol{c})^+ - (p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top\boldsymbol{c}) \ >\ \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - J\delta.$$

Step 2 ($\boldsymbol{\epsilon} \ge \mathbf{0}$): Recall that $\mathcal{V}_\emptyset(\boldsymbol{p},\delta) = \left\{\boldsymbol{v}\in\mathcal{V} :\ \mathbf{1}_{\mathcal{S}}^\top\boldsymbol{v} + \delta \le p_{\mathcal{S}},\ \forall\mathcal{S}\subseteq\mathcal{J} : \mathcal{S}\neq\emptyset\right\}$. In analogy to Step 1, we set $\hat{\boldsymbol{v}} = \frac{1}{2}\cdot(\overline{\boldsymbol{v}}+\boldsymbol{c}) - \delta\cdot\mathbf{1}$. By construction, $\mathbf{0}\le\hat{\boldsymbol{v}}\le\overline{\boldsymbol{v}}$ and thus $\hat{\boldsymbol{v}}\in\mathcal{V}$. Next, we prove the stronger statement that $\hat{\boldsymbol{v}}\in\mathcal{V}_\emptyset(\boldsymbol{p},\delta)$. Indeed, a direct calculation reveals that

$$\mathbf{1}_{\mathcal{S}}^\top\hat{\boldsymbol{v}} + \delta \ =\ \frac{1}{2}\cdot\mathbf{1}_{\mathcal{S}}^\top(\overline{\boldsymbol{v}}+\boldsymbol{c}) - (|\mathcal{S}|-1)\delta \ \le\ \frac{1}{2}\cdot\mathbf{1}_{\mathcal{S}}^\top(\overline{\boldsymbol{v}}+\boldsymbol{c}) \ =\ p_{\mathcal{S}}^\star \ \le\ p_{\mathcal{S}} \quad \forall\mathcal{S}\neq\emptyset.$$

Furthermore, the seller's regret under the value profile $\hat{\boldsymbol{v}}$ can be expressed as

$$\mathbf{1}^\top(\hat{\boldsymbol{v}}-\boldsymbol{c})^+ - (p_\emptyset - \mathbf{1}_\emptyset^\top\boldsymbol{c}) \ =\ \mathbf{1}^\top(\hat{\boldsymbol{v}}-\boldsymbol{c})^+ \ =\ \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - J\delta,$$

where the equality holds because $\delta \le \frac{1}{2}\min_{j\in\mathcal{J}}\overline{v}_j - c_j$. This concludes Step 2 because

$$\max_{\mathcal{S}\subseteq\mathcal{J}}\max_{\boldsymbol{v}\in\mathcal{V}_{\mathcal{S}}(\boldsymbol{p},\delta)} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - (p_{\mathcal{S}} - \mathbf{1}_{\mathcal{S}}^\top\boldsymbol{c}) \ \ge\ \mathbf{1}^\top(\hat{\boldsymbol{v}}-\boldsymbol{c})^+ - (p_\emptyset - \mathbf{1}_\emptyset^\top\boldsymbol{c}) \ \ge\ \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - J\delta.$$

Thus, the claim follows. ■

Theorem 2.4.1. *Define $\boldsymbol{p}^\star$ as in* (2.17)*, and choose any $s^\star \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}})$ such that $s^\star(\boldsymbol{v}) \in \arg\max_{\mathcal{S}\subseteq\mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top\boldsymbol{v} - p_{\mathcal{S}}^\star$ for all $\boldsymbol{v}\in\mathcal{V}$. Then, the posted price mechanism $(s^\star, \boldsymbol{p}^\star)$ is optimal in* (2.15)*, and the optimal value of* (2.15) *is given by $\frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$.*

Proof. Lemmas 2.4.3–2.4.5 imply that

$$\frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) \;\geq\; z'_{\mathrm{d}}(0) \;\geq\; z^\star_{\mathrm{d}} \;\geq\; \liminf_{\delta\downarrow 0} z'_{\mathrm{d}}(\delta) \;\geq\; \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}).$$

Hence, the optimal value of (2.15) equals $z^\star_{\mathrm{d}} = \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$. It is therefore sufficient to show that the worst-case regret of $(s^\star,\boldsymbol{p}^\star)$ is bounded below by $\frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$.

Using similar arguments as in the proof of Lemma 2.4.3, one can show that the objective value of $(s^\star,\boldsymbol{p}^\star)$ in (2.15) is bounded below by

$$\max_{\mathscr{S}\subseteq\mathscr{J}}\;\max_{\boldsymbol{v}\in\mathcal{V}_{\mathscr{S}}(\boldsymbol{p}^\star,\delta)} \mathbf{1}^\top(\boldsymbol{v}-\boldsymbol{c})^+ - (p^\star_{\mathscr{S}} - \mathbf{1}^\top_{\mathscr{S}}\boldsymbol{c}) \;\geq\; \frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c}) - J\delta$$

for any sufficiently small $\delta>0$, where the inequality follows from (2.19). By driving δ to 0, we may conclude that $(s^\star,\boldsymbol{p}^\star)$ attains the worst-case regret of at least $\frac{1}{2}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$, and thus the claim follows. ∎

Note that any optimal solution $(s^\star,\boldsymbol{p}^\star)$ of (2.15) gives rise to an optimal solution $(\boldsymbol{q}^\star,m^\star)$ of the original mechanism design problem (2.1) restricted to deterministic allocation rules, where $\boldsymbol{q}^\star(\boldsymbol{v}) = \mathbf{1}_{s^\star(\boldsymbol{v})}$ and $m^\star(\boldsymbol{v}) = p^\star_{s^\star(\boldsymbol{v})}$ for all $\boldsymbol{v}\in\mathcal{V}$.

Remark 2.4.4. *By Definition 2.2.3, under any posted price mechanism the buyer is forced to purchase one single (possibly empty) bundle. If the seller offers a menu of bundles, however, then this restriction would not apply, and the buyer could acquire multiple bundles. In this case the seller would have to ensure that the price of a bundle $\mathscr{S}\subseteq\mathscr{J}$ satisfies $p_{\mathscr{S}} \leq \sum_{i\in\mathscr{I}} p_{\mathscr{S}_i}$ for all possible partitions $\{\mathscr{S}_i : i\in\mathscr{I}\}$ of $\mathscr{S}$. Put differently, the seller might offer discounts but could never charge markups on bundles. Otherwise, the buyer would never choose a bundle that becomes cheaper when split into disjoint subsets. Even though the deterministic pricing problem* (2.15) *principally grants the seller the right to charge markups, the optimal mechanism is separable and does not capitalize on this flexibility.*

Theorems 2.3.1 and 2.4.1 immediately imply the following corollary.

Corollary 2.4.1. *The optimal deterministic mechanism provides an $\frac{e}{2}$-approximation to the optimal worst-case regret in* (2.1).

Corollary 2.4.1 suggests that a seller who implements the optimal posted price mechanism instead of the optimal randomized mechanism derived in Section 2.3 increases her worst-case regret by $\frac{e-2}{2}\approx 36\%$.

3 Regret Minimization and Separation in Multi-Bidder Multi-Item Auctions

We study a robust auction design problem with a minimax regret objective, where a seller seeks a mechanism for selling multiple items to multiple anonymous bidders with additive values. The seller knows that the bidders' values range over a box uncertainty set but has no information about their probability distribution. This auction design problem can be viewed as a zero-sum game between the seller, who chooses a mechanism, and a fictitious adversary or 'nature,' who chooses the bidders' values from within the uncertainty set with the aim to maximize the seller's regret. We characterize the Nash equilibrium of this game analytically. The Nash strategy of the seller is a mechanism that sells each item via a separate auction akin to a second price auction with a random reserve price. The Nash strategy of nature is mixed and constitutes a probability distribution on the uncertainty set under which each bidder's values for the items are comonotonic.

3.1 Introduction

Consider the problem of designing an auction for selling J items to I bidders. The bidders assign each item a private value, which captures the maximum amount of money they would be willing to pay for this item. The set of values that a bidder assigns to *all* items is referred to as his value profile. We assume that the bidders' preferences are quasilinear and additively separable, that is, the bidders assign any bundle of items a value equal to the sum of the values of its constituents.

In the standard Bayesian setting, the seller's beliefs about the bidders' value profiles are modeled via a commonly known probability distribution, and it is assumed that the seller aims to maximize her expected revenues. If there is only one item ($J = 1$), the optimal mechanism is well-understood under relatively general conditions, see, *e.g.*, Myerson (1981) and Cremer and McLean (1988). If there are multiple items ($J > 1$), on the other hand, computing the optimal mechanism is #P-hard even in unrealistically simple situations (Daskalakis et al., 2014). Even though Daskalakis et al. (2017) and Cai et al. (2019) recently proposed duality schemes for solving multi-item auction design problems, closed-form solutions remain limited to special probabilistic models and/or small numbers of items, see, *e.g.*, Daskalakis et al. (2013) or Giannakopoulos and Koutsoupias (2014).

Assuming that the probability distribution of the bidder's values is commonly known not only renders the mechanism design problem intractable, but it is also difficult to justify in practice. Instead, it is natural to seek mechanisms that are optimal under limited distributional information. When the probability distribution of the bidders' values is ambiguous, the term 'optimal' becomes ambiguous itself. The literature on (distributionally) robust mechanism design regards an auction as optimal if it maximizes the worst-case expected revenues in view of all possible distributions consistent with the information available. The bulk of this literature focuses on single-item auctions, see, *e.g.*, Bose et al. (2006), Bei et al. (2019), Koçyiğit et al. (2020) and Suzdaltsev (2020). As a notable exception, Bandi and Bertsimas (2014) propose a numerical procedure to solve a robust multi-item auction design problem with budget constraints. Carroll (2017) explicitly characterizes the optimal mechanism of a correlation-robust screening problem, where the marginal distributions of the agent's multidimensional type are precisely known to the principal, while their joint distribution remains unknown. The multidimensional monopoly pricing problem, which is equivalent to the single-bidder multi-item auction design problem, constitutes a special case of this screening problem. For this special case, Carroll (2017) shows that it is optimal to sell the items separately. Gravin and Lu (2018) then demonstrate that this separation result remains valid even if the bidder is subject to a budget constraint. Koçyiğit et al. (2018b) consider a variant of the multidimensional monopoly pricing problem with a minimax regret objective, where the seller has no knowledge of the value distribution apart from its support. They analytically characterize the best randomized as well as the best deterministic mechanism. In both cases, the optimal mechanism sells the items separately via single-item mechanisms that were first characterized by Bergemann and Schlag (2008).

The separation results reviewed above are not easily generalized to multi-bidder auctions. In this chapter, we consider the multi-bidder extension of the mechanism design problem studied by Koçyiğit et al. (2018b). Specifically, we assume that the seller perceives each bidder's value profile as an uncertain parameter that is only known to range over a rectangular uncertainty set spanned by the origin and a vector of nonnegative upper bounds. In addition, we assume that the bidders are anonymous, which implies that the vectors of item-wise upper bounds are identical for all bidders. When aiming to maximize the worst-case revenue, the seller faces a special case of the robust mechanism design problem studied by Bandi and Bertsimas (2014). Under the box uncertainty set considered here, however, the set of optimal mechanisms is very rich and contains naïve mechanisms that have little practical appeal. For example, it is optimal for the seller to keep all items. This prompts us to adopt a minimax regret objective, that is, we assume in this chapter that the seller seeks a mechanism that minimizes her worst-case regret. The regret of a mechanism is defined as the difference between the revenues that could have been achieved under full knowledge of the bidders' value profiles and the actual revenues generated by the mechanism. The worst-case regret is obtained by maximizing the realized regret across all possible value profiles of the bidders. Caldentey et al. (2017) as well as Poursoltani and Delage (2019) argue that, in a general robust optimization context, minimizing the worst-case regret results in less conservative decisions than maximizing the worst-case revenue. The main contributions of this chapter are listed below.

- We interpret the multi-bidder multi-item auction design problem with minimax regret objective as a zero sum game between the seller, who chooses a mechanism to auction the items, and a fictitious adversary or 'nature,' who chooses the bidders' value profiles from within a box uncertainty set with the aim to maximize the seller's regret. We characterize the Nash equilibrium of this game analytically and prove that the seller's Nash strategy is a mechanism under which each item is auctioned separately. The separate mechanisms for the individual items can be interpreted as second price auctions with random reserve prices. If there is only one bidder, these separate mechanisms reduce to randomized posted price mechanisms that were first described by Bergemann and Schlag (2008) in the context of the monopoly pricing of a single item.
- We show that nature's Nash strategy is mixed and thus represents a probability distribution on the uncertainty set. Under this distribution, each bidder's values for the items are comonotic, and any bidder's value profile can be non-zero only if all other bidders' value profiles vanish.

The mechanism design model studied in this chapter requires no distributional information except for an upper bound on each bidder's value for each item. This model is relevant if there is no trustworthy distributional information or if any distributional information is costly or time-consuming to acquire. Such a situation could arise, for example, when firms use auctions for initial public offerings. In this case, there is indeed no distributional information available about the bidders' values for the offered shares. On the other hand, the model studied here may be overly conservative when data is abundant, as is typically the case in online advertisement, where auctions for ad placements are held in real time within fractions of seconds. To our best knowledge, this chapter establishes the first non-trivial robust optimality guarantee for a separable mechanism involving multiple bidders as well as multiple items. We expect that the insights distilled in this chapter will pave the way towards more general separation results with a broader range of applications.

This chapter also relates to the literature on approximately optimal mechanism design, see, *e.g.*, Dhangwatnotai et al. (2015), Hart and Nisan (2017), Allouah and Besbes (2020) and the references therein. Under this modeling paradigm, the seller aims to identify a mechanism for which some objective function (*e.g.*, the expected revenue) is guaranteed to be close to a full information benchmark value (*e.g.*, the maximum expected revenue achievable) under every probability distribution consistent with the assumptions made. The vast majority of the existing approximation results critically rely on certain independence assumptions (*e.g.*, the values must be independent across bidders or items). In the context of a monopoly pricing problem with a single buyer it has been shown, for example, that if the buyer's values for the items are independent, then simple mechanisms (such as selling the goods separately or as a single grand bundle at deterministic posted prices) provide constant-factor approximations to the expected revenue of the unknown optimal mechanism (Hart and Nisan, 2017). However, if the buyer's values are correlated, these approximation guarantees cease to hold (Hart and Nisan, 2019). An important advantage of the robust approach adopted in this chapter is its ability to account for correlations and to provide optimality guarantees for simple mechanisms even if the bidders' values may be dependent.

Notation.

For any closed set $\mathscr{A} \subseteq \mathbb{R}^n$, we denote by $\Delta(\mathscr{A})$ the family of all probability distributions on $\mathscr{A}$, and for any $\mathbb{P} \in \Delta(\mathscr{A})$, $\mathrm{supp}(\mathbb{P})$ represents the support of $\mathbb{P}$. The set of all Borel-measurable functions from a Borel set $\mathscr{D} \subseteq \mathbb{R}^n$ to a Borel set $\mathscr{R} \subseteq \mathbb{R}^m$ is denoted by $\mathscr{L}(\mathscr{D},\mathscr{R})$. Random variables are designated by tildes (*e.g.*, $\tilde{\boldsymbol{v}}$), and their realizations are denoted by the same symbols without tildes (*e.g.*, $\boldsymbol{v}$). For a logical expression $\mathscr{E}$, we define $\mathbb{1}_{\mathscr{E}} = 1$ if $\mathscr{E}$ is true and $\mathbb{1}_{\mathscr{E}} = 0$ otherwise. Throughout the chapter, bidders are indexed by superscripts and items by subscripts.

3.2 Problem Setup

We consider the problem of designing a mechanism for selling J different items to $I \geq 2$ bidders. The sets of items and bidders are denoted by $\mathscr{J} = \{1, 2, \ldots, J\}$ and $\mathscr{I} = \{1, 2, \ldots, I\}$, respectively. Each bidder $i \in \mathscr{I}$ assigns each item $j \in \mathscr{J}$ a value v_j^i that reflects his willingness to pay. In the following we denote by $\boldsymbol{v}^i = (v_1^i, \ldots, v_J^i)$ the row vector of the values that bidder i assigns to all items and by $\boldsymbol{v}_j = (v_j^1, \ldots, v_j^I)^\top$ the column vector of all bidders' values for item j. In addition, we let $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_J)$ be the matrix of all bidders' values for all items. While bidder i has full knowledge of his value profile $\boldsymbol{v}^i$, the seller perceives the matrix $\boldsymbol{v}$ as uncertain. For each $j \in \mathscr{J}$, we assume that the seller only knows a common upper bound $\overline{v}_j > 0$ on the value v_j^i for all $i \in \mathscr{I}$. This assumption is a manifestation of the anonymity of the bidders. In the following we denote by $\overline{\boldsymbol{v}} = (\overline{v}_1, \ldots, \overline{v}_J)$ the vector of all item-wise upper bounds. The seller has no other information about the distribution of $\boldsymbol{v}$ or suspects that any available information is not trustworthy. For ease of exposition, we assume that the seller incurs no costs for supplying any of the items to any of the bidders. In the following, we denote by $\mathcal{V} = \times_{j\in\mathcal{J}}[0,\overline{v}_j]$ the uncertainty set of the value profiles of any fixed bidder $i \in \mathcal{I}$ and by $\mathcal{V}^I = \times_{j\in\mathcal{J}}[0,\overline{v}_j]^I$ the uncertainty set of the value profiles of all bidders, which is symmetric under permutations of the bidders. We also let $\mathcal{W}_j^i$, $i \in \mathcal{I}$ and $j \in \mathcal{J}$, be any partition of the hypercube $[0,\overline{v}_j]^I$ such that $\mathcal{W}_j^i$ contains only scenarios $\boldsymbol{v}_j$ for which bidder i is among the highest bidders for item j. In other words, $\boldsymbol{v}_j \in \mathcal{W}_j^i$ implies that $i \in \arg\max_{k\in\mathcal{I}} v_j^k$. If there are multiple highest bidders, an arbitrary tie-breaking rule is used (*e.g.*, the lexicographic tie-breaker assigns $\boldsymbol{v}_j$ to $\mathcal{W}_j^i$ if $i = \min\arg\max_{k\in\mathcal{I}} v_j^k$).

An auction mechanism $(\boldsymbol{q},\boldsymbol{m})$ consists of an allocation rule $\boldsymbol{q} \in \mathcal{L}(\mathcal{V}^I,\mathbb{R}_+^{I\times J})$ and a payment rule $\boldsymbol{m} \in \mathcal{L}(\mathcal{V}^I,\mathbb{R}^I)$. Given a matrix $\boldsymbol{v} \in \mathcal{V}^I$ of value profiles reported by all bidders, the mechanism $(\boldsymbol{q},\boldsymbol{m})$ outputs the allocation probabilities of the items to the bidders as well as the payments charged to the bidders. Specifically, in scenario $\boldsymbol{v}$, the seller allocates item j to bidder i with probability $q_j^i(\boldsymbol{v})$ and charges this bidder the amount $m^i(\boldsymbol{v})$. As a result, the utility of bidder i evaluates to $\sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{v})v_j^i - m^i(\boldsymbol{v})$ and is therefore quasilinear and additively separable across the items.

A (dominant strategy) incentive compatible and (ex-post) individually rational mechanism $(\boldsymbol{q},\boldsymbol{m})$ satisfies the following constraints.

$$\begin{aligned}
&\sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{v})v_j^i - m^i(\boldsymbol{v}) \geq \sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{w}^i,\boldsymbol{v}^{-i})v_j^i - m^i(\boldsymbol{w}^i,\boldsymbol{v}^{-i}) && \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I, \forall \boldsymbol{w}^i \in \mathcal{V} && \text{(IC)}\\
&\sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{v})v_j^i - m^i(\boldsymbol{v}) \geq 0 && \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I && \text{(IR)}\\
&\sum_{i\in\mathcal{I}} q_j^i(\boldsymbol{v}) \leq 1 && \forall j \in \mathcal{J}, \forall \boldsymbol{v} \in \mathcal{V}^I && \text{(Inv)}
\end{aligned}$$

The incentive compatibility constraint (IC) ensures that each bidder maximizes his utility by reporting his true value profile irrespective of the values reported by the other bidders. The individual rationality constraint (IR) ensures that the bidders earn nonnegative utilities from participating in the mechanism $(\boldsymbol{q},\boldsymbol{m})$ under truthful reporting. Finally, the inventory constraint (Inv) ensures that the seller allocates each item $j \in \mathcal{J}$ with a probability of at most one. Note that the inequality expresses the possibility that the seller may keep any item j to herself with a positive probability.

Remark 3.2.1. *Incentive compatibility and individual rationality constraints are routinely used in mechanism design and may be imposed essentially without any loss of generality thanks to the revelation principle (Krishna, 2009, Chapter 5). Throughout this chapter, we assume that the seller restricts attention to dominant strategy incentive compatible and ex-post individually* *rational mechanisms. In contrast, the Bayesian mechanism design literature typically studies Bayesian incentive compatibility and interim individual rationality, see,* e.g., *Myerson (1981). While less restrictive, these constraints can only be enforced if the seller is able to assign a crisp probability distribution to the bidders' values. In this chapter, we assume that the seller as well as the bidders lack the relevant information. It is thus natural to focus on dominant strategy incentive compatibility and ex-post individual rationality, which do not require any distributional information.*

The seller's ex-post regret is defined as the difference between the maximum profit that could have been realized under complete information about $\boldsymbol{v}$ and the profit earned with the mechanism $(\boldsymbol{q}, \boldsymbol{m})$. If the seller was fully aware of the bidders' values $\boldsymbol{v}$, she would sell item j at the price $\max_{i\in\mathcal{I}} v_j^i$ to any bidder $i \in \arg\max_{i\in\mathcal{I}} v_j^i$. The maximum profit under complete information can thus be expressed as $\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}} v_j^i)$. The profit earned with mechanism $(\boldsymbol{q}, \boldsymbol{m})$, on the other hand, amounts to $\sum_{i\in\mathcal{I}} m^i(\boldsymbol{v})$. In summary, the ex-post regret thus equals $\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} m^i(\boldsymbol{v})$, and the worst-case regret is obtained by maximizing the ex-post regret over all value profiles $\boldsymbol{v} \in \mathcal{V}^I$.

Throughout this chapter we assume that the seller aims to design an incentive compatible and individually rational mechanism that minimizes her worst-case regret. This mechanism design problem can be formalized as the following robust optimization problem.

$$
\begin{aligned}
z^\star = \inf_{\boldsymbol{q},\boldsymbol{m}} \quad & \sup_{\boldsymbol{v}\in\mathcal{V}^I} \sum_{j\in\mathcal{J}} (\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} m^i(\boldsymbol{v}) \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}_+^{I\times J}),\ \boldsymbol{m} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}^I) \\
& \text{(IC), (IR), (Inv)}
\end{aligned}
\tag{3.1}
$$

From now on, we use the shorthand $\mathcal{X}$ to denote the set of all mechanisms feasible in (3.1).

3.3 Optimal Mechanism

One particularly simple policy for the seller would be to auction each item individually. Any such mechanism is separable in view of the following definition.

Definition 3.3.1 (Separability)**.** *A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is called separable if there exists an item-wise allocation rule $\hat{\boldsymbol{q}}_j \in \mathcal{L}([0, \overline{v}_j]^I, \mathbb{R}_+^I)$ and an item-wise payment rule $\hat{\boldsymbol{m}}_j \in \mathcal{L}([0, \overline{v}_j]^I, \mathbb{R}^I)$ for all $j \in \mathcal{J}$ such that $\boldsymbol{q}(\boldsymbol{v}) = (\hat{\boldsymbol{q}}_1(\boldsymbol{v}_1), \ldots, \hat{\boldsymbol{q}}_J(\boldsymbol{v}_J))$ and $\boldsymbol{m}(\boldsymbol{v}) = \sum_{j\in\mathcal{J}} \hat{\boldsymbol{m}}_j(\boldsymbol{v}_j)$ for all $\boldsymbol{v} \in \mathcal{V}^I$.*

We now investigate the separable mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ consisting of the single-item mechanisms $(\hat{\boldsymbol{q}}_j, \hat{\boldsymbol{m}}_j)$, $j \in \mathcal{J}$, defined through

$$\hat{q}_j^i(\boldsymbol{v}_j) = \begin{cases} 1 + \log(\frac{v_j^i}{\overline{v}_j}) & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \geq \frac{\overline{v}_j}{e} \\ 0 & \text{otherwise} \end{cases} \tag{3.2a}$$

and

$$\hat{m}_j^i(\boldsymbol{v}_j) = \begin{cases} v_j^i + (\max_{k\neq i} v_j^k) \log(\max_{k\neq i} \frac{v_j^k}{\overline{v}_j}) & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \geq \max_{k\neq i} v_j^k \geq \frac{\overline{v}_j}{e} \\ v_j^i - \frac{\overline{v}_j}{e} & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \geq \frac{\overline{v}_j}{e} > \max_{k\neq i} v_j^k \\ 0 & \text{otherwise} \end{cases} \tag{3.2b}$$

for all $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$. Under this mechanism, the probability of allocating item $j \in \mathcal{J}$ to bidder $i \in \mathcal{I}$ can be strictly positive only if bidder i has the highest value for item j among all bidders and if that value exceeds $\frac{\overline{v}_j}{e}$. We emphasize that the payment rule $\hat{\boldsymbol{m}}_j$ is deterministic even though the allocation of item j is randomized under $\hat{\boldsymbol{q}}_j$. Specifically, bidder $i \in \mathcal{I}$ always pays a nonnegative amount for each item $j \in \mathcal{J}$, and this amount is strictly positive only when the corresponding allocation probability is strictly positive, *i.e.*, when $v_j^i > \frac{\overline{v}_j}{e}$.

In the remainder of the chapter we will show that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is optimal in (3.1). To this end, we will first construct a new mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ equivalent to $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, under which the seller implements a separate second price auction for each item $j \in \mathcal{J}$ with a random reserve price $\tilde{r}_j$ governed by the probability distribution $\mathbb{Q}_j \in \Delta([0, \overline{v}_j])$ defined via

$$\mathbb{Q}_j(\tilde{r}_j \leq x) = \begin{cases} 1 + \log(\frac{x}{\overline{v}_j}) & \text{if } \frac{\overline{v}_j}{e} \leq x \leq \overline{v}_j \\ 0 & \text{if } 0 \leq x < \frac{\overline{v}_j}{e}. \end{cases}$$

For each item $j \in \mathcal{J}$, the respective second price auction proceeds as follows. First, the reserve price r_j is sampled from the distribution $\mathbb{Q}_j$. Note that the smallest possible value of r_j under this distribution is $\frac{\overline{v}_j}{e}$. The seller then asks the bidders to report their bids for item j. After collecting all bids, the seller allocates item j to the highest bidder provided that his bid exceeds the reserve price r_j, and the winner pays an amount equal to the maximum of the second highest bid and r_j. In the case of ties, item j is given to the unique bidder whose index i satisfies $\boldsymbol{v}_j \in \mathcal{W}_j^i$.

By construction, the mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ is separable and can formally be described via the single-item mechanisms $(\hat{\boldsymbol{q}}'_j, \hat{\boldsymbol{m}}'_j)$, $j \in \mathcal{J}$, defined through

$$(\hat{q}')_j^i(\boldsymbol{v}_j, r_j) = \begin{cases} 1 & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \geq r_j \\ 0 & \text{otherwise} \end{cases} \tag{3.3a}$$

and

$$(\hat{m}')^i_j(\boldsymbol{v}_j, r_j) = \begin{cases} \max\{\max_{k\neq i} v^k_j, r_j\} & \text{if } \boldsymbol{v}_j \in \mathcal{W}^i_j \text{ and } v^i_j \geq r_j \\ 0 & \text{otherwise} \end{cases} \tag{3.3b}$$

for all $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$. Note that $(\boldsymbol{q}', \boldsymbol{m}')$ is manifestly randomized because it depends on the realizations of the random reserve prices. Note also that, unlike under $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, under $(\boldsymbol{q}', \boldsymbol{m}')$ a bidder pays for an item only if he receives it. We next show that the randomized mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ is equivalent to the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ if it is averaged across the random reserve prices.

Proposition 3.3.1. *We have* $\mathbb{E}_{\mathbb{Q}_j}[(\hat{q}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)] = \hat{q}^i_j(\boldsymbol{v}_j)$ *and* $\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)] = \hat{m}^i_j(\boldsymbol{v}_j)$ *for all* $i \in \mathcal{I}$, $j \in \mathcal{J}$ *and* $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$.

Proof. Fix an arbitrary $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$. If $\boldsymbol{v}_j \notin \mathcal{W}^i_j$, then both $\mathbb{E}_{\mathbb{Q}_j}[(\hat{q}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)]$ and $\hat{q}^i_j(\boldsymbol{v}_j)$ evaluate to 0 and are thus equal. If $\boldsymbol{v}_j \in \mathcal{W}^i_j$, on the other hand, one readily verifies that

$$\mathbb{E}_{\mathbb{Q}_j}[(\hat{q}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)] = \mathbb{E}_{\mathbb{Q}_j}[\mathbb{1}_{(\tilde{r}_j \leq v^i_j)}] = \mathbb{Q}_j(\tilde{r}_j \leq v^i_j) = \hat{q}^i_j(\boldsymbol{v}_j).$$

Consider now the expected payment $\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)]$ of bidder i for item j in scenario $\boldsymbol{v}_j$. If $\boldsymbol{v}_j \notin \mathcal{W}^i_j$, then both $\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)]$ and $\hat{m}^i_j(\boldsymbol{v}_j)$ evaluate to 0 and are thus equal. If $\boldsymbol{v}_j \in \mathcal{W}^i_j$ and $\max_{k\neq i} v^k_j \geq \frac{\overline{v}_j}{e}$, however, then we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)] &= \mathbb{E}_{\mathbb{Q}_j}\left[\max\{\max_{k\neq i} v^k_j, \tilde{r}_j\}\mathbb{1}_{(\tilde{r}_j \leq v^i_j)}\right] \\
&= \int_{\frac{\overline{v}_j}{e}}^{\max_{k\neq i} v^k_j} (\max_{k\neq i} v^k_j)\,\frac{1}{x}\,\mathrm{d}x + \int_{\max_{k\neq i} v^k_j}^{v^i_j} x\,\frac{1}{x}\,\mathrm{d}x \\
&= v^i_j + (\max_{k\neq i} v^k_j)\log\left(\max_{k\neq i}\frac{v^k_j}{\overline{v}_j}\right) = \hat{m}^i_j(\boldsymbol{v}_j).
\end{aligned}$$

Finally, if $\boldsymbol{v}_j \in \mathcal{W}^i_j$ and $v^i_j \geq \frac{\overline{v}_j}{e} > \max_{k\neq i} v^k_j$, then the reserve price exceeds the second highest bid with probability 1, which implies that

$$\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)] = \mathbb{E}_{\mathbb{Q}_j}\left[\max\{\max_{k\neq i} v^k_j, \tilde{r}_j\}\mathbb{1}_{(\tilde{r}_j \leq v^i_j)}\right] = \int_{\frac{\overline{v}_j}{e}}^{v^i_j} x\,\frac{1}{x}\,\mathrm{d}x = v^i_j - \frac{\overline{v}_j}{e} = \hat{m}^i_j(\boldsymbol{v}_j).$$

As the above arguments hold for each $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$, the claim follows. ■

Remark 3.3.1. *All second price auctions with deterministic reserve prices are (dominant strategy) incentive compatible (Krishna, 2009, Chapter 2). Thus,* $(\boldsymbol{q}', \boldsymbol{m}')$ *is incentive compatible for all realizations of the random reserve prices. As incentive compatibility is enforced via linear inequalities, it is preserved by averaging* $(\boldsymbol{q}', \boldsymbol{m}')$ *with respect to the distributions of the reserve* *prices, which then implies via Proposition 3.3.1 that* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *is incentive compatible, too. Under* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$*, the bidders thus have a weak preference to report their true values. All second price auctions with reserve prices are also (ex-post) individually rational because the bidders' utilities are always nonnegative. Indeed, under truthful bidding, a bidder pays at most his own bid. Using similar arguments as above, one can thus show that* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *is also individually rational. As the mechanism* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *ostensibly satisfies the inventory constraint (Inv), we thus conclude that it is feasible in* (3.1).

Proposition 3.3.2. *The worst-case regret of the separable mechanism* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *corresponding to the single-item mechanisms* (3.2) *is given by* $\sum_{j\in\mathcal{J}} \frac{\overline{v}_j}{e}$.

Proof. To evaluate the worst-case regret of $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, we will first compute the realized regret $(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j)$ of selling item $j \in \mathcal{J}$ in scenario $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$. To this end, fix an arbitrary $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$ and note that there is a unique $i' \in \mathcal{I}$ with $\boldsymbol{v}_j \in \mathcal{W}_j^{i'}$. If $\frac{\overline{v}_j}{e} > v_j^{i'}$, then $\hat{m}_j^i(\boldsymbol{v}_j) = 0$ for all $i \in \mathcal{I}$, that is, no bidder is charged. We thus have $(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) = v_j^{i'} < \frac{\overline{v}_j}{e}$. If $v_j^{i'} \geq \frac{\overline{v}_j}{e} > \max_{k\neq i'} v_j^k$, however, then $\hat{m}_j^{i'}(\boldsymbol{v}_j) = v_j^{i'} - \frac{\overline{v}_j}{e}$, and all other bidders pay nothing. Thus, $(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) = v_j^{i'} - (v_j^{i'} - \frac{\overline{v}_j}{e}) = \frac{\overline{v}_j}{e}$. Finally if $v_j^{i'} \geq \max_{k\neq i'} v_j^k \geq \frac{\overline{v}_j}{e}$, then $\hat{m}_j^{i'}(\boldsymbol{v}_j) = v_j^{i'} + (\max_{k\neq i'} v_j^k)\log(\max_{k\neq i'} \frac{v_j^k}{\overline{v}_j})$, and all other bidders pay nothing. This implies that

$$
\begin{aligned}
(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) &= v_j^{i'} - \left(v_j^{i'} + (\max_{k\neq i'} v_j^k)\log\left(\max_{k\neq i'} \frac{v_j^k}{\overline{v}_j}\right)\right) \\
&= -(\max_{k\neq i'} v_j^k)\log\left(\max_{k\neq i'} \frac{v_j^k}{\overline{v}_j}\right) \leq \max_{x\in[\overline{v}_j/e, \overline{v}_j]} -x\log\left(\frac{x}{\overline{v}_j}\right) = \frac{\overline{v}_j}{e},
\end{aligned}
$$

where the inequality follows from the assumption that $\max_{k\neq i'} v_j^k \geq \frac{\overline{v}_j}{e}$, and the last equality holds because $-x\log(\frac{x}{\overline{v}_j})$ is monotonically decreasing in $x \geq \frac{\overline{v}_j}{e}$.

The above reasoning implies that the worst-case regret of the individual mechanism $(\hat{\boldsymbol{q}}_j, \hat{\boldsymbol{m}}_j)$ for selling item $j \in \mathcal{J}$ amounts to $\sup_{\boldsymbol{v}_j\in\mathcal{V}} (\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) = \frac{\overline{v}_j}{e}$. This worst-case regret is attained by any scenario $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$ with $v_j^{i'} \geq \frac{\overline{v}_j}{e} \geq \max_{k\neq i'} v_j^k$. The worst-case regret of the separable mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is thus given by

$$
\begin{aligned}
\sup_{\boldsymbol{v}\in\mathcal{V}^I} \sum_{j\in\mathcal{J}} (\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} (m^\star)^i(\boldsymbol{v}) &= \sup_{\boldsymbol{v}\in\mathcal{V}^I} \sum_{j\in\mathcal{J}} \left((\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) \right) \\
&= \sum_{j\in\mathcal{J}} \left(\sup_{\boldsymbol{v}_j\in\mathcal{V}} (\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) \right) = \sum_{j\in\mathcal{J}} \frac{\overline{v}_j}{e},
\end{aligned}
$$

where the second equality follows from the rectangularity of the uncertainty set $\mathcal{V}^I$, which implies that $\boldsymbol{v}_j \in \mathcal{V}$ for all $j \in \mathcal{J}$ if and only if $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_J) \in \mathcal{V}^I$. ■

To show that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves (3.1), we first reformulate the minimax problem (3.1) as an infinite-dimensional convex program. To this end, note that problem (3.1) can be interpreted as a zero-sum game between the seller, who chooses the mechanism $(\boldsymbol{q}, \boldsymbol{m})$, and some fictitious adversary that one may think of as 'nature,' who chooses the bidders' value profiles $\boldsymbol{v}$ with the goal to inflict maximum damage to the seller. As the allocation probabilities may be fractional, the seller plays a mixed strategy chosen from the convex feasible set $\mathcal{X}$. Nature's feasible set coincides with the box uncertainty set $\mathcal{V}^I$ and is also convex. While the objective of the zero-sum game constitutes an affine function of the payment rule $\boldsymbol{m}$ for every fixed scenario $\boldsymbol{v}$, however, it is generically non-concave in $\boldsymbol{v}$ for fixed mechanisms $(\boldsymbol{q}, \boldsymbol{m})$. To convert this zero-sum game to a convex-concave saddle point problem, we should allow nature to play mixed strategies corresponding to distributions $\mathbb{P} \in \Delta(\mathcal{V}^I)$. With this standard trick, problem (1) can be reformulated as

$$\inf_{(\boldsymbol{q},\boldsymbol{m})\in\mathcal{X}} \quad \sup_{\mathbb{P}\in\Delta(\mathcal{V}^I)} \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}} (\max_{i\in\mathcal{I}} \tilde{v}^i_j) - \sum_{i\in\mathcal{I}} m^i(\tilde{\boldsymbol{v}})\right]. \tag{3.4}$$

We will now show that the zero-sum game (3.4) admits a Nash equilibrium in mixed strategies, which can be evaluated analytically. Specifically, we will prove that the seller's Nash strategy is given by the separable mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$. In order to construct nature's Nash strategy, we introduce a marginal distribution $\hat{\mathbb{P}} \in \Delta(\mathcal{V})$ for the value profile $\tilde{\boldsymbol{v}}^1$, which is defined through

$$\hat{\mathbb{P}}(\tilde{\boldsymbol{v}}^1 \le \boldsymbol{v}^1) = \begin{cases} \min_{j\in\mathcal{J}} \left(1 - \frac{1}{e}\left(\frac{\overline{v}_j}{v^1_j}\right)\right)^+ & \text{if } \boldsymbol{v}^1 \in \mathcal{V} \setminus \{(\overline{v}_1, \ldots, \overline{v}_J)\} \\ 1 & \text{if } \boldsymbol{v}^1 = (\overline{v}_1, \ldots, \overline{v}_J). \end{cases} \tag{3.5}$$

Note that $\hat{\mathbb{P}}$ is known to represent nature's Nash strategy in problem (3.4) if there is only one bidder, that is, in the special case where $I = 1$ and the auction design problem collapses to a monopoly pricing problem (Koçyiğit et al., 2018b, Theorem 2). Note also that the values $(\tilde{v}^1_1, \ldots, \tilde{v}^1_J)$ of bidder 1 for the different items under $\hat{\mathbb{P}}$ are comonotonic. In the following, we will use $\hat{\mathbb{P}}$ explicitly to construct a Nash strategy for nature when $I > 1$. To this end, define for each $i \in \mathcal{I}$ a probability distribution $\hat{\mathbb{P}}^i \in \Delta(\mathcal{V}^I)$ of the random matrix $\tilde{\boldsymbol{v}}$ through the relations

$$\hat{\mathbb{P}}^i(\tilde{\boldsymbol{v}}^i \le \boldsymbol{w}) = \hat{\mathbb{P}}(\tilde{\boldsymbol{v}}^1 \le \boldsymbol{w}) \;\; \forall \boldsymbol{w} \in \mathcal{V} \quad \text{and} \quad \hat{\mathbb{P}}^i(\tilde{\boldsymbol{v}}^k = \boldsymbol{0}) = 1 \;\; \forall k \ne i.$$

Note that under $\hat{\mathbb{P}}^i$, the marginal distribution of the value profile $\tilde{\boldsymbol{v}}^i$ coincides with $\hat{\mathbb{P}}$, while the marginal distributions of the value profiles $\tilde{\boldsymbol{v}}^k$, $k \ne i$, are all equal to the Dirac distribution that concentrates unit mass at $\boldsymbol{0}$. As $\hat{\mathbb{P}}$ constitutes a comonotonic distribution, the values of bidder i for the items are comonotonic under $\hat{\mathbb{P}}^i$. The support of the distribution $\hat{\mathbb{P}}^i$ is given by

$$\mathrm{supp}(\hat{\mathbb{P}}^i) = \left\{ \boldsymbol{v} \in \mathcal{V}^I : \boldsymbol{v}^i = s\overline{\boldsymbol{v}} \text{ for some } s \in [\tfrac{1}{e}, 1] \text{ and } \boldsymbol{v}^k = \boldsymbol{0} \ \ \forall k \neq i \right\}.$$

Finally, define $\mathbb{P}^\star \in \Delta(\mathcal{V}^I)$ as the average of the distributions $\hat{\mathbb{P}}^i$ across $i \in \mathscr{I}$, that is, set

$$\mathbb{P}^\star = \frac{1}{I} \sum_{i \in \mathscr{I}} \hat{\mathbb{P}}^i. \tag{3.6}$$

The support of $\mathbb{P}^\star$ can therefore be expressed as $\mathrm{supp}(\mathbb{P}^\star) = \cup_{i \in \mathscr{I}} \mathrm{supp}(\hat{\mathbb{P}}^i)$. Specifically, under $\mathbb{P}^\star$ the highest bidder's value profile exceeds the positive threshold $\frac{1}{e}\overline{\boldsymbol{v}}$ almost surely, while all other bidders' value profiles are almost surely equal to $\boldsymbol{0}$.

We will show that the separable mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ corresponding to the single-item mechanisms (3.2) and the probability distribution $\mathbb{P}^\star$ defined in (3.6) represent the Nash strategies of the seller and of nature in problem (3.4), respectively. To simplify the subsequent discussion, we denote by

$$z(\boldsymbol{m}, \mathbb{P}) = \mathbb{E}_{\mathbb{P}} \left[\sum_{j \in \mathscr{J}} (\max_{i \in \mathscr{I}} \tilde{v}^i_j) - \sum_{i \in \mathscr{I}} m^i(\tilde{\boldsymbol{v}}) \right]$$

the expected regret of the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ under the probability distribution $\mathbb{P} \in \Delta(\mathcal{V}^I)$.

Theorem 3.3.1 (Nash Equilibrium)**.** *The separable mechanism* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *corresponding to the single-item mechanisms* (3.2) *and the distribution* $\mathbb{P}^\star$ *defined in* (3.6) *satisfy the saddle point condition*

$$\sup_{\mathbb{P} \in \Delta(\mathcal{V}^I)} z(\boldsymbol{m}^\star, \mathbb{P}) \leq z(\boldsymbol{m}^\star, \mathbb{P}^\star) \leq \inf_{(\boldsymbol{q}, \boldsymbol{m}) \in \mathscr{X}} z(\boldsymbol{m}, \mathbb{P}^\star). \tag{3.7}$$

To prove Theorem 1, we will first show that the problem on the left-hand side of (3.7) is solved by $\mathbb{P}^\star$ and attains an optimal value of $\sum_{j \in \mathscr{J}} \frac{\overline{v}_j}{e}$. Next, we relax the problem on the right-hand side of (3.7) to a single-buyer multi-item pricing problem with the objective of minimizing the regret under the distribution $\hat{\mathbb{P}}$. This relaxation is facilitated by the symmetric construction of the distribution $\mathbb{P}^\star$. We know from Theorem 2 by Koçyiğit et al. (2018b) that the optimal value of the resulting pricing problem amounts to $\sum_{j \in \mathscr{J}} \frac{\overline{v}_j}{e}$. The observation that $z(\boldsymbol{m}^\star, \mathbb{P}^\star) = \sum_{j \in \mathscr{J}} \frac{\overline{v}_j}{e}$ then completes the proof.

Proof of Theorem 3.3.1. We first show that $\mathbb{P}^\star$ solves the problem on the left-hand side of (3.7) (Step 1), and then we prove that $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves the problem on the right-hand side of (3.7) (Step 2).

Step 1. By Proposition 3.3.2, the worst-case regret of the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is given by

$$\sup_{\boldsymbol{v} \in \mathcal{V}^I} \sum_{j \in \mathscr{J}} (\max_{i \in \mathscr{I}} v^i_j) - \sum_{i \in \mathscr{I}} (m^\star)^i(\boldsymbol{v}) = \sum_{j \in \mathscr{J}} \frac{\overline{v}_j}{e}. \tag{3.8}$$

The proof of Proposition 3.3.2 further reveals that the worst-case regret of the individual mechanism $(\hat{\boldsymbol{q}}_j, \hat{\boldsymbol{m}}_j)$ for selling item $j \in \mathscr{J}$ amounts to $\sup_{\boldsymbol{v}_j \in \mathcal{V}}(\max_{i\in\mathscr{I}} v_j^i) - \sum_{i\in\mathscr{I}} \hat{m}_j^i(\boldsymbol{v}_j) = \frac{\overline{v}_j}{e}$ and that this worst-case regret is attained by any scenario $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$ for which there exists $i \in \mathscr{I}$ with $v_j^i \geq \frac{\overline{v}_j}{e} \geq \max_{k\neq i} v_j^k$. As the uncertainty set $\mathcal{V}^I$ is rectangular, the worst-case regret (3.8) of the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is thus attained by any scenario $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_J) \in \mathcal{V}^I$ for which $\boldsymbol{v}_j$ satisfies the above condition for each $j \in \mathscr{J}$. This implies that the worst-case regret (3.8) is attained if

$$\boldsymbol{v} \in \left\{ \boldsymbol{v} \in \mathcal{V}^I : \exists i \in \mathscr{I} \text{ with } \boldsymbol{v}^i \geq \frac{1}{e}\overline{\boldsymbol{v}} \text{ and } \boldsymbol{v}^k = \boldsymbol{0} \;\; \forall k \neq i \right\} \supseteq \operatorname{supp}(\mathbb{P}^\star),$$

where the subset relation follows readily from the construction of $\mathbb{P}^\star$. Fix now an arbitrary distribution $\mathbb{P} \in \Delta(\mathcal{V}^I)$. Then, the expected regret of $\boldsymbol{m}^\star$ under $\mathbb{P}$ satisfies

$$z(\boldsymbol{m}^\star, \mathbb{P}) = \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathscr{J}} (\max_{i\in\mathscr{I}} \tilde{v}_j^i) - \sum_{i\in\mathscr{I}} (m^\star)^i(\tilde{\boldsymbol{v}}) \right] \leq \sum_{j\in\mathscr{J}} \frac{\overline{v}_j}{e},$$

where the inequality follows from (3.8). The inequality is tight for $\mathbb{P}^\star$ because the ex-post regret in any scenario $\boldsymbol{v} \in \operatorname{supp}(\mathbb{P}^\star)$ equals $\sum_{j\in\mathscr{J}} \frac{\overline{v}_j}{e}$. Thus, $\mathbb{P}^\star$ solves the problem on the left-hand side of (3.7).

Step 2. Consider now the expected regret minimization problem on the right-hand side of (3.7), which can be expressed more explicitly as

$$\begin{aligned} \inf_{\boldsymbol{q},\boldsymbol{m}} \quad & \mathbb{E}_{\mathbb{P}^\star}\left[\sum_{j\in\mathscr{J}} (\max_{i\in\mathscr{I}} \tilde{v}_j^i) - \sum_{i\in\mathscr{I}} m^i(\tilde{\boldsymbol{v}}) \right] \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathscr{L}(\mathcal{V}^I, \mathbb{R}_+^{I\times J}),\ \boldsymbol{m} \in \mathscr{L}(\mathcal{V}^I, \mathbb{R}^I) \\ & \text{(IC), (IR), (Inv).} \end{aligned} \tag{3.9}$$

To prove that $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves problem (3.9), we first relax this problem by replacing its objective function with a lower bound and by reducing the uncertainty sets of its robust constraints (Step 2.a). We then aggregate the constraints of the resulting problem across the bidders to obtain an even looser relaxation of (3.9), which turns out to be equivalent to a multi-item pricing problem involving a single bidder (Step 2.b). By leveraging Theorem 2 of Koçyiğit et al. (2018b), we then show that this problem's optimal value matches the objective value of $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ in (3.9).

Step 2.a. To construct a relaxation of problem (3.9), we first establish a lower bound on its objective function. Indeed, for any fixed mechanism $(\boldsymbol{q}, \boldsymbol{m})$ we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^\star}\left[\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}}\tilde{v}^i_j) - \sum_{i\in\mathcal{I}} m^i(\tilde{\boldsymbol{v}})\right] &= \frac{1}{I}\sum_{k\in\mathcal{I}}\mathbb{E}_{\hat{\mathbb{P}}^k}\left[\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}}\tilde{v}^i_j) - \sum_{i\in\mathcal{I}} m^i(\tilde{\boldsymbol{v}})\right] \\
= \frac{1}{I}\sum_{k\in\mathcal{I}}\mathbb{E}_{\hat{\mathbb{P}}^k}\left[\sum_{j\in\mathcal{J}}\tilde{v}^k_j - \sum_{i\in\mathcal{I}} m^i(\tilde{\boldsymbol{v}})\right] &\geq \frac{1}{I}\sum_{k\in\mathcal{I}}\mathbb{E}_{\hat{\mathbb{P}}^k}\left[\sum_{j\in\mathcal{J}}\tilde{v}^k_j - m^k(\tilde{\boldsymbol{v}})\right],
\end{aligned}$$

where the first equality follows from the definition of $\mathbb{P}^\star$, while the second equality holds because $\tilde{\boldsymbol{v}}^i = \boldsymbol{0}$ for all $i \neq k$ and $\tilde{\boldsymbol{v}}^k \geq \frac{1}{e}\overline{\boldsymbol{v}} > \boldsymbol{0}$ almost surely under $\hat{\mathbb{P}}^k$. The inequality exploits the individual rationality constraint (IR), which implies that $m^i(\tilde{\boldsymbol{v}}) \leq 0$ almost surely under $\hat{\mathbb{P}}^k$ for all $i \neq k$.

For each bidder $i \in \mathcal{I}$, define $\mathcal{S}^i = \{\boldsymbol{v} \in \mathcal{V}^I : \boldsymbol{v}^k = \boldsymbol{0} \quad \forall k \neq i\}$. Next, we relax the incentive compatibility constraint (IC) and the individual rationality constraint (IR) for any bidder $i \in \mathcal{I}$ by enforcing them only for scenarios $\boldsymbol{v} \in \mathcal{S}^i \subseteq \mathcal{V}^I$. The resulting relaxations are thus representable as

$$\begin{aligned}
&\sum_{j\in\mathcal{J}} q^i_j(\boldsymbol{v}^i, \boldsymbol{v}^{-i})v^i_j - m^i(\boldsymbol{v}^i, \boldsymbol{v}^{-i}) \\
&\geq \sum_{j\in\mathcal{J}} q^i_j(\boldsymbol{w}^i, \boldsymbol{v}^{-i})v^i_j - m^i(\boldsymbol{w}^i, \boldsymbol{v}^{-i}) \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{S}^i, \forall \boldsymbol{w}^i \in \mathcal{V}
\end{aligned} \tag{$\widehat{\text{IC}}$}$$

and

$$\sum_{j\in\mathcal{J}} q^i_j(\boldsymbol{v}^i, \boldsymbol{v}^{-i})v^i_j - m^i(\boldsymbol{v}^i, \boldsymbol{v}^{-i}) \geq 0 \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{S}^i. \tag{$\widehat{\text{IR}}$}$$

Similarly, we note that the original inventory constraint (Inv) implies the relaxation

$$\begin{aligned}
&\sum_{i\in\mathcal{I}} q^i_j(\boldsymbol{v}) \leq 1 \quad \forall j \in \mathcal{J}, \forall k \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{S}^k \\
&\implies \quad q^k_j(\boldsymbol{v}) \leq 1 \quad \forall j \in \mathcal{J}, \forall k \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{S}^k,
\end{aligned} \tag{$\widehat{\text{Inv}}$}$$

where the second implication holds because the allocation probabilities are nonnegative on $\mathcal{V}^I$.

In summary, we obtain the following relaxation of problem (3.9).

$$\begin{aligned}
\inf_{\boldsymbol{q},\boldsymbol{m}} \quad & \frac{1}{I}\sum_{k\in\mathcal{I}}\mathbb{E}_{\hat{\mathbb{P}}^k}\left[\sum_{j\in\mathcal{J}}\tilde{v}^k_j - m^k(\tilde{\boldsymbol{v}})\right] \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}^{I\times J}_+),\ \boldsymbol{m} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}^I) \\
& (\widehat{\text{IC}}),\ (\widehat{\text{IR}}),\ (\widehat{\text{Inv}})
\end{aligned} \tag{3.10}$$

Step 2.b. We now use constraint aggregation to construct a relaxation of problem (3.10), which constitutes another—even looser—relaxation of problem (3.9). To this end, define for any $i \in \mathscr{I}$ the linear embedding $E^i \in \mathscr{L}(\mathscr{V}, \mathscr{V}^I)$ via

$$E^i(\boldsymbol{v}) = (\underbrace{\boldsymbol{0}, \dots, \boldsymbol{0}}_{i-1}, \boldsymbol{v}^\top, \underbrace{\boldsymbol{0}, \dots, \boldsymbol{0}}_{I-i})^\top,$$

where, by slight abuse of notation, $\boldsymbol{v} \in \mathscr{V}$ denotes any value profile of a fixed bidder (recall also that the elements of $\mathscr{V}$ constitute row vectors). The proposed aggregation averages all constraint of (3.10) across the bidders and expresses the resulting optimization problem in terms of the new auxiliary variables $\boldsymbol{f} \in \mathscr{L}(\mathscr{V}, \mathbb{R}^J_+)$ and $g \in \mathscr{L}(\mathscr{V}, \mathbb{R})$ defined via

$$f_j(\boldsymbol{v}) = \frac{1}{I} \sum_{i \in \mathscr{I}} q^i_j(E^i(\boldsymbol{v})) \;\; \forall j \in \mathscr{J} \quad \text{and} \quad g(\boldsymbol{v}) = \frac{1}{I} \sum_{i \in \mathscr{I}} m^i(E^i(\boldsymbol{v})),$$

respectively, where $\boldsymbol{v} \in \mathscr{V}$ again denotes any value profile of a fixed bidder.

Thanks to the definition of the set $\mathscr{S}^i$ introduced in Step 2.a, the relaxed incentive compatibility constraint ($\widehat{\text{IC}}$) can be expressed as

$$\sum_{j \in \mathscr{J}} q^i_j(E^i(\boldsymbol{v}))v_j - m^i(E^i(\boldsymbol{v})) \geq \sum_{j \in \mathscr{J}} q^i_j(E^i(\boldsymbol{w}))v_j - m^i(E^i(\boldsymbol{w})) \quad \forall i \in \mathscr{I}, \forall \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V},$$

where we use again $\boldsymbol{v}$ and $\boldsymbol{w}$ to denote arbitrary value profiles of a fixed bidder. By averaging the above inequality across all bidders $i \in \mathscr{I}$, we obtain the following aggregate constraint, which can be reformulated in terms of the new decision variables $\boldsymbol{f}$ and g.

$$\begin{aligned} &\frac{1}{I} \sum_{i \in \mathscr{I}} \sum_{j \in \mathscr{J}} q^i_j(E^i(\boldsymbol{v}))v_j - \frac{1}{I} \sum_{i \in \mathscr{I}} m^i(E^i(\boldsymbol{v})) \geq \frac{1}{I} \sum_{i \in \mathscr{I}} \sum_{j \in \mathscr{J}} q^i_j(E^i(\boldsymbol{w}))v_j - \frac{1}{I} \sum_{i \in \mathscr{I}} m^i(E^i(\boldsymbol{w})) \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V} \\ \iff \quad &\sum_{j \in \mathscr{J}} f_j(\boldsymbol{v})v_j - g(\boldsymbol{v}) \geq \sum_{j \in \mathscr{J}} f_j(\boldsymbol{w})v_j - g(\boldsymbol{w}) \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V} \end{aligned} \tag{$\widehat{\text{IC}}'$}$$

Similarly, we can reformulate the relaxed individual rationality constraint ($\widehat{\text{IR}}$) as

$$\sum_{j \in \mathscr{J}} q^i_j(E^i(\boldsymbol{v}))v_j - m^i(E^i(\boldsymbol{v})) \geq 0 \quad \forall i \in \mathscr{I}, \forall \boldsymbol{v} \in \mathscr{V}.$$

By averaging the resulting inequality across all bidders $i \in \mathscr{I}$, we obtain the following aggregate constraint and its reformulation in terms of $\boldsymbol{f}$ and g.

$$\begin{aligned} &\frac{1}{I} \sum_{i \in \mathscr{I}} \sum_{j \in \mathscr{J}} q^i_j(E^i(\boldsymbol{v}))v_j - \frac{1}{I} \sum_{i \in \mathscr{I}} m^i(E^i(\boldsymbol{v})) \geq 0 \quad \forall \boldsymbol{v} \in \mathscr{V} \\ \iff \quad &\sum_{j \in \mathscr{J}} f_j(\boldsymbol{v})v_j - g(\boldsymbol{v}) \geq 0 \quad \forall \boldsymbol{v} \in \mathscr{V} \end{aligned} \tag{$\widehat{\text{IR}}'$}$$

Finally, the relaxed inventory constraint ($\widehat{\text{Inv}}$) can be formulated as

$$q_j^i(E^i(\boldsymbol{v})) \leq 1 \quad \forall j \in \mathscr{J}, \forall i \in \mathscr{I}, \forall \boldsymbol{v} \in \mathcal{V}.$$

Averaging the above inequality across all bidders $i \in \mathscr{I}$, we obtain

$$\begin{aligned} &\frac{1}{I}\sum_{i \in \mathscr{I}} q_j^i(E^i(\boldsymbol{v})) \leq 1 \quad \forall j \in \mathscr{J}, \forall \boldsymbol{v} \in \mathcal{V} \\ \iff \quad & f_j(\boldsymbol{v}) \leq 1 \quad \forall j \in \mathscr{J}, \forall \boldsymbol{v} \in \mathcal{V}. \end{aligned} \tag{$\widehat{\text{Inv}}'$}$$

We can also re-express the decision-dependent part of the objective function of problem (3.10) in terms of the new variables as

$$\frac{1}{I}\sum_{k \in \mathscr{I}} \mathbb{E}_{\hat{\mathbb{P}}^k}\left[m^k(\tilde{\boldsymbol{v}})\right] = \frac{1}{I}\sum_{k \in \mathscr{I}} \mathbb{E}_{\hat{\mathbb{P}}^k}\left[m^k(E^k(\tilde{\boldsymbol{v}}^k))\right] = \frac{1}{I}\sum_{k \in \mathscr{I}} \mathbb{E}_{\hat{\mathbb{P}}}\left[m^k(E^k(\tilde{\boldsymbol{v}}^1))\right] = \mathbb{E}_{\hat{\mathbb{P}}}\left[g(\tilde{\boldsymbol{v}}^1)\right],$$

where the first equality holds because $\tilde{\boldsymbol{v}}^i = \mathbf{0}$ for all $i \neq k$ almost surely under $\hat{\mathbb{P}}^k$, while the second equality holds because the marginal distribution of $\tilde{\boldsymbol{v}}^k$ under $\hat{\mathbb{P}}^k$ is given by $\hat{\mathbb{P}}$.

The resulting aggregation of problem (3.10) can now be represented as

$$\begin{aligned} \inf_{\boldsymbol{f}, g} \quad & \mathbb{E}_{\hat{\mathbb{P}}}\left[\sum_{j \in \mathscr{J}} \tilde{v}_j^1 - g(\tilde{\boldsymbol{v}}^1)\right] \\ \text{s.t.} \quad & \boldsymbol{f} \in \mathscr{L}(\mathcal{V}, \mathbb{R}_+^J),\ g \in \mathscr{L}(\mathcal{V}, \mathbb{R}) \\ & (\widehat{\text{IC}}'),\ (\widehat{\text{IR}}'),\ (\widehat{\text{Inv}}'). \end{aligned} \tag{3.11}$$

By construction, the problems (3.10) and (3.11) constitute two increasingly loose relaxations of problem (3.9). Moreover, problem (3.11) constitutes a multi-item pricing problem involving a single bidder ($I = 1$) that minimizes the expected regret under the distribution $\hat{\mathbb{P}}$. The decision variables $\boldsymbol{f}$ and g can be interpreted as the allocation and payment rules of the sales mechanism, respectively. The optimal value of this problem amounts to $\sum_{j \in \mathscr{J}} \frac{\overline{v}_j}{e}$ (Koçyiğit et al., 2018b, Theorem 2). As problem (3.11) is a relaxation of problem (3.9) and as $z(\boldsymbol{m}^\star, \mathbb{P}^\star) = \sum_{j \in \mathscr{J}} \frac{\overline{v}_j}{e}$ by Step 1, we can conclude that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves problem (3.9). This observation completes the proof. ■

By Theorem 3.3.1, the optimal value of (3.1) is given by $\sum_{j \in \mathscr{J}} \frac{\overline{v}_j}{e}$ and does therefore not depend on the number of bidders. This insight culminates in the following corollary.

Corollary 3.3.1. *The optimal value of* (3.1) *is independent of the number I of bidders.*

Corollary 3.3.1 implies that the seller incurs the same worst-case regret irrespective of the number of bidders participating in the auction.

3.4 Conclusion

We studied a robust auction design problem with minimax regret objective, where the seller only knows that the bidders' values range over a box uncertainty set. We interpreted this problem as a zero-sum game between the seller and nature, and we showed that this game admits a Nash equilibrium in mixed strategies that can be characterized in closed form. The seller's Nash strategy is a separable mechanism consisting of item-wise second price auctions with (random) reserve prices. Nature's Nash strategy is a distribution on the uncertainty set under which the values of the items are comonotonic for any fixed bidder and under which only the highest bidder assigns a positive value to any fixed item. Under this distribution the second highest bid for any item falls almost surely below the reserve price, which implies that the worst-case regret remains constant in the number of bidders. Our proof critically relies on the permutation symmetry of the bidders, which is a manifestation of their anonymity, and the rectangularity of the uncertainty set. We hope, however, that similar techniques can be used to solve auction design problems in which the seller has more information about the distribution of the bidders' values (*e.g.*, about their moments). Specifically, we hope that this chapter will provide insights and motivation to prove separation results for more general robust auction design problems with different informational assumptions.

Concluding Remarks

Mechanism design problems accommodate uncertain parameters due to information asymmetry between agents. The traditional Bayesian approach models all uncertainties via probabilistic beliefs. The respective probability distribution is assumed to be precisely known. Under such models, a decision is evaluated based on its performance under the underlying probability distribution. Unfortunately, this distribution is fundamentally unknown in many practical situations. Motivated by this fact, in this thesis, we investigated mechanism design problems under limited distributional information with a focus on auction design and pricing. Particularly, we adopted a distributionally robust approach, which evaluates decisions based on their performance under the most adverse distribution consistent with the information available.

One popular alternative to the distributionally robust mechanism design is the so-called approximately optimal mechanism design, which attracted a lot of attention over the last few years from different communities—especially from the computer science community—, see, *e.g.*, Dhangwatnotai et al. (2015), Hart and Nisan (2017), Allouah and Besbes (2020) and the references therein. Under this modeling paradigm, the mechanism designer aims to identify a mechanism that always (*i.e.*, under any realization of the probability distribution consistent with the assumptions made) guarantees an objective function value (*e.g.*, expected revenue) that is as close as possible to a full information benchmark value (*e.g.*, maximum expected revenue achievable).

The vast majority of the approximate optimality results available in the literature critically rely on certain independence assumptions (*e.g.*, independent values across bidders or items). For example, consider the monopoly pricing problem, where a seller endeavours to sell multiple items to a single buyer. If the buyer's values for the items are independent, simple mechanisms (such as selling the goods separately or as a single bundle at deterministic prices) provide constant-factor approximations to the expected revenue of the unknown optimal mechanism (Hart and Nisan, 2017). However, if the buyer's values are correlated, the optimal mechanism for selling more than one good may involve a menu of infinitely many price-lottery-pairs, and no deterministic mechanism can guarantee to extract any positive fraction of the optimal expected revenue (Hart and Nisan, 2019). This implies that the seller can be significantly worse off by setting deterministic posted prices for the bundles instead of implementing an optimal mechanism.

On the other hand, the assumption of independence of some or all uncertain parameters is rather difficult to handle by the distributionally robust approach because it may result in non-convex problem formulations. In contrast, allowing dependencies usually simplifies the distributionally robust mechanism design problems at hand. The presence of ambiguity and, moreover, correlation may dramatically change the optimal decisions and thus deserves further investigation, see, *e.g.*, Chapter 1 of this thesis. Distributionally robust models with dependent uncertain parameters can also be used to provide optimality guarantees for simple mechanisms. For example, in his seminal work, Carroll (2017) considers a multidimensinal screening problem, where the marginal distributions of the agent's multidimensional type are precisely known to the principal, while their dependence structure remains uncertain. Carroll shows that it is optimal for the principal to simply screen along each type separately. Chapters 2 and 3 of this thesis reveal that similar separation results persist in different variants of this mechanism design problem, namely, the robust multi-item pricing and multi-bidder multi-item auction design problems with minimax regret objective. Admittedly, the models studied in Chapters 2 and 3 may be overly conservative in situations with abundant data, and Carroll's setting is more appropriate for such situations. Carroll's separation result for the multidimensinal screening problem is seminal but unfortunately is limited to a single agent setting. The generalization to multi-agent settings seems to be very difficult even under symmetry assumptions across the agents. The difficulty of generalizing his technique originates from the core idea of the proof, which relies on a Markov chain argument. As a follow-up work, and motivated by the separation result presented in Chapter 3, we have been working on generalizing Carroll's result to the multi-item multi-bidder auction setting. This is an important open question as recognized by many authors in recent papers, see, *e.g.*, Gravin and Lu (2018).

As discussed, arguably, the most important advantage of the distributionally robust approach is the ability to incorporate potential dependencies. A promising future research avenue is to incorporate additional specific information about correlations rather than just allowing any form of it. Such distributionally robust models may admit tractable reformulations or strong tractable relaxations, see, *e.g.*, Delage and Ye (2010), and can thus contribute to a better understanding of mechanism design beyond independent values.

Even though the literature on approximately optimal mechanism design has studied models where the bidders' values fail to be additive across items, see, *e.g.*, Rubinstein and Weinberg (2018), the vast majority of the distributionally robust mechanism design problems focuses exclusively on the additive value setting. Chapter 2 and 3 of this thesis are no exceptions. The assumption of additive values is unfortunately not appropriate if supplements and complements are present. It is thus an intriguing research direction to investigate whether distributionally robust models can offer new insights for more general types of value settings.

Another interesting research direction is to investigate distributionally robust *dynamic* mechanism design problems. This research direction has already attracted attention, see, *eg.*, Balseiro et al. (2019) and the references therein. In an ongoing project, we work on a more application-tailored mechanism design problem, where we investigate how the decisions of a producer are affected by debt obligations and ambiguity with a focus on the dynamic asset selling problem. This problem is relevant to many industry branches including the agricultural industry, where farming operations are often financed by loans. We specifically investigate how the terms of the (debt) contract, that is, the initial debt value as well as the repayment schedule, and the incentives of the borrower and lender for an agreement to take place change with respect to the ambiguity level.

Finally, there exist many other relevant applications of mechanism design where robustness is critical, for example, in healthcare. Developing tailored distributionally robust models for such application-driven problems and to see whether these models are insightful in any way is another exciting future research avenue.

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