Waves at a fluid-solid interface: Explicit versus implicit formulation of boundary conditions using a discontinuous Galerkin method

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1 An accurate solution of the wave equation at a fluid-solid interface requires a correct
2 implementation of the boundary condition. Boundary conditions at acousto-elastic
3 interface require continuity of the normal component of particle velocity and traction,
4 whereas the tangential components vanish. The main challenge is to model interface
5 waves, namely the Scholte and leaky Rayleigh waves. In this study, we use a nodal
6 discontinuous Galerkin (dG) finite-element method with the medium discretized using
7 an unstructured uniform triangular meshes. The natural boundary conditions in the
8 dG method is implemented by 1) using an explicit upwind numerical flux and 2)
9 by using an implicit penalty flux and setting the modulus of rigidity of the acoustic
10 medium to zero. The accuracy of these methods are evaluated by comparing the
11 numerical solutions with analytical ones, with source and receiver at and away from
12 the interface. The study shows that the solution obtained from the explicit and
13 implicit boundary conditions produces the correct results. This is due to the fact
14 that the stability of the dG scheme is determined the numerical flux, which also
15 implements the boundary conditions by unifying the numerical solution at shared
16 edges of the elements in an energy stable manner.

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I. INTRODUCTION

The accurate computation of wave fields in realistic Earth models is an ongoing challenge in local and global seismology. The problem becomes more challenging in the presence of fluid-solid interfaces. Of particular interest, are applications in geophysics i.e., marine seismic exploration concerning the ocean bottom or global Earth seismology, using earthquake-generated seismic waves as a probing field, involving the core-mantle-boundary (CMB) and inner-core-boundary (ICB). In a fluid-solid setup, there exist two interface waves. First, the “Scholte waves” that mainly travels into the liquid (Cagniard, 1962), and, second is, the leaky-Rayleigh wave (Ash and Paige, 1985). The latter wave decays with depth in the solid while it behaves as a propagating mode in the fluid (Ash and Paige, 1985) with characteristics similar to the head waves. Surprisingly, the amplitude of the leaky-Rayleigh mode in fluid increases as it propagates away from the interface (Farnell, 1970).

The main physical aspects of the interface waves are discussed in (Glorieux *et al.*, 2002, 2001; Padilla *et al.*, 1999). In subsequent sections, the bulk velocities of the longitudinal waves in fluid and solid are represented by c_f and c_P , respectively, the shear-wave velocity of the solid is represented by c_S , and the phase velocity of the leaky Rayleigh wave, free Rayleigh waves in the solid and Scholte wave are denoted by c_{pR} , c_R and c_{Sc} , respectively. The findings of (Glorieux *et al.*, 2002, 2001; Padilla *et al.*, 1999) are summarized as:

1. In the case of $c_f < c_S < c_p$, the stiff solid condition, the leaky-Rayleigh wave propagates with a phase velocity (c_{pR}) slightly lower than c_S and leaks energy in the form of the head wave.
2. In the case of $c_s < c_f < c_P$, the soft solid condition, Padilla *et al.* (Padilla *et al.*, 1999) state that the leaky Rayleigh mode travels in the fluid without dissipation as it is described by a real root. This unphysical observation is refuted by (Glorieux *et al.*, 2001) with the argument that the root corresponding to the leaky Rayleigh mode is incorrectly identified.
3. If the density of the fluid ρ_f is close to zero, the leaky Rayleigh wave becomes a free Rayleigh wave.
4. In the case of a large contrast in the acoustic impedance of the solid and fluid, the phase velocity of the Scholte wave c_{Sc} is close to c_f .
5. For the case of a soft solid $c_S < c_f < c_p$, the phase velocity of the Scholte wave is less than c_f and the wave becomes more localized.
6. In the 2D case, the Scholte waves travel without attenuation whereas in 3D the dissipation of energy is caused by geometrical spreading.

In brief, for a stiff solid it holds that $c_{Sc} < c_R < c_{pR} < c_f < c_S < c_P$. The domain of existence of the leaky-Rayleigh mode is described by the Rayleigh window defined by the angle between the front of the Rayleigh wave and the interface (Carcione and Helle, 2004).

55 A variety of numerical methods are used to solve the acousto-elastic system. The dis-
 56 cretization is performed on (a) first-order system of partial differential equations (PDEs) in
 57 the stress-velocity formulation, or (b) in its original form of displacement, described by a
 58 system of second order PDEs. A detailed discussion on the numerical methods adopted for
 59 simulation of interface waves in a fluid-solid setup is discussed in a recent study by Car-
 60 cione *et al.* (Carcione *et al.*, 2018). The first to model the interface was Stephen (Stephen,
 61 1983). He used a second-order finite-difference method to solve the elastic wave equation
 62 and treated the boundary condition by approximating the interface with a transition zone
 63 where the velocity increases gradually. Stephen also treated the boundary condition by set-
 64 ting $c_S = 0$ in the acoustic media but obtained erroneous results. A spectral finite element
 65 method was used by Komatitsch *et al.*, (Komatitsch *et al.*, 2000) who treated the boundary
 66 conditions explicitly. Komatitsch *et al.* (Komatitsch *et al.*, 2000) do not show the propaga-
 67 tion of interface waves and lacks the comparison between numerical and analytical solutions.
 68 van Vossen *et al.* (van Vossen *et al.*, 2002) used an explicit method based on finite differences,
 69 where only five grid points per wavelength are required for an accurate modeling. Zhang
 70 (Zhang, 2004) used an integral-based approach to model the boundary conditions explicitly
 71 with a domain discretized by unstructured meshes. The study presented by van Vossen *et*
 72 *al.* (Zhang, 2004) and Zhang (Zhang, 2004) do not show the propagation of interface waves
 73 (Scholte and leaky Rayleigh waves) and also do not show the accuracy of the numerical so-
 74 lutions by comparison to the analytical solution. Carbajal-Romero *et al.* (Carbajal-Romero
 75 *et al.*, 2013) use a boundary element method to simulate the body and Scholte waves (ex-

cluding the leaky Rayleigh wave) and validate their results by comparing the results against solutions obtained by using a discrete wave-number method and a spectral-element method. Chaljub *et al.* (Chaljub *et al.*, 2003) solves the system for a heterogenous fluid-solid sphere but did not simulate the interface waves discussed here. Madec *et al.* (Madec *et al.*, 2009) solve the system by using a high-order finite element method and primarily simulated the body waves but do not consider the interface waves. DeBasabe and Sen (De Basabe and Sen, 2014) has modeled the Scholte waves and compared the solutions obtained with several algorithms such as finite differences and spectral-element methods, which show limited accuracy. They concluded that the single-grid method have spurious modes in the fluid subdomain, but they do not show any comparison or results obtained by an explicit method.

Recently, Carcione *et al.* (Carcione *et al.*, 2018) also simulated the interface waves using a direct-grid method, based on Chebyshev spatial differentiation, by incorporating natural boundary conditions using implicit and explicit approaches. In the implicit method, the natural boundary condition is incorporated by setting $c_S = 0$ in the acoustic medium (fluid). The explicit method is based on a domain decomposition technique (Carcione, 2014). They concluded that the implicit method gives erroneous numerical results by comparing the numerical and analytical solutions, whereas the explicit method worked well.

Unlike the methods discussed above, which recover the numerical solution globally, we consider the discontinuous Galerkin (dG) method. In the dG method, the global solution is obtained by unifying the solution at the shared edges of the elements by using a weak operator, known as a numerical flux (Hesthaven and Warburton, 2007, page 34). The

boundary and interface conditions are incorporated in the scheme through this operator. The numerical flux determines the stability of the numerical scheme and thus ensures a correct implementations of the interface boundary conditions. For a linear problem, the flux can be computed explicitly by solving a Riemann problem on the interior boundary of each element (Wilcox *et al.*, 2010), discretizing the acousto-elastic domain, resulting in an upwind flux. The computation of the upwind flux is expensive as it involves the eigen-decomposition, which becomes more cumbersome for anisotropic media and the extension is non-trivial. To circumvent this problem, Ye *et al.* (Ye *et al.*, 2016) proposed an explicit method to compute the numerical flux by adding a penalty term to the central flux (jump in the value of the field variables in the direction normal to the edge of the element), which eventually stabilizes the numerical scheme. In this study, we compute the numerical solutions using both fluxes, upwind and penalized central fluxes, and show that in both cases the dG method captures the interface waves accurately.

The paper is organized as follows. Section II outlines the acousto-elastic system and Section III reviews the dispersion equation for interface waves. Section IV describes the system of equations in matrix form. Section V formulates the dG scheme for a coupled acoustic-elastic system, and Section VI presents the results. In VII, we offer a few concluding remarks.

II. SYSTEM OF EQUATIONS DESCRIBING ACOUSTIC-ELASTIC WAVES

The two-dimensional velocity-stress equations for elastic wave propagation in the (x, z) plane can be expressed by:

i) Euler-Newton equations:

$$\dot{v}_1 = \frac{1}{\rho} \left(\frac{\partial \sigma_{xx}}{\partial x_1} + \frac{\partial \sigma_{xz}}{\partial x_3} \right), \quad (1)$$

$$\dot{v}_2 = \frac{1}{\rho} \left(\frac{\partial \sigma_{xz}}{\partial x_1} + \frac{\partial \sigma_{zz}}{\partial x_3} \right), \quad (2)$$

where v_1 and v_2 are the particle velocities along the x and z axis, σ_{xx} , σ_{zz} and σ_{xz} are stress components and ρ is the density. A dot above a variable denotes the time derivative.

ii) Constitutive equations:

$$\dot{\sigma}_{xx} = (\lambda + \mu) \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_3} \right) + \mu \left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_3} \right) + f_{xx}, \quad (3)$$

$$\dot{\sigma}_{zz} = (\lambda + \mu) \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_3} \right) - \mu \left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_3} \right) + f_{zz}, \quad (4)$$

$$\dot{\sigma}_{xz} = \mu \left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_2}{\partial x_1} \right) + f_{xz}, \quad (5)$$

where λ and μ are the Lamé's constants, and f_{ij} are moment forces. In 2D $\lambda = \rho(C_L^2 - 2C_S^2)$

and $\mu = \rho C_S^2$, where C_L and C_S are the compressional and shear wave velocities, respectively.

122 III. THE DISPERSION EQUATION

The dispersion equation is obtained by implementing the continuity of the normal components of the displacement and stress at the interface. This is expressed as (Carcione *et al.*, 2018; Carcione and Helle, 2004; Glorieux *et al.*, 2002, 2001; Scholte, 1942)

$$S(q) = 4\sqrt{(1-q)(1-aq)} - (2-q)^2 - \left(\frac{\rho_f}{\rho}\right) q^2 \sqrt{\frac{1-aq}{1-bq}}, \quad (6)$$

123 where $q = \frac{v^2}{c_S^2}$, $a = \frac{c_S^2}{c_L^2}$ and $b = \frac{c_S^2}{c_f^2}$ with v being the complex velocity of the interface waves.

124 The dispersion equation of the elastic Rayleigh wave is obtained by taking $\rho_f = 0$ in (6).

125 The Riemann surface of $S(q)$ has eight sheets, corresponding to different choices of the signs
 126 of the complex roots $\sqrt{1-q}$, $\sqrt{1-aq}$ and $\sqrt{1-bq}$ (Ansell, 1972). In the lossless case, the
 127 roots of the Scholte and leaky Rayleigh waves are real and complex, respectively. These
 128 surface waves are a superposition of inhomogeneous elastic waves and the attenuation vector
 129 makes an angle of 90° with respect to the propagation vector. The Scholte wave attenuates
 130 in the direction perpendicular to the interface, while the leaky Rayleigh wave attenuates
 131 along the interface. The Scholte waves travels without attenuation in 2D lossless media,
 132 whereas in 3D media the energy of Scholte waves decays due to geometrical spreading. The
 133 energy of the leaky Rayleigh wave decays both in 2D and 3D lossless media.

134 The existence of the leaky wave is subject to the condition that the sound velocity in the
 135 liquid must be less than the shear velocity in the solid. When the density of the liquid tends
 136 to zero, the leaky Rayleigh approaches the Rayleigh wave. Thus, for a stiff bottom (shear
 137 velocity in the solid higher than the velocity in water), the Rayleigh wave is not the Scholte

138 wave when the density of the liquid goes to zero. For the soft bottom case (shear wave
 139 velocity in the solid is smaller than the sound velocity in water), the Scholte wave converges
 140 to the Rayleigh wave as the density of the liquid goes to zero (Rauch, 1980). While, the
 141 free Rayleigh wave always exists, the leaky Rayleigh wave does not (Carcione *et al.*, 2018).
 142 For a stiff fluid-solid interface, the leaky Rayleigh wave has a velocity slightly lower than
 143 the body-wave shear velocity, whereas the Scholte wave velocity is lower than the velocity
 144 in the liquid. As the solid becomes stiffer, the velocity of the Scholte wave converges to the
 145 velocity of the liquid.

146 IV. SYMMETRIC FORM OF THE SYSTEM OF EQUATION IN MATRIX FORM

Equations (1)-(5) can be rewritten as a symmetric hyperbolic system of PDEs using the
 Voigt notation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \sum_{i=1}^2 \mathbf{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial x_i} \quad (7)$$

$$\mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} = \sum_{i=1}^2 \mathbf{A}_i \frac{\partial \mathbf{v}}{\partial x_i} + \mathbf{f} \quad (8)$$

$$147 \quad \text{where } \mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \mathbf{C}^{-1} = \frac{1}{\lambda\mu + 4\mu^2} \begin{bmatrix} \lambda + 2\mu & -\lambda & 0 \\ -\lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \frac{4\mu^2 + \mu\lambda}{\mu} \end{bmatrix}, \quad \mathbf{v} =$$

$$148 \quad [v_1, v_2]^T, \quad \boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}), \quad \mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = [f_{xx}, f_{zz}, f_{xz}]^T$$

149 Note that $\mathbf{C}$ is symmetric and positive-definite. Matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are constant but
 150 ρ , $\mathbf{C}$ and $\mathbf{C}^{-1}$ may vary spatially.

Combining equations (7) and (8) results

$$\mathbf{Q} \frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot (\mathbf{A} \mathbf{q}) = \mathbf{g}, \quad (9)$$

$$151 \quad \text{where } \mathbf{Q} = \left[\begin{array}{c|c} \mathbf{C}_{3 \times 3}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \rho \end{array} \right], \quad \mathbf{q} = [\sigma_{xx}, \sigma_{zz}, \sigma_{xz}, v_1, v_2]^T \quad \text{and} \quad \mathbf{A} = (\mathbf{A}_{12}, \mathbf{A}_{21}) \quad \text{with} \quad \mathbf{A}_{12} =$$

$$152 \quad \begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{2 \times 2} & \mathbf{A}_1^T \end{bmatrix} \quad \text{and} \quad \mathbf{A}_{21} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{2 \times 2} & \mathbf{A}_2^T \end{bmatrix}.$$

V. A DISCONTINUOUS GALERKIN FORMULATION FOR ACOUSTO-ELASTIC

WAVE PROPAGATION

We assume that the domain Ω is triangulated by a mesh Ω_h which consists of elements D^k which are images of a reference element $\hat{D}$ under the local affine mapping.

$$\mathbf{x}^k = \Phi^k \hat{\mathbf{x}},$$

where $\mathbf{x}^k = \{x^k, y^k\}$ for $d = 2$ denote the physical coordinates on D^k and $\hat{\mathbf{x}} = \{\hat{x}, \hat{y}\}$ for $d = 2$ and $\hat{\mathbf{x}} = \{\hat{x}, \hat{y}, \hat{z}\}$ for $d = 3$ denote coordinates on the reference element. We denote the determinant of the Jacobian of Φ^k as J .

Solutions over each element D^k are approximated from a local approximation space $V_h(D^k)$, which is defined as a composition of the mapping Φ^k and a reference approximation space $V_h(\hat{D})$

$$V_h(D^k) = \Phi^k \circ V_h(\hat{D}).$$

The global approximation space $V_h(\Omega_h)$ is defined as

$$V_h(\Omega_h) = \bigoplus_k V_h(D^k).$$

We take $V_h(\hat{D}) = P^N(\hat{D})$, with $P^N(\hat{D})$ being the space of polynomials of total degree N on the reference simplex. In two dimensions, P^N on a triangle is

$$P^N(\hat{D}) = \{\hat{x}^i \hat{y}^j, 0 \leq i + j \leq N\}.$$

The L^2 inner product and norm over D^k is represented as

$$(\mathbf{g}, \mathbf{h}) = \int_{D^k} \mathbf{g} \cdot \mathbf{h} \, d\mathbf{x} = \int_{\hat{D}} \mathbf{g} \cdot \mathbf{h} J \, d\hat{\mathbf{x}}, \quad \|\mathbf{g}\|_{L^2 \Omega}^2 = (\mathbf{g}, \mathbf{g})_{L^2(D^k)},$$

165 where $\mathbf{g}$ and $\mathbf{h}$ are vector-valued real functions. Global L^2 products and squared norms are
 166 defined as the sum of local L^2 inner products and squared norms over each element. The L^2
 167 inner product and norm over the boundary ∂D^k of an element are similarly defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(\partial D^k)} = \int_{\partial D^k} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} = \sum_{f \in \partial D^k} \int_{\hat{f}} \mathbf{u} \cdot \mathbf{v} J^f \, d\hat{\mathbf{x}}, \quad \|\mathbf{u}\|_{L^2(\partial D^k)}^2 = \langle \mathbf{u}, \mathbf{u} \rangle,$$

168 where J^f is the Jacobian of the mapping from a reference face $\hat{f}$ to a physical face f of an
 169 element.

170 Let f be a face of element D^k with neighboring element $D^{k,+}$ and unit outward normal
 171 $\mathbf{n}$. Let u be a piecewise smooth function with discontinuities across element interfaces. We
 172 define the interior value u^- and exterior value u^+ on face f of D^k ,

$$u^- = u|_{f \cap \partial D^k}, \quad u^+ = u|_{f \cap \partial D^{k,+}}.$$

173 The jump and average of a scalar function $u \in V_h(\Omega_h)$ over f are then defined as

$$[[u]] = u^+ - u^-, \quad \{\{u\}\} = \frac{u^+ + u^-}{2},$$

174 respectively. Jumps and averages of vector-valued functions $\mathbf{u} \in \mathbb{R}^m$ and matrix-valued
 175 functions $\tilde{\mathbf{S}} \in \mathbb{R}^{m \times n}$ are defined component-wise,

$$([[\mathbf{u}]])_i = [[\mathbf{u}_i]], \quad 1 \leq i \leq m \quad \left([[\tilde{\mathbf{S}}]] \right)_{ij} = [[\tilde{\mathbf{S}}]]_{ij}$$

The strong dG formulation on an element D^k is obtained by integration by parts of (9) twice, in the space on each element D^k ,

$$\int_{D^k} \mathbf{Q} \frac{\partial \mathbf{q}}{\partial t} \cdot \mathbf{p} d\mathbf{x} + \int_{D^k} (\nabla \cdot \mathbf{A} \mathbf{q}) \cdot \mathbf{p} d\mathbf{x} + \int_{\partial D^k} (\mathbf{n} \cdot ((\mathbf{A} \mathbf{q})^* - \mathbf{A}^- \mathbf{q}^-)) \cdot \mathbf{p} d\mathbf{x} = \int_{D^k} \mathbf{f} \cdot \mathbf{p} d\mathbf{x}$$

for all $\mathbf{p} \in \mathbf{V}_h(\Omega_h)$,

(10)

176 where $(\mathbf{A} \mathbf{q})^*$ is the numerical flux. The choice of the flux will determine the nature of the
177 scheme, i.e., explicit or implicit.

178 A. Explicit scheme

To compute the upwind flux, we solve a Riemann problem at shared edges of each element, which corresponds to an elastic-elastic, acoustic-acoustic, acoustic-elastic and elastic-acoustic interface. The Riemann problem to solve (10) is defined as

$$\mathbf{q}_0(\mathbf{x}) = \begin{cases} \mathbf{q}^- & \text{if } \mathbf{n} \cdot \mathbf{x} < 0 \\ \mathbf{q}^+ & \text{if } \mathbf{n} \cdot \mathbf{x} > 0. \end{cases} \quad (11)$$

179 The solution of (11) is described separately for elastic-elastic, acoustic-acoustic and elastic-
180 acoustic or acoustic elastic-interfaces

181 1. Elastic-elastic interfaces

Equation (9) is written as

$$\mathbf{Q} \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{A}_{12}}{\partial x_1} + \frac{\partial \mathbf{A}_{21}}{\partial x_2} = \mathbf{f}. \quad (12)$$

FIG. 1. Rankine-Hugonite jump conditions in the Riemann's problem

In the remainder of this section, $\mathbf{n} := \mathbf{n}^-$ denotes the outward interface unit normal vector and the flux in the normal direction is then $\mathbf{A}_n \mathbf{q}$ with $\mathbf{A}_n = n_1 \mathbf{A}_{12} + n_2 \mathbf{A}_{21}$. To construct an upwind flux requires the computations of the characteristics of the hyperbolic system defined in (9). To compute the characteristics, we need the eigen-decomposition of $\mathbf{Q}^{-1} \mathbf{A}_n$ expressed as

$$\mathbf{Q}^{-1} \mathbf{A} = \mathbf{R} \mathbf{\Lambda} \mathbf{R}^{-1} \quad (13)$$

with $\mathbf{\Lambda} = \text{diag}(-c_P, -c_S, 0, c_S, c_P)$ and the j^{th} column of $\mathbf{R}$ is an eigenvector of $\mathbf{Q}^{-1} \mathbf{A}$. For each wave speed c_i in our system, the Rankine-Hugonit jump condition is expressed as

$$-c_i \mathbf{Q}(\mathbf{q}^m - \mathbf{q}^p) + \mathbf{A}_n(\mathbf{q}^m - \mathbf{q}^p) = 0 \quad (14)$$

where, $\mathbf{q}^m$ and $\mathbf{q}^p$ are states in the negative and positive normal directions, travelling with speed c_i , across the discontinuity. Matrices $\mathbf{Q}$ and $\mathbf{A}_n$ are evaluated in the region where mode c_i travels, denoted by “-” and “+”. The Riemann problem has four unknown states

$(\mathbf{q}^a, \mathbf{q}^b, \mathbf{q}^c, \mathbf{q}^d)$ as described in Figure 1. A full set of Rakine-Hugonite jump conditions is given as

$$c_P^- \mathbf{Q}^- (\mathbf{q}^- - \mathbf{q}^a) + \mathbf{A}_n^- (\mathbf{q}^- - \mathbf{q}^a) = 0 \quad (15)$$

$$c_S^- \mathbf{Q}^- (\mathbf{q}^a - \mathbf{q}^b) + \mathbf{A}_n^- (\mathbf{q}^a - \mathbf{q}^b) = 0 \quad (16)$$

$$\mathbf{A}_n^- \mathbf{q}^b - \mathbf{A}_n^+ \mathbf{q}^c = 0 \quad (17)$$

$$-c_S^+ \mathbf{Q}^+ (\mathbf{q}^a - \mathbf{q}^b) + \mathbf{A}_n^+ (\mathbf{q}^c - \mathbf{q}^d) = 0 \quad (18)$$

$$-c_P^+ \mathbf{Q}^+ (\mathbf{q}^d - \mathbf{q}^+) + \mathbf{A}_n^+ (\mathbf{q}^d - \mathbf{q}^+) = 0 \quad (19)$$

Using the definitions of the eigenvalues and eigenvectors of $(\mathbf{Q}^-)^{-1} \mathbf{A}^-$ and $(\mathbf{Q}^+)^{-1} \mathbf{A}^+$, we obtain

$$\mathbf{q}^- - \mathbf{q}^a = \alpha_1 \mathbf{r}_1^-,$$

$$\mathbf{q}^a - \mathbf{q}^b = \alpha_2 \mathbf{r}_2^- + \alpha_3 \mathbf{r}_3^-,$$

$$\mathbf{q}^c - \mathbf{q}^d = \alpha_7 \mathbf{r}_7^+ + \alpha_8 \mathbf{r}_8^+,$$

$$\mathbf{q}^d - \mathbf{q}^+ = \alpha_9 \mathbf{r}_9^+.$$

Thus the upwind numerical flux $(\mathbf{A}_n \mathbf{q})^*$ is defined as (Hesthaven and Warburton, 2007)

$$\mathbf{n} \cdot ((\mathbf{A} \mathbf{q})^* - \mathbf{A}^- \mathbf{q}^-) = \alpha_1 c_P^- \mathbf{Q}^- \mathbf{r}_1^- + \alpha_2 c_S^- \mathbf{Q}^- \mathbf{r}_2^- + \alpha_3 c_S^- \mathbf{Q}^- \mathbf{r}_3^-. \quad (20)$$

The expression of α_1 , α_2 and α_3 is derived in detail by Wilcox et al. (Wilcox et al., 2010).

After substituting α_1 , α_2 and α_3 in (20), we recover

$$\begin{aligned} \mathbf{n} \cdot ((\mathbf{A}\mathbf{q})^* - \mathbf{A}^- \mathbf{q}^-) &= \frac{\mathbf{n} \cdot [\mathbf{S}] + \rho^+ c_p^+ [\mathbf{v}]}{\rho^+ c_p^+ + \rho^- c_p^-} \begin{bmatrix} \lambda^- \mathbf{I} + 2\mu^- \mathbf{n} \otimes \mathbf{n} \\ \rho^- c_p^- \mathbf{n} \end{bmatrix} \\ &\quad - \frac{1}{(\rho^- c_s^- + \rho^+ c_s^+)} \begin{bmatrix} 2\mu^- \text{sym}(\mathbf{n} \otimes (\mathbf{n} \times (\mathbf{n} \times [\mathbf{S}]))) \\ \rho^- c_s^- \mathbf{n} \times (\mathbf{n} \times [\mathbf{S}]) \end{bmatrix} \\ &\quad - \frac{\rho^+ c_s^+}{\rho^- c_s^- + \rho^+ c_s^+} \begin{bmatrix} 2\mu^- \text{sym}(\mathbf{n} \otimes (\mathbf{n} \times (\mathbf{n} \times [\mathbf{v}]))) \\ \rho^- c_s^- \mathbf{n} \times (\mathbf{n} \times [\mathbf{v}]) \end{bmatrix}, \end{aligned} \quad (21)$$

182 where $\mathbf{S} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$.

183 2. Acoustic-acoustic interface

Acoustic-acoustic interfaces are characterized by $\mu^- = \mu^+ = 0$ and thus $\mathbf{q}$ reduces to $\hat{\mathbf{q}} = [\sigma_{xx}, \sigma_{yy}, v_1, v_2]$. We can solve the Riemann problem to compute $\hat{\mathbf{q}}$ and keep the number of variables the same as in the case of the elastic-elastic interface, by using relation $\frac{\partial \sigma_{xy}}{\partial t} = \mu \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right)$. Rankine-Hugonite jump conditions for acoustic-acoustic interface are given

as

$$c_p^- \mathbf{Q}^- (\mathbf{q}^- - \mathbf{q}^a) + \mathbf{A}_n^- (\mathbf{q}^- - \mathbf{q}^a) = 0 \quad (22)$$

$$\mathbf{A}_n^- \mathbf{q}^a - \mathbf{A}_n^+ \mathbf{q}^b = 0 \quad (23)$$

$$-c_p^+ \mathbf{Q}^+ (\mathbf{q}^b - \mathbf{q}^+) + \mathbf{A}_n^+ (\mathbf{q}^b - \mathbf{q}^+) = 0. \quad (24)$$

Now, following the same approach as in the elastic-elastic case, the flux for the acoustic-acoustic interface is

$$\mathbf{n} \cdot ((\mathbf{A}\mathbf{q})^* - \mathbf{A}^- \mathbf{q}^-) = \frac{\mathbf{n} \cdot [\mathbf{S}] + \rho^+ c_p^+ [\mathbf{v}]}{\rho^+ c_p^+ + \rho^- c_p^-} \begin{bmatrix} \lambda^- \mathbf{I} \\ \rho^- c_p^- \mathbf{n} \end{bmatrix} \quad (25)$$

$$\text{where } \mathbf{S} = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{bmatrix}.$$

3. Elastic-acoustic and acoustic-elastic interfaces

For the elastic-acoustic interface ($\mu^- \neq 0$ and $\mu^+ = 0$), the solutions to the Riemann problem satisfy the Rankine-Hugonite jump conditions:

$$c_p^- \mathbf{Q}^- (\mathbf{q}^- - \mathbf{q}^a) + \mathbf{A}_n^- (\mathbf{q}^- - \mathbf{q}^a) = 0 \quad (26)$$

$$c_s^- \mathbf{Q}^- (\mathbf{q}^a - \mathbf{q}^b) + \mathbf{A}_n^- (\mathbf{q}^a - \mathbf{q}^b) = 0 \quad (27)$$

$$\mathbf{A}_n^- \mathbf{q}^b - \mathbf{A}_n^+ \mathbf{q}^c = 0 \quad (28)$$

$$-c_p^+ \widehat{\mathbf{Q}}^+ (\widehat{\mathbf{q}}^c - \widehat{\mathbf{q}}^+) + \widehat{\mathbf{A}}_n^+ (\widehat{\mathbf{q}}^c - \widehat{\mathbf{q}}^+) = 0 \quad (29)$$

where $\hat{\mathbf{q}} = [\sigma_{xx}, \sigma_{zz}, v_1, v_2]^T$ and $\hat{\mathbf{Q}} = \mathbf{Q}|_{\mu=0}$. The resulting numerical flux incorporates an upwind term from an incoming c_p characteristics and a boundary term from an incoming c_s characteristics (Wilcox *et al.*, 2010). The boundary condition for an incoming c_s characteristics is expressed as (Wilcox *et al.*, 2010)

$$\mathbf{s} \cdot \mathbf{v}^- - 2c_s^- \mathbf{s} \cdot \mathbf{S} \mathbf{n} = \mathbf{s} \cdot \mathbf{v}^+ \quad (30)$$

where $\mathbf{s}$ is the tangential vector along the interface. Solving equations (26)-(29) as in the elastic-elastic case and imposing the boundary condition (30) weakly, the numerical flux for the elastic-acoustic interface is

$$\begin{aligned} \mathbf{n} \cdot ((\mathbf{A}\mathbf{q})^* - \mathbf{A}^- \mathbf{q}^-) &= \frac{\mathbf{n} \cdot \llbracket \mathbf{S} \rrbracket + \rho^+ c_p^+ \llbracket \mathbf{v} \rrbracket}{\rho^+ c_p^+ + \rho^- c_P^-} \begin{bmatrix} \lambda^- \mathbf{I} + 2\mu^- \mathbf{n} \otimes \mathbf{n} \\ \rho^- c_P^- \mathbf{n} \end{bmatrix} \\ &\quad - \frac{1}{\rho^- c_S^-} \begin{bmatrix} 2\mu^- \text{sym}(\mathbf{n} \otimes (\mathbf{n} \times (\mathbf{n} \times \llbracket \mathbf{S} \rrbracket))) \\ \rho^- c_S^- \mathbf{n} \times (\mathbf{n} \times \llbracket \mathbf{S} \rrbracket) \end{bmatrix} \end{aligned} \quad (31)$$

Equation (31) could be interpreted as an elastic-elastic flux (21) for $\mu^+ = 0$. Using the same logic, the flux for the acoustic-elastic interface ($\mu^- = 0$ and $\mu^+ \neq 0$) is

$$\mathbf{n} \cdot ((\mathbf{A}\mathbf{q})^* - \mathbf{A}^- \mathbf{q}^-) = \frac{\mathbf{n} \cdot \llbracket \mathbf{S} \rrbracket + \rho^+ c_p^+ \llbracket \mathbf{v} \rrbracket}{\rho^+ c_p^+ + \rho^- c_P^-} \begin{bmatrix} \lambda^- \mathbf{I} \\ \rho^- c_P^- \mathbf{n} \end{bmatrix}. \quad (32)$$

186 The flux for the acoustic-elastic interface (32) is the same as that of the acoustic-acoustic
 187 interface (25). Thus, for the explicit scheme, the flux given in (21), (25), (31) and (32), is
 188 appropriately chosen by evaluating the values of μ^- and μ^+ of the medium.

189 B. Implicit scheme

The implicit scheme requires $\mu = 0$ in the numerical scheme. The dG scheme must be consistent locally as well as globally. The solid-solid, fluid-fluid, and solid-fluid boundary conditions are expressed as

$$\mathbf{v}^+ - \mathbf{v}^- = 0 \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma}^+ - \mathbf{n} \cdot \boldsymbol{\sigma}^- = 0 \quad \text{on} \quad \Gamma_{\text{SS}} \quad (33\text{a})$$

$$\mathbf{n} \cdot (\mathbf{v}^\pm - \hat{\mathbf{v}}^\mp) = 0 \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma}^\pm - \mathbf{n} \cdot \hat{\boldsymbol{\sigma}}^\mp = 0 \quad \text{on} \quad \Gamma_{\text{SF}} \text{ and } \Gamma_{\text{FS}} \quad (33\text{b})$$

$$\hat{\mathbf{v}}^+ - \hat{\mathbf{v}}^- = 0 \quad \text{and} \quad \hat{\boldsymbol{\sigma}}^+ - \hat{\boldsymbol{\sigma}}^- = 0 \quad \text{on} \quad \Gamma_{\text{FF}}. \quad (33\text{c})$$

To derive the implicit scheme, the matrix form of the acoustic wave equation is required and can be expressed as

$$\hat{\mathbf{Q}} \frac{\partial \hat{\mathbf{q}}}{\partial t} + \nabla \cdot (\hat{\mathbf{A}} \hat{\mathbf{q}}) = \hat{\mathbf{f}}, \quad (34)$$

190 where $\hat{\mathbf{Q}} = \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & \rho \mathbf{I}_{3 \times 3} \end{array} \right]$, $\hat{\mathbf{q}} = [\hat{\sigma}, v_1, v_2]$ with $\hat{\sigma} = -\lambda \hat{E}$, where $\hat{E}$ is the longitudinal strain.

The dG scheme in strong form for (34) is

$$\begin{aligned} \int_{D^k} \hat{\mathbf{Q}} \frac{\partial \hat{\mathbf{q}}}{\partial t} \cdot \hat{\mathbf{p}} d\mathbf{x} + \int_{D^k} (\nabla \cdot \hat{\mathbf{A}} \hat{\mathbf{q}}) \cdot \hat{\mathbf{p}} d\mathbf{x} + \int_{\partial D^k} \left(\mathbf{n} \cdot \left((\hat{\mathbf{A}} \hat{\mathbf{q}})^* - \hat{\mathbf{A}}^- \hat{\mathbf{q}}^- \right) \right) \cdot \hat{\mathbf{p}} d\mathbf{x} \\ = \int_{D^k} \hat{\mathbf{f}} \cdot \hat{\mathbf{p}} d\mathbf{x} \text{ for all } \hat{\mathbf{p}} \in \mathbf{V}_h(\Omega_h), \end{aligned} \quad (35)$$

To derive an implicit scheme, we consider a Lax-Friedrich flux ([Hesthaven and Warburton, 2007](#)), i.e.,

$$\mathbf{n} \cdot (\mathbf{A}\mathbf{q}^*) = n_x \{\{\mathbf{A}\mathbf{q}\}\} + n_y \{\{\mathbf{A}\mathbf{q}\}\} + \frac{c_{\max}}{2} [\mathbf{q}] \quad (36)$$

191 where $c_{\max}$ is the maximum phase velocity of the modes present in the system.

Substituting (36) in (10), the surface term is recovered as

$$\mathbf{n} \cdot ((\mathbf{A}\mathbf{q})^* - \mathbf{A}^- \mathbf{q}^-) = \frac{1}{2} \llbracket \mathbf{A}\mathbf{q} \rrbracket + \frac{c_{\max}}{2} [\mathbf{q}]. \quad (37)$$

192 In (37), the first term, $\frac{1}{2} \llbracket \mathbf{A}\mathbf{q} \rrbracket$ is responsible for the interface boundary conditions explained
 193 in (33a)-(33c), whereas the second term $\frac{c_{\max}}{2} [\mathbf{q}]$ is a penalty term that stabilizes the scheme.
 194 Implementation of the flux in (36) for the acousto-elastic system with $\mu = 0$, will produce
 195 the wrong results as the numerical scheme in (10) with the flux in (37) is consistent for
 196 purely elastic or acoustic systems, but not for a coupled acoustic-elastic system ([Wilcox
 197 et al., 2010](#)).

To accurately incorporate the interface conditions at solid-solid, fluid-fluid, solid-fluid and fluid-solid boundaries in (10) and (35), we express the dG formulation for the acoustic and elastic conditions as follows

$$\begin{aligned} \int_{D_S^k} \mathbf{Q} \frac{\partial \mathbf{q}}{\partial t} \cdot \mathbf{p} d\mathbf{x} + \int_{D_S^k} (\nabla \cdot \mathbf{A}\mathbf{q}) \cdot \mathbf{p} d\mathbf{x} + \frac{1}{2} \int_{\partial D_{SS}^k} \llbracket \mathbf{A}_n \mathbf{q} \rrbracket_{SS} \cdot \mathbf{p} d\mathbf{x} + \frac{1}{2} \int_{\partial D_{SF}^k} \llbracket \mathbf{A}_n \mathbf{q} \rrbracket_{SF} \cdot \mathbf{p} d\mathbf{x} \\ = \int_{D^k} \mathbf{f} \cdot \mathbf{p} d\mathbf{x} \text{ for all } \mathbf{p} \in \mathbf{V}_h(\Omega_h), \end{aligned} \quad (38)$$

$$\begin{aligned} \int_{D_F^k} \hat{\mathbf{Q}} \frac{\partial \hat{\mathbf{q}}}{\partial t} \cdot \hat{\mathbf{p}} d\mathbf{x} + \int_{D_F^k} (\nabla \cdot \hat{\mathbf{A}}\hat{\mathbf{q}}) \cdot \hat{\mathbf{p}} d\mathbf{x} + \frac{1}{2} \int_{\partial D_{FF}^k} \llbracket \hat{\mathbf{A}}_n \hat{\mathbf{q}} \rrbracket_{FF} \cdot \hat{\mathbf{p}} d\mathbf{x} + \frac{1}{2} \int_{\partial D_{FS}^k} \llbracket \hat{\mathbf{A}}_n \hat{\mathbf{q}} \rrbracket_{FS} \cdot \hat{\mathbf{p}} d\mathbf{x} \\ = \int_{D_F^k} \hat{\mathbf{f}} \cdot \hat{\mathbf{p}} d\mathbf{x} \text{ for all } \hat{\mathbf{p}} \in \mathbf{V}_h(\Omega_h), \end{aligned} \quad (39)$$

where we define

$$\begin{aligned} \llbracket \mathbf{A}_n \mathbf{q} \rrbracket_{SS} &= (\mathbf{A}_n \mathbf{q})^+ - (\mathbf{A}_n \mathbf{q})^-, & \llbracket \mathbf{A}_n \mathbf{q} \rrbracket_{SF} &= \mathbf{T}^T \left(\widehat{\mathbf{A}} \widehat{\mathbf{q}} \right)^+ - (\mathbf{A}_n \mathbf{q})^- \\ \llbracket \widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \rrbracket_{FF} &= \left(\widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \right)^+ - \left(\widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \right)^-, & \llbracket \widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \rrbracket_{FS} &= \mathbf{T} (\mathbf{A}_n \mathbf{q})^+ - \left(\widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \right) \end{aligned} \quad (40)$$

with the map $T : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ given by (Ye *et al.*, 2016)

$$T\mathbf{q} = \begin{bmatrix} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \\ (\mathbf{n} \cdot \mathbf{v}) \mathbf{n} \end{bmatrix} \quad \text{and} \quad T^T \widehat{\mathbf{q}} = \begin{bmatrix} (\mathbf{n} \mathbf{n} \widehat{\sigma}) \\ (\mathbf{n} \cdot \widehat{\mathbf{v}}) \mathbf{n} \end{bmatrix},$$

198 where

$$\mathbf{T} = \left[\begin{array}{ccc|cc} n_1 n_1 & n_2 n_2 & n_1 n_2 & & \mathbf{0} \\ \hline & & & n_1 n_1 & n_1 n_2 \\ \mathbf{0} & & & n_1 n_2 & n_2 n_2 \end{array} \right].$$

In (38) and (39), the numerical fluxes are central fluxes, which may be unstable. To stabilize these formulations, a penalization is used (Guo *et al.*, 2019; Warburton, 2013; Ye *et al.*, 2016). In this study, we used a penalization method of fluxes proposed by Ye *et al.* (Warburton, 2013). The penalized flux for the solid-solid is

$$\begin{aligned} F_E &= \frac{1}{2} \int_{\partial D_{SS}^k} \llbracket \mathbf{A}_n \mathbf{q} \rrbracket_{SS} \cdot \mathbf{p} d\mathbf{x} + \frac{1}{2} \int_{\partial D_{SF}^k} \llbracket \mathbf{A}_n \mathbf{q} \rrbracket_{SF} \cdot \mathbf{p} d\mathbf{x} + \frac{\alpha}{2} \left[\int_{\partial D_{SS}^k} (\mathbf{A}_n^T \llbracket \mathbf{A}_n \mathbf{q} \rrbracket) \cdot \mathbf{p} d\mathbf{x} \right. \\ &\quad \left. + \int_{\partial D_{SF}^k} (\mathbf{A}_n^T \llbracket \mathbf{A}_n \mathbf{q} \rrbracket) \cdot \mathbf{p} d\mathbf{x} \right], \end{aligned} \quad (41)$$

where $\alpha > 0$. Similarly, the penalized flux for the fluid-fluid and fluid-solid case is

$$\begin{aligned} F_A &= \frac{1}{2} \int_{\partial D_{FF}^k} \llbracket \widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \rrbracket_{FF} \cdot \widehat{\mathbf{p}} d\mathbf{x} + \frac{1}{2} \int_{\partial D_{FS}^k} \llbracket \widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \rrbracket_{FS} \cdot \widehat{\mathbf{p}} d\mathbf{x} + \frac{\alpha}{2} \left[\int_{\partial D_{FF}^k} \widehat{\mathbf{A}}_n^T \llbracket \widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \rrbracket \cdot \widehat{\mathbf{p}} d\mathbf{x} \right. \\ &\quad \left. + \int_{\partial D_{FS}^k} \widehat{\mathbf{A}}_n^T \llbracket \widehat{\mathbf{A}}_n \widehat{\mathbf{q}} \rrbracket \cdot \widehat{\mathbf{p}} d\mathbf{x} \right]. \end{aligned} \quad (42)$$

TABLE I. Material properties

Material	λ	μ	ρ
	(GPa)	(GPa)	(kg/m ³)
water	2.25	0	1000
plexiglass	4.3315	2.2799	1180
glass	25.253	28.157	2500

TABLE II. Velocities of body and surface waves

Interface	c_f	c_L	c_S	c_{Sc}	c_{pR}
	(m/s)	(m/s)	(m/s)	(m/s)	(m/s)
water-plexiglass	1500	2745	1390	1058	1363
water-glass	1500	5712	3356	1496	3091

C. Time discretization

We employ the low-storage explicit Runge-Kutta (LSERK) method ([Cockburn and Shu, 2001](#)). The LSERK method is a single-step method but comprises five intermediate stages. LSERK is preferred over other methods as it saves memory at the cost of computation time.

A stable CFL condition, depending on the polynomial degree N , is derived by Cockburn and Shu (Cockburn and Shu, 2001) and used here.

D. Forcing function

The forcing function $\mathbf{f}$, in (8), is the product of a Dirac delta in space ($\mathbf{x}_0$) and a Ricker wavelet in time,

$$\mathbf{f} = \delta(\mathbf{x} - \mathbf{x}_0)h(t) \quad (43)$$

where $h(t)$ is a wavelet, given by

$$h(t) = \exp \left[-2f_p^2 (t - t_0)^2 \right] \cos \left[2\pi f_p (t - t_0) \right],$$

with f_p being the central peak frequency and $t_0 = 3/(4f_p)$. In subsequent sections, f_p is taken to be 500 kHz with a dilatational moment force ($f_{xx} = f_{zz}$ and $f_{xy} = 0$).

VI. COMPUTATIONAL EXPERIMENTS

To perform the computational experiments, we consider a medium including water-plexiglass and water-glass interfaces. The material properties are given in Table I. The velocities of the body and surface or interface waves, computed from the dispersion relation(6), are given in Table II. The medium defined by the water-plexiglass combination represents the soft-solid system, whereas the water-glass interface is the stiff-solid condition. To determine the existence and evolution of the interface waves in soft and stiff solid conditions, we first solve the coupled acoustic-elastic system using the explicit numerical scheme,

(a) Horizontal particle velocity, v_1 at $t = 0.22 \mu s$

(b) Vertical particle velocity, v_3 at $t = 0.22 \mu s$

(c) Horizontal particle velocity, v_1 at $t = 0.33 \mu s$

(d) Vertical solid particle velocity, v_3 at $t = 0.33 \mu s$

FIG. 2. Snapshots of the simulated interface waves for a water-plexiglass interface, using explicit boundary-conditions, which shows the horizontal and vertical particle velocities at $0.22 \mu s$ (a, b) and $0.33 \mu s$ (c, d). The source is located in the solid at $38.4 \mu m$ below the interface, and it is an explosion ($fxx = fzz$), with a central frequency of 500 kHz .

defined in (10) in subsection V A. Figure 2 shows snapshots of the particle velocities computed for the water-plexiglass or soft-solid condition. The size of the computational domain is $[0.20 \text{ mm} \times 0.9 \text{ mm}]$. The minimum size of the edges of the equilateral triangles, used to mesh the domain, in the x and y directions, are 0.37 mm and 0.56 mm , respectively. The degree of polynomial (N) is 4 and the time step dt used for time integration is 0.217 ns .

(a) Horizontal particle velocity, v_1 at $t = 0.10 \mu s$

(b) Vertical particle velocity, v_3 at $t = 0.10 \mu s$

(c) Horizontal particle velocity, v_1 at $t = 0.16 \mu s$

(d) Vertical solid particle velocity, v_3 at $t = 0.16 \mu s$

FIG. 3. Snapshots of the simulated interface waves for a water-glass interface, using explicit boundary-conditions, which shows the horizontal and vertical particle velocities at $0.10 \mu s$ (a, b) and $0.16 \mu s$ (c, d). The source is located in the solid at $38.4 \mu m$ below the interface, and it is an explosion ($fx x = fz z$), with a central frequency of 500 kHz .

224 Figures 2a and b represent the snapshots of the horizontal (v_1) and vertical (v_2) particle
 225 velocities at $0.22 \mu\text{s}$. As discussed, the presence of the Scholte wave, is observed and marked
 226 in Figures 2a and b. To visualize the evolution of the Scholte wave, Figures 2c and d repre-
 227 sent the particle velocities (v_1, v_2) at $t = 0.33 \mu\text{s}$. The Scholte wave moves to a distance of
 228 1.1 mm (Figures 2c and 2d) at $t = 0.33 \mu\text{s}$ from 0.9836 mm (Figures 2a and 2b) measured
 229 at $t = 0.22 \mu\text{s}$. Thus the velocity of the Scholte wave is equal to 1058.18 m/s . This in
 230 agreement with the velocity computed from the dispersion relation, shown in Table II, with
 231 a relative error of 0.017% .

232 Figure 3 represents the snapshots of the horizontal (v_1) and vertical particle velocities
 233 (v_3) for the water-glass system, i.e., a stiff solid medium. The discretization parameters are
 234 the same as those used in Figure 1 except dt , which is $.104 \text{ ns}$. Figures 3a and 3b represent
 235 snapshot at $t = 0.10 \mu\text{s}$. The presence of a leaky Rayleigh mode is observed (also marked
 236 by a white box) at $x = 0.1085 \text{ mm}$. To see the evolution of the leaky Rayleigh wave, the
 237 snapshots of horizontal and particle velocities computed at $t = 0.16 \mu\text{s}$ are shown in Figures
 238 3c and 3d. The leaky Rayleigh mode is present $x = 0.1085 \text{ mm}$. Thus, the velocity of the
 239 leaky Rayleigh mode computed is 3091.67 m/s , with a relative error of 0.022% as compared
 240 to the velocity obtained from the dispersion relation.

241 Figures 4-8 show the comparison of the numerical solution against the analytical solution
 242 obtained by Berg et al. (Berg *et al.*, 1994) using the method of Cagniard-de Hoop (de Hoop
 243 and Van der Hijden, 1983). Figure 4 shows the simulation of the reflection response using
 244 an upwind flux (explicit boundary conditions) with a polynomial degree $N = 4$. Source and
 245 receiver are located in the fluid at 0.0158831 m above the interface, and are separated by
 246 0.016 m. The source is an explosion with a central frequency of 500 kHz. The discretization
 247 parameters for the water-plexiglass and water-glass models are the same as those used for
 248 results shown in Figures 2 and 3, respectively. The solid line and circles correspond to
 249 the analytical and numerical solutions, respectively. Figures 4a and 4b show the x and z
 250 components of the normalized particle velocities corresponding to the water-plexiglass model,
 251 respectively. Figures 4c and 4d show the x and z components of the normalized particle-
 252 velocity corresponding to the water-glass model, respectively. The direct and reflection
 253 modes are marked in the figures. The agreement between numerical and analytical solutions
 254 for both models is very good. Next, we verify the ability of the numerical scheme to model
 255 interface waves. Source and receiver are both located in the solid at $38.4 \mu\text{m}$ with an offset
 256 of 0.1 m. The discretization parameters are the same as those in Figure 4. Figure 5a and 5b
 257 show a comparison of the x and z components of the normalized particle velocities for the
 258 water-plexiglass model (soft solid condition). The solutions show a very good agreement.
 259 The dominance of the Scholte wave in both components is clearly observed. A comparison
 260 of the analytical and numerical solutions for the water-glass model (stiff-solid system) is
 261 shown in Figures 5c and 5d. The agreement between the solutions is satisfactory. The
 262

(a) x -component. Water-plexiglass interface

(b) z -component. Water-plexiglass interface

(c) x -component. Water-glass interface

(d) z component. Water-glass interface

FIG. 4. Explicit boundary-condition: Simulation of the reflection response using an upwind flux, where the solid line and circles correspond to the analytical and numerical solutions, respectively. The figure shows the normalized particle-velocity components corresponding to water-plexiglass (a and b) and water-glass (c and d). Source and receiver are located in the fluid at 0.0158831 m above the interface, and are separated by 0.016 m. The source is an explosion, with a central frequency of 500 kHz.

(a) x -component. Water-plexiglass interface

(b) z -component. Water-plexiglass interface

(c) x -component. Water-glass interface

(d) z -component. Water-glass interface

FIG. 5. Explicit boundary-condition: Simulation of interface-waves, where the solid line and dots correspond to the analytical and numerical solutions, respectively. The figure shows the normalized particle-velocity components corresponding to water-plexiglass (a and b) and water-glass (c and d). Source and receiver are both located in the solid at $38.4 \mu\text{m}$ below the interface, with a separation 0.1 m . The source is an explosion ($fx = fz$), with a central frequency of 500 kHz .

(a) x component. Water-plexiglass interface

(b) z component. Water-plexiglass interface

FIG. 6. Implicit boundary-condition: Simulation of the reflection response using the Lax-Friedrich (inconsistent) flux, where the solid line and circles correspond to the analytical and numerical solutions, respectively. The figure shows the normalized particle-velocity components corresponding to water-plexiglass (a and b). Source and receiver are located in the fluid at 0.0158831 m above the interface, and are separated by 0.016 m. The source is an explosion, with a central frequency of 500 kHz

presence of leaky Rayleigh and Scholte waves is clearly observed, along with the P and S waves. Next, we show the ability of the implicit numerical scheme with Lax-Friedrich flux (37) and setting the $\mu = 0$. Figures 6a and 6b show a comparison between the numerical and analytical solutions of the particle velocities for the water-plexiglass and water-glass models, respectively. It is clear that the reflection mode is not accurately simulated by the scheme. This results from the scheme not being consistent.

Now, we show results of the numerical scheme using the penalty fluxes expressed in (41) and (42). Figures 7 and 8 show the comparison of the analytical and numerical solutions

(a) x component. Water-plexiglass interface

(b) z component. Water-plexiglass interface

(c) x component. Water-glass interface

(d) z component. Water-glass interface

FIG. 7. Implicit boundary-condition: Simulation of the reflection response using the penalized consistent central flux, where the solid line and circles correspond to the analytical and numerical solutions, respectively. The figure shows the normalized particle-velocity components corresponding to water-plexiglass (a and b) and water-glass (c and d). Source and receiver are located in the fluid at 0.0158831 m above the interface, and are separated by 0.016 m. The source is an explosion, with a central frequency of 500 kHz

(a) x -component. Water-plexiglass interface

(b) z -component. Water-plexiglass interface

(c) x -component. Water-glass interface

(d) z -component. Water-glass interface

FIG. 8. Implicit boundary-condition: Simulation of interface-waves using a penalized consistent central flux, where the solid line and dots correspond to the analytical and numerical solutions, respectively. The figure shows the normalized particle-velocity components corresponding to water-plexiglass (a and b) and water-glass (c and d). Source and receiver are both located in the solid at $38.4 \mu\text{m}$ below the interface, with a separation 0.1 m . The source is an explosion ($fx = fz$), with a central frequency of 500 kHz .

for the reflection and interface problems. Discretization parameters and source-receiver geometry are same as those used for the results shown in Figure 4 and 5. Figures 6 and 7 show a very good agreement between the numerical and analytical solutions for the body (P and S waves) and interface waves, confirming that the implicit scheme is suitable provided a consistent flux is used.

VII. CONCLUSIONS

To our knowledge, past works dealing with the simulation of waves at fluid-solid interfaces, performed with the finite-element method (continuous or discontinuous Galerkin) with unstructured grids, do not show the complete set of waves (i.e. including the Scholte and leaky-Rayleigh waves) and a comparison with the analytical solution has not been performed. This test is essential to verify the correct simulation of wave propagation in the presence of interfaces, corresponding to soft and stiff fluid-solid systems. Here, we present analytic and numerical results for the simulation of waves propagating at the fluid-solid interface for two solids: plexiglass (soft) and glass (stiff). To perform the simulations, we use a discontinuous Galerkin method with upwind and penalty fluxes to implement the correct boundary conditions. We show that setting $\mu = 0$ in a dG formulation, corresponding to elastic wave equation, provides wrong results. The numerical solution obtained from upwind and penalty flux based methods produce the correct results and are in very good agreement with the analytical solutions.

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- Ansell, J. H. (1972). “The roots of the stoneley wave equation for solid-liquid interfaces,” *Pure and Applied Geophysics* **94**(1), 172–188.
- Ash, E. A., and Paige, E. G. (1985). *Rayleigh-Wave Theory and and Application: Proceedings of An International Symposium Organised by the Rank Prize Funds at the Royal Institution, London, 1985* (Springer-Verlag).
- Berg, P., If, F., Nielsen, P., and Skovgaard, O. (1994). *Analytical reference solutions, in Modeling The Earth For Oil Exploration*, (K. Helbig, ed., Pergamon Press), pp. 421–427.
- Cagniard, L. (1962). *Reflection and Refraction of Progressive Seismic Waves, Translation and Revision of Cagniard, L., 1939, Réflexion et Réfraction des Ondes Séismiques Progressives* (McGraw-Hill, NY).
- Carbajal-Romero, M., Flores-Mendez, E., Flores-Guzmán, N., Núñez-Farfán, J., Olivera-Villaseñor, E., and Sánchez-Sesma, F. J. (2013). “Scholte waves on fluid-solid interfaces by means of an integral formulation,” *Geofísica Internacional* **52**(1), 21–30.

311 Carcione, J. M. (**2014**). *Wave Fields in Real Media. Theory and numerical simulation of*
312 *wave propagation in anisotropic, anelastic, porous and electromagnetic media, 3rd edition*
313 (Elsevier).

314 Carcione, J. M., Bagaini, C., Ba, J., Wang, E., and Vesnaver, A. (**2018**). “Waves at fluid–
315 solid interfaces: explicit versus implicit formulation of the boundary condition,” *Geophys-*
316 *ical Journal International* **215**(1), 37–48.

317 Carcione, J. M., and Helle, H. B. (**2004**). “The physics and simulation of wave propagation
318 at the ocean bottom wave propagation at ocean bottom,” *Geophysics* **69**(3), 825–839.

319 Chaljub, E., Capdeville, Y., and Vilotte, J.-P. (**2003**). “Solving elastodynamics in a fluid–
320 solid heterogeneous sphere: a parallel spectral element approximation on non-conforming
321 grids,” *Journal of Computational Physics* **187**(2), 457–491.

322 Cockburn, B., and Shu, C.-W. (**2001**). “Runge–kutta discontinuous galerkin methods for
323 convection-dominated problems,” *Journal of Scientific Computing* **16**(3), 173–261.

324 De Basabe, J. D., and Sen, M. K. (**2014**). “A comparison of finite-difference and spectral-
325 element methods for elastic wave propagation in media with a fluid-solid interface,” *Geo-*
326 *physical Journal International* **200**(1), 278–298.

327 de Hoop, A. T., and Van der Hijden, J. H. (**1983**). “Generation of acoustic waves by an
328 impulsive line source in a fluid/solid configuration with a plane boundary,” *The Journal of*
329 *the Acoustical Society of America* **74**(1), 333–342.

330 Farnell, G. (**1970**). “Properties of elastic surface waves,” *Physical Acoustics* **6**, 109–166.

331 Glorieux, C., Van de Rostyne, K., Gusev, V., Gao, W., Lauriks, W., and Thoen, J. (2002).
332 “Nonlinearity of acoustic waves at solid–liquid interfaces,” The Journal of the Acoustical
333 Society of America **111**(1), 95–103.

334 Glorieux, C., Van de Rostyne, K., Nelson, K., Gao, W., Lauriks, W., and Thoen, J. (2001).
335 “On the character of acoustic waves at the interface between hard and soft solids and
336 liquids,” The Journal of the Acoustical Society of America **110**(3), 1299–1306.

337 Guo, K., Acosta, S., and Chan, J. (2019). “A weight-adjusted discontinuous galerkin method
338 for wave propagation in coupled elastic-acoustic media,” arXiv preprint:1905.09145 .

339 Hesthaven, J. S., and Warburton, T. (2007). *Nodal discontinuous Galerkin methods: algo-*
340 *rithms, analysis, and applications* (Springer Science & Business Media).

341 Komatitsch, D., Barnes, C., and Tromp, J. (2000). “Wave propagation near a fluid-solid
342 interface: A spectral-element approach,” Geophysics **65**(2), 623–631.

343 Madec, R., Komatitsch, D., and Diaz, J. (2009). “Energy-conserving local time stepping
344 based on high-order finite elements for seismic wave propagation across a fluid-solid inter-
345 face,” Computer Modeling in Engineering and Sciences (CMES) **14**(2), 163.

346 Padilla, F., de Billy, M., and Quentin, G. (1999). “Theoretical and experimental studies of
347 surface waves on solid–fluid interfaces when the value of the fluid sound velocity is located
348 between the shear and the longitudinal ones in the solid,” The Journal of the Acoustical
349 Society of America **106**(2), 666–673.

350 Rauch, D. (1980). “Experimental and theoretical studies of seismic interface waves in coastal
351 waters,” in *Bottom-Interacting Ocean Acoustics* (Springer), pp. 307–327.

352 Scholte, J. (**1942**). “On the stoneley wave equation,” Proceedings of the Koninklijke Neder-
 353 landse Akademie van Wetenschappen **45**(part 1), 20–25.
 354 Stephen, R. (**1983**). “A comparison of finite difference and reflectivity seismograms for ma-
 355 rine models,” Geophysical Journal International **72**(1), 39–57.
 356 van Vossen, R., Robertsson, J. O., and Chapman, C. H. (**2002**). “Finite-difference modeling
 357 of wave propagation in a fluid-solid configuration,” Geophysics **67**(2), 618–624.
 358 Warburton, T. (**2013**). “A low-storage curvilinear discontinuous galerkin method for wave
 359 problems,” SIAM Journal on Scientific Computing **35**(4), A1987–A2012.
 360 Wilcox, L. C., Stadler, G., Burstedde, C., and Ghattas, O. (**2010**). “A high-order discon-
 361 tinuous galerkin method for wave propagation through coupled elastic–acoustic media,”
 362 Journal of Computational Physics **229**(24), 9373–9396.
 363 Ye, R., de Hoop, M. V., Petrovitch, C. L., Pyrak-Nolte, L. J., and Wilcox, L. C. (**2016**).
 364 “A discontinuous galerkin method with a modified penalty flux for the propagation and
 365 scattering of acousto-elastic waves,” Geophysical Journal International **205**(2), 1267–1289.
 366 Zhang, J. (**2004**). “Wave propagation across fluid—solid interfaces: a grid method ap-
 367 proach,” Geophysical Journal International **159**(1), 240–252.