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MINIMAL POLYNOMIALS OF UNIPOTENT ELEMENTS OF NON-PRIME ORDER IN IRREDUCIBLE REPRESENTATIONS OF THE EXCEPTIONAL ALGEBRAIC GROUPS IN SOME GOOD CHARACTERISTICS

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Abstract. In a number of cases the minimal polynomials of the images of unipotent elements of non-prime order in irreducible representations of the exceptional algebraic groups in good characteristics are found. It is proved that if $p > 5$ for a group of type E_8 and $p > 3$ for other exceptional algebraic groups, then for irreducible representations of these groups in characteristic p with large highest weights with respect to p, the degree of the minimal polynomial of the image of a unipotent element is equal to the order of this element.

Keywords: exceptional algebraic groups, unipotent elements, irreducible representations

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МИНИМАЛЬНЫЕ ПОЛИНОМЫ УНИПОТЕНТНЫХ ЭЛЕМЕНТОВ НЕПРОСТОГО ПОРЯДКА В НЕПРИВОДИМЫХ ПРЕДСТАВЛЕНИЯХ ИСКЛЮЧИТЕЛЬНЫХ АЛГЕБРАИЧЕСКИХ ГРУПП В НЕКОТОРЫХ ХОРОШИХ ХАРАКТЕРИСТИКАХ

(Представлено академиком В. И. Янчевским)

Аннотация. В ряде случаев найдены минимальные многочлены образов унипотентных элементов непростого порядка в неприводимых представлениях исключительных алгебраических групп в хороших характеристиках. Установлено, что если $p > 5$ для группы типа E_8 и $p > 3$ для других исключительных алгебраических групп, то для неприводимых представлений этих групп в характеристике p с большими относительно характеристики старшими весами степень минимального многочлена образа унипотентного элемента равна порядку этого элемента.

Ключевые слова: исключительные алгебраические группы, унипотентные элементы, неприводимые представления

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Introduction. In a number of cases the minimal polynomials of the images of unipotent elements of non-prime order in irreducible representations of the exceptional algebraic groups in good characteristics are found. The problem is completely solved for the groups of types E_6 and G_2 (for the latter group, in characteristics 2 and 3 as well), for representations of the groups of type F_4 in characteristics 5 and 11, the groups of type E_7 in characteristics 5, 7, and 17, and the groups of type E_8 in characteristics 7 and 29. It is proved that if $p > 5$ for a group of type E_8 and $p > 3$ for other exceptional algebraic groups, then for irreducible representations of these groups in characteristic p with large highest weights with

respect to p, the degree of the minimal polynomial of the image of a unipotent element is equal to the order of this element.

The minimal polynomials of the images of unipotent elements of non-prime order in irreducible representations of the classical algebraic groups in odd characteristics have been found in [1], for unipotent elements of prime order and all simple algebraic groups, the problem was solved in [2]. In [1] one can find a motivation for this problem connected with recognizing representations and linear groups by the presence of particular matrices and a short discussion of some results on a similar problem for irreducible representations of finite groups close to simple.

The main part. Throughout the text $\mathbb{C}$ is the complex field, K is an algebraically closed field of an odd characteristic p, $\mathbb{Z}$ and $\mathbb{Z}^+$ are the sets of integers and nonnegative integers, respectively, G is a simply connected simple algebraic group of an exceptional type over K, $G_{\mathbb{C}}$ is the simply connected simple algebraic group over $\mathbb{C}$ of the same type as G, r is the rank of G, ω_i, $1 \le i \le r$, are the fundamental weights of G, $\omega(\varphi)$ is the highest weight of a representation φ. For an element x and a representation ρ of some algebraic group, the symbol $d_\rho(x)$ denotes the degree of the minimal polynomial of $\rho(x)$; $\langle \mu, \alpha \rangle$ is the value of a weight μ on a root α (the canonical pairing in the sense of [3, Section 1]). If φ is an irreducible representation of G, then $\varphi_{\mathbb{C}}$ is the irreducible representation of $G_{\mathbb{C}}$ with highest weight $\omega(\varphi)$. The characteristic p is called good for G if the maximal root of G is a linear combination of the simple roots with the coefficients smaller than p. Hence p is good for a group of type E_8 for $p > 5$ and p is good for other exceptional groups if $p > 3$. If p is good for G, there exists a canonical bijection f from the set of unipotent conjugacy classes of G onto the analogous set for $G_{\mathbb{C}}$ determined with the help of the distinguished parabolic subgroups in the Levi subgroups of G (see, for instance, comments in the Introduction of [4]). In what follows if $x \in G$ is a unipotent element from a class C, then $x_{\mathbb{C}} \in f(C) \subset G_{\mathbb{C}}$. Recall that an irreducible representation of a semisimple algebraic group over K is p-restricted if all coefficients of its highest weight (when expressed as a linear combination of the fundamental weights) are less than p.

It is well known that G contains unipotent elements of non-prime order in the following cases:

a) $G = E_6(K)$ or $F_4(K)$, $p < 13$;

b) $G = E_7(K)$, $p < 19$;

c) $G = E_8(K)$, $p < 31$;

d) $G = G_2(K)$, $p < 7$

(see [4]). In these situations if p is good for G, then the maximal order of a unipotent element in G is p^2.

T h e o r e m 1. *Let* $5 \le p \le 11$ *for* $G = E_6(K)$, $p \in \{5, 7, 17\}$ *for* $G = E_7(K)$, $p = 7$ *or* 29 *for* $G = E_8(K)$, $p = 5$ *or* 11 *for* $G = F_4(K)$, *and* $p = 5$ *for* $G = G_2(K)$. *Assume that* $x \in G$ *is an element of order* p^2 *and* φ *is a p-restricted irreducible representation of G. Then*

$$d_\varphi(x) = \min\{d_{\varphi_{\mathbb{C}}}(x_{\mathbb{C}}), pd_\varphi(x^p), p^2\} \tag{1}$$

or one of the following holds:

1) $p = 5$, $G = E_7(K)$, *x is a regular unipotent element*, $\omega(\varphi) = \omega_7$, $d_\varphi(x) = 24$;

2) $p = 7$, $G = E_7(K)$, *x is an element from the conjugacy class* $E_6(a_1)$, $\omega(\varphi) = \omega_1$, $d_\varphi(x) = 15$;

3) $p = 7$, $G = E_7(K)$, *x is an element from the conjugacy class* $E_7(a_1)$, $\omega(\varphi) = \omega_2$, $d_\varphi(x) = 36$;

4) $p = 7$, $G = E_7(K)$, *x is a regular unipotent element*, $\omega(\varphi) = \omega_2$, $d_\varphi(x) = 44$;

5) $p = 7$, $G = E_7(K)$, *x is an element from the conjugacy class* $E_6(a_1)$, $\omega(\varphi) = \omega_1 + \omega_7$, $d_\varphi(x) = 27$;

6) $p = 7$, $G = E_7(K)$, *x is an element from the conjugacy class* $E_6(a_1)$, $\omega(\varphi) = 2\omega_1$, $d_\varphi(x) = 29$;

7) $p = 7$, $G = E_7(K)$, *x is an element from the conjugacy class* $E_6(a_1)$, $\omega(\varphi) = 3\omega_1$, $d_\varphi(x) = 43$;

8) $p = 7$, $G = E_8(K)$, *x is an element from the conjugacy class* $E_6(a_1)$ *or* $E_6(a_1) + A_1$, *a regular unipotent element from a subsystem subgroup of type* A_7, *or an element from a subsystem subgroup of type* D_8 *that in the standard realization of this subgroup has Jordan blocks of sizes* 9 *and* 7, $\omega(\varphi) = \omega_8$, $d_\varphi(x) = 15$;

9) $p = 7$, $G = E_8(K)$, *x is a regular unipotent element from a subsystem subgroup of type* D_7, $\omega(\varphi) = \omega_8$, $d_\varphi(x) = 22$;

10) $p = 7$, $G = E_8(K)$, *x is a regular unipotent element from a subsystem subgroup of type* D_7 *or an element from a subsystem subgroup of type* D_8 *that in the standard realization of this subgroup has Jordan blocks of sizes* 13 *and* 3, $\omega(\varphi) = \omega_7$, $d_\varphi(x) = 43$;

11) $p = 7$, $G = E_8(K)$, *x is such as in Item* 8), $\omega(\varphi) = 2\omega_8$, $d_\varphi(x) = 29$;

12) $p = 7$, $G = E_8(K)$, *x is a regular unipotent element from a subsystem subgroup of type* D_7, $\omega(\varphi) = 2\omega_8$, $d_\varphi(x) = 43$;

13) $p = 7$, $G = E_8(K)$, *x is such as in Items* 8) *and* 11), $\omega(\varphi) = 3\omega_8$, $d_\varphi(x) = 43$.

Here the labeling of the unipotent conjugacy classes of G is such as in [2]. To find $d_{\varphi\mathbb{C}}(x_\mathbb{C})$ and $d_\varphi(x_p)$, one can apply results of [2, Theorem 1.1, Proposition 1.3, Algorithm 1.4, and Tables].

By the Steinberg tensor product theorem [5, Theorem 1.1], if ρ is an irreducible representation of a semisimple algebraic group over K, then $\rho \cong \bigotimes_{k=0}^{j} \rho_k \circ Fr^k$ where all ρ_k are p-restricted and Fr is the Frobenius morphism determined by raising the elements of the field to the power p. Set $\omega'(\rho) = \sum_{k=0}^{j} \omega(\rho_k)$. The weight $\omega'(\rho)$ is uniquely determined. We call an irreducible representation ρ of a simple algebraic group Γ over K *p-large* if $\langle \omega'(\rho), \beta \rangle \geq p$ for a maximal root β of Γ.

For p-large representations, the problem under consideration is solved for the exceptional algebraic groups in all good characteristics.

T h e o r e m 2. *Let* $p > 5$ *for* $G = E_8(K)$, $p > 3$ *otherwise, and* φ *be a p-large irreducible representation of G. Then* $d_\varphi(x) = |x|$ *for each unipotent element* $x \in G$.

Now state our results for $G = G_2(K)$ and $p = 2$ or 3. For $p = 2$, the group G has two conjugacy classes of unipotent elements of non-prime order. One of them consists of regular unipotent elements, another contains a regular unipotent element from a subsystem subgroup of type A_2.

P r o p o s i t i o n 1. *Let* $p = 2$, $x \in G$ *be a regular unipotent element,* $y \in G$ *be a regular unipotent element from a subsystem subgroup of type* A_2, *and* φ *be a nontrivial irreducible representation of G. Then* $d_\varphi(x) = 6$ *and* $d_\varphi(y) = 3$ *for* $\omega(\varphi) = 2^j\omega_1$. *Otherwise* $d_\varphi(x) = |x| = 8$ *and* $d_\varphi(y) = |y| = 4$.

For $p = 3$, only regular unipotent elements have order p^2, other unipotent elements have order p.

P r o p o s i t i o n 2. *Let* $p = 3$, $x \in G$ *be a regular unipotent element, and* φ *be a nontrivial irreducible representation of G. Then* $d_\varphi(x) = 7$ *for* $\omega(\varphi) = 3^j\omega_1$ *or* $3^j\omega_2$ *and* $d_\varphi(x) = |x| = 9$ *otherwise.*

We need some more notation. In what follows Γ is a simply connected simple algebraic group over $\mathbb{C}$ or K, $\Lambda(\Gamma)$, $\Lambda^+(\Gamma)$, $R(\Gamma)$, $R^+(\Gamma)$, and $R^-(\Gamma)$, respectively, are the sets of weights, dominant weights, roots, positive and negative roots of Γ, $\Pi(\Gamma)$ is a basis in $R(\Gamma)$; $\mathcal{X}_\beta$ and $x_\beta(t)$ are the root subgroup and the root element of Γ associated with a root β and an element t of the field; $\overline{cl(x)}$ is the Zarisky closure of the conjugacy class containing an element x; $\Gamma(\beta_1, \ldots, \beta_k)$ is the subgroup of Γ generated by root subgroups $\mathcal{X}_{\pm\beta_1}, \ldots, \mathcal{X}_{\pm\beta_k}$. We use the notation $\Gamma_\mathbb{C}$, ω_i, and $\omega(\varphi)$ as for G.

Throughout the text dim M (dim φ) is the dimension of a Γ-module M (a representation φ), $\Lambda(M)$ is the set of weights of M, and $d_M(x)$ is the degree of the minimal polynomial of an element x acting on M. If $\omega \in \Lambda^+(\Gamma)$, then $M(\omega)$, $V(\omega)$, and $\varphi(\omega)$ are the irreducible module, the Weyl module, and the irreducible representation of Γ with highest weight ω; $\omega(m)$ is the weight of a weight vector m from some module. If H is a subgroup of Γ, then $M \mid H$ is the restriction of a Γ-module M to H. We assume that the weights and the roots of Γ are considered with respect to a fixed maximal torus T. If $T \cap H$ is a maximal torus in H, then $\omega \mid H$ is the restriction of a weight ω to $T \cap H$. In this case for a weight vector m from some Γ-module, we set $\omega_H(m) = \omega(m)|H$. If M is an irreducible Γ-module, then $v \in M$ is a nonzero highest weight vector.

The following facts are used intensively in the proofs of the main results.

P r o p o s i t i o n 3 [1, a part of Proposition 2.5]. *Let M be a* Γ*-module,* $x \in \Gamma$ *be a unipotent element, and* $|x| = p^{s+1} > p$.

(a) *Assume that* $l \leq s$ *and* $z = x^{p^l}$. *Then* $p^l(d_M(z) - 1) < d_M(z) \leq p^l d_M(z)$.

(b) *Let* $y = x^{p^s}$, $d_M(y) = a + 1$, $M_y = (y - 1)^a M$, *and* $d_{M_y}(x) = b$. *Then* $b \leq p^s$, $d_M(x) = ap^s + b$, *and* $\dim(x-1)^{ap^s+b-1} M = \dim(x-1)^{b-1} M_y$.

Denote by C_i^j the number of combinations of j elements chosen from i elements.

L e m m a 1 [6, Lemma 22.4]. *Let* $a = \sum_{j=0}^{s} a_j p^j$ *and* $b = \sum_{j=0}^{s} b_j p^j$ *where* a_j, $b_j \in \mathbb{Z}^+$. *Suppose that* $a_j + b_j < p$, $0 \leq j \leq s$. *Then* $C_{a+b}^a \not\equiv 0 \pmod{p}$.

In Lemma 2 and Theorem 3 J_a is a unipotent Jordan block of size a, $d(u)$ is the degree of the minimal polynomial of a unipotent element u regarded as an element of $GL_t(K)$ where t is clear from the context. We shall write $J_a \otimes J_b \cong J_{i_1} \oplus ... \oplus J_{i_t}$ if $J_{i_1}, \ldots, J_{i_t}$ is the complete collection of blocks in the canonical Jordan form of the matrix $J_a \otimes J_b$ (multiplicities are taken into account); in this case we also write kJ_a instead of the sum $J_a \oplus \ldots \oplus J_a$ (k times).

L e m m a 2 [7, Chapter VIII, Theorem 2.7]. *Let* $1 \le s \le t \le p$. *Then*

$$J_f \otimes J_g \cong \bigoplus_{i=0}^{h-1} J_{g-f+2i+1} \oplus NJ_p,$$

where $h = \min\{f, p-g\}$, $N = 0$ *for* $f+g \le p$, *and* $N = f+g-p$ *for* $f+g > p$. *In particular,* $\mathrm{d}(J_f \otimes J_g) = f+g-1$ *for* $f+g \le p$ *and* $\mathrm{d}(J_f \otimes J_g) = p$ for $f+g > p$.

T h e o r e m 3 [8, Lemma 6.14 and Theorem 6.4]. *Set* $q = p^s$, $s \ge 1$. *Assume that* $0 < g, h \le q$,

$$J_g \otimes J_h \cong \bigoplus_{i=1}^{l} J_{n_i} \oplus NJ_q,$$

and all $n_i < q$. *Then* $l = \min\{g, h, q-g, q-h\}$.

Let $a = uq + g$ *and* $b = vq + h$ *with* $0 \le u \le v \le p-1$. *For* $0 \le j \le u$ *put* $f_i = v - u + 2j$. *If* $a + b \le pq$, *then*

$$J_a \otimes J_b \cong \bigoplus_{i=1}^{l}\bigoplus_{j=0}^{u-1} J_{f_j q+n_i} \oplus \bigoplus_{i=1}^{l}\bigoplus_{j=0}^{u-1} J_{(f_j+2)q-n_i} \oplus \bigoplus_{j=0}^{u-1} |g-h|\, J_{(f_j+2)q} \oplus \bigoplus_{j=0}^{u-1} |q-g-h|\, J_{(f_j+1)q} \oplus P,$$

where

$$P = \begin{cases} 0 & \text{for } l = g, \\ (g-h)J_{(v-u)q} & \text{for } l = h, \\ (g+h-q)J_{(u+v+1)q} & \text{for } l = q-h, \\ (g-h)J_{(v-u)q} \oplus (g+h-q)J_{(u+h-q)} & \text{for } l = q-g. \end{cases}$$

Hence $d(J_a \otimes J_b) < pq$.

Now let $a + b > pq$. *Set* $a_1 = pq - a$, $b_1 = pq - b$. *Then* $a_1 + b_1 < pq$ *and*

$$J_a \otimes J_b \cong J_{a1} \otimes J_{b1} \oplus (a+b-pq)J_{pq}.$$

Therefore $d(J_a \otimes J_b) = pq$.

L e m m a 3 [2, Lemma 2.20]. *Let* Γ *be a semisimple algebraic group,* $x, y \in \Gamma$ *be unipotent, and* $y \in \overline{cl(x)}$. *Then* $d_\varphi(y) \le d_\varphi(x)$ *for each representation* φ *of* Γ.

In the following lemma the symbol $\mathbb{F}_p$ denotes the field of p elements.

L e m m a 4 [1, Lemma 2.38]. *Let* Γ *be a semisimple algebraic group over* K, $\beta_1, \ldots, \beta_k \in R^+(\Gamma)$, $t_1, \ldots, t_k \in \mathbb{Z}$, *and* $\overline{t_j}$ *be the image of* t_j *under the natural homomorphism* $\mathbb{Z} \to \mathbb{F}_p$. *Let* $x = \prod_{j=1}^{k} x_{\beta_j}(\overline{t_j}) \in \Gamma$, $x_{\mathbb{C}} = \prod_{j=1}^{k} x_{\beta_j}(t_j) \in \Gamma_{\mathbb{C}}$, *and* φ *be an irreducible representation of* Γ. *Then* $d_\varphi(x) \le d_{\varphi\mathbb{C}}(x_{\mathbb{C}})$.

L e m m a 5 [1, Lemma 2.42]. *Let* Γ *be a semisimple algebraic group over* $\mathbb{C}$ *or* K, $I \subset \Pi(\Gamma)$ *be a proper subset, and* M *be a* Γ*-module. Denote by* Σ_I *the set of integer linear combinations of the simple roots from* I. *Set* $R_I = R(\Gamma) \cap \Sigma_I$ *and* $R' = R^+(\Gamma) \setminus (R_I \cap R^+(\Gamma))$. *Let* $m = m_1 + \ldots + m_k \in M$ *and* m_j, $1 \le j \le k$, *be the weight components of* m. *If* $k > 1$, *assume that* $\omega(m_i) - \omega(m_j) \in \Sigma_I$ *for* $1 \le i < j \le k$. *Suppose that* $x' \in \langle \mathcal{X}_\alpha \mid \alpha \in R' \rangle$, $x_I \in \langle \mathcal{X}_\alpha \mid \alpha \in R_I \cap R^+(\Gamma) \rangle$, $x = x'x_I$, *and* $(x_I - 1)^a m \ne 0$. *Then* $(x-1)^a \ne 0$. *In particular,* $d_M(x) \ge d_M(x_I)$.

L e m m a 6 [1, Lemma 2.51]. *Let* λ_1 *and* $\lambda_2 \in \Lambda^+(\Gamma)$, $\omega = \lambda_1 + \lambda_2$, $M_j = M(\lambda_j)$, $M = M(\omega)$, *and* $x \in \Gamma$ *be a unipotent element. Then* $d_M(x) \le d_{M_1}(x) + d_{M_2}(x) - 1$.

L e m m a 7. *Let* M *be a* Γ*-module,* N *be a composition factor of* M, *and* $x \in \Gamma$ *be a unipotent element. Then* $d_M(x) \ge d_N(x)$.

Lemma 7 follows easily from [9, Lemma 2.4].

P r o p o s i t i o n 4 [2, Proposition 2.15]. *Let* ω, λ_i, M_i, *and* M *be such as in Lemma 6 and* $x \in \langle \mathcal{X}_\alpha \mid \alpha \in R^-(\Gamma)\rangle$. *For* $l \in \mathbb{Z}^+$, *put* $\Lambda_l = \left\{\mu \in \Lambda(M) \mid \omega - \mu = \sum_{i=1}^{r} c_i\alpha_i, \sum_{i=1}^{r} c_i \geq l\right\}$.

Let $v_j \in M_j$ *and* $v \in M$ *be nonzero highest weight vectors, and let* $(x-1)^{f_j} v_j \neq 0$ *for some* $f_j \in \mathbb{Z}^+$, $j = 1, 2$. *Assume that* $f = f_1 + f_2$, $C_f^{f_1} \not\equiv 0 \pmod p$, *the vectors* $(x-1)^{f_j} v_j$ *have nonzero weight components of weight* μ_j, *and* $\dim V(\omega)_\mu = \dim M_\mu$ *for* $\mu = \mu_1 + \mu_2$. *Then* $(x-1)^f v \neq 0$. *In particular, the inequality holds if* $\dim V(\omega)_\mu = \dim M_\mu$ *for all* $\mu \in \Lambda_f$ (*for instance, if* Λ_f *consists of the lowest weight of* M *or the module* $V(\omega)$ *is irreducible*).

T h e o r e m 4 [1, Theorem 1.1]. *For a p-large representation* ρ *of a simple algebraic group* Γ *of a classical type over* K *and a unipotent element* $x \in \Gamma$, *the degree of the minimal polynomial of* $\rho(x)$ *is equal to the order of* x.

The length of this article does not permit us to include the proofs even for the principal results, so we present the general outline of these proofs. Let p be good for G. It is well known that for every unipotent conjugacy class of G, one can choose for representatives x and $x_{\mathbb{C}}$ products of the "same" root elements with the coefficients from the prime field or the ring of integers, respectively. Hence we can apply Lemma 4 which yields that $d_\varphi(x) \leq d_{\varphi_{\mathbb{C}}}(x_{\mathbb{C}})$ for each irreducible representation φ. One has to consider separately the groups of all types in different characteristics because of the differences in the structure of centralizers of unipotent elements from a fixed conjugacy class. If for a fixed p, the groups of type E_i have elements of order p^2 for different i, we start our analysis from the groups of the smallest possible rank.

Results of [2] imply that in the proof of Theorem 2 it suffices to consider the cases where G has unipotent elements of non-prime order.

First assume that one of the following holds:

1) $G = E_6(K)$, $p = 5$ or 7,
2) $G = E_7(K)$, $p \in \{5, 7, 11\}$,
3) $G = E_8(K)$, $p \in \{7, 11, 13, 17\}$,
4) $G = F_4(K)$, $p = 5$ or 7.

In this situation the proof of Theorem 2 is based on the following

P r o p o s i t i o n 5. *Let* $H \subset G$ *be a subsystem subgroup of type* A_5 *or* D_5 *for* $G = E_6(K)$, *of type* A_6, D_6, *or* E_6 *for* $G = E_7(K)$, *of type* A_8, D_8, *or* E_7 *for* $G = E_8(K)$, *and of type* C_3 *or* B_4 *for* $G = F_4(K)$, *and* φ *be a p-large irreducible representation of* G. *Then the restriction* $\varphi \mid H$ *has a p-large composition factor.*

Spaltenstein's results [10] imply that under our assumptions for each element $x \in G$ of order p^2, the set $\overline{cl(x)}$ contains an element u of order p^2 from one of the groups H from Proposition 5. Therefore Theorem 2 for such groups follows from Theorem 4, Proposition 5, and Lemmas 3 and 7.

Let φ be a p-restricted irreducible representation of G. Assume that none of Conditions 1–4 holds or φ is not p-large. Let $x \in G$ be an element of order p^2 and $y = x^p$. One can find $d_\varphi(y)$ using the results of [2]. Suppose that $x \in H$ where $H \subset G$ is a subsystem subgroup with simple components of classical types. The majority of unipotent conjugacy classes in G contain elements from such subgroups. The minimal polynomials of the images of unipotent elements in irreducible representations of the classical algebraic groups in odd characteristics are found in [1, Theorems 1.1, 1.3, 1.9, and 1.10]. In the restriction $\varphi \mid H$ we construct a composition factor ψ such that $d_\psi(x) = \min\{d_{\varphi_{\mathbb{C}}}(x_{\mathbb{C}}), pd_\varphi(y), p^2\}$. Then by Lemmas 4 and 7 and Proposition 3, $d_\varphi(x) = d_\psi(x)$. Observe that if it is proved that $d_\varphi(x) = p^2$ or $d_\varphi(x) = pd_\varphi(y)$, then by Lemma 3 and Proposition 3, in the first case $d_\varphi(z) = p^2$ for each unipotent element z with $x \in \overline{cl(z)}$, and in the second one $d_\varphi(u) = d_\varphi(x)$ for each element u such that $x \in \overline{cl(u)}$ and $u^p = y$. For $G = E_7(K)$ or $E_8(K)$, we apply this approach also to elements that lie in subsystem subgroups one of whose simple components is of type E_6 or E_7, respectively, and use already available results for the groups of smaller ranks and the same characteristic.

Now let x be an element which is not conjugate to any element of a proper subsystem subgroup of G, and let one cannot deduce from Lemma 3 that $d_\varphi(x) = d_\varphi(q)$ for some element q from such subgroup. Assume that $d_\varphi(y) = a + 1$. Let M be a module affording the representation φ and $M_y = (y-1)^a M$. Often we can represent x in the form $x = x_1 x'$ where x_1 and x' satisfy the assumptions of Lemma 5, $|x_1| = p$, $x_1 \in S \subset C_G(y)$,

and S is a semisimple subgroup. We construct a weight vector $m \in M_y$ invariant with respect to a maximal unipotent subgroup of S. Using results of [2], one can find the minimal polynomial of x_I on some composition factor of KSm. Then we apply Lemma 5 and Proposition 3 to estimate $d_{M_y}(x)$ and $d_M(x)$, respectively. If in this situation we can show that $d_M(x) \geq \min\{d_{M\mathbb{C}}(x_{\mathbb{C}}), pd_M(y), p^2\}$, then one can find $d_M(x)$ using Lemma 4 and Proposition 3. Sometimes Lemma 6 yields that $d_M(x) \leq d$, and the approach described above allows one to prove that $d_M(x) \geq d$. Then $d_M(x) = d$.

In some situations we use Proposition 4 and an explicit construction of certain representations. Lemma 1 is applied to check whether the assumption of Proposition 4 on the binomial coefficient holds.

If for fixed G and p the problem is solved for all p-restricted representations, one can solve it in the general case applying Lemma 2 and Theorem 3.

Finally, let $G = G_2(K)$ and $p = 2$ or 3. In the proof of Proposition 1 Lawther's results on the block structure of the images of unipotent elements of G in the irreducible representations with highest weights ω_1 and ω_2 [4], well-known properties of the Steinberg module, and Theorem 3 and Lemma 2 are used.

The proof of Proposition 2 is based on the following: Lawther's results mentioned above, the information on morphisms and properties of representations of a group of type G_2 in characteristic 3 from [3, § 10 and § 12], the irreducibility of the restriction of the irreducible representation of a group of type B_3 in an odd characteristic with highest weight $2\omega_1$ to a subgroup of type G_2 [11, Theorem 17.1], the theorem on the minimal polynomials of the images of regular unipotent elements in the irreducible representations of the classical algebraic groups in odd characteristics [1, Theorem 1.7], Theorem 3, and Lemma 2.

Conclusion. The minimal polynomials of the images of unipotent elements of non-prime order in irreducible representations of the exceptional algebraic groups in some good characteristics are found. In the majority of cases these polynomials are determined by Formula (1), the exceptions are indicated in Theorem 1.

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