Nonlinear Gyrokinetic Coulomb Collision Operator

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A gyrokinetic Coulomb collision operator is derived, which is particularly useful to describe the plasma dynamics at the periphery region of magnetic confinement fusion devices. The derived operator is able to describe collisions occurring in distribution functions arbitrarily far from equilibrium with variations on spatial scales at and below the particle Larmor radius. A multipole expansion of the Rosenbluth potentials is used in order to derive the dependence of the full Coulomb collision operator on the particle gyroangle. The full Coulomb collision operator is then expressed in gyrocentre phase-space coordinates, and a closed formula for its gyroaverage in terms of the moments of the gyrocenter distribution function in a form ready to be numerically implemented is provided. Furthermore, the collision operator is projected onto a Hermite-Laguerre velocity space polynomial basis and expansions in the small electron-to-ion mass ratio are provided.

1. Introduction

The plasma periphery, which encompasses the edge and the scrape-off layer regions, plays a central role in determining the overall performance of a fusion device, as it regulates the overall plasma confinement, it controls the plasma-wall interactions, it is responsible for power exhaust, and it governs the plasma refueling and the removal of fusion ashes (Ricci 2015). Understanding the plasma dynamics in the periphery is therefore crucial for the success of the whole fusion program (Connor *et al.* 1998).

While the plasma dynamics in the scrape-off layer has been described mainly using drift-reduced fluid models valid at low frequencies compared to the ion cyclotron frequency, $\omega \ll \Omega_i$, and in the limits $k_\parallel \lambda_{mfpe} \ll 1$ and $k_\perp \rho_i \ll 1$, i.e. short electron mean free path in comparison to the parallel wavelength and long perpendicular wavelengths with respect to the Larmor radius (Dudson *et al.* 2009; Tamain *et al.* 2009; Ricci *et al.* 2012; Halpern *et al.* 2016; Stegmeir *et al.* 2016; Zhu *et al.* 2018; Paruta *et al.* 2018), these approximations are often marginal near the separatrix and inside it, i.e. in the edge region. In fact, even though turbulence is still dominated by low-frequency fluctuations, the plasma in the edge is hotter and less collisional than in the scrape-off layer and the use of a fluid model becomes questionable. Moreover, in the edge region, small scale $k_\perp \rho_i \sim 1$ fluctuations are important (Hahm *et al.* 2009). This is especially relevant in the high-temperature tokamak H-mode regime (Zweben *et al.* 2007), the regime of operation relevant for ITER and future devices. Despite recent progress (Chang *et al.* 2017; Shi *et al.* 2017; Pan *et al.* 2018), overcoming the limitation of the drift-reduced fluid models in the description of

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the tokamak periphery region by using a gyrokinetic model valid at $k_\perp \rho_i \sim 1$ has proven to be exceptionally demanding. Among the numerous challenges, the effort is undermined by the lack of a proper collisional gyrokinetic model for the periphery.

In fact, with respect to the core, due to the lower temperature values and associated high collisionality, the use of a gyrokinetic model to simulate the plasma dynamics in the tokamak periphery requires an accurate collision operator. This is necessary as collisions set the level of neoclassical transport and strongly influence the turbulent dynamics by affecting the linear growth rate and nonlinear evolution of turbulent modes (Barnes *et al.* 2009).

Since the first formulations of the gyrokinetic theory, there have been significant research efforts to take collisions into account (Catto & Tsang 1977; McCoy *et al.* 1981; Start 2002; Brizard 2004; Abel *et al.* 2008*b*; Barnes *et al.* 2009; Li & Ernst 2011; Dorf *et al.* 2012; Estève *et al.* 2015; Hakim *et al.* 2019; Pan & Ernst 2019). The first effort devoted to a gyrokinetic collision operator can be traced back to the work of Catto & Tsang (1977), later improved by Abel *et al.* (2008*a*) by adding the necessary terms needed to ensure non-negative entropy production. The result of this effort is a linearized gyrokinetic collision operator that contains pitch-angle scattering effects and retains important conservation properties. A linearized gyrokinetic Coulomb collision operator derived from first principles was then presented in Li & Ernst (2011) and Madsen (2013). However, as turbulence in the tokamak periphery is essentially nonlinear, the relative level of fluctuations in this region being of order unity (Scott 2002), and the level of collisions is not sufficient for a local thermalization, the distribution function may significantly deviate from a local Maxwellian distribution (Tskhakaya 2012). Therefore, a nonlinear formulation of the gyrokinetic Coulomb collision operator is crucial to adequately describe the dynamics in the periphery. Only recently, several theoretical studies have emerged in order to derive non-linearized collisional gyrokinetic operators that keep conservation laws in their differential form. In particular, we mention the recent Poisson bracket formulations of the full nonlinear Coulomb collision operator (Brizard 2004; Sugama *et al.* 2015; Burby *et al.* 2015). While the formulation of these operators represent significant progress, the presence of a six-dimensional phase-space integral in these expressions makes their numerical implementation still extremely difficult.

In this work, the Coulomb gyrokinetic collision operator is derived in a form that can be efficiently implemented in numerical simulation codes as it involves only integrals over the two gyrokinetic velocity coordinates. The derivation of the full Coulomb collision operator is based on a multipole expansion of the Rosenbluth potentials. This allows us to write the Coulomb collision operator in terms of moments of the distribution function and apply the gyroaverage operator to the resulting expansion. The Coulomb collision operator is then expressed in terms of two-dimensional velocity integrals of the distribution function. We show that the gyroangle dependence of the expansion coefficients, given in terms of scalar spherical harmonics, allows for analytical gyroaveraging integrations at arbitrary values of the perpendicular wavevector. Furthermore, motivated by recent work based on a pseudo-spectral approach to the gyrokinetic equation (Mandell *et al.* 2018; Frei *et al.* 2019), the collision operator is projected onto a Hermite-Laguerre polynomial basis, and is expressed in terms of moments of the distribution function on the same basis. The set of moment-hierarchy equations can then be rigorously closed by using systematic techniques [such as the semi-collisional closure (Zocco & Schekochihin 2011; Jorge *et al.* 2017)] without requiring ad-hoc truncation of infinite series.

This paper is organized as follows. Section 2 derives the gyrokinetic equation and Section 3 presents the multipole expansion of the Coulomb collision operator. In Section 4, the Coulomb operator is ported to a gyrocenter coordinate system, while Section 5 projects the collision operator onto a Hermite-Laguerre polynomial basis to obtain a closed-form expression of the Coulomb collision operator in terms of the moments of the distribution function on the same basis. The gyrokinetic collision operator for unlike-species is presented in Section 6 using an expansion based on the smallness of the electron-to-ion mass ratio. The conclusions follow.

2. Gyrokinetic Model

The evolution of the distribution function $f_a = f_a(\mathbf{x}, \mathbf{v})$ is given by the Boltzmann equation

$$\frac{\partial f_a}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f_a}{\partial \mathbf{x}} + \dot{\mathbf{v}} \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b), \tag{2.1}$$

with $C(f_a, f_b)$ the Coulomb (also known as Landau) collision operator (Landau 1936; Rosenbluth *et al.* 1957). This is an operator of the Fokker-Planck type, derived from first principles, and valid in the common case where small-angle Coulomb collisions dominate. Its expression is given by

$$C(f_a, f_b) = L_{ab} \sum_{j,k} \frac{\partial}{\partial v_k} \left[\frac{\partial}{\partial v_j} \left(\frac{\partial^2 G_b}{\partial v_k \partial v_j} f_a \right) - 2 \left(1 + \frac{m_a}{m_b} \right) \frac{\partial H_b}{\partial v_k} f_a \right], \tag{2.2}$$

with

$$H_b = \int \frac{f_b(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} d\mathbf{v}', \tag{2.3}$$

and

$$G_b = \int f_b(\mathbf{v}') |\mathbf{v} - \mathbf{v}'| d\mathbf{v}', \tag{2.4}$$

the Rosenbluth potentials satisfying $\nabla_v^2 G_b = 2 H_b$ and $\nabla_v^2 H_b = -4\pi f_b$ (Rosenbluth *et al.* 1957). In Eq. (2.2), $L_{ab} = q_a^2 q_b^2 \lambda_{ab} / (8\pi \epsilon_0^2 m_a^2) = \nu_{ab} v_{tha}^3 / n_b$ is introduced, where λ_{ab} and ν_{ab} are the Coulomb logarithm and the collision frequency between species a and b respectively, $v_{tha} = \sqrt{2T_a/m_a}$ is the thermal speed, and q_a and m_a are the charge and the mass of particles of species a, $a = e, i$.

In the present paper we consider a plasma with properties that satisfy the gyrokinetic ordering (Brizard & Mishchenko 2009; Frei *et al.* 2019). More precisely, denoting typical turbulent frequencies as $\omega \sim |\partial_t \log n| \sim |\partial_t \log T_e|$ with n_e and T_e the electron density and temperature respectively, and typical wavenumbers $\mathbf{k} = k_\parallel \mathbf{b} + \mathbf{k}_\perp$, being $\mathbf{k} \sim |\nabla \log n_e| \sim |\nabla \log T_e|$ and $\mathbf{b} = \mathbf{B}/B$ the magnetic field unit vector, we assume

$$\epsilon \sim \frac{|\mathbf{v_E}|}{c_s} \sim \frac{k_\parallel}{k_\perp} \ll 1, \tag{2.5}$$

where $c_s = \sqrt{T_e/m_i}$ is the sound speed, $\rho_s = c_s/\Omega_i$ the sound Larmor radius, $\Omega_i = eB/m_i$ the ion gyrofrequency, and $\mathbf{v_E} = \mathbf{E} \times \mathbf{B}/B^2$ the $\mathbf{E} \times \mathbf{B}$ drift velocity with $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$ the electric field. As strong radial electric fields are known to play a role in the tokamak edge (particularly in the H-mode pedestal), and large scale fluctuations are the ones at play in the tokamak scrape-off layer, we split the electrostatic potential as $\phi = \phi_0 + \phi_1$ (Dimits *et al.* 1992; Qin *et al.* 2006; Frei *et al.* 2019), i.e. into a possibly large-scale drift-kinetic component, ϕ_0, satisfying

$$\frac{e\phi_0}{T_e} \sim 1, \tag{2.6}$$

and its small-amplitude gyrokinetic, ϕ_1, component

$$\frac{\phi_1}{\phi_0} \sim \epsilon_\delta \ll 1. \tag{2.7}$$

A similar decomposition into large and small scale fluctuations is applied to the magnetic vector potential $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$, with $|\mathbf{A}_1|/|\mathbf{A}_0| \sim \epsilon_\delta$. Both ϕ_0 and ϕ_1 are assumed to yield a similar contribution to the total electric field

$$\mathbf{E} \sim \nabla_\perp \phi_0 \sim \nabla_\perp \phi_1. \tag{2.8}$$

Therefore, by ordering typical gradient lengths of ϕ_1 to be comparable to ρ_s, we set

$$\rho_s \left| \frac{\nabla_\perp \phi_1}{\phi_1} \right| \sim 1, \tag{2.9}$$

which, using Eqs. (2.7) and (2.8), constraints typical gradient lengths of ϕ_0 to be much larger than ρ_s, as

$$\rho_s \left| \frac{\nabla_\perp \phi_0}{\phi_0} \right| \sim \epsilon_\delta. \tag{2.10}$$

In the following, we set $\epsilon_\delta \sim \epsilon$, which, using Eq. (2.5), yields

$$k_\perp \rho_s \frac{e\phi_0}{T_e} \sim \frac{e\phi_1}{T_e} \sim \epsilon. \tag{2.11}$$

The scale length $L_B \sim R_0$ of the equilibrium magnetic field (with R_0 the major radius of the tokamak device), is ordered by the small parameter $\epsilon_B \sim \rho_s/L_B$. We note that the collision operator developed here is valid for both $\epsilon_B \sim \epsilon^2$ and $\epsilon_B \sim \epsilon^3$, with the second case being more of interest for the periphery since the plasma temperature is lower than the tokamak core (Hahm *et al.* 2009). Finally, the collision frequency is ordered as

$$\frac{\nu_i}{\Omega_i} \sim \epsilon_\nu \sim \epsilon^2, \tag{2.12}$$

with $\nu_i = \nu_{ii}$ the ion-ion collision frequency. For $T_e \sim T_i$, the ordering in Eq. (2.12) implies that $k_\parallel \lambda_{mfpe} \sim k_\parallel \lambda_{mfpi} \sim k_\perp \rho_s/\epsilon$, with $\lambda_{mfpa} = v_{tha}/\nu_a$.

By taking advantage of the ordering in Eqs. (2.5-2.7) and Eq. (2.12), the gyrokinetic model effectively removes the fast time scale associated with the cyclotron motion and reduces the dimensionality of the kinetic equation from six phase-space variables, i.e. $(\mathbf{x}, \mathbf{v})$, to five. While linear and nonlinear gyrokinetic equations of motion were originally derived using recursive techniques (Taylor & Hastie 1968; Rutherford & Frieman 1968; Catto 1978), more recent derivations of the gyrokinetic equation based on Hamiltonian Lie perturbation theory (Cary 1981) ensure the existence of phase-space volume and magnetic moment conservation laws (Hahm 1988; Brizard & Hahm 2007; Hahm *et al.* 2009; Frei *et al.* 2019). The Hamiltonian derivations are carried out in two steps. In the first step, small-scale electromagnetic fluctuations with perpendicular wavelengths comparable to the particle Larmor radius (ϕ_1 and $A_{\parallel 1}$) are neglected (Cary & Brizard 2009). Within this approximation, the coordinate transformation from particle phase-space coordinates $(\mathbf{x}, \mathbf{v})$ to guiding-center coordinates $\mathbf{Z} = (\mathbf{R}, v_\parallel, \mu, \theta)$ is derived, where $\mathbf{R}$ is the guiding-center, $v_\parallel$ the parallel velocity, μ the magnetic moment, and θ the gyroangle. The second step introduces small-scale and small-amplitude electromagnetic fluctuations, ϕ_1 and $\mathbf{A}_1$. For this purpose, a gyrocenter coordinate system $\overline{\mathbf{Z}} = (\overline{\mathbf{R}}, \overline{v}_\parallel, \overline{\mu}, \overline{\theta})$ is constructed perturbatively from the guiding-center coordinates $\mathbf{Z}$ via a transformation T of the form

$$\overline{\mathbf{Z}} = T\mathbf{Z} = \mathbf{Z} + \epsilon_\delta \mathbf{Z}_1 + ..., \tag{2.13}$$

where $\mathbf{Z}_1$ contains terms proportional to ϕ_1 and $\mathbf{A}_1$, such that $\overline{\mu} = T\mu = \mu + \epsilon_\delta \mu_1 + ...$, remains an adiabatic invariant [see, e.g., Brizard & Hahm (2007)]. This allows us to reduce the number of phase-space variables in the kinetic Boltzmann equation describing the evolution of the particle distribution function from six to five, simplifying the analytical and numerical treatment of magnetized plasma systems.

More precisely, in order to simplify and remove the gyroangle dependence of the Boltzmann equation, the distribution function $f_a(\mathbf{x}, \mathbf{v})$ is first expressed in terms of the guiding-center coordinates $\mathbf{Z}$ by defining the guiding centre distribution function $F_a(\mathbf{Z})$ as

$$F_a(\mathbf{Z}) = f_a(\mathbf{x}(\mathbf{Z}), \mathbf{v}(\mathbf{Z})). \tag{2.14}$$

The coordinate transformation $\mathbf{v}(\mathbf{Z})$ is given by

$$\mathbf{v} = v_\| \mathbf{b} + \mathbf{v}_{E0} + v_\perp{}'(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2) = v\left[\cos\varphi \mathbf{b} + \sin\varphi(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2)\right], \tag{2.15}$$

with $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$ a fixed right-handed coordinate set, $v_{E0} = -\nabla\phi_0$ the drift-kinetic $\mathbf{E}\times\mathbf{B}$ drift, $\cos\varphi = v_\|/v$ the cosine of the pitch angle and θ the gyroangle. The magnetic moment μ is defined as

$$\mu = \frac{m_a v_\perp'^2}{2B}, \tag{2.16}$$

whereas the particle position $\mathbf{x}(\mathbf{Z})$ is written as

$$\mathbf{x} = \mathbf{R} + \rho_a, \tag{2.17}$$

with

$$\rho_a = \rho_a(\mathbf{R}, \mu, \theta) = \sqrt{\frac{2m_a\mu}{q_a^2 B}}(-\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2) \tag{2.18}$$

the Larmor radius and $\mathbf{R}$ the guiding-center of the particle. The Jacobian of the guiding-center transformation of Eqs. (2.15-2.17) is given by $B_\|^*/m_a = (B/m_a)(1 + \mathbf{b}\cdot\nabla\times\mathbf{v}_E/\Omega_a + v_\|\mathbf{b}\cdot\nabla\times\mathbf{b}/\Omega_a)$ (Cary & Brizard 2009). We note that, in a weak-flow regime where $\mathbf{v}_E$ is absent from Eq. (2.15), the calculation that follows remains valid, except for the Jacobian of the guiding-center transformation $B_\|^*$, which should be replaced by $B_\|^* = (B/m_a)(1 + v_\|\mathbf{b}\cdot\nabla\times\mathbf{b}/\Omega_a)$.

To account for the small scale fluctuations and magnetic inhomogeneity, the gyrokinetic distribution function $\overline{F}_a(\overline{\mathbf{Z}})$ is then defined as (Brizard & Hahm 2007)

$$\overline{F_a}(\overline{\mathbf{Z}}) = F_a(\mathbf{Z}) = T\overline{F}_a(\mathbf{Z}), \tag{2.19}$$

with the coordinate transformation between $\overline{\mathbf{Z}}$ and $\mathbf{Z}$ given perturbatively by Eq. (2.13). Indeed, using the chain rule to rewrite the Boltzmann equation, Eq. (2.1), in terms of gyrocenter $\overline{\mathbf{Z}}$ coordinates, we obtain

$$\frac{\partial \overline{F_a}}{\partial t} + \dot{\overline{\mathbf{Z}}}\cdot\frac{\partial \overline{F_a}}{\partial \overline{\mathbf{Z}}} = \sum_b C(\overline{F_a}, \overline{F_b}). \tag{2.20}$$

We now introduce the gyroaverage operator $\langle ... \rangle_{\overline{\mathbf{R}}}$ defined by

$$\langle \chi \rangle_{\overline{\mathbf{R}}} = \frac{1}{2\pi}\int_0^{2\pi} \chi(\overline{\mathbf{Z}}) d\overline{\theta}, \tag{2.21}$$

where all gyrocenter coordinates $\overline{\mathbf{Z}}$ but $\overline{\theta}$ are kept fixed during the integration. By applying the gyroaverage operator to Eq. (2.20), the gyrokinetic equation is obtained

$$\frac{\partial}{\partial t}\left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}} + \left\langle \dot{\overline{\mathbf{Z}}} \cdot \frac{\partial \overline{F_a}}{\partial \overline{\mathbf{Z}}}\right\rangle_{\overline{\mathbf{R}}} = \sum_b \left\langle C(\overline{F_a}, \overline{F_b})\right\rangle_{\overline{\mathbf{R}}}. \tag{2.22}$$

Equation (2.22) can be further simplified by noting that, in the gyrokinetic framework, the transformation in Eq. (2.13) is constructed in such a way that the gyrocenter equations of motion, i.e. the equations that determine $\dot{\overline{\mathbf{Z}}} = (\dot{\overline{\mathbf{R}}}, \dot{\overline{v_\parallel}}, \dot{\overline{\mu}}, \dot{\overline{\theta}})$, are gyroangle independent and that $\overline{\mu}$ is an adiabatic invariant satisfying $\dot{\overline{\mu}} = 0$ (Brizard & Hahm 2007). Therefore, the gyrokinetic equation in Eq. (2.22) can be written as

$$\frac{\partial}{\partial t}\left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}} + \dot{\overline{\mathbf{R}}} \cdot \frac{\partial \left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mathbf{R}}} + \dot{\overline{v_\parallel}} \frac{\partial \left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{v_\parallel}} = \sum_b \left\langle C(\overline{F_a}, \overline{F_b})\right\rangle_{\overline{\mathbf{R}}}. \tag{2.23}$$

In order to further simplify Eq. (2.23), we estimate the order of magnitude of the gyrophase dependent part of the distribution function $\widetilde{\overline{F_a}} = \overline{F_a} - \left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}}$, where $\overline{F_a}$ obeys Eq. (2.20) and $\left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}}$ obeys Eq. (2.23). For this purpose, we note that the equation for the evolution of $\widetilde{\overline{F_a}} = \overline{F_a} - \left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}}$ can be obtained by subtracting Eq. (2.23) from Eq. (2.20), yielding

$$\frac{\partial \widetilde{\overline{F_a}}}{\partial t} + \dot{\overline{\mathbf{R}}} \cdot \frac{\partial \widetilde{\overline{F_a}}}{\partial \overline{\mathbf{R}}} + \dot{\overline{v_\parallel}} \frac{\partial \widetilde{\overline{F_a}}}{\partial \overline{v_\parallel}} + \dot{\overline{\theta}} \frac{\partial \widetilde{\overline{F_a}}}{\partial \overline{\theta}} = \sum_b C(\overline{F_a}, \overline{F_b}) - \left\langle C(\overline{F_a}, \overline{F_b})\right\rangle_{\overline{\mathbf{R}}}. \tag{2.24}$$

To lowest order, $\dot{\overline{\theta}} \partial_{\overline{\theta}} \widetilde{\overline{F_a}} \sim \Omega_a \widetilde{\overline{F_a}}$ and $\partial_t \sim \dot{\overline{\mathbf{R}}} \cdot \nabla_{\overline{\mathbf{R}}} \sim \dot{\overline{v_\parallel}} \partial_{\overline{v_\parallel}} \sim \epsilon \Omega_i$. Therefore, the leading order estimate of Eq. (2.24) gives

$$\widetilde{\overline{F_a}} \simeq \frac{1}{\Omega_a} \sum_b \int_0^{\overline{\theta}} \left[C\left(\left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}}, \left\langle \overline{F_b}\right\rangle_{\overline{\mathbf{R}}}\right) - \left\langle C\left(\left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}}, \left\langle \overline{F_b}\right\rangle_{\overline{\mathbf{R}}}\right)\right\rangle_{\overline{\mathbf{R}}} \right] d\overline{\theta}'. \tag{2.25}$$

Using the fact that $C(\overline{F_a}, \overline{F_b}) \sim \nu_a \overline{F}_a$, together with Eq. (2.12), and expanding $\overline{F_a}$ as $\overline{F_a} = \left\langle \overline{F_a}\right\rangle_{\overline{\mathbf{R}}} + \epsilon_\nu \overline{F}_{a1} + ...$, we find that

$$\frac{\widetilde{\overline{F_e}}}{\left\langle \overline{F_e}\right\rangle_{\overline{\mathbf{R}}}} \sim \left(\frac{T_i}{T_e}\right)^{3/2} \sqrt{\frac{m_e}{m_i}} \epsilon_\nu \sim \left(\frac{T_i}{T_e}\right)^{3/2} \sqrt{\frac{m_e}{m_i}} \epsilon^2, \tag{2.26}$$

and

$$\frac{\widetilde{\overline{F}_i}}{\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}} \sim \frac{\nu_i}{\Omega_i} \sim \epsilon_\nu \sim \epsilon^2. \tag{2.27}$$

showing that, up to second order in ϵ, the gyroangle dependence of the distribution function can be neglected in Eq. (2.23). We remark that a similar estimate for the gyrophase dependent part of the guiding-center distribution function was found in Jorge *et al.* (2017).

We now evaluate the magnitude of the collisional term in Eq. (2.24). Using the expansion $C(\overline{F_a}, \overline{F_b}) = C_0(\overline{F_a}, \overline{F_b}) + \epsilon_\delta C_1(\overline{F_a}, \overline{F_b}) + ...$ with $C_0(\overline{F_a}, \overline{F_b}) \sim \nu_a \overline{F_a}$, and noting that the first order gyrocenter transformation $\mathbf{Z}_1 = \mathbf{Z} - \overline{\mathbf{Z}} + O(\epsilon_\delta^2)$ in Eq. (2.13) is mass dependent, i.e. $\overline{\mathbf{Z}_{1e}} \sim \sqrt{m_e/m_i} \overline{\mathbf{Z}_{1i}}$ [see, e.g., Brizard & Hahm (2007)], the magnitude of the Coulomb collision operator for electrons can be estimated as

$$C(\overline{F}_e, \overline{F_b}) \sim \nu_e \overline{F}_e \sim \sqrt{\frac{m_i}{m_e}} \epsilon_\nu \Omega_i \overline{F}_e + O\left(\epsilon_\nu \epsilon_\delta \Omega_i \overline{F}_e\right) \sim \sqrt{\frac{m_i}{m_e}} \epsilon^2 \Omega_i \overline{F}_e + O\left(\epsilon^3 \Omega_i \overline{F}_e\right). \tag{2.28}$$

Thus, the third order term in the expansion in Eq. (2.28) does not contain a $\sqrt{m_i/m_e}$ factor, in contrast with its lower order counterpart. A similar argument holds for the ions, yielding

$$C(\overline{F}_i, \overline{F_b}) \sim \nu_i \overline{F}_i \sim \epsilon_\nu \Omega_i \overline{F}_i + O\left(\epsilon_\nu \epsilon_\delta \Omega_i \overline{F}_i\right) \sim \epsilon^2 \Omega_i \overline{F}_i + O\left(\epsilon^3 \Omega_i \overline{F}_i\right). \tag{2.29}$$

Equations (2.28) and (2.29) show that the lowest order collision operator $C_0(\overline{F_a}, \overline{F_b})$ is, in fact, $O(\epsilon^2)$. Therefore, the gyrokinetic equation valid up to second order in ϵ, considered in a large number of edge gyrokinetic models [see, e.g., Qin *et al.* (2006, 2007); Hahm *et al.* (2009); Frei *et al.* (2019)], can thus be written as

$$\frac{\partial}{\partial t} \left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}} + \dot{\overline{\mathbf{R}}} \cdot \frac{\partial}{\partial \overline{\mathbf{R}}} \left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}} + \dot{\overline{v_\|}} \frac{\partial}{\partial \overline{v_\|}} \left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}} = \sum_b \left\langle C_0(\left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}}, \left\langle \overline{F_b} \right\rangle_{\overline{\mathbf{R}}}) \right\rangle_{\overline{\mathbf{R}}}, \tag{2.30}$$

assuming $\dot{\overline{\mathbf{R}}}$ and $\dot{\overline{v_\|}}$ to be at least $O(\epsilon^2)$ accurate. We note that although only the lowest order in ϵ_δ collision operator $C_0(\left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}}, \left\langle \overline{F_b} \right\rangle_{\overline{\mathbf{R}}})$ is used in Eq. (2.30), all orders in $k_\perp \rho_s$ are kept.

3. Multipole Expansion of the Coulomb Collision Operator

The goal of this section is to find a suitable basis to expand f_a such that the Coulomb operator in Eq. (2.2) can be cast as a function of moments of f_a. This first step considerably simplifies the derivation of the gyrokinetic collision operator. We start by noting that the Rosenbluth potential H_b in Eq. (2.3) is analogous to the expression of the electrostatic potential due to a charge distribution, a similarity already noted by Rosenbluth *et al.* (1957). This fact allows us to make use of known electrostatic expansion techniques (Jackson 1998) to perform a multipole expansion of the Rosenbluth potentials. We first Taylor expand the factor $1/|\mathbf{v} - \mathbf{v}'|$ in Eq. (2.3) around $\mathbf{v} = 0$ if $v \leqslant v'$ or around $\mathbf{v}' = 0$ if $v > v'$, yielding

$$\frac{1}{|\mathbf{v} - \mathbf{v}'|} = \begin{cases} \displaystyle\sum_{l=0}^{\infty} \frac{(-\mathbf{v}')^l}{l!} \cdot \frac{\partial^l}{\partial \mathbf{v}^l} \left(\frac{1}{v}\right), & v' \leqslant v, \\ \displaystyle\sum_{l=0}^{\infty} \frac{(-\mathbf{v})^l}{l!} \cdot \frac{\partial^l}{\partial (\mathbf{v}')^l} \left(\frac{1}{v'}\right), & v < v'. \end{cases} \tag{3.1}$$

where we used the identity $\partial_{\mathbf{v}}(1/|\mathbf{v} - \mathbf{v}'|)_{v=0} = -\partial_{\mathbf{v}'}(1/v')$ and where we denote the inner product between all the l indices of two l-rank tensors, $\mathbf{T}_1^l$ and $\mathbf{T}_2^l$, as $\mathbf{T}_1^l \cdot \mathbf{T}_2^l$. Both $v \leqslant v'$ and $v > v'$ cases are included in order to take into account the fact that $f_b(\mathbf{v}')$ is, in general, finite over the entire velocity space $\mathbf{v}'$. Denoting $\mathbf{Y}^l(\mathbf{v})$ the spherical harmonic tensor (Weinert 1980)

$$\mathbf{Y}^l(\mathbf{v}) = \frac{(-1)^l v^{2l+1}}{(2l-1)!!} \left(\frac{\partial}{\partial \mathbf{v}}\right)^l \frac{1}{v}, \tag{3.2}$$

we obtain the following expression for H_b

$$H_b = 2\sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} \left(\int_{v>v'} f_b(\mathbf{v}') \frac{(\mathbf{v}')^l}{v^{2l+1}} \cdot \mathbf{Y}^l(\mathbf{v}) d\mathbf{v}' + \int_{v' \geqslant v} f_b(\mathbf{v}') \frac{(\mathbf{v})^l}{(v')^{2l+1}} \cdot \mathbf{Y}^l(\mathbf{v}') d\mathbf{v}' \right). \tag{3.3}$$

In order to simplify Eq. (3.3), we note that the tensor $\mathbf{Y}^l(\mathbf{v}) = Y^l_{\alpha\beta\ldots\gamma}(\mathbf{v})$ is symmetric and totally traceless, i.e. traceless between any combination of two of its indices. Symmetry arises from the fact that any couple of indices in $Y^l_{\alpha\beta\ldots\gamma}(\mathbf{v})$ is interchangeable as the velocity derivatives commute for $v \neq 0$. The totally traceless feature (i.e. $\sum_\alpha Y^l_{\alpha\alpha\ldots\gamma}(\mathbf{v}) = 0$ and the same for any other pair of indices), stems from the fact that the contraction between any two indices in $Y^l_{\alpha\beta\ldots\gamma}(\mathbf{v})$ leads to the multiplicative factor $\nabla^2_{\mathbf{v}} = \partial_{\mathbf{v}} \cdot \partial_{\mathbf{v}}(1/v)$, which vanishes for $v \neq 0$ (we note that $Y^l(\mathbf{v})$ vanishes in the limit $\mathbf{v} = 0$). Furthermore, by defining the tensor $(\mathbf{v})^l_{\mathrm{TS}}$ as the traceless symmetric counterpart of $(\mathbf{v})^l$ [e.g., $(\mathbf{v})^2_{\mathrm{TS}} = \mathbf{v}\mathbf{v} - \mathbf{I}v^2/3$ with $\mathbf{I}$ the identity matrix], we replace the tensors $(\mathbf{v}')^l$ and $(\mathbf{v})^l$ in Eq. (3.3) by their traceless symmetric counterpart $(\mathbf{v}')^l_{\mathrm{TS}}$ and $(\mathbf{v})^l_{\mathrm{TS}}$ respectively

$$H_b = 2\sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} \left(\int_{v>v'} f_b(\mathbf{v}') \frac{(\mathbf{v}')^l_{\mathrm{TS}}}{v^{2l+1}} \cdot \mathbf{Y}^l(\mathbf{v}) d\mathbf{v}' + \int_{v' \geqslant v} f_b(\mathbf{v}') \frac{(\mathbf{v})^l_{\mathrm{TS}}}{(v')^{2l+1}} \cdot \mathbf{Y}^l(\mathbf{v}') d\mathbf{v}' \right), \tag{3.4}$$

as they differ only by terms proportional to the identity matrix that vanish when summed with $\mathbf{Y}^l(\mathbf{v})$ and $\mathbf{Y}^l(\mathbf{v}')$. In fact, for the $l = 2$ case we have $(\mathbf{v}^2 - (\mathbf{v})^2_{\mathrm{TS}}) \cdot \mathbf{Y}^2(\mathbf{v}) = (v^2/3)\mathbf{I} \cdot \mathbf{Y}^2(\mathbf{v}) = (v^2/3)\sum_\alpha Y^2_{\alpha\alpha} = 0$, and similarly for $l > 2$. In addition, following the convention in Snider (2017), scalars ($l = 0$) and vectors ($l = 1$) are considered to be traceless symmetric quantities. Finally, we relate the tensors $(\mathbf{v})^l_{\mathrm{TS}}$ and $\mathbf{Y}^l(\mathbf{v})$. For $l = 0$ and $l = 1$, we have $\mathbf{Y}^0(\mathbf{v}') = 1 = (\mathbf{v}')^0_{\mathrm{TS}}$ and $\mathbf{Y}^1(\mathbf{v}') = \mathbf{v}' = (\mathbf{v}')^1_{\mathrm{TS}}$. For $l = 2$, Eq. (3.2) gives

$$\mathbf{Y}^2(\mathbf{v}') = \mathbf{v}'\mathbf{v}' - \frac{v'^2}{3}\mathbf{I} = (\mathbf{v}')^2_{TS}. \tag{3.5}$$

The results obtained for $l = 0, 1$, and 2 can be generalized, i.e. $(\mathbf{v}')^l_{\mathrm{TS}} = \mathbf{Y}^l(\mathbf{v}')$ as proved by induction (Weinert 1980). The Rosenbluth potential H_b can therefore be written as

$$H_b = 2\sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} \mathbf{Y}^l(\mathbf{v}) \cdot \left[\frac{1}{(v^2)^{l+1/2}} \int_{v'<v} f_b(\mathbf{v}') \mathbf{Y}^l(\mathbf{v}') d\mathbf{v}' + \int_{v' \geqslant v} f_b(\mathbf{v}') \frac{\mathbf{Y}^l(\mathbf{v}')}{[(v')^2]^{l+1/2}} d\mathbf{v}' \right]. \tag{3.6}$$

The first term in Eq. (3.6) can be regarded as the potential due to the charge distribution $f_b(\mathbf{v}')$ inside a sphere of radius v, while the second term is the potential due to a finite charge distribution $f_b(\mathbf{v}')$ at $v' \geqslant v$.

We now look for an expansion of f_b that allows us to perform the integrals in Eq. (3.6) analytically by writing H_b as a sum of velocity moments of f_b. We consider the basis functions (Hirshman & Sigmar 1976)

$$\mathbf{Y}^{lk}(\mathbf{v}) = \mathbf{Y}^l(\mathbf{v}) L_k^{l+1/2}(v^2), \tag{3.7}$$

with $L_k^{l+1/2}(v)$ an associated Laguerre polynomial (Abramowitz *et al.* 1965), i.e.

$$L_k^{l+1/2}(v) = \sum_{m=0}^{k} L^l_{km} v^m, \tag{3.8}$$

where

$$L^l_{km} = \frac{(-1)^m (l+k+1/2)!}{(k-m)!(l+m+1/2)!m!}. \tag{3.9}$$

The basis $\mathbf{Y}^{lk}(\mathbf{v})$ is orthogonal, being the orthogonality relation given by (Banach & Piekarski 1989; Snider 2017)

$$\int e^{-v^2}\mathbf{Y}^{l'k'}(\mathbf{v})\mathbf{Y}^{lk}(\mathbf{v})d\mathbf{v}\cdot\mathbf{T}^{lk} = \delta_{ll'}\delta_{kk'}\pi^{3/2}\sigma^l_k\mathbf{T}^{lk}, \tag{3.10}$$

with $\mathbf{T}^{lk}$ an arbitrary symmetric and traceless tensor, and σ^l_k the normalization constant

$$\sigma^l_k = \frac{l!(l+k+1/2)!}{2^l(l+1/2)!k!}. \tag{3.11}$$

A proof that $\mathbf{Y}^{lk}(\mathbf{v})$ is a complete basis, i.e. that each l and k element of $\mathbf{Y}^{lk}(\mathbf{v})$ is linearly independent and that a linear combination of its elements spans any smooth function $f(\mathbf{v})$, can be found in Banach & Piekarski (1989), where the equivalence between Grad's moment expansion in tensorial Hermite polynomials (which forms a complete basis) and $\mathbf{Y}^{lk}(\mathbf{v})$ is shown. We then write f_b as

$$f_b = f_{Mb}\sum_{l,k=0}^{\infty}\mathbf{Y}^{lk}\left(\frac{\mathbf{v}}{v_{thb}}\right)\cdot\frac{\mathbf{M}^{lk}_b}{\sigma^l_k}, \tag{3.12}$$

with f_{Mb} a Maxwellian distribution function

$$f_{Mb} = \frac{n_b}{v^3_{thb}\pi^{3/2}}e^{-\frac{v^2}{v^2_{thb}}}. \tag{3.13}$$

According to Eq. (3.10), with $\mathbf{T}^{lk} = \mathbf{M}^{lk}_b$, the coefficients $\mathbf{M}^{lk}_b$ are obtained by taking velocity moments of f_b of the form

$$\mathbf{M}^{lk}_b = \frac{1}{n_b}\int f_b(\mathbf{v})\mathbf{Y}^{lk}\left(\frac{\mathbf{v}}{v_{thb}}\right)d\mathbf{v}. \tag{3.14}$$

Finally, we note that Eq. (3.12) allows us to retain only the $l=k=0$ moment when the plasma is in thermal equilibrium.

Plugging the expansion for f_b given by Eq. (3.12) into Eq. (3.6), we obtain the following expression

$$\begin{aligned}H_b = &\frac{n_b}{v_{thb}\pi^{3/2}}\sum_{l,l',k}\frac{(2l-1)!!}{l!\sigma^{l'}_k}\\ &\times\left(\frac{\mathbf{Y}^l(\hat{v})}{x_b^{(l+1)/2}}\cdot\int_0^{x_b}e^{-x}L^{l'+1/2}_k(x)x^{(l+l'+1)/2}dx\int\mathbf{Y}^l(\hat{v}')\mathbf{Y}^{l'}(\hat{v}')d\sigma'\cdot\mathbf{M}^{l'k}_b\right.\\ &\left.+x_b^{l/2}\mathbf{Y}^l(\hat{v})\cdot\int_{x_b}^{\infty}e^{-x}L^{l+1/2}_k(x)dx\int\mathbf{Y}^l(\hat{v}')\mathbf{Y}^{l'}(\hat{v}')d\sigma'\cdot\mathbf{M}^{l'k}_b\right),\end{aligned} \tag{3.15}$$

where we define the normalized velocity $x_b = v^2/v^2_{thb}$, the solid angle σ such that $d\mathbf{v} = v^2 dv d\sigma$, and use the relation $\mathbf{Y}^l(v) = v^l\mathbf{Y}^l(\hat{v})$ with $\mathbf{v} = v\hat{v}$ (Weinert 1980). Applying the orthogonality relation of Eq. (3.10) for $k=0$, and expanding the associated

Laguerre polynomials using Eq. (3.8), we write H_b as

$$H_b = \frac{2n_b}{v_{thb}} \sum_{l,k} \sum_{m=0}^{k} \frac{L_{km}^l}{\sigma_k^l} \frac{\mathbf{Y}^l(\hat{v}) \cdot \mathbf{M}_b^{lk}}{2l+1} \times \frac{1}{\sqrt{\pi}} \left(\frac{1}{x_b^{(l+1)/2}} \int_0^{x_b} e^{-x} x^{m+l+1/2} dx + x_b^{l/2} \int_{x_b}^{\infty} e^{-x} x^m dx \right), \tag{3.16}$$

where the identity

$$\frac{(2l-1)!!}{2^l (l+1/2)!} = \frac{2}{\sqrt{\pi}} \frac{1}{2l+1}, \tag{3.17}$$

is used to simplify Eq. (3.16).

We note that the expression of H_b in Eq. (3.16) corresponds to the one in Ji & Held (2006), having replaced the $\mathbf{Y}^l(\mathbf{v})$ tensors by the $\mathbf{P}^l(\mathbf{v})$ tensors defined by the recursion relation [see Eq. (14) of Ji & Held (2006)]

$$\mathbf{P}^{l+1}(\mathbf{v}) = \mathbf{v}\mathbf{P}^l(\mathbf{v}) - \frac{v^2}{2l+1} \frac{\partial}{\partial \mathbf{v}} \mathbf{P}^l(\mathbf{v}), \tag{3.18}$$

with $\mathbf{P}^0(\mathbf{v}) = 1$ and $\mathbf{P}^1(\mathbf{v}) = \mathbf{v}$. We can indeed prove that $\mathbf{Y}^l(\mathbf{v}) = \mathbf{P}^l(\mathbf{v})$ by deriving the tensor $\mathbf{Y}^l(\mathbf{v})$ using Eq. (3.2). This yields

$$\begin{aligned} \frac{\partial}{\partial \mathbf{v}} \mathbf{Y}^l(\mathbf{v}) &= \frac{(-1)^l}{(2l-1)!!} \left[(2l+1) v^{2l-1} \mathbf{v} \frac{\partial^l}{\partial \mathbf{v}^l} \frac{1}{v} + v^{2l+1} \frac{\partial^{l+1}}{\partial \mathbf{v}^{2l+1}} \frac{1}{v} \right] \\ &= \frac{2l+1}{v^2} \left[\mathbf{v} \frac{v^{2l+1}(-1)^l}{(2l-1)!!} \frac{\partial^l}{\partial \mathbf{v}^l} \frac{1}{v} - \frac{(-1)^{l+1} v^{2(l+1)+1}}{(2l+1)!!} \frac{\partial^{l+1}}{\partial \mathbf{v}^{l+1}} \frac{1}{v} \right] \\ &= \frac{2l+1}{v^2} \left[\mathbf{v}\mathbf{Y}^l(\mathbf{v}) - \mathbf{Y}^{l+1}(\mathbf{v}) \right], \end{aligned} \tag{3.19}$$

which is the same recursion relation present in Eq. (3.18). Since $\mathbf{Y}^0(\mathbf{v}) = \mathbf{P}^0(\mathbf{v})$ and $\mathbf{Y}^1(\mathbf{v}) = \mathbf{P}^1(\mathbf{v})$, the proof is complete.

The integrals in Eq. (3.16) can be put in terms of upper

$$I_+^k = \frac{1}{\sqrt{\pi}} \int_0^{x_b} dx e^{-x} x^{(k-1)/2}, \tag{3.20}$$

and lower

$$I_-^k = \frac{1}{\sqrt{\pi}} \int_{x_b}^{\infty} dx e^{-x} x^{(k-1)/2}, \tag{3.21}$$

incomplete gamma functions (Abramowitz *et al.* 1965), yielding

$$H_b = \frac{2n_b}{v_{thb}} \sum_{l,k} \sum_{m=0}^{k} \frac{L_{km}^l}{\sigma_k^l} \frac{\mathbf{Y}^l(\hat{v}) \cdot \mathbf{M}_b^{lk}}{2l+1} \left(\frac{I_+^{2l+2m+2}}{x_b^{(l+1)/2}} + x_b^{l/2} I_-^{2m+1} \right). \tag{3.22}$$

A procedure similar to the one used to obtain Eq. (3.22) can be followed for the second Rosenbluth potential G_b by expanding the distribution function f_b appearing in G_b

according to Eq. (3.12), therefore obtaining

$$G_b = \frac{2n_b}{v_{thb}} \sum_{l,k} \sum_{m=0}^{k} \frac{L_{km}^l}{\sigma_k^l} \frac{\mathbf{Y}^l(\hat{v}) \cdot \mathbf{M}_b^{lk}}{2l+1} \left[\frac{1}{2l+3} \left(\frac{I_+^{2l+2m+4}}{x_b^{(l+1)/2}} + x_b^{l/2+1} I_-^{2m+1} \right) - \frac{1}{2l-1} \left(\frac{I_+^{2l+2m+2}}{x_b^{(l-1)/2}} + x_b^{l/2} I_-^{2m+3} \right) \right]. \tag{3.23}$$

Having derived a closed-form expression for the Rosenbluth potentials, we now turn to the full Coulomb collision operator. We first note that, although the Rosenbluth potentials H_b and G_b are linear functions of f_b, the Coulomb collision operator is, in fact, bilinear in f_a and f_b. In order to rewrite the Coulomb collision operator in Eq. (2.2) in terms of a single spherical harmonic tensor $\mathbf{Y}^l(\mathbf{v})$, we make use of the following identity between symmetric traceless tensors (Ji & Held 2009)

$$[\mathbf{Y}^{l-u}(\hat{v}) \cdot \mathbf{M}_a^{lk}] \cdot^u [\mathbf{Y}^{n-u}(\hat{v}) \cdot \mathbf{M}_a^{nk}] = \sum_{j=0}^{min(l,n)-u} d_j^{l-u,n-u} \mathbf{Y}^{l+n-2(j+u)}(\hat{v}) \cdot \left(\mathbf{M}_a^{lk} \cdot^{j+u} \mathbf{M}_b^{nq} \right)_{TS}, \tag{3.24}$$

where $\cdot^n$ is the n-fold inner product [e.g., for the matrix $\mathbf{A} = A_{ij}$, $(\mathbf{A} \cdot^1 \mathbf{A})_{ij} = \sum_k A_{ki} A_{kj}$]. The $d_j^{l,n}$ coefficient can be written in terms of the $t_j^{l,n}$ coefficient

$$t_j^{l,n} = \frac{l!n!(-2)^j (2l+2n-2j)!(l+n)!}{(2l+2n)!j!(l-j)!(n-j)!(l+n-j)!}, \tag{3.25}$$

as

$$d_j^{l,n} = \sum_{j_k | \sum_{k=1}^h j_k = j} (-1)^h \prod_{k=1}^{h} t_{j_k}^{l - \sum_{g=1}^{k-1} j_g, n - \sum_{g=1}^{k-1} j_g}. \tag{3.26}$$

In Eq. (3.26), the summation range involves the integer partitions of j, i.e., the decomposition of j into different sums of h positive integers, here labeled as j_k, with k ranging from 1 to h (e.g. for $j = 3$, we obtain for $h = 2$ the terms $j_1 = 2$ and $j_2 = 1$, and for $h = 3$ we obtain $j_1 = j_2 = j_3 = 1$). Expanding f_a and f_b using Eq. (3.12), the expression for the Rosenbluth potentials in Eqs. (3.22) and (3.23), and the identity in Eq. (3.24), the collision operator in Eq. (2.2) can be rewritten in terms of products of $\mathbf{M}_a^{lk}$ and $\mathbf{M}_b^{lk}$ as

$$C(f_a, f_b) = f_{aM} \sum_{l,k,n,q=0}^{\infty} \sum_{m=0}^{k} \sum_{r=0}^{q} \frac{L_{km}^l}{\sigma_k^l} \frac{L_{qr}^n}{\sigma_q^n} c_{ab}^{lkmnqr}, \tag{3.27}$$

with

$$c_{ab}^{lkmnqr} = \sum_{u=0}^{\min(2,l,n)} \nu_{*abu}^{lm,nr}(v^2) \sum_{i=0}^{min(l,n)-u} d_i^{l-u,n-u} \mathbf{Y}^{l+n-2(i+u)}(\hat{v}) \cdot \left(\mathbf{M}_a^{lk} \cdot^{i+u} \mathbf{M}_b^{nq} \right)_{TS}. \tag{3.28}$$

The quantity $\nu_{*abu}^{lm,nr}$ consists of a linear combination of I_+^l and I_-^l integrals and its derivatives, which can be written as linear combinations of the error function and its derivatives. Their expressions are reported in Ji & Held (2009).

4. Gyrokinetic Coulomb Collision Operator

In Section 3, the Coulomb collision operator is cast in terms of velocity moments of the multipole expansion of the particle distribution functions f_a and f_b. We now express it in terms of the gyrokinetic distribution functions $\langle \overline{F_a} \rangle$ and $\langle \overline{F_b} \rangle$. As a first step, the gyroangle dependence of the basis functions $\mathbf{Y}^{lk}$ is found explicitly by using a coordinate transformation from the particle phase-space coordinates $(\mathbf{x}, \mathbf{v})$ to the guiding-center coordinate system $\mathbf{Z}$. This allows us to decouple the fast gyromotion time associated with the gyroangle θ from the typical turbulence time scales. The multipole moments $\mathbf{M}_a^{lk}$ and $\mathbf{M}_b^{lk}$ can then be written in terms of moments of the guiding-center distribution function $\langle F_a \rangle$ and $\langle F_b \rangle$ for arbitrary values of $k_\perp \rho_s$. As a second step, the gyrocenter coordinate system $\overline{\mathbf{Z}}$ is introduced by using the coordinate transformation T in Eq. (2.13). As shown in Section 2, for a gyrokinetic equation up to second order accurate in ϵ_δ, only the lowest order collision operator C_0 needs to be retained. This allows us to straightforwardly obtain the gyrokinetic collision operator from the guiding-center one by a simple coordinate relabeling.

We first derive the polar and azimuthal angle (gyroangle) dependence of the $\mathbf{Y}^l(\mathbf{v})$ tensor in terms of scalar spherical harmonics. This is useful to analytically perform the gyroaverage of the collision operator in the Boltzmann equation, Eq. (2.30). For this purpose, as a first step, we show that the Laplacian of $\mathbf{Y}^l(\mathbf{v})$ vanishes, i.e. that $\mathbf{Y}^l(\mathbf{v})$ are harmonic tensors. By applying the operator $\nabla_\mathbf{v}^2$ to $\mathbf{Y}^l(\mathbf{v})$, and recalling that $\nabla_\mathbf{v}^2(1/v) = 0$ for $v \neq 0$, we obtain

$$\nabla_\mathbf{v}^2 \mathbf{Y}^l(\mathbf{v}) = \frac{2(-1)^l (2l+1) v^{2l+1}}{(2l-1)!!} \left[(l+1) \left(\frac{\partial}{\partial \mathbf{v}} \right)^l \frac{1}{v} + \mathbf{v} \cdot \left(\frac{\partial}{\partial \mathbf{v}} \right)^{l+1} \frac{1}{v} \right] = 0, \tag{4.1}$$

since

$$\mathbf{v} \cdot \left(\frac{\partial}{\partial \mathbf{v}} \right)^{l+1} \frac{1}{v} = -(l+1) \left(\frac{\partial}{\partial \mathbf{v}} \right)^l \frac{1}{v}, \tag{4.2}$$

as can be proved by induction (Weinert 1980). The angular dependence of $\mathbf{Y}^l(\mathbf{v})$ can now be found by expressing the Laplacian of Eq. (4.1) in spherical coordinates. Using the fact that $\mathbf{Y}^l(\mathbf{v}) = v^l \mathbf{Y}^l(\hat{v})$, we obtain

$$\begin{aligned} 0 = \nabla_\mathbf{v}^2 \mathbf{Y}^l(\mathbf{v}) &= \nabla_\mathbf{v}^2 [v^l \mathbf{Y}^l(\hat{v})] \\ &= \mathbf{Y}^l(\hat{v}) \left(\frac{\partial^2}{\partial v^2} + \frac{2}{v} \frac{\partial}{\partial v} \right) v^l - v^{l-2} L^2 \mathbf{Y}^l(\hat{v}), \end{aligned} \tag{4.3}$$

where L^2 is the angular part of the operator $\nabla_\mathbf{v}^2$ multiplied by v^2

$$L^2 = \frac{1}{\sin\varphi} \frac{\partial}{\partial \varphi} \left(\sin\varphi \frac{\partial}{\partial \varphi} \right) + \frac{1}{\sin\varphi^2} \frac{\partial^2}{\partial \theta^2}, \tag{4.4}$$

with φ and θ chosen, respectively, as the pitch angle and the gyroangle variables, both defined in Eq. (2.15). Evaluating the v derivatives in Eq. (4.3), the following differential equation for $\mathbf{Y}^l(\mathbf{v})$ is obtained

$$L^2 \mathbf{Y}^l(\hat{v}) = l(l+1) \mathbf{Y}^l(\hat{v}). \tag{4.5}$$

We identify Eq. (4.5) as the eigenvalue equation for the scalar spherical harmonics $Y_{lm}(\varphi, \theta)$ (Arfken *et al.* 2013), which can be written in terms of associated Legendre polynomials

$P_l^m(\cos\varphi)$ as (Abramowitz *et al.* 1965)

$$Y_{lm}(\varphi,\theta) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\varphi) e^{im\theta}. \tag{4.6}$$

with

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}[P_l(x)], \tag{4.7}$$

and $P_l(x) = (d^l/dx^l)[(x^2-1)^l]/(2^l l!)$ a Legendre polynomial. Therefore, using Eq. (4.5), and denoting $\mathbf{e}^{lm}$ the basis elements of $\mathbf{Y}^l(\mathbf{v})$ (an elementary derivation of the basis tensors $\mathbf{e}^{lm}$ is shown in Appendix A), we write $\mathbf{Y}^l(\mathbf{v})$ as

$$\mathbf{Y}^l(\mathbf{v}) = v^l \sqrt{\frac{2\pi^{3/2} l!}{2^l (l+1/2)!}} \sum_{m=-l}^{l} Y_{lm}(\varphi,\theta)\mathbf{e}^{lm}. \tag{4.8}$$

Having derived the gyroangle dependence of the $\mathbf{Y}^l(\mathbf{v})$ tensors, we now compute the fluid moments $\mathbf{M}_a^{lk}$ in terms of $v_\parallel$ and μ moments of the guiding-center distribution function $\langle F_a \rangle$. In order to perform the velocity integration in the definition of the moments $\mathbf{M}^{lk}$ in Eq. (3.14) at arbitrary $k_\perp \rho$ in guiding-center phase-space coordinates, we use the identity $f(\mathbf{x},\mathbf{v}) = \int f(\mathbf{x}',\mathbf{v})\delta(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$. By imposing $\mathbf{x}' = \mathbf{R} + \rho$, writing the volume element in phase-space as $d\mathbf{x}'d\mathbf{v} = (B_\parallel^*/m)d\mathbf{R}dv_\parallel d\mu d\theta$, and using Eq. (2.14), we obtain

$$n_a \mathbf{M}_a^{lk}(\mathbf{x}) = \int F_a(\mathbf{R}, v_\parallel, \mu, \theta)\mathbf{Y}^{lk}\left(\frac{\mathbf{v}}{v_{tha}}\right)\delta(\mathbf{x}-\mathbf{R}-\rho_a)\frac{B_\parallel^*}{m} d\mathbf{R}dv_\parallel d\mu d\theta. \tag{4.9}$$

from Eq. (3.14). Expressing $\mathbf{v} = \mathbf{v}(\mathbf{Z})$, as shown by Eq. (2.15), and performing the integral over $\mathbf{R}$ in Eq. (4.9), it follows that

$$n_a \mathbf{M}_a^{lk}(\mathbf{x}) = \int F_a(\mathbf{x}-\rho_a, v_\parallel, \mu, \theta)\mathbf{Y}^{lk}\left[\frac{\mathbf{v}(\mathbf{x}-\rho_a, v_\parallel, \mu, \theta)}{v_{tha}}\right]\frac{B_\parallel^*}{m} dv_\parallel d\mu d\theta. \tag{4.10}$$

The orderings in Eqs. (2.26) and (2.27) for the guiding-center distribution function F_a allows us to approximate $F_a \simeq \langle F_a \rangle_\mathbf{R}$ (Jorge *et al.* 2017), effectively neglecting ϵ^2 effects in $\mathbf{M}_a^{lk}$, hence in the collision operator $C(f_a, f_b)$. To make further analytical progress, and in line with previous gyrokinetic literature (Li & Ernst 2011; Pan & Ernst 2019), we represent $F_a(\mathbf{R}, v_\parallel, \mu, \theta)$ by its Fourier transform $F_a(\mathbf{k}, v_\parallel, \mu, \theta) = \int F_a(\mathbf{R}, v_\parallel, \mu, \theta)e^{-i\mathbf{k}\cdot\mathbf{R}}d\mathbf{R}$, and write

$$n_a \mathbf{M}_a^{lk}(\mathbf{x}) = \int \left\langle F_a(\mathbf{k}, v_\parallel, \mu, \theta)\right\rangle_\mathbf{R} \mathbf{Y}^{lk}\left[\frac{\mathbf{v}(\mathbf{x}-\rho_a, v_\parallel, \mu, \theta)}{v_{tha}}\right] e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\rho_a}\frac{B_\parallel^*}{m} d\mathbf{k}dv_\parallel d\mu d\theta. \tag{4.11}$$

By aligning the $\mathbf{k}$ coordinate system in the integral of Eq. (4.11) with the axes $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$, i.e. $\mathbf{k} = k_\parallel \mathbf{b} + k_\perp(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2)$, we write $\exp(-i\mathbf{k}\cdot\rho) = \exp(-ik_\perp \rho\cos\theta)$. We then use the Jacobi-Anger expansion (Andrews 1992)

$$e^{-ik_\perp\rho\cos\theta} = \sum_{p=-\infty}^{\infty}(-i)^p J_p(k_\perp\rho)e^{-ip\theta}, \tag{4.12}$$

with J_p the Bessel function of order p, and rewrite Eq. (4.11) as

$$
\begin{aligned}
n_a \mathbf{M}_a^{lk}(\mathbf{x}) = \sum_{p=-\infty}^{\infty} (-1)^p \int J_p(k_\perp \rho) \left\langle F_a(\mathbf{k}, v_\parallel, \mu, \theta) \right\rangle_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{x}} \\
\times \mathbf{Y}^{lk} \left[\frac{\mathbf{v}(\mathbf{x} - \rho_a, v_\parallel, \mu, \theta)}{v_{tha}} \right] e^{-ip\theta} \frac{B_\parallel^*}{m} d\mathbf{k} dv_\parallel d\mu d\theta.
\end{aligned}
\tag{4.13}
$$

The velocity $\mathbf{v}$ in the argument of $\mathbf{Y}^{lk}$ in Eq. (4.13) is then expanded as

$$
\mathbf{v}(\mathbf{x} - \rho_a, v_\parallel, \mu, \theta) = \mathbf{v}(\mathbf{x}, v_\parallel, \mu, \theta) + O(\epsilon_B). \tag{4.14}
$$

The second term in Eq. (4.14) introduces $\epsilon_B \ll \epsilon$ terms in the collision operator and is therefore neglected. An example of a numerical implementation using a similar Fourier representation can be found in Pan & Ernst (2019).

Using Eq. (4.8) to express $\mathbf{Y}^{lk}(\mathbf{v})$ in terms of spherical harmonics, we perform the gyroangle integration in Eq. (4.13). By rewriting the spherical harmonics $Y_{lm}(\varphi, 0)$ in terms of associated Legendre polynomials $P_l^m(\cos\varphi)$ using Eq. (4.6), the gyroaverage of the product $Y^{lk}(\mathbf{v}/v_{tha})e^{-ip\theta}$ can be performed, yielding

$$
\begin{aligned}
n_a \mathbf{M}_a^{lk}(\mathbf{x}) = \sum_{p=-\infty}^{\infty} (-1)^p \int J_p(k_\perp \rho_a) \left\langle F_a(\mathbf{k}, v_\parallel, \mu, \theta) \right\rangle_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{x}} \\
\times \left\langle \mathbf{Y}^{lk} \left(\frac{\mathbf{v}}{v_{tha}} \right) e^{-ip\theta} \right\rangle \frac{B_\parallel^*}{m} d\mathbf{k} dv_\parallel d\mu 2\pi,
\end{aligned}
\tag{4.15}
$$

with

$$
\begin{aligned}
\left\langle \mathbf{Y}^{lk} \left(\frac{\mathbf{v}}{v_{tha}} \right) e^{-ip\theta} \right\rangle = L_k^{l+1/2} \left(\frac{v}{v_{tha}} \right) \left(\frac{v}{v_{tha}} \right)^l \sqrt{\frac{\pi^{1/2} l!}{2^l (l-1/2)!}} \\
\times \sum_{m=-l}^{l} (-1)^m \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\varphi) \mathbf{e}^{lm} \delta_{m,p}.
\end{aligned}
\tag{4.16}
$$

We note that, the $p = 0$ case of Eq. (4.16) corresponds to the gyroaveraged formulas in Ji *et al.* (2009, 2013); Ji & Held (2014) used to derive closures for fluid models at zeroth order in ϵ. Finally, by defining the Bessel-Fourier operator

$$
j_m[F_a] \equiv \int J_m(k_\perp \rho_a) \left\langle F_a(\mathbf{k}, v_\parallel, \mu, \theta) \right\rangle_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \tag{4.17}
$$

the expression for the fluid moments $\mathbf{M}_a^{lk}$ in terms of coupled $v_\parallel$ and μ moments of the guiding-center distribution function $\langle F_a \rangle_{\mathbf{R}}$ is obtained

$$
n_a \mathbf{M}_a^{lk}(\mathbf{x}) = \sqrt{\frac{2\pi^{5/2} l!}{2^l (l+1/2)!}} \sum_{m=-l}^{l} \mathbf{e}^{lm} (-1)^m \mathcal{M}_{am}^{lk}(\mathbf{x}), \tag{4.18}
$$

with

$$
\mathcal{M}_{am}^{lk}(\mathbf{x}) = \int j_m[F_a] v^l L_k^{l+1/2}(x_a^2) Y_{lm}(\varphi, 0) \frac{B_\parallel^*}{m} dv_\parallel d\mu. \tag{4.19}
$$

Equation (4.18) can now be used to express the collision operator $C(f_a, f_b)$ in terms of $v_\parallel$ and μ integrals of $\langle F_a \rangle$. Using Eqs. (4.8), (3.27) and (3.28), and defining

$$
E_{jv}^{lsnt} = \mathbf{e}^{l+n-2j\ v} \cdot (\mathbf{e}^{ls} \cdot^j \mathbf{e}^{nt})_{TS}, \tag{4.20}
$$

we can write the collision operator in Eqs. (3.27) and (3.28) as a function of the $\mathcal{M}_{am}^{lk}$ moments, i.e.

$$c_{ab}^{lkmnqr} = \sum_{u=0}^{\min(2,l,n)} \sum_{j=0}^{\min(l,n)-u} d_j^{l-u,n-u} a_{j+u}^{ln} \sum_{s=-l}^{l} \sum_{t=-n}^{n} \sum_{v=-(l+n-2j-2u)}^{l+n-2j-2u} E_{j+u\ v}^{lsnt} \times Y_{l+n-2j-2u\ v}(\varphi,\theta) \frac{\nu_{*abu}^{lm,nr}(v^2)}{n_a n_b} \mathcal{M}_{as}^{lk}(\mathbf{x}) \mathcal{M}_{bt}^{nq}(\mathbf{x}), \tag{4.21}$$

with

$$a_j^{ln} = \frac{1}{2^{l+n-j}} \sqrt{\frac{8\pi^{13/2} l! n! (l+n-2j)!}{(l+1/2)!(n+1/2)!(l+n-2j+1/2)!}}. \tag{4.22}$$

We now focus on the gyroaverage of the collision operator in Eq. (4.21) with the gyroaverage operation performed at constant $\mathbf{R}$. We first note that the gyroangle θ dependence in c_{ab}^{lkmnqr} is present through the spherical harmonic $Y_{l+n-2j-2u\ v}(\varphi,\theta)$ and through the fluid moments $\mathcal{M}_{as}^{lk}$ and $\mathcal{M}_{bt}^{nq}$ as the latter are functions of $\mathbf{x} = \mathbf{R} + \rho_a$. To make the gyroangle dependence explicit, we write both $\mathcal{M}_{as}^{lk}$ and $\mathcal{M}_{bt}^{nq}$ in Fourier space as

$$\mathcal{M}_{as}^{lk}(\mathbf{x}) \mathcal{M}_{bt}^{nq}(\mathbf{x}) = \int d\mathbf{k} d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}} \mathcal{M}_{as}^{lk}(\mathbf{k}) \mathcal{M}_{bt}^{nq}(\mathbf{k}') e^{i(\mathbf{k}\cdot\rho_a + \mathbf{k}'\cdot\rho_a)}. \tag{4.23}$$

Using the Jacobi-Anger expansion of Eq. (4.12), we find that

$$\left\langle Y_{lm}(\varphi,\theta) \mathcal{M}_{as}^{lk}(\mathbf{x}) \mathcal{M}_{bt}^{nq}(\mathbf{x}) \right\rangle_{\mathbf{R}} = \int d\mathbf{k} d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}} \mathcal{M}_{as}^{lk}(\mathbf{k}) \mathcal{M}_{bt}^{nq}(\mathbf{k}') \times \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\varphi) \sum_{p=-\infty}^{\infty} (-1)^{p+m} e^{i(p+m)\alpha} J_p(k_\perp \rho_a) J_{p+m}(k'_\perp \rho_a), \tag{4.24}$$

with α the azimuthal angle of the $\mathbf{k}'$ vector, i.e., the angle between $\mathbf{k}'_\perp$ and $\mathbf{k}_\perp$. The gyroaveraged collision operator at arbitrary $k_\perp \rho$ is therefore given by

$$\langle C(F_a, F_b) \rangle_{\mathbf{R}} = f_{aM} \sum_{l,k,n,q=0}^{\infty} \sum_{m=0}^{k} \sum_{r=0}^{q} L_{km}^{l} L_{qr}^{n} \left\langle c_{ab}^{lkmnqr} \right\rangle_{\mathbf{R}}, \tag{4.25}$$

with

$$\left\langle c_{ab}^{lkmnqr} \right\rangle_{\mathbf{R}} = \sum_{u=0}^{\min(2,l,n)} \sum_{j=0}^{\min(l,n)-u} d_j^{l-u,n-u} a_{j+u}^{ln} \sum_{s=-l}^{l} \sum_{t=-n}^{n} \sum_{v=-(l+n-2j-2u)}^{l+n-2j-2u} E_{j+u\ v}^{lsnt} \times b_{j+u}^{l+nv} P_{l+n-2j-2u}^{v}(\cos\varphi) \nu_{*abu}^{lm,nr}(v^2) \int \mathcal{M}_{as}^{lk}(\mathbf{k}) \mathcal{M}_{bt}^{nq}(\mathbf{k}') e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}} \times \sum_{p=-\infty}^{\infty} (-1)^{p+v} e^{i(p+v)\alpha} J_p(k_\perp \rho_a) J_{p+v}(k'_\perp \rho_a) d\mathbf{k} d\mathbf{k}', \tag{4.26}$$

and

$$b_j^{lv} = i^v \sqrt{\frac{2l-4j}{4\pi} \frac{(l-2j-v)!}{(l-2j+v)!}} \tag{4.27}$$

We note that, if only first order $k_\perp \rho$ terms are kept in the Fourier-Bessel operator of Eq. (4.17), the collision operator in Eq. (4.25) reduces to the drift-kinetic collision operator found in Jorge *et al.* (2017).

In Eq. (4.25), the gyroaveraged collision operator is cast in terms of $v_\parallel$ and μ moments of the guiding-center distribution function $\langle F_a \rangle$ for arbitrary values of $k_\perp \rho$. We now apply the transformation T, introduced in Eq. (2.13), to Eq. (4.25) in order to write the gyroaveraged collision operator in terms of $\overline{v}_\parallel$ and $\overline{\mu}$ moments of the gyrocenter distribution function $\langle \overline{F_a} \rangle_{\overline{\mathbf{R}}}$. As shown in Section 2, only the zeroth order terms in the ϵ_δ expansion of $\langle C(\overline{F_a}, \overline{F_b}) \rangle_{\overline{\mathbf{R}}}$ are needed in order to adequately describe collisional processes in the gyrokinetic framework. Therefore, using Eq. (2.13), we apply the zeroth order transformations $\mathbf{Z} \simeq \overline{\mathbf{Z}}$ and $F_a(\mathbf{Z}) = T\overline{F_a}(\mathbf{Z}) \simeq \overline{F_a}(\mathbf{Z})$ to the collision operator $\langle C(F_a, F_b) \rangle$ in Eq. (4.25), yielding

$$\left\langle C(\overline{F_a}, \overline{F_b}) \right\rangle_{\overline{\mathbf{R}}} \simeq f_{aM} \sum_{l,k,n,q=0}^{\infty} \sum_{m=0}^{k} \sum_{r=0}^{q} L_{km}^{l} L_{qr}^{n} \left\langle \overline{c}_{ab}^{lkmnqr} \right\rangle_{\overline{\mathbf{R}}}, \tag{4.28}$$

with

$$\begin{aligned} \left\langle \overline{c}_{ab}^{lkmnqr} \right\rangle_{\overline{\mathbf{R}}} = & \sum_{u=0}^{\min(2,l,n)} \sum_{j=0}^{\min(l,n)-u} d_j^{l-u,n-u} a_{j+u}^{ln} \sum_{s=-l}^{l} \sum_{t=-n}^{n} \sum_{v=-(l+n-2j-2u)}^{l+n-2j-2u} E_{j+u\ v}^{lsnt} \\ & \times b_{j+u}^{l+nv} P_{l+n-2j-2u}^{v}(\overline{v}_\parallel/\overline{v}) \nu_{*abu}^{lm,nr}(\overline{v}^2) \int \overline{\mathcal{M}}_{as}^{lk}(\mathbf{k}) \overline{\mathcal{M}}_{bt}^{nq}(\mathbf{k}') e^{i(\mathbf{k}+\mathbf{k}')\cdot\overline{\mathbf{R}}} \\ & \times \sum_{p=-\infty}^{\infty} (-1)^{p+v} e^{i(p+v)\alpha} J_p(k_\perp \overline{\rho}_a) J_{p+v}(k'_\perp \overline{\rho}_a) d\mathbf{k} d\mathbf{k}', \end{aligned} \tag{4.29}$$

where $\overline{\rho_a} = \rho_a(\overline{\mathbf{R}}, \overline{\mu}, \overline{\theta})$, $\overline{v}^2 = \overline{v}_\parallel^2 + 2B\overline{\mu}/m$, and

$$\overline{\mathcal{M}}_{am}^{lk} = \int j_m[\overline{F_a}] \overline{v}^l L_k^{l+1/2}(\overline{v}^2) Y_{lm}(\overline{\varphi}, 0) \frac{B_\parallel^*}{m} d\overline{v}_\parallel d\overline{\mu}. \tag{4.30}$$

The collision operator in Eq. (4.28) represents the gyrokinetic full Coulomb collision operator that can be used in gyrokinetic models that are up to $O(\epsilon_\delta^2)$ accurate. In Eq. (4.28), the integral-differential character of the $C(f_a, f_b)$ operator is replaced by two-dimensional integrals of the gyrocenter distribution function over the velocity coordinates $\overline{v}_\parallel$ and $\overline{\mu}$. We note that, in practice, a truncation of the series present in Eqs. (4.28) and (4.29) requires a numerical study in order to assess their convergence.

5. Hermite-Laguerre Expansion of the Coulomb Operator

In this section, we expand the distribution function into an orthogonal Hermite-Laguerre polynomial basis and compute the Hermite-Laguerre moments of the Coulomb collision operator in Eq. (4.28). An expansion of the drift-kinetic (Jorge *et al.* 2017) and gyrokinetic (Mandell *et al.* 2018; Frei *et al.* 2019) equation in Hermite-Laguerre polynomials has been recently introduced, showing that this is an advantageous approach to the study of plasma waves and instabilities (Jorge *et al.* 2018, 2019). A key reason for using a basis of Hermite-Laguerre polynomials in gyrokinetics is that these polynomials are orthogonal with respect to a Maxwellian, and can be directly related to the Bessel functions used in evaluating gyroaverage operators such as the ones present in Eqs. (4.29) and (4.30). We therefore expand $\langle \overline{F_a} \rangle_{\overline{\mathbf{R}}}$ as

$$\langle \overline{F_a} \rangle_{\overline{\mathbf{R}}} = f_{Ma} \sum_{p,j} \frac{\overline{N}_a^{pj}}{\sqrt{2^p p!}} H_p\left(\overline{s}_{\parallel a}\right) L_j\left(\overline{s}_{\perp a}^2\right), \tag{5.1}$$

where H_p are *physicists'* Hermite polynomials of order p defined by the Rodrigues' formula

$$H_p(x) = (-1)^p e^{x^2} \frac{d^p}{dx^p} e^{-x^2}, \tag{5.2}$$

and normalized via

$$\int_{-\infty}^{\infty} dx H_p(x) H_{p'}(x) e^{-x^2} = 2^p p! \sqrt{\pi} \delta_{pp'}, \tag{5.3}$$

and L_j the Laguerre polynomials of order j defined by the Rodrigues' formula

$$L_j(x) = \frac{e^x}{j!} \frac{d^j}{dx^j} (e^{-x} x^j), \tag{5.4}$$

and orthonormal with respect to the weight e^{-x}

$$\int_0^{\infty} dx L_j(x) L_{j'}(x) e^{-x} = \delta_{jj'}. \tag{5.5}$$

In Eq. (5.1), we introduce the normalized parallel velocity

$$\overline{s}_{\parallel a} = \frac{\overline{v}_\parallel}{v_{tha}}, \tag{5.6}$$

and the perpendicular velocity coordinate

$$\overline{s}_{\perp a}^2 = \frac{\overline{\mu} B}{T_a}. \tag{5.7}$$

Due to the orthogonality of the Hermite-Laguerre polynomial basis, the coefficients N_a^{pj} of the expansion in Eq. (5.1) can be computed as

$$\overline{N}_a^{pj} = \int \frac{H_p\left(\overline{s}_{\parallel a}\right) L_j\left(\overline{s}_{\perp a}^2\right) \left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}}}{\sqrt{2^p p!}} \frac{B}{m_a} d\overline{v}_\parallel d\overline{\mu} d\overline{\theta}. \tag{5.8}$$

We note that the integrand of $\overline{N}_a^{pj}$ in Eq. (5.8) contains the multiplicative factor B/m_a, as opposed to the Jacobian containing the factor $B_\parallel^*/m_a$. In the following, we also use the Hermite-Laguerre moments of $\left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}}$ with $B_\parallel^*$ in the integrand instead of B. These are denoted as $\overline{N}_a^{*pj}$, i.e.

$$\begin{aligned}\overline{N}_a^{*pj} &= \int \frac{H_p\left(\overline{s}_{\parallel a}\right) L_j\left(\overline{s}_{\perp a}^2\right) \left\langle \overline{F_a} \right\rangle_{\overline{\mathbf{R}}}}{\sqrt{2^p p!}} \frac{B_\parallel^*}{m_a} d\overline{v}_\parallel d\overline{\mu} d\overline{\theta} \\ &= \overline{N}_a^{pj} \left(1 + \frac{\mathbf{b} \cdot \nabla \times \mathbf{v_E}}{\Omega_a}\right) + \frac{v_{tha}}{\sqrt{2} \Omega_a} \mathbf{b} \cdot \nabla \times \mathbf{b} \left(\sqrt{p+1} \overline{N}_a^{p+1\, j} + \sqrt{p} \overline{N}_a^{p-1\, j}\right),\end{aligned} \tag{5.9}$$

while in the weak flow regime the term proportional to $\mathbf{b} \cdot \nabla \times \mathbf{v_E}/\Omega_a$ in Eq. (5.9) is set to zero.

In order to express the collision operator in terms of the moments $\overline{N}_a^{pj}$ given in Eq. (5.8) and evaluate its Hermite-Laguerre moments, we first consider the gyrokinetic moments $\overline{\mathcal{M}}_{am}^{lk}$ and write the integral that defines them in Eq. (4.30) as a function of the gyrocenter moments $\overline{N}_a^{pj}$ of Eq. (5.8). As a first step, we project both the Fourier-Bessel operator $j_m[\overline{F_a}]$ and the spherical harmonics Y_{lm} on the Hermite-Laguerre basis. We remark that the $\overline{\mu}$ and $k_\perp$ dependence in the Fourier-Bessel operator j_m, Eq. (4.17), can be decomposed by introducing $\rho_{tha} = v_{tha}/\Omega_a$ and noting that $|\rho_a| = \sqrt{\overline{\mu} B / T_a} \rho_{tha} = \overline{s}_{\perp a} \rho_{tha}$. This allows the use of the following identity between Bessel and Legendre functions (Gradshteyn & Ryzhik 2007)

$$J_m(2b_a\overline{s}_{\perp a}) = \sigma_m b_a^{|m|}\overline{s}_{\perp a}^{|m|}e^{-b_a^2}\sum_{r=0}^{\infty}\frac{L_r^{|m|}(\overline{s}_{\perp a}^2)}{(|m|+r)!}b_a^{2r}. \tag{5.10}$$

with $b_a = k_\perp \rho_{tha}/2$, $\sigma_0 = 1$ and $\sigma_m = \mathrm{sgn}(m)^m$ for $m \neq 0$. The Fourier-Bessel operator in Eq. (4.17), with the identity in Eq. (5.10) and the Hermite-Laguerre expansion of Eq. (5.1), can then be written as

$$j_m[\overline{F_a}] = f_{Ma}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{H_p(\overline{s}_{\parallel a})L_j(\overline{s}_{\perp a}^2)}{\sqrt{2^p p!}}\frac{L_r^m(\overline{s}_{\perp a}^2)\overline{s}_{\perp a}^m}{(m+r)!}\int \overline{N}_a^{pj}(\mathbf{k})b_a^{m+2r}e^{-b_a^2}e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}. \tag{5.11}$$

As a second step, we consider

$$Y_{lm}(\varphi,0) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\varphi). \tag{5.12}$$

which are used in the definition in Eq. (4.6). In order to expand the associated Legendre polynomials $P_l^m(\cos\varphi)$ appearing in Eq. (5.12) in a Hermite-Laguerre basis, we generalize the basis transformation between a Legendre-Associated Laguerre and a Hermite-Laguerre basis presented in Jorge *et al.* (2017) to a transformation between an Associated Legendre-Associated Laguerre and a Hermite-Laguerre basis, that is

$$\frac{\overline{v}^l}{v_{tha}^l}P_l^m\left(\frac{\overline{v}_\parallel}{\overline{v}}\right)L_k^{l+1/2}\left(\frac{\overline{v}^2}{v_{tha}^2}\right) = \sum_{p=0}^{l+2k}\sum_{j=0}^{k+\lfloor l/2\rfloor}T_{lkm}^{pj}H_p\left(\frac{\overline{v}_{\parallel a}}{v_{tha}}\right)L_j\left(\frac{\overline{\mu}B}{T_a}\right)\left(\frac{\overline{\mu}B}{T_a}\right)^{m/2}. \tag{5.13}$$

For the derivation and expression of the T_{lkm}^{pj} coefficients, see Appendix B. The inverse transformation coefficients $\left(T^{-1}\right)_{pj}^{lkm}$ are defined as

$$H_p\left(\frac{\overline{v}_{\parallel a}}{v_{tha}}\right)L_j\left(\frac{\overline{\mu}B}{T_a}\right)\left(\frac{\overline{\mu}B}{T_a}\right)^{m/2} = \sum_{l=0}^{p+2j}\sum_{k=0}^{j+\lfloor p/2\rfloor}\left(T^{-1}\right)_{pj}^{lkm}\frac{\overline{v}^l}{v_{tha}^l}P_l^m\left(\frac{\overline{v}_\parallel}{\overline{v}}\right)L_k^{l+1/2}\left(\frac{\overline{v}^2}{v_{tha}^2}\right). \tag{5.14}$$

The gyrocenter moments $\overline{\mathcal{M}}_{am}^{lk}$ in Eq. (4.30) can then be rewritten using the identities in Eqs. (5.11) and (5.13) and

$$L_r^m(x)L_j(x)x^m = \sum_{s=0}^{m+r+j}d_{rjs}^m L_s(x), \tag{5.15}$$

with the d_{mjs}^r coefficients given by

$$d_{rjs}^m = \sum_{r_1=0}^{r}\sum_{j_1=0}^{j}\sum_{s_1=0}^{s}L_{rr_1}^{-1/2}L_{jj_1}^{m-1/2}L_{ss_1}^{-1/2}(r_1+j_1+s_1+m)!, \tag{5.16}$$

yielding the following expression

$$\overline{\mathcal{M}}_{am}^{lk}(\mathbf{k}) = \sum_{g=0}^{\infty}\sum_{h=0}^{l+2k}\sum_{u=0}^{k+\lfloor l/2\rfloor}\sum_{s=0}^{m+r+u}M_{lkmg}^{hus}\overline{N}_a^{*hs}(\mathbf{k})b_a^{2g+m}e^{-b_a^2}. \tag{5.17}$$

where we defined

$$M_{lkmg}^{hus} = (-1)^m \frac{T_{lkm}^{hu} d_{gus}^m \sqrt{2^p p!}}{(m+g)!} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}. \tag{5.18}$$

Using the form for $\overline{\mathcal{M}}_{am}^{lk}$ in Eq. (5.17), the collision operator in Eq. (4.28) can therefore be expressed in terms of Hermite-Laguerre moments N^{pj} of the distribution function. We note that the moments $\overline{\mathcal{M}}_{am}^{lk}$ in Eq. (5.17) reduce to the ones in Jorge *et al.* (2017) in the lowest order drift-kinetic limit $k_\perp \rho_{tha} = 0$.

We now take Hermite-Laguerre moments of the collision operator $\left\langle C(\overline{F_a}, \overline{F_b}) \right\rangle$, i.e. we evaluate

$$C_{ab}^{pj}(\overline{\mathbf{R}}) = \int \left\langle C(\overline{F_a}, \overline{F_b}) \right\rangle_{\overline{\mathbf{R}}} \frac{H_p(\overline{s}_{\parallel a}) L_j(\overline{s}_{\perp a^2})}{\sqrt{2^p p!}} \frac{B_\parallel^*}{m_a} d\overline{v}_\parallel d\overline{\mu} d\overline{\theta}. \tag{5.19}$$

Writing the gyroaveraged collision operator $\left\langle C(\overline{F_a}, \overline{F_b}) \right\rangle$ in Eq. (4.28) using Eqs. (5.17) and (4.29), and expanding the Bessel functions $J_p(k_\perp \rho_a)$ and $J_{p+m}(k'_\perp \rho_a)$ using Eq. (5.10), the following form for the $\left\langle \overline{c}_{ab}^{lkmnqr} \right\rangle_{\overline{\mathbf{R}}}$ term appearing in $\left\langle C(\overline{F_a}, \overline{F_b}) \right\rangle_{\overline{\mathbf{R}}}$ is obtained

$$\begin{aligned}\left\langle \overline{c}_{ab}^{lkmnqr} \right\rangle_{\overline{\mathbf{R}}} = &\sum_{u=0}^{min(2,l,n)} \sum_{i=0}^{min(l,n)-u} \sum_{d=-l-n+2i+2u}^{l+n-2i-2u} \sum_{z=0}^{\infty} \sum_{p,p'=0}^{\infty} \int D_{abuidzpp'}^{lkmnqr}(\mathbf{k}, \mathbf{k}') \\ &\times P_{l+n-2i-2u}^{d}\left(\frac{\overline{v}_\parallel}{\overline{v}}\right) \overline{s}_{\perp a}^{d+2z} L_p^z(\overline{s}_{\perp a}^2) L_{p'}^{z+d}(\overline{s}_{\perp a}^2) \nu_{*abu}^{lm,nr}(\overline{v}^2) e^{i(\mathbf{k}+\mathbf{k}')\mathbf{R}} d\mathbf{k} d\mathbf{k}'.\end{aligned} \tag{5.20}$$

In Eq. (5.20), we defined the $D_{abuidzpp'}^{lkmnqr}$ term

$$D_{abuidzpp'}^{lkmnqr}(\mathbf{k}, \mathbf{k}') = \sum_{s=-l}^{l} \sum_{t=-n}^{n} E_{i+ud}^{lsnt} B_a^{zdp'} \frac{d_i^{l-u,n-u} a_{i+u}^{ln} e^{-b_a^2 - b_a'^2}}{(p+z)!(z+d+p')!} \mathcal{N}_{abuidz}^{lkmnqr}(\mathbf{k}, \mathbf{k}'), \tag{5.21}$$

with $B_a^{pvz'} = b_a^{p+2z} b_a^{p+v+2z'}$ and $b'_a = k'_\perp \rho_{tha}/2$, while the convolution operator $\mathcal{N}_{abuidz}^{lkmnqr}(\mathbf{k}, \mathbf{k}')$ is given by

$$\mathcal{N}_{abuidz}^{lkmnqr}(\mathbf{k}, \mathbf{k}') = (-1)^{z+d} e^{i(z+d)\alpha} b_{i+u}^{l+nd} \overline{\mathcal{M}}_{as}^{lk}(\mathbf{k}) \overline{\mathcal{M}}_{bt}^{nq}(\mathbf{k}'), \tag{5.22}$$

with $\overline{\mathcal{M}}_{as}^{lk}$ the moments of the distribution function defined in Eq. (5.17).

Finally, the result in Eq. (5.20) is used in Eq. (5.19) in order to find the Hermite-Laguerre moments C_{ab}^{pj} of the full Coulomb collision operator expressed in Eq. (4.28). This yields

$$C_{ab}^{pj} = \sum_{l,k,n,q=0}^{\infty} \sum_{m=0}^{k} \sum_{r=0}^{q} \frac{L_{km}^l L_{qr}^n}{\sqrt{2^p p!}} C_{ab,lkm}^{pj,nqr}, \tag{5.23}$$

with

$$C_{ab,lkm}^{pj,nqr}(\mathbf{k}, \mathbf{k}') = \sum_{u=0}^{min(2,l,n)} \sum_{i=0}^{min(l,n)-u} \sum_{d=-l-n+2i+2u}^{l+n-2i-2u} \sum_{z,p,p'=0}^{\infty} D_{abuidzpp'}^{lkmnqr}(\mathbf{k}, \mathbf{k}') I, \tag{5.24}$$

and

$$I = \int f_{aM} P^d_{l+n-2i-2u}(\overline{v}_\parallel/\overline{v}) \nu^{lm,nr}_{*abu}(\overline{v}^2) H_p(\overline{s}_{\parallel a}) L_j(\overline{s}^2_{\perp a}) \overline{s}^{d+2z}_{\perp a} L^z_p(\overline{s}^2_{\perp a}) L^{z+d}_{p'}(\overline{s}^2_{\perp a}) \frac{B^*_\parallel}{m_a} d\overline{v}_\parallel d\overline{\mu}. \tag{5.25}$$

The integral factor I can be performed analytically by first rewriting the product of two Laguerre polynomials as a single one using

$$L^m_r(x) L_j(x) = \sum_{s=0}^{r+j} \overline{d}^m_{rjs} L_s(x), \tag{5.26}$$

with

$$\overline{d}^m_{rjs} = \sum_{r_1=0}^{r} \sum_{j_1=0}^{j} \sum_{s_1=0}^{s} L^{-1/2}_{rr_1} L^{m-1/2}_{jj_1} L^{-1/2}_{ss_1} (r_1 + j_1 + s_1)!, \tag{5.27}$$

expressing the resulting Hermite-Laguerre basis in terms of Legendre-Associated Laguerre using Eq. (5.14), and writing the phase-space volume $(B^*_\parallel/m) d\overline{v}_\parallel d\overline{\mu}$ as $\overline{v}^2 d\overline{v} d\overline{\xi}$ with $\overline{\xi} = \overline{v}_\parallel/\overline{v}$. This yields

$$I = \sum_{h=0}^{p+z+j} \sum_{g=0}^{g+p'} \sum_{s=0}^{p+2g} \sum_{t=0}^{g+\lfloor p/2 \rfloor} d^{std}_{pjh} \overline{d}^{z+d}_{p'hg} \left(T^{-1}\right)^{std}_{pg} C^{st,lm,nr}_{*abu} \frac{(s+d)!}{(s-d)!} \frac{\delta_{l+n-2i-2u,s}}{4\pi(s+1/2)}. \tag{5.28}$$

Ji & Held (2009) present an analytical closed expression ready to be numerically implemented of the factor $C^{st,lm,nr}_{*abu} = \int f_{Ma} \nu^{lm,nr}_{*abu}(v^2) L^{s+1/2}_t(v^2) v^s d\mathbf{v}$. We note that the long-wavelength limit can be found by setting $d = z = 0$ in the collision operator Eq. (5.24). This yields the Hermite-Laguerre moments of the collision operator moments found in Jorge *et al.* (2017).

6. Small Mass-Ratio Approximation

In this section, we simplify the electron-ion and the ion-electron collision operator in Eq. (2.2) by taking advantage of the small electron-to-ion mass ratio m_e/m_i, and derive their expressions in the gyrokinetic regime. We first consider the electron-ion collision operator.

In $(\mathbf{x}, \mathbf{v})$ phase-space coordinates, the electron-ion Coulomb collision operator can be greatly simplified by taking advantage of the fact that the ion thermal speed, is small in comparison to the electron thermal speed, for $T_e \sim T_i$. To first order in m_e/m_i, the electron-ion Coulomb collision operator, also called Lorentz pitch-angle scattering operator, can be written as (Helander & Sigmar 2005)

$$C_{ei} = \frac{n_i L_{ei}}{v^3_{the}} \frac{\partial}{\partial \mathbf{c}_e} \cdot \left[\frac{1}{c_e} \frac{\partial f_e}{\partial \mathbf{c}_e} - \frac{\mathbf{c}_e}{c^3_e} \left(\mathbf{c}_e \cdot \frac{\partial f_e}{\partial \mathbf{c}_e} \right) \right], \tag{6.1}$$

with $\mathbf{c}_e = \mathbf{v}/v_{the}$. We expand f_e according to Eq. (3.12). We note that the expansion in Eq. (3.12) is an eigenbasis of the pitch-angle scattering operator C_{ei} with eigenvalue $l(l+1)$ (Ji & Held 2008). Therefore, we write

$$C_{ei} = -f_{eM} \sum_{l,k} \frac{n_i L_{ei}}{v^3_{the} c^3_e} \frac{l(l+1)}{\sqrt{\sigma^l_k}} L^{l+1/2}_k \left(c^2_e\right) \mathbf{Y}^l(\mathbf{c}_e) \cdot \mathbf{M}^{lk}_e(\mathbf{x}). \tag{6.2}$$

We now Fourier transform the moments $\mathbf{M}^{lk}_e$ in Eq. (6.2) as $\mathbf{M}^{lk}_e(\overline{\mathbf{R}}) = \int \mathbf{M}^{lk}_e(\mathbf{k}) e^{i\mathbf{k}\cdot\overline{\mathbf{R}}} d\mathbf{k}$

and write the gyroaveraged collision operator C_{ei} as

$$\langle C_{ei}\rangle_{\overline{\mathbf{R}}} = -\int d\mathbf{k} e^{i\mathbf{k}\cdot\overline{\mathbf{R}}} f_{eM} \sum_{l,k} \frac{n_i L_{ei}}{v_{the}^3 c_e^3} \frac{l(l+1)}{\sqrt{\sigma_k^l}} L_k^{l+1/2}\left(c_e^2\right)\left\langle \mathbf{Y}^l(\mathbf{c}_e) e^{i\mathbf{k}\cdot\rho_e}\right\rangle_{\overline{\mathbf{R}}} \cdot \mathbf{M}_e^{lk}(\mathbf{k}). \quad (6.3)$$

Using the Jacobi-Anger expansion of Eq. (4.12), Eq. (5.10), the inverse basis transformation Eq. (5.14), and the identities $J_{-p}(x) = (-1)^p J_p(x)$ and

$$L_r^m(x) = \sum_{j=0}^{r} \binom{m+r-j-1}{r-j} L_j(x), \quad (6.4)$$

we obtain

$$\begin{aligned}\left\langle \mathbf{Y}^l(\mathbf{v}) e^{i\mathbf{k}\cdot\rho_e}\right\rangle_{\mathbf{R}} = &\sum_{m=-l}^{l}\sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{s=0}^{2j}\sum_{t=0}^{j} \sqrt{\frac{\pi^{1/2} l!}{2^l (l-1/2)!}\frac{(l-m)!}{(l+m)!}} \frac{i^m \mathbf{e}^{lm}}{(m+r)!}\frac{(m+r-j-1)!}{(r-j)!(m-1)!} \\ &\times v_{the}^l (T^{-1})_{0j}^{stm} b_e^{2r+m} e^{-b_e^2} c_e^{l+s} P_l^m(\cos\varphi) P_s^m(\cos\varphi) L_t^{s+1/2}(c_e^2), \end{aligned}\quad (6.5)$$

with $b_e = k_\perp \rho_{the}/2$. The collision operator in Eq. (6.3) represents the gyrokinetic electron-ion collision operator.

Equation (6.5) provides an expression of the pitch-angle scattering operator $\langle C_{ei}\rangle$ in Eq. (6.3) suitable for projection onto a Hermite-Laguerre basis, i.e.

$$C_{ei}^{pj} = \int \langle C_{ei}\rangle \frac{H_p\left(\frac{\overline{v}_\parallel}{v_{tha}}\right) L_j\left(\frac{\overline{\mu}B}{T_a}\right)}{\sqrt{2^p p!}} \frac{B_\parallel^*}{m_a} d\overline{v}_\parallel d\overline{\mu} d\overline{\theta} = 2\pi \sum_{l=0}^{p+2j}\sum_{k=0}^{j+\lfloor p/2\rfloor} \frac{(T^{-1})_{pj}^{lk0} v_{the}^3}{\sqrt{2^p p!}} I_{ei}^{lk}, \quad (6.6)$$

where we define

$$I_{ei}^{lk} = \int \langle C_{ei}\rangle\, c_e^l P_l(\cos\varphi) L_k^{l+1/2}(c_e^2) c_e^2 dc_e d\cos\varphi. \quad (6.7)$$

An analytical form for the integral factor I_{ei}^{lk} can be derived using the expression for $\langle C_{ei}\rangle$, Eq. (6.3), and Eq. (6.5), yielding

$$\begin{aligned} I_{ei}^{lk}(\mathbf{k}) = &-\sum_{u,v} \frac{n_e n_i L_{ei}}{v_{the}^{6-u}} \frac{u(u+1)}{\pi\sqrt{\sigma_v^u}} \sum_{m=-u}^{u} \mathbf{M}_e^{lk}(\mathbf{k})\cdot\mathbf{e}^{um} \sum_{r=0}^{\infty}\sum_{i=0}^{r}\sum_{s=0}^{2i}\sum_{t=0}^{i} (T^{-1})_{0i}^{stm} e^{-b_e^2} \\ &\times \sqrt{\frac{u!}{2^u(u-1/2)!}\frac{(u-m)!}{(u+m)!}} \frac{i^m b_e^{2r+m}}{(m+r)!}\frac{(m+r-i-1)!}{(r-i)!(m-1)!} I_{Lkt}^{lsuv} I_{Pm}^{lus}, \end{aligned}\quad (6.8)$$

with I_{Lkt}^{lsuv} and I_{Pm}^{lus} defined by

$$I_{Lkt}^{lsuv} = \int L_k^{l+1/2}(x) L_t^{s+1/2}(x) e^{-x} x^{(l+u+v)/2-1} dx, \quad (6.9)$$

and

$$I_{Pm}^{lus} = \int_{-1}^{1} P_l(x) P_u^m(x) P_s^m(x) \frac{dx}{2}, \quad (6.10)$$

respectively. The electron fluid moments $\mathbf{M}_e^{lk}$ can be cast in terms of Hermite-Laguerre moments $\overline{N}_e^{lk}$ using the expressions in Eqs. (4.18), (5.9), and (5.17). The factor I_{Lkt}^{lsuv} can be analytically evaluated by expanding the associated Laguerre polynomials using

Eq. (3.8), which leads to

$$I^{lsuv}_{Lkt} = \sum_{m_1=0}^{k} \sum_{m_2=0}^{t} L^l_{km_1} L^s_{tm_2} (m_1 + m_2 + (l+u+v)/2 - 1)!. \tag{6.11}$$

Similarly, the factor integral I^{lus}_{Pm} can be calculated using an extended version of Gaunt's formula (Gaunt 1929), yielding (Mavromatis & Alassar 1999)

$$I^{lus}_{Pm} = (-1)^m \begin{pmatrix} l & u & s \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & u & s \\ 0 & m & -m \end{pmatrix} \sqrt{\frac{(s+m)!(u+m)!}{(s-m)!(u-m)!}}. \tag{6.12}$$

We note that, in Eq. (6.12), the Wigner 3-j symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is related to the Clebsch-Gordan coefficients $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$ via (Olver *et al.* 2010)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}} \langle j_1 m_1 j_2 m_2 | j_3 (-m_3) \rangle, \tag{6.13}$$

with the Clebsch-Gordan coefficients given by

$$\begin{aligned} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle &= \delta_{m_3, m_1 + m_2} \sqrt{\frac{(2j_3+1)(j_3+j_1-j_2)!(j_3-j_1+j_2)!(j_1+j_2-j_3)!}{(j_1+j_2+j_3+1)!}} \\ &\times \sqrt{(j_3+m_3)!(j_3-m3)!(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_3-m_3)!} \\ &\times \sum_k \frac{(-1)^k}{k!(j_1+j_2-j_3-k)!(j_1-m_1-k)!(j_2+m_2-k)!} \\ &\times \frac{1}{(j_3-j_2+m_1+k)!(j_3-j_1-m_2+k)!}, \end{aligned} \tag{6.14}$$

where the summation in Eq. (6.14) is extended over all integers k that make every factorial in the sum nonnegative (Bohm & Loewe 1993).

We now turn to the ion-electron collision operator C_{ie}. To first order in m_e/m_i, this is given by (Helander & Sigmar 2005)

$$C_{ie} = \nu_{ei} \frac{m_e}{m_i} \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{v} f_i + \frac{T_e}{m_i} \frac{\partial f_i}{\partial \mathbf{v}} \right), \tag{6.15}$$

where the electron-ion friction force is neglected for simplicity. We simplify Eq. (6.15) by using Eq. (2.27), therefore approximating the distribution function f_i by its gyroaveraged component $f_i \simeq \langle \overline{F}_i \rangle_{\overline{\mathbf{R}}}$, and retaining the lowest-order terms in the ϵ_δ expansion. This allows us to convert the C_{ie} operator in Eq. (6.15) to the gyrocenter variables $\overline{\mathbf{Z}}$ using the chain rule at lowest order in ϵ_δ, i.e. to express Eq. (6.15) in $\mathbf{Z}$ coordinates using the guiding-center transformation in Eqs. (2.15) to (2.17) and approximate $\mathbf{Z} \simeq \overline{\mathbf{Z}}$. The velocity derivatives can be written as a function of $\overline{\mathbf{Z}}$ using the chain rule, yielding

$$\frac{\partial \langle \overline{F}_i \rangle_{\overline{\mathbf{R}}}}{\partial \mathbf{v}} = \mathbf{b} \frac{\partial \langle \overline{F}_i \rangle_{\overline{\mathbf{R}}}}{\partial \overline{v}_\parallel} + \mathbf{c} \left(\sqrt{\frac{2 m_a \overline{\mu}}{B}} \frac{\partial \langle \overline{F}_i \rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}} - \frac{1}{\Omega_i} \mathbf{a} \cdot \nabla_{\overline{\mathbf{R}}} \langle \overline{F}_i \rangle_{\overline{\mathbf{R}}} \right), \tag{6.16}$$

where we define $\mathbf{c} = (\cos\overline{\theta} \mathbf{e}_1 + \sin\overline{\theta} \mathbf{e}_2)$ and $\mathbf{a} = \mathbf{c} \times \mathbf{b} = (-\sin\overline{\theta} \mathbf{e}_1 + \cos\overline{\theta} \mathbf{e}_2)$. The ion-electron collision operator can therefore be written as

$$
\begin{aligned}
C_{ie} = \nu_{ei}\frac{m_e}{m_i}\Bigg[&3\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}} + v_\parallel \frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{v}_\parallel} + 2\overline{\mu}\frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}} - \sqrt{\frac{2B\overline{\mu}}{m_i}}\frac{\mathbf{a}\cdot\nabla_{\overline{\mathbf{R}}}}{\Omega_i}\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}\\
&+\frac{T_e}{m_i}\left(\frac{\partial^2 \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{v}_\parallel^2} + \frac{2m_i\overline{\mu}}{B}\frac{\partial^2 \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}^2} + \frac{\mathbf{a}\cdot\nabla_{\overline{\mathbf{R}}}\mathbf{a}\cdot\nabla_{\overline{\mathbf{R}}}}{\Omega_i^2}\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}\right.\\
&\left.-2\sqrt{\frac{2m_i\overline{\mu}}{B}}\mathbf{a}\cdot\nabla_{\overline{\mathbf{R}}}\frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}} + \frac{m_i}{B}\frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}}\right)\Bigg].
\end{aligned}
\tag{6.17}
$$

We now Fourier transform both T_e and $\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}$ and gyroaverage C_{ie}, yielding

$$
\begin{aligned}
\langle C_{ie}\rangle_{\overline{\mathbf{R}}} = \nu_{ei}\frac{m_e}{m_i}\int e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}}\Bigg[&\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}} + \frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{v}_\parallel} + 2\overline{\mu}\frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}}\\
&+J_0(k_\perp'\overline{\rho_i})\frac{T_e(\mathbf{k}')}{m_i}\left(\frac{\partial^2 \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{v}_\parallel^2} + \frac{2m_i\overline{\mu}}{B}\frac{\partial^2 \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}^2} + \frac{m_i}{B}\frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}}\right).\\
&+\frac{T_e(\mathbf{k}')}{m_i}i\frac{\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{2\Omega_i^2}\left[J_0(k_\perp'\overline{\rho_i})k_\perp^2 + J_2(k_\perp'\overline{\rho_i})\mathbf{k}\mathbf{k}:(\mathbf{e}_1\mathbf{e}_1-\mathbf{e}_2\mathbf{e}_2)\right]\\
&-\frac{T_e(\mathbf{k}')}{m_i}\mathbf{k}\cdot\mathbf{e}_2 iJ_1(k_\perp'\overline{\rho_i})\frac{2m_i\overline{v}_\perp}{B\Omega_i}\frac{\partial \left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}}{\partial \overline{\mu}}\Bigg],
\end{aligned}
\tag{6.18}
$$

where we have used the identities $\left\langle \mathbf{a}e^{i\mathbf{k}'\cdot\rho_i}\right\rangle_{\overline{\mathbf{R}}} = iJ_1(k_\perp'\overline{\rho_i})\mathbf{e}_2$ and $\left\langle \mathbf{a}\mathbf{a}e^{i\mathbf{k}'\cdot\rho_i}\right\rangle_{\overline{\mathbf{R}}} = (1/2)[J_0(k_\perp'\overline{\rho_i})(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2)+J_2(k_\perp'\overline{\rho_i})(\mathbf{e}_1\mathbf{e}_1-\mathbf{e}_2\mathbf{e}_2)]$. Finally, we take Hermite-Laguerre moments of the gyroaveraged ion-electron collision operator $\langle C_{ie}\rangle_{\overline{\mathbf{R}}}$ in Eq. (6.18), using the expansion of $\left\langle \overline{F}_i\right\rangle_{\overline{\mathbf{R}}}$ in Eq. (5.1), yielding

$$
\begin{aligned}
C_{ie}^{pj} = \nu_{ei}\frac{m_e}{m_i}\int e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}}\sum_{l,k}\Bigg[&A_{lk}^{pj} + e^{-b_i^2}\frac{T_e(k_\perp')}{T_i}\sum_{r=0}^{\infty}\frac{b_i^{2r}}{r!}\left(\sum_{s=0}^{r+j}d_{rjs}^0 B_{lkrs}^{pj}\right.\\
&\left.+\sum_{v=0}^{r}\sum_{s=0}^{v+j+1}\frac{d_{vjs}^1 i\rho_{thi}^2 b_i^2\delta_{lp}\delta_{ks}}{4(r+1)(r+2)}\mathbf{k}\mathbf{k}:(\mathbf{e}_1\mathbf{e}_1-\mathbf{e}_2\mathbf{e}_2)\right)\Bigg]\overline{N}_i^{lk}(\mathbf{k}),
\end{aligned}
\tag{6.19}
$$

with A_{lk}^{pj} given by

$$
A_{lk}^{pj} = 2j\delta_{lp}\delta_{kj-1} - (p+2j)\delta_{lp}\delta_{kj} - \sqrt{p(p-1)}\delta_{lp-2}\delta_{kj},
\tag{6.20}
$$

and B_{lkrs}^{pj} by

$$
\begin{aligned}
B_{lkrs}^{pj} = &\sqrt{p(p-1)}\delta_{lp-2}\delta_{ks} + \frac{T_i}{m_i}\frac{ik_\perp^2}{2\Omega_i^2} - \sum_{i=0}^{s-1}(3+2s)\delta_{lp}\delta_{ki} + \sum_{i=0}^{s-2}2s\delta_{lp}\delta_{ki}\\
&+i\mathbf{k}\cdot\mathbf{e}_2\frac{2v_{thi}}{\Omega_i}\frac{b_i}{r+1}[(1+s)\delta_{lp}\delta_{ks} - s\delta_{lp}\delta_{ks-1}].
\end{aligned}
\tag{6.21}
$$

7. Conclusion

In this work, a formulation of the nonlinear gyrokinetic Coulomb collision operator is derived, providing an extension of a previously derived nonlinear Coulomb drift-kinetic collision operator to the gyrokinetic regime. This constitutes a key element necessary to perform quantitative studies of turbulence, flows, and, in general, of the plasma dynamics in the periphery of magnetized fusion devices. The gyroaveraged collision operator is cast in terms of parallel and perpendicular velocity integrals of the gyroaveraged distribution function at arbitrary $k_\perp \rho_s$, yielding the formula in Eq. (4.28). In order to provide an analytical formulation of the Coulomb collision operator ready to be used in pseudospectral formulations of the gyrokinetic equation for distribution functions arbitrarily far from equilibrium and for an arbitrary collisionality regime, the Hermite-Laguerre moments of the gyroaveraged collision operator are evaluated, yielding Eq. (5.23). Furthermore, the electron-to-ion mass ratio is used to simplify the form of the electron-ion and ion-electron collision operators, yielding Eq. (6.6) and Eq. (6.19), respectively.

We conclude by noting that the present collision operator is derived by porting the Coulomb operator to the gyrocenter phase-space by using a framework valid up to second order in the expansion parameter ϵ, yielding second order accurate momentum and energy conservation laws. The use of the techniques developed here to analytically gyroaverage the Coulomb operator and obtain its projection onto an orthogonal polynomial basis should, in principle, be applicable to collision operators of the Fokker-Planck type that add the necessary correction terms in order to ensure exact conservation laws (Brizard 2004; Sugama *et al.* 2015; Burby *et al.* 2015).

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Appendix A. Basis Tensors

In this appendix, we derive the form of the basis tensors $\mathbf{e}^{lm}$ used in the definition of $\mathbf{Y}^l(\mathbf{v})$ in Eq. (4.8). We start with the $l = 1$ case, for which Eq. (4.8) yields

$$\mathbf{Y}^1(\mathbf{v}) = \mathbf{v} = \sqrt{\frac{4\pi}{3}} v \sum_{m=-1}^{1} Y_{1m}(\phi, \theta)\mathbf{e}^{1m}. \tag{A 1}$$

The spherical basis vectors $\mathbf{e}^{1m}$ can then be derived from Eq. (A 1) by expressing the vector $\mathbf{v}$ in spherical coordinates as

$$\mathbf{v} = v\left(\sin\phi\cos\theta\mathbf{e}_x + \sin\phi\sin\theta\mathbf{e}_y + \cos\phi\mathbf{e}_z\right), \tag{A 2}$$

and using the identities for the spherical harmonics

$$Y_{1m}(\phi,\theta) = \begin{cases} \sqrt{\frac{3}{8\pi}}\sin\phi e^{-i\theta}, & m=-1,\\ \sqrt{\frac{3}{4\pi}}\cos\phi, & m=0,\\ -\sqrt{\frac{3}{8\pi}}\sin\phi e^{i\theta}, & m=1, \end{cases} \tag{A 3}$$

therefore obtaining

$$\mathbf{e}^{1m} = \begin{cases} \frac{\mathbf{e}_x - i\mathbf{e}_y}{\sqrt{2}}, & m=-1,\\ \mathbf{e}_z, & m=0,\\ -\frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}}, & m=1. \end{cases} \tag{A 4}$$

We now construct spherical basis tensors $\mathbf{e}^{lm}$ from the spherical basis vectors $\mathbf{e}^{1m}$ leveraging the techniques developed for the angular momentum formalism in quantum mechanics (Zettili & Zahed 2009; Snider 2017). As a first step, we note that the basis vectors $\mathbf{e}^{1m}$ are eigenvectors of the angular momentum matrix G_z

$$G_z = i\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \tag{A 5}$$

with eigenvalue m, that is

$$G_z \cdot \mathbf{e}^{1m} = m\mathbf{e}^{1m}. \tag{A 6}$$

As a second step, we note that the relationship between the basis vectors $\mathbf{e}_\alpha$ for $\alpha = (x, y, z)$ and the angular momentum matrices G_α is given by

$$G_\alpha = -i\mathbf{e}_\alpha \cdot \epsilon, \tag{A 7}$$

with ϵ the standard Levi-Civita tensor. In index notation, Eq. (A 7) can be written as

$$(G_\alpha)_{kl} = -i\sum_{j=1}^{3}(e_\alpha)_j\,\epsilon_{jkl}. \tag{A 8}$$

The raising G_+ and lowering G_- operators (corresponding to the ladder operators in quantum mechanics), defined by

$$G_\pm = G_x \pm iG_y. \tag{A 9}$$

allows us to obtain the basis vectors $\mathbf{e}^{1\pm1}$ from $\mathbf{e}^{10}$ using

$$G_\pm\mathbf{e}^{10} = \mathbf{e}^{1\pm1}. \tag{A 10}$$

In addition, we have that

$$\mathbf{e}^{1-1} = (G_-)^2\mathbf{e}^{11}. \tag{A 11}$$

We can now define the spherical tensor basis $\mathbf{e}^{lm}$ that define the irreducible tensors $\mathbf{Y}^l$. We start with the spherical basis tensor

$$\mathbf{e}^{ll} = \mathbf{e}^{11}\mathbf{e}^{11}...\mathbf{e}^{11}, \tag{A 12}$$

formed by the product of l basis vectors $\mathbf{e}^{11}$. Similarly to $\mathbf{Y}^l(\mathbf{v})$, this tensor is of rank l, symmetric, and totally traceless, as $\mathbf{e}^{11}\cdot\mathbf{e}^{11} = 0$. Furthermore, we note that $\mathbf{e}^{ll}$ is an eigenvector with eigenvalue l of the angular momentum tensor G_z^l, with G_n^l a tensor of rank $2l$ defined by

$$\left[G^l_\alpha\right]_{a_1a_2...a_lb_1b_2...b_l} = \sum_{j'k'...l'} \left\{ \left[G_\alpha\right]_{a_1b_1} \delta_{a_2b_2}...\delta_{a_lb_l} + \delta_{a_1b_1}\left[G_\alpha\right]_{a_2b_2}...\delta_{a_lb_l} + ... + \delta_{a_1b_1}\delta_{a_2b_2}...\left[G_\alpha\right]_{a_lb_l} \right\}. \tag{A 13}$$

The remaining basis tensor elements $\mathbf{e}^{lm}$ can be obtained by applying the tensorial lowering operator $G^l_- = G^l_x - iG^l_y$ to $\mathbf{e}^{ll}$, namely

$$\mathbf{e}^{lm} = \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}} \left(G^l_- \cdot^l\right)^{l-m} \mathbf{e}^{ll}, \tag{A 14}$$

with $m = -l, -l+1, ..., -1, 0, 1, ..., l$ and $\left(G^l_- \cdot^l\right)^{l-m}\mathbf{e}^{ll}$ a tensor of order l built by the application of the $G^l_- \cdot^l$ operator to $\mathbf{e}^{ll}$ $l-m$ times. The normalization factor in Eq. (A 14) is obtained by requiring that the contravariant $\mathbf{e}^{lm}$ and the covariant $\mathbf{e}^l_m$ basis tensors form an orthonormal basis, i.e.

$$\mathbf{e}^{lm} \cdot \mathbf{e}^l_{m'} = \delta_{m,m'}. \tag{A 15}$$

In order to find a covariant basis e^l_m, we start with the case $l=1$ and note that the set of vectors $\mathbf{e}^1_m = (\mathbf{e}^{1m})^* = (-1)^m \mathbf{e}^{1-m}$, with $(\mathbf{e}^1_m)^*$ the complex conjugate of $\mathbf{e}^1_m$ satisfies Eq. (A 15). We therefore define $\mathbf{e}^l_m = (\mathbf{e}^{lm})^*$, and use Eq. (A 15) to normalize $\mathbf{e}^{lm}$. For computational purposes, we note that the tensor $\mathbf{e}^{lm}$ can also be written as a function of the basis vectors $\mathbf{e}^{1m}$ as (Snider 2017)

$$\mathbf{e}^{lm} = N_{lm} \sum_{n=0}^{\lfloor \frac{l+m}{2} \rfloor} a^{lm}_n \left\{ (\mathbf{e}^{11})^{m+n} (\mathbf{e}^{1-1})^n (\mathbf{e}^{10})^{l-m-2n} \right\}_{TS}, \tag{A 16}$$

where $N_{lm} = \sqrt{(l+m)!(l-m)!2^{l-m}/(2l)!}$ and $a^{lm}_n = l!/[2^n n!(m+n)!(l-m-2n)!]$.

Appendix B. Basis Transformation

In this section, we derive a closed-form expression for the T^{pj}_{lkm} and $(T^{-1})^{lkm}_{pj}$ coefficients defined in Eqs. (5.13) and (5.14). By multiplying Eq. (5.13) by a Hermite and a Laguerre polynomial and by an exponential of the form $e^{-\overline{v}^2}$, and integrating over the whole $\overline{v}_\parallel$ and $\overline{\mu}$ space, we obtain the following integral expression for T^{pj}_{lkm}

$$T^{pj}_{lkm} = \frac{v^{m-l}_{tha}}{2^p p! \sqrt{\pi}} \int \frac{\overline{v}^l}{\overline{v}^m_\perp} P^m_l\left(\frac{\overline{v}_\parallel}{\overline{v}}\right) L^{l+1/2}_k\left(\frac{\overline{v}^2}{v^2_{tha}}\right) H_p\left(\frac{\overline{v}_{\parallel a}}{v_{tha}}\right) L_j\left(\frac{\overline{v}^2_\perp}{v^2_{tha}}\right) e^{-\frac{v^2}{v^2_{tha}}} \frac{d\mathbf{v}}{2\pi}. \tag{B 1}$$

We first write the integrand in Eq. (B 1) in terms of $\overline{\xi} = \overline{v}_\parallel/\overline{v}$ and $\overline{v}$ coordinates using the basis transformation in Eq. (5.14), yielding

$$T^{pj}_{lkm} = \sum_{l'=0}^{p+2j} \sum_{k'=0}^{j+\lfloor p/2 \rfloor} \frac{(l+1/2)k!}{(l+k+1/2)!} T^{pj}_{l'k'} \times \int_{-1}^{1} \frac{P^m_l(\overline{\xi}) P_{l'}(\overline{\xi})}{(1-\overline{\xi})^2} d\overline{\xi} \int_0^\infty x_a^{(l+l'-m+1)/2} L^{l+1/2}_k(x_a) L^{l'+1/2}_{k'}(x_a) dx_a, \tag{B 2}$$

where we used the fact that $(T^{-1})^{pj}_{lk} = T^{pj}_{lk}\sqrt{\pi}2^p p! k!(l+1/2)/(k+l+1/2)!$ (Jorge *et al.* 2017). The ξ integral in Eq. (B 2) is performed by expanding P_l as a finite sum of the form

$$P_l(x) = \sum_{s=0}^{l} c_s^l x^s, \tag{B 3}$$

with the coefficients $c_s^l = 2^l[(l+s-1)/2]!/[s!(l-s)!((s-l-1)/2)!]$, and using the relation between associated Legendre functions $P_l^m(x)$ and Legendre polynomials $P_l(x)$

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m P_l(x)}{dx^m}. \tag{B 4}$$

The x integral in Eq. (B 2) is performed by using the expansion of the associated Laguerre polynomials in Eq. (3.8). The T_{lkm}^{pj} coefficient can then be written as

$$T_{lkm}^{pj} = \sum_{l'=0}^{p+2j}\sum_{k'=0}^{j+\lfloor p/2\rfloor} T_{l'k'}^{pj}\frac{(l'+1/2)k'!}{(l'+k'+1/2)!}\sum_{m_1=0}^{k}\sum_{m_2=0}^{k'}\sum_{s_1=0}^{l}\sum_{s_2=0}^{l'} L_{km_1}^{l}L_{k'm_2}^{l'}$$
$$\times\frac{c_{s_1}^l c_{s_2}^{l'}}{2}\frac{s_1!}{(s_1-m)!}\frac{[1+(-1)^{s_1+s_2-m}]}{s_1+s_2+1-m}\left(m_1+m_2+\frac{l+l'-m+1}{2}\right)!. \tag{B 5}$$

The inverse transformation coefficients $(T^{-1})_{pj}^{lkm}$ defined by Eq. (5.14) can be found similarly, yielding

$$(T^{-1})_{pj}^{lkm} = \frac{2^p p!\sqrt{\pi}k!(l+1/2)(l-m)!}{(k+l+1/2)!(l+m)!}T_{lkm}^{pj}. \tag{B 6}$$

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