An efficient treatment of the full Coulomb collision operator with applications

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Outline

- Role of Collisions
- Moment Hierarchy Approach
- Part 1: Linearized Coulomb Collision Operator
  - Electron-Plasma Waves
  - Drift-Waves
- Part 2: Non-Linear Coulomb Collision Operator
  - Small Larmor Radius
  - Arbitrary Larmor Radius
- Closure

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Collisions play an important role in many plasma systems

Laser-Plasma Interactions, Inertial Fusion

- ❑ Collisional damping losses for electron-plasma (Langmuir) waves.

While $\nu_{coll}/\omega_{pe} \sim 10^{-4}$ the time scales for collisionless damping to occur are $\geq 10^3 \omega_{pe}^{-1}$

- ❑ Collisions lead to particle detrapping, even at low/intermediate collisionalities

Need to incorporate collisional effects on

linear and non-linear wave-particle resonances

Collisions play an important role in many plasma systems

Magnetic Fusion

- Collisions influence the level of particle confinement and turbulent transport
- Future fusion reactors will have plasmas with different collisionality regimes

ITER: $T_e \sim 10 \text{ eV} - 10 \text{ keV} \quad n \sim 10^{18} - 10^{20} \text{ m}^{-3}$ → $\dfrac{\lambda_{\text{mfp}}}{R} \sim 10^{-1} - 10^{3}$

Need for a predictive model with collisions at

arbitrary collisionalities

Why are collisions so hard to model?

Full Coulomb Collision Operator

$$\frac{df}{dt} = C(f_a, f_b) = \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{A}_{ab} f_a + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{D}_{ab} f_a) \right]$$

Friction and diffusion coefficients

$$\mathbf{A}_{ab} = -\left(1 + \frac{m_a}{m_b}\right) \frac{\partial \mathbf{D}_{ab}}{\partial \mathbf{v}}$$

$$\mathbf{D}_{ab} = -\nu_{ab} \frac{\partial^2 G_b}{\partial \mathbf{v} \partial \mathbf{v}}$$

Rosenbluth potentials

$$\nabla_v^2 G_b = H_b$$

$$H_b = -\frac{1}{4\pi} \int \frac{f_b}{\mathbf{v} - \mathbf{v}'} d\mathbf{v}'$$

Why are collisions so hard to model?

$$C(f_a, f_b) = \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{A}_{ab} f_a + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{D}_{ab} f_a) \right]$$

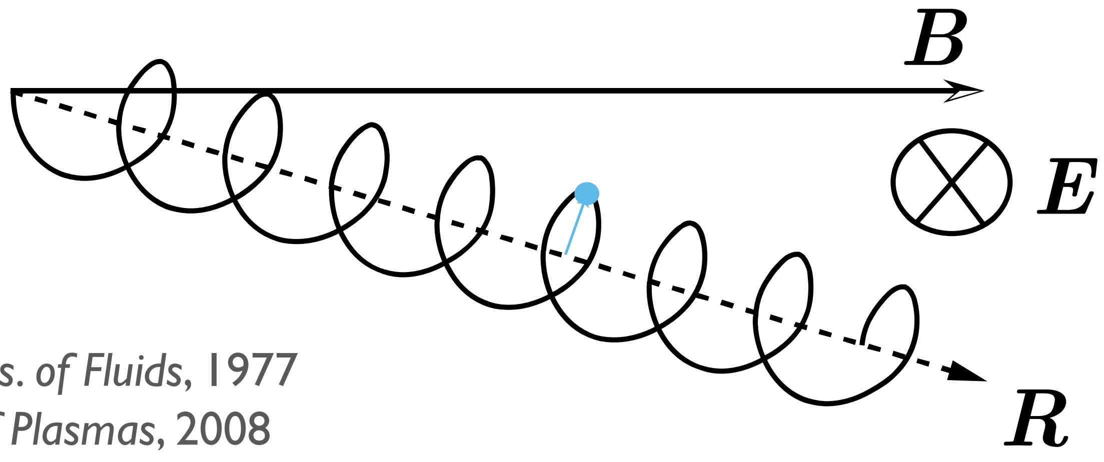
Difficulties associated with the Coulomb operator

- ❑ Bilinear
- ❑ Integro-differential
- ❑ Tensorial
- ❑ $1/\mathrm{v}$ dependence

Often leads us to consider simplified models (fluid equations, linearized operators, simplified/ad-hoc collision operators, etc..)

Additional difficulty: Magnetized Plasmas

- ❑ Particles in a magnetized plasma have drifts of the guiding center position R



- ❑ Effort on Linearized Operators

Catto & Tsang, *Phys. of Fluids*, 1977
Abel et. al., *Phys. of Plasmas*, 2008
Li & Ernst, *Phys. Rev. Lett.*, 2011

How to describe particle collisions non-linearly as a function of R with different Larmor radii at arbitrary collisionalities?

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Simplification of the Collision Operator

Expand the Rosenbluth potentials

$$H(F) \sim \int \frac{F(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} d\mathbf{v}'$$

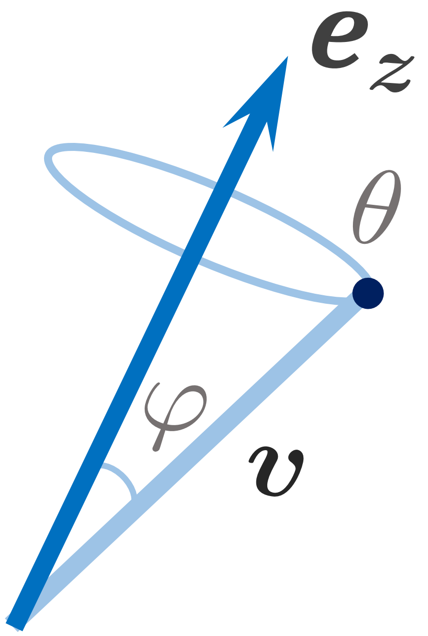
Electrostatic potential of a charge distribution F in velocity space

in a Taylor series

$$\frac{1}{|\mathbf{v}-\mathbf{v}'|} = \begin{cases} \sum_l \frac{(-\mathbf{v}')^l}{l!} \cdot \partial^l_{\mathbf{v}} \left(\frac{1}{v}\right), & v' \leq v \\ \sum_l \frac{(-\mathbf{v})^l}{l!} \cdot \partial^l_{\mathbf{v}'} \left(\frac{1}{v'}\right) & v < v' \end{cases}$$

and use a cylindrical coordinate system

$$(-1)^l v^{l+1} \left(\frac{\partial}{\partial \mathbf{v}}\right)^l \frac{1}{v} \sim \sum_{m=-l}^{l} Y_{lm}(\varphi, \theta) \hat{\mathbf{e}}^{lm}$$



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Multipole expansion of f

$$f \sim \sum_{m,l} N^{lm}(\mathbf{r}) Y_{lm}(\varphi, \theta)$$

and use a cylindrical coordinate system

$$(-1)^l v^{l+1} \left(\frac{\partial}{\partial \mathbf{v}}\right)^l \frac{1}{v} \sim \sum_{m=-l}^{l} Y_{lm}(\varphi, \theta) \hat{\mathbf{e}}^{lm}$$

Method – Turn Kinetic Eq. Into a Hierarchy of Fluid-Like Eqs.

Solve for the multipoles of f using

3D Moment Hierarchy Equations

$$\frac{dn}{dt} = \ldots$$

$$\frac{dv}{dt} = \ldots$$

$$\frac{dT}{dt} = \ldots$$

$$\ldots$$

$$\frac{df}{dt} = C$$

Retain necessary kinetic effects and no more

Advantages of a Moment Hierarchy Model

Set of fluid-like equations with reasonable computational cost

Tune the number of moments according to the desired level of accuracy (function of collisionality)

Reduce to the fluid model at high collisionality

3 Steps To Build a Moment Hierarchy Model

1. Choose an orthogonal polynomial basis for F

$$F = F_M \sum_{p,j} N^{pj}(\mathbf{R}) H_p(v_{\|}) L_j(\mu)$$

2. Project kinetic equation with collisions onto basis

$$\int (\text{Kinetic Eq.}) H_p L_j dv_{\|} d\mu$$

3. Convert spherical harmonics to new basis and obtain moment-hierarchy

$$\frac{dN^{pj}}{dt} = \int (\text{Collision op.}) H_p L_j dv_{\|} d\mu = C^{pj}$$

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Linear Theory of Electron-Plasma Waves (EPW)

$$f = f_M + \delta f$$ Small electron perturbations from a homogeneous equilibrium

$$\delta f_i = 0$$ Maxwellian background of ions

Boltzmann Equation (Fourier transformed and averaged over θ)

$$\frac{\partial \langle \delta f_k \rangle}{\partial t} + ikv_z \langle \delta f_k \rangle + ikv_z \frac{4\pi e^2 \delta n_k}{k^2 T_0} = \frac{\langle C(f_M \delta f_k) \rangle}{f_M}$$

How is Landau damping affected by collisions?

Do the roots of the collisionless dispersion relation change significantly?

Build an EPW moment-hierarchy model using the 3 steps

EPW Moment Hierarchy Equation

$$i\frac{\partial N^{pj}}{\partial t} = \sqrt{\frac{p+1}{2}}N^{p+1j} + \sqrt{\frac{p}{2}}N^{p-1j} + \frac{N^{00}}{k^2\lambda_D^2}\frac{\delta_{p,1}\delta_{j,0}}{\sqrt{2}} + iC^{pj}$$

Time Evolution | Phase-Mixing | Electric Force | Collisions

R. Jorge et. al., JPP **85** (2), 2019

t normalized to kv_{the}
λ_D = Debye length

EPW Collision Term

Expression for the moments of the linearized Coulomb operator

Multipole Moments

$$C^{pj} = \sum_b \nu_b \sum_{s,l,t} \sigma^{pj}_{slt} \left(N^{ls}_a A^{lts}_{ab} + N^{ls}_b B^{lts}_{ab} \right)$$

Collision Frequency

Numerical Coefficient

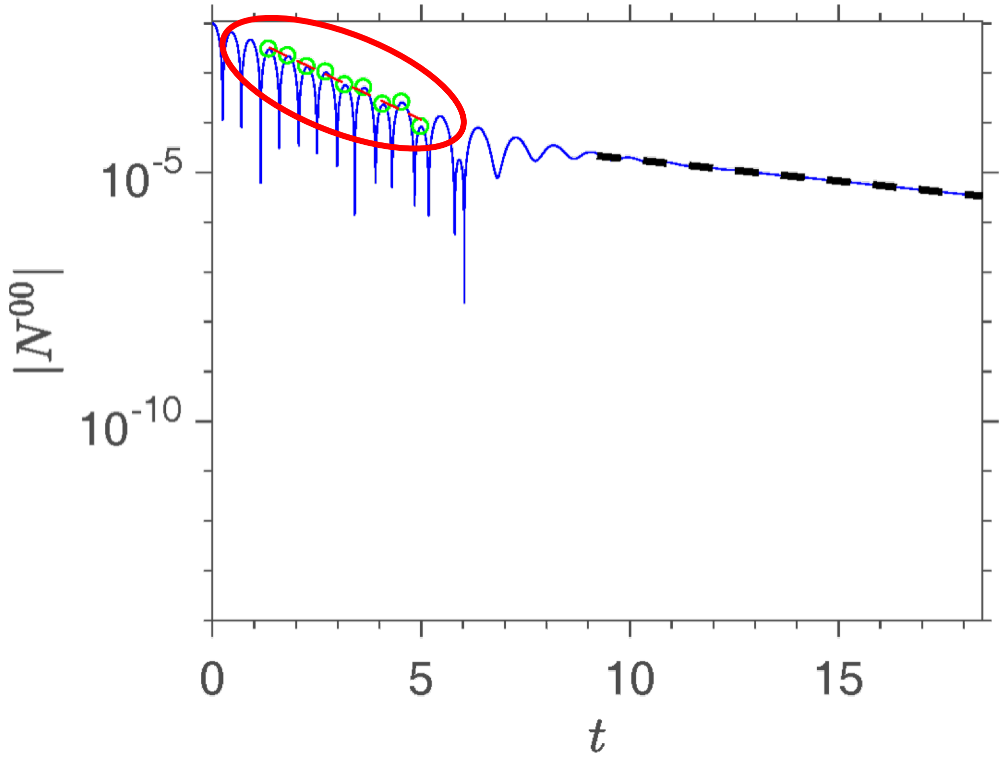
Test-Particle Coefficient

Field-Particle Coefficient

Valid for zero or small Larmor radius up to $O(k_\perp^2 \rho_i^2)$

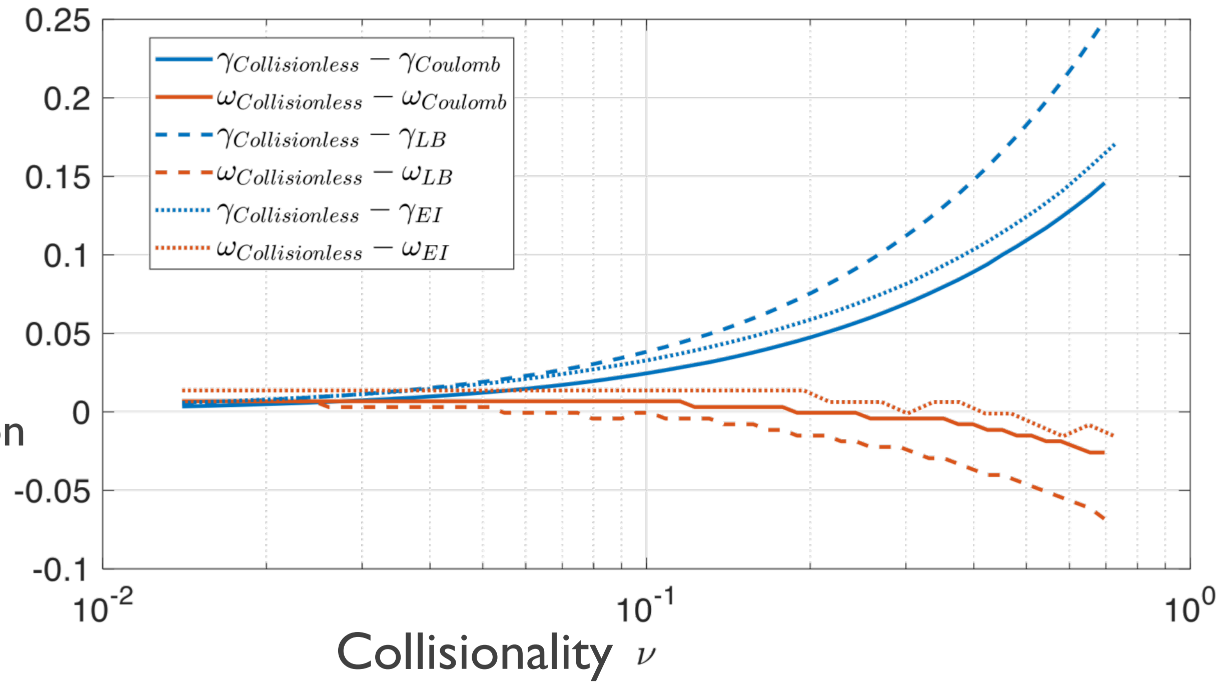
Conserves particle, momentum and energy

Damping at Arbitrary Collisionality



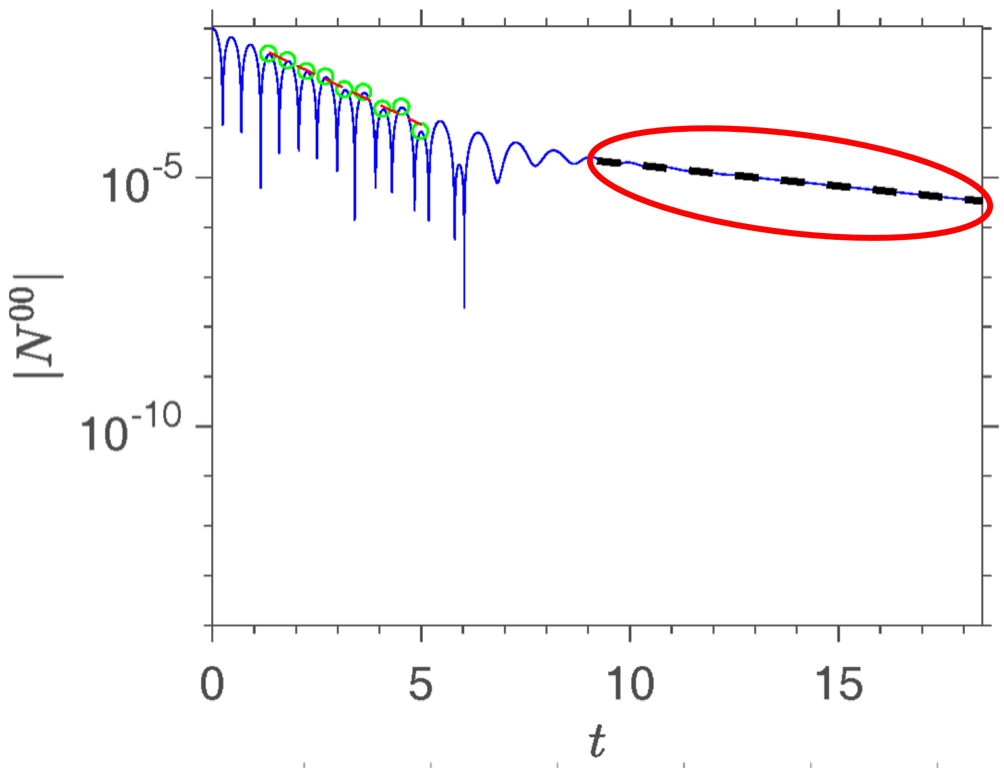
Parameter scan

Three collision operators

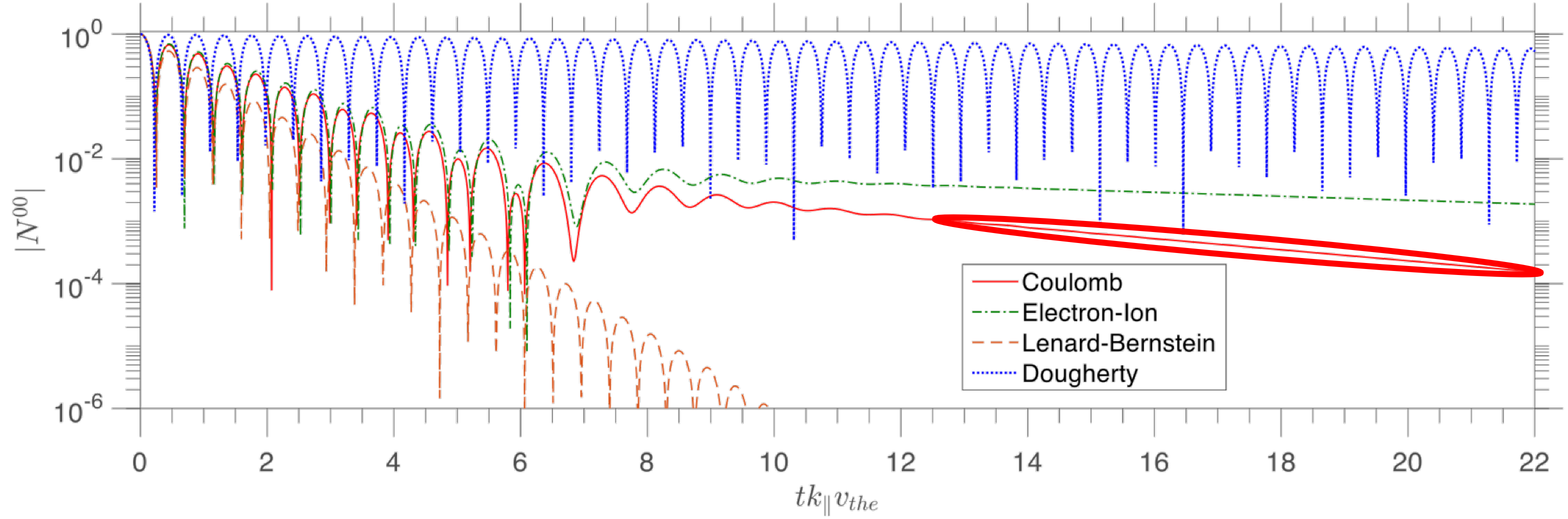


- Collisional and Landau Damping computed for the first time with the full Coulomb collision operator
- Only ~20 Hermite and ~2 Laguerre polynomials needed

Entropy mode at High Collisionality



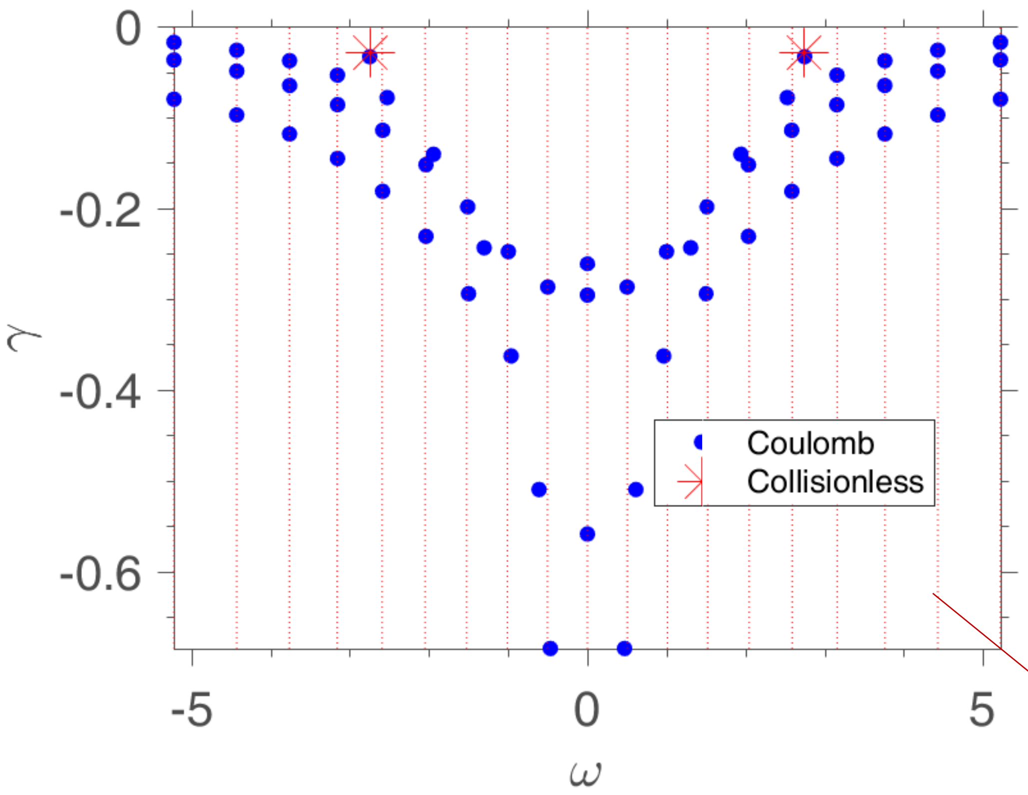
- Electron-ion collisions known to yield long-time zero-frequency behaviour
- Full Coulomb collisions increase the entropy mode damping rate



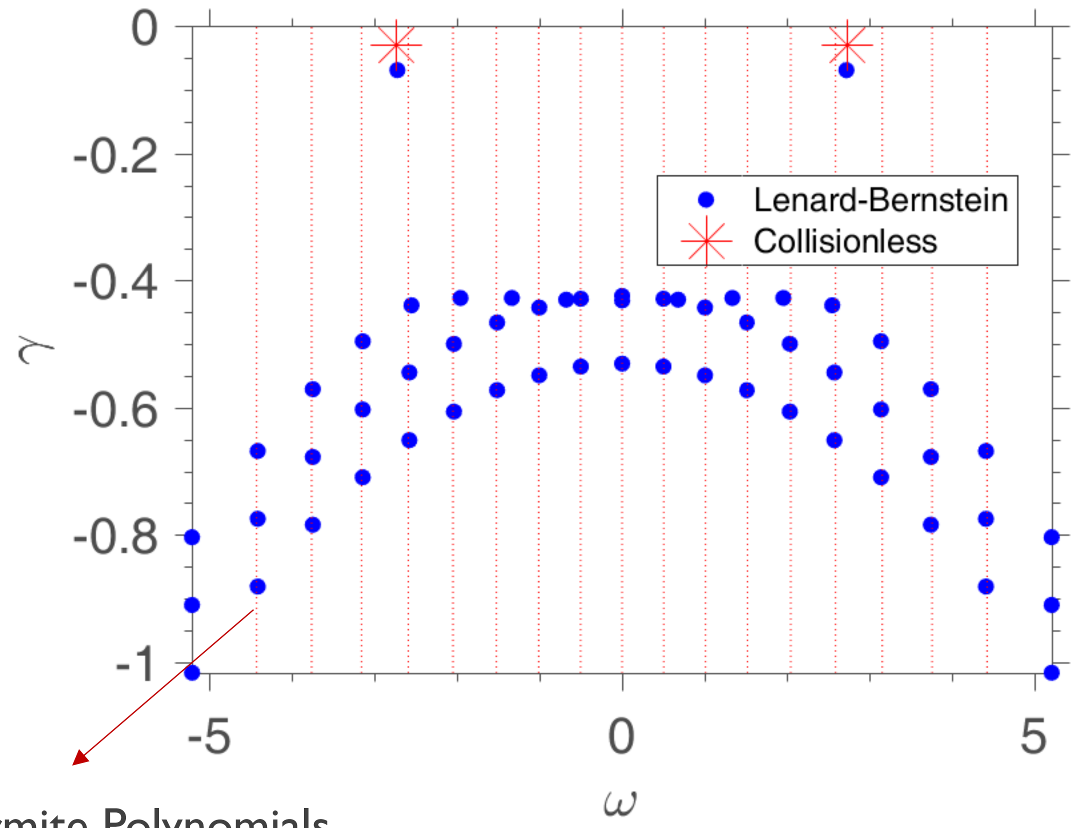
Entropy mode not present in simplified collision operators

EPW Eigenmode Spectrum

Full Coulomb Collision Operator



Simplified Lenard-Bernstein Operator



Zeros of Hermite Polynomials

Difference between eigenmode spectra may lead to different nonlinear evolution of EPW

Linear Theory of the Drift-Wave Instability

- Two spatial and two velocity dimensions in slab geometry
- Electron and ion perturbations
- Finite background density gradient

Boltzmann Equation

$$\frac{\partial F_a}{\partial t} + \left(v_\parallel \mathbf{b} + \mathbf{v}_E\right) \cdot \nabla F_a - q_a \nabla_\parallel \phi \frac{m_i}{m_a} \frac{\partial F_a}{\partial v_\parallel} = \sum_b \langle C_{ab} \rangle$$

What is the peak growth rate at finite collisionality?

How many moments do we need to describe the DW instability?

Build a DW moment-hierarchy model using the 3 steps

DW Moment Hierarchy Equation

ExB Advection

$$\underbrace{i\frac{\partial N_a^{pj}}{\partial t}}_{\text{Time Evolution}} = \underbrace{k_\parallel \frac{\sqrt{\tau_a}}{\sigma_a}\left(\sqrt{p+1}N_a^{p+1j} + \sqrt{p}N_a^{p-1j}\right)}_{\text{Phase-Mixing}} - \phi\left(k_\perp \delta_{p,0} - \frac{q_a k_\parallel}{\sqrt{\tau_a}\sigma_a}\delta_{p,1}\right)\delta_{j,0} + \underbrace{iC^{pj}}_{\text{Collisions}}$$

Time Evolution Phase-Mixing Electric Force Collisions

R. Jorge *et al.*, PRL **121** (16), 2018

t normalized to c_s/L_n
$\tau_a = T_a/T_e$
$\sigma_a = m_a/m_i$

Build a DW moment-hierarchy model using the 3 steps

DW Moment Hierarchy Equation

ExB Advection

$$i\frac{\partial N_a^{pj}}{\partial t} = k_\parallel \frac{\sqrt{\tau_a}}{\sigma_a}\left(\sqrt{p+1}N_a^{p+1j} + \sqrt{p}N_a^{p-1j}\right) - \phi\left(k_\perp \delta_{p,0} - \frac{q_a k_\parallel}{\sqrt{\tau_a}\sigma_a}\delta_{p,1}\right)\delta_{j,0} + iC^{pj}$$

Time Evolution

Phase-Mixing

Electric Force

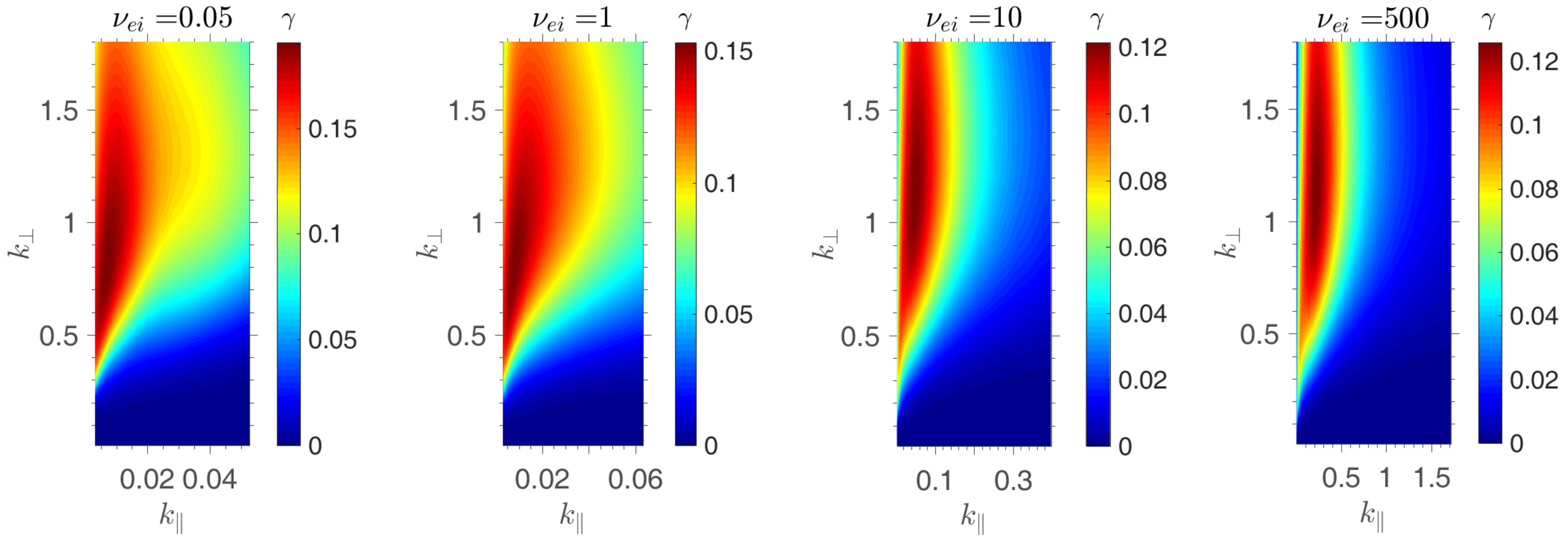
Collisions

Precomputed in the EPW case

Add ion perturbations

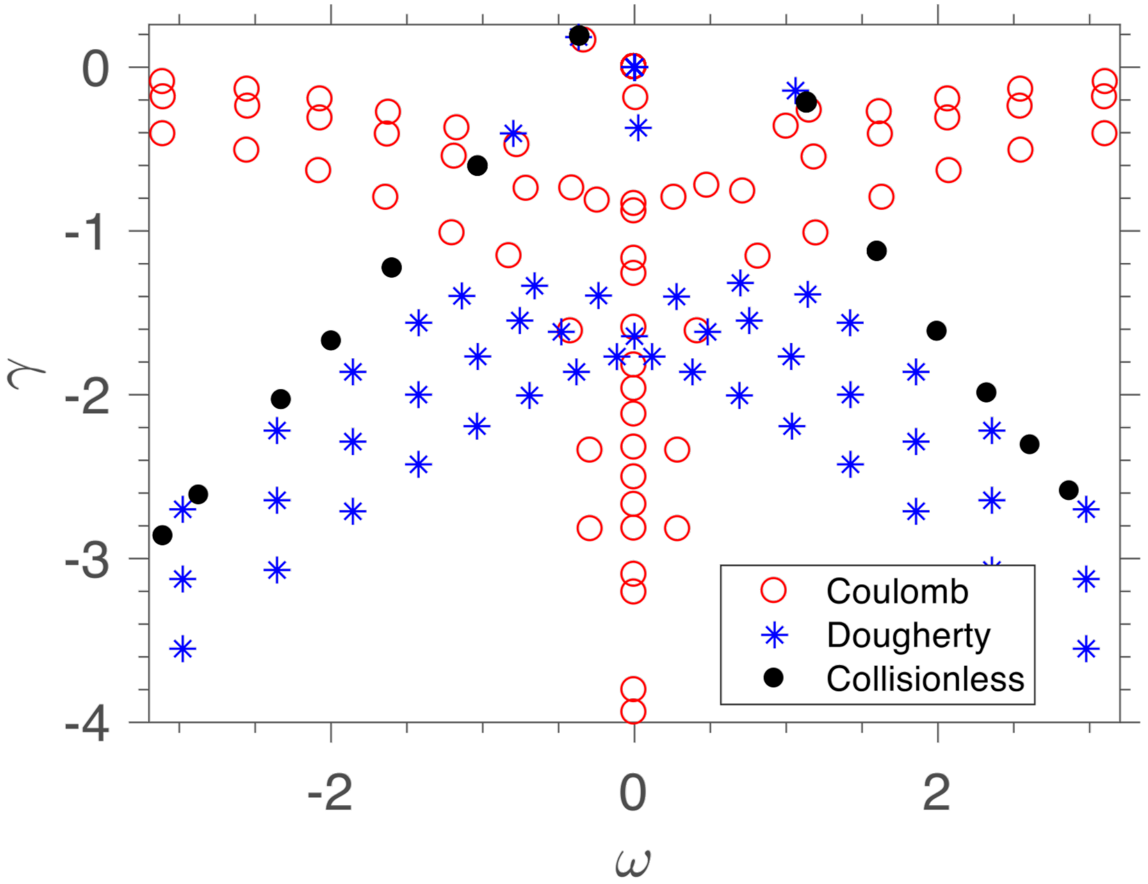
First Study on Coulomb Collision Effects on the Drift-Wave Instability

Peak growth rate γ at arbitrary collisionality



- Saturation of $k_\perp$ and increase of $k_\parallel$ for higher collisionality

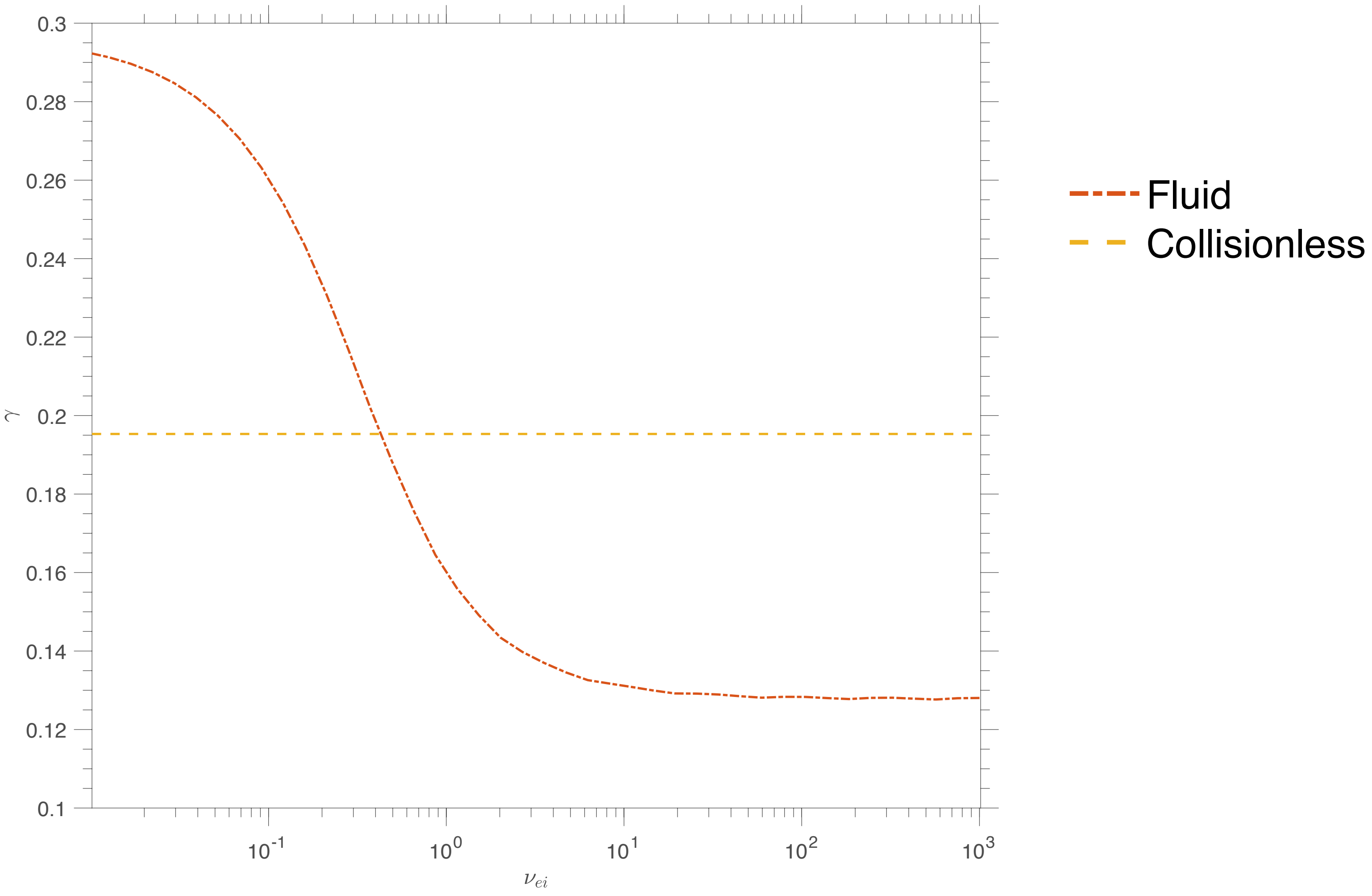
Eigenmode Spectra



Important deviations between full Coulomb collision operator and presently used collision operators

Convergence Study

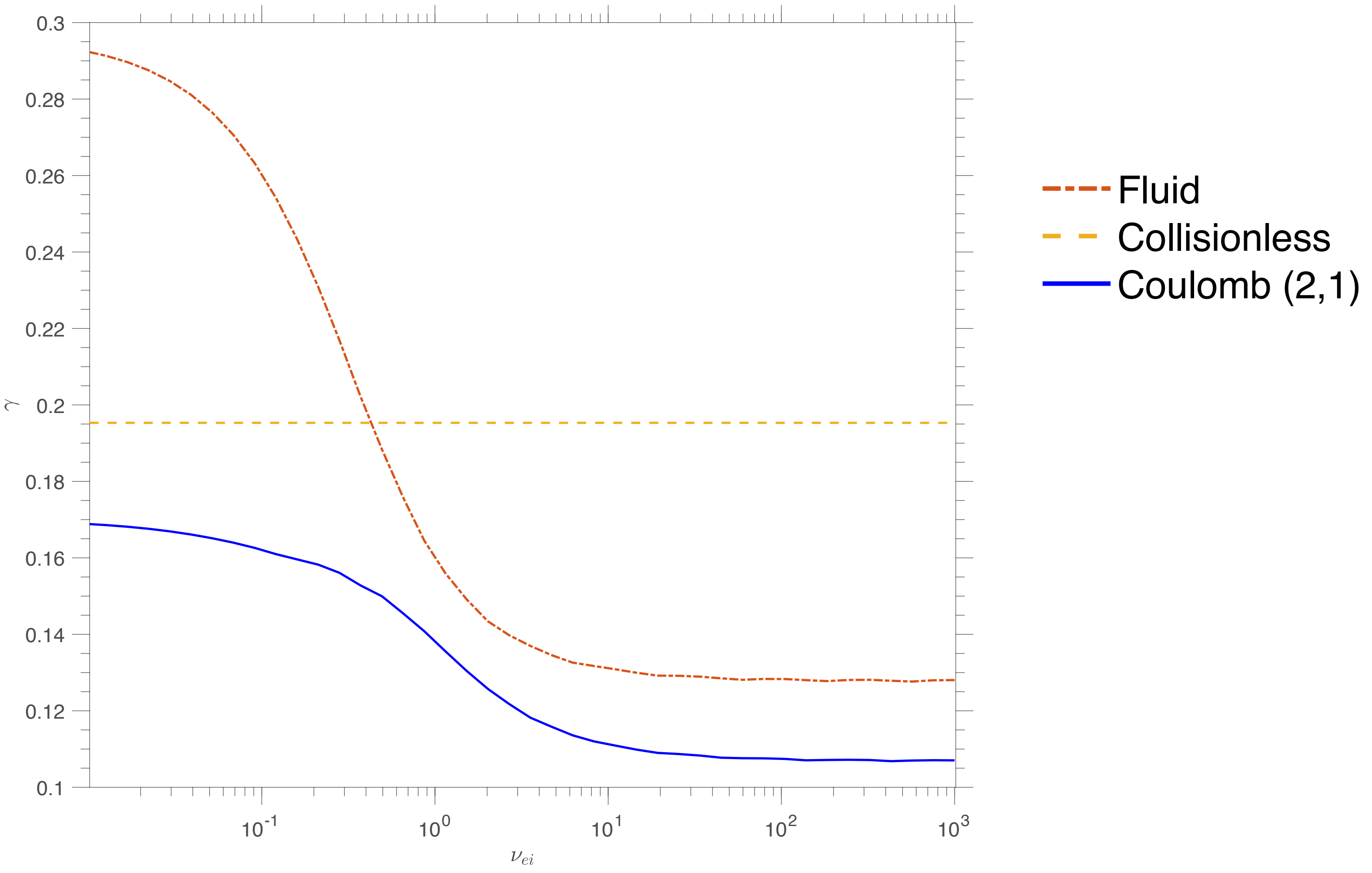
Peak Growth Rate



Collision Frequency

Convergence Study

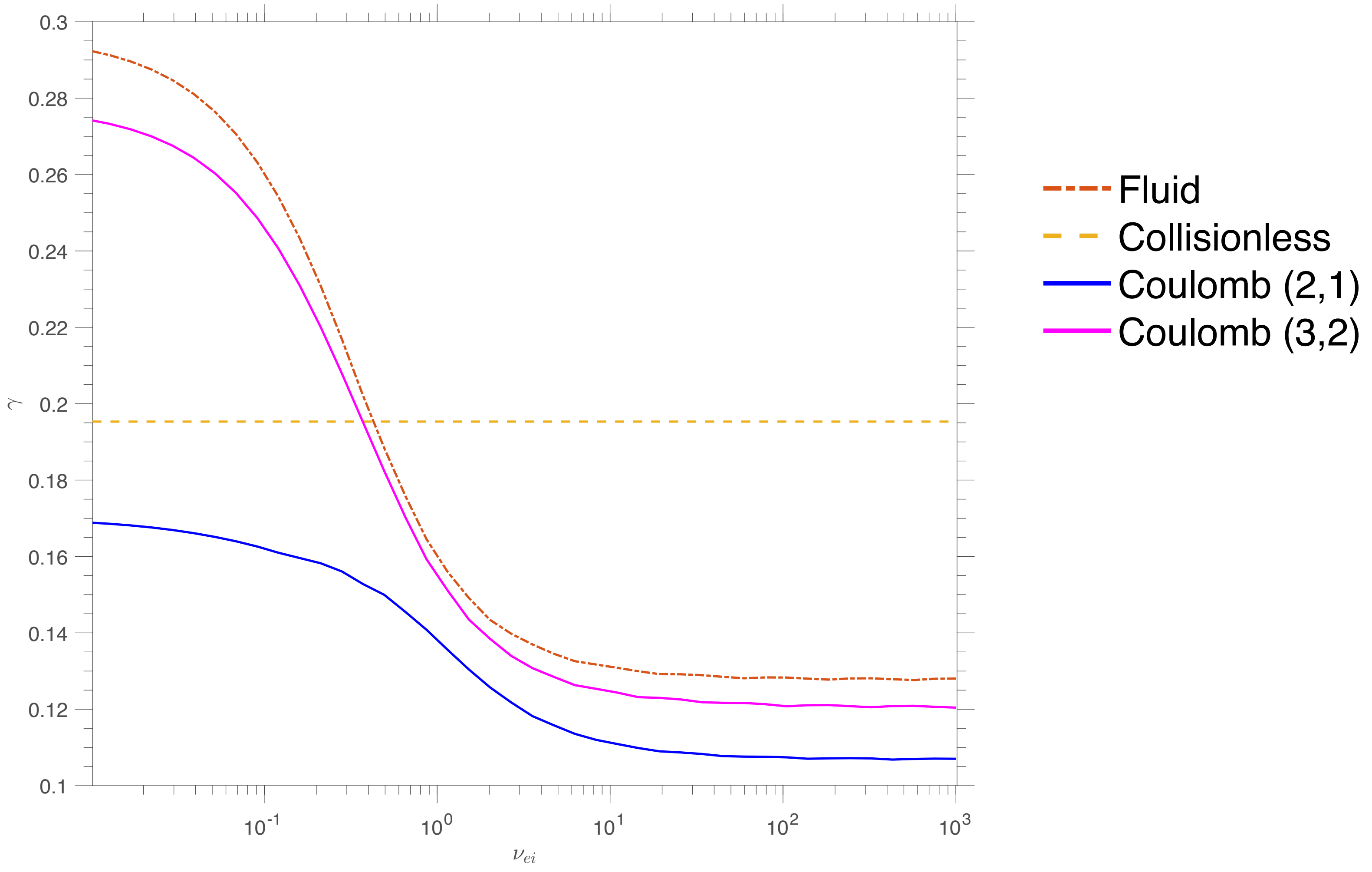
Peak Growth Rate



Collision Frequency

Convergence Study

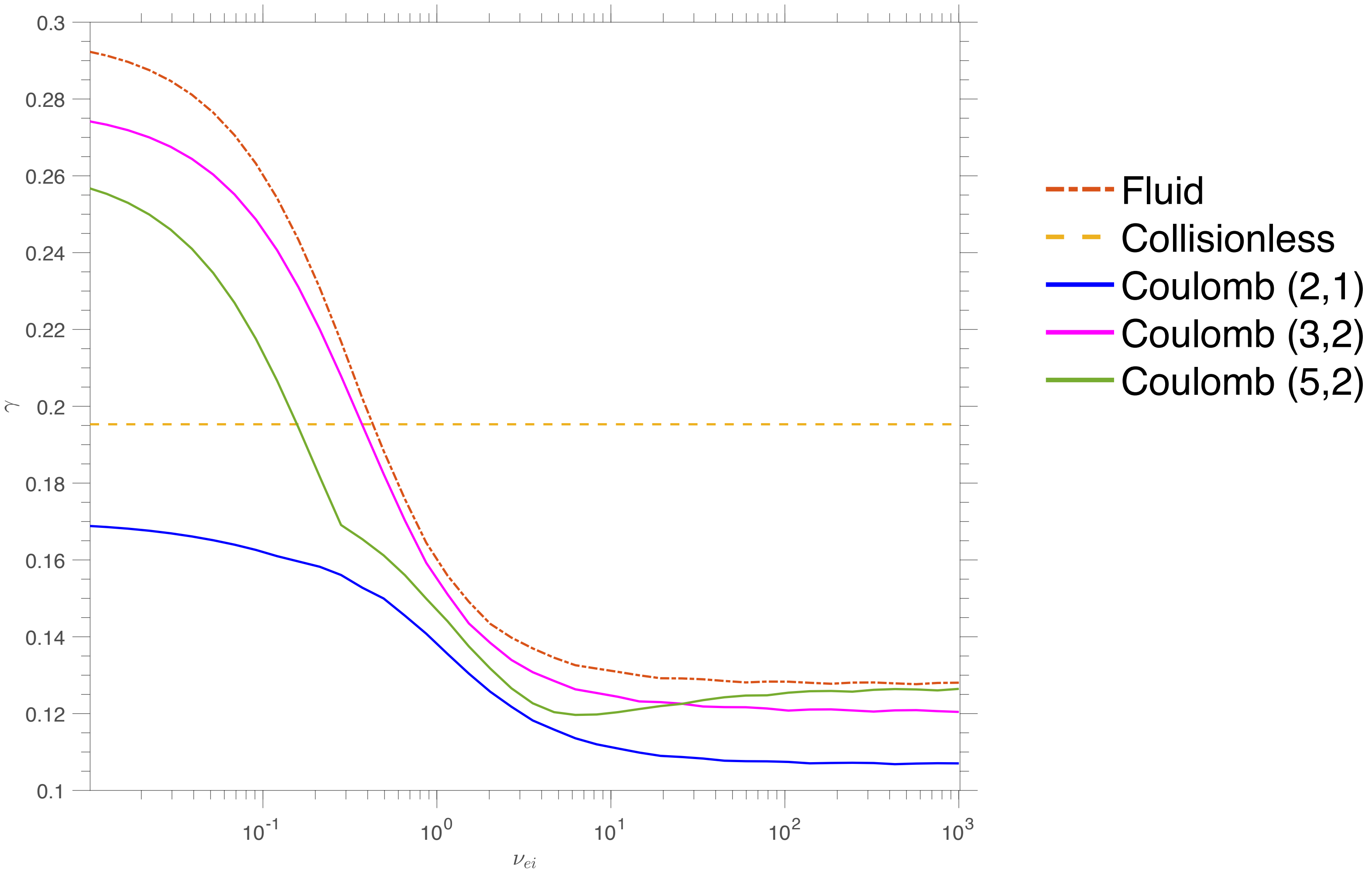
Peak Growth Rate



Collision Frequency

Convergence Study

Peak Growth Rate



Collision Frequency

Convergence Study

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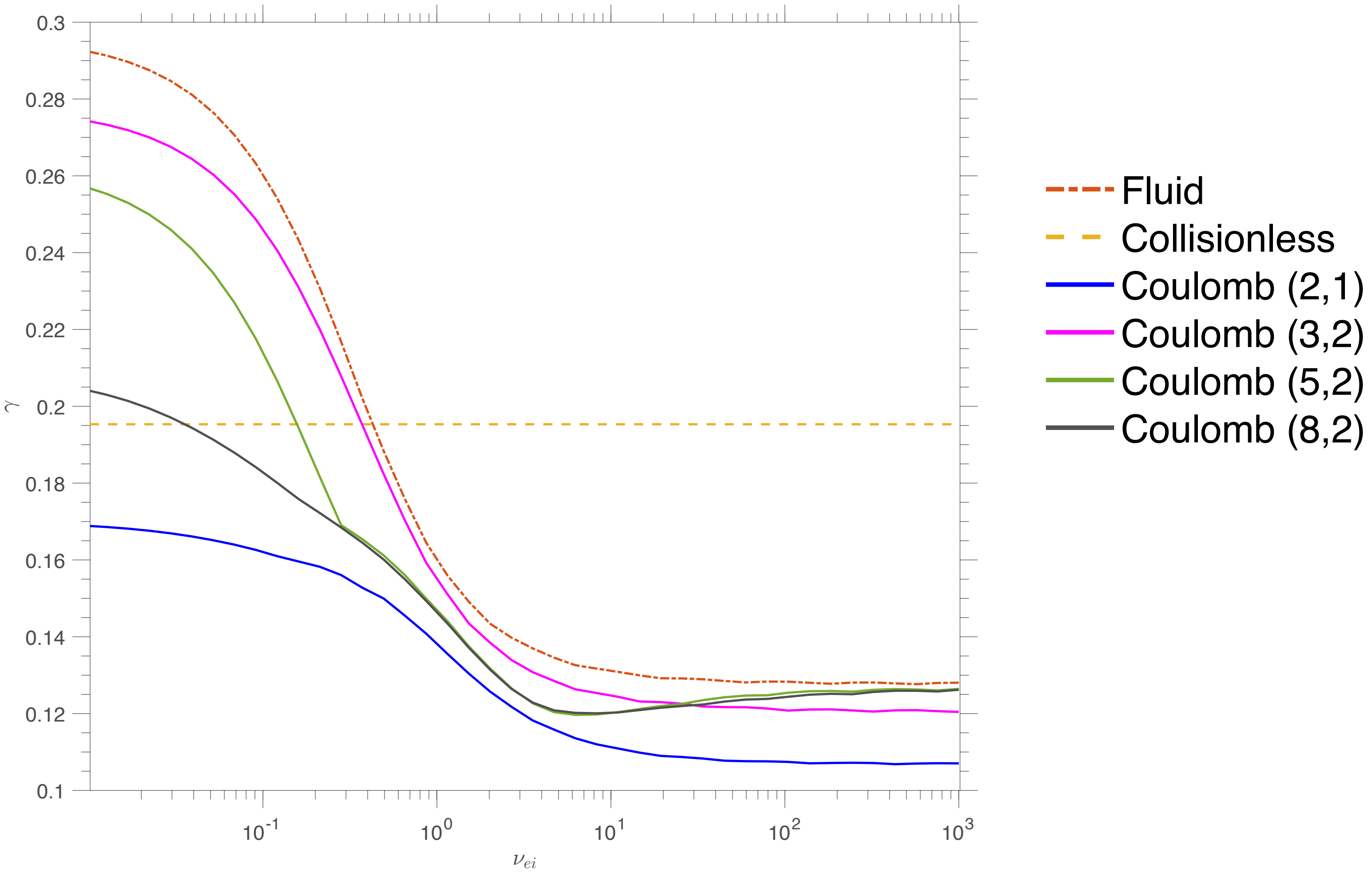

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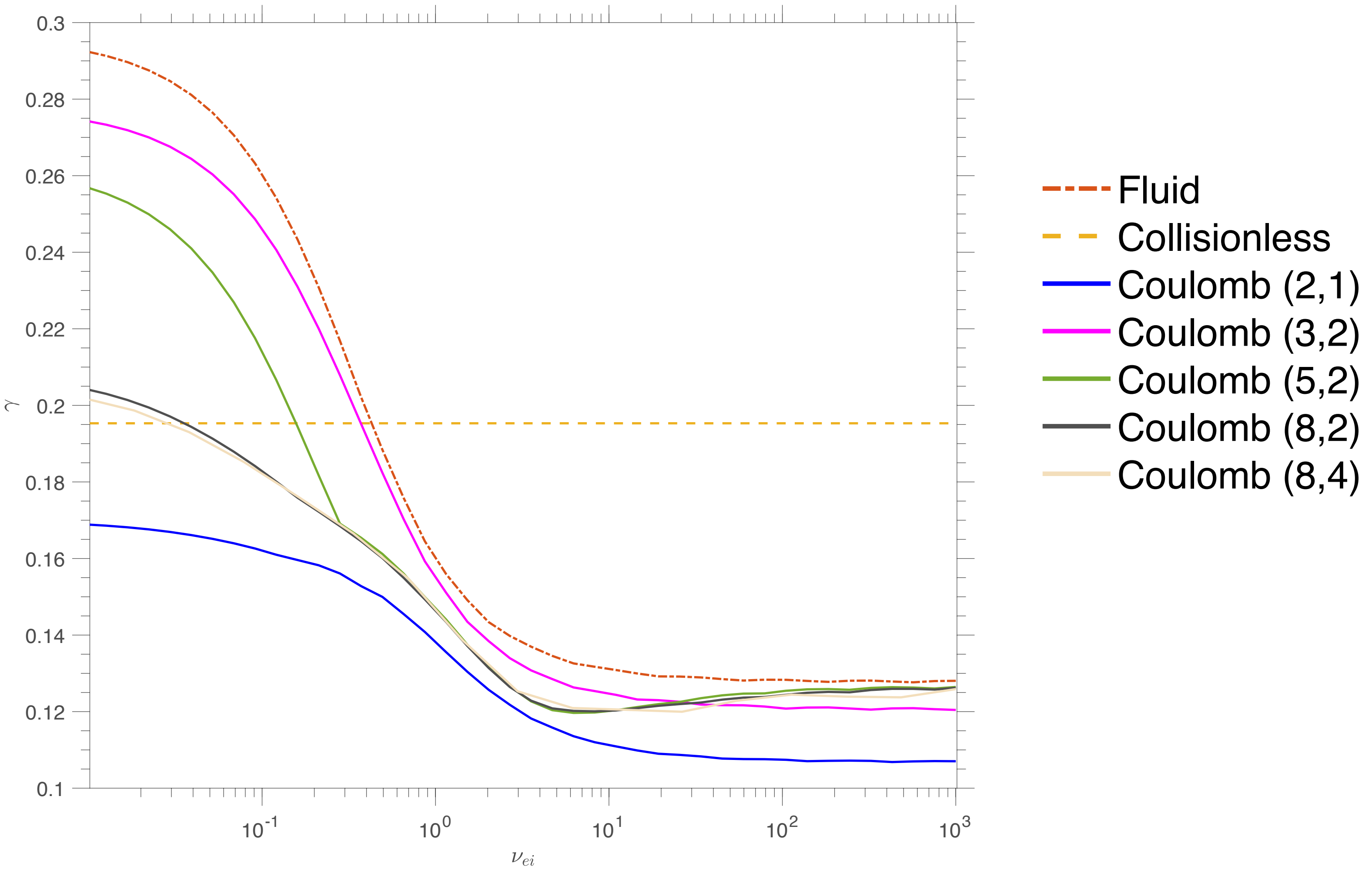

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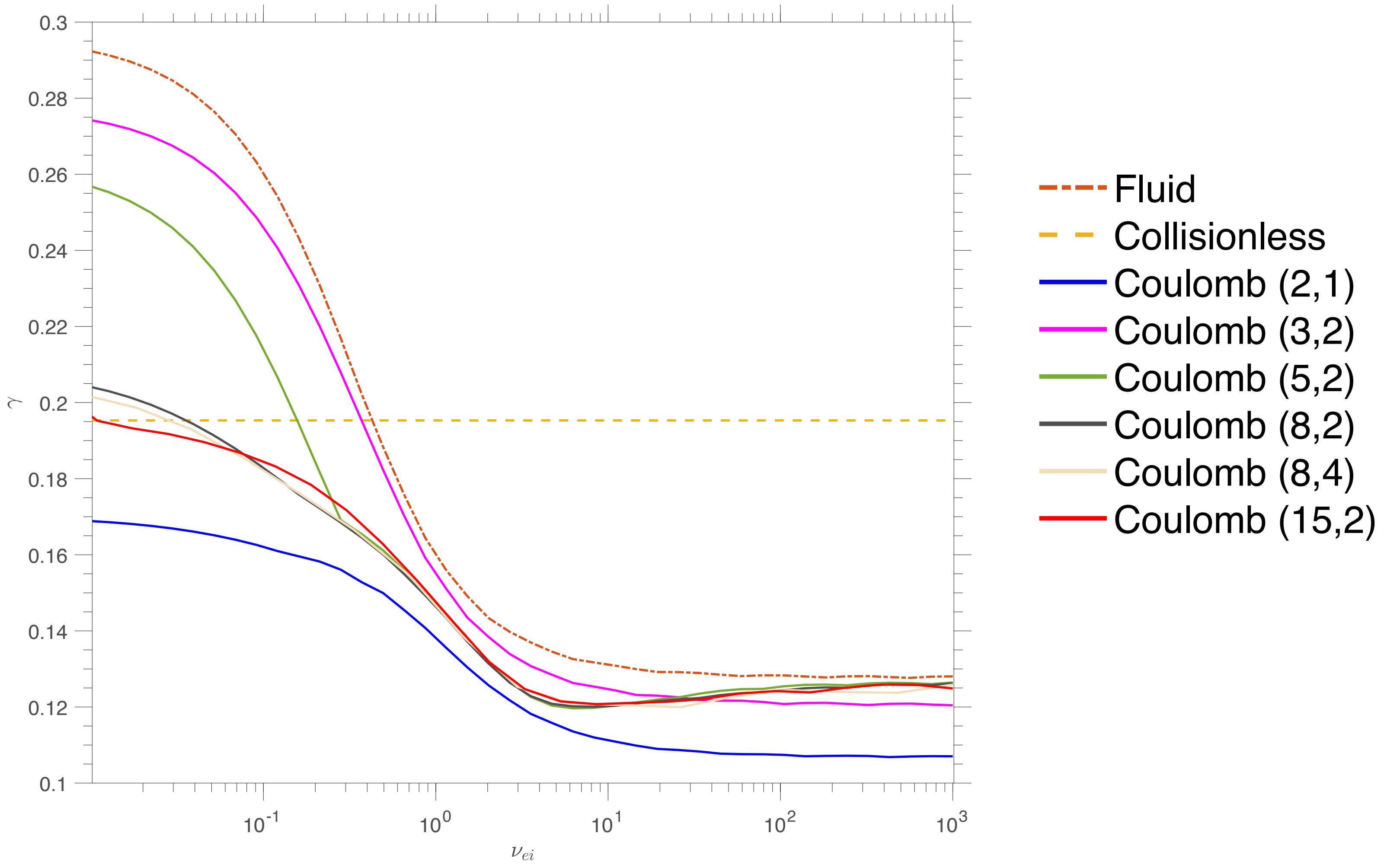

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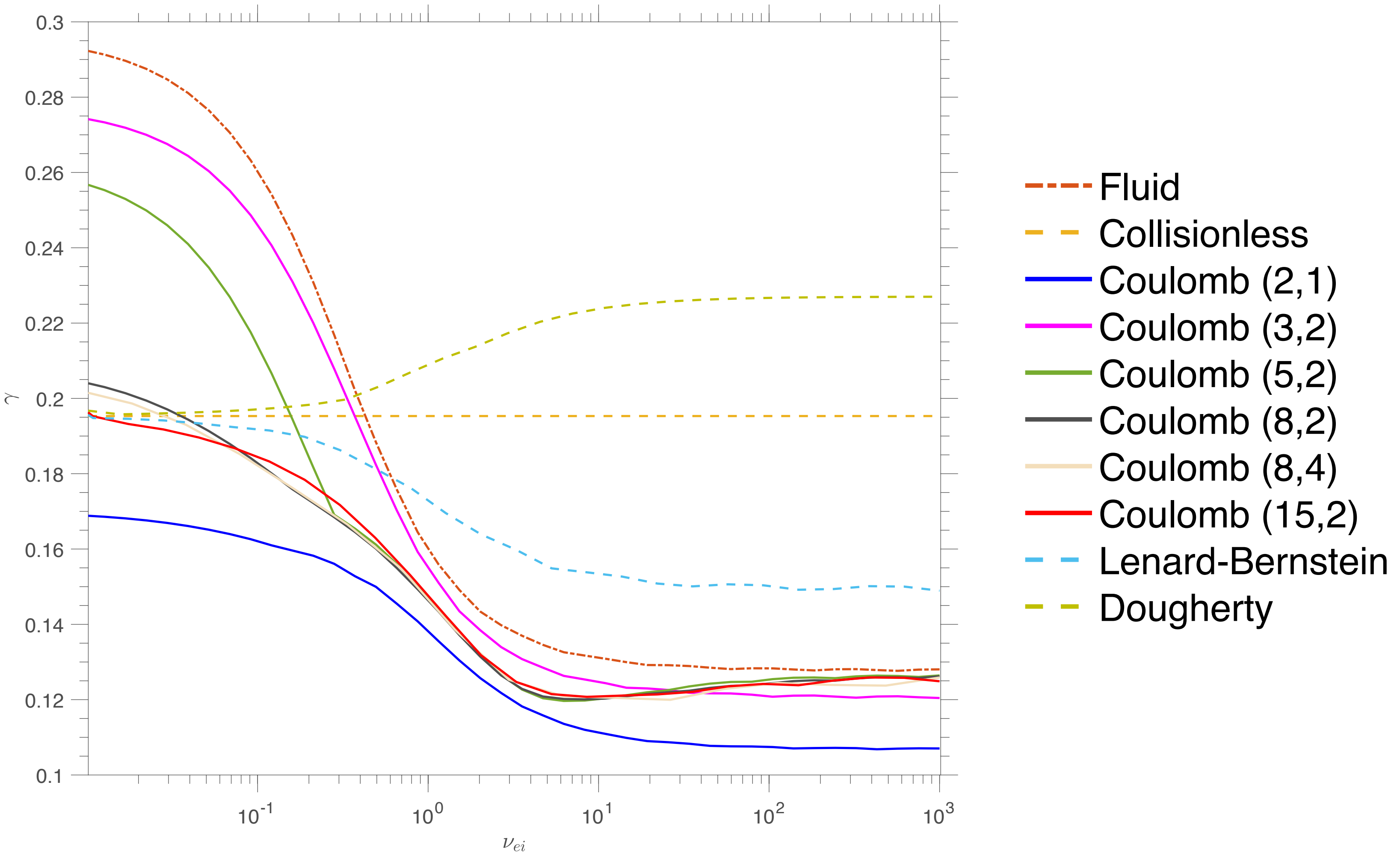

Collision Frequency

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Part 2: Non-Linear Model

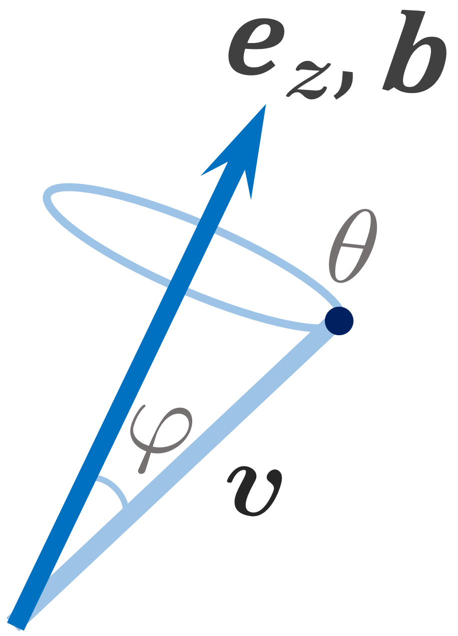
Expression for the moments of the non-linear Coulomb operator

Multipole Moments

$$C^{pj} = \sum_b \nu_b \sum_{ls} \sum_{tu} C^{pj}_{ablstu} \boxed{N_a^{ls} N_b^{tu}}$$

Collision Frequency

Function of the Larmor radius, mass and temperature ratios



Valid at arbitrary Larmor radii

R. Jorge et. al., JPP *accepted*, 2019, arXiv:1906.03252

Small or Zero Larmor Radius

$$C[F] \sim C[Y_{lm}] \sim Y_{lm}$$

Averaging Procedure

$$\langle Y_{lm}(\varphi, \theta) \rangle \longrightarrow H_p(v_{||}) L_j(\mu)$$

After integration

$$\downarrow$$

$C^{pj} =$ moments N^{pj} of F

We are now able to

- Model scrape-off layer of fusion devices in the H-mode regime
- Study non-linear collision effects in unmagnetized plasmas

R. Jorge et. al., JPP **83** (6), 2017

Large Larmor Radius

Averaging Procedure in Fourier Space

$$\langle f(\mathbf{x})\rangle = \int d\mathbf{k} \left(\int d\theta e^{i\mathbf{k}\cdot\mathbf{x}} \right) f(\mathbf{k}) \longrightarrow J_0(k_\perp \rho) f(\mathbf{k})$$

Basis Transformation

$$J_0(k_\perp \rho) = \sum_j K_j(k_\perp) L_j(\mu)$$

The kernel function K_j allows us to accurately model finite Larmor radius effects

$$K_j(x) = \frac{1}{j!}\left(\frac{x}{2}\right)^{2j} e^{-x^2/4}$$

Small contribution from large x and large j

Large Larmor Radius

$$\langle Y_{lm}(\varphi,\theta)\rangle \longrightarrow K_j(k_\perp\rho)H_p(v_{||})L_j(\mu)$$

After integration

$$C^{pj} = K_j(k_\perp\rho)\times\text{moments } N^{pj}$$

- Able to model the whole periphery of fusion devices
- First non-linearized gyrokinetic Coulomb collision operator

R. Jorge et. al., JPP *accepted*, 2019
arXiv:1906.03252

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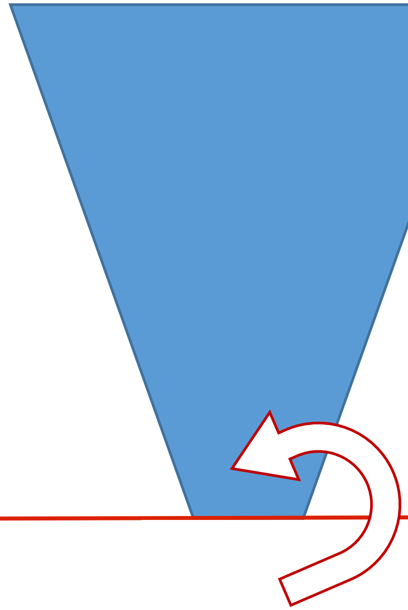
How to close the moment-hierarchy?

Compare $\dfrac{\partial N^{pj}}{\partial t}$ with C^{pj}

$(0,0)$
$(1,0)$
$(0,1)$
$(2,0)$
$\vdots$



The closed formula for C^{pj} allows us to estimate the collisional terms at arbitrary collisionalities

Dissipation range

Use result of dissipation range in previous phase-mixing terms

Semi-collisional closure

Closure at high-collisionality

$$F = F_M(1 + \delta F) \quad \text{with} \quad \delta F = \sum_{p,j} N^{pj} H_p L_j$$

Semi-collisional closure

$$\delta F \sim \frac{\lambda_{\text{mfp}}}{L_{\|}}$$

Drift-Reduced Braginskii

equations retrieved

$$N^{30} \sim q_{\|} = -\chi_{\|} \nabla_{\|} T$$

Conclusions

- Now able to model arbitrary collisionalities with unmagnetized and magnetized plasmas
- Retained small and large Larmor radii effects in the collision operator
- Numerical efficiency shown for the linear case

Future

- Study collisional effects on non-linear dynamics of unmagnetized systems (EPW and IAW)
- Model turbulence in magnetized plasmas using a full Coulomb collision operator

R. Jorge et. al., JPP **83** (6), 2017

R. Jorge *et al.*, PRL **121** (16), 2018

R. Jorge et. al., JPP **85** (2), 2019

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