# Moments of L-functions and exponential sums

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Raphaël Pierre ZACHARIAS

Acceptée sur proposition du jury

Prof. D. Testerman, présidente du jury
Prof. Ph. Michel, directeur de thèse
Prof. E. Kowalski, rapporteur
Prof. V. Blomer, rapporteur
Prof. M. Viasovska, rapporteuse

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# Abstract

Let $q > 2$ be a prime number. We obtained in this thesis asymptotic formulae for two different moments of $L$-functions.

In the first part, we study a twisted fourth moment of Dirichlet $L$-functions of the form $\sum_{\chi(q)} |L(\chi, \frac{1}{2})|^4 \chi(\ell_1)\overline{\chi}(\ell_2)$, where the sum runs over characters of $\mathbf{F}_q^\times$ and $\ell_1, \ell_2$ are natural numbers less than $q$. The principal tool is a careful analysis of a shifted convolution problem involving the divisor function and it is made through the spectral theory of automorphic forms.

In the second part, we analyse a generalized cubic moment $\sum_{\chi(q)} L(f \otimes \chi, \frac{1}{2}) L(\chi, \frac{1}{2}) \chi(\ell)$,where $\ell < q$, $f$ is either a cuspidal Hecke eigenform with trivial central character and level dividing $q$, or an Eisenstein series $\chi_1 \boxplus \chi_2$ associated to an arbitrary pair $(\chi_1, \chi_2)$ of multiplicative characters modulo $q$. This requires an extention of the work of Fouvry, Kowalski and Michel on correlation between Hecke eigenvalues of modular forms and a certain class of $q$-periodic functions called algebraic trace functions modulo $q$.

Finally in the last chapter, we combine the first two parts together with the mollification method to obtain some simultaneous nonvanishing results for these families of $L$-functions.

# Résumé

Soit $q > 2$ un nombre premier. Dans cette thèse, nous obtenons des formules asymptotiques pour deux différent moments de fonctions $L$.
Dans la première partie, nous étudions un quatrième moment tordu de fonctions $L$ de Dirichlet de la forme $\sum_{\chi(q)} |L(\chi, \frac{1}{2})|^4 \chi(\ell_1)\overline{\chi}(\ell_2)$, où $\chi$ parcourt les charactères de $\mathbf{F}_q^\times$ and $\ell_1, \ell_2$ sont des nombres naturels plus petit que $q$. L'outil principal est une analyse fine d'un problème de convolution impliquant la fonction diviseur et est réalisé grâce à la théorie spectrale des formes automorphes.
Dans la deuxième partie, nous analysons un moment cubique généralisé de la forme $\sum_{\chi(q)} L(f \otimes \chi, \frac{1}{2})L(\chi, \frac{1}{2})\chi(\ell)$, où $\ell < q$, $f$ est soit une forme cuspidale de Hecke avec charactère central trivial et de niveau divisant $q$, soit une série d'Eisenstein $\chi_1 \boxplus \chi_2$ associée à une paire de charactères multiplicatif $(\chi_1, \chi_2)$ de module $q$. Cela requiert une extension du travail de Fouvry, Kowalski et Michel sur la correlation entre les valeurs propres de Hecke de formes modulaires et une certaine classe de fonctions $q$-périodique appelé fonctions trace modulo $q$.
Finallement, nous combinons dans le dernier chapitre les deux premières parties à la méthode de mollification pour obtenir certains resultats de non-annulation simultanée pour ces familles de fonctions $L$.

# Contents

# Introduction

The zeros of automorphic $L$-functions on the critical line have received considerable attention these last years [12, 17, 48, 43, 35]. In particular, at the central point $s = \frac{1}{2}$, an $L$-function is expected to vanish only for either a good reason or a trivial reason. For example, if $E$ is an elliptic curve defined over $\mathbf{Q}$ and $L(E,s)$ is its associated $L$-function, then according to the Birch and Swinnerton-Dyer conjecture, $L(E, \frac{1}{2}) = 0$ if and only if the group of $\mathbf{Q}$-points $E(\mathbf{Q})$ has positive rank. A trivial reason is for instance when the sign of the functional equation is $-1$, which is the case if the $L$-function is attached to an odd Hecke-Maass form.

A typical approach in the study of nonvanishing problems is to consider a family of $L$-functions $\{L(\pi, \frac{1}{2})\}$ for $\pi$ varying in some finite set $\mathcal{A}$ and try to give a lower bound for the proportion of $\pi \in \mathcal{A}$ such that $L(\pi, \frac{1}{2}) \neq 0$ as $|\mathcal{A}| \to \infty$. In [31], H. Iwaniec and P. Sarnak examined $L(\chi, s)$ at $s = \frac{1}{2}$ as $\chi$ ranges over all primitive Dirichlet characters modulo an integer $q > 2$. They proved that at least $\frac{1}{3}$ of the central values $L(\chi, \frac{1}{2})$ are not zero as $q \to \infty$. This proportion has been slightly improved to 0.3411 by H.M. Bui [11] and finally to $\frac{3}{8}$ by R. Khan and H.T. Ngo [40] with the restriction to prime moduli $q$.

## 0.1 Simultaneous nonvanishing and the mollification method

In [50], P. Michel and J. Vanderkam considered simultaneous nonvanishing problems : given three distinct Dirichlet characters $\chi_1, \chi_2, \chi_3$ of fixed modulus $D_1, D_2, D_3$ (satisfying some technical conditions), they proved that a positive proportion of holomorphic primitive Hecke cusp forms $f$ of weight 2, prime level $q$ and trivial nebentypus are such that the product $L(f \otimes \chi_1, \frac{1}{2}) L(f \otimes \chi_2, \frac{1}{2}) L(f \otimes \chi_3, \frac{1}{2})$ is not zero for sufficiently large $q$ (in terms of $D_1, D_2, D_3$). They derived various arithmetic applications, especially the existence of quotients of $J_0(q)$ of large dimension satisfying the Birch and Swinnerton-Dyer conjecture over cyclic number fields of degree less than 5.

In this thesis, we let $q > 2$ a prime number, $\chi_1, \chi_2$ be arbitrary Dirichlet characters of modulus $q$ [1], $\chi_0$ be the trivial character modulo $q$, $\mathcal{D}(q)$ (resp. $\mathcal{D}_{\chi_1,\chi_2}(q)$) the set of primitive characters modulo $q$ (resp. different from $\overline{\chi}_1, \overline{\chi}_2$) and $f$ a cuspidal Hecke eigenform for $\mathrm{SL}_2(\mathbf{Z})$ (holomorphic or Maass). We are interested in the distribution of the values of the two families

[1] We point out that there is no additional difficulty by considering fixed $\chi_1, \chi_2$ of conductors $D_1, D_2 < q$.

$$\{L(\chi,\tfrac{1}{2})\,L(\chi\chi_1,\tfrac{1}{2})\,L(\chi\chi_2,\tfrac{1}{2})\;:\;\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)\} \tag{0.1}$$

and

$$\{L(f\otimes\chi,\tfrac{1}{2})\,L(\chi,\tfrac{1}{2})\;:\;\chi\in\mathcal{D}(q)\}, \tag{0.2}$$

as $q\to\infty$ through the prime numbers. Set

$$\phi^*(q):=|\mathcal{D}(q)| \text{ and } \phi^*_{\chi_1,\chi_2}(q):=|\mathcal{D}_{\chi_1,\chi_2}(q)|$$

and observe that $\phi^*(q)=q-2$ and $|\phi^*_{\chi_1,\chi_2}(q)-q|\leqslant 4$. We state the two main results of this thesis :

**Theorem 1.** *Let $\varepsilon>0$ be a real number. There exists an explicit absolute constant $c_1>0$ and $Q=Q(\varepsilon)>2$ such that for any prime $q\geqslant Q$ and every Dirichlet characters $\chi_1,\chi_2$ of modulus $q$, we have the lower bound*

$$\left|\left\{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)\;:\;|L(\chi\chi_i,\tfrac{1}{2})|\geqslant\frac{1}{\log q},\; i=0,1,2\right\}\right|\geqslant(c_1-\varepsilon)\phi^*_{\chi_1,\chi_2}(q),$$

*where $\chi_0$ is the trivial character modulo $q$.*

**Theorem 2.** *Let $f$ be a Hecke cusp form for $\mathrm{SL}_2(\mathbf{Z})$ which we assume to satisfy the Ramanujan-Petersson conjecture and let $t_f$ be its spectral parameter. Let $\varepsilon>0$ be a real number. Then there exists an explicit absolute constant $c_2>0$ and $Q=Q(\varepsilon,t_f)>2$ such that for any prime $q\geqslant Q$,*

$$\left|\left\{\chi\in\mathcal{D}(q)\;:\;|L(f\otimes\chi,\tfrac{1}{2})|\geqslant\frac{1}{\log^2 q},\;|L(\chi,\tfrac{1}{2})|\geqslant\frac{1}{\log q}\right\}\right|\geqslant(c_2-\varepsilon)\phi^*(q).$$

We now discuss the general principle that has made the success of many of the papers cited above. It starts with the so called method of moments. Assume that we are interested in the non-zero values of the family (0.1), we can consider the following moment

$$\mathbf{M}^3(\chi_1,\chi_2;q):=\frac{1}{\phi^*_{\chi_1,\chi_2}(q)}\sum_{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)}L(\chi,\tfrac{1}{2})\,L(\chi\chi_1,\tfrac{1}{2})\,L(\chi\chi_2,\tfrac{1}{2}).$$

Noting that the summation can be restricted to the $\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)$ such that the triple product is not zero, we obtain using the generalized Hölder inequality

$$|\mathbf{M}^3(\chi_1,\chi_2;q)|\leqslant\left(\mathbf{M}^4(q)\right)^{3/4}\left(\frac{1}{\phi^*_{\chi_1,\chi_2}(q)}\sum_{\chi:L(\chi,\frac{1}{2})L(\chi\chi_1,\frac{1}{2})L(\chi\chi_2,\frac{1}{2})\neq 0}1\right)^{1/4},$$

where $\mathbf{M}^4(q)$ is the fourth moment

$$\mathbf{M}^4(q) := \frac{1}{\phi^*(q)} \sum_{\chi\in\mathcal{D}(q)} \left|L\left(\chi, \tfrac12\right)\right|^4.$$

It follows that the proportion of $\chi \in \mathcal{D}_{\chi_1,\chi_2}(q)$ such that $L\left(\chi, \frac12\right) L\left(\chi\chi_1, \frac12\right) L\left(\chi\chi_2, \frac12\right) \neq 0$ is at least given by the ratio

$$\frac{|\mathbf{M}^3(\chi_1,\chi_2;q)|^4}{\mathbf{M}^4(q)^3}. \tag{0.3}$$

However, we know from [29] that $\mathbf{M}_4(q) \sim c \log^4 q$ for some explicit positive constant $c$ and $\mathbf{M}_3(\chi_1,\chi_2;q) \sim 1$ as we will see in Theorem 4 with $\ell = 1$. Therefore, the proportion (0.3) tends to zero when $q$ goes to infinity !

The mollification method is usually used to remedy this situation. The origin of the method goes back to the works of Bohr and Landau [10] and of Selberg [54] on zeros of the Riemann zeta function. The starting idea is to attach to the supposedly nonvanishing value $L(\chi, \frac12)$ a quantity $\mathrm{M}(\chi)$, called the "mollifier", which, on average, approximates its inverse. The goal is to choose a mollifier such that the two mollified moments $\mathscr{M}^3(\chi_1,\chi_2;q)$ and $\mathscr{M}^4(q)$ are comparable; that is, we want

$$\mathscr{M}^3(\chi_1,\chi_2;q) := \frac{1}{\phi^*_{\chi_1,\chi_2}(q)} \sum_{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)} \prod_{i=0}^{2} L(\chi\chi_i, \tfrac12)\mathrm{M}(\chi\chi_i) \asymp 1 \tag{0.4}$$

$$\mathscr{M}^4(q) := \frac{1}{\phi^*(q)} \sum_{\chi\in\mathcal{D}(q)} \left|L(\chi, \tfrac12)\mathrm{M}(\chi)\right|^4 \asymp 1, \tag{0.5}$$

where $\chi_0$ is the trivial character modulo $q$. From this a positive nonvanishing proportion can be inferred, namely at least

$$\frac{|\mathscr{M}^3(\chi_1,\chi_2;q)|^4}{\mathscr{M}^4(q)^3} \gg 1.$$

In [31], Iwaniec and Sarnak introduced a systematic technique that has since served as a model for other families of $L$-functions. They took the mollifier

$$\mathrm{M}(\chi) = \sum_{\ell\leqslant L} \frac{\chi(\ell)\boldsymbol{x}(\ell)}{\ell^{1/2}} P\left(\frac{\log\left(\frac{L}{\ell}\right)}{\log L}\right),$$

where $L$ is a small power of $q$, $(\boldsymbol{x}(\ell))_\ell$ is a sequence of complex numbers and $P(X) \in \mathbf{C}[X]$ is a polynomial satisfying $P(0) = 0$ and $P(1) = 1$.

Given integers $1 \leqslant \ell, \ell_1, \ell_2 < q$ such that $(\ell_1,\ell_2) = 1$, the above treatment suggests us to study three twisted moments

$$\mathscr{T}^3(\chi_1,\chi_2,\ell;q) := \frac{1}{\phi^*_{\chi_1,\chi_2}(q)} \sum_{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)} L\big(\chi,\tfrac12\big)\,L\big(\chi\chi_1,\tfrac12\big)\,L\big(\chi\chi_2,\tfrac12\big)\,\chi(\ell) \tag{0.6}$$

$$\mathscr{T}^3(f,\ell;q) := \frac{1}{\phi^*(q)} \sum_{\chi\in\mathcal{D}(q)} L\big(f\otimes\chi,\tfrac12\big)\,L\big(\chi,\tfrac12\big)\,\chi(\ell) \tag{0.7}$$

$$\mathscr{T}^4(\ell_1,\ell_2;q) := \frac{1}{\phi^*(q)} \sum_{\chi\in\mathcal{D}(q)} \big|L\big(\chi,\tfrac12\big)\big|^4\,\chi(\ell_1)\overline{\chi}(\ell_2). \tag{0.8}$$

We will prove

**Theorem 3.** *Let $q>2$ be a prime number, $\ell_1,\ell_2\in\mathbf{N}$ be cubefree integers such that $(\ell_1,\ell_2)=(\ell_1\ell_2,q)=1$, $\ell_1,\ell_2\leqslant L$ and $\varepsilon>0$. Then the twisted fourth moment defined in* (0.8) *admits the following decomposition*

$$\mathscr{T}^4(\ell_1,\ell_2;q) = \mathrm{MT}_D(\ell_1,\ell_2;q) + \mathrm{MT}_{OD}(\ell_1,\ell_2;q) + O\left(\frac{(\ell_1\ell_2)^{3/2}L^5}{q^{\eta-\varepsilon}}\right), \tag{0.9}$$

*where $\mathrm{MT}_D(\ell_1,\ell_2;q)$, $\mathrm{MT}_{OD}(\ell_1,\ell_2;q)$ are main terms given respectively by* (3.10), (3.87) *and $\eta = 1/14-3\theta/7$ with $\theta=7/64$.*

The cubefree assumption is not essential but it simplifies a lot our treatment. Since the primary goal of this paper is mollification, we did not concentrate our efforts on the optimization of the power of $L$ and on the value of $\eta$, but rather on the computation of the main terms.

**Theorem 4.** *Let $q>2$ be a prime number, $\chi_1,\chi_2$ be Dirichlet characters of modulus $q$, $f$ a primitive Hecke cusp form of level* 1 *or $q$ and trivial central character. Assume that $f$ satisfies the Ramanujan-Petersson conjecture, then for any $1\leqslant\ell\leqslant q^{3/13}$ and $\varepsilon>0$, we have*

$$\mathscr{T}^3(\chi_1,\chi_2,\ell;q) = \delta_{\ell=1} + O\Big(q^{-\frac{1}{64}+\varepsilon}\Big), \tag{0.10}$$

$$\mathscr{T}^3(f,\ell;q) = \delta_{\ell=1} + O\Big(q^{-\frac{1}{52}+\varepsilon}\Big), \tag{0.11}$$

*where the implied constant only depends on $\varepsilon>0$ and polynomially on the Archimedean parameters of $f$ (the weight or the Laplace eigenvalue) in* (0.11).

In Chapter 4, we use Theorem 3 to compute an asymptotic formula for the mollified fourth moment of Dirichlet $L$-functions $\mathscr{M}^4(q)$ defined in (0.5). More precisely, if $L=q^\lambda$ with $\lambda>0$, we will obtain

**Theorem 5.** *Let* $\boldsymbol{x}_\ell = \mu(\ell)$ *be the Möbius function and* $P(X) = X^2$. *Then for any* $0 < \lambda < \frac{11}{8064}$, *we have the asymptotic formula*

$$\mathscr{M}_4(q) = \sum_{i=0}^{4} c_i \lambda^{-i} + O_\lambda\left(\frac{1}{\log q}\right),$$

*for some calculable coefficients* $c_i \in \mathbf{R}$.

## 0.2 Structure of the twisted fourth moment

In 2010, Young established in a breakthrough paper [58] the following asymptotic formula for the fourth moment of Dirichlet $L$-functions at $s = 1/2$ and for prime moduli $q$ with a power saving error term

$$\frac{1}{\phi^*(q)} \sum_{\chi \in \mathcal{D}(q)} \left| L\left(\chi, \tfrac{1}{2}\right)\right|^4 = P(\log q) + O\left(q^{-\frac{5}{512}+\varepsilon}\right), \tag{0.12}$$

for any $\varepsilon > 0$ and where $P$ is a degree four polynomial with leading coefficient $(2\pi^2)^{-1}$ and $5/512 = (1-2\theta)/80$ with $\theta = 7/64$ is the best known approximation towards the Ramanujan-Petersson conjecture and it is due to Kim and Sarnak [41].

More recently, Blomer, Fouvry, Kowalski, Michel and Milićević revisited the problem in [5] by considering more general moments, namely of the form

$$\mathbf{M}^4_{f,g}(q) := \frac{1}{\phi^*(q)} \sum_{\chi \in \mathcal{D}(q)} L(f \otimes \chi, \tfrac{1}{2}) \overline{L(g \otimes \chi, \tfrac{1}{2})},$$

where $f$ and $g$ can be cuspidal Hecke eigenforms (holomorphic or Maass) or $E(z)$, the central derivative of the unique non-holomorphic Eisentsein series for the full modular group $\mathrm{PSL}_2(\mathbf{Z})$. They obtained various asymptotic formulae depending on the nature of $f, g$ (see [5, Theorems 1.1,1.2,1.3]). In particular, the case $f = g = E$ corresponds to (0.12) since the twisted $L$-function associated to $E$ is given by

$$L(E \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{\tau(n)\chi(n)}{n^s} = L(\chi, s)^2, \ \Re e(s) > 1,$$

where $\tau(n) = \sum_{d|n} 1$ is the divisor function. [5, Theorem 1.1] gives a significant improvement in the error term by passing to an exponent $-1/32$. They used on one hand, powerful results coming from algebraic geometry concerning general bilinear forms involving trace functions associated to $\ell$-adic sheaves on the projective line $\mathbf{P}^1_{\mathbf{F}_q}$ [23, 46]. On the other hand, they managed to almost eliminate the dependence in $\theta$ in their error bound by using an average result concerning Hecke eigenvalues (see [51, Lemma 2.4] and [5, § 3.5]). More recently, the five authors lowered the exponent to $-1/20$ in [2] using a smooth version of a Theorem of Shparlinski and Zhang [55, Theorem 3.1] where the trace function corresponds to rank 2 Kloosterman sums.

The mollification method and its view toward nonvanishing results is an example of why we may consider more general moments, called twisted moments. Indeed, asymptotic formulae for (0.8) can also be applied to the resonance method, as in the work of Bob Hough on the angle of large values of L-functions [30] where he established a formula for the same moment (see Theorem 4 and the proof is in the Appendix) by adapting the method of M.P. Young. We mention that our present approach is different and allows us to deal with not necessarily squarefree integers $\ell_1, \ell_2$, which is crucial for our application.

### 0.2.1 Outline of the proof of Theorem 3

In this section, we outline the proof of Theorem 3. Using the functional equation for $|L(\chi,1/2)|^4$ (c.f. (1.28)), we represent the twisted central values as a convergent series

$$\left|L\left(\chi,\tfrac{1}{2}\right)\right|^4 \chi(\ell_1)\overline{\chi}(\ell_2) = 2\sum\sum_{n,m\geqslant 1} \frac{\tau(n)\tau(m)}{(nm)1/2}\chi(n\ell_1)\overline{\chi}(m\ell_2)V\left(\frac{nm}{q^2}\right),$$

for some function $V(t)$ which depends on the archimedean factor $L_\infty(\chi,s)$ and decays rapidly for $t \geqslant q^\varepsilon$. An important fact is that $V$ depends on the character $\chi$ only through its parity. It is therefore natural to separate the average into even and odd characters. Assuming we are dealing with the even case, the orthogonality relations (c.f. (3.2)) gives

$$\frac{1}{\phi^*(q)}\sideset{}{^+}\sum_{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)} \left|L\left(\chi,\tfrac{1}{2}\right)\right|^4 \chi(\ell_1)\overline{\chi}(\ell_2) = \frac{1}{\phi^*(q)}\sum_{d|q}\phi(d)\mu\left(\frac{q}{d}\right)\sum\sum_{\ell_1 n\equiv\pm\ell_2 m \ (\mathrm{mod}\ d)} \frac{\tau(n)\tau(m)}{(nm)^{1/2}}V\left(\frac{nm}{q^2}\right),$$

where the + on the summation means that we restrict to even characters. A first main term $\mathrm{MT}_D(\ell_1,\ell_2,q)$ is extracted from the diagonal contribution $\ell_1 n = \ell_2 m$ and is computed in section 3.1. Putting this part away, applying a partition of unity and we are reduced to the evaluation of the following expression

$$\frac{1}{(NM)^{1/2}\phi^*(q)}\sum_{d|q}\phi(d)\mu\left(\frac{q}{d}\right)\sum\sum_{\substack{\ell_2 n\equiv\pm\ell_1 m \ (\mathrm{mod}\ d)\\ \ell_1 n\neq\ell_2 m}} \tau(n)\tau(m)W\left(\frac{n}{N}\right)W\left(\frac{m}{M}\right), \tag{0.13}$$

where $1\leqslant M\leqslant N$ (up to switch $\ell_1$ and $\ell_2$), $NM\leqslant q^{2+\varepsilon}$ and $W$ is a smooth and compactly supported function on $\mathbf{R}_{>0}$ satisfying $W^{(j)}\ll_j 1$. Since $q$ is prime, the arithmetical sum over $d|q$ could be separated into two terms corresponding to $d=1$ and $d=q$. However, as expected by the beautiful work of Young, an off-diagonal main term $\mathrm{MT}^{\pm}_{OD}(\ell_1,\ell_2,N,M;q)$ arises when $N\asymp M$ and this sum facilitates its calculation since it cancels some poles whose contributions seem to be big (c.f. Section 3.5.2 and Lemma 3.9). This technical step allows us to rebuild the partition of unity and to express the second main term as a contour integral of the form

$$\mathrm{MT}_{OD}(\ell_1,\ell_2;q) = \frac{1}{2\pi i}\int_{(\varepsilon)}\mathcal{F}(s,\ell_1,\ell_2;q)\frac{ds}{s},$$

for some function $\mathcal{F}(s,\ell_1,\ell_2;q)$ (c.f. (3.73)). A critical feature of this term is that the $q^s$ has disappeared, making it impossible to evaluate the integral by standard contour shift on the left. The situation is similar to that of § 4.3 in [44] where they study mollification of automorphic $L$-functions. Fortunately, using the functional equation for the Riemann zeta function, a crucial trigonometric identity for the gamma function (c.f. (3.85)) and a careful analysis of a Dirichlet series involving the $\ell_i'$s variables, we show that the integrand is odd and therefore, we are able to evaluate explicitly this integral through a residue at $s=0$ (see Lemma 3.14 and Proposition 3.15). Computing such contour integral exactly using the symmetry properties of the integrand was also observed in the work of Soundararajan [56], Blomer-Harcos [7] and Blomer [4].

For the rest of this outline, we only consider the case $d=q$ in (0.13) and we put this off-diagonal main term aside by writing

$$\begin{aligned}\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q)=\frac{1}{(MN)^{1/2}}&\sum\sum_{\substack{\ell_1 n\equiv\pm\ell_2 m \ (\mathrm{mod}\ q)\\ \ell_1 n\neq\ell_2 m}}\tau(n)\tau(m)W\left(\frac{n}{N}\right)W\left(\frac{m}{M}\right)\\ &-\mathrm{MT}_{OD}^{\pm}(\ell_1,\ell_2,N,M;q).\end{aligned}\tag{0.14}$$

We mention that $\mathrm{MT}_{OD}$ becomes small when $N\gg M$. More precisely, we prove in Lemma 3.13

$$\mathrm{MT}_{OD}^{\pm}(\ell_1,\ell_2,N,M;q)\ll L^2q^{\varepsilon}\left(\frac{M}{N}\right)^{1/2}.$$

The conclusion of Theorem 3 will follow as soon as we prove that

$$\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q)\ll L^A q^{-\eta}$$

for some absolute constants $A,\eta>0$. By the trivial bound

$$\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q)\ll Lq^{\varepsilon}\frac{(NM)^{1/2}}{q},$$

we may assume that $NM$ is close to $q^{2+\varepsilon}$. We will treat (0.14) by different methods depending on the relative ranges of $N$ and $M$.

### The shifted convolution problem

When $M$ and $N$ are relatively close to each other, we interpret the congruence condition $\ell_1 n\equiv\pm\ell_2 m \pmod q$ ($\ell_1 n\neq\ell_2 m$) as $\ell_1 n\mp\ell_2 m-hq=0$ with $h\neq 0$. Hence for each $h\neq 0$, we need to analyze the shifted convolution problem for the divisor function. This problem is interesting in its own right and has a long history. Blomer and Milićević in [9] used Jutila's variant of the circle method. This has the main advantage to have a certain degree of freedom with respect to the choice of the moduli and one can deal directly with the congruence subgroup $\Gamma_0(\ell_1\ell_2)$ in the trace formula. Unfortunately, this method is useless here essentially because the uniform estimate for Hecke eigenvalues of cuspidal forms (Wilton's bound) fails for the divisor function.

In the case $\ell_1 = \ell_2 = 1$, Young used an approximate functional equation for the divisor function [58, Lemma 5.4] to separate the variables $n$ and $m$. Adapting this technique to our case involves the choice of a lift of a multiplicative inverse $\overline{\ell_2}$ (mod $q$) whose location is hard to control.

Another possibility would be to use a recent method of Topacogullari [57], but in the end we would face similar issue .

The first thing to do is to smooth the condition $\ell_1 n \mp \ell_2 m - hq = 0$. We chose to return to the classical $\delta$-symbol method which was developed by Duke, Friedlander and Iwaniec in [18, 19]. We follow closely the first steps of [20] and are reduced to estimate sums of the shape (see (3.29))

$$\frac{Q^{-1}}{(MN)^{1/2}} \sum_{d_i|\ell_i} \sum_{h} \sum_{n,m} \tau(n)\tau(m) \sum_{\substack{(c,\ell_1'\ell_2')=1 \\ c \asymp Q}} \frac{S(hq, d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m; cd_1d_2)}{c} G(n,m,cd_1d_2), \quad (0.15)$$

where $Q$ is the parameter of the delta symbol, $G$ is a weight function, $\ell_i' = \ell_i/d_i$ and the inverse of $\ell_1'$ (resp $\ell_2'$) have to be taken modulo $cd_2$ (resp $cd_1$). For this, we exploit cancellations in the Kloosterman sums using spectral theory of automorphic forms. Returning to (0.15), we focus on the quantity

$$\sum_{(c,\ell_1'\ell_2')=1} \frac{S(hq, d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m; cd_1d_2)}{c} G(n,m,cd_1d_2).$$

At this step, we cannot apply directly the usual Kuznetsov formula because of the different inverses $\overline{\ell_1'}$, $\overline{\ell_2'}$ which are not with respect to the modulus (they are mod $cd_2$ (resp $cd_1$)) and we need first to transform the Kloosterman sum. Inspired by [57], we factor in a unique way $d_i = d_i^* d_i'$ with $(d_i^*, \ell_i') = 1$, $d_i' | (\ell_i')^\infty$ and use the twisted multiplicativity to obtain the factorization (we set $v := d_1'd_2'$),

$$\begin{aligned} S(hq, d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m; cd_1d_2) &= S(hq, \overline{v}^2(d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m); cd_1^*d_2^*) \\ &\times S(hq, \overline{(cd_1^*d_2^*)}^2(d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m); v), \end{aligned}$$

where all multiplicative inverses are this time modulo the modulus of the Kloosterman sum. We then exploit an idea of Blomer and Milićević [9] to separate the variable $c$

$$S(hq, \overline{(cd_1^*d_2^*)}^2(d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m); v) = \frac{1}{\phi(v)} \sum_{\chi(v)} \overline{\chi}(cd_1^*d_2^*) \hat{S}_v(\overline{\chi}, n, m, \ell_i, hq),$$

with

$$\hat{S}_v(\chi, n, m, \ell_i, hq) := \sum_{\substack{y(v) \\ (y,v)=1}} \overline{\chi}(y) S(hq\overline{y}, (d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m)\overline{y}; v).$$

In this way we obtain sum of Kloosterman sums twisted by Dirichlet characters which we can evaluate using Kuznetsov formula for automorphic forms with non trivial nebentypus. We finally obtain the bound

$$\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q) \ll L^A q^{\varepsilon-1/2+\theta}\left(\frac{N}{M}\right)^{1/2},$$

for some $A>0$ and it is exactly the expected error term (modulo the power of $L$) according to the treatment of Young. We thus obtain Theorem 3 as long as

$$\frac{N}{M} \leqslant q^{1-2\theta-2\eta}.$$

The analysis of the shifted convolution sum in the complementary range has already been done first in [5] and then improved in [2]. We briefly recall their work in Section 3.2.1.

## 0.3 Structure of the twisted cubic moment

Before going in the sketch of the proof of Theorem 4, we mention the work of S. Das and R. Khan [13] who evaluated a moment of the form

$$\frac{1}{q-1}\sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi\neq 1}} L\big(f\otimes\chi,\tfrac{1}{2}\big)\overline{L\big(\chi,\tfrac{1}{2}\big)}.$$

As the authors explained, the complex conjugation above $L(\chi,1/2)$ was introduced to avoid difficulties connected to the oscillations of Gauss sums. What we show here is that these difficulties are resolved using variants of the methods of [24] [46]. We point out that there is also an advantage in considering the two moments without the complex conjugation. In this case we obtain a main term only when $\ell=1$ and this main term is 1 (in particular independent of $q,\chi_i$ or $f$), which greatly facilitates the average over $\ell$ in the mollification method (see Section 7.1).

After an application of the approximate functional equation to (0.6) and (0.7), which expresses the central value of automorphic $L$-functions as a convergent series, and an average over the characters, we isolate a main term which appears only if $\ell=1$ (c.f. § 6.1.1-6.1.3).

The treatment of the error term passes by the analysis of sums of the shape

$$\mathcal{S}(\chi_1,\chi_2;q)=\frac{1}{(qN_0N_1N_2)^{1/2}}\sum\sum\sum_{\substack{n_0\sim N_0,n_1\sim N_1\\ n_2\sim N_2}}\overline{\chi}_1(n_1)\overline{\chi}_2(n_2)\mathrm{Kl}_3(n_0n_1n_2,\chi_1,\chi_2,1;q), \tag{0.16}$$

$$\mathcal{C}(f;q)=\frac{1}{(qMN)^{1/2}}\sum\sum_{n\sim N,m\sim M}\lambda_f(n)\mathrm{Kl}_3(nm;q), \tag{0.17}$$

where $\mathrm{Kl}_3$ is the 2-dimensional normalized hyper-Kloosterman sum, $\mathrm{Kl}_3(\chi_1,\chi_2,1;q)$ is the twisted version as defined in (2.9), $\{\lambda_f(n)\}_{n\geqslant 1}$ are the Hecke eigenvalues of $f$ and $N_0,N_1,N_2,N,M$ are parameters satisfying

$$1\leqslant N_i,N,M,\ \ N_0N_1N_2\leqslant q^{3/2+\varepsilon} \text{ and } MN\leqslant q^{3/2+\varepsilon}.$$

The ultimate goal is to obtain a bound of the form

$$\mathcal{S}(\chi_1,\chi_2;q),\mathcal{C}(f;q)=O\left(q^{-\delta}\right),$$

for some absolute constant $\delta>0$. Using Poisson summation in the three variables in (0.16), or Voronoi formula in the $n$-variable in (0.17) (followed by Poisson on $m$) allows us to get rid of the cases where the product of the variables is larger than $q$; namely in Sections 6.1.4 and 6.2.3, we prove

$$\mathcal{S}(\chi_1,\chi_2;q)\ll q^{\varepsilon}\left(\frac{q}{N_0N_1N_2}\right)^{1/2} \text{ and } \mathcal{C}(f;q)\ll q^{\varepsilon}\left(\frac{q}{NM}\right)^{1/2}.$$

Combining these two estimates with the trivial bounds

$$\mathcal{S}(\chi_1,\chi_2;q)\ll\left(\frac{N_0N_1N_2}{q}\right)^{1/2} \text{ and } \mathcal{C}(f;q)\ll q^{\varepsilon}\left(\frac{NM}{q}\right)^{1/2},$$

we can assume for the rest of this outline that

$$N_0N_1N_2=NM=q.$$

We treat these sums differently according to the relative size of the various parameters. If $N_1\sim 1$ (say) and $M\sim 1$, we exploit the $n_0,n_2$-sum (resp. the $n$-sum) in (0.16) (resp. in (0.17)) and average trivially over the others. Grouping $n_0n_2$ into a long variable $n$ and we need to analyze roughly

$$\frac{1}{q}\sum_{n\sim q}\lambda_{\overline{\chi}_2}(n,0)\mathrm{Kl}_3(nn_1,\chi_1,\chi_2;q) \text{ and } \frac{1}{q}\sum_{n\sim q}\lambda_f(n)\mathrm{Kl}_3(nm;q),$$

where for any $t\in\mathbf{R}$,

$$\lambda_{\overline{\chi}_2}(n,it)=\sum_{n_0n_2=n}\overline{\chi}_2(n_2)\left(\frac{n_2}{n_0}\right)^{it}.$$

In [23] and [24], Fouvry, Kowalski and Michel studied these sums when $f$ is a fixed cusp form for the group $\mathrm{SL}_2(\mathbf{Z})$, $\lambda_{\overline{\chi}_2}(n,it)$ is replaced by the generalized divisor function $d_{it}(n)=\sum_{ab=n}a^{it}b^{-it}$ and for a general Frobenius trace function modulo $q$ instead of $\mathrm{Kl}_3$. We will show in Chapter 5 that their theorems [24, Theorem 1.2] [23, Theorem 1.15] can be extended to a Hecke eigenforms (cuspidal or not) of level $q$ and arbitrary central character $\omega$. More precisely, for $V$ a smooth and compactly supported function on $\mathbf{R}_+^{\times}$, we consider the sums

$$\mathcal{S}_V(f,K;q) := \sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{q}\right), \tag{0.18}$$

$$\mathcal{S}_V(\omega,it,K;q) := \sum_{n\geqslant 1} \lambda_\omega(n,it)K(n)V\left(\frac{n}{q}\right). \tag{0.19}$$

**Theorem 6.** *Let $q>2$ be a prime number, $\omega$ a Dirichlet character of modulus $q$, $f$ a primitive Hecke cusp form of level $q$, nebentypus $\omega$ and spectral parameter $t_f$. Let $K$ be an isotypic trace function modulo $q$ of conductor* $\mathrm{cond}(K)$ *such that its Fourier transform is not $\omega$-exceptional, as defined in* (2.3) *and* (2.5). *Let $V$ be a function satisfying $V(C,P,Q)$* (*see Definition* (2.1)). *Then there exists constants $s\geqslant 1$ and $A\geqslant 1$ such that*

$$\begin{aligned}\mathcal{S}_V(f,K;q) &\ll_{C,\delta} (1+|t_f|)^A \mathrm{cond}(K)^s q^{1-\delta}(PQ)^{1/2}(P+Q)^{1/2},\\ \mathcal{S}_V(\omega,it,K;q) &\ll_{C,\delta} (1+|t|)^A \mathrm{cond}(K)^s q^{1-\delta}(PQ)^{1/2}(P+Q)^{1/2},\end{aligned}$$

*for any $\delta<1/16$, where $A$ depends on $\varepsilon$ and $s$ is absolute.*

Therefore, Theorem 6 provides the desired power saving (set $P=Q=1$) in the special case where one of the variable is very small in (0.16) and $M\sim 1$ in (0.17). Assume now that $N_0,N_1,N_2\geqslant q^\eta$ and $M\geqslant q^\eta$ for some small real number $\eta>0$. From now, we need to take care of the different nature of expressions (0.16) and (0.17). Indeed, for (0.16), the fact of having three free variables allows us to factorize two of them (say $n_0n_2$) in such a way that $N_0N_2\geqslant q^{1/2+\eta}$. In this case, we can form a bilinear sum and use a general version of Polyá-Vinogradov (see Theorem 2.19) to obtain a power saving in the error term. The same method also works for (0.17), as long as $M\leqslant q^{1/2-\eta}$ or $N\leqslant q^{1/2-\eta}$, because in this case $N\geqslant q^{1/2+\eta}$ or $M\geqslant q^{1/2+\eta}$. Hence the critical range for the second sum, i.e. when Polyá-Vinogradov is useless, appears when $M\sim q^{1/2}$ and $N\sim q^{1/2}$ and here we apply the general result of Kowalski, Michel and Sawin concerning bilinear forms involving classical Kloosterman sums [46, Theorem 1.3].

# 1 Preliminaries I : Automorphic Forms

## 1.1 Summation formulae

We give here two versions of the Voronoi summation formula. The first one concerns the divisor function $\tau(n)$ and it is provided by [36, Theorem 1.7]. The second one involves Hecke eigenvalues of cusp forms of prime level and a reference for this result can be found in [45, Appendix A.3-A.4].

**Proposition 1.1.** *Let $g$ be a smooth and compactly supported function in $\mathbf{R}_+^\times$ and $d,\ell \geqslant 1$ be integers such that $(d,\ell)=1$. Then*

$$\sum_{n=1}^{\infty}\tau(n)e\left(\frac{dn}{\ell}\right)g(n)=\frac{1}{\ell}\int_0^{\infty}(\log x+2\gamma-2\log\ell)g(x)dx+\sum_{\pm}\sum_{n=1}^{\infty}\tau(n)e\left(\frac{\pm\overline{d}n}{\ell}\right)g^{\pm}(n),$$

*where $\overline{d}$ denotes the inverse of $d$ modulo $\ell$ and $g^{\pm}$ are the Bessel transforms of $g$ defined by*

$$g^{+}(y):=\frac{4}{\ell}\int_0^{\infty}g(x)K_0\left(\frac{4\pi\sqrt{xy}}{\ell}\right)dx$$

*and*

$$g^{-}(y):=-\frac{2\pi}{\ell}\int_0^{\infty}g(x)Y_0\left(\frac{4\pi\sqrt{xy}}{\ell}\right)dx,$$

*where $K_0$, $Y_0$ are the usual Bessel functions.*

**Proposition 1.2.** *Let $q>2$ be a prime number, $\omega$ a Dirichlet character of modulus $q$ and $f$ a primitive Hecke cusp form of level $q$ and nebentypus $\omega$ with associated Hecke eigenvalues $(\lambda_f(n))_{n\geqslant 1}$. Let $a$ be an integer coprime with $q$ and $g:\mathbf{R}_+^\times\to\mathbf{C}$ a smooth and compactly supported function. Then we have the identity*

$$\sum_{n\geqslant 1}\lambda_f(n)e\left(\frac{an}{q}\right)g(n)=\frac{\overline{\omega}(a)}{q}\sum_{\pm}\sum_{n\geqslant 1}\lambda_f(n)e\left(\mp\frac{\overline{a}n}{q}\right)g_{\pm}\left(\frac{n}{q^2}\right), \tag{1.1}$$

*where*

$$g_\pm(y) = \int_0^\infty g(x)\mathcal{J}_\pm(4\pi\sqrt{xy})\,dx,$$

*with*

$$\mathcal{J}_+(x) = 2\pi i^k J_{k-1}(x)\ \ ,\ \ \mathcal{J}_-(x) = 0$$

*if $f$ is holomorphic of weight $k$ and*

$$\mathcal{J}_+(x) = -\frac{\pi}{\sin(\pi i t_f)}\left(J_{2it_f}(x) - J_{-2it_f}(x)\right)\ ,\ \mathcal{J}_-(x) = \varepsilon_f 4\cos(\pi i t_f)K_{2it_f}(x)$$

*if $f$ is a Maass form of parity $\varepsilon_f \in \{-1,+1\}$ and spectral parameter $t_f$.*

Finally, we consider the decay properties of the Bessel transforms $g_\pm$ (see [5, Lemma 2.4]).

**Lemma 1.3.** *Let $g : \mathbf{R}_+^* \to \mathbf{C}$ be a smooth and compactly supported function satisfying*

$$x^i g^{(i)}(x) \ll_{i,\varepsilon} q^{\varepsilon i} \tag{1.2}$$

*for any $\varepsilon > 0$ and $i \geqslant 0$. In the cuspidal and non-holomorphic case set $\vartheta = \Re e(it_f)$, otherwise set $\vartheta = 0$. Then for any $\varepsilon > 0$, for any $i, j \geqslant 0$ and all $y > 0$, we have*

$$y^j g_\pm^{(j)}(y) \ll_{i,j,\varepsilon} \frac{(1+y)^{j/2}}{\left(1+(yq^{-\varepsilon})^{1/2}\right)^i}\left(1+y^{-2\vartheta-\varepsilon}\right).$$

## 1.2 Automorphic forms

In this section, we briefly compile the main results from the theory of automorphic forms. An exhaustive account of the theory can be found in [33] and [32] from which we borrow much of the notations.

### 1.2.1 Hecke eigenbases

Let $N \geqslant 1$ be an integer, $\omega$ a Dirichlet character of modulus $N$, $\kappa = \frac{1-\omega(-1)}{2} \in \{0,1\}$ and $k \geqslant 2$ satisfying $k \equiv \kappa \pmod 2$. We denote by $\mathcal{B}_k(N,\omega)$ (resp. $\mathcal{B}(N,\omega)$) a Hecke basis of the Hilbert space of holomorphic cusp forms of weight $k$ (resp. of Maass cusp forms of weight $\kappa$) with respect to the Hecke congruence subgroup $\Gamma_0(N)$ and with nebentypus $\omega$. The continuous spectrum is continuously spanned by the Eisenstein series $E_\mathfrak{a}(\cdot, 1/2+it)$ where $\mathfrak{a}$ runs over the singular cusps of $\Gamma_0(N)$ with respect to $\omega$.

Sometimes it is useful to employ another basis of Eisenstein series formed of Hecke eigenforms: the adelic reformulation of the theory of modular forms provides a natural spectral expansion of the Eisenstein spectrum in which the basis of Eisenstein series is indexed by a set of parameters of the form

$$\{(\omega_1,\omega_2,f) \mid \omega_1\omega_2 = \omega,\ f \in \mathcal{B}(\omega_1,\omega_2)\}, \tag{1.3}$$

where $(\omega_1,\omega_2)$ ranges over the pairs of characters of modulus $N$ such that $\omega_1\omega_2 = \omega$ and $\mathcal{B}(\omega_1,\omega_2)$ is a finite orthonormal basis in some induced representation. We do not need to be more explicit here and we refer to [26] for a precise definition of these parameters. The main advantage of such a basis is that the Eisenstein series are eigenforms of the Hecke operators $T_n$ with $(n,N) = 1$ : we have

$$T_n E_{\omega_1,\omega_2,f}(z,1/2+it) = \lambda_{\omega_1,\omega_2}(n,t)E_{\omega_1,\omega_2,f}(z,1/2+it),$$

with

$$\lambda_{\omega_1,\omega_2}(n,t) := \sum_{ab=n} \omega_1(a)a^{it}\omega_2(b)b^{-it}. \tag{1.4}$$

**The Eisenstein series in the special case $N = 2q$**

Let $q > 2$ be a prime number. For some technical reasons, it is convenient for the proof of Theorem 6 to embed our form $f$ of level $q$ in an orthonormal basis of forms of level $2q$ (see the beginning of Section 4.1 and Section 5.5 in [24]). For arbitrary level $N$, the Eisenstein series $E_{\mathfrak{a}}(\cdot,1/2+it)$ are usually not eigenfunctions of the Hecke operators. In the special case where $N = 2q$, there are exactly four inequivalent cups for $\Gamma_0(2q)$ which are

$$\mathfrak{a} = 1\ ,\ \frac{1}{2}\ ,\ \frac{1}{q}\ ,\ \frac{1}{2q},$$

see for example [32, Proposition 2.6] and all are singular. The main advantage in this situation is that these Eisentein series are eigenforms of the Hecke operators $T_n$ for $(n,2q) = 1$. More precisely, if $\mathfrak{a} = 1/v$ with $v \in \{1,2,q,2q\}$, then we have for $(n,2q) = 1$,

$$T_n E_{\mathfrak{a}}(\cdot,1/2+it) = \lambda_{\mathfrak{a}}(n,it)E_{\mathfrak{a}}(\cdot,1/2+it),$$

with explicitly

$$\lambda_{\mathfrak{a}}(n,it) = \begin{cases} \sum_{ab=n}\omega(a)\left(\frac{a}{b}\right)^{it} & \text{if} \quad v = q,2q \\ \\ \sum_{ab=n}\omega(b)\left(\frac{a}{b}\right)^{it} & \text{if} \quad v = 1,2, \end{cases} \tag{1.5}$$

see [21, (6.16)-(6.17)].

**Remark 1.4.** In the case $N = q$, there are exactly two inequivalent cusps $\mathfrak{a} = 1, 1/q$ and the two Eisenstein series are eigenfunctions of the Hecke operators $T_n$ for $(n,q) = 1$ with eigenvalues given by (1.5). Moreover, they are also Eisenstein series of level $2q$ after the normalization by $1/\sqrt{3}$.

**Remark 1.5.** It can be shown more generally that Eisenstein series are always Hecke eigenfunctions for squarefree level.

### 1.2.2 Hecke eigenvalues, Fourier coefficients and boundedness properties

Let $f$ be a Hecke eigenform for $\Gamma_0(N)$ with central character $\omega$ and Hecke eigenvalues $\lambda_f(n)$ for all $(n,N)=1$. We write the Fourier expansion of $f$ at a singular cusp $\mathfrak{a}$ as follows ($z=x+iy$) :

$$f_{|^k\sigma_{\mathfrak{a}}}(z)=\sum_{n\geqslant 1}\rho_{f,\mathfrak{a}}(n)n^{\frac{k-1}{2}}e(nz) \;\text{ for } f\in\mathcal{B}_k(N,\omega),$$

$$f_{|_\kappa\sigma_{\mathfrak{a}}}(z)=\sum_{n\neq 0}\rho_{f,\mathfrak{a}}(n)|n|^{-1/2}W_{\frac{n}{|n|}\frac{\kappa}{2},it_f}(4\pi|n|y)e(nx) \;\text{ for } f\in\mathcal{B}(N,\omega),$$

where $\sigma_{\mathfrak{a}}$ is a scaling matrix of $\mathfrak{a}$, i.e. $\sigma_{\mathfrak{a}}\in \mathrm{SL}_2(\mathbf{R})$ is such that $\sigma_{\mathfrak{a}}\infty=\mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_0(N)_{\mathfrak{a}}\sigma_{\mathfrak{a}}=B:=\left\{\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\,:\,b\in\mathbf{Z}\right\}$ where $\Gamma_0(N)_{\mathfrak{a}}$ denotes the isotropy subgroup of $\mathfrak{a}$, $W_{\frac{n}{|n|}\frac{\kappa}{2},it_f}$ is the Whittaker function and $t_f$ is the spectral parameter of $f$. i.e. $\lambda_f=1/4+t_f^2$ with $\lambda_f$ the eigenvalue for the hyperbolic Laplace operator. For any $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbf{R})$, the two slash operators $|^k\gamma$ and $|_\kappa\gamma$ of weights $k$ and $\kappa$ are defined by

$$f_{|^k\gamma}(z):=(cz+d)^{-k}f(\gamma z) \;\text{ and }\; f_{|_\kappa\gamma}(z):=\left(\frac{cz+d}{|cz+d|}\right)^{-\kappa}f(\gamma z).$$

For an Eisenstein series $E_{\Diamond}(\cdot,1/2+it)$ indexed by $\Diamond=(\omega_1,\omega_2,f)$ or $\Diamond=\mathfrak{b}$ is a singular cusp for $\Gamma_0(N)$, we write its Fourier expansion as

$$\begin{aligned}E_{\Diamond|\sigma_{\mathfrak{a}}}(z,1/2+it)&=c_{1,\Diamond,\mathfrak{a}}(t)y^{1/2+it}+c_{2,\Diamond,\mathfrak{a}}(t)y^{1/2-it}\\&\quad+\sum_{n\neq 0}\rho_{\Diamond,\mathfrak{a}}(n,it)|n|^{-1/2}W_{\frac{|n|}{n}\frac{\kappa}{2},it}(4\pi|n|y)e(nx).\end{aligned}$$

When $f$ is a Hecke eigenform and $\mathfrak{a}=\infty$ is the usual cusp, there is a closed relation between the Fourier coefficients and the Hecke eigenvalues $\lambda_f(n)$ ; for $(m,N)=1$ and $n\geqslant 1$, one has

$$\lambda_f(m)\rho_f(n)=\sum_{d|(m,n)}\omega(d)\rho_f\left(\frac{mn}{d^2}\right). \tag{1.6}$$

In particular, for all $(m,q)=1$,

$$\lambda_f(m)\rho_f(1)=\rho_f(m). \tag{1.7}$$

If $f$ is primitive, the relations (1.6) and (1.7) are valid for every $m\geqslant 1$. In particular its first Fourier coefficient is not zero and we have the following lower bounds for $\rho_f(1)$ : for any $\varepsilon>0$

$$|\rho_f(1)|^2\gg_\varepsilon\begin{cases}\frac{\cosh(\pi t_f)}{N(1+|t_f|)^\kappa(N+|t_f|)^\varepsilon} & \text{if } \quad f\in\mathcal{B}(N,\omega)\\[2ex] \frac{(4\pi)^{k-1}}{(k-1)!N^{1+\varepsilon}k^\varepsilon} & \text{if } \quad f\in\mathcal{B}_k(N,\omega),\end{cases} \tag{1.8}$$

see [21, (6.22),(7.16)] and [49, Lemma 2.2 and (2.23)]. For an Eisenstein series $E_{\mathfrak{b}}(\cdot, 1/2+it)$, we have

$$|\rho_{\mathfrak{b}}(1,it)|^2 \gg \frac{\cosh(\pi t)}{N(1+|t|)^{\kappa}(\log(N+|t|))^2}, \tag{1.9}$$

see [21, (6.23),(7.15)].

The Hecke eigenavlues $\lambda_f(n)$ satisfy the following multiplicative property : for $(nm, N) = 1$, we have

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(n,m)} \omega(d)\lambda_f\Big(\frac{nm}{d^2}\Big), \tag{1.10}$$

and also

$$\overline{\lambda_f(n)} = \overline{\omega}(n)\lambda_f(n). \tag{1.11}$$

Note that if $f$ is holomorphic, then by the work of Deligne and Serre, we have the Ramanujan-Petersson conjecture, namely

$$|\lambda_f(n)| \leqslant \tau(n). \tag{1.12}$$

The same bound is of course trivial if $f$ is an Eisenstein series with eigenvalues (1.4) or (1.5) in the case where $N|2q$. In the case of a Maass cusp form $f$, the best result is due to Kim and Sarnak [41] and it is given by

$$|\lambda_f(n)| \leqslant \tau(n)n^{\theta}\ ,\ \theta = \frac{7}{64}, \tag{1.13}$$

with an analogous bound for the spectral parameter

$$|\Im m(t_f)| \leqslant \theta, \tag{1.14}$$

However, the conjecture is true on average, in the sense that for every $X \geqslant 1$,

$$\sum_{n\leqslant X} |\lambda_f(n)|^2 \ll (N(1+|t_f|))^{\varepsilon} X, \tag{1.15}$$

with an implied constant depending only on $\varepsilon$ [21, Proposition 19.6]. We will also need later similar bound for the fourth-power on average and it is enough for our purpose to restrict to prime numbers $p$ not dividing the level $N$,

$$\sum_{\substack{p\leqslant X\\(p,N)=1}} |\lambda_f(p)|^4 \ll (XN(1+|t_f|))^{\varepsilon} X, \tag{1.16}$$

for any $\varepsilon > 0$ and the constant only depends on $\varepsilon$. This bound is a consequence of the automorphy of the symmetric square $\mathrm{Sym}^2 f$ [25] and Rankin-Selberg theory.

## 1.3 Spectral identities

### 1.3.1 Bessel transforms

We collect here some facts about Bessel functions and their integral transforms. Let $\phi : [0,\infty) \to \mathbf{C}$ be a smooth function satisfying $\phi(0) = \phi'(0) = 0$ and $\phi^{(i)} \ll_i (1+x)^{-3}$ for all $0 \leqslant i \leqslant 3$. For $\kappa \in \{0,1\}$, we define the following three integral transforms

$$\begin{aligned}
\dot{\phi}(k) &:= i^k \int_0^\infty \phi(x) J_{k-1}(x) \frac{dx}{x}, \\
\widehat{\phi}(t) &:= \frac{\pi i t^\kappa}{2\sinh(\pi t)} \int_0^\infty (J_{2it}(x) - (-1)^\kappa J_{-2it})\phi(x) \frac{dx}{x}, \\
\check{\phi}(t) &:= 2i^{-\kappa} \int_0^\infty f(x)\cosh(\pi t) K_{2it}(x) \frac{dx}{x}.
\end{aligned} \tag{1.17}$$

Here are some useful estimates concerning the above Bessel transforms.

**Lemma 1.6.** *Let $\phi$ be a smooth and compactly supported function in $(X,2X)$ satisfying*

$$\phi^{(i)} \ll_i X^{-i}$$

*for any $i \geqslant 0$. Then for all $t \geqslant 0$ and real $k > 1$, we have*

$$\frac{\widehat{\phi}(t)}{(1+t)^\kappa}, \check{\phi}(t), \dot{\phi}(t) \ll \frac{1+|\log X|}{1+X}, \quad t \geqslant 0, \tag{1.18}$$

$$\frac{\widehat{\phi}(t)}{(1+t)^\kappa}, \check{\phi}(t), \dot{\phi}(t) \ll_k \left(\frac{1}{t}\right)^k \left(\frac{1}{t^{1/2}} + \frac{X}{t}\right), \quad t \geqslant \max(2X,1), \tag{1.19}$$

*where all implied constants are absolute.*

*Proof.* The case $\kappa = 0$ is covered in [3, Lemma 2.1]. The proof carry over to the case $\kappa = 1$ with minimal changes. □

### 1.3.2 The Petersson formula

For $k \geqslant 2$ an integer such that $k \equiv \kappa \pmod 2$, the Petersson trace formula expresses an average of product of Fourier coefficients over $\mathcal{B}_k(N,\omega)$ in terms of sums of Kloosterman sums (see [33, Theorem 9.6] and [34, Proposition 14.5]) : for any integers $n, m > 0$ and $\mathfrak{a}, \mathfrak{b}$ two singular cusps, we have

$$\frac{(k-2)!}{(4\pi)^{k-1}} \sum_{g \in \mathcal{B}_k(N,\omega)} \overline{\rho_{g,\mathfrak{a}}(n)} \rho_{g,\mathfrak{b}}(m) = \delta(n,m) + 2\pi i^{-k} \sum_c^{\Gamma_0(N)} \frac{1}{\gamma} S_\omega^{\mathfrak{a}\mathfrak{b}}(n,m;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{1.20}$$

where $S_\omega^{\mathfrak{ab}}(n,m;c)$ is the generalized twisted Kloosterman sum and it is defined by

$$S_\omega^{\mathfrak{ab}}(n,m;c) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in B\backslash \sigma_{\mathfrak{a}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}/B} \overline{\omega}\left(\sigma_{\mathfrak{a}}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\sigma_{\mathfrak{b}}^{-1}\right) e\left(\frac{na+md}{c}\right).$$

The notation $\sum_c^{\Gamma_0(N)}$ means that we sum over all positive $c$ such that $S_\omega^{\mathfrak{ab}}(n,m;c)$ is not empty.

**Remark 1.7.** If $\mathfrak{a}=\mathfrak{b}=\infty$, the Kloosterman sum becomes

$$S_\omega(n,m;) = \sum_{\substack{d \ (\mathrm{mod}\ c) \\ (d,c)=1}} \overline{\omega}(d) e\left(\frac{m\overline{d}+nd}{c}\right)$$

and the sum $\sum_c^{\Gamma_0(N)}$ is taken over $N|c$.

### 1.3.3 The Kuznetsov formula

Let $\phi : \mathbf{R}_+ \to \mathbf{C}$ be a smooth function satisfying $\phi(0)=\phi'(0)=0$ and $\phi^{(j)}(x) \ll_\varepsilon (1+x)^{-2-\varepsilon}$ for $0 \leqslant j \leqslant 3$ and every $\varepsilon > 0$. Let $\blacklozenge$ be a finite set indexing the basis of the continuous spectrum ($\blacklozenge$ is the set of singular cusps or (1.3)).Then for every integers $m,n>0$ and every pair of singular cusps $\mathfrak{a},\mathfrak{b}$, we have the following spectral decomposition of the Kloosterman sums [33, Theorem 9.4 and 9.8].

$$\begin{aligned}\sum_c^{\gamma_0(N)} \frac{1}{c} S_\omega^{\mathfrak{ab}}(n,m;c)\phi\left(\frac{4\pi\sqrt{mn}}{c}\right) = &\sum_{\substack{k\equiv\kappa \bmod 2 \\ k>\kappa}}\sum_{g\in\mathcal{B}_k(N,\omega)} \dot{\phi}(k)\frac{(k-1)!}{\pi(4\pi)^{k-1}}\overline{\rho_{g,\mathfrak{a}}(n)}\rho_{g,\mathfrak{b}}(m) \\ &+ \sum_{g\in\mathcal{B}(N,\omega)} \tilde{\phi}(t_g)\frac{4\pi}{\cosh(\pi t_g)}\overline{\rho_{g,\mathfrak{a}}(n)}\rho_{g,\mathfrak{b}}(m) \\ &+ \sum_{\lozenge\in\blacklozenge}\int_0^\infty \frac{\tilde{\phi}(t)}{\cosh(\pi t)}\overline{\rho_{\lozenge,\mathfrak{a}}(n,it)}\rho_{\lozenge,\mathfrak{b}}(m,it)dt,\end{aligned} \tag{1.21}$$

and

$$\begin{aligned}\sum_c^{\gamma_0(N)} \frac{1}{c} S_\omega^{\mathfrak{ab}}(n,-m;c)\phi\left(\frac{4\pi\sqrt{mn}}{c}\right) = &\sum_{g\in\mathcal{B}(N,\omega)} \check{\phi}(t_g)\frac{4\pi}{\cosh(\pi t_g)}\overline{\rho_{g,\mathfrak{a}}(n)}\rho_{g,\mathfrak{b}}(-m) \\ &+ \sum_{\lozenge\in\blacklozenge}\int_0^\infty \frac{\check{\phi}(t)}{\cosh(\pi t)}\overline{\rho_{\lozenge,\mathfrak{a}}(n,it)}\rho_{\lozenge,\mathfrak{b}}(-m,it)dt.\end{aligned} \tag{1.22}$$

### 1.3.4 The spectral large sieve inequality

Often the Kuznetsov formula is used hand in hand with the spectral large sieve inequality. Before stating the result, we denote by $N_0$ the conductor of $\omega$ and we also recall that each cusp for $\Gamma_0(N)$ (not necessarily singular) is equivalent to a fraction of the form $u/v$, where $v \geqslant 1$,

$v|N$ and $(v,u)=1$. We define the following quantity :

$$\mu(\mathfrak{a}) := N^{-1}\left(v, \frac{N}{v}\right). \tag{1.23}$$

Furthermore, if $(a_m)$ is a sequence a complex numbers and $M$ is a real number such that $M \geqslant 1/2$, we set

$$\begin{aligned}
||a_m||_M^2 &:= \sum_{M<n\leqslant 2M} |a_m|^2, \\
\Sigma^{(\mathrm{H})}(k,f,M) &:= \sqrt{\frac{(k-1)!}{(4\pi)^{k-1}}} \sum_{M<m\leqslant 2M} a_m \rho_{f,\mathfrak{a}}(m), \\
\Sigma^{(\mathrm{M})}_{\pm}(f,M) &:= \frac{(1+|t_f|)^{\pm\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t_f)}} \sum_{M<m\leqslant 2M} a_m \rho_{f,\mathfrak{a}}(\pm m), \\
\Sigma^{(\mathrm{E})}_{\pm}(\Diamond, it, M) &:= \frac{(1+|t|)^{\pm\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t)}} \sum_{M<m\leqslant 2M} a_m \rho_{\Diamond,\mathfrak{a}}(\pm m, it).
\end{aligned}$$

Then the following bounds are known as the spectral large sieve inequalities.

**Proposition 1.8.** *Let $T \geqslant 1$ and $M \geqslant 1/2$ be real numbers, $(a_m)$ a sequence of complex numbers and $\mathfrak{a}$ a singular cusp for the group $\Gamma_0(N)$. Then*

$$\begin{aligned}
\sum_{\substack{2\leqslant k\leqslant T \\ k\equiv\kappa\ (2)}} \sum_{f\in\mathcal{B}_k(N,\omega)} \left|\Sigma^{(\mathrm{H})}(k,f,M)\right|^2 &\ll \left(T^2 + N_0^{\frac{1}{2}}\mu(\mathfrak{a})M^{1+\varepsilon}\right)||a_m||_M^2, \\
\sum_{|t_f|\leqslant T} \left|\Sigma^{(\mathrm{M})}_{\pm}(f,M)\right|^2 &\ll \left(T^2 + N_0^{\frac{1}{2}}\mu(\mathfrak{a})M^{1+\varepsilon}\right)||a_m||_M^2, \\
\sum_{\Diamond\in\blacklozenge} \int_{-T}^{T} \left|\Sigma^{(\mathrm{E})}_{\pm}(\Diamond, it, M)\right|^2 dt &\ll \left(T^2 + N_0^{\frac{1}{2}}\mu(\mathfrak{a})M^{1+\varepsilon}\right)||a_m||_M^2,
\end{aligned}$$

*with all implied constants depending only on $\varepsilon$.*

*Proof.* We refer to [16]. □

We can improve the above Proposition in the case where $\mathfrak{a}=\infty$ is the usual cusp and the conductor of $\omega$ is odd and squarefree.

**Lemma 1.9.** *Let $N \geqslant 1$ be an integers and $\omega$ a Dirichlet character of modulus $N$ whose conductor is odd and squarefree. Then for any $m,n \in \mathbf{Z}$ and $N|c$, we have the Weil bound*

$$|S_\omega(m,n;c)| \leqslant \tau(c)(m,n,c)^{1/2}c^{1/2}.$$

*Proof.* The proof is a consequence of the twisted multiplicativity of Kloosterman sums and [42, Propositions 9.7 and 9.8]. □

**Proposition 1.10.** *Let $N \geqslant 1$ be an integer, $\omega$ a Dirichlet character of modulus $N$ with squarefree and odd conductor. Let $T \geqslant 1$, $M \geqslant 1/2$ and $(a_m)_{M<m\leqslant 2M}$ a sequence of complex numbers. Assume that $\mathfrak{a} = \infty$ and let $\varepsilon > 0$. Then each of the three quantities appearing in Proposition 1.8 is bounded, up to a constant depending only on $\varepsilon$, by*

$$\left(T^2 + \frac{M^{1+\varepsilon}}{N}\right)||a_m||_M^2.$$

*Proof.* The extra factor $N_0^{1/2}$ in the conslusion of Proposition 1.8 comes from [16, Lemma 4.6, (4.20)] and is a consequence of the general estimation [42, Theorem 9.3]

$$S_\omega(m,n;c) \ll \tau(c)^{O(1)}(m,n,c)^{1/2}(N_0c)^{1/2}.$$

By the hypothesis on the conductor of $\omega$ (that it is squarefree and odd), we can apply Lemma 1.9 whose consequence is the cancellation of the factor $N_0^{1/2}$ in [16, Lemma 4.6, (4.20)] and the rest of the proof is completely similar. □

## 1.4 $L$-functions and functional equations

### 1.4.1 Dirichlet $L$-functions

Let $\chi$ be a non-principal Dirichlet character of modulus $q > 2$ with $q$ prime, $\kappa \in \{0,1\}$ satisfying $\chi(-1) = (-1)^\kappa$ and define the complete $L$-function

$$\Lambda(\chi, s) := q^{s/2} L_\infty(\chi, s) L(\chi, s),$$

where

$$L_\infty(\chi, s) := \pi^{-s/2}\Gamma\left(\frac{s+\kappa}{2}\right). \tag{1.24}$$

It is well known that $\Lambda(\chi, s)$ admits an analytic continuation to the whole complex plane and satisfies the functional equation [34, Theorem 4.15]

$$\Lambda(\chi, s) = i^\kappa \varepsilon(\chi)\Lambda(\overline{\chi}, 1-s), \tag{1.25}$$

where $\varepsilon(\chi)$ is the normalized Gauss sum defined by

$$\varepsilon(\chi) := \frac{1}{q^{1/2}} \sum_{x\in\mathbf{F}_q^\times} \chi(x) e\left(\frac{x}{q}\right). \tag{1.26}$$

Using (1.25), we can express the central value of a Dirichlet $L$-function as a convergent series [34, Theorem 5.3] and thus, extend in an easy way the proof to a product of three or four $L$-functions.

**Lemma 1.11.** *Let $\chi, \chi_1, \chi_2$ be Dirichlet characters. Let $\kappa$ (resp $\kappa_1$, $\kappa_2$) $\in \{0,1\}$ be such that*

$\chi(-1) = (-1)^{\kappa}$ (*resp* $\chi_1(-1) = (-1)^{\kappa_1}$, $\chi_2(-1) = (-1)^{\kappa_2}$).

1) *If* $\chi \neq 1, \overline{\chi}_1, \overline{\chi}_2$, *we have*

$$L\left(\chi, \tfrac{1}{2}\right) L\left(\chi\chi_1, \tfrac{1}{2}\right) L\left(\chi\chi_2, \tfrac{1}{2}\right) = \sum\sum\sum_{n_0,n_1,n_2\geqslant 1} \frac{\chi(n_0n_1n_2)\chi_1(n_1)\chi_2(n_2)}{(n_0n_1n_2)^{1/2}} \mathbf{V}_{\chi,\chi_1,\chi_2}\left(\frac{n_0n_1n_2}{q^{3/2}}\right)$$
$$+ \chi(-1) i^{\kappa+\kappa_1+\kappa_2} \varepsilon(\chi)\varepsilon(\chi\chi_1)\varepsilon(\chi\chi_2) \sum\sum\sum_{n_0,n_1,n_2\geqslant 1} \frac{\overline{\chi}(n_0n_1n_2)\overline{\chi}_1(n_1)\overline{\chi}_2(n_2)}{(n_0n_1n_2)^{1/2}} \mathbf{V}_{\chi,\chi_1,\chi_2}\left(\frac{n_0n_1n_2}{q^{3/2}}\right),$$

*where*

$$\mathbf{V}_{\chi,\chi_1,\chi_2}(x) := \frac{1}{2\pi i}\int_{(2)} \frac{L_\infty\left(\chi, \frac{1}{2}+s\right) L_\infty\left(\chi\chi_1, \frac{1}{2}+s\right) L_\infty\left(\chi\chi_2, \frac{1}{2}+s\right)}{L_\infty\left(\chi, \frac{1}{2}\right) L_\infty\left(\chi\chi_1, \frac{1}{2}\right) L_\infty\left(\chi\chi_2, \frac{1}{2}\right)} x^{-s} Q(s) \frac{ds}{s}. \tag{1.27}$$

2) *If* $\chi \neq 1$, *we have*

$$\left|L\left(\chi, \tfrac{1}{2}\right)\right|^4 = 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tau(n)\tau(m)}{(nm)^{1/2}} \chi(n)\overline{\chi}(m) \mathbf{V}_\chi\left(\frac{mn}{q^2}\right), \tag{1.28}$$

*with*

$$\mathbf{V}_\chi(x) := \frac{1}{2\pi i}\int_{(2)} \frac{L_\infty^2(\chi, 1/2+s) L_\infty^2(\overline{\chi}, 1/2+s)}{L_\infty^2(\chi, 1/2) L_\infty^2(\overline{\chi}, 1/2)} x^{-s} Q(s) \frac{ds}{s}. \tag{1.29}$$

*In* (1) *and* (2), $Q(s)$ *is an entire and even function having exponential decay in vertical strips and satisfying* $Q(0) = 1$.

**Remark 1.12.** Observe that for $\mathbf{V} = \mathbf{V}_\chi$ or $\mathbf{V} = \mathbf{V}_{\chi,\chi_1,\chi_2}$, the function $\mathbf{V}$ depends on $\chi$ only through its parity. Shifting the $s$-contour to the right in (1.27) or (1.29) and we see that for $x \geqslant 1$ and any $A \geqslant 0$, we have the estimation

$$\mathbf{V}(x) \ll_A x^{-A}.$$

Now moving the $s$-line to $\Re e(s) = -\frac{1}{2} + \varepsilon$, we pass a simple pole at $s = 0$ of residu 1 and thus, we obtain for $0 < x \leqslant 1$

$$\mathbf{V}(x) = 1 + O_\varepsilon\left(x^{1/2-\varepsilon}\right).$$

### 1.4.2 Twisted $L$-functions

Let $q > 2$ be a prime number, $\omega$ a Dirichlet character of modulus $q$ and $f$ a primitive Hecke cusp of level $q$ or 1 and nebentypus $\omega$ ($\omega = 1$ if the level is 1). For $\chi$ a non trivial character modulo $q$, we can construct the twisted modular form $f \otimes \chi$ whose $n$-th Fourier coefficient is given by $\rho_f(n)\chi(n)$. This form is a Hecke eigenform of level $q^2$ with nebentypus $\omega\chi^2$ and eigenvalues $\lambda_f(n)\chi(n)$ for $(n, q) = 1$ (see [32, Chapter 7]). The following proposition says when $f \otimes \chi$ stills a primitive form and derives the functional equation of the associated $L$-function $L(f \otimes \chi, s)$ (see for example [27] and [28]).

**Proposition 1.13.** *Let $q > 2$ be a prime number, $f, \omega$ and $\chi$ as above and assume further that $\chi \neq 1, \overline{\omega}$. Then the twisted modular form $f \otimes \chi$ is a primitive Hecke cusp form of level $q^2$ and nebentypus $\chi^2\omega$ with associated Hecke eigenvalues $\chi(n)\lambda_f(n)$ for every $n \geqslant 1$. If*

$$L(f \otimes \chi, s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}, \ \Re e(s) > 1 \tag{1.30}$$

*is its associated $L$-function, then there exists a factor $L_\infty(f \otimes \chi, s)$ such that the product*

$$\Lambda(f \otimes \chi, s) := q^s L_\infty(f \otimes \chi, s) L(f \otimes \chi, s)$$

*extends holomorphically to* $\mathbf{C}$ *and satisfies the functional equation*

$$\Lambda(f \otimes \chi, s) = \varepsilon_\infty(f, \chi)\varepsilon(f \otimes \chi)\Lambda(\overline{f} \otimes \overline{\chi}, 1 - s), \tag{1.31}$$

*where $\varepsilon(f \otimes \chi) = \varepsilon(\chi)\varepsilon(\omega\chi)$, $\varepsilon(\chi), \varepsilon(\omega\chi)$ are defined by* (1.26)*, the infinite factor $\varepsilon_\infty(f, \chi)$ satisfies $|\varepsilon_\infty(f, \chi)| = 1$ and both $L_\infty(f \otimes \chi, s)$ and $\varepsilon(\chi), \varepsilon(\omega\chi)$ depend on $\chi$ only through its parity.*

**Remark 1.14.** The infinite factor presents as a product of Gamma functions

$$L_\infty(f \otimes \chi, s) = \pi^{-s}\Gamma\left(\frac{s - \mu_{1,f\otimes\chi}}{2}\right)\Gamma\left(\frac{s - \mu_{2,f\otimes\chi}}{2}\right),$$

where $\mu_{i,f\otimes\chi}$ are the local parameters at the infinite place (encodes the weight in the holomorphic setting or the Laplace eigenvalue if $f$ is a Maass form) and we recall that they depend on $\chi$ at most through its parity. In any case, a consequence of the work of Kim and Sarnak [41] toward the Ramanujan-Petersson conjecture is that

$$\Re e(\mu_{i,f\otimes\chi}) \leqslant \frac{7}{64}. \tag{1.32}$$

We finally state the analogous of Lemma 1.11 for the product $L(f \otimes \chi, s)L(\chi, s)$ on the critical point $s = 1/2$.

**Proposition 1.15.** *Let $q > 2$ be a prime number, $\omega$ a Dirichlet character of modulus $q$ and $f$ a primitive Hecke cusp form of level $q$ or $1$ and nebentypus $\omega$ with associated Hecke eigenvalues $\lambda_f(n)$ for all $n \geqslant 1$. Then for every character $\chi$ modulo $q$ such that $\chi \neq 1, \overline{\omega}$, $\chi(-1) = (-1)^\kappa$ with $\kappa \in \{0, 1\}$, we have*

$$\begin{aligned} L\left(f \otimes \chi, \tfrac{1}{2}\right) L\left(\chi, \tfrac{1}{2}\right) &= \sum\sum_{n,m\geqslant 1} \frac{\lambda_f(n)\chi(n)\chi(m)}{(nm)^{1/2}} \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right) \\ &\quad + i^\kappa \varepsilon_\infty(f, \chi)\varepsilon(f \otimes \chi)\varepsilon(\chi) \sum\sum_{n,m\geqslant 1} \frac{\overline{\lambda_f(n)}\overline{\chi}(n)\overline{\chi}(m)}{(nm)^{1/2}} \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right), \end{aligned} \tag{1.33}$$

*where*

$$\mathbf{V}_{f,\chi}(x) := \frac{1}{2\pi i}\int_{(2)} \frac{L_\infty\left(f\otimes\chi, s+\frac{1}{2}\right)L_\infty\left(\chi, s+\frac{1}{2}\right)}{L_\infty\left(f\otimes\chi, \frac{1}{2}\right)L_\infty\left(\chi, \frac{1}{2}\right)} x^{-s} Q(s)\frac{ds}{s}, \tag{1.34}$$

*for any entire even function $Q(s)$ with exponential decay in vertical strips and satisfying $Q(0) = 1$.*

**Remark 1.16.** Shifting the $s$-contour on the right in (1.34) and we obtain that for every $x \geqslant 1$ and any $A > 0$,

$$\mathbf{V}_{f,\chi}(x) \ll_A x^{-A}.$$

By (1.32), moving the $s$-line to $\Re e(s) = -1/4$, we catch a simple pole at $s = 0$ of residue 1 and thus

$$\mathbf{V}_{f,\chi}(x) = 1 + O(x^{1/4}) \text{ for } 0 < x \leqslant 1.$$

# 2 Preliminaries II : Algebraic Trace Functions

## 2.1 $\ell$-adic twists of modular forms

In this chapter, we fix $q > 2$ a prime number, $\omega$ a Dirichlet character modulo $q$, $f$ a primitive Hecke cusp form of level $q$ and nebentypus $\omega$ and we denote by $\{\lambda_f(n)\}_{n\geqslant 1}$ the Hecke eigenvalues of $f$. For any $t \in \mathbf{R}$, we also define the twisted divisor function $\lambda_\omega(n, it)$ by

$$\lambda_\omega(n,it) := \sum_{ab=n} \omega(a)\left(\frac{a}{b}\right)^{it}, \tag{2.1}$$

which, for $(n,q) = 1$, appears as Hecke eigenvalues of Eisenstein series $E_{\mathfrak{a}}(\cdot, 1/2+it)$ of level $q$ and nebentypus $\omega$ for a suitable choice of cusp $\mathfrak{a}$ (c.f. Section 1.2.1 and (1.5)).

As announced in Section 0.3, for $K : \mathbf{Z} \longrightarrow \mathbf{C}$ a $q$-periodic function, a crucial step in the proof of Theorem 6 requires non trivial estimates for sums of the shape

$$\mathcal{S}_V(f,K;q) = \sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{q}\right), \tag{2.2}$$

$$\mathcal{S}_V(\omega,it,K;q) = \sum_{n\geqslant 1} \lambda_\omega(n,it)K(n)V\left(\frac{n}{q}\right), \tag{2.3}$$

where $V$ is a smooth and compactly supported function on $\mathbf{R}_+^*$. Assuming that $|K(n)| \leqslant M$ for every $n \in \mathbf{Z}$, we obtain by Cauchy-Schwarz inequality and (1.15),

$$\mathcal{S}_V(\omega,it,K;q), \mathcal{S}_V(f,K;q) \ll Mq^{1+\varepsilon}, \tag{2.4}$$

with an implied constant depending only on $V$, $\varepsilon$ and the spectral parameter $t_f$ and this bound can be seen as the trivial one. Theorem 6 improves on (2.4) with a power saving in the $q$-aspect, namely

$$\mathcal{S}_V(\omega,it,K;q), \mathcal{S}_V(f,K;q) \ll q^{1-\frac{1}{16}+\varepsilon},$$

for any $\varepsilon > 0$ and with an implied constant depending on $\varepsilon, V, t_f$ (or $t$) and controlled by some invariant of $K$, called the conductor (see Definition (2.5)). As in [24, Definition 1.1], we make the following definition about the test function $V$.

**Definition 2.1** (Condition $V(C,P,Q)$)**.** Let $P > 0$ and $Q \geqslant 1$ be real numbers and let $C = (C_\nu)_{\nu\geqslant 0}$ be a sequence of non-negative real numbers. A smooth compactly supported function $V$ on $\mathbf{R}$ satisfies Condition $(V(C,P,Q))$ if

(1) The support of $V$ is contained in the dyadic interval $[P,2P]$;

(2) For all $x > 0$ and all integer $\nu \geqslant 0$, we have the inequality

$$\left|x^\nu V^{(\nu)}(x)\right| \leqslant C_\nu Q^\nu.$$

In practice, the test function $V$ is not compactly supported, but rather in the Schwartz class. We give here a simple Corollary of Theorem 6.

**Corollary 2.2.** *Let $q > 2$ be a prime number, $f,\omega$ and $K$ as in Theorem 6. Let $Q \geqslant 1$, $C = (C_\nu)_\nu$ a sequence of non-negative real numbers and $V$ a smooth function on $\mathbf{R}$ with the property that for any $M > 0$,*

$$V(x) \ll_A \frac{1}{(1+|x|)^M} \text{ and } \left|x^\nu V^{(\nu)}(x)\right| \leqslant C_\nu Q^\nu, \;\; \nu \geqslant 0. \tag{2.5}$$

*Then for every $X > 0$ and every $\varepsilon > 0$, we have*

$$\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right) \ll_{C,\varepsilon} (qX)^\varepsilon(1+|t_f|)^A \mathrm{cond}(K)^s XQ^2\left(1+\frac{q}{X}\right)^{1/2} q^{-1/16},$$

$$\sum_{n\geqslant 1} \lambda_\omega(n,it)K(n)V\left(\frac{n}{X}\right) \ll_{C,\varepsilon} (qX)^\varepsilon(1+|t|)^A \mathrm{cond}(K)^s XQ^2\left(1+\frac{q}{X}\right)^{1/2} q^{-1/16},$$

*where $A$ is the constant appearing in Theorem 6.*

*Proof.* We consider the cuspidal case since the other is completely similar. Applying a partition of unity to $[1,\infty)$ leads to the decomposition

$$\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right) = \sum_N \sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right)W\left(\frac{n}{N}\right),$$

where $W$ is a smooth and compactly supported function on $(1/2,2)$ satisfying $|x^j W^{(j)}(x)| \leqslant \tilde{c}_j$ for some sequence $\tilde{c} = (\tilde{c}_j)$ of non-negative real numbers and $N$ runs over real numbers of the form $2^i$ with $i \geqslant 0$. Since $V$ has fast decay at infinity, we can focus on the contribution of $1 \leqslant N \leqslant q^\varepsilon X$ at the cost of an error of size $O(q^{-10})$, (say). Hence, we just need to bound $O(\log(qX))$ sums of the form

$$\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right)W\left(\frac{n}{N}\right).$$

By Mellin inversion formula, we have for any $\varepsilon > 0$

$$\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right)W\left(\frac{n}{N}\right) = \frac{1}{2\pi i}\int_{(\varepsilon)}\left(\frac{X}{N}\right)^s \widetilde{V}(s)\left(\sum_{n\geqslant 1}\lambda_f(n)K(n)W_s\left(\frac{n}{N}\right)\right)ds,$$

where the function $W_s(x) := x^{-s}W(x)$ satisfies

$$x^j W_s^{(j)}(x) \ll_{\tilde{c},j} (1+|s|)^j . \tag{2.6}$$

For fixed $s$ with $\Re e(s) = \varepsilon$, we apply Theorem 6 to the inner sum with the function $V(x) = W_s(xq/N)$ which satisfies condition $V(\tilde{C}, N/q, 1+|s|)$ for some $\tilde{C}$ depending on $\tilde{c}$, obtaining the bound

$$(1+|t_f|)^A \mathrm{cond}(K)^s q^{\frac{1}{2}-\frac{1}{16}+\varepsilon}\left(\frac{X}{N}\right)^{\varepsilon}\int_{(\varepsilon)}|\widetilde{V}(s)|(N(1+|s|))^{1/2}\left(\frac{N}{q}+1+|s|\right)^{1/2} ds.$$

Using the fact that the Mellin transform $\widetilde{V}(s)$ satisfies

$$\widetilde{V}(s) \ll \left(\frac{Q}{1+|s|}\right)^B,$$

for every $B > 0$ with an implied constant depending on $B$ and $\Re e(s)$, we see that we can restrict the integral to $|s| \leqslant q^{\varepsilon}Q$. Hence replacing $1+|s|$ by its maximal value, maximizing over $N \leqslant q^{\varepsilon}X$ and average trivially over $|s| \leqslant q^{\varepsilon}Q$ in the integral yields the desire result. □

## 2.2 Trace functions of $\ell$-adic sheaves

The functions to which we will apply Theorem 6 are called *trace functions* modulo $q$, which we now define formally.

Let $\ell \neq q$ be an auxiliary prime number. To any constructible $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbf{A}^1_{\mathbf{F}_q}$ and any point $x \in \mathbf{A}^1_{\mathbf{F}_q}$, we have a linear action of a geometric Frobenius $\mathrm{F}_x$ acting on a finite dimensional $\overline{\mathbf{Q}}_\ell$-vector space $\mathcal{F}_{\overline{x}}$. We can thus consider the trace $\mathrm{Tr}(\mathrm{F}_x|\mathcal{F}_{\overline{x}})$. Because this trace takes values in $\overline{\mathbf{Q}}_\ell$, we also fix a field isomorphism

$$\iota : \overline{\mathbf{Q}}_\ell \xrightarrow{\simeq} \mathbf{C},$$

and we consider functions of the shape

$$K(x) = \iota\left(\mathrm{Tr}(\mathrm{F}_x|\mathcal{F}_{\overline{x}})\right), \tag{2.7}$$

as defined in [39, (7.3.7)].

**Definition 2.3** (Trace sheaves)**.** 1) A constructible $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbf{A}^1_{\mathbf{F}_q}$ is called a *trace sheaf* if it is a middle extension sheaf (in the sense of [22, Section 1]) whose restriction on any non empty open subset $U \subset \mathbf{A}^1_{\mathbf{F}_q}$ where $\mathcal{F}$ is lisse is pointwise pure of weight zero.

2) A trace sheaf is called a *Fourier trace sheaf* if in addition, it is a Fourier sheaf in the sense of [38, Definition 8.2.2].

3) We say that $\mathcal{F}$ is an *isotypic trace sheaf* if it is a Fourier trace sheaf and if for every non empty open subset $U$ as in (1), the associated $\ell$-adic representation

$$\pi_1(U \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q, \overline{\eta}) \longrightarrow \mathrm{GL}(\mathcal{F}_{\overline{\eta}}),$$

of the geometric etale fondamental group is an isotypic representation [38, Chapter 2]. We define similarly an *irreducible trace sheaf*.

**Definition 2.4.** Let $q$ be a prime number. A function $K : \mathbf{F}_q \longrightarrow \mathbf{C}$ is called a *trace function* (resp. *Fourier trace function, isotypic trace function*) if there exists a trace sheaf (resp Fourier trace sheaf, isotypic trace sheaf) $\mathcal{F}$ such that $K$ is given by (2.7).

There is an important invariant which measures the complexity of a trace function which we define now.

**Definition 2.5** (Conductor)**.** Let $\mathcal{F}$ be a constructible $\overline{\mathbf{Q}}_\ell$-sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$ with $n(\mathcal{F})$ singularities in $\mathbf{P}^1_{\mathbf{F}_q}$. The *conductor* of $\mathcal{F}$ is the integer defined by

$$\mathrm{cond}(\mathcal{F}) := \mathrm{rank}(\mathcal{F}) + n(\mathcal{F}) + \sum_{x \in \mathbf{P}^1_{\mathbf{F}_q}} \mathrm{Swan}_x(\mathcal{F}),$$

where $\mathrm{Swan}_x(\mathcal{F}) = 0$ if $\mathcal{F}$ is lisse at $x$ (see for example [37, Section 4.6]). If $K$ is a trace function, the conductor $\mathrm{cond}(K)$ of $K$ is the smallest conductor of a trace sheaf $\mathcal{F}$ with trace function $K$.

Let $\mathcal{F}$ be a trace sheaf with associated trace function $K : \mathbf{F}_q \to \mathbf{C}$. The normalized Fourier transform of $K$, denoted by $\widehat{K}$ or $\mathrm{FT}(K)$, is the function on $\mathbf{F}_q$ defined by

$$\widehat{K}(y) := \frac{1}{q^{1/2}} \sum_{x \in \mathbf{F}_q} K(x) e\left(\frac{xy}{q}\right).$$

When $\mathcal{F}$ is a Fourier sheaf, there is a deep interpretation of the Fourier transform at the level of sheaves that was discovered by Deligne and developed by Laumon, especially in [47]. To be precise, there exists a Fourier sheaf $\mathcal{G}$ whose conductor satisfies

$$\mathrm{cond}(\mathcal{G}) \leqslant 10\, \mathrm{cond}(\mathcal{F})^2, \tag{2.8}$$

and with the property that

$$\iota\left(\mathrm{Tr}\left(\mathrm{F}_x | \mathcal{G}_{\overline{x}}\right)\right) = -\widehat{K}(x).$$

Moreover, the sheaf $\mathcal{G}$ is geometrically isotypic (resp. geometrically irreducible) if and only if $\mathcal{F}$ has this property [24, Lemma 8.1].

### 2.2.1 Kloosterman sheaves

Let $k \geqslant 2$ be an integer, $\chi_1, ..., \chi_k$ be multiplicative characters on $\mathbf{F}_q^\times$. The twisted rank $k$ Kloosterman sum $\mathrm{Kl}_k(\chi_1, ..., \chi_k; q)$ is the function on $\mathbf{F}_q^\times$ defined by

$$\mathrm{Kl}_k(a, \chi_1, ..., \chi_k; q) := \frac{1}{q^{\frac{k-1}{2}}} \sum_{\substack{x_1, ..., x_k \in \mathbf{F}_q^\times \\ x_1 \cdots x_k = a}} \chi_1(x_1) \cdots \chi_k(x_k) e\left(\frac{x_1 + ... + x_k}{q}\right), \tag{2.9}$$

for every $a \in \mathbf{F}_q^\times$. If $\chi_1 = ... = \chi_k = 1$, we write $\mathrm{Kl}_k(a; q)$ instead of $\mathrm{Kl}_k(a, 1, ..., 1; q)$. The main result is the existence of Kloosterman sheaves and it is due to Deligne [38, Theorem 4.1.1].

**Theorem 2.6** (Kloosterman sheaves)**.** *For every prime $\ell \neq q$, there exists a constructible $\overline{\mathbf{Q}}_\ell$-sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$, denoted by $\mathcal{K}\ell_k(\chi_1, ..., \chi_k; q)$ (or simply $\mathcal{K}\ell$), of rank $k$ satisfying the following properties :*

(1) *For every $a \in \mathbf{F}_q^\times$, we have the equality*

$$\iota\left(\mathrm{Tr}\left(\mathrm{F}_a | (\mathcal{K}\ell)_{\overline{a}}\right)\right) = (-1)^{k-1} \mathrm{Kl}_k(a, \chi_1, ..., \chi_k; q);$$

(2) *$\mathcal{K}\ell$ is geometrically irreducible, lisse on $\mathbf{G}_{m,\mathbf{F}_q}$ and pointwise pure of weight zero;*

(2) *$\mathcal{K}\ell$ has wild ramification at $\infty$ with* $\mathrm{Swan}_\infty(\mathcal{K}\ell) = 1$*, tamely ramified at* $0$ *and has conductor $k+3$.*

*In particular, $\mathcal{K}\ell$ is an irreducible trace sheaf in the sense of Definition 2.3* (3).

**Corollary 2.7.** *For every $a \in \mathbf{F}_q^\times$, we have the sharp bound*

$$|\mathrm{Kl}_k(a, \chi_1, ..., \chi_k; q)| \leqslant k. \tag{2.10}$$

It will be convenient in Chapter 6, specially because of the Poisson summation and Fourier inversion formula, to present the Klososterman sum Kl as a Fourier transform of a function defined on $\mathbf{F}_q$. For this, we set

$$\mathrm{Kl}_1(a, \chi_1; q) := \chi_1(a) e\left(\frac{a}{q}\right),$$

and we see that for any $k \geqslant 2$ and $a \in \mathbf{F}_q^\times$, $\mathrm{Kl}_k(a, \chi_1, ..., \chi_k; q)$ is given by the formula

$$\mathrm{Kl}_k(a, \chi_1, ..., \chi_k; q) = \chi_k(a) \mathrm{FT}\Big(\mathbf{F}_q \ni x \mapsto \chi_k(x) \mathbf{K}_{k-1}(x, \chi_1, ..., \chi_{k-1}; q)\Big)(a), \tag{2.11}$$

where the function $\mathbf{K}_{k-1}$ is defined by

$$\mathbf{K}_{k-1}(x, \chi_1, ..., \chi_{k-1}; q) := \begin{cases} \mathrm{Kl}_{k-1}(\overline{x}, \chi_1, ..., \chi_{k-1}; q) & \text{if} \quad x \in \mathbf{F}_q^\times, \\ \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{2.12}$$

**Remark 2.8.** There are several ways to extend the function $\mathrm{Kl}_k$ to $a = 0$. One can choose for example the extension by zero. We choose here the middle extension, i.e. that $\mathrm{Kl}_k(0, \chi_1, ..., \chi_k; q)$ coincides with the trace of the Frobenius at $x = 0$. It is a deep result of Deligne that the estimate (2.10) remains valid for $a = 0$ (see [14, (1.8.9)]).

## 2.3 Twisted Correlation Sums and the $\omega$-Möbius Group

The strategy in the proof of Theorem 6 is to estimate an amplified second moment of $\mathcal{S}_V(g, K; q)$ for $g$ varying in a basis of cusp forms of level $q$ and nebentypus $\omega$. After completing the spectral sum, applying Kuznetsov-Petersson and Poisson formula, we have to confront some sums that we call *twisted correlation sums*, which we now define.

We let $\mathrm{PGL}_2(\mathbf{F}_q)$ acts on the projective line $\mathbf{P}^1(\mathbf{F}_q)$ by fractional linear transformations

$$\gamma z = \frac{az+b}{cz+d},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{F}_q).$$

**Definition 2.9** (Twisted correlation sum)**.** Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbf{F}_q)$. For $\omega$ a multiplicative character modulo $q$ and $K : \mathbf{F}_q \to \mathbf{C}$, we define the twisted correlation sum $\mathcal{C}(K, \omega; \gamma)$ by

$$\mathcal{C}(K, \omega; \gamma) := \sum_{\substack{z \in \mathbf{F}_q \\ z \neq -d/c}} \overline{\omega}(cz+d) K(\gamma z) \overline{K(z)}.$$

**Remark 2.10.** Note that for $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q)$, $\mathcal{C}(K, \omega; \gamma)$ is well defined up to multiplication by $\omega(-1) \in \{-1, +1\}$. This is in fact not a problem since only the complex modulus $|\mathcal{C}(K, \omega; \gamma)|$ will be considered later. We also mention that unlike the definition of correlation sum that the authors made for the original Theorem (see [24, eq. (1.10)]), we have the presence here of a twist by the nebentypus of the modular form $f$. This is because the Kloosterman sums that we obtain after the application of Kuznetsov trace formula are also twisted by $\omega$.

Note that for $K$ a trace function, we have the bound $||K||_\infty \leqslant \mathrm{cond}(K)$. Hence using Cauchy-Schwarz and Parseval identity, we get

$$|\mathcal{C}(K, \omega; \gamma)| \leqslant \mathrm{cond}(K)^2 q. \tag{2.13}$$

In order to obtain better bounds, we introduce a geometric object associated to the correlation sum $\mathcal{C}(K, \omega; \gamma)$.

**Definition 2.11.** Let $q$ be prime number and $\mathcal{F}$ an isotypic trace sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$. Let $\omega$ be a multiplicative character modulo $q$ and $\mathcal{L}_\omega$ the associated Kummer sheaf. The $\omega$-Möbius group $\mathbf{G}_{\mathcal{F},\omega}$ is the subgroup of $\mathrm{PGL}_2(\mathbf{F}_q)$ defined by

$$\mathbf{G}_{\mathcal{F},\omega} := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{F}_q) \mid \mathcal{F} \simeq_{\mathrm{geom}} \gamma^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)} \right\}.$$

**Remark 2.12.** Note that Definition 2.11 makes sense in the sense that if $\gamma, \gamma' \in \mathrm{GL}_2(\mathbf{F}_q)$ are equal in $\mathrm{PGL}_2(\mathbf{F}_q)$, then $\gamma = \pm \mathrm{I}_2 \gamma'$ and thus

$$\gamma^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)} = \gamma'^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(-c'X-d')} \simeq_{\text{geom}} \gamma'^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(c'X+d')} \otimes \mathcal{L}_{\overline{\omega}(-1)} \simeq_{\text{geom}} \gamma'^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(c'X+d')}.$$

The crucial property here is that $\mathbf{G}_{\mathcal{F},\omega}$ is indeed a subgroup of $\mathrm{PGL}_2(\mathbf{F}_q)$.

**Proposition 2.13.** $\mathbf{G}_{\mathcal{F},\omega}$ *is a subgroup of* $\mathrm{PGL}_2(\mathbf{F}_q)$.

*Proof.* Let $\mathscr{F}$ be the set of geometric isomorphism classes of trace sheaves. To show that $\mathbf{G}_{\mathcal{F},\omega}$ is a subgroup, it is enough to prove that the map $\mathscr{F} \times \mathrm{PGL}_2(\mathbf{F}_q) \longrightarrow \mathscr{F}$ given by

$$(\mathcal{F}, \gamma) \longmapsto \mathcal{F} \cdot \gamma := \gamma^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)}, \tag{2.14}$$

defines a right group action because $\mathbf{G}_{\mathcal{F},\omega}$ will be the stabilizer of $\mathcal{F}$. For this, we will use the fact that we have geometric isomorphisms (we use the notation $\simeq$ instead of $\simeq_{\text{geom}}$)

$$\mathcal{L}_{\overline{\omega}(d)} \simeq \overline{\mathbf{Q}}_\ell \simeq \mathcal{L}_{\overline{\omega}(cX+d)} \otimes \mathcal{L}_{\omega(cX+d)}, \tag{2.15}$$

where $\overline{\mathbf{Q}}_\ell$ denotes the constant sheaf. The first isomorphism implies that the identity matrix acts trivially. For the second part, note that for $\gamma_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbf{F}_q)$, we have

$$\mathcal{L}_{\overline{\omega}(cX+d)} \simeq j(\gamma)^* \mathcal{L}_\chi, \;\; j(\gamma) := \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix},$$

and

$$j(\gamma_1\gamma_2) = \begin{pmatrix} 0 & 1 \\ ca' + dc' & cb' + dd' \end{pmatrix} \;\; , \;\; j(\gamma_1)\gamma_2 = \begin{pmatrix} c' & d' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

Combining the above equality with the second isomorphism in (2.15) leads to

$$j(\gamma_1\gamma_2)^* \mathcal{L}_\omega \simeq (j(\gamma_1)\gamma_2)^* \mathcal{L}_\omega \otimes j(\gamma_2)^* \mathcal{L}_\omega.$$

Hence we obtain

$$\begin{aligned} \mathcal{F} \cdot (\gamma_1\gamma_2) &\simeq (\gamma_1\gamma_2)^* \mathcal{F} \otimes j(\gamma_1\gamma_2)^* \mathcal{L}_\omega \simeq \gamma_2^* \gamma_1^* \mathcal{F} \otimes \gamma_2^* j(\gamma_1)^* \mathcal{L}_\omega \otimes j(\gamma_2)^* \mathcal{L}_\omega \\ &\simeq \gamma_2^* \Big( \gamma_1^* \mathcal{F} \otimes j(\gamma_1)^* \mathcal{L}_\omega \Big) \otimes j(\gamma_2)^* \mathcal{L}_\omega \\ &\simeq (\mathcal{F} \cdot \gamma_1) \cdot \gamma_2, \end{aligned}$$

which completes the proof of this Proposition. □

We will also need the following fact about the conductor of $\mathcal{F} \cdot \gamma$.

**Lemma 2.14.** *Let $\mathcal{F}$ be a trace sheaf and $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q)$. Then*

$$\left|\mathrm{cond}(\mathcal{F}\cdot\gamma) - \mathrm{cond}(\mathcal{F})\right| \leqslant 2.$$

*Proof.* Since the Kummer sheaves are of rank one and tamely ramified at the singularities, we have for any $x \in \mathbf{P}^1_{\mathbf{F}_q}$ (by definition (2.14) of the action of $\gamma$ on $\mathcal{F}$),

$$\mathrm{rank}(\gamma^*\mathcal{F}\otimes\mathcal{L}_{\overline{\omega}(cX+d)}) = \mathrm{rank}(\gamma^*\mathcal{F}) = \mathrm{rank}(\mathcal{F}),\ \ \mathrm{Swan}_x(\mathcal{F}\cdot\gamma) = \mathrm{Swan}_x(\mathcal{F}),$$

see [37, 4.6 (iv)]. Moreover, if $n(\mathcal{F})$ (resp $n(\mathcal{L}_{\overline{\omega}(cX+d)})$) denotes the number of singularities of $\mathcal{F}$ (resp. of $\mathcal{L}_{\overline{\omega}(cX+d)}$), the tensor product $\gamma^*\mathcal{F}\otimes\mathcal{L}_{\overline{\omega}(cX+d)}$ satisfies (see for example [53, Proposition 1.23])

$$\begin{aligned} n(\mathcal{F}) - n(\mathcal{L}_{\overline{\omega}(cX+d)}) &= n(\gamma^*\mathcal{F}) - n(\mathcal{L}_{\overline{\omega}(cX+d)}) \\ &\leqslant n(\gamma^*\mathcal{F}\otimes\mathcal{L}_{\overline{\omega}(cX+d)}) \\ &\leqslant n(\gamma^*\mathcal{F}) + n(\mathcal{L}_{\overline{\omega}(cX+d)}) = n(\mathcal{F}) + n(\mathcal{L}_{\overline{\omega}(cX+d)}), \end{aligned}$$

which completes the proof since $n(\mathcal{L}_{\overline{\omega}(cX+d)}) = 0$ or $2$ depending on whether $c = 0$ or not. □

The following proposition establishes the link between the correlation sum $\mathcal{C}(K,\omega;\gamma)$ and the $\omega$-Möbius group $\mathbf{G}_{\mathcal{F},\omega}$. The proof uses deep results on $\ell$-adic cohomology for varieties over finite fields.

**Proposition 2.15.** *Let $q > 2$ be a prime number, $\mathcal{F}$ an isotypic trace sheaf with associated trace function $K$ modulo $q$ of conductor $\mathrm{cond}(K)$. Let $\omega : \mathbf{F}_q^\times \to \mathbf{C}^\times$ be a multiplicative character. Then there exists absolute constants $A, s \geqslant 1$ such that for any $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q) - \mathbf{G}_{\mathcal{F},\omega}$,*

$$\left|\mathcal{C}(K,\omega;\gamma)\right| \leqslant A(\mathrm{cond}(K))^s q^{1/2}.$$

*Proof.* See [24, Theorem 9.1]. □

The last part of this section is devoted to the structure of the group $\mathbf{G}_{\mathcal{F},\omega} \subset \mathrm{PGL}_2(\mathbf{F}_q)$. For some technical reasons due to the amplification method and the fact that we are dealing with forms of level $q$, we want to avoid the presence of unipotent elements in our group $\mathbf{G}_{\mathcal{F},\omega}$ because in contrast to [24, Theorem 1.2], parabolic matrices could appear in our case and their contribution seems to be big. We therefore impose an additional hypothesis on our sheaf $\mathcal{F}$ and prove that under this extra assumption, the group $\mathbf{G}_{\mathcal{F},\omega}$ does not contain any unipotent elements. Before doing this, we introduce the following notation :

– For $x \neq y$ in $\mathbf{P}^1$, the pointwise stabilizer of $x$ and $y$ in $\mathrm{PGL}_2$ is denoted by $\mathrm{T}^{x,y}$ (this is a maximal torus) and its stabilizer in $\mathrm{PGL}_2$ (or the stabilizer of the set $\{x, y\}$) is denoted by $\mathrm{N}^{x,y}$.

**Definition 2.16.** Let $\mathcal{F}$ be an isotypic trace sheaf. We say that $\mathcal{F}$ is $\omega$-exceptional if its geometric irreducible component is of the form $\mathcal{L}_\psi \cdot \gamma = \gamma^*\mathcal{L}_\psi \otimes \mathcal{L}_{\omega(cX+d)}$ for some Artin-Schreier sheaf $\mathcal{L}_\psi$ and some $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q)$.

**Proposition 2.17.** *Let $q > 2$ be a prime number, $\mathcal{F}$ an isotypic trace sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$ and $\omega : \mathbf{F}_q^\times \to \mathbf{C}^\times$. Assume that $\mathcal{F}$ is not $\omega$-exceptional and that $q$ is large enough compared to the conductor* $\mathrm{cond}(K)$. *Then $\mathbf{G}_{\mathcal{F},\omega}$ satisfies one of the following :*

(*a*) $|\mathbf{G}_{\mathcal{F},\omega}| \leqslant 60$ *and the non trivial elements of* $\mathbf{G}_{\mathcal{F},\omega}$ *belong to at most* 59 *different tori.*

(*b*) $\mathbf{G}_{\mathcal{F},\omega}$ *is cyclic and is contained in the normalizer* $\mathrm{N}^{x,y}$ *of a certain maximal torus* $\mathrm{T}^{x,y}$ *for* $x \neq y$ *in* $\mathbf{P}^1$.

(*c*) $\mathbf{G}_{\mathcal{F},\omega}$ *is dihedral and its cyclic subgroup is contained in a maximal torus* $\mathrm{T}^{x,y}$ *and any element not contained in it is in the normalizer* $\mathrm{N}^{x,y}$ *($x \neq y$).*

*In particular,* $\mathbf{G}_{\mathcal{F},\omega}$ *does not contain parabolic elements.*

*Proof.* If the order of $\mathbf{G}_{\mathcal{F},\omega}$ is coprime with $q$, the first paragraph in the proof of [24, Theorem 1.14] says that $\mathbf{G}_{\mathcal{F},\omega}$ is one of the three types of groups cited above.

We now show that the order of $\mathbf{G}_{\mathcal{F},\omega}$ cannot be divisible by $q$. Assume by contradiction that it is the case and fix an element $\gamma_0 \in \mathbf{G}_{\mathcal{F},\omega}$ of order $q$. Then $\gamma_0$ is necessarily parabolic, so it has a unique fixed point $x \in \mathbf{P}^1(\mathbf{F}_q)$. Let $\sigma \in \mathrm{PGL}_2(\mathbf{F}_q)$ be such that

$$\sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma^{-1} = \gamma_0,$$

and define

$$\mathcal{G} := \mathcal{F} \cdot \sigma = \sigma^*\mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)}, \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since geometrically we have $[+1]^*\mathcal{F} \simeq \mathcal{F} \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we see that we have a geometric isomorphism

$$[+1]^*\mathcal{G} \simeq \mathcal{G}.$$

Suppose first that $\mathcal{G}$ is ramified at some $y \in \mathbf{A}^1(\overline{\mathbf{F}}_q)$, then by the above, $\mathcal{G}$ is also ramified at $y+1, \dots, y+p-1$ and we obtain by Lemma 2.14

$$\mathrm{cond}(\mathcal{F}) \geqslant \mathrm{cond}(\mathcal{G}) - 2 \geqslant q - 2 + \mathrm{rank}(\mathcal{G}) \geqslant q - 1,$$

which is a contradiction with the fact that $\mathrm{cond}(\mathcal{F}) < q-1$. Assume now that $\mathcal{G}$ is lisse on $\mathbf{A}^1(\overline{\mathbf{F}}_q)$. Since $\mathcal{F}$ is geometrically isotypic, the same is true for $\mathcal{G}$ and the geometrically irreducible component $\mathcal{G}_1$ of $\mathcal{G}$ also satisfies $[+1]^*\mathcal{G}_1 \simeq \mathcal{G}_1$. Using [22, Lemma 5.4, (2)] with $G = \mathbf{F}_q$ and $P_h = 0$, we have either

$$\mathrm{cond}(\mathcal{G}) \geqslant \mathrm{Swan}_\infty(\mathcal{G}_1) \geqslant q + \mathrm{rank}(\mathcal{G}_1)$$

and in this case we are done as before, or $\mathcal{G}_1$ is geometrically isomorphic to some Artin-Schreier sheaf $\mathcal{L}_\psi$ for some additive character $\psi$. It follows that $\mathcal{G}$ is geometrically isomorphic to a direct sum of copies of $\mathcal{L}_\psi$ and thus, by definition of $\mathcal{G}$, we have a geometric isomorphism

$$\mathcal{F} \simeq \left(\bigoplus \mathcal{L}_\psi\right)\cdot\sigma^{-1} = \bigoplus \mathcal{L}_\psi \cdot \sigma^{-1},$$

which contradicts the fact that $\mathcal{F}$ is not $\omega$-exceptional. □

## 2.4 Bilinear forms involving trace functions

We begin with a classical result, which is a simple consequence of the Poisson summation formula.

**Proposition 2.18** (Polyá-Vinogradov method)**.** *Let $q$ be a prime number and $\mathcal{F}$ be a Fourier trace sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$ with corresponding trace function $K$ modulo $q$. Let $f$ be a smooth and compactly supported function on $\mathbf{R}$ and $N > 0$ be a real number. Then for any $\varepsilon > 0$, we have*

$$\sum_{n\in\mathbf{Z}} K(n) f\left(\frac{n}{N}\right) \ll \min\left\{N, \frac{N}{q^{1/2}}\left(1+\frac{q^{1+\varepsilon}}{N}\right)\right\},$$

*where the implied constant depends on $\varepsilon$, $f$ and the conductor of $\mathcal{F}$.*

A more elaborate treatment of the Polyá-Vinogradov method can be used to obtain bounds for bilinear sums [23, Theorem 1.17].

**Theorem 2.19.** *Let $K$ be an isotypic trace function modulo $q$ associated to an isotypic $\ell$-adic sheaf $\mathcal{F}$ such that $\mathcal{F}$ does not contain a sheaf of the form $\mathcal{L}_\omega \otimes \mathcal{L}_\psi$ in his irreducible component. Let $M, N \geqslant 1$ be parameters and $(\alpha_m)_m, (\beta_n)_n$ two sequences of complex numbers supported on $[M/2, 2M]$ and $[N/2, 2N]$ respectively.*

(1) *We have*

$$\sum_{\substack{n,m \\ (m,q)=1}}\sum \alpha_m \beta_n K(mn) \ll ||\alpha||_2 ||\beta||_2 (NM)^{1/2}\left(\frac{1}{q^{1/4}} + \frac{1}{M^{1/2}} + \frac{q^{1/4}\log^{1/2} q}{N^{1/2}}\right),$$

*with*

$$||\alpha||_2^2 = \sum_m |\alpha_m|^2 \ , \ ||\beta||_2^2 = \sum_n |\beta_n|^2.$$

(2) *We have*

$$\sum_{(m,q)=1} \alpha_m \sum_{n\leqslant N} K(mn) \ll \left(\sum_m |\alpha_m|\right) N \left(\frac{1}{q^{1/2}} + \frac{q^{1/2}\log q}{N}\right).$$

*In both estimates, the implicit constants depend only, and at most polynomially, on the conductor of $\mathcal{F}$.*

The above theorem beats the trivial bound and gives a power saving in the error term as long as $\max(N, M) \geqslant q^{1/2+\delta}$ and $\min(M, N) \geqslant q^{\delta}$ for some $\delta > 0$. In the critical case where $N \sim M \sim q^{1/2}$, we have the powerful result of Kowalski, Michel and Sawin, which still saves a small power of $q$, but has been proved in the special case of classical hyper-Kloosterman sums [46, Theorem 1.3].

**Theorem 2.20.** *Let $q$ be a prime number and $a$ an integer coprime with $q$. Let $M, N \geqslant 1$ be such that*

$$1 \leqslant M \leqslant N^2, \ \ N < q, \ \ MN < q^{3/2}. \tag{2.16}$$

*Let $(\alpha_m)_{m \leqslant M}$ be a sequence of complex numbers and $\mathcal{N} \subset [1, q-1]$ be an interval of length $N$. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \in \mathcal{N}} \sum_{1 \leqslant m \leqslant M} \alpha_m \mathrm{Kl}_k(anm; q) \ll q^{\varepsilon} ||\alpha||_1^{1/2} ||\alpha||_2^{1/2} M^{1/4} N \left( \frac{M^2 N^5}{q^3} \right)^{-1/12}, \tag{2.17}$$

*with*

$$||\alpha||_1 = \sum_{1 \leqslant m \leqslant M} |\alpha_m|$$

*where the implied constant in* (2.17) *only depends on $\varepsilon$ and $k$.*

# 3 Proof of Theorem 3

Let $q > 2$ be a prime number and $\ell_1, \ell_2$ be two cubefree integers such that $(\ell_1, \ell_2) = 1$, $(\ell_1\ell_2, q) = 1$ and $\ell_i \leqslant L$ with $L$ a small power of $q$. The fundamental quantity that we will study in this chapter is the twisted fourth moment defined in (0.8). It is in fact more natural to split the family $\{\chi \pmod q\}$ separately into even characters and odd characters because they have different gamma factors in their functional equations. In this chapter, we concentrate almost exclusively on the even characters because the case of the odd characters is similar (we could treat both cases simultaneously but it would clutter the notation). We briefly describe the necessary changes to treat the odd characters at the end of Section 3.5.3 since we need to take them in account for the symmetry of a certain function (see Section 3.5.3). We thus study

$$\mathscr{T}^4(\ell_1, \ell_2; q) := \frac{2}{\phi^*(q)} \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi \neq 1}} \left|L\left(\chi, \tfrac{1}{2}\right)\right|^4 \chi(\ell_1)\overline{\chi}(\ell_2), \tag{3.1}$$

where the symbol + over the summation means that we restrict ourselves to the case of even characters and we recall that $\phi^*(q)$ denotes the number of primitive characters modulo $q$ (here $q-2$).

Using the approximate functional equation (1.28) from Lemma 1.11 (we omit the dependence in $\chi$ in the definition of $V_\chi$ since we deal with even characters and $V_\chi$ depends on $\chi$ only through its parity) we can rewrite (3.1) as

$$\mathscr{T}_4(\ell_1, \ell_2; q) = \frac{4}{\phi^*(q)} \sum_{n,m} \frac{\tau(n)\tau(m)}{(nm)^{1/2}} \mathbf{V}\left(\frac{nm}{q^2}\right) \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi \neq 1}} \chi(n)\overline{\chi}(m)\chi(\ell_1)\overline{\chi}(\ell_2).$$

We now use the following identity which allows us to average the sum over the characters and it is valid for $(m, q) = 1$ (see for instance [31, (3.1)-(3.2)])

$$\sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi \neq 1}} \chi(m) = \frac{1}{2} \sum_{\pm} \sum_{\substack{d \mid q \\ m \equiv \pm 1 (d)}} \phi(d)\mu\left(\frac{q}{d}\right). \tag{3.2}$$

Hence we obtain $\mathscr{T}^4(\ell_1,\ell_2;q)=\sum_{\pm}\mathscr{T}^{4,\pm}(\ell_1,\ell_2;q)$ with

$$\mathscr{T}^{4,\pm}(\ell_1,\ell_2;q):=\frac{2}{\phi^*(q)}\sum_{d|q}\phi(d)\mu\left(\frac{q}{d}\right)\sum_{\substack{\ell_1 n\equiv\pm\ell_2 m\ (d)\\ (mn,q)=1}}\sum\frac{\tau(n)\tau(m)}{(nm)^{1/2}}V\left(\frac{nm}{q^2}\right). \tag{3.3}$$

We now decompose $\mathscr{T}^4(\ell_1,\ell_2;q)$ into a diagonal part and a off-diagonal term by writing

$$\mathscr{T}^4(\ell_1,\ell_2;q)=\sum_{\pm}\mathscr{T}^{4,\pm}_{OD}(\ell_1,\ell_2;q)+\mathscr{T}^4_D(\ell_1,\ell_2;q), \tag{3.4}$$

where $\mathscr{T}^{4,\pm}_{OD}(\ell_1,\ell_2;q)$ is the same as in (3.3) but with the extra condition that $n\ell_1\neq m\ell_2$ and the diagonal part is given by

$$\mathscr{T}^4_D(\ell_1,\ell_2;q):=2\sum_{\substack{\ell_1 n=\ell_2 m\\ (nm,q)=1}}\sum\frac{\tau(n)\tau(m)}{(nm)^{1/2}}V\left(\frac{nm}{q^2}\right). \tag{3.5}$$

## 3.1 Computation of the Diagonal Part

In this section, we extract a main term coming from the diagonal part $\mathscr{T}^4_D(\ell_1,\ell_2;q)$. We use the standard technique consisting in shifting the contour of integration. We first remark that up to an error of size $O(L^{1/2}q^{-1+\varepsilon})$, we can remove the primality condition $(nm,q)=1$. Once we have done this, we write $V$ as inverse Mellin transform (see definition (1.29)), obtaining (up to an error term of $O(L^{1/2}q^{-1+\varepsilon})$)

$$\mathscr{T}^4_D(\ell_1,\ell_2;q)=\frac{2}{2\pi i}\int_{(2)}G(s)q^{2s}\left(\sum_{\ell_1 n=\ell_2 m}\sum\frac{\tau(n)\tau(m)}{(nm)^{1/2+s}}\right)\frac{ds}{s}, \tag{3.6}$$

where $G(s)$ is the integrand in **V**, i.e. (see (1.29) and recall that $\kappa=0$ here)

$$G(s)=\pi^{-2s}\frac{\Gamma\left(\frac{\frac{1}{2}+s}{2}\right)^4}{\Gamma(\frac{1}{4})^4}Q(s). \tag{3.7}$$

**Lemma 3.1.** *We have the factorization*

$$\sum_{\ell_1 n=\ell_2 m}\sum\frac{\tau(n)\tau(m)}{(nm)^{1/2+s}}=\frac{f(\ell_1\ell_2;1+2s)}{(\ell_1\ell_2)^{1/2+s}}\frac{\zeta^4(1+2s)}{\zeta(2+4s)}, \tag{3.8}$$

*where $n\mapsto f(n;s)$ is a multiplicative function supported on cubefree integers whose values on $p$ and $p^2$ are given by*

$$f(p;s)=\frac{2}{1+p^{-s}}\ ,\ f(p^2;s)=\frac{3-p^{-s}}{1+p^{-s}}. \tag{3.9}$$

*Proof.* Since $(\ell_1,\ell_2)=1$, the condition $\ell_1 n=\ell_2 m$ is equivalent to $n=\ell_2 j$ and $m=\ell_1 j$ with

$j \geqslant 1$. Thus, the lefthand side of (3.8) can be written as

$$\frac{1}{(\ell_1\ell_2)^{1/2+s}}\sum_{j\geqslant 1}\frac{\tau(\ell_1 j)\tau(\ell_2 j)}{j^{1+2s}}.$$

Using the fact that the $\ell_i'$s are cubefree, we factorize the above sum as an infinite product over the primes

$$\prod_{p\nmid \ell_1\ell_2} L_p \prod_{p\|\ell_1} L_p \prod_{p^2|\ell_1} L_p \prod_{p\|\ell_2} L_p \prod_{p^2|\ell_2} L_p,$$

with

$$L_p = \begin{cases} \sum_{\alpha\geqslant 0}\frac{(\alpha+2)(\alpha+1)}{p^{\alpha(1+2s)}} & \text{if} \quad p\|\ell_i, \\ \sum_{\alpha\geqslant 0}\frac{(\alpha+3)(\alpha+1)}{p^{\alpha(1+2s)}} & \text{if} \quad p^2|\ell_i, \\ \sum_{\alpha\geqslant 0}\frac{(\alpha+1)^2}{p^{\alpha(1+2s)}} & \text{if} \quad p\nmid \ell_1\ell_2. \end{cases}$$

Using

$$\sum_{\alpha\geqslant 0}\frac{(\alpha+1)}{p^{\alpha(1+2s)}} = \frac{1}{(1-p^{-1-2s})^2} \quad \text{and} \quad \sum_{\alpha\geqslant 0}\frac{(\alpha+1)^2}{p^{\alpha(1+2s)}} = \frac{1+p^{-1-2s}}{(1-p^{-1-2s})^3}$$

and we get for $p\|\ell_i$

$$\begin{aligned} L_p &= \sum_{\alpha\geqslant 0}\frac{(\alpha+1)}{p^{\alpha(1+2s)}} + \sum_{\alpha\geqslant 0}\frac{(\alpha+1)^2}{p^{\alpha(1+2s)}} = \frac{1}{(1-p^{-1-2s})^2} + \frac{1+p^{-1-2s}}{(1-p^{-1-2s})^3} \\ &= \left(\frac{1-p^{-1-2s}}{1+p^{-1-2s}}+1\right)\frac{1+p^{-1-2s}}{(1-p^{-1-2s})^3} = \frac{2}{1+p^{-1-2s}}\frac{1+p^{-1-2s}}{(1-p^{-1-2s})^3}. \end{aligned}$$

We proceed in a similar way for $p^2|\ell_i$ and we obtain

$$L_p = \frac{3-p^{-1-2s}}{1+p^{-1-2s}}\frac{1+p^{-1-2s}}{(1-p^{-1-2s})^3}.$$

We conclude the lemma by the well known identity

$$\sum_{n\geqslant 1}\frac{\tau(n)^2}{n^s} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^3} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

□

We insert the factorization (3.8) in (3.6), obtaining

$$\mathscr{T}_D^4(\ell_1,\ell_2,q) = \frac{2}{2\pi i}\int_{(2)}\frac{G(s)q^{2s}}{\zeta(2+4s)}\frac{f(\ell_1\ell_2;1+2s)}{(\ell_1\ell_2)^{1/2+s}}\zeta^4(1+2s)\frac{ds}{s},$$

Moving the $s$-line on the left to $s=-1/4+\varepsilon$, we pass a pole of order five at $s=0$. Note that for $\Re e(s)=\delta>-1/2$, we have uniformly $f(\ell_1\ell_2,1+2s)\ll_{\delta,\varepsilon}(\ell_1\ell_2)^{\varepsilon}$ and thus, we can bound the remaining integral by $O(q^{-1/2+\varepsilon}(\ell_1\ell_2)^{-1/4})$. Hence we obtain

**Proposition 3.2.** *The diagonal part given by* (3.5) *can be written as*

$$\mathscr{T}_D^4(\ell_1,\ell_2;q)=\mathrm{MT}_D(\ell_1,\ell_2;q)+O\left(\frac{q^{\varepsilon-1/2}}{(\ell_1\ell_2)^{1/4}}\right), \tag{3.10}$$

*where* $\mathrm{MT}_D(\ell_1,\ell_2,q)$ *is given by the residue*

$$2\mathrm{Res}_{s=0}\left\{\frac{G(s)q^{2s}}{s\zeta(2+4s)}\frac{f(\ell_1\ell_2;1+2s)}{(\ell_1\ell_2)^{1/2+s}}\zeta^4(1+2s)\right\},$$

*and* $f(\ell_1\ell_2,1+2s)$ *is defined in Lemma 3.1.*

## 3.2 The off-diagonal term

We evaluate in this section the off-diagonal part in the decomposition (3.4). Removing the primality condition $(mn,q)=1$ in (3.3) for an error cost of $O(Lq^{-1/2+\varepsilon})$ and we are reduced to analyze the following quantity

$$\mathscr{T}_{OD}^{4,\pm}(\ell_1,\ell_2;q)=\frac{2}{\phi^*(q)}\sum_{d|q}\phi(d)\mu\left(\frac{q}{d}\right)\sum\sum_{\substack{\ell_1 n\equiv\pm\ell_2 m \ (\mathrm{mod}\ d)\\ \ell_1 n\neq \ell_2 m}}\frac{\tau(n)\tau(m)}{(nm)^{1/2}}V\left(\frac{nm}{q^2}\right). \tag{3.11}$$

It is convenient for the analysis of (3.11) to localize the variables $n$ and $m$ by applying a partition of unity. We choose a partition on $\mathbf{R}_{>0}\times\mathbf{R}_{>0}$ as in the work of Young (c.f. [58]), namely, of the form $\{W_{N,M}(x,y)\}_{N,M}$ where $N,M$ runs over power (positive and negative) of 2. In consequence, the numbers of such $N,M$ such that $1\leqslant N,M\leqslant X$ is $O(\log^2 X)$. The functions $W_{N,M}(x,y)$ are of the form $W_N(x)W_M(y)$ with $W_N$ a smooth function supported on $[N,2N]$. Moreover, it is possible to take $W_N(x)=W(x/N)$ with $W$ a fixed, smooth and compactly supported function on $\mathbf{R}_{>0}$ satisfying $W^{(j)}\ll_j 1$. Applying this partition to (3.11), we obtain $\mathscr{T}_{OD}^{4,\pm}(\ell_1,\ell_2;q)=\sum_{N,M}\mathscr{T}_{OD}^{4,\pm}(\ell_1,\ell_2,N,M;q)$ with

$$\begin{aligned}\mathscr{T}_{OD}^{4,\pm}(\ell_1,\ell_2,N,M,q):=&\frac{2}{(NM)^{1/2}\phi^*(q)}\sum_{d|q}\phi(d)\mu\left(\frac{q}{d}\right)\\ &\times\sum\sum_{\substack{\ell_2 n\equiv\pm\ell_1 m\ (\mathrm{mod}\ d)\\ \ell_1 n\neq\ell_2 m}}\tau(n)\tau(m)W\left(\frac{n}{N}\right)W\left(\frac{m}{M}\right)V\left(\frac{nm}{q^2}\right),\end{aligned} \tag{3.12}$$

where we made the substitution

$$W(x)\leftrightarrow x^{-1/2}W(x). \tag{3.13}$$

Because of the fast decay of the function $V(y)$ as $y\to+\infty$ (see Remark 1.12) we can assume that $NM\leqslant q^{2+\varepsilon}$ for arbitrary fixed $\varepsilon>0$ at the cost of an error term $O(q^{-100})$. Furthermore, since each dependency in $\ell_1,\ell_2$ which will appear in the error terms will be of the form $(\ell_1\ell_2)^A$ or $L^B$, we can also assume that $N\geqslant M$. We will treat differently (3.12) according to the relative size of $M$ and $N$. We also note that the trivial bound is given by

$$\mathscr{T}_{OD}^{4,\pm}(\ell_1,\ell_2,N,M;q) \ll q^{\varepsilon} L \frac{(MN)^{1/2}}{q}. \tag{3.14}$$

### 3.2.1 The off-diagonal term when $N \gg M$

In this section, we treat the shifted convolution sum (3.12) when $N$ and $M$ have relatively different sizes (see (3.17)). As a first step, we replace $\phi^*(q)$ by $\phi(q)$ for an error cost of $O(Lq^{-1+\varepsilon})$. Once we have done this, we separate the arithmetical sum over $d|q$. When $d = q$, since $(\ell_1, q) = 1$, we detect the congruence condition $n \equiv \overline{\ell_1}\ell_2 m \pmod{q}$ using additive characters. We thus get (up to $O(Lq^{-1+\varepsilon})$)

$$\begin{aligned}\mathscr{T}_{OD}^{4,\pm}(\ell_1,\ell_2,N,M;q) = \frac{2}{q(MN)^{1/2}} \sum_m \tau(m) W\left(\frac{m}{M}\right) \sum_{\substack{a \pmod q\\(a,q)=1}} e\left(\frac{\pm a\overline{\ell_1}\ell_2 m}{q}\right)& \\ \times \sum_{\substack{n\\ \ell_1 n\neq \ell_2 m}} \tau(n) e\left(\frac{an}{q}\right) W\left(\frac{n}{N}\right) V\left(\frac{nm}{q^2}\right)& \quad (3.15)\\ +2\frac{q^{-1}-\phi^*(q)^{-1}}{(MN)^{1/2}} \sum_{\substack{n,m\\ \ell_1 n\neq \ell_2 m}} \tau(n)\tau(m) W\left(\frac{n}{N}\right) W\left(\frac{m}{M}\right) V\left(\frac{nm}{q^2}\right),& \quad (3.16)\end{aligned}$$

where the line (3.16) is the contribution of the trivial additive character and the case $d = 1$ (the minus sign comes from the Möbius function) and is of size at most $O(q^{-1+\varepsilon})$. Hence we are reduced to the estimation of (3.15). It is also convenient to separate the variables $n, m$ in the test function $V$. This technical step can be achieved using the integral representation of $V$ (see for example [58, Section 4.1]). Hence, we are reduced to bound sums of the shape

$$\mathcal{K}^{\pm}(N,M;q) = \frac{1}{q(NM)^{1/2}} \sum_{\substack{a \pmod q\\(a,q)=1}} \sum_{\substack{n,m\geqslant 1\\ \ell_1 n\neq \ell_2 m}} \tau(n)\tau(m) e\left(\frac{\pm a\overline{\ell_1}\ell_2 m}{q} e\left(\frac{an}{q}\right)\right) W_1\left(\frac{n}{N}\right) W_2\left(\frac{m}{M}\right),$$

where the functions $W_i$ are smooth, compactly supported on $\mathbf{R}_{>0}$ and satisfy $W_i^{(j)} \ll_{\varepsilon,j} q^{\varepsilon j}$ for every $\varepsilon > 0$ and every $j \geqslant 0$.

Let $N = q^{\nu}$, $M = q^{\mu}$ and let $\eta > 0$ be a small real number. By the fast decay of $V(y)$ as $y \to +\infty$ and the bound (3.14), we can assume that $2 - 2\eta \leqslant \nu + \mu \leqslant 2 + \varepsilon$. Anticipating the results of Section 3.3, we also make the additional assumption that

$$\nu - \mu \geqslant 1 - 2\theta - 2\eta, \tag{3.17}$$

where $\theta = 7/64$ is the current best approximation toward the Ramanujan-Petersson conjecture (c.f. (1.13)). Following [2, Section 4], we obtain

**Proposition 3.3.** *Assume that we are in the range* (3.17)*. Then for any* $\varepsilon > 0$*, we have*

$$\mathcal{K}^{\pm}(N, M; q) \ll q^{-\eta+\varepsilon},$$

*where the implied constant depends only on* $\varepsilon$ *and*

$$\eta = \frac{1-6\theta}{14} = \frac{11}{448}. \tag{3.18}$$

*Proof.* We sketch the proof of this proposition. Applying Voronoi summation formula in the $n$-variable and opening the divisor functions reduces the problem of bounding sums of the shape

$$S^{\pm}(M_1, M_1, M_3, M_4) = \frac{1}{(qMN^*)^{1/2}} \sum_{m_1,\dots,m_4} \prod_{i=1}^{4} W_i\left(\frac{m_i}{M_i}\right) \mathrm{Kl}_2(\pm m_1 m_2 m_3 m_4 \ell; q),$$

where $\ell$ is an integer coprime with $q$, $W_i$ are smooth and compactly supported functions on $[1,2]$ satisfying $W_i^{(j)} \ll_j q^{\varepsilon j}$ and $N^* = q^2/N$, $M_i = q^{\mu_i}$, $M = M_1 + M_2$ satisfying

$$0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \mu_4,\ \sum \mu_i = \mu + \nu',\ \nu' \leqslant \nu^*,\ \nu^* = 2 - \nu.$$

Using the trivial bound for the Kloosterman sum and (3.17), we can assume that

$$1 - 2\eta \leqslant \mu + \nu' \leqslant \mu + \nu^* \leqslant 1 + 2\theta + 2\eta. \tag{3.19}$$

The strategy is the following : if the largest variables $m_3, m_4$ are large enough, we apply [2, Proposition 1.2] to them (fixing $m_1, m_2$); otherwise, we find it more beneficial to group variables differently producing a bilinear sum of Kloosterman sums to which we apply Theorem 2.19.
Explicitly, using [2, Proposition 1.2], we obtain that

$$\begin{aligned} S^{\pm}(M_1, M_2, M_3, M_4) &\ll q^{\varepsilon} \frac{M_1 M_2}{(qMN^*)^{1/2}} \left(q^{1/2} + \frac{M_3 M_4}{q^{1/2}}\right) \\ &\ll q^{\varepsilon}\left(\left(\frac{M_1 M_2}{M_3 M_4}\right)^{1/2} + \frac{(MN')^{1/2}}{q}\right) \ll q^{\varepsilon}\left(\left(\frac{M_1 M_2}{M_3 M_4}\right)^{1/2} + q^{-\eta}\right), \end{aligned}$$

assuming that $\frac{1}{2}(1 + 2\theta + 2\eta) - 1 \leqslant -\eta$. We may therefore assume that

$$\mu_3 + \mu_4 - (\mu_1 + \mu_2) \leqslant 2\eta. \tag{3.20}$$

We now apply Theorem 2.19 with $M = M_4$ and $N = M_1 M_2 M_3$ so that $NM = q^{\mu+\nu'} \leqslant MN^*$ and derive

$$S^{\pm}(M_1, M_2, M_3, M_4) \ll q^{\varepsilon}\left(q^{\frac{\mu_1+\mu_2+\mu_3-1}{2}} + q^{-\frac{1}{4}+\frac{\mu_4}{2}}\right).$$

For the first term, we have since $\mu_4 \geqslant \mu_i$ for $i = 1,2,3$,

$$\left(1+\frac{1}{3}\right)(\mu_1+\mu_2+\mu_3+\mu_4) \leqslant \sum_{i=1}^{4} \mu_i \leqslant 1+2\theta+2\eta \Rightarrow \mu_1+\mu_2+\mu_3 \leqslant \frac{3}{4}+\frac{3}{2}(\theta+\eta),$$

so that

$$\frac{\mu_1+\mu_2+\mu_3-1}{2} \leqslant -\frac{1}{8}+\frac{3}{4}(\theta+\eta) \leqslant -\eta$$

assuming that

$$\eta \leqslant \frac{1}{14}-\frac{3\theta}{7}. \tag{3.21}$$

For the second term, we use (7.6) and $\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \mu_4$ to get

$$\mu_4 \leqslant 2\eta+\mu_1+\mu_2-\mu_3 \leqslant 2\eta+\mu_1 \leqslant 2\eta+\frac{1}{3}(1+2\theta+2\eta-\mu_4).$$

Hence

$$\mu_4 \leqslant \frac{5}{4}\eta+\frac{1}{4}+\frac{\theta}{2}$$

and

$$-\frac{1}{4}+\frac{\mu_4}{2} \leqslant -\frac{1}{8}+\frac{5}{8}\eta+\frac{\theta}{4} \leqslant -\eta,$$

as long as $\eta \leqslant \frac{1-2\theta}{13}$, which is true assuming (3.21). $\square$

## 3.3 The off-diagonal using spectral theory of automorphic forms

We analyze in this section the shifted convolution problem when $N, M$ are relatively close. More precisely, by the trivial bound (3.14) and the Proposition 3.3, we can assume that $N = q^\nu$ and $M = q^\mu$ are located in the range

$$2-2\eta \leqslant \nu+\mu \leqslant 2+\varepsilon \quad \text{and} \quad \nu-\mu \leqslant 1-2\theta-2\eta. \tag{3.22}$$

In particular, this restriction implies that

$$\mu \geqslant \frac{1}{2}+\theta \ \text{ and } \ 1-\eta \leqslant \nu \leqslant \frac{3}{2}-\theta-\eta+\varepsilon/2. \tag{3.23}$$

After an application of the Voronoi summation formula, we will see that the off-diagonal part given by (3.12) decomposes as

$$\mathscr{T}^4_{OD}, \pm(\ell_1,\ell_2,N,M;q) = \mathrm{MT}^{\pm}_{OD}(\ell_1,\ell_2,N,M,q)+\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q),$$

where the first is a main term and the second is an error term. We treat here the error term $\mathrm{Err}^{\pm}$ and evaluate $\mathrm{MT}^{\pm}_{OD}$ in Section 3.5.

### 3.3.1 The $\delta$-symbol

We follow [20]. Let $Q \geqslant 1$ be a real number and choose a smooth, even and compactly supported function $w$ in $[Q, 2Q]$ satisfying $w(0) = 0$, $w^{(i)} \ll Q^{-1-i}$ and $\sum_{r=1}^{\infty} w(r) = 1$. We can express the delta function in terms of additives characters in the following way

$$\delta(n) = \sum_{\ell=1}^{\infty} \sum_{k(\ell)}{}^{*} e\left(\frac{kn}{\ell}\right) \Delta_\ell(n),$$

where the superscript $^*$ means that we restrict the summation to primitive classes modulo $\ell$ and

$$\Delta_\ell(u) := \sum_{r=1}^{\infty} (r\ell)^{-1} \left( w(r\ell) - w\left(\frac{u}{r\ell}\right)\right).$$

The function $\Delta_\ell$ satisfies the following bound [20, Lemma 2]

$$\Delta_\ell(u) \ll \min\left(\frac{1}{Q^2}, \frac{1}{\ell Q}\right) + \min\left(\frac{1}{|u|}, \frac{1}{\ell Q}\right), \tag{3.24}$$

and we also have a good control on its derivatives [1, Lemma 4.1]

$$\Delta_\ell^i(u) \ll_i \frac{1}{(\ell Q)^{i+1}}.$$

To keep partial track that $\ell_1 n \pm \ell_2 m - hd$ is not too large, it is also convenient to pick $\varphi$ a smooth function such that $\varphi(0) = 1$, $\varphi(u) = 0$ for $|u| \geqslant U$ and $\varphi^{(i)} \ll U^{-i}$ for some $U$ satisfying $U \leqslant Q^2$. We thus remark that $\Delta_\ell(u) = 0$ if $|u| \leqslant U$ and $\ell > 2Q$ (the parameters $U$ and $Q$ will be explicit in Lemma 3.5). We now return to the expression (3.12) and write the congruence condition $\ell_1 n \equiv \pm \ell_2 m \pmod{d}$ as $\ell_1 n \mp \ell_2 m = hd$ for $h \neq 0$ (since $\ell_1 n \neq \ell_2 m$). We see that if $d = q$, we can assume that $(h, q) = 1$ for a cost of $O(Lq^{-1+\varepsilon})$, an extra condition that will be used only in Section 3.3.3 and will not be srated under each $h$-summation until there. It follows that (3.12) can be written as

$$\begin{aligned} \mathscr{T}_{OD}^{4,\pm}(\ell_1, \ell_2, N, M; q) = \frac{2}{(MN)^{1/2}\phi^*(q)} \sum_{d|q} \phi(q)\mu\left(\frac{q}{d}\right) \sum_{\ell \leqslant 2Q} \sum_{h \neq 0} \sum_{k(\ell)}{}^{*} e\left(\frac{-khd}{\ell}\right) \\ \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tau(n)\tau(m) e\left(\frac{k(\ell_1 n \mp \ell_2 m)}{\ell}\right) E^{\mp}(n, m, \ell), \end{aligned} \tag{3.25}$$

with (omitting the dependence in $d$ and $\ell_i$ in these definitions)

$$E^{\pm}(x, y, \ell) := F^{\pm}(x, y) \Delta_\ell(\ell_1 x \pm \ell_2 y - hd), \tag{3.26}$$

and

$$F^{\pm}(x, y) := W\left(\frac{x}{N}\right) W\left(\frac{y}{M}\right) \varphi(\ell_1 x \pm \ell_2 y - hd) V\left(\frac{xy}{q^2}\right).$$

### 3.3.2 Application of the Voronoi summation formula

We apply the Voronoi summation formula twice (c.f. Proposition 1.1) on the $(m,n)$-sum in (3.25) and get eight error terms plus a principal term [20, (23)]. We write explicitly the principal term in Section 3.5 (c.f. eq (3.49)). All error terms can be treated similarly, so we only focus here on the one which is of the form (recall that $(\ell_1,\ell_2)=1$)

$$\begin{aligned}\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q) &:= \frac{1}{(MN)^{1/2}\phi^*(q)}\sum_{d\mid q}\phi(d)\mu\left(\frac{q}{d}\right)\sum_{\ell\leqslant 2Q}\frac{(\ell_1\ell_2,\ell)}{\ell^2}\sum_{h\neq 0}\sideset{}{^*}\sum_{k(\ell)}\\ &\times e\left(\frac{-khd}{\ell}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tau(n)\tau(m)e\left(-n\frac{\overline{\ell_1' k}}{\ell'}\right)e\left(\pm m\frac{\overline{\ell_2' k}}{\ell''}\right)I^{\mp}(n,m,\ell),\end{aligned} \tag{3.27}$$

where $\ell_i'=\ell_i/(\ell_i,\ell)$, $\ell'=\ell/(\ell_1,\ell)$, $\ell''=\ell/(\ell_2,\ell)$, the overlines denote the inverse modulo the respective denominators and where $I^{\mp}(n,m,\ell)$ involves the $Y_0$ Bessel function :

$$I^{\mp}(n,m,\ell):=4\pi^2\int_0^{\infty}\int_0^{\infty}E^{\mp}(x,y,\ell)Y_0\left(\frac{4\pi d_1\sqrt{nx}}{\ell}\right)Y_0\left(\frac{4\pi d_2\sqrt{my}}{\ell}\right)dxdy, \tag{3.28}$$

where we also set $d_i:=(\ell_i,\ell)$. The main result of this section is the following non-trivial bound.

**Theorem 3.4.** *The quantity defined by* (3.27) *satisfies*

$$\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q)\ll q^{\varepsilon-1/2+\theta}(\ell_1\ell_2)^{3/2}L^5\left(\frac{N}{M}\right)^{1/2}+q^{\varepsilon}L^8\left(\frac{N}{q^2}\right)^{1/4}.$$

*where the implied constant only depends on* $\varepsilon$.

From now on, we only consider the case $\mathrm{Err}^{+}(\ell_1,\ell_2,N,M,q)$ since the other treatment is completly identical and we write $I(n,m,\ell)$ and Err instead of $I^{-}(n,m,\ell)$ and $\mathrm{Err}^{+}$. As in Section 3.2.1, we can also remove the test function $V$ in the definition of $E(x,y,\ell)$ using its integral representation for an error cost of $q^{\varepsilon}$ and a minor change on the function $W$. To not clutter further notations and computations, we will assume that $W^{(j)}\ll 1$ instead of $\ll q^{\varepsilon j}$. The following Lemma allows us to assume that $\ell$ is not too small and further that $n,m$ are not too big for a suitable choice of the parameters $Q$ and $U$.

**Lemma 3.5.** *Set* $Q^{-}:=N^{1/2-\varepsilon}$, $Q=LN^{1/2+\varepsilon}$ *and* $U=LN$.

(*a*) *The* $\ell$*-sum is very small* ($\ll_C q^{-C}$ *for any* $C>0$) *unless*

$$Q^{-}\leqslant\ell\leqslant 2Q.$$

(*b*) *If* $Q^{-}\leqslant\ell\leqslant Q$, *then the integral is negligible unless*

$$n\leqslant\mathcal{N}_0:=\frac{Q^{2+\varepsilon}}{N}\quad,\quad m\leqslant\mathcal{M}_0:=\frac{Q^{2+\varepsilon}}{M}.$$

*Proof.* The lemma is proved by successive integration by parts. We refer to [1, Lemmas 4.1,4.2] for the details. □

### 3.3.3 Preparation for the Kuznetsov formula

We go back to (3.27) (remember that we are dealing with $\mathrm{Err}^{+}_{OD}$) and multiply the arguments of the exponential in the $n$-sum (resp the $m$-sum) to obtain the numerators $-nd_1\overline{\ell'_1}\overline{k}$ (resp $md_2\overline{\ell'_2}\overline{k}$) over the same denominator $\ell$. Once we have done this, we execute the $k$-summation over primitive class modulo $\ell$, obtaining the complete Kloosterman sums. Applying finally a partition of unity to the interval $[Q^-, 2Q]$, we are reduced to estimate $O(\log q)$ sums of the shape

$$\begin{aligned}\frac{1}{\mathcal{Q}(NM)^{1/2}}\sum_{d_i|\ell_i}\frac{1}{\phi^*(q)}\sum_{d|q}\phi(d)\mu\Big(\frac{q}{d}\Big)\sum_{(c,\ell'_1\ell'_2)=1}c^{-1}\vartheta\Big(\frac{cd_1d_2}{\mathcal{Q}}\Big)\sum_{h\neq 0}\sum_{n\geqslant 1}\sum_{m\geqslant 1}\\ \times\tau(n)\tau(m)S(hd, d_1\overline{\ell'_1}n-d_2\overline{\ell'_2}m; cd_1d_2)I(n,m,cd_1d_2),\end{aligned}\tag{3.29}$$

where $Q^- \leqslant \mathcal{Q} \leqslant Q$ and $\vartheta$ is a smooth and compactly supported function on $\mathbf{R}_{>0}$ such that $\vartheta^{(j)} \ll_j 1$ for all $j \geqslant 0$. The first obstruction for the application of the trace formula is the presence of inverses in the Kloosterman sums which are not with respect to its modulus. Indeed, $\overline{\ell'_2}$ (resp $\overline{\ell'_1}$) need to be understood modulo $cd_1$ (resp $cd_2$). We note that if the original $\ell'_i$s were squarefree, then one could take these inverses to be modulo $cd_1d_2$.

To solve this problem, we factor in a unique way $d_i = d_i^* d_i'$ with $(d_i^*, \ell_i') = 1$ and $d_i'|(\ell_i')^\infty$. Now since $(cd_1^*d_2^*, d_1'd_2') = 1$, we may apply the twisted multiplicativity of the Kloosterman sums, getting

$$\begin{aligned}S(hd, d_1\overline{\ell'_1}n-d_2\overline{\ell'_2}m; cd_1d_2) &= S(hd, \overline{(d_1'd_2')^2}(d_1\overline{\ell'_1}n-d_2\overline{\ell'_2}m); cd_1^*d_2^*)\\ &\quad\times S(hd, \overline{(cd_1^*d_2^*)^2}(d_1\overline{\ell'_1}n-d_2\overline{\ell'_2}m); d_1'd_2'),\end{aligned}$$

where the inverse of $d_1'd_2'$ (resp $cd_1^*d_2^*$) is taken modulo $cd_1^*d_2^*$ (resp $d_1'd_2'$). In the first line, both $\ell_i'$ are coprime with $cd_1^*d_2^*$ and therefore, we can take the inverse to be with respect to this modulus. In the second line, the quantity $d_1\overline{\ell'_1}n-d_2\overline{\ell'_2}m$ does not depend anymore on $c$ since we are modulo $d_1'd_2'$. Following an idea of Blomer and Milićević [9], also used by Topacogullari in [57], we separate the dependency in $c$ in the second Kloosterman sum by exploiting the orthogonality of Dirichlet characters, namely writing $v := d_1'd_2'$, we have

$$S(hd, \overline{(cd_1^*d_2^*)^2}(d_1\overline{\ell'_1}n-d_2\overline{\ell'_2}m); v) = \frac{1}{\phi(v)}\sum_{\chi(v)}\overline{\chi}(cd_1^*d_2^*)\hat{S}_v(\overline{\chi}, n, m, \ell_i, hd),$$

with

$$\hat{S}_v(\chi, n, m, \ell_i, hd) := \sum_{\substack{y(v)\\(y,v)=1}}\overline{\chi}(y)S(hd\overline{y}, (d_1\overline{\ell'_1}n-d_2\overline{\ell'_2}m)\overline{y}; v),\tag{3.30}$$

and where the inverse of $\ell_1'$ (resp $\ell_2'$) has to be taken modulo $d_2'$ (resp $d_1'$). We note that the trivial bound for $\hat{S}_v$ is (recall that $d=1$ or $q$ and $(v,q)=1$ since $(\ell_i,q)=1$)

$$\hat{S}_v \ll q^{\varepsilon}(h,v)^{1/2}v^{3/2}.$$

Altough we do not really need it in our treatment, it is in fact possible to do better. In [57, (3.6)], he obtained

$$\hat{S}_v \ll q^{\varepsilon}\left(h, \frac{v}{\mathrm{cond}(\chi)}\right)v. \tag{3.31}$$

Inserting the previous factorization of the Kloosterman sums in (3.29), we obtain

$$\begin{aligned}
&\frac{1}{\mathcal{Q}(NM)^{1/2}}\sum_{d_i|\ell_i}\frac{1}{\phi(v)}\sum_{\chi(v)}\overline{\chi}(d_1^*d_2^*)\frac{1}{\phi^*(q)}\sum_{d|q}\phi(d)\mu\left(\frac{q}{d}\right)\\
&\times\sum_{h\neq 0}\sum_{n,m\geqslant 1}\tau(n)\tau(m)\hat{S}_v(\overline{\chi},n,m,\ell_i,hd)\\
&\times\sum_{(c,\ell_1'\ell_2')=1}\overline{\chi}(c)\frac{S(hd,\overline{v^2\ell_1'\ell_2'}(d_1\ell_2'n-d_2\ell_1'm);cd_1^*d_2^*)}{c}\vartheta\left(\frac{cd_1d_2}{\mathcal{Q}}\right)I(n,m,cd_1d_2).
\end{aligned}\tag{3.32}$$

The strategy is to analyze carefully the two last lines of (3.32) and then to average trivially over the first line. It is convenient from now on to localize the variables $n,m$ and $h$ by applying a partition of unity. Inspired by [9], we also localize $b := d_1\ell_2'n - d_2\ell_1'm$ and are therefore reduced to estimate $O(\log^4 q)$ sums of the form

$$\begin{aligned}
\mathscr{D}(\mathcal{N},\mathcal{M},B,H;d,\chi) &:= \sum_{h\asymp H}\sum_{|b|\asymp B}\sum\sum_{\substack{d_1\ell_2'n-d_2\ell_1'm=b\\ n\asymp\mathcal{N},m\asymp\mathcal{M}}}\tau(n)\tau(m)\hat{S}_v(\overline{\chi},n,m,\ell_i,hd)\\
&\times\sum_{(c,\ell_1'\ell_2')=1}\overline{\chi}(c)\frac{S(hd,\overline{v^2\ell_1'\ell_2'}b;cd_1^*d_2^*)}{c}\vartheta\left(\frac{cd_1d_2}{\mathcal{Q}}\right)\mathscr{I}(n,m,b,h,cd_1d_2)\\
&=: \mathscr{D}^+ + \mathscr{D}^- + \mathscr{D}^0,
\end{aligned}\tag{3.33}$$

where $1\leqslant\mathcal{N}\leqslant\mathcal{N}_0$, $1\leqslant\mathcal{M}\leqslant\mathcal{M}_0$, $1\leqslant H\leqslant LN/d$, and where $\mathscr{D}^0$ (respectively $\mathscr{D}^+$, $\mathscr{D}^-$) denotes the contribution of $b=0$ (respectively $b>0$, $b<0$) and the function $\mathscr{I}(n,m,b,h,cd_1d_2)$ is given by the product $G(n,m,|b|,h)I(n,m,cd_1d_2)$ with $G$ a smooth and compactly supported function on $[\mathcal{N},2\mathcal{N}]\times[\mathcal{M},2\mathcal{M}]\times[B,2B]\times[H,2H]$ satisfying

$$G^{(i,j,k,p)} \ll \mathcal{N}^{-i}\mathcal{M}^{-j}B^{-k}H^{-p}.$$

**Remark 3.6.** The size of $B$ depends on the sign of $d_1\ell_2'n - d_2\ell_1'm = b$. If $b>0$, then $B\leqslant d_1\ell_2'\mathcal{N}\leqslant L^2\mathcal{N}$ while for $b<0$, $B\leqslant L^2\mathcal{M}$ which is much larger (c.f. Lemma 3.5 (b)).

**Evaluation of $\mathscr{D}^0$**

To estimate the contribution of $b=0$, we use the bound $Y_0(z) \ll z^{-1/2}$ for $z>0$ [45, Lemma C.2], the fact that $x \asymp N, y \asymp M, n \asymp \mathcal{N}, m \asymp \mathcal{M}$ and (3.24) for the delta function which allows us to bound the integral :

$$\begin{aligned} I(n,m,cd_1d_2) &\ll \frac{c(d_1d_2)^{1/2}}{(MN\mathcal{M}\mathcal{N})^{1/4}} \int_0^\infty \int_0^\infty |E(x,y,cd_1d_2)|dxdy \\ &\ll \frac{(MN)^{3/4}}{(\mathcal{M}\mathcal{N})^{1/4}(d_1d_2)^{1/2}Q}. \end{aligned}$$

We now use (3.31), the Weil bound for Kloosterman sums and $H \leqslant LN/d$, $\mathcal{N} \leqslant \mathcal{N}_0$ to obtain (recall that $v = d_1' d_2'$)

$$\begin{aligned} \mathscr{D}^0 &\ll q^\varepsilon \frac{(NM\mathcal{N})^{3/4} v (d_1^* d_2^*)^{1/2}}{\mathcal{M}^{1/4}Q} \sum_{h \asymp H} \sum_{c \leqslant \frac{\mathcal{Q}}{d_1 d_2}} \frac{(h,v)(hd, cd_1^*d_2^*)^{1/2}}{c^{1/2}} \\ &\ll q^\varepsilon L d^{-1} \frac{(NM\mathcal{N})^{3/4} v^{1/2} N \mathcal{Q}^{1/2}}{(d_1d_2)^{1/2}\mathcal{M}^{1/4}Q} \ll q^\varepsilon L d^{-1} \frac{v^{1/2} M^{3/4} N \mathcal{Q}^2}{(d_1d_2)^{1/2}Q}. \end{aligned} \tag{3.34}$$

We come back to (3.33) and we write $\mathscr{D}^\pm$ in a uniform way (recall that $v|(\ell_1'\ell_2')^\infty$ and $d_i = d_i^* d_i'$)

$$\begin{aligned} \mathscr{D}^\pm = 4\pi^2 d_1 d_2 \sqrt{\ell_1'\ell_2'} \sum_{h \asymp H} \sum_{b \asymp B} \sum_{\substack{d_1\ell_2' n - d_2 \ell_1' m = b \\ n \asymp \mathcal{N}, m \asymp \mathcal{M}}} \sum \tau(n)\tau(m) \hat{S}_v(\overline{\chi}, n, m, \ell_i, hd) \\ \times \sum_{(c, \ell_1'\ell_2' v^2)=1} \overline{\chi}(c) \frac{S(hd, \overline{v^2 \ell_1'\ell_2'} b; cd_1^* d_2^*)}{cd_1d_2\sqrt{\ell_1'\ell_2'}} \Phi\left(\frac{4\pi\sqrt{|b|hd}}{cd_1d_2\sqrt{\ell_1'\ell_2'}}\right), \end{aligned} \tag{3.35}$$

where the function $\Phi$ depends also on the variables $n, m, b$ and $h$ and is defined by

$$\begin{aligned} \Phi(z,n,m,b,h) := G(n,m,|b|,h) \vartheta\left(\frac{4\pi\sqrt{|b|hd}}{z\mathcal{Q}\sqrt{\ell_1'\ell_2'}}\right) \int_0^\infty \int_0^\infty E\left(x, y, \frac{4\pi\sqrt{|b|hd}}{z\sqrt{\ell_1'\ell_2'}}\right) \\ \times Y_0\left(zd_1\sqrt{\frac{\ell_1'\ell_2'}{|b|hd}nx}\right) Y_0\left(zd_2\sqrt{\frac{\ell_1'\ell_2'}{|b|hd}my}\right) dxdy. \end{aligned}$$

**Remark 3.7.** We can always assume that we are treating the case where $h \asymp H$ is positive since otherwise, we write $h \leftrightarrow -h$ and use $S(-hq, \overline{\ell_1'\ell_2'}b; cd_1d_2) = S(hq, \overline{\ell_1'\ell_2'}(-b); cd_1d_2)$.

**Proposition 3.8.** *The function $\Phi$ is $C_c^\infty(\mathbf{R}^5)$ with each variable supported in*

$$z \asymp Z := \frac{\sqrt{BHd}}{\mathcal{Q}\sqrt{\ell_1'\ell_2'}},\ n \asymp \mathcal{N},\ m \asymp \mathcal{M},\ b \asymp B,\ h \asymp H.$$

*Further, its partial derivatives satisfies the following bound*

$$\Phi^{(\boldsymbol{\alpha})} \ll_{\boldsymbol{\alpha}} \frac{M^{3/4}N^{1/4}}{L(d_1d_2)^{1/2}(\mathcal{M}\mathcal{N})^{1/4}} Z^{-\alpha_1}\mathcal{N}^{-\alpha_2}\mathcal{M}^{-\alpha_3}B^{-\alpha_4}H^{-\alpha_5}, \tag{3.36}$$

*for any multi-index* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_5) \in \mathbb{N}^5$.

*Proof.* Setting $\xi := 4\pi\sqrt{|b|hd/\ell_1'\ell_2'}$, using the bound $Y_0 \ll z^{-1/2}$, the fact $\Delta_\ell(u) \ll (\ell Q)^{-1}$ provides by (3.24), the ranges $x \asymp N$, $y \asymp M$ and the value $Q = LN^{1/2+\varepsilon}$ lead to

$$\begin{aligned}\Phi &\ll \frac{\xi}{z}(d_1d_2)^{-1/2}(MN\mathcal{N}\mathcal{M})^{-1/4}\int_0^\infty\int_0^\infty |E(x,y,\xi/z)|dxdy \\ &\ll \frac{(d_1d_2)^{-1/2}(MN)^{3/4}}{(\mathcal{M}\mathcal{N})^{1/4}Q} \ll \frac{M^{3/4}N^{1/4}}{L(d_1d_2)^{1/2}(\mathcal{N}\mathcal{M})^{1/4}}.\end{aligned}$$

For the second part, we take the derivatives under the sign of the integral and we use the following estimations [45, Lemma C.2]

$$z^i Y_0^{(i)}(z) \ll_j \frac{1+|\log z|}{(1+z)^{1/2}},$$

and

$$\frac{\partial^i}{\partial \ell^i}E(x,y,\ell) \ll_i \frac{1}{\ell^i(\ell Q)}.$$

We mention that when we differentiate $\varphi(\ell_1x-\ell_2y-hd)\Delta_\ell(\ell_1x-\ell_2y-hd)$ with respect to $h$, we catch a factor $d/LN$ but since $H \ll LN/d$, we get the desired $1/H$. □

### 3.3.4 Applying the trace formula

Before applying the Kuznetsov trace formula to the second line in (3.35), we need the following identity (c.f. [15, (9.1)-(9.2)] or [57, (2.3)]) which allows us to get rid of the inverses in the Kloosterman sum by moving to a suitable cusp (apply this identity with $r = \ell_1'\ell_2'v^2$, $s = d_1^*d_2^*$ and $c = c$) :

$$\begin{aligned}&\sum_{(c,\ell_1'\ell_2'v^2)=1} \overline{\chi}(c)\frac{S(hd,\overline{v^2\ell_1'\ell_2'}b;cd_1^*d_2^*)}{cd_1^*d_2^*\sqrt{v^2\ell_1'\ell_2'}}\Phi\left(\frac{4\pi\sqrt{|b|hd}}{cd_1^*d_2^*\sqrt{v^2\ell_1'\ell_2'}},n,m,b,h\right) \\ &\qquad = e\left(-\frac{b\overline{d_1^*d_2^*}}{v^2\ell_1'\ell_2'}\right)\sum_{\gamma}^{\Gamma_0(v\ell_1\ell_2)}\frac{S^\chi_{\infty\mathfrak{a}}(hd,b;\gamma)}{\gamma}\Phi\left(\frac{4\pi\sqrt{|b|hd}}{\gamma},n,m,b,h\right),\end{aligned}$$

where $\mathfrak{a} := 1/d_1^* d_2^*$ is a singular cusp for the congruence group $\Gamma_0(v\ell_1\ell_2)$. We now apply the Kuznetsov formula (1.21) to the $\gamma$-sum and we write separatly

$$\mathscr{D}^+ = \frac{4\pi\sqrt{\ell_1'\ell_2'}}{(d_1d_2)^{-1}} \sum_{h\asymp H}\sum_{b\asymp B} e\left(-\frac{b\overline{d_1^* d_2^*}}{v^2\ell_1'\ell_2'}\right) \sum_{\substack{d_1\ell_2' n - d_2\ell_1' m = b \\ n\asymp \mathcal{N}, m\asymp \mathcal{M}}}\sum \tau(n)\tau(m)\hat{S}_v(\overline{\chi}, n, m, hd)$$
$$\times \left(\mathscr{H}(n,m,b,h) + \mathscr{M}^+(n,m,b,h) + \mathscr{E}^+(n,m,b,h)\right),$$
$$\mathscr{D}^- = \frac{4\pi\sqrt{\ell_1'\ell_2'}}{(d_1d_2)^{-1}} \sum_{h\asymp H}\sum_{b\asymp B} e\left(-\frac{b\overline{d_1^* d_2^*}}{v^2\ell_1'\ell_2'}\right) \sum_{\substack{d_1\ell_2' n - d_2\ell_1' m = b \\ n\asymp \mathcal{N}, m\asymp \mathcal{M}}}\sum \tau(n)\tau(m)\hat{S}_v(\overline{\chi}, n, m, hd)$$
$$\times \left(\mathscr{M}^-(n,m,b,h) + \mathscr{E}^-(n,m,b,h)\right),$$

where $\mathscr{H}$, $\mathscr{M}$ and $\mathscr{E}$ denote the contribution of the holomorphic part, the Maass cusp forms and the Eisenstein spectrum and are given respectively by

$$\mathscr{H}^+(n,m,b,h) = \sum_{\substack{k\geqslant 2 \\ k\equiv \kappa\ (2)}} \dot{\Phi}_{n,m,b,h}(k)\frac{(k-1)!}{\pi(4\pi)^{k-1}} \sum_{f\in\mathcal{B}_k(v\ell_1\ell_2,\chi)} \overline{\rho_{f,\infty}}(hd)\rho_{f,\mathfrak{a}}(b),$$
$$\mathscr{M}^+(n,m,b,h) = \sum_{f\in\mathcal{B}(v\ell_1\ell_2,\chi)} \widehat{\Phi}_{n,m,b,h}(t_f)\frac{1}{\cosh(\pi t_f)}\overline{\rho_{f,\infty}}(hd)\rho_{f,\mathfrak{a}}(b),$$
$$\mathscr{E}^+(n,m,b,h) = \sum_{\substack{\chi_1\chi_2=\chi \\ f\in\mathcal{B}(\chi_1,\chi_2)}} \frac{1}{4\pi}\int_{\mathbb{R}} \widehat{\Phi}_{n,m,b,h}(t)\frac{1}{\cosh(\pi t)}\overline{\rho_{f,\infty}}(hd,t)\rho_{f,\mathfrak{a}}(b,t)\,dt.$$

We have the same expressions for $\mathscr{M}^-$ and $\mathscr{E}^-$, but with $\check{\Phi}_{n,m,b,h}$ instead of $\widehat{\Phi}_{n,m,b,h}$ (see (1.22)). We will analyze in detail $\mathscr{D}^-$, whose contribution is bigger than the plus case. This is due to the fact that if $b>0$, then $B$ is at most $\mathcal{N} \ll q^\varepsilon L^2$ while for $b<0$, $B$ could be of size $\mathcal{M} \ll q^\varepsilon L^2 N/M$ (c.f. Remark 3.6). Furthermore, the holomorphic setting and the continuous spectrum will give a better bound than the discrete part since the Ramanujan-Petersson conjecture is true for both of them. Finally, since the treatment of these three terms is similar, we only focus on the Maass cusp forms in $\mathscr{D}^-$.

### 3.3.5 Spectral analysis of $\mathscr{D}^-$

As said in the previous paragraph, we only focus on the discrete spectrum, writing $\mathscr{D}^{-,\mathrm{M}}$ for its contribution to $\mathscr{D}^-$. By $\mathscr{D}_K^{-,\mathrm{M}}$, we mean that we restrict the spectral parameter to the dyadic interval $K \leqslant t_f < 2K$ in the definition of $\mathscr{D}^{-,\mathrm{M}}$. Using Proposition 3.8 and Lemma 1.6 (eq (1.19)), we see that we can restrict our attention to $K \leqslant q^\varepsilon(1+Z)$ at the cost of $O(q^{-100})$. We now separate the variables in $\check{\Phi}_{n,m,b,h}(t)$ using the Mellin inversion formula in $n,m,b,h$ :

$$\check{\Phi}_{n,m,b,h}(t) = \frac{1}{(2\pi i)^4}\int_{(0)}\int_{(0)}\int_{(0)}\int_{(0)} \frac{\widetilde{\check{\Phi}}(t)(s_1,\dots,s_4)}{n^{s_1}m^{s_2}|b|^{s_3}h^{s_4}}\,ds_4\,ds_3\,ds_2\,ds_1,$$

where the Mellin transform equals

$$\tilde{\check{\Phi}}(t)(s_1,...,s_4) = \int_{(\mathbb{R}_{>0})^4} \check{\Phi}_{\mathfrak{n},\mathfrak{m},\mathfrak{b},\mathfrak{h}}(t)\mathfrak{n}^{s_1}\mathfrak{m}^{s_2}\mathfrak{b}^{s_3}\mathfrak{h}^{s_4}\frac{d\mathfrak{n}d\mathfrak{m}d\mathfrak{b}d\mathfrak{h}}{\mathfrak{n}\mathfrak{m}\mathfrak{b}\mathfrak{h}}. \tag{3.37}$$

By virtue of Proposition 3.8 (the bound (3.36)), we see that we can restrict the supports of the integrals to $|\Im m(s_i)| \leqslant (Kq)^{\varepsilon}$ for a cost of $O((Kq)^{-100})$. We have therefore

$$\mathscr{D}_K^{-,\mathrm{M}} = \frac{4\pi^2 d_1 d_2\sqrt{\ell_1'\ell_2'}}{(4\pi i)^4}\iiiint\limits_{|\Im m(s_i)|\leqslant (Kq)^{\varepsilon}} \mathscr{B}_K^{-,\mathrm{M}}(s_1,...,s_4)ds_4ds_3ds_2ds_1 + O\big((Kq)^{-100}\big), \tag{3.38}$$

where we defined

$$\begin{aligned}\mathscr{B}_K^{-,\mathrm{M}}(s_1,...,s_4) := &\sum_{\substack{f\in\mathcal{B}(v\ell_1\ell_2,\chi)\\ K\leqslant|t_f|<2K}}\frac{\tilde{\check{\Phi}}(t_f)(s_1,...,s_4)}{\cosh(\pi t_f)}\sum_{h\asymp H}h^{-s_4}\\ &\times\sum_{b\asymp B}|b|^{-s_3}\alpha(b,h,s_1,s_2)\overline{\rho_{f,\infty}}(hd)\rho_{f,\mathfrak{a}}(b),\end{aligned} \tag{3.39}$$

and

$$\alpha(b,h,s_1,s_2) := e\left(-\frac{b\overline{d_1^* d_2^*}}{v^2\ell_1'\ell_2'}\right)\sum\sum_{\substack{d_1\ell_2' n - d_2\ell_1' m = b\\ n\asymp\mathcal{N}\\ m\asymp\mathcal{M}}}\frac{\tau(n)\tau(m)}{n^{s_1}m^{s_2}}\hat{S}_v(\overline{\chi},n,m,h). \tag{3.40}$$

Since we want to apply Cauchy-Schwarz in (3.39) to make the square of the $h$ and $b$ sum appear in order to use the large sieve inequality, we need to separate $h$ from $b$ in $\alpha(b,h)$. Using the Definition (3.30) of $\hat{S}_v(\overline{\chi},n,m,h)$ and opening the Kloosterman sum, we have

$$\mathscr{B}_K^{-,\mathrm{M}}(s_1,...,s_4) = \sum_{\substack{x,y(v)\\ (xy,v)=1}}\chi(y)\mathscr{A}_K(x,y,s_1,...,s_4), \tag{3.41}$$

with this time

$$\begin{aligned}\mathscr{A}_K(x,y,s_1,...,s_4) := &\sum_{\substack{f\in\mathcal{B}(v\ell_1\ell_2,\chi)\\ K\leqslant|t_f|<2K}}\frac{\tilde{\check{\Phi}}(t_f)(s_1,...,s_4)}{\cosh(\pi t_f)}\sum_{h\asymp H}\delta(h,s_4)\\ &\times\sum_{b\asymp B}|b|^{-s_3}\omega(b,s_1,s_2)\overline{\rho_{f,\infty}}(hd)\rho_{f,\mathfrak{a}}(b),\end{aligned} \tag{3.42}$$

where

$$\delta(h,s_4) := h^{-s_4}e\left(\frac{hd\overline{y}x}{v}\right),$$

and

$$\omega(b,s_1,s_2,s_3) := e\left(-\frac{b\overline{d_1^* d_2^*}}{v^2\ell_1'\ell_2'}\right)\sum_{\substack{d_1\ell_2' n - d_2\ell_1' m = b \\ n \asymp \mathcal{N} \\ m \asymp \mathcal{M}}}\sum \frac{\tau(n)\tau(m)}{n^{s_1}m^{s_2}} e\left(\frac{\left(d_1\overline{\ell_1'}n - d_2\overline{\ell_2'}m\right)\overline{xy}}{v}\right).$$

Since the supports of the integrals in (3.38) are restricted to $|\Im m(s_i)| \leqslant (Kq)^{\varepsilon}$, we can just estimate the quantity (3.39) and then average trivially over the $s_i$-integrals for an error cost of $(Kq)^{4\varepsilon}$. In fact, we will analyze $\mathscr{A}_K(x,y,s_1,...,s_4)$ and then apply the trivial bound $\mathscr{B}_K \leqslant \phi(v)^2 \sup_{x,y,s_i}|\mathscr{A}_K(x,y,s_1,...,s_4)|$. Using Cauchy-Schwarz inequality, we infer

$$\begin{aligned}\left|\mathscr{A}_K(x,y,s_1,...,s_4)\right| &\leqslant \sup_{\substack{K\leqslant t<2K \\ \Re e(s_i)=0}}\left|\tilde{\check{\Phi}}(t)(s_1,...,s_4)\right| \\ &\times \left(\sum_{\substack{f\in\mathcal{B}(v\ell_1\ell_2,\chi)\\ K\leqslant|t_f|<2K}}\frac{(1+|t_f|)^{\kappa}}{\cosh(\pi t_f)}\left|\sum_{h\asymp H}\delta(h,s_4)\overline{\rho_{f,\infty}}(hd)\right|^2\right)^{1/2} \\ &\times \left(\sum_{\substack{f\in\mathcal{B}(v\ell_1\ell_2,\chi)\\ K\leqslant|t_f|<2K}}\frac{(1+|t_f|)^{-\kappa}}{\cosh(\pi t_f)}\left|\sum_{b\asymp B}|b|^{-s_3}\omega(b,s_1,s_2)\rho_{f,\mathfrak{a}}(b)\right|^2\right)^{1/2},\end{aligned} \tag{3.43}$$

where $\kappa\in\{0,1\}$ satisfies $\chi(-1)=(-1)^{\kappa}$. We mention that we implicitly used the fact that $\cosh(\pi t_f)$ is always positive since $|\Im m(t_f)| \leqslant \theta = 7/64$ by (1.14) (it is enough to have $|\Im m(t_f)| < 1/2$). Before applying the spectral large sieve, we need to control the size of the Mellin-Kuznetsov transform

$$t \mapsto \tilde{\check{\Phi}}(t)(s_1,...,s_4).$$

To do this, we return to Definitions (3.37) and (1.17) and note (by permutation of integrals) that this is in fact the Bessel transform of the function

$$z \mapsto \Psi(z,s_1,...,s_4) := \iiiint_{(\mathbf{R}_{>0})^4}\Phi(z,\mathfrak{n},\mathfrak{m},\mathfrak{b},\mathfrak{h})\mathfrak{n}^{s_1}\mathfrak{m}^{s_2}\mathfrak{b}^{s_3}\mathfrak{h}^{s_4}\frac{d\mathfrak{n}d\mathfrak{m}d\mathfrak{b}d\mathfrak{h}}{\mathfrak{n}\mathfrak{m}\mathfrak{b}\mathfrak{h}}.$$

Using again Proposition 3.8, we see that the support of $\Psi$ is $z \asymp Z$ and that it satisfies the uniform bound (recall that $\Re e(s_i) = 0$)

$$\Psi^{(i)} \ll q^{\varepsilon}\frac{M^{3/4}N^{1/4}}{L(d_1d_2)^{1/2}(\mathcal{M}\mathcal{N})^{1/4}}Z^{-i}.$$

Therefore, it follows from Lemma 1.6 (the bound (1.18)) that

$$\tilde{\check{\Phi}}(t)(s_1,...,s_4) \ll q^{\varepsilon}\frac{M^{3/4}N^{1/4}}{ZL(d_1d_2)^{1/2}(\mathcal{M}\mathcal{N})^{1/4}}. \tag{3.44}$$

We substitute the bound above in the first line of (3.43). In the second line, we exploit the fact that we are at the cusp $\infty$ by mean of the Hecke relation between the Fourier coefficients and the eigenvalues (c.f. (1.6)), namely (we recall that we assumed throughout that $(h,q)=1$ and we use this hypothesis only here)

$$\rho_{f,\infty}(hd) = \lambda_f(d)\rho_{f,\infty}(h).$$

Note that we also used the fact that $q$ is coprime to the level of the group $\Gamma_0(v\ell_1\ell_2)$ since $(\ell_1\ell_2, q) = 1$ and $v|\ell_1\ell_2$. We now use the bound $|\lambda_f(d)| \leqslant 2d^\theta$ (c.f (1.13) and recall that either $d=1$ or $d=q$ is prime), the large sieve inequality (c.f Proposition 1.8), the fact that $\mu(\mathfrak{a}) = (v\ell_1\ell_2)^{-1}$ (c.f (1.23)), $\mathrm{cond}(\chi) \leqslant v$, the two bounds

$$||\delta||_H \leqslant H^{1/2}\,,\ ||\omega||_B \ll q^\varepsilon \mathcal{N} B^{1/2}$$

and we obtain

$$\mathscr{A}_K \ll q^\varepsilon d^\theta \frac{M^{3/4}N^{1/4}(BH)^{1/2}\mathcal{N}^{3/4}}{ZL(d_1d_2)^{1/2}\mathcal{M}^{1/4}}\left(K + \frac{v^{-1/4}B^{1/2}}{(\ell_1\ell_2)^{1/2}}\right)\left(K + \frac{v^{-1/4}H^{1/2}}{(\ell_1\ell_2)^{1/2}}\right).$$

For $K$, we have since $\mathcal{Q} \geqslant N^{1/2-\varepsilon}$ and $H \leqslant NL/d$,

$$K \leqslant q^\varepsilon Z = q^\varepsilon \frac{(BHd)^{1/2}}{\mathcal{Q}(\ell_1'\ell_2')^{1/2}} \ll q^\varepsilon \frac{(LN)^{1/2}}{\mathcal{Q}}\frac{B^{1/2}}{(\ell_1'\ell_2')^{1/2}} \ll q^\varepsilon \frac{(LB)^{1/2}}{(\ell_1'\ell_2')^{1/2}}.$$

Hence,

$$\mathscr{A}_K \ll q^\varepsilon d^\theta L^{-1/2}\frac{M^{3/4}N^{1/4}BH^{1/2}\mathcal{N}^{3/4}}{Z(\ell_1\ell_2)^{1/2}\mathcal{M}^{1/4}(\ell_1'\ell_2')^{1/2}}\left((LB)^{1/2} + H^{1/2}\right).$$

Using $Z \asymp \mathcal{Q}^{-1}(\ell_1'\ell_2')^{-1/2}(BHd)^{1/2}$ leads to

$$\begin{aligned}\mathscr{A}_K &\ll q^\varepsilon d^{-1/2+\theta}L^{-1/2}\frac{M^{3/4}N^{1/4}B^{1/2}\mathcal{N}^{3/4}\mathcal{Q}}{\mathcal{M}^{1/4}(\ell_1\ell_2)^{1/2}}\left((LB)^{1/2} + H^{1/2}\right)\\ &=: \mathscr{A}_K(B) + \mathscr{A}_K(H).\end{aligned}$$

For the first expression, we have using $B \leqslant L^2\mathcal{M}$ and the maximum values of $\mathcal{M}$ and $\mathcal{N}$ given by Lemma 3.5 (b)

$$\begin{aligned}\mathscr{A}_K(B) &\ll q^\varepsilon d^{-1/2+\theta}L^2\frac{(M\mathcal{M}\mathcal{N})^{3/4}N^{1/4}\mathcal{Q}}{(\ell_1\ell_2)^{1/2}}\\ &\ll q^\varepsilon d^{-1/2+\theta}L^2\frac{Q^3\mathcal{Q}}{(\ell_1\ell_2N)^{1/2}} \ll q^\varepsilon d^{-1/2+\theta}L^5\frac{N\mathcal{Q}}{(\ell_1\ell_2)^{1/2}}.\end{aligned}$$

For the second term, we have using also $H \leqslant LN/d$

$$\begin{aligned}\mathscr{A}_K(H) &= q^{\varepsilon} d^{-1/2+\theta} L^{-1/2} \frac{M^{3/4} N^{1/4} (BH)^{1/2} \mathcal{N}^{3/4} \mathcal{Q}}{\mathcal{M}^{1/4} (\ell_1 \ell_2)^{1/2}} \\ &\ll q^{\varepsilon} d^{-1+\theta} L \frac{(NM\mathcal{N})^{3/4} \mathcal{M}^{1/4} \mathcal{Q}}{(\ell_1 \ell_2)^{1/2}} \ll q^{\varepsilon} d^{-1+\theta} L \frac{Q^2 M^{1/2} \mathcal{Q}}{(\ell_1 \ell_2)^{1/2}} \\ &\ll q^{\varepsilon} d^{-1+\theta} L^3 \frac{N M^{1/2} \mathcal{Q}}{(\ell_1 \ell_2)^{1/2}}.\end{aligned}$$

### 3.3.6 Conclusion of Theorem 3.4

We insert these two estimations first in (3.41), so that it will be multiplied by $\phi(v)^2$. We next multiply by $d_1 d_2 \sqrt{\ell_1' \ell_2'}$ as in (3.38). Finally, we replace the two last lines of (3.32) by these bounds and execute the first line summation, obtaining that the contribution of $\mathscr{D}^-$ to $\mathrm{Err}(\ell_1, \ell_2, N, M; q)$ is

$$q^{\varepsilon-1/2+\theta} (\ell_1 \ell_2)^{3/2} \left( \left( \frac{L^6 N}{q} \right)^{1/2} + \left( \frac{L^{10} N}{M} \right)^{1/2} \right) \ll q^{\varepsilon-1/2+\theta} L^5 (\ell_1 \ell_2)^{3/2} \left( \frac{N}{M} \right)^{1/2}, \tag{3.45}$$

since $M/q \leqslant 1$. We do exactly the same thing with $\mathscr{D}^0$ (see (3.34)), getting that its contribution to Err is at most

$$q^{\varepsilon} L^8 \frac{M^{1/4} N^{1/2}}{q} \ll q^{\varepsilon} L^8 \left( \frac{N}{q^2} \right)^{1/4}, \tag{3.46}$$

which completes the proof of Theorem 3.4.

## 3.4 Combining the error terms of sections 3.2.1 and 3.3

We combine now the error terms coming from Sections 3.2.1 and 3.3. We remind that $N = q^{\nu}$ and $M = q^{\mu}$ with $\nu \geqslant \mu$. If we are in the case $\nu + \mu \leqslant 2 - 2\eta$, then we can apply the trivial bound (3.14), obtaining

$$\mathscr{T}_{OD}^{4,\pm}(\ell_1, \ell_2, N, M; q) \ll L q^{\varepsilon - \eta}. \tag{3.47}$$

Assume now that we are in the range $2 - 2\eta \leqslant \nu + \mu$. If $\nu - \mu \geqslant 1 - 2\theta - 2\eta$, we apply Proposition 3.3, getting the same as (3.47) (without the factor $L$). In the complementary case $\nu - \mu \leqslant 1 - 2\theta - 2\eta$, we apply Theorem 3.4 to the error term Err and we obtain

$$\mathrm{Err}^{\pm}(\ell_1, \ell_2, N, M; q) \ll q^{\varepsilon} (\ell_1 \ell_2)^{3/2} \frac{L^5}{q^{\eta}} + q^{\varepsilon} L^8 q^{\frac{1}{4}(\nu - 2)}.$$

Finally, applying the second part of (3.23) on the second term yields

$$\mathrm{Err}^{\pm}(\ell_1,\ell_2,N,M;q) \ll q^{\varepsilon}\left(\frac{L^5(\ell_1\ell_2)^{3/2}}{q^{\eta}} + \frac{L^8}{q^{\frac{1}{4}(\frac{1}{2}+\theta+\eta)}}\right). \tag{3.48}$$

Setting $L = q^{\lambda}$, the first term is automatically bigger than the second if $\lambda < \eta$, a condition we henceforth assume to hold and that gives the desired error term of Theorem 3.

## 3.5 The Off-Diagonal Main Term

We return to the main term that we left besides in the beginning of Section 3.3.2. This expression corresponds to the product of the two constants terms after the application of Voronoi summation formula to (3.25) and is given by (see also [20, Section 5])

$$\mathrm{MT}^{\pm}_{OD}(\ell_1,\ell_2,N,M;q) := \frac{2\phi^*(q)^{-1}}{(MN)^{1/2}}\sum_{d\mid q}\phi(d)\mu\left(\frac{q}{d}\right)\sum_{h\neq 0}\sum_{\ell\leqslant 2Q}\frac{(\ell_1\ell_2,\ell)}{\ell^2}S(hd,0;\ell)I_{\pm}, \tag{3.49}$$

and $I_{\pm}$ is the integral defined by

$$I_{\pm} := \int_0^{\infty}\int_0^{\infty}\left(\log\ell_1 x - \lambda_{\ell_1,\ell}\right)\left(\log\ell_2 y - \lambda_{\ell_2,\ell}\right)E^{\pm}(x,y,\ell)\,dx\,dy, \tag{3.50}$$

with $E^{\pm}$ given by (3.26) and

$$\lambda_{\ell_i,\ell} := \log\left(\frac{\ell_i\ell^2}{(\ell_i,\ell)^2}\right) - 2\gamma.$$

As a first step, we need to remove the delta function $\Delta_\ell$ in our integral because this is an obstruction for the calculation. This can be done as follows : we make a first change of variables $\ell_1 x \mapsto x$ and $\ell_2 y \mapsto y$ and then, $x = \mp y + hd + u$, getting

$$I_{\pm} = \frac{1}{\ell_1\ell_2}\int_0^{\infty}\int_{\mathbf{R}} C(\mp y + hd + u, y)\Delta_\ell(u)\,du\,dy, \tag{3.51}$$

where we defined

$$C(x,y) := (\log x - \lambda_{\ell_1,\ell})(\log y - \lambda_{\ell_2,\ell})F\left(\frac{x}{\ell_1},\frac{y}{\ell_2}\right).$$

For the inner integral in (3.51), we use equation [20, (18)] and obtain

$$\int_{\mathbf{R}} C(\mp y + hq + u, y)\Delta_\ell(u)\,du = C(\mp y + hd, y) + O\left(\left(\frac{\ell Q}{\ell_1 N}\right)^j\right),$$

where the implied constant depends on $j \geqslant 1$. Assuming $\ell < (\ell_1 N/\ell Q)^{1-\varepsilon}$, we make the error term above very small by choosing $j$ large enough. Therefore, we have for $\ell$ in this range

$$I_{\pm} = \frac{1}{\ell_1\ell_2}\int_0^{\infty} C(\mp y + hd, y)\,dy + O\left(q^{-100}\right).$$

On the other hand, we also have the bound (c.f. (30), [20]) $I_\pm \ll M\log Q$ which is valid for all $\ell$. Hence, using $|h| \leqslant LN/d$, the bound for the Ramanujan sum $S(hd,0;\ell) \ll (hd,\ell)$ and the definition of $Q = LN^{1/2+\varepsilon}$, we get

$$\begin{aligned} \mathrm{MT}^{\pm}_{OD}(\ell_1,\ell_2,N,M;q) = \frac{2(\phi^*(q)\ell_1\ell_2)^{-1}}{(MN)^{1/2}} \sum_{d|q}\phi(d)\mu\Big(\frac{q}{d}\Big)\sum_{h\neq 0}\sum_{\ell=1}^{\infty}\frac{(\ell_1\ell_2,\ell)}{\ell^2}S(hd,0;\ell) \\ \times \int_0^{\infty} C(\mp x+hd,x)\,dx + O\big(L^2q^{\varepsilon-1/2}\big), \end{aligned} \tag{3.52}$$

where the error term takes care of the tail of the $\ell$-sum. We now recall that we have made the substitution $W(x) \leftrightarrow x^{-1/2}W(x)$ (c.f. (3.13)), so up to an error term of $O(L^2q^{\varepsilon-1/2})$, we have to compute the following expression

$$\begin{aligned} \mathrm{MT}^{\pm}_{OD}(\ell_1,\ell_2,N,M;q) = \frac{2}{\phi^*(q)(\ell_1\ell_2)^{1/2}} \sum_{d|q}\phi(d)\mu\Big(\frac{q}{d}\Big)\sum_{h\neq 0}\sum_{\ell=1}^{\infty}\frac{(\ell_1\ell_2,\ell)c_\ell(hd)}{\ell^2} \\ \times \int_0^{\infty}\Lambda^{\pm}(x,hd,\ell_1,\ell_2,\ell;q)\,dx, \end{aligned}$$

where $c_\ell(hd) = S(hd,0;\ell)$ and where the function $\Lambda^\pm$ is defined by

$$\begin{aligned} \Lambda^{\pm}(x,hd,\ell_1,\ell_2,\ell;q) := \frac{(\log(\mp x+hd)-\lambda_{\ell_1,\ell})(\log x-\lambda_{\ell_2,\ell})}{(x(\mp x+hd))^{1/2}} \\ \times \mathbf{V}\left(\frac{x(\mp x+hd)}{\ell_1\ell_2q^2}\right)W\left(\frac{\mp x+hd}{\ell_1 N}\right)W\left(\frac{x}{\ell_2 M}\right). \end{aligned} \tag{3.53}$$

Before evaluating the $x$-integral, we use the well known formula for the Ramanujan sum

$$c_\ell(hd) = \sum_{\substack{ab=\ell \\ b|hd}}\mu(a)b,$$

to get

$$\begin{aligned} \mathrm{MT}^{\pm}_{OD}(\ell_1,\ell_2,N,M;q) = \frac{2}{\phi^*(q)(\ell_1\ell_2)^{1/2}} \sum_{d|q}\phi(d)\mu\Big(\frac{q}{d}\Big)\sum_{a\geqslant 1}\frac{\mu(a)}{a^2}\sum_{b\geqslant 1}\frac{(\ell_1\ell_2,ab)}{b} \\ \times \sum_{\substack{h\neq 0 \\ b|hd}}\int_0^{\infty}\Lambda^{\pm}(x,hd,\ell_1,\ell_2,ab;q)\,dx. \end{aligned} \tag{3.54}$$

It is convenient to replace the condition $b|hd$ by $b|h$. If $d=1$, there is nothing to do. Now if $d=q$, we use the fact that the integral is supported on $x \asymp \ell_2 M$, the $h$-summation to $|h| \leqslant LN/q$ and $b \leqslant LN$ to obtain that up to an error term of $O(L^2q^{-1+\varepsilon})$, we can assume that $(b,q)=1$. Once we have done this, we can also remove the condition $(b,q)=1$ for the same cost.

### 3.5.1 Evaluation of the $x$-Integral

For this evaluation, we need to separate the $\pm$ case. Using the integral representation for $\mathbf{V}$ from (1.29) and the Mellin inversion formula for $W$, we have for the minus case

$$\begin{aligned}\int_0^\infty \Lambda^-(x,hd,\ell_1,\ell_2,ab;q)\,dx &= \frac{1}{(2\pi i)^3}\int_{(*)}\int_{(*)} \widetilde{W_{N,M}}(w_1,w_2)\\ &\times \int_{(*)} \frac{G(s)q^{2s}}{\ell_1^{-s-w_1}\ell_2^{-s-w_2}}\int_0^\infty x^{1/2-s-w_2}\frac{\left(\log x-\lambda_{\ell_2,ab}\right)\left(\log(x+hd)-\lambda_{\ell_1,ab}\right)}{(x+hd)^{1/2+s+w_1}}\\ &\times (\delta_{h>0}+\delta_{h<0}\delta_{x>-hd})\frac{dx}{x}\frac{ds}{s}dw_2dw_1,\end{aligned} \tag{3.55}$$

while for the plus case, we clearly have zero if $h<0$ and otherwise

$$\begin{aligned}\int_0^\infty \Lambda^+(x,hd,\ell_1,\ell_2,ab;q)\,dx &= \frac{1}{(2\pi i)^3}\int_{(*)}\int_{(*)} \widetilde{W_{N,M}}(w_1,w_2)\int_{(*)} \frac{G(s)q^{2s}}{\ell_1^{-s-w_1}\ell_2^{-s-w_2}}\\ &\times \int_0^\infty x^{1/2-s-w_2}\frac{\left(\log x-\lambda_{\ell_2,ab}\right)\left(\log(hd-x)-\lambda_{\ell_1,ab}\right)}{(hd-x)^{1/2+s+w_1}}\delta_{x<hd}\frac{dx}{x}\frac{ds}{s}dw_2dw_1.\end{aligned} \tag{3.56}$$

Here $(*)$ means that we choose contours such that the $x$-integral is convergent. More precisely, for (3.55), we need to impose

$$\begin{aligned}&\Re e(s+w_2)<1/2\,,\ \Re e(2s+w_1+w_2)>0 \quad \text{if} \quad h>0\\ &\Re e(s+w_1)<1/2\,,\ \Re e(2s+w_1+w_2)>0 \quad \text{if} \quad h<0,\end{aligned} \tag{3.57}$$

and for (3.56), we must have

$$\Re e(s+w_1)<1/2 \ \text{ and } \ \Re e(s+w_2)<1/2. \tag{3.58}$$

In order to perform this computation and to deal later with real Dirichlet series, we put the logarithm factors in a more appropriate form, namely

$$(\log x-\lambda_{\ell_2,ab})(\log(\pm x+hd)-\lambda_{\ell_1,ab}) = \mathfrak{D}_\gamma\cdot\left(\frac{x(\ell_2,ab)^2}{\ell_2(ab)^2}\right)^{u_2}\left(\frac{(\pm x+hd)(\ell_1,ab)^2}{\ell_1(ab)^2}\right)^{u_1},$$

where

$$\mathfrak{D}_\gamma := (\partial_{u_1}+2\gamma)(\partial_{u_2}+2\gamma)|_{u_1=u_2=0}.$$

Assuming that (3.57) and (3.58) hold, we can rewrite the two last lines of (3.55) in the form

$$\mathfrak{D}_\gamma\cdot\left\{\frac{(\ell_1,ab)^{2u_1}(\ell_2,ab)^{2u_2}}{\ell_1^{u_1}\ell_2^{u_2}a^{2u_1+2u_2}b^{2u_1+2u_2}}\int_0^\infty x^{1/2-s-w_2+u_2}\frac{\delta_{h<0}\delta_{x>-hd}+\delta_{h>0}}{(x+hd)^{1/2+s+w_1-u_1}}\frac{dx}{x}\right\}, \tag{3.59}$$

and the last line of (3.56) equals

$$\mathfrak{D}_\gamma \cdot \left\{ \frac{(\ell_1, ab)^{2u_1}(\ell_2, ab)^{2u_2}}{\ell_1^{u_1}\ell_2^{u_2} a^{2u_1+2u_2} b^{2u_1+2u_2}} \int_0^\infty x^{1/2-s-w_2+u_2} \frac{\delta_{x<hd}}{(hd-x)^{1/2+s+w_1-u_1}} \frac{dx}{x} \right\}. \tag{3.60}$$

Now using [52, (2.21)&(2.19)], we obtain that the Mellin transform (3.59) equals

$$\begin{aligned} &\mathfrak{D}_\gamma \cdot \left\{ \frac{(\ell_1, ab)^{2u_1}(\ell_2, ab)^{2u_2}}{\ell_1^{u_1}\ell_2^{u_2} a^{2u_1+2u_2} b^{2u_1+2u_2}(|h|d)^{2s+w_1+w_2-u_1-u_2}} \right. \\ &\times \begin{cases} \frac{\Gamma(2s+w_1+w_2-u_1-u_2)\Gamma(\frac{1}{2}-s-w_2+u_2)}{\Gamma(\frac{1}{2}+s+w_1-u_1)} & \text{if} \quad h>0, \\ \\ \frac{\Gamma(2s+w_1+w_2-u_1-u_2)\Gamma(\frac{1}{2}-s-w_1+u_1)}{\Gamma(1/2+s+w_2-u_2)} & \text{if} \quad h<0. \end{cases} \end{aligned} \tag{3.61}$$

Similarly, we use [52, (2.20)] for (3.60) and we obtain

$$\begin{aligned} &\mathfrak{D}_\gamma \cdot \left\{ \frac{(\ell_1, ab)^{2u_1}(\ell_2, ab)^{2u_2}}{\ell_1^{u_1}\ell_2^{u_2} a^{2u_1+2u_2} b^{2u_1+2u_2}(|h|d)^{2s+w_1+w_2-u_1-u_2}} \right. \\ &\times \begin{cases} \frac{\Gamma(\frac{1}{2}-s-w_1+u_1)\Gamma(\frac{1}{2}-s-w_2+u_2)}{\Gamma(1-2s-w_1-w_2+u_1+u_2)} & \text{if} \quad h>0, \\ \\ 0 & \text{if} \quad h<0. \end{cases} \end{aligned} \tag{3.62}$$

According to (3.57) and (3.58), we choose finally the following contours

$$\Re e(s), \Re e(w_1), \Re e(w_2) = \begin{cases} \varepsilon\,, 0\,, \varepsilon & \text{if} \quad h>0 \\ \\ \varepsilon\,, \varepsilon, 0 & \text{if} \quad h<0, \end{cases} \tag{3.63}$$

assuming of course that the $u_i'$ s variables are sufficiently small compared to $\varepsilon$.

### 3.5.2 Assembling the Partition of Unity

The partition of unity is an obstruction for the computation of the second main term, so we need to rebuild it. This step requires some preparations. We return to Expression (3.54) (recall that we have removed $b|hd \leftrightarrow b|h$) and separate the case $h<0$ from $h>0$ by writing $\mathrm{MT}_{OD}^{\pm}(\ell_1,\ell_2,N,M;q) = \mathrm{MT}_{OD}^{\pm,h>0}(\ell_1,\ell_2,N,M;q) + \mathrm{MT}_{OD}^{\pm,h<0}(\ell_1,\ell_2,N,M;q)$, recalling that $\mathrm{MT}_{OD}^{+,h<0} = 0$. In order to have a symmetric situation, we may group the terms as follow :

$$\begin{aligned} \mathrm{MT}_{OD} := \sum_{\pm} \mathrm{MT}_{OD}^{\pm} &= \left( \mathrm{MT}_{OD}^{-,h>0} + \frac{1}{2}\mathrm{MT}_{OD}^{+,h>0} \right) + \left( \mathrm{MT}_{OD}^{-,h<0} + \frac{1}{2}\mathrm{MT}_{OD}^{+,h>0} \right) \\ &=: \mathcal{C}_1(\ell_1,\ell_2,N,M;q) + \mathcal{C}_2(\ell_1,\ell_2,N,M;q). \end{aligned} \tag{3.64}$$

In $\mathrm{MT}_{OD}^{+,h>0}$ appearing in the second term, we just change the $w_1, w_2$-contours to have the same as $\mathrm{MT}_{OD}^{-,h<0}$ (see (3.63)). Inserting the results (3.61) and (3.62), we obtain

$$\begin{aligned} \mathcal{C}_1(\ell_1,\ell_2,N,M;q) &= \frac{2}{\phi^*(q)}\sum_{d|q}\phi(d)\mu\Big(\frac{q}{d}\Big)\sum_{a\geqslant 1}\frac{\mu(a)}{a^2}\sum_{b\geqslant 1}\frac{(\ell_1\ell_2,ab)}{b}\sum_{\substack{h\geqslant 1\\ b|h}} \\ &\times\frac{1}{(2\pi i)^3}\int_{(0)}\int_{(\varepsilon)}\widetilde{W_{N,M}}(w_1,w_2)\int_{(\varepsilon)}\frac{G(s)q^{2s}}{\ell_1^{1/2-s-w_1}\ell_2^{1/2-s-w_2}} \\ &\times\mathfrak{D}_\gamma\cdot\left\{\frac{(\ell_1,ab)^{2u_1}(\ell_2,ab)^{2u_2}H_1(s,w_1,w_2,u_1,u_2)}{\ell_1^{u_1}\ell_2^{u_2}(ab)^{2u_1+2u_2}(hd)^{2s+w_1+w_2-u_1-u_2}}\right\}\frac{ds}{s}dw_2dw_1. \end{aligned} \tag{3.65}$$

where

$$\begin{aligned} H_1(s,w_1,w_2,u_1,u_2) := &\frac{\Gamma(2s+w_1+w_2-u_1-u_2)\Gamma(\frac12-s-w_2+u_2)}{\Gamma(\frac12+s+w_1-u_1)} \\ &+\frac12\frac{\Gamma(\frac12-s-w_1+u_1)\Gamma(\frac12-s-w_2+u_2)}{\Gamma(1-2s-w_1-w_2+u_1+u_2)}. \end{aligned} \tag{3.66}$$

The definition of $\mathcal{C}_2(\ell_1,\ell_2,N,M)$ is the same, but with $\Re e(w_1)=\varepsilon$, $\Re e(w_2)=0$ and $H_2$ instead of $H_1$ with

$$\begin{aligned} H_2(s,w_1,w_2,u_1,u_2) := &\frac{\Gamma(2s+w_1+w_2-u_1-u_2)\Gamma(\frac12-s-w_1+u_1)}{\Gamma(\frac12+s+w_2-u_2)} \\ &+\frac12\frac{\Gamma(\frac12-s-w_1+u_1)\Gamma(\frac12-s-w_2+u_2)}{\Gamma(1-2s-w_1-w_2+u_1+u_2)}. \end{aligned} \tag{3.67}$$

#### Shifting the $s$-contour

The goal here is to move the $s$-line on the right to make the $h$-summation absolutely convergent and bring up the zeta function. We will see that we catch some poles whose contributions seem to be big. Fortunately, the arithmetical sum over $d|q$ cancels these extra factors and this is the reason why we did not separate it at the beginning of Chapter 3. We treat here only $\mathcal{C}_1(\ell_1,\ell_2,N,M;q)$ since the other is completely similar by changing $w_1\leftrightarrow w_2$ in our arguments. Since we deal with the $s,w_1,w_2$-integrals, we can put the differential operator $\mathfrak{D}_\gamma$ outside and only focus on the following quantity :

$$\begin{aligned} \mathcal{I} := &\frac{1}{\phi^*(q)}\sum_{d|q}\phi(d)\mu\Big(\frac{q}{d}\Big)\sum_{b|h}\frac{1}{(2\pi i)^3}\int_{(0)}\int_{(\varepsilon)}\widetilde{W}_{N,M}(w_1,w_2) \\ &\times\int_{(\varepsilon)}\frac{G(s)q^{2s}H_1(s,w_1,w_2,u_1,u_2)}{\ell_1^{1/2-s-w_1+u_1}\ell_2^{1/2-s-w_2+u_2}(hd)^{2s+w_1+w_2-u_1-u_2}}\frac{ds}{s}dw_2dw_1. \end{aligned} \tag{3.68}$$

We now move the $s$-line of integration to $\Re e(s)=1/2-\varepsilon/3$, passing a simple pole at $s=1/2-w_2+u_2$ coming from the factor $\Gamma(1/2-s-w_2+u_2)$ in the function $H_1(s,w_1,w_2,u_1,u_2)$.

Note that since we moved to the right, the residue has to be taken with the minus sign. Hence we obtain that $\mathcal{I} = -\mathcal{R} + \mathcal{I}'$ where $\mathcal{I}'$ is the same as (3.68) but with $\Re e(s) = 1/2 - \varepsilon/3$ and the residue part is given by

$$\mathcal{R} = -\frac{3}{2\phi^*(q)} \sum_{d|q} \phi(d)\mu\Big(\frac{q}{d}\Big) \sum_{b|h} \frac{1}{(2\pi i)^2} \int_{(0)} \int_{(\varepsilon)} \widetilde{W_{N,M}}(w_1, w_2) \times \frac{G(1/2 - w_2 + u_2) q^{1-2w_2+2u_2}}{\ell_1^{w_2-w_1+u_1-u_2} (hd)^{1+w_1-w_2+u_2-u_1}} dw_2 dw_1. \tag{3.69}$$

In $\mathcal{R}$, we shift the $w_1$-contour to $\Re e(w_1) = 2\varepsilon$, passing no pole. Since $\Re e(1 + w_1 - w_2 + u_2 - u_1) > 1$, we can switch the $h$-summation with these two integrals, obtaining

$$\mathcal{R} = -\frac{3}{2\phi^*(q)} \sum_{d|q} \phi(d)\mu\Big(\frac{q}{d}\Big) \frac{1}{(2\pi i)^2} \int_{(2\varepsilon)} \int_{(\varepsilon)} \widetilde{W_{N,M}}(w_1, w_2) \times \zeta(1 + w_1 - w_2 + u_2 - u_1) \frac{G(1/2 - w_2 + u_2) q^{1-2w_2+2u_2}}{\ell_1^{w_2-w_1+u_1-u_2} (bd)^{1+w_1-w_2+u_2-u_1}} dw_2 dw_1. \tag{3.70}$$

Now we deal with $\mathcal{I}'$. Since $\Re e(2s + w_1 + w_2 - u_1 - u_2) > 1$, we can also switch the $h$-summation with the three integrals. Once we have done this, we move the $s$-line to $\Re e(s) = \varepsilon$, passing two poles : one at $2s + w_1 + w_2 - u_1 - u_2 = 1$ coming from the new factor $\zeta(2s + w_1 + w_2 - u_1 - u_2)$ and the other again at $s = 1/2 - w_2 + u_2$. Hence we have (this time the residues have to be taken with positive signs) $\mathcal{I}' = \mathcal{I}'' + \mathcal{R}' + \mathscr{R}$ where $\mathcal{R}'$ is the same as (3.70), but with $\Re e(w_1) = 0$ instead of $2\varepsilon$ and $\mathscr{R}$ is the residue at $2s + w_1 + w_2 - u_1 - u_2 = 1$. In summary, we obtained the following decomposition of (3.68) :

$$\mathcal{I} = \mathcal{I}'' + \mathcal{R}' - \mathcal{R} + \mathscr{R}. \tag{3.71}$$

**Lemma 3.9.** *With the above notations, we have*

$$|\mathcal{R}' - \mathcal{R}| + |\mathscr{R}| = O((qb)^{-1+\varepsilon}).$$

*Proof.* We begin with $\mathcal{R}' - \mathcal{R}$. Since the only difference between these two expressions is the $w_1$-contour, we want to shift it in $\mathcal{R}$ to $\Re e(w_1) = 0$. Before doing this, we switch the arithmetic sum over $d$ with the $w_i$-integrals, obtaining

$$\mathcal{R} = -\frac{3}{2\phi^*(q)} \frac{1}{(2\pi i)^2} \int_{(2\varepsilon)} \int_{(\varepsilon)} \widetilde{W_{N,M}}(w_1, w_2) \frac{G(1/2 - w_2 + u_2) q^{1-2w_2+2u_2}}{\ell_1^{w_2-w_1+u_1-u_2} b^{1+w_1-w_2+u_2-u_1}} \times \zeta(1 + w_1 - w_2 + u_2 - u_1) \left( \frac{\phi(q)}{q^{1+w_1-w_2+u_2-u_1}} - 1 \right) dw_2 dw_1.$$

From the obvious observation that

$$\frac{\phi(q)}{q^{1+w_1-w_2+u_2-u_1}} - 1 = \left( q^{w_2-w_1+u_1-u_2} - 1 \right) - \frac{1}{q^{1+w_1-w_2+u_2-u_1}},$$

we can separate $\mathcal{R}$ as a sum of two terms $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, according to the above decomposition. In the second expression, we can average trivially over the $w_i$-integrals, obtaining the bound $O((qb)^{-1+\varepsilon})$. In $\mathcal{R}_1$, since the pole of the zeta function at $w_1 - u_1 = w_2 - u_2$ is cancelled by the factor $q^{w_2-w_1+u_1-u_2} - 1$, we can shift the $w_1$-line to $\Re e(w_1) = 0$. Writing the same decomposition for $\mathcal{R}'$, namely $\mathcal{R}' = \mathcal{R}'_1 + \mathcal{R}'_2$, we obtain that $\mathcal{R}'_1 = \mathcal{R}_1$ and $\mathcal{R}'_2 = O((qb)^{-1+\varepsilon})$. We play the same game for $\mathscr{R}$. We have, after summing over $d|q$,

$$\mathscr{R} = \frac{1}{2\phi^*(q)(2\pi i)^2} \int_{(0)} \int_{(\varepsilon)} \widetilde{W_{N,M}}(w_1, w_2) \frac{G(\frac{1-w_1-w_2+u_1+u_2}{2}) q^{1-w_1-w_2+u_1+u_2}}{\ell_1^{\frac{w_2-w_1+u_1-u_2}{2}} \ell_2^{\frac{w_1-w_2+u_2-u_1}{2}} b}$$
$$\times H_1\left(\frac{1-w_1-w_2+u_1+u_2}{2}, w_1, w_2, u_1, u_2\right)\left(\frac{\phi(q)}{q} - 1\right) dw_2 dw_1,$$

which is bounded by $O((qb)^{-1+\varepsilon})$. □

We substitute the decomposition (3.71) of $\mathcal{I}$ together with Lemma 3.9 in the expression (3.65) of $\mathcal{C}_1(\ell_1, \ell_2, N, M; q)$. After doing this, we only retain in the $d$-summation the case where $d = q$ ; the other contributes $O(q^{-1+\varepsilon})$. We collect the previous computations in the following proposition.

**Proposition 3.10.** *The quantity defined by* (3.65) *is equal, up to* $O(q^{-1+\varepsilon})$, *to*

$$\mathcal{C}_1(\ell_1, \ell_2, N, M; q) = 2 \sum_{a,b \geqslant 1} \frac{\mu(a)(\ell_1\ell_2, ab)}{(\ell_1\ell_2)^{1/2} a^2 b} \frac{1}{(2\pi i)^3} \int_{(0)} \int_{(0)} \frac{\widetilde{W_{N,M}}(w_1, w_2)}{\ell_1^{-w_1} \ell_2^{-w_2}}$$
$$\times \int_{(\varepsilon)} \frac{G(s)}{(\ell_1\ell_2)^{-s}} \mathfrak{D}_\gamma \cdot \Bigg\{ \frac{(\ell_1, ab)^{2u_1} (\ell_2, ab)^{2u_2}}{a^{2u_1+2u_2}}$$
$$\times \frac{\zeta(2s + w_1 + w_2 - u_1 - u_2) H_1(s, w_1, w_2, u_1, u_2)}{\ell_1^{u_1} \ell_2^{u_2} b^{2s+w_1+w_2+u_1+u_2} q^{w_1+w_2-u_1-u_2}} \Bigg\} \frac{ds}{s} dw_2 dw_1.$$

**Remark 3.11.** The previous Proposition is also valid for $\mathcal{C}_2(\ell_1, \ell_2, N, M; q)$ but with $H_2$ replaced by $H_1$.

**Adding the missing pairs** $(N, M)$

We recall that at this step, the variables $N$ and $M$ belong to the set

$$\left\{ (N, M) \mid 1 \leqslant M \leqslant N,\ \frac{N}{M} \leqslant q^{1-2\theta-2\eta},\ NM \leqslant q^{2+\varepsilon} \right\}.$$

If we could add all the other pairs $(N, M)$ to complete the partition of unity, then we could use the following lemma (see [58, Section 6])

**Lemma 3.12.** *Let* $F(s_1, s_2)$ *be a holomorphic function in the strip* $a < \Re e(s_i) < b$ *with* $a < 0 < b$

*that decays rapidly to zero in each variable (in the imaginary direction). Then we have*

$$\sum_{N,M}\frac{1}{(2\pi i)^2}\int_{(*)}\int_{(*)}\widetilde{W_{N,M}}(s_1,s_2)F(s_1,s_2)ds_2ds_1=F(0,0).$$

*Proof.* Let $f(x,y)$ be the inverse Mellin transform of $F(s_1,s_2)$ ; then the left handside equals

$$\begin{aligned}\sum_{N,M}\int_0^\infty\int_0^\infty f(x,y)\left(\frac{1}{(2\pi i)^2}\int_{(*)}\int_{(*)}\widetilde{W_{N,M}}(s_1,s_2)x^{s_1}y^{s_2}ds_2ds_1\right)\frac{dxdy}{xy}\\ =\sum_{N,M}\int_0^\infty\int_0^\infty f(x,y)W_{N,M}(x^{-1},y^{-1})\frac{dxdy}{xy}\\ =\int_0^\infty\int_0^\infty f(x,y)\frac{dxdy}{xy}=F(0,0),\end{aligned}$$

and the lemma is proved. □

In order to apply Lemma 3.12, we have the following result which allows us to add all the missing pairs $(N,M)$ at the cost of a negligible error.

**Lemma 3.13.** *The quantity defined in Proposition 3.10 satisfies the following bound*

$$\mathcal{C}_i(\ell_1,\ell_2,N,M;q)\ll q^\varepsilon L^2\min\left\{\left(\frac{M}{N}\right)^{1/2},\left(\frac{q^2}{MN}\right)^C,\frac{(NM)^{1/2}}{q}\right\},$$

*where in the second estimation, the implied constant depends on* $C$*.*

*Proof.* This lemma is obtained by moving suitably the different lines of integration. By suitably, we mean that we need to avoid the poles coming from the three different factors (we focus on $\mathcal{C}_1$)

$$\zeta(2s+w_1+w_2-u_1-u_2)\ ,\ \Gamma(2s+w_1+w_2-u_1-u_2)\ ,\ \Gamma(\frac{1}{2}-s-w_2+u_2).$$

In other words, after each manipulation, we must have (recall that $u_i$ are arbitrarily small)

$$0<\Re e(2s+w_1+w_2)<1\ \textbf{ and }\ \Re e(s+w_i)<1/2,\ i=1,2.$$

For the first bound, we just shift the $w_2$-contour to $\Re e(w_2)=1/2-2\varepsilon$ and then, the $w_1$-contour to $\Re e(w_1)=-1/2+2\varepsilon$.

For the second bound, we fix a constant $C>1$ and we shift the $w_i$-contours to $\Re e(w_i)=-\varepsilon/4^{2C}$. The first step is to move to $\Re e(s)=1/2$ and then to $\Re e(w_i)=-1/2+\varepsilon/4^{2C}$. The second step is : $\Re e(s)=1-2\varepsilon/4^{2C}$ and then $\Re e(w_i)=-1+4\varepsilon/4^{2C}$. Again, the third step is $\Re e(s)=3/2-8\varepsilon/4^{2C}$ and $\Re e(w_i)=-3/2+16\varepsilon/4^{2C}$. It follows that after the $j^{th}$ step, we are at ($j\geqslant 2$)

$$\Re e(s)=\frac{j}{2}-\frac{4^{j-1}\varepsilon}{2\cdot 4^{2C}}\quad\text{and}\quad\Re e(w_i)=-\frac{j}{2}+\frac{4^{j-1}\varepsilon}{4^{2C}}.$$

Taking $j = [2C]$ finishes the proof.

The last part is obtained by shifting $\Re e(w_i)$ to $1/2 - 2\varepsilon$. $\square$

This Lemma allows us to sum over all $(N, M)$, getting (recall Decomposition (3.64))

$$\begin{aligned} \mathrm{MT}_{OD}(\ell_1, \ell_2, q) &:= \sum_{N,M} \mathrm{MT}_{OD}(\ell_1, \ell_2, N, M, q) = \sum_{i=1}^{2} \sum_{N,M} \mathcal{C}_i(\ell_1, \ell_2, N, M, q) \\ &= \frac{2}{2\pi i} \int_{(\varepsilon)} \mathcal{F}(s, \ell_1, \ell_2, q) \frac{ds}{s}, \end{aligned} \tag{3.72}$$

where the function $s \mapsto \mathcal{F}(s, \ell_1, \ell_2; q)$ is defined by

$$\mathcal{F}(s, \ell_1, \ell_2; q) := \mathfrak{D}_\gamma \cdot \left\{ \frac{G(s) H(s, u_1, u_2) \zeta(2s - u_1 - u_2) \mathcal{L}(s, u_1, u_2, \ell_1, \ell_2)}{(\ell_1 \ell_2)^{1/2} q^{-u_1 - u_2}} \right\}, \tag{3.73}$$

with

$$H(s, u_1, u_2) := H_1(s, 0, 0, u_1, u_2) + H_2(s, 0, 0, u_1, u_2),$$

$$\mathcal{L}(s, u_1, u_2; \ell_1, \ell_2) = \ell_1^{s - u_1} \ell_2^{s - u_2} L(s, u_1, u_2; \ell_1, \ell_2),$$

and $L(s, u_1, u_2; \ell_1, \ell_2)$ is the Dirichlet series given by

$$L(s, u_1, u_2; \ell_1, \ell_2) := \sum_{a \geqslant 1} \sum_{b \geqslant 1} \frac{\mu(a) (\ell_1, ab)^{1 + 2u_1} (\ell_2, ab)^{1 + 2u_2}}{a^{2 + 2u_1 + 2u_2} b^{1 + 2s + u_1 + u_2}}. \tag{3.74}$$

### 3.5.3 A Symmetry for the Function $\mathcal{F}(s, \ell_1, \ell_2; q)$

As we note actually in Expression (3.73), we have completely removed powers of $q$ in the $s$-aspect in (3.72). Thus the usual method of evaluation consisting in shifting the $s$-contour to the left (as in section 3.1) to get a negative power of $q$ and taking the residues passed along way cannot work here. As it turns out, we will be able to evaluate explicitly the $s$-part through a residue at zero since the function $s \mapsto \mathcal{F}(s, \ell_1, \ell_2; q)$ is even in $s$. This affirmation does not follow directly from definitions (3.73) and (3.74) and requires a finer analysis on the Dirichlet series $\mathcal{L}$, the functional equation for the Riemann zeta function and a crucial identity for the function $H$.

**Analysis of $\mathcal{L}(s, u_1, u_2, \ell_1, \ell_2)$**

We will use the following notations

$$r = 2 + 2u_1 + 2u_2 \quad \text{and} \quad t = 1 + 2s + u_1 + u_2 \tag{3.75}$$

and factorize $\mathcal{L}$ as an infinite product (recall that $\ell_1, \ell_2$ are cubefree and coprime)

$$\mathcal{L} = \prod_{p\nmid \ell_1\ell_2} \mathcal{L}_p \prod_{p\|\ell_1} \mathcal{L}_p \prod_{p^2|\ell_1} \mathcal{L}_p \prod_{p\|\ell_2} \mathcal{L}_p \prod_{p^2|\ell_2} \mathcal{L}_p, \tag{3.76}$$

where for each prime $p$, we have

$$\mathcal{L}_p = p^{v_p(\ell_1)(s-u_1)} p^{v_p(\ell_2)(s-u_2)} \sum_{\substack{0\leqslant a\leqslant 1\\ b\geqslant 0}} \frac{(-1)^a p^{\sum_{i=1}^2 \min(a+b, v_p(\ell_i))(1+2u_i)}}{p^{ar+bs}}.$$

We will compute the above expression according to the different cases appearing in decomposition (3.76). When $p\nmid \ell_1\ell_2$, we easily get

$$\mathcal{L}_p = \left(1 - \frac{1}{p^r}\right)\left(1 - \frac{1}{p^t}\right)^{-1}. \tag{3.77}$$

If $p\|\ell_i$, we have

$$\mathcal{L}_p = \left(\frac{1}{p^{-s+u_i}} - \frac{1}{p^{t-s+u_i}} + \frac{1}{p^{t-s-1-u_i}} - \frac{1}{p^{r-s-1-u_i}}\right)\left(1 - \frac{1}{p^t}\right)^{-1}. \tag{3.78}$$

Finally, assuming $p^2|\ell_i$, we obtain

$$\begin{aligned}&\left(\frac{1}{p^{-2s+2u_i}} + \frac{1}{p^{t-2s-1}} - \frac{1}{p^{r-2s-1}} - \frac{1}{p^{t-2s+2u_i}} - \frac{1}{p^{2t-2s-1}}\right.\\ &\left.+\frac{1}{p^{2t-2s-2-2u_i}} + \frac{1}{p^{t+r-2s-1}} - \frac{1}{p^{t+r-2s-2-2u_i}}\right)\left(1 - \frac{1}{p^t}\right)^{-1}.\end{aligned} \tag{3.79}$$

From (3.77), (3.78), (3.79) and the change of variables (3.75), we infer the following factorization

$$\mathcal{L}(s, u_1, u_2; \ell_1, \ell_2) = \frac{\zeta(1+2s+u_1+u_2)}{\zeta(2+2u_1+2u_2)} \boldsymbol{\delta}(\ell_1; s, u_1, u_2)\boldsymbol{\delta}(\ell_2; s, u_2, u_1), \tag{3.80}$$

where $n \mapsto \boldsymbol{\delta}(n; s, u_1, u_2)$ is the multiplicative function supported on cubefree integers and whose values on $p$ and $p^2$ are given by

$$\begin{aligned}\boldsymbol{\delta}(p; s, u_1, u_2) := &\left\{\frac{1}{p^{s+u_2}}\left(1 - \frac{1}{p^{1+2u_1}}\right) + \frac{1}{p^{-s+u_1}}\left(1 - \frac{1}{p^{1+2u_2}}\right)\right\}\\ &\times \left(1 - \frac{1}{p^{2+2u_1+2u_2}}\right)^{-1},\end{aligned} \tag{3.81}$$

$$\begin{aligned}\boldsymbol{\delta}(p^2;s,u_1,u_2) := &\left\{\frac{1}{p^{2s+2u_2}}\left(1-\frac{1}{p^{1+2u_1}}\right)+\frac{1}{p^{-2s+2u_1}}\left(1-\frac{1}{p^{1+2u_2}}\right)\right.\\ &\left.+\frac{1}{p^{u_1+u_2}}\left(1+\frac{1}{p^{2u_1+2u_2}}-\frac{1}{p^{1+2u_1}}-\frac{1}{p^{1+2u_2}}\right)\right\}\\ &\times\left(1-\frac{1}{p^{2+2u_1+2u_2}}\right)^{-1}.\end{aligned} \tag{3.82}$$

**Parity of** $\mathcal{F}(s,\ell_1,\ell_2;q)$

We split the differential operator $\mathfrak{D}_\gamma=\sum_{i=0}^2\mathfrak{D}_\gamma^i$ with $\mathfrak{D}_\gamma^0=4\gamma^2$, $\mathfrak{D}_\gamma^1=2\gamma(\partial_{u_1}+\partial_{u_2})|_{u_i=0}$, $\mathfrak{D}_\gamma^2=\partial^2_{u_i=0}$ and separate the function $\mathcal{F}(s,\ell_1,\ell_2;q)=\sum_{i=0}^2\mathcal{F}_i(s,\ell_1,\ell_2;q)$ according to this decomposition. We will show that each $\mathcal{F}_i$ is even in $s$. For this, we exploit the factorization (3.80) and define

$$\begin{aligned}A(s,u_1,u_2;q) &:= \frac{G(s)H(s,u_1,u_2)\zeta(2s-u_1-u_2)\zeta(1+2s+u_1+u_2)}{\zeta(2+2u_1+2u_2)q^{-u_1-u_2}},\\ B(s,u_1,u_2;\ell_1,\ell_2) &:= \frac{\boldsymbol{\delta}(\ell_1;s,u_1,u_2)\boldsymbol{\delta}(\ell_2;s,u_2,u_1)}{(\ell_1\ell_2)^{1/2}},\end{aligned} \tag{3.83}$$

in order to have

$$\mathcal{F}(s,\ell_1,\ell_2;q)=\mathfrak{D}_\gamma\cdot\left\{A(s,u_1,u_2;q)B(s,u_1,u_2;\ell_1,\ell_2)\right\}.$$

We also mention the functional equation for the Riemann zeta function

$$\zeta(1+2s)=\pi^{1/2+2s}\zeta(-2s)\frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)}, \tag{3.84}$$

and a crucial identity for the function $H(s,u_1,u_2)$ (see [58, (8.5)&(8.6)])

$$H(s,u_1,u_2)=\pi^{1/2}\frac{\Gamma\left(\frac{-u_1-u_2+2s}{2}\right)}{\Gamma\left(\frac{1+u_1+u_2-2s}{2}\right)}\frac{\Gamma\left(\frac{\frac{1}{2}+u_1-s}{2}\right)\Gamma\left(\frac{\frac{1}{2}+u_2-s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-u_1+s}{2}\right)\Gamma\left(\frac{\frac{1}{2}-u_2+s}{2}\right)}. \tag{3.85}$$

**Lemma 3.14.** *Each of the following functions are even in $s$ : $A(s,0,0;q)$, $B(s,0,0;\ell_1,\ell_2)$, $(\partial_{u_1}+\partial_{u_2})|_{u_i=0}B$, $\partial_{u_i=0}A$, $\partial^2_{u_1u_2=0}A$ and $\partial^2_{u_1u_2=0}B$.*

*Proof.* We begin with $A(s,0,0;q)$. Recalling Definition (3.7) of $G(s)$ and using the Identity (3.85), we have

$$A(s,0,0;q)=Q(s)\frac{\pi^{1/2-2s}\Gamma\left(\frac{\frac{1}{2}+s}{2}\right)^2\Gamma\left(\frac{\frac{1}{2}-s}{2}\right)^2}{\zeta(2)\Gamma(1/4)^4}\zeta(2s)\zeta(1+2s)\frac{\Gamma(s)}{\Gamma(\frac{1}{2}-s)}.$$

Applying now the functional equation (3.84) to $\zeta(1+2s)$, we obtain

$$A(s,0,0;q) = Q(s)\pi \frac{\Gamma\left(\frac{\frac{1}{2}+s}{2}\right)^2 \Gamma\left(\frac{\frac{1}{2}-s}{2}\right)^2}{\zeta(2)\Gamma(1/4)^4} \zeta(2s)\zeta(-2s)\frac{\Gamma(s)\Gamma(-s)}{\Gamma(\frac{1}{2}-s)\Gamma(\frac{1}{2}+s)},$$

which is of course even. For the function $B(s,0,0;\ell_1,\ell_2)$, we easily see from Definitions (3.81) and (3.82) that it is even, since each local factor is even. To compute the others, it will be very convenient to express them as logarithmic derivatives. To be more precise, we can express $\partial_{u_i=0}A$ as

$$\partial_{u_i=0}A = A(s,0,0;q)\left(\frac{H_{u_i}}{H} + \frac{\zeta'(1+2s)}{\zeta(1+2s)} - \frac{\zeta'(2s)}{\zeta(2s)} - 2\frac{\zeta'(2)}{\zeta(2)} + \log q\right).$$

On one hand, we have using (3.85),

$$2\frac{H_{u_i}}{H} = -\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(\frac{1}{2}-s)}{\Gamma(\frac{1}{2}-s)} + \frac{\Gamma'\left(\frac{\frac{1}{2}-s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-s}{2}\right)} + \frac{\Gamma'\left(\frac{\frac{1}{2}+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+s}{2}\right)}.$$

On the other hand, we have by applying the logarithm derivative to (3.84),

$$2\frac{\zeta'(1+2s)}{\zeta(1+2s)} = 2\log(\pi) - 2\frac{\zeta'(-2s)}{\zeta(-2s)} - \frac{\Gamma'(-s)}{\Gamma(-s)} - \frac{\Gamma'(\frac{1}{2}+s)}{\Gamma(\frac{1}{2}+s)}.$$

It follows that $\partial_{u_i}A$ is even. Similarily, we can compute $\partial^2_{u_1u_2=0}A$ in a fancy way :

$$\begin{aligned}\partial^2_{u_1u_2=0}A = A(s,0,0;q)\Bigg\{&\left(\frac{H_{u_1}}{H} + \frac{\zeta'(1+2s)}{\zeta(1+2s)} - \frac{\zeta'(2s)}{\zeta(2s)} - 2\frac{\zeta'(2)}{\zeta(2)} + \log q\right)^2 \\ &+ \partial_{u_2=0}\left(\frac{H_{u_1}(s,0,u_2)}{H(s,0,u_2)} + \frac{\zeta'(1+2s+u_2)}{\zeta(1+2s+u_2)} - \frac{\zeta'(2s-u_2)}{\zeta(2s-u_2)} - \frac{\zeta'(2+2u_2)}{\zeta(2+2u_2)}\right)\Bigg\}.\end{aligned}$$

We already know that the first line is even. For the second, we have by (3.85),

$$\frac{H_{u_1}(s,0,u_2)}{H(s,0,u_2)} = -\frac{1}{2}\frac{\Gamma'\left(\frac{-u_2+2s}{2}\right)}{\Gamma\left(\frac{-u_2+2s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1+u_2-2s}{2}\right)}{\Gamma\left(\frac{1+u_2-2s}{2}\right)} + \frac{1}{2}\frac{\Gamma'\left(\frac{\frac{1}{2}-s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-s}{2}\right)} + \frac{1}{2}\frac{\Gamma'\left(\frac{\frac{1}{2}+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+s}{2}\right)},$$

and using again (3.84), we infer

$$\frac{\zeta'(1+2s+u_2)}{\zeta(1+2s+u_2)} = \log(\pi) - \frac{\zeta'(-2s-u_2)}{\zeta(-2s-u_2)} - \frac{1}{2}\frac{\Gamma'(\frac{-2s-u_2}{2})}{\Gamma(\frac{-2s-u_2}{2})} - \frac{1}{2}\frac{\Gamma'(\frac{1+2s+u_2}{2})}{\Gamma(\frac{1+2s+u_2}{2})}.$$

Hence the parenthesis in the second line of $\partial^2_{u_1u_2=0}$ is preserved under the action of $\partial_{u_2=0}$. It remains to evaluate $(\partial_{u_1}+\partial_{u_2})|_{u_i=0}B$ and $\partial^2_{u_1u_2=0}B$. In this case precisely, it is very useful to express as logarithm derivatives since $B(s,u_1,u_2;\ell_1,\ell_2)$ can be written as a product of the primes dividing $\ell_1\ell_2$ and the logarithm derivative transforms this product into a sum in which each term will be even. Indeed, we compute

$$(\partial_{u_1}+\partial_{u_2})|_{u_i=0}B = B(s,0,0;\ell_1,\ell_2)\sum_{i=1}^{2}\sum_{p|\ell_i}\frac{(\partial_{u_1}+\partial_{u_2})|_{u_i=0}\boldsymbol{\delta}(p^{v_p(\ell_i)};s,u_1,u_2)}{\boldsymbol{\delta}(p^{v_p(\ell_i)};s,0,0)},$$

and it is easy to check that each term appearing in the sum above is even. We mention that the $\partial_{u_i=0}\boldsymbol{\delta}$ are not individually even; it is $(\partial_{u_1}+\partial_{u_2})|_{u_i=0}$ that creates the symmetry (see (3.86)). Finally, we have for the last one (recall that $u_1$ and $u_2$ are swapped when we deal with $\ell_2$)

$$\begin{aligned}\partial^2_{u_1u_2=0}B = B(s,0,0;\ell_1,\ell_2)&\left\{\left(\frac{\boldsymbol{\delta}_{u_1}(\ell_1;s,0,0)}{\boldsymbol{\delta}(\ell_1;s,0,0)}+\frac{\boldsymbol{\delta}_{u_2}(\ell_2;s,0,0)}{\boldsymbol{\delta}(\ell_2;s,0,0)}\right)\right.\\ &\times\left(\frac{\boldsymbol{\delta}_{u_2}(\ell_1;s,0,0)}{\boldsymbol{\delta}(\ell_1;s,0,0)}+\frac{\boldsymbol{\delta}_{u_1}(\ell_2;s,0,0)}{\boldsymbol{\delta}(\ell_2;s,0,0)}\right)+\partial_{u_2}|_{u_2=0}\left(\frac{\boldsymbol{\delta}_{u_1}(\ell_1;s,0,u_2)}{\boldsymbol{\delta}(\ell_1;s,0,u_2)}\right)\\ &\left.+\,\partial_{u_1}|_{u_1=0}\left(\frac{\boldsymbol{\delta}_{u_2}(\ell_2;s,u_1,0)}{\boldsymbol{\delta}(\ell_2;s,u_1,0)}\right)\right\}.\end{aligned}$$

Using the symmetry

$$\frac{\boldsymbol{\delta}_{u_1}(\ell_i;-s,0,0)}{\boldsymbol{\delta}(\ell_i;-s,0,0)}=\frac{\boldsymbol{\delta}_{u_2}(\ell_i;s,0,0)}{\boldsymbol{\delta}(\ell_i,s,0,0)},\tag{3.86}$$

we remark that the product of the two parentheses is invariant under $s\leftrightarrow -s$ since it just switches the two factors. We conclude this Lemma by checking that the local value (at a prime $p$) of the two order two terms is given by

$$\sum_{i=1}^{2}\left(\frac{\boldsymbol{\delta}_{u_1,u_2}(p^{v_p(\ell_i)};s,0,0)}{\boldsymbol{\delta}(p^{v_p(\ell_i)};s,0,0)}-\frac{\boldsymbol{\delta}_{u_1}(p^{v_p(\ell_i)};s,0,0)\boldsymbol{\delta}_{u_2}(p^{v_p(\ell_i)};s,0,0)}{\boldsymbol{\delta}(p^{v_p(\ell_i)};s,0,0)^2}\right)$$

and each individual term is even by a direct computation and (3.86) (using of course (3.81) and (3.82)). □

**Proposition 3.15.** *The function $\mathcal{F}_i(s,\ell_1,\ell_2;q)$ is even for $i=0,1,2$.*

*Proof.* We do not mention the arguments of the functions and write $A_{u_i}$ instead of $\partial_{u_i=0}A$. Since $A_{u_1}=A_{u_2}$, we have

$$\mathcal{F}_0=4\gamma^2AB,\ \mathcal{F}_1=2\gamma(A_{u_1}+A_{u_2})B+A(B_{u_1}+B_{u_2}),$$

$$\mathcal{F}_2=A_{u_1u_2}B+(B_{u_1}+B_{u_2})A_{u_i}+AB_{u_1u_2},$$

and the conclusion follows directly from Lemma 3.14. □

**Corollary 3.16.** *The off-diagonal main term* (3.72) *equals*

$$\mathrm{MT}_{OD}(\ell_1,\ell_2;q)=\sum_{i=0}^{2}\mathrm{Res}_{s=0}\left\{\frac{\mathcal{F}_i(s,\ell_1,\ell_2;q)}{s}\right\}.\tag{3.87}$$

### A note on odd characters

In this chapter, we concentrated exclusively on even characters. The contribution of the odd characters carries through in the same way with slight changes that we mention now. First of all, the function $G(s)$ defined in (3.7) becomes

$$G(s) = \pi^{-2s} \frac{\Gamma\left(\frac{3/2+s}{2}\right)^4}{\Gamma(3/4)^4} Q(s),$$

so we need to remove the original $G(s)$ in the diagonal main term (3.10). The estimations of the error terms as did in Sections 3.2.1 and 3.3 carry through as before. The most significant change appears in the treatment of the off-diagonal main term. Here, the last gamma factor coming from the dual terms (c.f (3.62)) is subtracted in the definition of $H$ instead to be added. Fortunately, the parity of the function $\mathcal{F}(s, \ell_1, \ell_2; q)$ is preserved trough a similar identity (apply [58, Lemma 8.4] with $a = 1/2 - s + u_1$ and $b = 1/2 - s + u_2$)

$$H(s, u_1, u_2) = \pi^{1/2} \frac{\Gamma\left(\frac{2s-u_1-u_2}{2}\right)}{\Gamma\left(\frac{1-2s+u_1+u_2}{2}\right)} \frac{\Gamma\left(\frac{\frac{3}{2}-s+u_1}{2}\right)\Gamma\left(\frac{\frac{3}{2}-s+u_2}{2}\right)}{\Gamma\left(\frac{\frac{3}{2}+s-u_1}{2}\right)\Gamma\left(\frac{\frac{3}{2}+s-u_2}{2}\right)}. \tag{3.88}$$

# 4 The Mollified Fourth Moment

In this chapter, we explain how to use Theorem 3 to prove Theorem 5, i.e. to establish an asymptotic formula for a mollified fourth moment of the form

$$\mathscr{M}^4(q) := \frac{1}{\phi^*(q)} \sum_{\chi \in \mathcal{D}(q)} |L(\chi, \tfrac{1}{2})\mathrm{M}(\chi)|^4, \tag{4.1}$$

where $\mathrm{M}(\chi)$ is our mollifier which is expressed as a short linear form

$$\mathrm{M}(\chi) := \sum_{\ell \geqslant 1} \frac{\boldsymbol{x}(\ell)\chi(\ell)}{\ell^{1/2}}, \tag{4.2}$$

and the coefficients $\boldsymbol{x}(\ell)$ are given by

$$\boldsymbol{x}(\ell) := \mu(\ell)\delta_{\ell \leqslant L} P\left(\frac{\log\left(\frac{L}{\ell}\right)}{\log L}\right), \tag{4.3}$$

for some suitable polynomial $P \in \mathbf{R}[X]$ that satisfies $P(0) = 0$ and $P(1) = 1$. The parameter $L$ will be a small power of $q$ ($L = q^\lambda$ with $\lambda > 0$) and $\mu$ is the Möbius function. Now for $P(X) = \sum_{k \geqslant 1} a_k X^k \in \mathbf{R}[X]$, we define

$$\widehat{P_L}(s) := \sum_{k \geqslant 1} a_k \frac{k!}{(s \log L)^k}. \tag{4.4}$$

Then we have the following integral representation which can be easily deduced using contour shift.

**Lemma 4.1.** *For $L > 0$ not an integer and $\ell \in \mathbf{N}$, we have*

$$\delta_{\ell \leqslant L} P\left(\frac{\log\left(\frac{L}{\ell}\right)}{\log L}\right) = \frac{1}{2\pi i} \int_{(2)} \frac{L^s}{\ell^s} \widehat{P_L}(s) \frac{ds}{s}.$$

## 4.1 Reduction to the twisted fourth moment

Opening the fourth power in (4.1), we obtain

$$\mathscr{M}^4(q) = \sum\sum_{a,b,c,d} \frac{\boldsymbol{x}(a)\boldsymbol{x}(b)\boldsymbol{x}(c)\boldsymbol{x}(d)}{(abcd)^{1/2}} \mathscr{T}^4(ab,cd;q),$$

where $\mathscr{T}^4$ is the twisted fourth moment defined in (0.8). To ensure the coprimality condition between $ab$ and $cd$, we use the definition of the coefficients $\boldsymbol{x}$ and then use the integral representation provided by Lemma 4.1

$$\begin{aligned} \mathscr{M}^4(q) = &\frac{1}{(2\pi i)^4} \int_{(2)}\int_{(2)}\int_{(2)}\int_{(2)} \prod_{i=1}^{4} L^{z_i} \widehat{P_L}(z_i) \\ &\times \sum\sum_{a,b,c,d} \frac{\mu(a)\mu(b)\mu(c)\mu(d)}{a^{1/2+z_1} b^{1/2+z_2} c^{1/2+z_3} d^{1/2+z_4}} \mathscr{T}^4(ab,cd;q) \frac{dz_1 dz_2 dz_3 dz_4}{z_1 z_2 z_3 z_4}. \end{aligned} \tag{4.5}$$

For the sum in the second line, we group the variables $ab = \ell_1$, $cd = \ell_2$ and then set $d = (\ell_1, \ell_2)$, obtaining

$$\sum_{\substack{d\geqslant 1 \\ (d\ell_1\ell_2,q)=1}}\sum\sum_{(\ell_1,\ell_2)=1} \frac{\mu_{2,z_1-z_2}(d\ell_1)\mu_{2,z_3-z_4}(d\ell_2)}{d^{1+z_1+z_3} \ell_1^{1/2+z_1} \ell_2^{1/2+z_3}} \mathscr{T}^4(\ell_1,\ell_2;q), \tag{4.6}$$

where for any complex number $\nu \in \mathbb{C}$, $\mu_{2,\nu}$ is the inverse of the generalized divisor function $\sigma_\nu(n) = \sum_{d|n} d^\nu$ for the Dirichlet convolution, namely

$$\mu_{2,\nu}(n) = \sum_{ab=n} \mu(a)\mu(b)b^\nu. \tag{4.7}$$

In particular this is a multiplicative function supported on cubefree integers and whose values on prime powers are given by

$$\mu_{2,\nu}(p) = -1 - p^\nu \ , \ \mu_{2,\nu}(p^2) = p^\nu \quad \mu_{2,\nu}(p^j) = 0 \ , \ \forall\ j \geqslant 3. \tag{4.8}$$

Inserting (4.6) in (4.5), we see (by shifting the $z_i$-line to $\Re e(z_i) = C > 1$) that we can assume that $\ell_i \leqslant L^{2+\varepsilon}$ for an error cost of $O(q^{-100})$ because $L$ is a positive power of $q$. We are now in position to apply Theorem 3 to $\mathscr{T}^4(\ell_1,\ell_2;q)$. Once we applied the Theorem, we can again remove the condition $\ell_i \leqslant L^{2+\varepsilon}$ for the same cost and sum over all $\ell_i$. The decomposition into a diagonal, off-diagonal and error term leads to the following decomposition

$$\mathscr{M}^4(q) = \mathscr{M}^4_D(q) + \mathscr{M}^4_{OD}(q) + \mathscr{E}(L,q), \tag{4.9}$$

where

$$\begin{aligned}\mathscr{M}_D^4(q) := \frac{1}{(2\pi i)^4}\int_{(2)}\int_{(2)}\int_{(2)}\int_{(2)}\prod_{i=1}^4 L^{z_i}\widehat{P_L}(z_i)\sum_{\substack{d\geqslant 1}}\sum_{\substack{(\ell_1,\ell_2)=1\\(d\ell_1\ell_2,q)=1}}\sum\frac{\mu_{2,z_1-z_2}(d\ell_1)\mu_{2,z_3-z_4}(d\ell_2)}{d^{1+z_1+z_3}\ell_1^{1/2+z_1}\ell_2^{1/2+z_3}}\\ \times\frac{1}{2}\left(\mathscr{T}_D^4(\ell_1,\ell_2;q)+\mathscr{T}_D^{4,-}(\ell_1,\ell_2;q)\right)\frac{dz_1dz_2dz_3dz_4}{z_1z_2z_3z_4},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\mathscr{M}_{OD}^4(q) := \frac{1}{(2\pi i)^4}\int_{(2)}\int_{(2)}\int_{(2)}\int_{(2)}\prod_{i=1}^4 L^{z_i}\widehat{P_L}(z_i)\sum_{\substack{d\geqslant 1}}\sum_{\substack{(\ell_1,\ell_2)=1\\(d\ell_1\ell_2,q)=1}}\sum\frac{\mu_{2,z_1-z_2}(d\ell_1)\mu_{2,z_3-z_4}(d\ell_2)}{d^{1+z_1+z_3}\ell_1^{1/2+z_1}\ell_2^{1/2+z_3}}\\ \times\frac{1}{2}\left(\mathrm{MT}_{OD}(\ell_1,\ell_2;q)+\mathrm{MT}_{OD}^{-}(\ell_1,\ell_2;q)\right)\frac{dz_1dz_2dz_3dz_4}{z_1z_2z_3z_4},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\mathscr{E}(L,q) := \frac{1}{(2\pi i)^4}\int_{(2)}\int_{(2)}\int_{(2)}\int_{(2)}\prod_{i=1}^4 L^{z_i}\widehat{P_L}(z_i)\sum_{\substack{d\geqslant 1}}\sum_{\substack{(\ell_1,\ell_2)=1\\(d\ell_1\ell_2,q)=1}}\sum\frac{\mu_{2,z_1-z_2}(d\ell_1)\mu_{2,z_3-z_4}(d\ell_2)}{d^{1+z_1+z_3}\ell_1^{1/2+z_1}\ell_2^{1/2+z_3}}\\ \times O\left(q^{\varepsilon}\frac{(\ell_1\ell_2)^{3/2}L^{10}}{q^{\eta}}\right)\frac{dz_1dz_2dz_3dz_4}{z_1z_2z_3z_4},\end{aligned} \tag{4.12}$$

where $\mathscr{T}_D^4(\ell_1,\ell_2;q)$, $\mathrm{MT}_{OD}(\ell_1,\ell_2,q)$ are respectively given by (3.10), (3.87) and $\mathscr{T}_D^{4,-},\mathrm{MT}_{OD}^{-}$ are the contribution of the odd characters (see the end of Section 3.5.3 for the necessary changes). We can immediately evaluate the error term $\mathcal{E}(L,q)$. For this, we move the $z_i$-lines to $\Re e(z_i)=2+\varepsilon$, making all summations absolutely convergent, obtaining therefore

$$\mathscr{E}(L,q) = O\left(q^{\varepsilon}\frac{L^{18}}{q^{\eta}}\right), \tag{4.13}$$

which is non-trivial as long as

$$\lambda < \frac{\eta}{18} = \frac{1-6\theta}{18\cdot 14} = \frac{11}{8064} \approx \frac{1}{733}. \tag{4.14}$$

## 4.2 Evaluation of $\mathscr{M}_D^4(q)$

We focus on $\mathscr{T}_D(\ell_1,\ell_2;q)$ since the other term gives the same result. Indeed, the change is on the function $G(s)$ but we will see that the terms which contribute in the asymptotic formula only involve $G(0)$, which is equal to 1 in any case. We now recall that $\mathscr{T}_D(\ell_1,\ell_2;q)$ is given by the following residue (up to some error term, see Proposition 3.2)

$$2\mathrm{Res}_{s=0}\left\{\frac{G(s)q^{2s}}{16s^5\zeta(2+4s)}F(\ell_1\ell_2;s)H(s)\right\},$$

where we factorize $\zeta(1+2s) = (2s)^{-1}H(s)$ with $H(0) = 1$. Since it is a pole of order five, this residue can be expressed as a linear combination in which the sum of the order of derivation of each function (except $s^{-5}$) is four, but it turns out that only the terms where $G(s)H(s)\zeta(2+4s)^{-1}$ are not differentiated that contribute in our asymptotic formula; the contribution of the others are at most $O_\lambda(\log^{-1} q)$. Hence we infer

$$\mathscr{M}_D^4(q) = \frac{1}{8\zeta(2)} \sum_{i+j=4} (i!j!)^{-1}(2\log q)^j \mathscr{M}_D^4(i) + O_\lambda\left(\frac{1}{\log q}\right), \tag{4.15}$$

where

$$\mathscr{M}_D^4(i) := \frac{1}{(2\pi i)^4} \int_{(2)}\int_{(2)}\int_{(2)}\int_{(2)} \prod_{k=1}^{4} L^{z_k}\widehat{P_L}(z_k)\partial_{s=0}^i \mathcal{L}(s,z_1,z_2,z_3,z_4)\frac{dz_1dz_2dz_3dz_4}{z_1z_2z_3z_4}, \tag{4.16}$$

and $\mathcal{L}(s,z_1,...,z_4)$ is the Dirichlet series defined by (recall the definition of $F(\ell_1\ell_2;s)$ given in Proposition 3.2)

$$\mathcal{L}(s,z_1,z_2,z_3,z_4) := \sum_{d\geqslant 1}\sum\sum_{(\ell_1,\ell_2)=1} \frac{\mu_{2,z_1-z_2}(d\ell_1)\mu_{2,z_3-z_4}(d\ell_2)f(\ell_1;1+2s)f(\ell_2;1+2s)}{d^{1+z_1+z_3}\ell_1^{1+z_1+s}\ell_2^{1+z_3+s}}. \tag{4.17}$$

Writing $\mathcal{L}(s,z_1,z_2,z_3,z_4)$ as an Euler product using (4.8) and (3.9) and examining the polar parts leads to (see also [6, Lemma 2.24 & Corollary 2.25])

**Lemma 4.2.** *The Dirichlet series $\mathcal{L}(s,z_1,z_2,z_3,z_4)$ factorizes as*

$$\mathcal{L}(s,z_1,z_2,z_3,z_4) = \mathscr{P}(s,z_1,z_2,z_3,z_4)\prod_{i=1}^{2}\left(\frac{\zeta(1+z_1+z_{i+2})\zeta(1+z_2+z_{i+2})}{\zeta^2(1+z_i+s)\zeta^2(1+z_{i+2}+s)}\right), \tag{4.18}$$

*where $\mathscr{P}(s,z_1,z_2,z_3,z_4)$ is an explicit Euler product which is absolutely convergent in the region $\Re e(s),\Re e(z_i) \geqslant -\kappa$ for some $\kappa > 0$.*

It will also be convenient to isolate the polar parts of the various zeta functions appearing in Lemma 4.2. Namely, we write

$$\mathcal{L}(s,z_1,z_2,z_3,z_4) = \frac{(z_1+s)^2(z_2+s)^2(z_3+s)^2(z_4+s)^2}{(z_1+z_3)(z_1+z_4)(z_2+z_3)(z_2+z_4)}\mathscr{F}(s,z_1,z_2,z_3,z_4), \tag{4.19}$$

where this time $\mathscr{F}(s,z_1,z_2,z_3,z_4)$ is a holomorphic function which does not vanish in a domain that we describe now. From the Prime Number Theorem, we know that there exists an absolute constant $c > 0$ such that the Riemann zeta function does not vanish in

$$\Omega = \left\{s \in \mathbb{C} \mid \Re e(s) \geqslant 1 - \frac{c}{\log(2+|\Im m(s)|)}\right\}.$$

The function $\mathscr{F}$ is therefore holomorphic in the domain

$$\{\Re e(s),\Re e(z_i) \geqslant -\kappa\} \cap \{1+z_i+s \in \Omega\,, i = 1,...,4\}.$$

We insert now the factorization (4.19) in (4.16) and apply the operator $\partial_{s=0}^i$. In this linear combination, we again retain only the terms where $\mathscr{F}$ still not differentiated since the others will also not contribute in the the formula of Theorem 5. This is of course not obvious right now, but it is enough to convince yourself to apply exactly the same calculations that follow from now on, but with $j_s + j_1 + ... + j_4 < 4$ and with at least one derivative of $\mathscr{F}$ at $s = 0$ in expressions (4.20), (4.21) below. It follows that (4.15) can be written in the form

$$\mathscr{M}_D^4(q) = \frac{1}{8\zeta(2)} \sum\sum\sum\sum_{\substack{j_s+j_1+j_2+j_3+j_4=4 \\ 0\leqslant j_k\leqslant 2\ ,\ k=1,...,4}} (2\log q)^{j_s} \mathscr{C}_{j_s,j_1,...,j_4} \mathscr{M}_D^4(j_1,...,j_4;\mathscr{F}) + O_\lambda\left(\frac{1}{\log q}\right), \tag{4.20}$$

where

$$\mathscr{C}_{j_s,j_1,...,j_4} := \boldsymbol{\alpha}(j_1,...,j_4) C_{j_s,j_1,...,j_4},$$
$$C_{j_s,j_1,...,j_4} := \frac{1}{j_s!j_1!j_2!j_3!j_4!},$$
$$\boldsymbol{\alpha}(j_1,...,j_4) := \prod_{i=1}^4 \alpha(j_i)\ \ ,\ \alpha(0) = 1\ \ \alpha(1) = \alpha(2) = 2,$$

and

$$\begin{aligned}\mathscr{M}_D^4(j_1,...,j_4;\mathscr{F}) := \frac{1}{(2\pi i)^4} \int_{(*)}\int_{(*)}\int_{(*)}\int_{(*)} \prod_{i=1}^4 L^{z_i} \widehat{P_L}(z_i) \mathscr{F}(0, z_1, z_2, z_3, z_4) \\ \times \frac{z_1^{2-j_1} z_2^{2-j_2} z_3^{2-j_3} z_4^{2-j_4}}{(z_1+z_3)(z_1+z_4)(z_2+z_3)(z_2+z_4)} \frac{dz_4 dz_3 dz_2 dz_1}{z_4 z_3 z_2 z_1}.\end{aligned} \tag{4.21}$$

By $(*)$ under the integrals, we mean that $1 + z_i \in \Omega$ with $\Re e(z_i) > 0$ and furthermore, we want that the real parts sufficiently small so that all future manipulations are justified, for example $1 + z_1 + ... + z_4$ also belongs to $\Omega$.

### 4.2.1 Shifting the $z_i$-contours

We focus now on the calculation of $\mathscr{M}_D^4(j_1,...,j_4;\mathscr{F})$ for a fixed multi-index $(j_1,...,j_4)$ such that $j_1 + ... + j_4 \leqslant 4$. We also choose the polynomial in [1] in (4.3) to be $P(X) = X^2$. Using the

[1] Of course we could take a general polynomial $P(X) = \sum_{k\geqslant 1} a_k X^k$ and try to minimize the coefficients at the end. In [6, Prop 6.5], the authors found $P(X) = X$ for the mollification of the second moment of twisted $L$-functions $L(f \otimes \chi)$, where $f$ is a cuspidal Hecke eigenform. This polynomial does not work in the case where $f = E$ is the Eisenstein series mentioned in Section 0.2 because of the pole of the zeta function. The choice $P(X) = X^2$ appears to be the simplest adaptation to our treatment

Definition (4.4) of $\widehat{P_L}$ and we get

$$\mathscr{M}_D^4(j_1,\dots,j_4;\mathscr{F}) = \frac{16}{(\log L)^8(2\pi i)^4}\int_{(*)}\int_{(*)}\int_{(*)}\int_{(*)} \frac{L^{z_1+z_2+z_3+z_4}\mathscr{F}(0,z_1,\dots,z_4)}{(z_1+z_3)(z_1+z_4)(z_2+z_3)(z_2+z_4)} \times \frac{dz_4\,dz_3\,dz_3\,dz_1}{z_4^{1+j_4}z_3^{1+j_3}z_2^{1+j_2}z_1^{1+j_1}}. \tag{4.22}$$

We start by shifting the $z_4$-contour left to zero such that $\Re e(z_1+z_2+z_3+z_4)<0$, passing three poles : one of order $1+j_4$ at $z_4=0$ and two of order one at $z_4=-z_1$ and $z_4=-z_2$. Since $\Re e(z_1+z_2+z_3+z_4)<0$, the resulting integral is at most $O(\log^{-8} L)$. We will analyze separately the three poles and find out that each of them contributes.

**The pole at** $z_4=-z_1$

Since it is a simple pole, the residue at $z_4=-z_1$ is given by

$$\frac{16(-1)^{1+j_4}}{(\log L)^8(2\pi i)^3}\int_{(*)}\int_{(*)}\int_{(*)} \frac{L^{z_2+z_3}\mathscr{F}(0,z_1,z_2,z_3,-z_1)}{(z_1+z_3)(z_2+z_3)(z_2-z_1)}\frac{dz_3\,dz_2\,dz_1}{z_3^{1+j_3}z_2^{1+j_2}z_1^{2+j_1+j_4}}. \tag{4.23}$$

In this integral, we move the $z_3$-line such that $\Re e(z_2+z_3)<0$, passing a pole of order $1+j_3$ at $z_3=0$ and one of order one at $z_3=-z_2$. We immediately see that the one at $z_3=-z_2$ contributes at most $O(\log^{-8} L)$. The residue at $z_3=0$ is given by the following linear combination (again we do not take in account the $z_3$-derivatives of $\mathscr{F}$

$$16(-1)^{1+j_4}\sum_{k+\ell+n=j_3}\sum \boldsymbol{\beta}(k,\ell,n)\frac{(\log L)^{k-8}}{(2\pi i)^2}\int_{(*)}\int_{(*)} \frac{L^{z_2}\mathscr{F}(0,z_1,z_2,0,-z_1)}{(z_2-z_1)z_1^{3+\ell+j_1+j_4}z_2^{2+n+j_2}}dz_2\,dz_1,$$

where for any $(a,b,c)\in\mathbb{N}^3$, we defined

$$\boldsymbol{\beta}(a,b,c) := \frac{(-1)^{b+c}}{a!}. \tag{4.24}$$

We fix $k+n+\ell=j_3$ and we move the $z_2$-line to $\Re e(z_2)<0$, passing two poles : one at $z_2=z_1$ of order 1 and the other at $z_2=0$ of order $2+n+j_2$. The last one is easily see to be bounded by $O((\log L)^{1+k+n+j_2-8}) = O((\log L)^{1+j_2+j_3-\ell-8})$ and thus, will contribute at the end at most $O(\log^{-3} q)$ (recall that $L=q^\lambda$). The residue at $z_2=z_1$ equals

$$16(-1)^{1+j_4}\sum_{k+\ell+n=j_3}\sum \boldsymbol{\beta}(k,\ell,n)\frac{(\log L)^{k-8}}{2\pi i}\int_{(*)} \frac{L^{z_1}\mathscr{F}(0,z_1,z_1,0,-z_1)}{z_1^{5+j_1+j_2+j_4+\ell+n}}dz_1.$$

Finally shifting to $\Re e(z_1)<0$ and we obtain that the above sum is

$$16\mathscr{F}(0,0,0,0,0)\boldsymbol{\gamma}(j_1,j_2,j_3,j_4)(\log L)^{j_1+j_2+j_3+j_4-4} + O\Big((\log L)^{j_1+j_2+j_3+j_4-5}\Big), \tag{4.25}$$

with

$$\boldsymbol{\gamma}(j_1, j_2, j_3, j_4) := (-1)^{1+j_4} \sum\sum_{k+\ell+n=j_3} \frac{\boldsymbol{\beta}(k,\ell,n)}{(4+j_1+j_2+j_4+\ell+n)!}. \tag{4.26}$$

**The pole at $z_4 = -z_2$**

It is not difficult to see that in fact, the pole at $z_4 = -z_2$ has the same main term as in the previous case. In fact, applying the changes $z_1 \leftrightarrow z_2$ and we see that this residue is given by (4.23), but with the first two variables switched in $\mathscr{F}$. This is not a real problem since the main term only involves $\mathscr{F}(0,0,0,0,0)$.

**The pole at $z_4 = 0$**

We return to Expression (4.22). The residue at $z_4 = 0$ is given by the linear combination (we do not mention the derivatives of $\mathscr{F}$)

$$16 \sum\sum_{k+\ell+n=j_4} \boldsymbol{\beta}(k,\ell,n)(\log L)^{k-8}\mathcal{A}(j_1,\dots,j_4,k,\ell,n),$$

where $\boldsymbol{\beta}(k,\ell,n)$ is defined by (4.24) and

$$\mathcal{A}(j_1,\dots,j_4,k,\ell,n) := \frac{1}{(2\pi i)^3}\int_{(*)}\int_{(*)}\int_{(*)} \frac{L^{z_1+z_2+z_3}\mathscr{F}(0,z_1,z_2,z_3,0)dz_3dz_2dz_1}{(z_1+z_3)(z_2+z_3)z_3^{1+j_3}z_2^{2+j_2+n}z_1^{2+j_1+\ell}}.$$

We now move the $z_3$-line such that $\Re e(z_3) < -\Re e(z_1+z_2)$, passing three poles : one at $z_3 = 0$ of order $1 + j_3$, one at $z_3 = -z_1$ of order 1 and the last at $z_3 = -z_2$, that is also simple. We thus get the decomposition

$$\mathcal{A}(j_1,\dots,j_4,k,\ell,n) = \sum_{i=0}^{2} \mathcal{R}_i(j_1,\dots,j_4,k,\ell,n) + O(1).$$

It is the routine now to see that

$$\mathcal{R}_0 = \sum\sum_{a+b+c=j_3} \boldsymbol{\beta}(a,b,c)\frac{(\log L)^a}{(2\pi i)^2}\int_{(*)}\int_{(*)} \frac{L^{z_1+z_2}\mathscr{F}(0,z_1,z_2,0,0)}{z_1^{3+j_1+\ell+b}z_2^{3+j_2+n+c}}dz_2dz_1.$$

Now moving the $z_2$-line to $\Re e(z_2) < -\Re e(z_1)$ and then the $z_1$-contour to $\Re e(z_1) < 0$ and we obtain that

$$\begin{aligned}\mathcal{R}_0 = \sum\sum_{a+b+c=j_3} \boldsymbol{\beta}(a,b,c)&\frac{\mathscr{F}(0,0,0,0,0)(\log L)^{4+j_1+j_2+j_3+\ell+n}}{(2+j_2+n+c)!(2+j_1+\ell+b)!} \\ &+ O\Big((\log L)^{3+j_1+j_2+j_3+\ell+n}\Big).\end{aligned} \tag{4.27}$$

For $\mathcal{R}_1$, we just obsere that

$$\mathcal{R}_1 = \frac{(-1)^{1+j_3}}{(2\pi i)^2} \int_{(*)} \int_{(*)} \frac{L^{z_2} \mathscr{F}(0, z_1, z_2, -z_1, 0)}{(z_2 - z_1) z_1^{3+j_1+j_3+\ell} z_2^{2+j_2+n}} dz_2 dz_1,$$

and we can proceed as in the previous case (the pole at $z_4 = -z_1$), obtaining

$$\begin{aligned}\mathcal{R}_1 = & \frac{(-1)^{1+j_3} \mathscr{F}(0,0,0,0,0)(\log L)^{4+j_1+j_2+j_3+\ell+n}}{(4 + j_1 + j_2 + j_3 + \ell + n)!} \\ & + O\Big((\log L)^{3+j_1+j_2+j_3+\ell+n}\Big).\end{aligned} \tag{4.28}$$

Finally, we see that $\mathcal{R}_2 = \mathcal{R}_1$.

### Assembling the main terms

We define

$$\boldsymbol{\eta}(j_1, j_2, j_3, j_4) := \sum_{\substack{k+\ell+n=j_4 \\ a+b+c=j_3}}\sum\sum \frac{\boldsymbol{\beta}(k, \ell, n) \boldsymbol{\beta}(a, b, c)}{(2 + j_2 + n + c)!(2 + j_1 + \ell + b)!}, \tag{4.29}$$

and

$$\mathfrak{S}(j_1, j_2, j_3, j_4) := 2\boldsymbol{\gamma}(j_1, ..., j_4) + 2\boldsymbol{\gamma}(j_1, j_2, j_4, j_3) + \boldsymbol{\eta}(j_1, ..., j_4). \tag{4.30}$$

Then we obtain

**Proposition 4.3.** *The quantity defined by* (4.21) *equals*

$$\mathscr{M}_D^4(i_1, ..., j_4; \mathscr{F}) = 16 \frac{\mathscr{F}(0, ..., 0) \mathfrak{S}(j_1, ..., j_4)}{(\log L)^{4-(j_1+...j_4)}} + O\left(\frac{1}{(\log L)^{5-(j_1+...+j_4)}}\right),$$

*where* $\mathfrak{S}(j_1, ..., j_4)$ *is defined by* (4.30).

In order to finalize our computation, we have

**Lemma 4.4.** *The value of* $\mathscr{F}(s, z_1, ..., z_4)$ *at* $(0, ..., 0)$ *is* $\zeta(2)$.

*Proof.* Examining (4.18) and (4.19) and we see that

$$\mathscr{F}(0,0,0,0,0) = \mathscr{P}(0,0,0,0,0).$$

To prove the result, it is enough to show that for each prime $p$, we have

$$\mathscr{P}_p(0, ..., 0) = \left(1 - \frac{1}{p^2}\right)^{-1}.$$

By (4.18), the local factor at $p$ of $\mathscr{P}(s,z_1,\dots,z_4)$ is given by the local factor of $\mathcal{L}(s,z_1,\dots,z_4)$ divided by the one of the righthand side involving the zeta functions. Since we evaluate at $(0,\dots,0)$, we obtain

$$\mathscr{P}_p(0,\dots,0)=\mathcal{L}_p(0,\dots,0)\left(1-\frac{1}{p}\right)^{-4}.$$

Thus, it is enough to show that

$$\mathcal{L}_p(0,\dots,0)\left(1-\frac{1}{p^2}\right)=\left(1-\frac{1}{p}\right)^4.$$

Writing $\mu_2$ for $\mu_{2,0}$, we have

$$\begin{aligned}\mathcal{L}_p(0,\dots,0)&=\sum_{\substack{0\leqslant d+\ell_i\leqslant 2\\ \ell_1\ell_2=0}}\sum\frac{\mu_2(p^{d+\ell_1})\mu_2(p^{d+\ell_2})f(p^{\ell_1};1)f(p^{\ell_2};1)}{p^{d+\ell_1+\ell_2}}\\&=2\sum_{0\leqslant d+\ell\leqslant 2}\sum\frac{\mu_2(p^{\ell+d})\mu_2(p^d)f(p^\ell;1)}{p^{d+\ell}}-\sum_{0\leqslant d\leqslant 2}\frac{\mu_2(p^d)^2}{p^d}.\end{aligned}$$

Using (3.9) and (4.8) (with $\nu=0$), we obtain that this expression is

$$1+\frac{4}{p}-\frac{8}{p+1}-\frac{8}{p(p+1)}+\frac{1}{p^2}+\frac{2}{p}\left(\frac{3-p^{-1}}{1+p}\right).$$

Finally, multiplying by $(1-p^{-2})=p^{-2}(p^2-1)$ leads to the desired factor $(1-p^{-1})^4$. $\square$

We now replace $\mathscr{F}(0,\dots,0)$ by $\zeta(2)$ in Proposition 4.3 and then insert the value of $\mathcal{M}^4_D(j_1,\dots,j_4)$ in (4.20). Writing $\log q=\lambda^{-1}\log L$, we get

**Proposition 4.5.** *We have the following asymptotic formula for the diagonal main term appearing in decomposition* (4.9)

$$\mathscr{M}^4_D(q)=2\sum_{\substack{j_s+j_1+\dots+j_4=4\\ 0\leqslant j_k\leqslant 2,\ k=1,\dots,4}}\sum\sum\sum\sum\left(\frac{2}{\lambda}\right)^{j_s}\mathscr{C}_{j_s,j_1,\dots,j_4}\mathfrak{S}(j_1,\dots,j_4)+O_\lambda\left(\frac{1}{\log q}\right).\tag{4.31}$$

## 4.3 Evaluation of $\mathscr{M}^4_{OD}(q)$

We proceed in a completely analogous way as in the previous section. First of all, we also restrict the computation to $\mathrm{MT}_{OD}(\ell_1,\ell_2,q)$ since the dual term gives the same result. We will begin by evaluating the residue (3.87) up to some terms that will not contribute. After that, we will return to (4.11) and find an appropriate expression for a certain Dirichlet series in order to localize the various poles. Finally, we will see that the resulting expression matches perfectly with (4.21) whose value has already been established in Proposition 4.3.

### 4.3.1 Computation of the residue of $\mathcal{F}(s,\ell_1,\ell_2;q)$ at $s=0$

We recall that

$$\mathrm{MT}_{OD}(\ell_1,\ell_2,q)=\sum_{i=0}^{2}\mathrm{Res}_{s=0}\left\{\frac{\mathcal{F}_i(s,\ell_1,\ell_2;q)}{s}\right\}.$$

A very pleasant fact is that $\mathrm{Res}_{s=0}\mathcal{F}_i$ will not contribute in the final asymptotic formula unless $i=2$; the two others will be at most $O_\lambda(\log^{-1}q)$. The heuristic reason is the following : in Section 4.3.2, we will express our main term as the $z_i$-integral in which a certain differential operator (see (4.38)) depending on $s,u_1,u_2$ and $\log q$ acts on a function. If we look at this operator, we remark that for each term, the sum of the order of differentiation plus the power of $\log q$ is 4. If we take in count the residue of $\mathcal{F}_i$ with $i\leqslant 1$, then we just add some lower 'order' terms to (4.38). We therefore focus on $i=2$. By Proposition 3.15 and Lemma 3.14, we see that each part of $\mathcal{F}_2$ is even, so we can take the residue at $s=0$ for each of them separately. We first isolate the polar part in the function $A$ around $s=0$ by writing

$$A(s,u_1,u_2;q)=\frac{q^{u_1+u_2}\mathscr{A}(s,u_1,u_2)}{(2s+u_1+u_2)(2s-u_1-u_2)},\tag{4.32}$$

where $\mathscr{A}(s,u_1,u_2)$ is entire and does not vanish in a neighborhood of $\Re e(s)=\Re e(u_i)=0$. We now easily get

$$\partial_{u_1=0}A=\frac{\mathscr{A}(s,0,0)}{4s^2}\left(\log q+\frac{\mathscr{A}_{u_1}(s,0,0)}{\mathscr{A}(s,0,0)}\right),$$

and

$$\partial^2_{u_1u_2=0}A=\frac{\mathscr{A}(s,0,0)}{4s^2}\left\{\left(\log q+\frac{\mathscr{A}_{u_1}(s,0,0)}{\mathscr{A}(s,0,0)}\right)\left(\log q+\frac{\mathscr{A}_{u_2}(s,0,0)}{\mathscr{A}(s,0,0)}\right)\right.$$
$$\left.+\ \frac{1}{2s^2}+\partial_{u_2=0}\left(\frac{\mathscr{A}_{u_1}(s,u_1,0)}{\mathscr{A}(s,u_1,0)}\right)\right\}.$$

Writing

$$\begin{aligned}
&B_0(s;\ell_1,\ell_2):=B(s,0,0;\ell_1,\ell_2),\\
&B_1(s;\ell_1,\ell_2):=(\partial_{u_1=0}+\partial_{u_2=0})B(s,u_1,u_2;\ell_1,\ell_2),\\
&B_2(s;\ell_1,\ell_2):=\partial^2_{u_1u_2=0}B(s,u_1,u_2;\ell_1,\ell_2),
\end{aligned}\tag{4.33}$$

we infer that the contribution of $\mathcal{F}_2(s,\ell_1,\ell_2;q)$ to our final asymptotic formula comes from the residue at $s=0$ of the following quantity (in fact we drop out all factors where we derive $\mathscr{A}$)

$$\frac{\mathscr{A}(s,0,0)}{4s^3}\left\{B_0(s;\ell_1,\ell_2)\left(\frac{1}{2s^2}+\log^2q\right)+B_1(s;\ell_1,\ell_2)\log q+B_2(s;\ell_1,\ell_2)\right\},\tag{4.34}$$

which is

$$\begin{aligned}\frac{\mathscr{A}(0,0,0)}{8}&\left(\frac{1}{4!}\partial^4_{s=0}B_0(s;\ell_1,\ell_2)+\log^2(q)\partial^2_{s=0}B_0(s;\ell_1,\ell_2)\right.\\
&\left.+\log(q)\partial^2_{s=0}B_1(s;\ell_1,\ell_2)+\partial^2_{s=0}B_2(s;\ell_1,\ell_2)\right)+E(\ell_1,\ell_2;q),\end{aligned}\tag{4.35}$$

where the error term $E(\ell_1,\ell_2;q)$ is such that when we average it over the $\ell_i, d$ in (4.11), we obtain $O((\log q)^{-1})$ (here $E(\ell_1,\ell_2;q)$ contains all term where $\mathscr{A}$ is derived at least one time). By construction, we have $\mathscr{A}(0,0,0)=2\zeta(0)\zeta(2)^{-1}=-\zeta(2)^{-1}$ (see (3.83), (3.85), (4.32)) and thus, taking into account the error coming from $\mathcal{F}_i$, $i\leqslant 1$, we can summarize all previous computation in

**Proposition 4.6.** *Let* $\mathrm{MT}_{OD}(\ell_1,\ell_2;q)$ *defined by* (3.87). *Then we have the formula*

$$\mathrm{MT}_{OD}(\ell_1,\ell_2;q)=c(\ell_1,\ell_2;q)+\mathcal{E}(\ell_1,\ell_2;q),$$

*where* $c(\ell_1,\ell_2;q)$ *is given by* (4.35) *with* $\mathscr{A}(0,0,0)=-\zeta(2)^{-1}$ *and the error term* $\mathcal{E}(\ell_1,\ell_2;q)$ *is such that when we average it over* $\ell_i$ *in* (4.11), *we get* $O_\lambda((\log q)^{-1})$.

### 4.3.2 Averaging over $\ell_i$

We come back to (4.11) and insert the quantity $\mathrm{MT}_{OD}$ given by (4.35). We can remove the primality condition $(k\ell_1\ell_2,q)=1$ for an acceptable error term and we obtain our off-diagonal main term

$$\mathscr{M}^4_{OD}(q)=\frac{-1}{8\zeta(2)(2\pi i)^4}\int_{(2)}\int_{(2)}\int_{(2)}\int_{(2)}\prod_{i=1}^4 L^{z_i}\widehat{P_L}(z_i)\mathfrak{D}^4_q\cdot\{\mathcal{L}(s,u_1,u_2,z_1,\dots,z_4)\}\times\frac{dz_4dz_3dz_2dz_1}{z_4z_3z_2z_1},\tag{4.36}$$

where $\mathcal{L}(s,u_1,u_2,z_1,\dots,z_4)$ is the Dirichlet series defined by (recall Definitions (4.33) and (3.83))

$$\begin{aligned}&\mathcal{L}(s,u_1,u_2,z_1,\dots,z_4)\\&:=\sum_{k\geqslant 1}\sum_{(\ell_1,\ell_2)=1}\frac{\mu_{2,z_1-z_2}(k\ell_1)\mu_{2,z_3-z_4}(k\ell_2)\boldsymbol{\delta}(\ell_1;s,u_1,u_2)\boldsymbol{\delta}(\ell_2;s,u_2,u_1)}{k^{1+z_1+z_3}\ell_1^{1+z_1}\ell_2^{1+z_3}},\end{aligned}\tag{4.37}$$

and $\mathfrak{D}^4_q$ is an order four differential operator given by

$$\mathfrak{D}^4_q:=\left(\frac{1}{4!}\partial_s^4+\log^2(q)\partial_s^2+\log(q)\partial_s^2(\partial_{u_1}+\partial_{u_2})+\partial_s^2\partial^2_{u_1u_2}\right)\bigg|_{s=u_i=0}.\tag{4.38}$$

It is very important now to have an adequate expression for (4.37) in a way to locate the poles and their orders for future contour shift in the $z_i$-integrals. The classical method, as in Section 4.2, is to compute for each prime $p$ the local factor $\mathcal{L}_p$ at $p$. This is a quite tedious calculation, but is not difficult since all arithmetic functions are cubefree and we already computed their values on prime powers (see (4.8), (3.81) and (3.82)). We do not want to figure out all details, but by close examination of the polar part in the local factor, we can conclude that $\mathcal{L}$ admits the following factorization

$$\begin{aligned}\mathcal{L} = \mathscr{P}(s,u_1,u_2,z_1,\dots,z_4)&\prod_{i=1}^{2}\left(\frac{\zeta(1+z_i+z_3)\zeta(1+z_i+z_4)}{\zeta(1+s+z_i+u_2)\zeta(1-s+z_i+u_1)}\right)\\ \times&\prod_{i=3}^{4}\left(\frac{1}{\zeta(1+s+z_i+u_1)\zeta(1-s+z_i+u_2)}\right),\end{aligned} \tag{4.39}$$

where $\mathscr{P}$ is an Euler product absolutely convergent in a "good" neighborhood of the domain of holomorphy of the above product. Furthermore, if we factorize now the poles of the zeta functions, then we can rewrite (4.39) as

$$\mathcal{L} = \mathscr{F}(s,u_1,u_2,z_1,\dots,z_4)Q(s,u_1,u_2,z_1,\dots,z_4), \tag{4.40}$$

where $\mathscr{F}$ is an entire function in a neighborhood of $(0,\dots,0)$ which does not vanish and $Q \in \mathbf{C}(s,u_1,u_2,z_1,\dots,z_4)$ is the rational function defined by

$$\begin{aligned}Q(s,u_1,u_2,z_1,\dots,z_4) :=\ &(z_1+s+u_2)(z_1-s+u_1)(z_2+s+u_2)(z_2-s+u_1)\\ &\times\frac{(z_3+s+u_1)(z_3-s+u_2)(z_4+s+u_1)(z_4-s+u_2)}{(z_1+z_3)(z_1+z_4)(z_2+z_3)(z_2+z_4)}.\end{aligned} \tag{4.41}$$

Using our classical argument concerning the derivatives of $\mathscr{F}$ and we obtain

$$\begin{aligned}\mathscr{M}_{OD}^4(q) &= \frac{-1}{8\zeta(2)(2\pi i)^4}\int\limits_{(2)}\int\limits_{(2)}\int\limits_{(2)}\int\limits_{(2)}\prod_{i=1}^{4}L^{z_i}\widehat{P_L}(z_i)\mathscr{F}(0,0,0,z_1,\dots,z_4)\\ &\quad\times\mathfrak{D}_q^4\cdot\{Q(s,u_1,u_2,z_1,\dots,z_4)\}\frac{dz_4dz_3dz_2dz_1}{z_4z_3z_2z_1}+O_\lambda\left(\frac{1}{\log q}\right)\\ &=: \sum_{i=1}^{4}\mathscr{M}_{OD}^4(i;q)+O_\lambda\left(\frac{1}{\log q}\right),\end{aligned}$$

where this decomposition takes care of the separation of the operator $\mathfrak{D}_q^4$ in (4.38). We will compute each term separately.

### 4.3.3 Computation of $\mathscr{M}_{OD}^4(i;q)$

We compute in this last section $\mathscr{M}_{OD}^4(i;q)$ for $i = 1,\dots,4$. Fortunately, we will see that these main terms can be expressed as the same integral as (4.22), which has already been computed. We start with $i = 1$; first of all, we have

$$\frac{1}{4!}\partial_{s=0}^4(\mathrm{Num}(Q(s,0,0,z_1,\dots,z_4))) = \sum_{\substack{i_1+j_1+\dots+i_4+j_4=4\\ 0\leqslant i_k,j_k\leqslant 1}}\sum\sum\sum\mathscr{B}_{i_1,j_1,\dots,i_4,j_4}\prod_{k=1}^{4}z_k^{2-i_k-j_k},$$

where

$$\mathscr{B}_{i_1,j_1,\dots,i_4,j_4} = \frac{(-1)^{j_1+\dots+j_4}}{i_1!j_1!\cdots i_4!j_4!}. \tag{4.42}$$

Hence, writing explicitly $\widehat{P_L}(z_i)$ and we obtain

$$\mathscr{M}_{OD}^4(1;q) = \frac{-2}{\zeta(2)} \sum\sum\sum\sum_{\substack{i_1+j_1+\dots+i_4+j_4=4 \\ 0\leqslant i_k,j_k\leqslant 1}} \mathscr{B}_{i_1,j_1,\dots,i_4,j_4} \mathscr{M}(i_1+j_1,\dots,i_4+j_4;\mathscr{F}), \tag{4.43}$$

where for any $(a,b,c,d)\in\mathbb{N}^4$, we define

$$\mathscr{M}(a,b,c,d;\mathscr{F}) := \frac{(\log L)^{-8}}{(2\pi i)^4} \int_{(*)}\int_{(*)}\int_{(*)}\int_{(*)} \frac{L^{z_1+z_2+z_3+z_4}\mathscr{F}(0,0,0,z_1,\dots,z_4)}{(z_1+z_3)(z_1+z_4)(z_2+z_3)(z_2+z_4)} \times \frac{dz_4dz_3dz_2dz_1}{z_1^{1+a}z_2^{1+b}z_3^{1+c}z_4^{1+d}}.$$

We remark that this integral is exactly the expression $\mathscr{M}_D^4(j_1,\dots,j_4;\mathscr{F})$ in (4.22) (modulo the factor 16), then its value is given by (see Proposition 4.3)

$$\begin{aligned}\mathscr{M}(i_1+j_1,\dots,i_4+j_4;\mathscr{F}) &= \mathscr{F}(0,\dots,0)\mathfrak{S}(i_1+j_1,i_2+j_2,i_3+j_3,i_4+j_4) \\ &\times (\log L)^{-4+\sum_{k=1}^4 i_k+j_k} + O\Big((\log L)^{-5+\sum_{k=1}^4 i_k+j_k}\Big),\end{aligned} \tag{4.44}$$

where $\mathfrak{S}$ is given by (4.30).
Let $\mathscr{M}_{OD}^4(2;q)$ denotes the contribution of $\log^2(q)\partial_{s=0}^2$, then we get

$$\mathscr{M}_{OD}^4(2;q) = \frac{-4}{\zeta(2)}\log^2(q) \sum\sum\sum\sum_{\substack{i_1+j_1+\dots+i_4+j_4=2 \\ 0\leqslant i_k,j_k\leqslant 1}} \mathscr{B}_{i_1,j_1,\dots,i_4,j_4}\mathscr{M}(i_1+j_1,\dots,i_4+j_4;\mathscr{F}). \tag{4.45}$$

We now compute the part with $\log(q)\partial_s^2(\partial_{u_1}+\partial_{u_2})|_{s=u_i=0}$. We remark that the action of $(\partial_{u_1}+\partial_{u_2})|_{u_i=0}$ consists of a sum of eight terms in which one of the eight factors is missing. We have therefore

$$\mathscr{M}_{OD}^4(3;q) = \frac{-4}{\zeta(2)}\log(q)\sum_{\ell=1}^4 \sum\sum\sum\sum_{\substack{i_1+j_1+\dots i_4+j_4=2 \\ 0\leqslant i_k,j_k\leqslant 1 \\ i_\ell j_\ell=0}} \mathscr{B}_{i_1,\dots,j_4}\mathscr{M}^{(\ell)}(i_1+j_1,\dots,i_4+j_4;\mathscr{F}), \tag{4.46}$$

where $\mathscr{M}^{(\ell)}$ means that we add 1 at the $\ell^{th}$ component.

Finally, we can focus on the action of $\partial_s^2\partial_{u_1u_2}^2|_{s=u_i=0}$. We see first that $\partial_{u_i=0}^2(\mathrm{NUM}(Q))$ consists of a sum of sixteen terms where there are two missing factors (one indexed by $i_n$ and the other by $i_\ell$). It follows that

$$\mathscr{M}^4_{OD}(4;q) = \frac{-4}{\zeta(2)} \sum_{n,\ell=1}^{4} \sum_{\substack{i_1+j_1+\dots i_4+j_4=2\\ 0\leqslant i_k,j_k\leqslant 1\\ i_n=j_\ell=0}}\sum\sum\sum \mathscr{B}_{i_1,\dots,j_4}\mathscr{M}^{(n,\ell)}(i_1+j_1,\dots,i_4+j_4;\mathscr{F}), \tag{4.47}$$

where this time, $\mathcal{M}^{(n,\ell)}$ means that we add 1 to $i_n$ and 1 to $j_\ell$. To finalize the calculations, we have

**Lemma 4.7.** *The value of $\mathscr{F}(0,\dots,0)$ is given by the infinite product*

$$\mathscr{F}(0,\dots,0) = \zeta(2)\prod_p \left(1+2\frac{1+p^{-1}}{p^2(1-p^{-1})^3}\right).$$

*Proof.* As in Lemma 4.4, we have by examining (4.39) and (4.40),

$$\mathscr{F}(0,\dots,0) = \mathscr{P}(0,\dots,0)$$

and for each prime $p$,

$$\mathscr{P}_p(0,\dots,0) = \mathcal{L}_p(0,\dots,0)\left(1-\frac{1}{p}\right)^{-4}.$$

By (4.37), we see that

$$\begin{aligned}\mathcal{L}_p(0,\dots,0) &= \sum_{\substack{0\leqslant k+\ell_i\leqslant 2\\ \ell_1\ell_2=0}}\sum \frac{\mu_2(p^{k+\ell_1})\mu_2(p^{k+\ell_2})\boldsymbol{\delta}(p^{\ell_1};0,0,0)\boldsymbol{\delta}(p^{\ell_2};0,0,0)}{p^{k+\ell_1+\ell_2}}\\ &= 2\sum_{0\leqslant k+\ell\leqslant 2}\sum \frac{\mu_2(p^{k+\ell})\mu_2(p^k)\boldsymbol{\delta}(p^\ell;0,0,0)}{p^{k+\ell}} - \sum_{0\leqslant k\leqslant 2}\frac{\mu_2(p^k)^2}{p^k},\end{aligned}$$

with (see (3.81) and (3.82))

$$\boldsymbol{\delta}(p;0,0,0) = 2\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1} \text{ and } \boldsymbol{\delta}(p^2;0,0,0) = 4\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}.$$

Hence (recall (4.8))

$$\begin{aligned}\mathcal{L}_p(0,\dots,0) &= 1+\frac{4}{p}+\frac{1}{p^2}-\frac{8}{p}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}\\ &= \left\{1-\frac{1}{p^4}-\frac{4}{p}-\frac{4}{p^3}+\frac{8}{p^2}\right\}\left(1-\frac{1}{p^2}\right)^{-1}\\ &= \left\{\left(1-\frac{1}{p}\right)^4+\frac{2}{p^2}\left(1-\frac{1}{p^2}\right)\right\}\left(1-\frac{1}{p^2}\right)^{-1}.\end{aligned}$$

Multiplying by the factor $(1-p^{-1})^{-4}$ finishes the proof. □

# 5 Proof of Theorem 6

This chapter is devoted to the proof of Theorem 6. For the cuspidal case, we will indicate the necessary changes in [24, Sections 4,5,6] due to the level $q$ and the presence of a nebentypus. Finally, we will explain in Section 5.2 how to adapt [**?**, Section 2] and put together with Section 5.1 to obtain the conclusion of Theorem 6 in the Eisenstein case.

## 5.1 The cuspidal case

Let $q>2$ be a prime number, $\omega$ a Dirichlet character of modulus $q$ and $\kappa\in\{0,1\}$ such that $\omega(-1)=(-1)^{\kappa}$. Let $f$ be a $L^2$-normalized primitive Hecke cusp form of weight $k_f\equiv\kappa \pmod 2$ (resp. with Laplace eigenvalue $1/4+t_f^2$) if $f$ is holomorphic (resp. if $f$ is a Maass form) of level $q$ and nebentypus $\omega$. For some technical reasons, it is convenient to view $f$ as a modular form of level $2q$ (see the beginning of § 1.2.1) under the isometric embedding (with respect to the Petersson inner product)

$$f(z)\mapsto\frac{f(z)}{[\Gamma_0(q):\Gamma_0(2q)]^{1/2}}=\frac{f(z)}{\sqrt{3}},$$

which can be embedded in a suitable orthonormal basis of modular cusp forms of level $2q$, i.e. either $\mathcal{B}_{k_f}(2q,\omega)$ or $\mathcal{B}(2q,\omega)$.

### 5.1.1 The amplification method

The strategy is to estimate an amplified second moment of the sum $\mathcal{S}_V(g,K;q)$ where $g$ runs over a basis of $\mathcal{B}_{k_f}(2q,\omega)$ and $\mathcal{B}(2q,\omega)$.

To be precise, let $L\geqslant 1$ and $(b_\ell)$ a sequence of coefficients supported on $1\leqslant\ell\leqslant 2L$. For any modular form $g$, we let

$$B(g):=\sum_{1\leqslant\ell\leqslant 2L}b_\ell\lambda_g(\ell).$$

For an Eisenstein series $E_{\mathfrak{a}}(\cdot, 1/2+it)$, we set

$$B(\mathfrak{a}, it) := \sum_{1\leqslant \ell \leqslant 2L} b_\ell \lambda_{\mathfrak{a}}(\ell, it),$$

where for any singular cusp $\mathfrak{a}$, $\lambda_{\mathfrak{a}}(\ell, it)$ is given by (1.5). Since the original form is of level $q$ and $L$ will be at the end a small power of $q$, we cannot choose the standard coefficients $b_\ell = \overline{\lambda_f(\ell)}$ for $\ell$ a prime $p \sim L$, but rather the less obvious amplifier found by Iwaniec in [**?**],

$$b_\ell = \begin{cases} \lambda_f(p)\overline{\omega}(p) & \text{if} \quad \ell = p \sim L^{1/2} \text{ and } (p, 2q) = 1, \\ -\overline{\omega}(p) & \text{if} \quad \ell = p^2 \sim L \text{ and } (p, 2q) = 1, \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

Since we will apply the trace formula, it is also better to consider the Fourier coefficients $\rho_g(n)$ instead of the Hecke eigenvalues $\lambda_g(n)$ in the definition of $\mathcal{S}_V(g, k; q)$. For this, we define

$$\tilde{\mathcal{S}}_V(g, K; q) = \sum_n \rho_g(n) K(n) V\left(\frac{n}{q}\right)$$

and note that for $g$ primitive, it is related to the original sum $\mathcal{S}_V(g, K; q)$ by the simple relation (c.f. (1.7)),

$$\tilde{\mathcal{S}}_V(g, K; q) = \rho_g(1)\mathcal{S}_V(g, K; q). \tag{5.2}$$

We then let

$$\begin{aligned} M(L) := & \sum_{\substack{k\equiv\kappa \ (\mathrm{mod}\ 2)\\ k>\kappa}} \dot{\phi}(k)(k-1)M(L;k) \\ & + \sum_{g\in\mathcal{B}(2q,\omega)} \tilde{\phi}(t_g)\frac{4\pi}{\cosh(\pi t_g)}|B(g)|^2 \left|\tilde{\mathcal{S}}_V(g,K;q)\right|^2 \\ & + \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \tilde{\phi}(t)\frac{1}{\cosh(\pi t)}|B(\mathfrak{a}, it)|^2 \left|\tilde{\mathcal{S}}_V(E_{\mathfrak{a}}(\cdot, 1/2+it), K; q)\right|^2 dt, \end{aligned} \tag{5.3}$$

where for any $k \equiv \kappa \pmod 2$ with $k > \kappa$,

$$M(L;k) := \frac{(k-2)!}{\pi(4\pi)^{k-1}} \sum_{g\in\mathcal{B}_k(2q,\omega)} |B(g)|^2 \left|\tilde{\mathcal{S}}_V(g,K;q)\right|^2, \tag{5.4}$$

and we refer to [24, Section 3.2] or [8, (2.9)] for the choice and properties of the test function $\phi = \phi_{a,b}$. The key Proposition is the following [24, Proposition 4.1].

**Proposition 5.1.** *Let $K: \mathbf{F}_q \to \mathbf{C}$ and $V$ as in Theorem 6. Let $(b_\ell)$ be the sequence of complex numbers defined by* (5.1). *Then for any $\varepsilon > 0$, there exists $k(\varepsilon) > \kappa$ such that for any $k \geqslant k(\varepsilon)$*

*and any integers $a > b > 2$ satisfying*

$$a - b \geqslant k(\varepsilon), \quad a - b \equiv \kappa \ (\mathrm{mod}\ 2),$$

*we have the bound*

$$M(L),\ M(L;k) \ll \mathrm{cond}(K)^s \left\{ q^{1+\varepsilon} L^{1/2} P(P+Q) + q^{1/2} L^2 P Q^2 (P+Q) \right\}, \tag{5.5}$$

*for some absolute constant $s \geqslant 1$, provided*

$$q^{\varepsilon} L Q < q^{1/4} \tag{5.6}$$

*and where the implied constant depends on $C, \varepsilon, a, b, k$ and polynomially on the archimedean parameter of $f$.*

Theorem 6 can be deduced from Proposition 5.1 exactly in the same way as in [24, Section 4.2]. The only changes is to use (5.2) to pass from $|B(f)|^2 |\mathbb{S}_V(f,K;q)|^2$ to $|B(f)|^2 |\tilde{\mathbb{S}}_V(f,K;q)|^2$ and then (1.8) for the upper bound on $|\rho_f(1)|^{-2}$. Finally, since for any prime $p$ different from $q$ we have the elementary relation

$$\lambda_f(p)^2 - \lambda_f(p^2) = \omega(p),$$

we obtain the lower bound

$$B(f) \gg \frac{L^{1/2}}{\log L},$$

simply using the prime number Theorem. Hence it remains to prove Proposition 5.1.

Expanding the square in $|B(g)|^2$ and $|\tilde{\mathbb{S}}_V(g,K;q)|$ (choosing the variables $\ell_1, \ell_2$ for those comming from the amplifier), we get a first decomposition of $M(L)$ and $M(L;k)$

$$M(L) = M_d(L) + M_{nd}(L) \ \text{ and } \ M(L;k) = M_d(L;k) + M_{nd}(L;k), \tag{5.7}$$

depending on weither $(\ell_1, \ell_2) > 1$ (the diagonal term) or not. For the diagonal term, we have the following lemma which is the analogous of [24, Lemma 5.1].

**Lemma 5.2.** *For any $\varepsilon > 0$, we have*

$$M_d(L), M_d(L;k) \ll \mathrm{cond}(K)^2 q^{1+\varepsilon} L^{1/2} P(P+1),$$

*where the implied constant depends only on $\varepsilon$.*

*Proof.* We consider $M_d(L)$ which decomposes as a sum of the holomorphic, Maass and Eisenstein contributions

$$M_d(L) = M_{d,\mathrm{hol}}(L) + M_{d,\mathrm{Maass}}(L) + M_{d,\mathrm{Eis}}(L).$$

We treat only $M_{d,\mathrm{Maass}}(L)$ since the other contributions are the same and even simpler. For instance, we have

$$M_{d,\mathrm{Maass}}(L) = \sum_{g\in\mathcal{B}(2q,\omega)} \tilde{\phi}(t_g)\frac{4\pi}{\cosh(\pi t_g)}\mathscr{C}(g,L)\left|\sum_n K(n)\rho_g(n)V\left(\frac{n}{q}\right)\right|^2,$$

with

$$\mathscr{C}(g,L) := \sum_{(\ell_1,\ell_2)>1} b_{\ell_1}\overline{b_{\ell_2}}\lambda_g(\ell_1)\overline{\lambda_g(\ell_2)}.$$

By definition of the coefficients $b_\ell$ (c.f. (5.1)), the case $(\ell_1,\ell_2)>1$ appears when $\ell_1=\ell_2=p\sim L^{1/2}$, $\ell_1=p^2=\ell_2^2\sim L$ (or the inverse) and $\ell_1=\ell_2=p^2\sim L$. We write $\mathscr{C}(g,L)=\mathscr{C}_1(g,L)+\mathscr{C}_2(g,L)+\mathscr{C}_3(g,L)$ according to the different possibilities and we estimate the three quantities individually. We first have by Cauchy-Schwarz inequality and (1.16),

$$\begin{aligned}
\mathscr{C}_1(g,L) &= \sum_{\substack{p\sim L^{1/2}\\ p \text{ prime}}} |\lambda_f(p)|^2|\lambda_g(p)|^2\\
&\leqslant \left(\sum_{p\sim L^{1/2}}|\lambda_f(p)|^4\right)^{1/2}\left(\sum_{p\sim L^{1/2}}|\lambda_g(p)|^4\right)^{1/2}\\
&\ll (qL(1+|t_f|)(1+t_g))^{\varepsilon}L^{1/2},
\end{aligned}$$

where the implied constant only depends on $\varepsilon$. For the second case, we have using $|\lambda(p^2)|\leqslant 1+|\lambda(p)|^2$ (c.f. (1.10)), Hölder and again (1.16),

$$\begin{aligned}
|\mathscr{C}_2(g,L)| &\leqslant \sum_{\substack{p\sim L^{1/2}\\ p \text{ prime}}} |\lambda_f(p)||\lambda_g(p)||\lambda_g(p^2)| \leqslant \sum_{\substack{p\sim L^{1/2}\\ p \text{ prime}}} |\lambda_f(p)||\lambda_g(p)|(1+|\lambda_g(p)|^2)\\
&\leqslant \left(\sum_{p\sim L^{1/2}}|\lambda_f(p)|^4\right)^{1/4}\left(\sum_{p\sim L^{1/2}}|\lambda_g(p)|^4\right)^{1/4}\left(\sum_{p\sim L^{1/2}}(1+|\lambda_g(p)|^2)^2\right)^{1/2}\\
&\ll (qL(1+|t_f|)(1+t_g))^{\varepsilon}L^{1/2}.
\end{aligned}$$

Using the inequality $|\lambda_g(p^2)|^2\leqslant 2(1+|\lambda_g(p)|^4)$, we treat in the same way $\mathscr{C}_3(g,L)$. The rest of the proof is exactly the same as [24, Lemma 5.1], except that we must use Proposition 1.10 for the spectral large sieve (possible since the conductor of $\omega$ is either 1 or a prime $q$) instead of the original version of Deshouillers-Iwaniec [**?**, Theorem 2, (1.29)]. □

Now comes the contribution of the $\ell_1,\ell_2$ such that $(\ell_1,\ell_2)=1$. We first change the complex conjugate $\overline{\lambda_g(\ell_2)}=\overline{\omega}(\ell_2)\lambda_g(\ell_2)$ in $M_{nd}(L)$ and $M_{nd}(L;k)$ appearing in the decomposition (5.7) (c.f. (1.11)). By the primality condition, we use the multiplicativity of the Hecke eigenvalues (1.10) followed by the relation (1.6) to obtain

$$\lambda_g(\ell_1\ell_2)\rho_g(n_1) = \sum_{d|(\ell_1\ell_2,n_1)} \omega(d)\rho_g\left(\frac{\ell_1\ell_2 n_1}{d^2}\right).$$

Once we have done this, we apply the Petersson trace formula (1.20) to $M_{nd}(L;k)$ in (5.7), obtaining

$$\pi M_{nd}(L;k) = M_1(L;k) + M_2(L;k),$$

where $M_1(L;k)$ corresponds to the diagonal term $\delta(\ell_1\ell_2 n_1 d^{-2}, n_2)$. Similarily, we apply Kuznetsov formula (1.21) to $M_{nd}(L)$ and since there is no diagonal term, we write $M_{nd}(L) = M_2(L)$. The treatment of the diagonal term $M_1(L;k)$ is contained in [24, Lemma 5.3], with the appropriate changes using (1.15) for the coefficients of the amplifier,

$$M_1(L;k) \ll \mathrm{cond}(K)^2 (q(1+|t_f|))^{\varepsilon} q L^{1/2} P, \tag{5.8}$$

with an implied constant depending only on $\varepsilon$.

### 5.1.2 The off-diagonal terms

This is the most important case of $M_2(L)$ and $M_2(L;k)$ and thus we write explicitly the quantities to study. For $\phi$ an arbitrary function, we write

$$\begin{aligned} M_2[\phi] = \frac{1}{2q} \sum_{(\ell_1,\ell_2)=1} b_{\ell_1} \overline{b_{\ell_2}} \overline{\omega}(\ell_2) \sum_{d \mid \ell_1\ell_2} \omega(d) \sum_{\substack{n_1,n_2 \\ d \mid n_1}} K(n_1)\overline{K(n_2)} V\left(\frac{n_1}{q}\right) V\left(\frac{n_2}{q}\right) \\ \times \sum_{c \geqslant 1} c^{-1} S_\omega(\ell_1\ell_2 n_1 d^{-2}, n_2; 2cq) \phi\left(\frac{4\pi}{2cq}\sqrt{\frac{\ell_1\ell_2 n_1 n_2}{d^2}}\right), \end{aligned} \tag{5.9}$$

in order to have

$$M_2(L) = M_2[\phi_{a,b}] \text{ and } M_2(L;k) = M_2[\phi_k]$$

where $\phi_k = 2\pi i^{-k} J_{k-1}$ is the Bessel function. We transform the sum as

$$M_2[\phi] = \sum_{(\ell_1,\ell_2)=1} b_{\ell_1} \overline{b_{\ell_2}} \overline{\omega}(\ell_2) \sum_{de=\ell_1\ell_2} \omega(d) M_2[\phi; d, e], \tag{5.10}$$

where

$$M_2[\phi; d, e] = \frac{1}{2q} \sum_{c \geqslant 1} c^{-1} \mathcal{E}_\phi(c, d, e)$$

and

$$\begin{aligned} \mathcal{E}_\phi(c,d,e) &= \sum_{n_1,n_2} S_\omega(en_1, n_2; 2cq) K(dn_1) \overline{K(n_2)} \phi\left(\frac{4\pi\sqrt{en_1 n_2}}{2cq}\right) V\left(\frac{dn_1}{q}\right) V\left(\frac{n_2}{q}\right) \\ &= \sum_{n_1,n_2} S_\omega(en_1, n_2; 2cq) K(dn_1) \overline{K(n_2)} H_\phi(n_1, n_2), \end{aligned}$$

with

$$H_\phi(x,y) = \phi\left(\frac{4\pi\sqrt{exy}}{2cq}\right) V\left(\frac{dx}{q}\right) V\left(\frac{y}{q}\right). \tag{5.11}$$

As in [24, Section 5.4], we truncate the parameter $c$ in $M_2[\phi; d, e]$ by writing $M_2[\phi; d, e] = M_{2,C}[\phi; d, e] + M_3[\phi; d, e]$ where $M_{2,C}[\phi; d, e]$ denotes the contribution of ther terms with $c > C$ for some $C = C(d, e) \geqslant 1/2$ and correspondingly

$$M_2[\phi] = M_{2,\mathrm{tail}}[\phi] + M_3[\phi]. \tag{5.12}$$

It turns out that with the choice

$$C = \max\left(\frac{1}{2}, q^{\delta} P \sqrt{\frac{e}{d}}\right) \ll q^{\delta} LP, \tag{5.13}$$

the contribution of $c > C$ is negligible (see [24, (5.9)]), so we focus on the complementary sum, which is given by

$$M_3[\phi; d, e] = \frac{1}{2q} \sum_{1 \leqslant c \leqslant C} c^{-1} \mathcal{E}_{\phi}(c, d, e). \tag{5.14}$$

In particular, the above expression is zero if $C < 1$.

Recall that we factored the product $\ell_1 \ell_2$ as $de = \ell_1 \ell_2$. Since we allow $\ell_1$ and $\ell_2$ to be square of primes, there are more types of factorization to consider. We distinguish three types :

- Type I (balanced case) : this is when both $d$ and $e$ are $\neq 1$ and $d/e \sim 1$, so $d$ and $e$ are either primes $\sim L^{1/2}$ (type $(L^{1/2}, L^{1/2})$) or square of primes $\sim L$ (type $(L, L)$) with $(d, e) = 1$ in each case.
- Type II (unbalanced case) : this is when $(d, e)$ satisfies $e/d \gg L^{1/2}$, i.e. is of type $(1, L), (1, L^{3/2}), (1, L^2)$, $(L^{1/2}, L)$ and $(L^{1/2}, L^{3/2})$.
- Type III (unbalanced case) : this is when $(d, e)$ satisfies $d/e \gg L^{1/2}$, so is of type

$$(L, 1), (L^{3/2}, 1), (L^2, 1), (L, L^{1/2}) \text{ and } (L^{3/2}, L^{1/2}).$$

Assuming the harmless condition

$$q^{\delta} P \ll L^{1/2}, \tag{5.15}$$

we obtain by (5.13) :

**Lemma 5.3.** *Suppose that* $(d, e)$ *is of Type III and that* (5.15) *is satisfied. Then we have the equality*

$$M_3[\phi; d, e] = 0.$$

It remains to deal with the types I and II. The goal now is to transform the sums $\mathcal{E}_{\phi}(c, d, e)$ to connect them with the correlation sums $\mathcal{C}(\widehat{K}, \omega; \gamma)$ of the Fourier transform of $K$ defined in 2.9 for suitable matrices $\gamma$. This is the content of [24, Section 5.5] and it is achieved using principally twisted multiplicativity of the Kloosterman sums and Poisson summation formula. The only difference here is the appearance of the nebentypus $\omega$ when we open the Kloosterman sum. We also mention that it is in this treatment that we use the fact that the level is $2q$ and not $q$. The result is that for any $c \geqslant 1$, we have the identity

$$\mathcal{E}_{\phi}(c,d,e) = \frac{\omega(-d)}{q} \sum\sum_{\substack{n_1 n_2 \neq 0,\ (n_2,2c)=1 \\ n_1 n_2 \equiv e (\mathrm{mod}\ 2c)}} \widehat{H}_{\phi}\left(\frac{n_1}{2cq}, \frac{n_2}{2cq}\right) \mathcal{C}(\widehat{K},\omega;\gamma), \tag{5.16}$$

where $\widehat{H}_{\phi}$ is the Fourier transform of $H_{\phi}$ and

$$\gamma = \gamma(c,d,e,n_1,n_2) := \begin{pmatrix} n_1 & \frac{n_1 n_2 - e}{2c} \\ 2cd & dn_2 \end{pmatrix} \in \mathrm{M}_2(\mathbf{Z}) \cap \mathrm{GL}_2(\mathbf{Q}). \tag{5.17}$$

**Remark 5.4.** Observe that $\det(\gamma) = de$ which is coprime with $q$. Hence the reduction of $\gamma$ modulo $q$ provides a well defined element of $\mathrm{PGL}_2(\mathbf{F}_q)$.

### 5.1.3 Analysis of $\mathcal{E}_{\phi}(c,d,e)$

The first step in the analysis of (5.16) passes by the study of the Fourier transform of $H_{\phi}(x,y)$. This is the content of [24, Sections 5.6-5.7] and it is contained in Lemmas 5.7,5.9. One of the consequences is that it allows to truncate the $n_1, n_2$-sum in $\mathcal{E}_{\phi}(c,d,e)$ to

$$0 \neq |n_1| \leqslant \mathcal{N}_1 := q^{\varepsilon} cd \frac{\left(Q + \frac{P}{2c}\sqrt{\frac{e}{d}}\right)}{P},\ \ 0 \neq |n_2| \leqslant \mathcal{N}_2 := \frac{\mathcal{N}_1}{d}, \tag{5.18}$$

(see [24, (5.21)]). The final strategy is to separate the terms in (5.16) (with the restriction (5.18) on $n_1, n_2$) according to whether the reduction of $\gamma$ modulo $q$ belongs to $\mathbf{G}_{\widehat{\mathcal{F}},\omega}$ or not (see Definition 2.11). In the first case, we use the bound (see (2.13) and (2.8))

$$|\mathcal{C}(\widehat{K},\omega;\gamma)| \leqslant \mathrm{cond}(\widehat{K})^2 q \leqslant 100\,\mathrm{cond}(K)^4 q,$$

while for $\gamma$ not in $\mathbf{G}_{\widehat{\mathcal{F}},\omega}$, we have by Proposition 2.15

$$|\mathcal{C}(\widehat{K},\omega;\gamma)| \leqslant A(\mathrm{cond}(\widehat{K}))^s q^{1/2} \ll \mathrm{cond}(K)^{2s} q^{1/2}.$$

We thus write

$$\mathcal{E}_{\phi}(c,d,e) = \mathcal{E}^{c}_{\phi}(c,d,e) + \mathcal{E}^{n}_{\phi}(c,d,e),$$

where $\mathcal{E}^{c}_{\phi}(c,d,e)$ is the subsum of (5.16) where we restrict to the variables $n_1, n_2$ such that the reduction modulo $q$ of $\gamma(c,d,e,n_1,n_2)$ belongs to $\mathbf{G}_{\widehat{\mathcal{F}},\omega}$ and $\mathcal{E}^{n}_{\phi}(c,d,e)$ is the contribution of the remaining terms. According to (5.14), (5.12) and (5.10), we also write

$$\begin{aligned} M_3[\phi; d, e] &= \frac{1}{2q}\sum_{c\leqslant C} c^{-1}\left(\mathcal{E}^c_\phi(c,d,e) + \mathcal{E}^n_\phi(c,d,e)\right) \\ &= M_3^c[\phi; d, e] + M_3^n[\phi; d, e], \end{aligned}$$

and

$$\begin{aligned} M_3[\phi] &= \sum_{(\ell_1,\ell_2)=1} b_{\ell_1}\overline{b_{\ell_2}}\overline{\omega}(\ell_2)\sum_{de=\ell_1\ell_2}\omega(d)\left(M_3^c[\phi; d, e] + M_3^n[\phi; d, e]\right) \\ &= M_3^c[\phi] + M_3^n[\phi]. \end{aligned}$$

**Lemma 5.5.** *With the above notations, we have*

$$M_3^n[\phi_{a,b}] \ll \mathrm{cond}(K)^4 q^{1/2+\varepsilon} L^2 P Q^2 (P+Q), \quad M_3^n[\phi_k] \ll \mathrm{cond}(K)^4 k^3 q^{1/2+\varepsilon} L^2 P Q^2 (P+Q),$$

*for any $\varepsilon > 0$ where the implied constant depends on $\varepsilon, a, b$ for $\phi = \phi_{a,b}$ and on $\varepsilon$ for $\phi = \phi_k$.*

*Proof.* This is the content of [24, pp. 625-626], with minimal changes due to the different nature of pairs $(d, e)$ of type I and II. □

To conclude the proof of Proposition 5.1, it remains to evaluate the contribution $M_3^c[\phi; d, e]$ corresponding to the matrices whose reduction modulo $q$ is in $\mathbf{G}_{\widehat{\mathcal{F}},\omega}$. The final lemma is the following:

**Lemma 5.6.** *Under the assumption*

$$q^{3\varepsilon} L Q < q^{1/4}, \tag{5.19}$$

*we have*

$$M_3^c[\phi_k] \ll \mathrm{cond}(K)^{2s} k^3 q^{1+\varepsilon} L^{1/2} P Q, \quad M_3^c[\phi_{a,b}] \ll \mathrm{cond}(K)^{2s} q^{1+\varepsilon} L^{1/2} P Q,$$

*where $s \geqslant 2$ and the implied constant depends on $\varepsilon, a, b$.*

*Proof.* The proof is [24, Sections 6.1,6.3,6.5] (recall that here there are no parabolic elements by Proposition 2.17). Various arguments use the fact that the discriminant of certain binary quadratic form is not zero. For example, if $\gamma = \gamma(c, d, e, n_1, n_2)$ is a toric matrix, then we need to have $(n_1 + dn_2)^2 - 4de \neq 0$ and we cannot say that $de = \ell_1\ell_2$ is squarefree since we allow square of primes in the amplifier. This is not a problem here because if $(n_1 + dn_2)^2 = 4de$, then we would get (see (5.17))

$$\mathrm{Tr}(\gamma)^2 - 4\det(\gamma) = 0 \text{ in } \mathbf{F}_q.$$

This means that $\gamma$ has only one distinct eigenvalue, so it is necessarily scalar since not parabolic by assumption. But for $\gamma$ scalar, we have $cd \equiv 0 \pmod q$, which is not possible by (5.19) and (5.13).

Another argument uses the fact that $dn_2^2 - e \neq 0$ in a situation where $n_1 - dn_2 = 0$ (c.f. [24, Section 6.3, p.p 629]). Again, $d$ and $e$ are not necessarily coprime here so we cannot argue in the same way. However, if $dn_2^2 = e$, then since $n_1 = dn_2$, we obtain $n_1 n_2 - e = dn_2^2 - e = 0$ and the matrix $\gamma(c,d,e,n_1,n_2)$ takes the form

$$\gamma(c,d,e,n_1,n_2) = \begin{pmatrix} n_1 & 0 \\ 2cd & dn_2 \end{pmatrix}.$$

Since $dn_2 = n_1$, this matrix is parabolic with single fixed point $z = 0$, which contradicts the fact that $\mathbf{G}_{\widehat{\mathcal{F}},\omega}$ does not contain parabolic elements. $\square$

## 5.2 The Eisenstein case

We recall the notations from Section 2.1. For $q > 2$ prime, $\omega$ a Dirichlet character modulo $q$ and $t \in \mathbf{R}$, we set

$$\lambda_\omega(n, it) = \sum_{ab=n} \omega(a) \left(\frac{a}{b}\right)^{it},$$

and

$$\mathcal{S}_V(\omega, it, K; q) = \sum_{n \geqslant 1} \lambda_\omega(n, it) K(n) V\left(\frac{n}{q}\right),$$

for $K$ an isotypic trace function such that its Fourier transform is not $\omega$-exceptional (see Definition 2.16) and $V$ satisfying condition $V(C,P,Q)$. Since $\lambda_\omega(n, it)$ appears as Hecke eigenvalues (for $(n,q) = 1$) of the Eisenstein series $E_1(\cdot, 1/2 + it)$ (the cusp $\mathfrak{a} = 1$) lying in the continuous spectrum of the Laplacian on the space of modular forms of level $q$ (and thus also of level $2q$ after a normalization) and nebentypus $\omega$ (see Section 1.2.1 and Remark 1.4), we may estimate an amplified second moment of $\mathcal{S}_V(\omega, it, K; q)$ by embedding in the Eisenstein spectrum and using Kuznetsov trace formula as in the cuspidal case.

For $\tau \in \mathbf{R}$, we define as in (5.1)

$$b_\ell(\tau) := \begin{cases} \lambda_\omega(\ell, i\tau)\overline{\omega}(\ell) & \text{if} \quad \ell = p \sim L^{1/2} \text{ and } (p, 2q) = 1 \\ -\overline{\omega}(\ell) & \text{if} \quad \ell = p^2 \sim L \text{ and } (p, 2q) = 1 \\ 0 & \text{else,} \end{cases} \tag{5.20}$$

and for $g$ a cuspidal form, we set

$$B_\tau(g) = \sum_{1 \leqslant \ell \leqslant 2L} b_\ell(\tau) \lambda_g(\ell).$$

For an Eisenstein series $E_{\mathfrak{a}}(\cdot, 1/2+it)$, we let

$$B_\tau(\mathfrak{a}, it) = \sum_{1\leqslant \ell \leqslant 2L} b_\ell(\tau)\lambda_{\mathfrak{a}}(\ell, it)$$

and we also write $B_\tau(\omega, it)$ so that it corresponds to the Eisenstein series $E_1(\cdot, 1/2+it)$ having $\lambda_\omega(n, it)$ as Hecke eigenvalues. Since the trace formula involves Fourier coefficients instead of Hecke eigenvalues, we define as in Section 5.1

$$\tilde{\mathcal{S}}_V(\omega, it, K; q) = \sum_{n\geqslant 1} \rho_\omega(n, it) K(n) V\left(\frac{n}{q}\right),$$

with the relation

$$\mathcal{S}_V(\omega, it, K; q) = \rho_\omega(1, it)^{-1}\tilde{\mathcal{S}}_V(\omega, it, K; q). \tag{5.21}$$

**Remark 5.7.** Actually, the relation (5.21) is true if we restrict the $n$-summation in $\mathcal{S}_V(\omega, it, K; q)$ to $(n, q) = 1$. However, we could consider directly this restriction at the beginning since the error to pass from one to the other is given by

$$\mathcal{S}_V(\omega, it, K; q) = \sum_{(n,q)=1} \lambda_\omega(n, it) K(n) V\left(\frac{n}{q}\right) + O\big(q^\varepsilon M(P+1)\big).$$

Using the lower bound for $\rho_\omega(1, it)$ given by (1.9) and $\tilde{\phi}_{a,b}(t) \asymp (1+|t|)^{\kappa-2b-2}$ (c.f. [8, (2.21)]), we obtain exactly as in Section 5.1 (see Proposition 5.1),

$$\begin{aligned}\int_{\mathbf{R}} \frac{\left|\mathcal{S}_V(\omega, it, K; q)\right|^2}{(1+|t|)^{2b+2-\varepsilon}} |B_\tau(\omega, it)|^2\, dt &\ll q^{1+\varepsilon}\int_{\mathbf{R}} \frac{(1+|t|)^{\kappa-2b-2}}{\cosh(\pi t)}\left|\tilde{\mathcal{S}}_V(\omega, it, K; q)\right|^2 |B_\tau(\omega, it)|^2\, dt \\ &\ll q^{1+\varepsilon}\int_{\mathbf{R}} \frac{\tilde{\phi}_{a,b}(t)}{\cosh(\pi t)}\left|\tilde{\mathcal{S}}_V(\omega, it, K; q)\right|^2 |B_\tau(\omega, it)|^2\, dt \\ &\ll \operatorname{cond}(K)^{2s}\left\{q^{2+\varepsilon}L^{1/2}P(P+Q) + q^{3/2}L^2PQ^2(P+Q)\right\}.\end{aligned} \tag{5.22}$$

In order to apply (5.22), the following Lemma gets a suitable lower bound for the amplifier $B_\tau(\omega, it)$ when $\tau$ is close enough to $t$ (see [**?**, Lemma 2.4]).

**Lemma 5.8.** *For L large enough, we have*

$$B_\tau(\omega, it) \gg \frac{L^{1/2}}{\log L},$$

*uniformly in $t, \tau \in \mathbf{R}$ satisfying*

$$|t-\tau| \leqslant \frac{1}{\log^2 L}.$$

*Proof.* Observe that since $\ell$ has at most three divisors, we have $|b_\ell(\tau)| \leqslant 3$ and thus

$$\begin{aligned}
|B_\tau(\omega,it)-B_\tau(\omega,i\tau)| &\leqslant \sum_\ell |b_\ell(\tau)||\lambda_\omega(\ell,it)-\lambda_\omega(\ell,i\tau)| \\
&\leqslant 3 \sum_{\substack{p\sim L^{1/2}\\ p \text{ prime}}} \{|\lambda_\omega(p,it)-\lambda_\omega(p,i\tau)|+|\lambda_\omega(p^2,it)-\lambda_\omega(p^2,i\tau)|\} \\
&\leqslant 6 \sum_{\substack{p\sim L^{1/2}\\ p \text{ prime}}} \left\{|p^{it}-p^{i\tau}|+|p^{2it}-p^{2i\tau}|\right\} \\
&\leqslant 36|t-\tau| \sum_{\substack{p\sim L^{1/2}\\ p \text{ prime}}} \log(p) \ll \frac{L^{1/2}}{\log^2 L}.
\end{aligned}$$

It is therefore enough to prove the result for $t=\tau$. But this is a consequence of the elementary relation

$$\lambda_\omega(p,it)^2\overline{\omega}(p)-\overline{\omega}(p)\lambda_\omega(p^2,it)=1,$$

valid for $(p,q)=1$, and the prime number Theorem. □

The above Lemma combining with the average bound (5.22) allows us to deduce a first upper-bound for short averages of twists of Eisenstein series. For this, we introduce the notation

$$I(\tau,q) := \left\{ t\in \mathbf{R} \mid |t-\tau| \leqslant \frac{1}{\log^2 q}\right\}.$$

and

$$\mathcal{K}(P,Q;q) := \operatorname{cond}(K)^s q^{1-\frac{1}{16}}(PQ)^{1/2}(P+Q)^{1/2},$$

so that Theorem 6 claims that

$$\mathcal{S}_V(\omega,it,K;q) \ll_\varepsilon q^\varepsilon (1+|t|)^A \mathcal{K}(P,Q;q)$$

for any $\varepsilon>0$ and some $A\geqslant 1$ depending on $\varepsilon$.

**Proposition 5.9.** *For any $\varepsilon>0$, there exists $B\geqslant 1$, depending only on $\varepsilon$, such that for any $\tau\in\mathbf{R}$ we have*

$$\int_{I(\tau,q)} \left|\mathcal{S}_V(\omega,it,K;q)\right|^2 dt \ll_\varepsilon q^\varepsilon (1+|\tau|)^B \mathcal{K}(P,Q;q)^2, \tag{5.23}$$

*where the implied constant depends only on $\varepsilon$.*

*Proof.* Using Lemma 5.8 and (5.22), we obtain

$$\frac{L}{\log^2 L}\int_{I(\tau,q)} \left|\mathcal{S}_V(\omega,it,K;q)\right|^2 dt \ll \int_{I(\tau,q)} \frac{(1+|\tau|)^{2b+2}}{(1+|t|)^{2b+2-\varepsilon}} \left|\mathcal{S}_V(\omega,it,K;q)\right|^2 |B_\tau(\omega,it)|^2 dt$$
$$\ll (1+|\tau|)^{2b+2}\mathrm{cond}(K)^{2s}\left\{q^{2+\varepsilon}L^{1/2}P(P+Q)+q^{3/2}L^2PQ^2(P+Q)\right\},$$

and we conclude as in [24, Section 4.2] by choosing

$$L=\frac{1}{2}q^{1/4-\varepsilon}Q^{-1},$$

and $B=2b+2$ which depends on $\varepsilon$. □

The last step is to derive a pointwise bound for $\mathcal{S}_V(\omega,it,K;q)$. For this, we separate the variables $n,m$ in the twisted divisor function $\lambda_\omega(n,it)$ and using a partition of unity, we can decompose $\mathcal{S}_V(\omega,it,K;q)$ into $O(\log Pq)$ sums of the shape

$$\mathcal{S}_{V,M,N}(\omega,it,K;q)=\sum_{n,m\geqslant 1}K(mn)\omega(m)\left(\frac{m}{n}\right)^{it}W_1\left(\frac{m}{M}\right)W_2\left(\frac{n}{N}\right)V\left(\frac{nm}{q}\right),$$

where the parameters $(M,N)$ belong to the set

$$\mathbf{P}:=\left\{(M,N)\mid \frac{Pq}{4}\leqslant NM\leqslant 4Pq\,,\ 1\leqslant N,M\right\} \tag{5.24}$$

and $W_1,W_2$ are smooth and compactly supported functions on $[-1/2,2]$ satisfying $x^jW_i^{(j)}(x)\ll_j 1$ for every $j\geqslant 0$. It follows that

$$\mathcal{S}_V(\omega,it,K;q)\ll\log(Pq)\max_{(M,N)\in\mathbf{P}}\left|\mathcal{S}_{V,M,N}(\omega,it,K;q)\right|. \tag{5.25}$$

The relation between $\mathcal{S}_{V,M,N}(\omega,it,K;q)$ and an average of $\mathcal{S}_V(\omega,it,K;q)$ is given through the Mellin transform (see [**?**, Lemma 2.1]).

**Lemma 5.10.** *Given $s\in\mathbf{C}$ and $x>0$, we define*

$$V_s(x):=V(x)x^{-s}.$$

*Then for every $\varepsilon>0$, we have*

$$\mathcal{S}_{V,M,N}(\omega,it,K;q)\ll_\varepsilon \iint_{|t_1|,|t_2|\leqslant q^\varepsilon}\left|\mathcal{S}_{V_{it_1}}(\omega,it_2+it,K;q)\right|dt_1dt_2+O\left(q^{-100}\right). \tag{5.26}$$

*Proof.* Using Mellin inversion formula for $W_1$ and $W_2$, we can write

$$\mathcal{S}_{V,M,N}(\omega,it,K;q)=\frac{1}{(2\pi i)^2}\int_{(0)}\int_{(0)}\widehat{W_1}(s_1)\widehat{W_2}(s_2)\mathcal{T}_V(s_1,s_2)M^{s_1}N^{s_2}ds_1ds_2,$$

where $\widehat{W_1}, \widehat{W_2}$ denote the Mellin transform of the smooth functions $W_1, W_2$ and

$$\mathcal{T}_V(s_1,s_2) = \sum_{n,m\geqslant 1} K(nm)\omega(m)m^{it-s_1}n^{-it-s_2}V\left(\frac{nm}{q}\right).$$

Note that this sum can be expressed as a twist of Eisenstein series, namely

$$\mathcal{T}_V(s_1,s_2) = q^{-\theta_1}\mathcal{S}_{V_{\theta_1}}(\omega,\theta_2+it,K;q),$$

with

$$\theta_1 = \frac{s_1+s_2}{2},\quad \theta_2 = \frac{-s_1+s_2}{2}.$$

For $\Re e(\theta_1)$, the smooth function $V_{\theta_1}$ satisfies condition $V(C,P,Q(\theta_1))$ with

$$Q(\theta_1) \ll Q + |\theta_1|, \tag{5.27}$$

where the implied constant is absolute. Thus by a change of variables, we get

$$\begin{aligned}\mathcal{S}_{V,M,N}(\omega,it,K;q) = \frac{1}{(2\pi i)^2}\int_{(0)}\int_{(0)} &\widehat{W_1}(\theta_1-\theta_2)\widehat{W_2}(\theta_1+\theta_2)\left(\frac{M}{N}\right)^{\theta_2} \\ &\times\left(\frac{MN}{q}\right)^{\theta_1}\mathcal{S}_{V_{\theta_1}}(\omega,\theta_2+it,K;q)\,d\theta_1 d\theta_2.\end{aligned} \tag{5.28}$$

Because we have the estimations

$$\widehat{W_1}(s), \widehat{W_2}(s) \ll \frac{1}{(1+|s|)^C},$$

with an implied constant depending on $C$ and $\Re e(s)$, we can truncate the integral (5.28) to $|\theta_1| \leqslant q^{\varepsilon}$, $|\theta_2| \leqslant q^{\varepsilon}$ for a cost of $O(q^{-100})$ by taking $C$ large enough in term of $\varepsilon$ and using the trivial bound for $\Re e(\theta_1) = \Re e(\theta_2) = 0$

$$\mathcal{S}_{V_{\theta_1}}(\omega,\theta_2+it,K;q) \ll MPq\log q.$$

□

### 5.2.1 Conclusion

We are now in position to obtain the conclusion of Theorem 6 in the Eisenstein case. Indeed, fix $\varepsilon > 0$ and take $B = B(\varepsilon)$ as in Proposition 5.9. By (5.25), it is enough to estimate $\mathcal{S}_{V,M,N}(\omega,it,K;q)$ for $(M,N) \in \mathbf{P}$. Now let $\varepsilon' = \varepsilon/B$ such that we have the estimate (5.26) of Lemma 5.10. We thus get

$$\mathcal{S}_{V,M,N}(\omega,it,K;q) \ll_{\varepsilon'} q^{\varepsilon'} \max_{|t_1|\leqslant q^{\varepsilon'}}\int_{|t_2|\leqslant q^{\varepsilon'}}\left|\mathcal{S}_{V_{it_1}}(\omega,it_2+it,K;q)\right|dt_1dt_2 + O\left(q^{-100}\right).$$

We split the above integral into $O(q^{\varepsilon'})$ integrals over intervals of length $\log^{-2} q$. For such interval $I$ centered at $\tau$, we obtain by Proposition 5.9, the value (5.27) and Cauchy-Schwarz inequality, the bound

$$\mathcal{S}_{V,M,N}(\omega, it, K; q) \ll q^{\varepsilon}(1+|\tau|)^{B/2}\mathcal{K}(P, Q+q^{\varepsilon'}; q),$$

(the function $Q \mapsto \mathcal{K}(P, Q; q)$ is increasing). Finally, taking the maximal value $|\tau| \leqslant |t| + q^{\varepsilon'}$ yields the desired result.

# 6 Proof of Theorem 4

## 6.1 The triple product

As in Chapter 3, we separate the sum (0.6) into even and odd primitives characters and we treat only the case of even characters since the odd case is completely similar. We therefore consider

$$\mathscr{T}^3(\chi_1,\chi_2,\ell;q) := \frac{2}{q-1}\sideset{}{^+}\sum_{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)} L\big(\chi,\tfrac{1}{2}\big)\,L\big(\chi\chi_1,\tfrac{1}{2}\big)\,L\big(\chi\chi_2,\tfrac{1}{2}\big)\,\chi(\ell), \tag{6.1}$$

where $\sum^+$ means that we sum over even characters and $\mathcal{D}_{\chi_1,\chi_2}(q)$ denotes the set of primitive characters modulo $q$ different from $\overline{\chi}_1$ and $\overline{\chi}_2$.

### 6.1.1 Applying the approximate functional equation

Applying the approximate function equation provided by Lemma 1.11, we decompose (6.1) into two terms

$$\mathscr{T}^3(\chi_1,\chi_2,\ell;q) = \mathcal{S}_1(\chi_1,\chi_2,\ell;q) + i^{\kappa_1+\kappa_2}\mathcal{S}_2(\chi_1,\chi_2,\ell;q),$$

with

$$\mathcal{S}_1(\chi_1,\chi_2,\ell;q) := \frac{2}{q-1}\sideset{}{^+}\sum_{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)}\ \sideset{}{^*}\sum_{n_0,n_1,n_2\geqslant 1}\frac{\chi(n_0n_1n_2\ell)\chi_1(n_1)\chi_2(n_2)}{(n_0n_1n_2)^{1/2}}\mathbf{V}_{\chi,\chi_1,\chi_2}\left(\frac{n_0n_1n_2}{q^{3/2}}\right), \tag{6.2}$$

and

$$\begin{aligned}\mathcal{S}_2(\chi_1,\chi_2,\ell;q) := \frac{2}{q-1}\sideset{}{^+}\sum_{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)}\ \sideset{}{^*}\sum_{n_0,n_1,n_2\geqslant 1}&\frac{\chi(\overline{n_0n_1n_2}\ell)\overline{\chi}_1(n_1)\overline{\chi}_2(n_2)}{(n_0n_1n_2)^{1/2}}\\ &\times\varepsilon(\chi)\varepsilon(\chi\chi_1)\varepsilon(\chi\chi_2)\mathbf{V}_{\chi,\chi_1,\chi_2}\left(\frac{n_0n_1n_2}{q^{3/2}}\right),\end{aligned} \tag{6.3}$$

where the symbol $^*$ over the $n_i's$ sum means that we restrict to $(n_0n_1n_2,q)=1$ and the function $\mathbf{V}_{\chi,\chi_1,\chi_2}$ is defined in (1.27). In particular, since we sum over even characters, this function is constant on the average and we write $\mathbf{V}$ instead of $\mathbf{V}_{\chi,\chi_1,\chi_2}$.

**Remark 6.1.** The function $\mathbf{V}$ has rapid decay at infinity by Remark 1.12, so that the sums (6.2)-(6.3) are essentially supported on $1\leqslant n_0n_1n_2\leqslant q^{3/2+\varepsilon}$. It follows that the sum over $n_0,n_1,n_2$ is trivially bounded by $O(q^{3/4+\varepsilon})$, so we can remove as it suits us the contribution of $\chi=1,\overline{\chi}_1$ or $\chi=\overline{\chi}_2$ for an error of size $O(q^{-1/4+\varepsilon})$.

### 6.1.2 Average over the primitive and even characters

We need to average the sum over the characters in (6.2)-(6.3). For this, we use the orthogonality relations asserting that for any prime $q>2$ and any integer $a$ coprime with $q$, we have (c.f. [31, (3.2)-(3.4)])

$$\sideset{}{^+}\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq 1}}\chi(a)=\frac{q-1}{2}\delta_{a\equiv\pm1(q)}-1, \tag{6.4}$$

and for $\kappa\in\{0,1\}$,

$$\sideset{}{^\kappa}\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq 1}}\chi(m)\varepsilon(\chi)=\frac{q-1}{2q^{1/2}}\sum_{\pm}(\pm1)^\kappa\left(e\left(\pm\frac{\overline{m}}{q}\right)+\frac{1}{(q-1)}\right), \tag{6.5}$$

where the supscript $\kappa$ means that we sum over $\chi$ such that $\chi(-1)=(-1)^\kappa$. In (6.2), we remove the contribution of $\chi=\overline{\chi}_1,\overline{\chi}_2$ (see Remark 6.1) and after applying (6.4), we get $\mathcal{S}_1=\mathcal{S}_1^++\mathcal{S}_1^-+O(q^{-1/4+\varepsilon})$ with

$$\mathcal{S}_1^\pm(\chi_1,\chi_2,\ell;q)=\sum\sum\sum_{\substack{n_0,n_1,n_2\geqslant1\\ n_0n_1n_2\ell\equiv\pm1\ (\mathrm{mod}\ q)}}\frac{\chi_1(n_1)\chi_2(n_2)}{(n_0n_1n_2)^{1/2}}\mathbf{V}\left(\frac{n_0n_1n_2}{q^{3/2}}\right). \tag{6.6}$$

For (6.3), we remove the contribution of $\chi=1,\overline{\chi}_2$ and note that for $(m,q)=1$ we have, opening the Gauss sum $\varepsilon(\chi\chi_2)$ and using (6.5),

$$\begin{aligned}\frac{2}{q-1}\sideset{}{^+}\sum_{\chi\neq\overline{\chi}_1}\chi(m)\varepsilon(\chi\chi_1)\varepsilon(\chi\chi_2)&=\frac{1}{q^{1/2}}\sum_{a\in\mathbf{F}_q^\times}\chi_2(a)e\left(\frac{a}{q}\right)\left(\frac{2}{q-1}\sideset{}{^+}\sum_{\chi\neq\overline{\chi}_1}\chi(am)\varepsilon(\chi\chi_1)\right)\\&=\frac{\overline{\chi}_1(m)}{q^{1/2}}\sum_{a\in\mathbf{F}_q^\times}\overline{\chi}_1\chi_2(a)e\left(\frac{a}{q}\right)\left(\frac{2}{q-1}\sideset{}{^{\kappa_1}}\sum_{\chi\neq1}\chi(am)\varepsilon(\chi)\right)\\&=\frac{1}{q^{1/2}}\sum_{\pm}\overline{\chi}_1(\pm m)\left(\frac{1}{q^{1/2}}\sum_{a\in\mathbf{F}_q^\times}\overline{\chi}_1\chi_2(a)e\left(\frac{a}{q}\right)\left(e\left(\frac{\pm\overline{am}}{q}\right)+\frac{1}{q-1}\right)\right).\end{aligned} \tag{6.7}$$

The second expression in the above parenthesis is easily computed as a Gauss sum. For the first term, we have

$$\begin{aligned}\frac{1}{q^{1/2}}\sum_{a\in\mathbf{F}_q^\times}\overline{\chi}_1\chi_2(a)e\left(\frac{a}{q}\right)e\left(\frac{\pm\overline{am}}{q}\right)&=\chi_1\overline{\chi}_2(\pm m)\frac{1}{q^{1/2}}\sum_{a\in\mathbf{F}_q^\times}\overline{\chi}_1\chi_2(a)e\left(\frac{\overline{a}}{q}\right)e\left(\frac{\pm\overline{m}a}{q}\right)\\&=\chi_1(\pm m)\mathrm{Kl}_2(\pm\overline{m},\chi_1,\chi_2;q),\end{aligned}$$

where the twisted Kloosterman sum is defined by (2.9) (see also (2.11)). Hence we see that (6.7) is equal to

$$\frac{2}{q-1}\sum_{\chi\neq\overline{\chi}_1}^{+}\chi(m)\varepsilon(\chi\chi_1)\varepsilon(\chi\chi_2)=\frac{1}{q^{1/2}}\mathrm{Kl}_2(\pm\overline{m},\chi_1,\chi_2;q)+\frac{\varepsilon(\overline{\chi}_1\chi_2)\overline{\chi}_1(m)(1+(-1)^{\kappa_1})}{q^{1/2}(q-1)}. \quad (6.8)$$

Now opening the Gauss sum $\varepsilon(\chi)$ and using (6.8), we obtain for every $(m,q)=1$,

$$\begin{aligned}\frac{2}{q-1}\sum_{\chi\neq\overline{\chi}_1}^{+}\chi(m)\varepsilon(\chi)\varepsilon(\chi\chi_1)\varepsilon(\chi\chi_2)&=\frac{1}{q^{1/2}}\sum_{a\in\mathbf{F}_q^\times}e\left(\frac{a}{q}\right)\left(\frac{2}{q-1}\sum_{\chi\neq\overline{\chi}_1}^{+}\chi(am)\varepsilon(\chi\chi_1)\varepsilon(\chi\chi_2)\right)\\&=\frac{1}{q}\sum_{a\in\mathbf{F}_q^\times}\mathrm{Kl}_2(\pm\overline{am},\chi_1,\chi_2;q)e\left(\frac{a}{q}\right)+O\big(q^{-3/2}\big) \quad (6.9)\\&=\frac{1}{q^{1/2}}\mathrm{Kl}_3(\pm\overline{m},\chi_1,\chi_2,1;q)+O\big(q^{-3/2}\big).\end{aligned}$$

Finally, using (6.9) in (6.3) with $m=\overline{n_0n_1n_2\ell}$ yields $\mathcal{S}_2=\mathcal{S}_2^{+}+\mathcal{S}_2^{-}+O(q^{-1/4+\varepsilon})$ (recall Remark 6.1) with

$$\mathcal{S}_2^{\pm}(\chi_1,\chi_2,\ell;q)=\frac{1}{q^{1/2}}\sum_{n_0,n_1,n_2\geqslant1}^{*}\frac{\overline{\chi}_1(n_1)\overline{\chi}_2(n_2)}{(n_0n_1n_2)^{1/2}}\mathrm{Kl}_3(\pm n_0n_1n_2\overline{\ell},\chi_1,\chi_2,1;q)\mathbf{V}\left(\frac{n_0n_1n_2}{q^{3/2}}\right). \quad (6.10)$$

We will evaluate each of the two terms ((6.6) and (6.10)) separately and find that a main term appears only in $\mathcal{S}_1^{+}(\chi_1,\chi_2,\ell;q)$ when $\ell=1$. The others will contribute as an error term.

### 6.1.3 The main term

The main contribution comes from $n_0=n_1=n_2=\ell=1$ in (6.6). Indeed, assuming $n_0n_1n_2\ell=1$, we obtain by the Remark 1.12

$$\mathbf{V}\left(\frac{1}{q^{3/2}}\right)=1+O\big(q^{-3/4+\varepsilon}\big).$$

When $n_0n_1n_2\ell\equiv\pm1 \pmod q$ with $n_0n_1n_2\ell\neq1$, we write the congruence equation in the form $n_0n_1n_2\ell=\pm1+kq$ with $1\leqslant k\leqslant \ell q^{1/2+\varepsilon}+1$. Therefore, we get that the contribution of $n_0n_1n_2\ell\neq1$ is at most

$$\ell^{1/2}q^{\varepsilon-1/2}\sum_{1\leqslant k\leqslant \ell q^{1/2+\varepsilon}+1}\frac{1}{k^{1/2}}\ll \ell q^{-1/4+\varepsilon}.$$

We conclude with

$$\mathcal{S}_1^+(\chi_1,\chi_2,\ell;q)=\delta_{\ell=1}+O\big(\ell q^{-1/4+\varepsilon}\big)\ \text{ and }\ \mathcal{S}_1^-(\chi_1,\chi_2,\ell;q)=O\big(\ell q^{-1/4+\varepsilon}\big),$$

which gives the desired main term and error term of Theorem 4 (0.10) provided

$$\ell\leqslant q^{\frac{1}{4}-\frac{1}{64}}=q^{\frac{15}{64}}. \tag{6.11}$$

### 6.1.4 The error term

In this section, we analyze the expression (6.10) and will find that it contributes as an error term. Applying a partition of unity to $[1,\infty)$ for each variable in order to locate $n_0,n_1,n_2$ and we obtain $\mathcal{S}_2^{\pm}(\chi_1,\chi_2,\ell;q)=\sum_{N_0,N_1,N_2}\mathcal{S}_2^{\pm}(\ell,N_0,N_1,N_2;q)$ with

$$\begin{aligned}\mathcal{S}_2^{\pm}(\ell,N_0,N_1,N_2;q)=\frac{1}{(qN_0N_1N_2)^{1/2}}&\sideset{}{^*}\sum_{n_0,n_1,n_2\in\mathbf{Z}}\overline{\chi}_1(n_1)\overline{\chi}_2(n_2)f_1\left(\frac{n_1}{N_1}\right)f_2\left(\frac{n_2}{N_2}\right)\\&\times\mathrm{Kl}_3(\pm n_0n_1n_2\overline{\ell},\chi_1,\chi_2,1;q)f_0\left(\frac{n_0}{N_0}\right)\mathbf{V}\left(\frac{n_0n_1n_2}{q^{3/2}}\right),\end{aligned} \tag{6.12}$$

where the functions $f_i$ are smooth and compactly supported on $(1/2,2)$ and the $N_i'$s runs over real numbers of the form $2^i$, $i\geqslant 0$. By the fast decay at infinity of $\mathbf{V}$, we can assume that

$$1\leqslant N_0,N_1,N_2\ \text{ and }\ N_0N_1N_2\leqslant q^{3/2+\varepsilon}.$$

Hence it remains to bound $O(\log^3 q)$ sums of the shape (6.12). It is also convenient to separate the variables $n_0n_1n_2$ in the test function $\mathbf{V}$. This can be done using its integral representation (1.27) as in Section 3.2.1. We keep the same notation $\mathcal{S}_2^{\pm}(\ell,N_0,N_1,N_2;q)$, but with the factor $\mathbf{V}$ removed, and also for the functions $f_i$, i.e.

$$\begin{aligned}\mathcal{S}_2^{\pm}(\ell,N_0,N_1,N_2;q)=\frac{1}{(qN_0N_1N_2)^{1/2}}&\sideset{}{^*}\sum_{n_0,n_1,n_2\in\mathbf{Z}}\overline{\chi}_1(n_1)\overline{\chi}_2(n_2)\mathrm{Kl}_3(\pm n_0n_1n_2\overline{\ell},\chi_1,\chi_2,1;q)\\&\times f_0\left(\frac{n_0}{N_0}\right)f_1\left(\frac{n_1}{N_1}\right)f_2\left(\frac{n_2}{N_2}\right),\end{aligned} \tag{6.13}$$

with

$$x^jf_i^{(j)}(x)\ll_j q^{\varepsilon j}. \tag{6.14}$$

Note finally that the trivial estimate is

$$\mathcal{S}_2^{\pm}(\ell,N_0,N_1,N_2;q)\ll\left(\frac{N_0N_1N_2}{q}\right)^{1/2}. \tag{6.15}$$

**Pólya-Vinogradov bound**

We show here that (6.13) is very small if we assume that one of the parameters $N_i$ is bigger than $q$. Indeed, since the argument is the same, we suppose that $N_1 \geqslant q/2$. In this case, for fixed $(n_0 n_2, q) = 1$ we focus on the $n_1$-sum

$$\mathcal{P}(N_1; q) = \sum_{n_1 \in \mathbf{Z}}^{*} \overline{\chi}(n_1) \mathrm{Kl}_3(\pm n_0 n_1 n_2 \overline{\ell}, \chi_1, \chi_2, 1; q) f_1\left(\frac{n_1}{N_1}\right).$$

By Remark 2.8, we can add the contribution of $q|n_1$ for an error of size $O(N_1/q)$ (since $N_1 \geqslant q/2$). Hence, applying Proposition 2.18 with the Fourier trace sheaf

$$\mathcal{L}_{\overline{\chi}} \otimes \left[\times\left(\pm n_0 n_2 \overline{\ell}\right)\right]^{*} \mathcal{K}\ell_3(\chi_1, \chi_2, 1; q),$$

we get

$$\mathcal{P}(N_1; q) = O\left(q^{\varepsilon} \frac{N_1}{q^{1/2}} + \frac{N_1}{q}\right) = O\left(q^{\varepsilon} \frac{N_1}{q^{1/2}}\right).$$

Finally, averaging trivially over $n_0$ and $n_2$ in (6.13) yields

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll q^{\varepsilon} \left(\frac{N_0 N_1 N_2}{q^2}\right)^{1/2}.$$

Since $N_0 N_1 N_2 \leqslant q^{3/2+\varepsilon}$, we obtain

**Proposition 6.2.** *Assume that $N_i \geqslant q/2$ for some $i \in \{0,1,2\}$. Then for any $\varepsilon > 0$, we have*

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = O\left(q^{-1/4+\varepsilon}\right),$$

*with an implied constant depending only on $\varepsilon$.*

**Applying Poisson summation in the three variables**

In this section, we obtain an estimate for $\mathcal{S}_2^{\pm}$ which is satisfactory if the product of the three variables $N_0 N_1 N_2$ is greater that $q$. This can be done using successive applications of Poisson summation formula. Before this, we just note that by Proposition 6.2, we can assume that $N_i < q/2$ for $i = 0,1,2$, which allows us to ignore the primality condition $(n_0 n_1 n_2, q) = 1$ in (6.13) since we also have $N_i \geqslant 1$. We begin with the $n_0$-variable. In (6.13), we write the Kloosterman sum as the Fourier transform of the function

$$\mathbf{F}_q \ni x \mapsto \mathbf{K}_2(x) := \mathbf{K}_2(x, \chi_1, \chi_2; q)$$

defined in (2.12). Hence, an application of Poisson summation in $n_0$ and Fourier inversion formula gives (recall that $(n_1n_2, q) = 1$)

$$\sum_{n_0\in\mathbf{Z}} \widehat{\mathbf{K}_2}(\pm n_0 n_1 n_2 \overline{\ell}) f_0\left(\frac{n_0}{N_0}\right) = \frac{N_0}{q^{1/2}} \sum_{n_0\in\mathbf{Z}} \widehat{\widehat{\mathbf{K}_2}}(\pm n_0 \overline{n_1 n_2}\ell) \widehat{f}_0\left(\frac{n_0 N_0}{q}\right)$$
$$= \frac{N_0}{q^{1/2}} \sum_{n_0\in\mathbf{Z}} \mathbf{K}_2(\mp n_0 \overline{n_1 n_2}\ell) \widehat{f}_0\left(\frac{n_0 N_0}{q}\right).$$

Since by Definition $\mathbf{K}_2(x) = 0$ for $q|x$, we obtain

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = \frac{N_0^{1/2}}{q(N_1N_2)^{1/2}} \sum_{\substack{n_0,n_1,n_2\in\mathbf{Z}\\ (n_0,q)=1}} \overline{\chi}_1(n_1)\overline{\chi}_2(n_2)\mathrm{Kl}_2(\mp n_1 n_2 \overline{n_0 \ell}, \chi_1, \chi_2; q)$$
$$\times \widehat{f}_0\left(\frac{n_0 N_0}{q}\right) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right). \tag{6.16}$$

We continue with the $n_2$-variable. As before, since the argument of $\mathrm{Kl}_2$ is non zero modulo $q$, we can express the Kloosterman sum as suitable Fourier transform, namely (see (2.12))

$$\mathrm{Kl}_2(\mp n_1 n_2 \overline{\ell}, \chi_1, \chi_2; q) = \chi_2(\mp n_1 n_2 \overline{n_0 \ell})\big[\widehat{\chi_2 \mathbf{K}_1(\chi_1; q)}\big](\mp n_1 n_2 \overline{n_0 \ell}). \tag{6.17}$$

Using exactly the same argument as before, after replacing $\mathrm{Kl}_2$ by (6.17) in (6.16), we get

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = \frac{\overline{\chi}_2(\mp\ell) N_0^{1/2}}{q(N_1N_2)^{1/2}} \sum_{\substack{n_0,n_1,n_2\in\mathbf{Z}\\ (n_0 n_2,q)=1}} \overline{\chi}_1\chi_2(n_1)\overline{\chi}_2(n_0)[\widehat{\chi_2 \mathbf{K}_1(\chi_1)}](\mp n_1 n_2 \overline{n_0 \ell})$$
$$\times \widehat{f}_0\left(\frac{n_0 N_0}{q}\right) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right).$$

Applying Poisson in the $n_2$-variable yields

$$\sum_{n_2\in\mathbf{Z}} \widehat{\chi_2 \mathbf{K}_1(\chi_1)}(\mp n_1 n_2 \overline{n_0 \ell}) f_2\left(\frac{n_2}{N_2}\right) = \frac{N_2}{q^{1/2}} \sum_{n_2\in\mathbf{Z}} \widehat{\widehat{\chi_2 \mathbf{K}_1(\chi_1)}}(\mp n_2 \overline{n_1} n_0 \ell) \widehat{f}_2\left(\frac{n_2 N_2}{q}\right)$$
$$= \frac{N_2}{q^{1/2}} \sum_{n_2\in\mathbf{Z}} \chi_2(\pm n_2 \overline{n_1} n_0 \ell) \mathbf{K}_1(\pm n_2 \overline{n_1} n_0 \ell, \chi_1; q) \widehat{f}_2\left(\frac{n_2 N_2}{q}\right)$$
$$= \frac{N_2}{q^{1/2}} \sum_{(n_2,q)=1} \overline{\chi}_1\chi_2(\pm n_2 \overline{n_1} n_0 \ell) e\left(\frac{\pm n_1 \overline{n_2 n_0 \ell}}{q}\right) \widehat{f}_2\left(\frac{n_2 N_2}{q}\right).$$

Hence

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = \overline{\chi}_1(\pm\ell)\chi_2(-1)\left(\frac{N_0 N_2}{q^3 N_1}\right)^{1/2} \sum_{\substack{n_0,n_1,n_2\in\mathbf{Z}\\ (n_0 n_2,q)=1}} \overline{\chi}_1(n_2 n_0)\chi_2(n_2) e\left(\frac{\pm n_1 \overline{n_2 n_0 \ell}}{q}\right)$$
$$\times f_1\left(\frac{n_1}{N_1}\right) \widehat{f}_0\left(\frac{n_0 N_0}{q}\right) \widehat{f}_2\left(\frac{n_2 N_2}{q}\right). \tag{6.18}$$

It remains to do Poisson in the $n_1$-variable. Let $a \in \mathbf{F}_q$, we denote by $\boldsymbol{\delta}_a$ the Dirac function on $\mathbf{F}_q$ defined by $\boldsymbol{\delta}_a(x) = 1$ if $x = a$ and zero else. Then the exponential map

$$n_1 \mapsto e\left(\frac{\pm n_1 \overline{n_0 n_2 \ell}}{q}\right)$$

is the additive Fourier transform of the Dirac function $x \mapsto q^{1/2}\boldsymbol{\delta}_{\pm\overline{n_0 n_2 \ell}}(x)$. It follows that the $n_1$ sum in (6.18) is equal to

$$\sum_{n_1 \in \mathbf{Z}} \widehat{q^{1/2}\boldsymbol{\delta}_{\pm\overline{n_0 n_2 \ell}}}(n_1) f_1\left(\frac{n_1}{N_1}\right) = N_1 \sum_{n_1 \in \mathbf{Z}} \boldsymbol{\delta}_{\pm\overline{n_0 n_2 \ell}}(-n_1) \widehat{f}_1\left(\frac{n_1 N_1}{q}\right).$$

Summarizing all the previous computations yields the bound

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll \left(\frac{N_0 N_1 N_2}{q^3}\right)^{1/2} \sum_{n_0 n_1 n_2 \ell \equiv \mp 1 (\mathrm{mod}\ q)} \left| \widehat{f}_0\left(\frac{n_0 N_0}{q}\right) \widehat{f}_1\left(\frac{n_1 N_1}{q}\right) \widehat{f}_2\left(\frac{n_2 N_2}{q}\right)\right|.$$

Finally, using the fact that all these Fourier transform have fast decay at infinity, we see that the above sum is essentially supported on $|n_i| \leqslant \frac{q^{1+\varepsilon}}{N_i}$ and thus, a trivial estimate leads to

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll q^{\varepsilon} \left(\frac{N_0 N_1 N_2}{q^3}\right)^{1/2} \left(\frac{q^2}{N_0 N_1 N_2} + 1\right) \ll q^{\varepsilon} \left(\frac{q}{N_0 N_1 N_2}\right)^{1/2}. \tag{6.19}$$

### Estimation of $\mathcal{S}_2^{\pm}$ using Theorems 6 and 2.19

We return to expression (6.13). The combination of (6.15) and (6.19) shows that it remains to deal with the case where the product $N_0 N_1 N_2$ is of length about $q$. The strategy is as follow : if one of the variables $N_i$ is very small, then we factorize the two others to form a new long variable and apply Theorem 6 for the twist of Eisenstein series. If none of the $N_i$'s are too small, then it is possible to factorize two variables and form a bilinear sum in such a way that an application of Theorem 2.19 is beneficial.

We prove in this section :

**Proposition 6.3.** *Let* $N = \max(N_0, N_1, N_2)$, $M = \min(N_0, N_1, N_2)$ *and write* $D$ *for the remaining parameter, i.e.* $M \leqslant D \leqslant N$. *Then for every* $\varepsilon > 0$, *we have*

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll_{\varepsilon} q^{\varepsilon} \left(\frac{N_0 N_1 N_2}{q}\right)^{1/2} \begin{cases} \left(1 + \frac{qM}{N_0 N_1 N_2}\right)^{1/2} q^{-1/16} \\ \\ \frac{1}{q^{1/4}} + \frac{1}{D^{1/2}} + \frac{q^{1/4}}{(NM)^{1/2}}. \end{cases} \tag{6.20}$$

*Proof.* To fix the ideas, we assume that

$$M = N_0 \leqslant N_1 = D \leqslant N_2 = N,$$

and we leave it to the reader to ensure that the other cases can be treated with minimal changes. We first focus on the $n_1,n_2$-sum in (6.13) and write it in the form (recall that $N_1,N_2<q/2$ so the primality condition is satisfied)

$$\sum_{n_1,n_2\geqslant 1}\overline{\chi}_1\chi_2(n_1)\overline{\chi}_2(n_1n_2)\mathrm{Kl}_3(n_0n_1n_2\overline{\ell},\chi_1,\chi_2;q)f_1\left(\frac{n_1}{N_1}\right)f_2\left(\frac{n_2}{N_2}\right). \tag{6.21}$$

We show now how to transform this expression in order to obtain the same as in Corollary 2.2. To simplify notations, we define

$$K(n):=\overline{\chi}_2(n)\mathrm{Kl}_3(\pm nn_0\overline{\ell},\chi_1,\chi_2;q). \tag{6.22}$$

Using Mellin inversion on $f_1$ and $f_2$ in (6.21) leads to

$$\frac{1}{(2\pi i)^2}\int_{(0)}\int_{(0)}\widetilde{f}_1(s_1)\widetilde{f}_2(s_2)N_1^{s_1}N_2^{s_2}\sum_{n_1,n_2\geqslant 1}\overline{\chi}_1\chi_2(n_1)K(n_1n_2)n_1^{-s_1}n_2^{-s_2}ds_1ds_2.$$

Making the change of variables

$$\theta_1=\frac{s_1+s_2}{2}\ ,\ \ \theta_2=\frac{-s_1+s_2}{2},$$

and we see that the above integral takes the form

$$\begin{aligned}\frac{2}{(2\pi i)^2}&\int_{(0)}\int_{(0)}\widetilde{f}_1(\theta_1-\theta_2)\widetilde{f}_2(\theta_1+\theta_2)\left(\frac{N_2}{N_1}\right)^{\theta_2}\\&\times\sum_{n_1,n_2\geqslant 1}\overline{\chi}_1\chi_2(n_1)\left(\frac{n_1}{n_2}\right)^{\theta_2}K(n_1n_2)\left(\frac{N_1N_2}{n_1n_2}\right)^{\theta_1}d\theta_1d\theta_2\\&=\frac{2}{(2\pi i)^2}\int_{(0)}\left(\frac{N_2}{N_1}\right)^{\theta_2}\sum_{n\geqslant 1}\lambda_{\overline{\chi}_1\chi_2}(n,\theta_2)K(n)V\left(\frac{n}{N_1N_2},\theta_2\right)d\theta_2,\end{aligned} \tag{6.23}$$

where for any $x\geqslant 0$ and $\Re e(\theta_2)=0$, we defined

$$V(x,\theta_2):=\int_{(0)}\widetilde{f}_1(\theta_1-\theta_2)\widetilde{f}_2(\theta_1+\theta_2)x^{-\theta_1}d\theta_1. \tag{6.24}$$

Because the Mellin transforms satisfy (c.f. (6.14))

$$\widetilde{f}_1(s),\widetilde{f}_2(s)\ll\left(\frac{q^{\varepsilon}}{1+|s|}\right)^B, \tag{6.25}$$

with an implied constant depending on $\varepsilon$, $B$ and $\Re e(s)$, the function $V(x,\theta_2)$ satisfies

$$V(x,\theta_2)\ll_B\frac{1}{(1+x)^B}\ \text{ and }\ x^{\nu}V^{(\nu)}(x,\theta_2)\ll_{\nu,\varepsilon}q^{\nu\varepsilon},$$

uniformly in $\Re e(\theta_2)=0$. Since we want to estimate the inner sum in (6.23) using Theorem 6 and then average trivially over the $\theta_2$-integral, we also need to control the function $V(x,\theta_2)$ with respect to the $\theta_2$-variable. By (6.25), for any $B \geqslant 1$, we have uniformly on $x > 0$ and with an implied constant depending only on $B$,

$$V(x,\theta_2) \ll \int_{(0)} \left( \frac{q^{\varepsilon}}{(1+|\theta_1-\theta_2|)(1+|\theta_1+\theta_2|)} \right)^B d\theta_1.$$

Note the identity

$$\begin{aligned}(1+|\theta_1-\theta_2|)(1+|\theta_1+\theta_2|) &= 1+|\theta_1^2-\theta_2^2|+|\theta_1-\theta_2|+|\theta_1+\theta_2| \\ &\geqslant 1+|\theta_1^2-\theta_2^2|+2\max(|\theta_1|,|\theta_2|).\end{aligned}$$

Hence, splitting the integral depending on whether $|\theta_1| \leqslant |\theta_2|$ or not and we get

$$\begin{aligned} V(x,\theta_2) &\ll \int\limits_{\substack{\Re e(\theta_1)=0 \\ |\theta_1|\geqslant|\theta_2|}} \left( \frac{q^{\varepsilon}}{1+|\theta_1^2-\theta_2^2|+2|\theta_1|} \right)^B d\theta_1 + \int\limits_{\substack{\Re e(\theta_1)=0 \\ |\theta_1|\leqslant|\theta_2|}} \left( \frac{q^{\varepsilon}}{1+|\theta_1^2-\theta_2^2|+2|\theta_2|} \right)^B d\theta_1 \\ &\leqslant \int_{|t|\geqslant|\theta_2|} \left( \frac{q^{\varepsilon}}{1+2|t|} \right)^B dt + \int_{|t|\leqslant|\theta_2|} \left( \frac{q^{\varepsilon}}{1+2|\theta_2|} \right)^B dt \ll \left( \frac{q^{\varepsilon}}{1+|\theta_2|} \right)^{B-1}. \end{aligned}$$

Therefore, for any $\varepsilon' > 0$, we obtain that (6.23) is bounded, up to a constant which depends only on $\varepsilon'$, by

$$q^{\varepsilon'} \max_{\substack{|\theta_2|\leqslant q^{\varepsilon'} \\ \Re e(\theta_2)=0}} \left| \sum_{n\geqslant 1} \lambda_{\overline{\chi}_1\chi_2}(n,\theta_2) K(n) V\left( \frac{n}{N_1N_2}, \theta_2 \right) \right|. \tag{6.26}$$

We now apply Corollary 2.2 with the Schwartz function $V(x,\theta)$ and with the sheaf

$$\mathcal{F} := \mathcal{L}_{\overline{\chi}_2} \otimes [\pm n_0\overline{\ell}]^* \mathcal{K}\ell_3(\chi_1,\chi_2,1;q)$$

having trace function (6.22). Note that since $\mathrm{Kl}_3(\cdot,\chi_1,\chi_2,1;q)$ is invariant under permutation of the triple $(\chi_1,\chi_2,1)$, we have by (2.11) a geometric isomorphism

$$\mathcal{F} \simeq [\times(\pm n_0\overline{\ell})]^* \mathrm{FT}\left( \mathcal{L}_{\chi_2} \otimes [x \mapsto x^{-1}]^* \mathcal{K}\ell_2(\chi_1,1;q) \right)$$

and hence $\mathcal{F}$ is not Fourier-exceptional since by Fourier inversion, its $\ell$-adic Fourier transform is a rank 2 irreducible sheaf. It follows that for any $\varepsilon > 0$, we can estimate (6.26) by

$$q^{\varepsilon'} \max_{|\theta_2|\leqslant q^{\varepsilon'}} (qN_1N_2)^{\varepsilon} (1+|\theta_2|)^A N_1N_2 \left( 1+\frac{q}{N_1N_2} \right)^{1/2} q^{-1/16}.$$

Choosing $\varepsilon' = \varepsilon/A$, maximizing the above quantity by setting $\theta_2 = q^{\varepsilon'}$, replacing the obtained bound in (6.21) and finally, averaging trivially over $n_0$ in (6.13) yields the first estimate of (6.20)

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll_\varepsilon q^\varepsilon \left(\frac{N_0 N_1 N_2}{q}\right)^{1/2} \left(1 + \frac{q}{N_1 N_2}\right)^{1/2} q^{-1/16}. \tag{6.27}$$

For the second bound, we group together the variables $n_0 n_2 = m$ in (6.13) and we obtain

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = \frac{1}{(q N_0 N_1 N_2)^{1/2}} \sum_{n, n_1} \alpha_m \beta_{n_1} \mathrm{Kl}_3(\pm n n_1 \overline{\ell}, \chi_1, \chi_2, 1; q), \tag{6.28}$$

with

$$\alpha_m := \sum_{n_0 n_2 = m} \overline{\chi}_2(n_2) f_0\left(\frac{n_0}{N_0}\right) f_2\left(\frac{n_2}{N_2}\right) \text{ and } \beta_{n_1} := \overline{\chi}_1(n_1) f_1\left(\frac{n_1}{N_1}\right).$$

Applying Theorem 2.19 (1) with $N = N_0 N_2$ and $M = N_2$ gives

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll q^\varepsilon \left(\frac{N_0 N_1 N_2}{q}\right)^{1/2} \left(\frac{1}{q^{1/4}} + \frac{1}{N_1^{1/2}} + \frac{q^{1/4}}{(N_0 N_2)^{1/2}}\right), \tag{6.29}$$

as wishes. □

### 6.1.5 Conclusion of the triple product

Write $N_i = q^{\mu_i}$ with $\mu_i \geqslant 0$ and let $\eta > 0$ be a parameter. If $\mu_0 + \mu_1 + \mu_2 < 1 - 2\eta$ or $\mu_0 + \mu_1 + \mu_2 > 1 + 2\eta$, we use the trivial bound (6.15) or the estimate (6.19) to obtain

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = O\left(q^{-\eta+\varepsilon}\right).$$

We therefore assume that we are in the range

$$1 - 2\eta \leqslant \mu_0 + \mu_1 + \mu_2 \leqslant 1 + 2\eta. \tag{6.30}$$

Let $\delta > 0$ be an auxiliary parameter. As we already see, there is no loss of generality assuming that $\mu_0 \leqslant \mu_1 \leqslant \mu_2$. Suppose first that

$$\mu_0 \leqslant \delta. \tag{6.31}$$

In this case, we apply (6.27) which, combining with (6.30) and (6.31) gives

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll_\varepsilon q^\varepsilon \left(q^{\eta - \frac{1}{16}} + q^{\frac{\delta}{2} - \frac{1}{16}}\right) \ll_\varepsilon q^{-\eta+\varepsilon},$$

provided

$$\eta \leqslant \frac{1}{32} \text{ and } \delta \leqslant \frac{1}{8} - 2\eta, \tag{6.32}$$

which condition we henceforth assume to hold.

Suppose now that we are in the case

$$\mu_0 \geqslant \delta. \tag{6.33}$$

The estimate (6.29) leads to

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll q^{\varepsilon}\left(q^{\eta-\frac{1}{4}} + q^{\frac{1}{2}(\mu_0+\mu_2-1)} + q^{\frac{1}{2}(\mu_1-\frac{1}{2})}\right).$$

The first term is clearly smaller than $q^{-\eta+\varepsilon}$ by (6.32). For the second, note that $\mu_1 \geqslant \mu_0 \geqslant \delta$ and thus, by (6.30)

$$\mu_0 + \mu_2 - 1 \leqslant 2\eta - \delta.$$

It follows that

$$q^{\varepsilon+\frac{1}{2}(\mu_0+\mu_2-1)} \leqslant q^{\varepsilon+\eta-\frac{\delta}{2}} \leqslant q^{-\eta+\varepsilon},$$

under the assumption that

$$\delta \geqslant 4\eta. \tag{6.34}$$

Finally, the combination of (6.30), $\mu_1 \leqslant \mu_2$ and (6.33) gives

$$\mu_1 \leqslant \frac{1}{2} + \eta - \frac{\delta}{2}$$

and hence

$$q^{\varepsilon+\frac{1}{2}(\mu_1-\frac{1}{2})} \leqslant q^{\varepsilon+\frac{\eta}{2}-\frac{\delta}{4}} \leqslant q^{-\eta+\varepsilon},$$

provided

$$\delta \geqslant 6\eta, \tag{6.35}$$

which is more restrictive than (6.34). To finalize the computations, we just note that the second condition in (6.32) and (6.35) are simultaneously satisfied as long as $\eta \leqslant \frac{1}{64}$, which gives the correct exponent of the error term in Theorem 4.

## 6.2 The cuspidal case

We consider as in Section 6.1 the average over primitive and even characters (recall that the nebentypus is trivial)

$$\mathscr{T}^3(f, \ell; q) := \frac{2}{q-1} \sideset{}{^+}\sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq 1}} L\left(f \otimes \chi, \tfrac{1}{2}\right) L\left(\chi, \tfrac{1}{2}\right) \chi(\ell). \tag{6.36}$$

### 6.2.1 Applying the approximate functional equation

Using Proposition 1.15, we can write (6.36) in the form

$$\mathscr{T}^3_{\mathrm{even}}(f,\ell;q) = \mathcal{C}_1(f,\ell;q) + \varepsilon_\infty(f,+1)\mathcal{C}_2(f,\ell;q)$$

with

$$\mathcal{C}_1(f,\ell;q) = \frac{2}{q-1} \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi\neq 1}} \sum_{n.m\geqslant 1} \frac{\lambda_f(n)\chi(nm\ell)}{(nm)^{1/2}} \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right),$$

$$\mathcal{C}_2(f,\ell;q) = \frac{2}{q-1} \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi\neq 1}} \sum_{n.m\geqslant 1} \frac{\overline{\lambda_f(n)}\overline{\chi}(nm)\chi(\ell)}{(nm)^{1/2}} \varepsilon(\chi)^3 \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right),$$

where we recall that $\mathbf{V}_{f,\chi}$ depends on $\chi$ only through its parity. Since we assume that $f$ satisfies the Ramanujan-Petersson conjecture, we have $|\lambda_f(n)| \leqslant \tau(n)$. Hence, proceeding as in Section 6.1.2 for the average over the characters and writing $\mathbf{V} = \mathbf{V}_{f,\chi}$, we find

$$\mathcal{C}_i(f,\ell;q) = \sum_{\pm} \mathcal{C}_i^{\pm}(f,\ell;q) + O\left(q^{-1/4+\varepsilon}\right),$$

where

$$\mathcal{C}_1^{\pm}(f,\ell;q) = \sideset{}{^*}\sum_{nm\ell\equiv\pm 1 \pmod q} \frac{\lambda_f(n)}{(nm)^{1/2}} \mathbf{V}\left(\frac{nm}{q^{3/2}}\right), \tag{6.37}$$

and

$$\mathcal{C}_2^{\pm}(f,\ell;q) = \frac{1}{q^{1/2}} \sideset{}{^*}\sum_{n,m\geqslant 1} \frac{\overline{\lambda_f(n)}}{(nm)^{1/2}} \mathrm{Kl}_3(\pm nm\overline{\ell};q) \mathbf{V}\left(\frac{nm}{q^{3/2}}\right). \tag{6.38}$$

### 6.2.2 The main term

The extraction of the main term is done is a similar way as in Section 6.1.3. We just conclude with

$$\mathcal{C}_1^{+}(f,\ell;q) = \delta_{\ell=1} + O\left(\ell q^{-1/4+\varepsilon}\right), \quad \mathcal{C}_1^{-}(f,\ell;q) = O\left(\ell q^{-1/4+\varepsilon}\right).$$

Note that the error terms are $O(q^{-\frac{1}{52}+\varepsilon})$ (c.f. Theorem 4) if we assume that

$$\ell \leqslant q^{\frac{1}{4}-\frac{1}{52}} = q^{\frac{3}{13}}. \tag{6.39}$$

### 6.2.3 The error term

Applying a partition of unity to (6.38), removing the test function $\mathbf{V}$ using its integral representation (see § 6.1.4), we are reduced to analyze $O(\log^2 q)$ sums of the shape

$$\mathcal{C}_2^{\pm}(f,N,M;q) := \frac{1}{(qNM)^{1/2}} \sum_{n,m\in\mathbf{Z}}^{*} \overline{\lambda_f(n)}\mathrm{Kl}_3(\pm nm\overline{\ell};q) W_1\left(\frac{m}{M}\right) W_2\left(\frac{n}{N}\right), \tag{6.40}$$

where $W_i$ are smooth and compactly supported functions on $(1/2,2)$ such that $x^j W_i^{(j)}(x) \ll_{\varepsilon,j} q^{\varepsilon j}$ for all $j \geqslant 0$ and $M,N$ are reals numbers with the standard restriction due to the fast decay of $\mathbf{V}$ at infinity

$$1 \leqslant M,N \text{ and } NM \leqslant q^{3/2+\varepsilon}.$$

Note that the trivial bound is

$$\mathcal{C}_2^{\pm}(f,N,M;q) \ll \left(\frac{NM}{q}\right)^{1/2}. \tag{6.41}$$

Moreover, if $M \geqslant q/2$, then an application of Polyá-Vinograov method in the $m$-variable (see Proposition 2.18 and Section 6.1.4) leads to

$$\mathcal{C}_2^{\pm}(f,N,M;q) \ll_{\varepsilon} q^{\varepsilon}\left(\frac{NM}{q^2}\right)^{1/2} \ll q^{-1/4+\varepsilon}.$$

Hence we can suppose from now on that $M < q/2$ in such a way that the condition $(m,q)=1$ under the summation in (6.38) is automatically satisfied.

#### Application of Poisson/Voronoi summation formula

The first step is to apply Voronoi summation formula in the $n$-variable. To get in a good position, we write the Kloosterman sum $\mathrm{Kl}_3$ for $(a,q)=1$ in the form

$$\mathrm{Kl}_3(a;q) = \frac{1}{q^{1/2}} \sum_{x\in\mathbf{F}_q^{\times}} \mathrm{Kl}_2(\overline{x};q) e\left(\frac{ax}{q}\right). \tag{6.42}$$

Note that this definition can be extended to $a=0$ with the value

$$\mathrm{Kl}_3(0;q) = \frac{1}{q}\left(\sum_{x\in\mathbf{F}_q^{\times}} e\left(\frac{x}{q}\right)\right)^2 = \frac{1}{q}.$$

It follows that after writing $\mathrm{Kl}_3(\pm nm\overline{\ell};q)$ in the form (6.42) and adding the contribution of $q|n$ for negligible error term (of size at most $q^{-3/4+\varepsilon}$), we get

$$\mathcal{C}_2^{\pm}(f,N,M;q) = \frac{1}{(qNM)^{1/2}} \frac{1}{q^{1/2}} \sum_{x\in\mathbf{F}_q^{\times}} \mathrm{Kl}_2(\overline{x};q) \sum_{m\in\mathbf{Z}} W_1\left(\frac{m}{M}\right) \sum_{n\geqslant 1} \overline{\lambda_f(n)} e\left(\frac{\pm nm\overline{\ell}x}{q}\right) W_2\left(\frac{n}{N}\right).$$

(6.43)

Assuming we are dealing with the plus case and applying Voronoi formula (c.f. Proposition 1.2) to the inner sum in (6.43), we obtain

$$\mathcal{C}_2^+(f,N,M;q) = \left(\frac{N}{q^3M}\right)^{1/2}\frac{1}{q^{1/2}}\sum_{\pm}\sum_{x\in\mathbf{F}_q^\times}\mathrm{Kl}_2(\overline{x};q)\sum_{n\geqslant 1}\overline{\lambda_f(n)}W_2^{\pm}\left(\frac{nN}{q^2}\right) \times \sum_{m\in\mathbf{Z}} e\left(\frac{\mp n\overline{mx}\ell}{q}\right)W_1\left(\frac{m}{M}\right).$$

Changing the order of summation, making the change of variable $\overline{x}\leftrightarrow xm$ (recall that $(m,q)=1$) allows us to write

$$\mathcal{C}_2^+(f,N,M;q) = \left(\frac{N}{q^3M}\right)^{1/2}\frac{1}{q^{1/2}}\sum_{\pm}\sum_{x\in\mathbf{F}_q^\times}\sum_{n\geqslant 1}\overline{\lambda_f(n)}e\left(\frac{\mp nx\ell}{q}\right)W_2^{\pm}\left(\frac{nN}{q^2}\right) \times \sum_{m\in\mathbf{Z}}\mathrm{Kl}_2(xm;q)W_1\left(\frac{m}{M}\right). \tag{6.44}$$

By Poisson formula and since $\mathrm{Kl}_2$ is the Fourier transform of the function defined by (2.12), we see that the $m$-sum in (6.44) is equal to

$$\frac{M}{q^{1/2}}\sum_{(m,q)=1} e\left(-\frac{x\overline{m}}{q}\right)\widehat{W_1}\left(\frac{mM}{q}\right).$$

Replacing this identity in (6.44) yields

$$\mathcal{C}_2^+(f,N,M;q) = \left(\frac{NM}{q^3}\right)^{1/2}\sum_{\pm}\sum_{n\geqslant 1}\sum_{(m,q)=1}\overline{\lambda_f(n)}\widehat{W_1}\left(\frac{mM}{q}\right)W_2^{\pm}\left(\frac{nN}{q^2}\right) \times\frac{1}{q}\sum_{x\in\mathbf{F}_q^\times} e\left(x\frac{\mp n\ell-\overline{m}}{q}\right), \tag{6.45}$$

with the same expression for the minus case $\mathcal{C}_2^-$, but with $\mp$ replaced by $\pm$ in the exponential. Because of the fast decay of $\widehat{W_1}$ and $W_2^{\pm}$ at infinity (c.f. Lemma 1.3), the $n,m$-sum (6.45) is essentially supported on $|m|\leqslant q^{1+\varepsilon}/M$ and $|n|\leqslant q^{2+\varepsilon}/N$. In this range, we use the estimate $|\lambda_f(n)|\leqslant\tau(n)\ll_\varepsilon n^\varepsilon$ and we apply Lemma 1.3 with $\vartheta=0$ (recall that $f$ satisfies the Ramanujan-Petersson Conjecture) to bound $W_2^{\pm}$ by $q^\varepsilon$. Adding the contribution of $x=0$, estimating this extra factor trivially and executing the complete $x$-summation gives

$$\mathcal{C}_2^+(f,N,M;q) = \left(\frac{NM}{q^3}\right)^{1/2}\sum_{\pm}\sum\sum_{nm\ell\equiv\mp 1\ (\mathrm{mod}\ q)}\overline{\lambda_f(n)}\widehat{W_1}\left(\frac{mM}{q}\right)W_2^{\pm}\left(\frac{nN}{q^2}\right) + O\left(q^\varepsilon\left(\frac{q}{NM}\right)^{1/2}\right).$$

Therefore using $|\lambda_f(n)| \leqslant \tau(n)$, we obtain as in § 6.1.4,

$$\mathcal{C}_2^{\pm}(f,N,M;q) \ll_{\varepsilon} q^{\varepsilon}\left(\frac{q}{NM}\right)^{1/2}, \tag{6.46}$$

uniformly on $\ell < q$.

**Estimation of $\mathcal{C}_2$ using bounds for bilinear forms and Theorem 6**

We finally state the analogous of Proposition 6.3 which is an immediate application of Theorem 2.19 (1)-(2), Theorem 2.20 and Corollary 2.2.

**Proposition 6.4.** *For any $\varepsilon > 0$, the quantity defined in* (6.40) *satisfies*

$$\mathcal{C}_2^{\pm}(f,N,M;q) \ll q^{\varepsilon}\left(\frac{NM}{q}\right)^{1/2}\begin{cases} \frac{1}{q^{1/4}}+\frac{1}{M^{1/2}}+\frac{q^{1/4}}{N^{1/2}} \\ \frac{1}{q^{1/2}}+\frac{q^{1/2}}{M} \\ \left(\frac{N^2M^5}{q^3}\right)^{-1/12} \\ \left(1+\frac{q}{N}\right)^{1/2}q^{-1/16}, \end{cases}$$

*where the implied constant depends on $\varepsilon$ and polynomially on $t_f$ in the last bound and the third bound is valid in the case where $1 \leqslant N \leqslant M^2$, $M < q$ and $NM < q^{3/2}$* (*c.f.* (2.16)).

### 6.2.4 Conclusion of the cuspidal case

Fix $\eta > 0$ a parameter and write $M = q^{\mu}$, $N = q^{\nu}$ with $\mu,\nu \geqslant 0$. By the trivial bound (6.41) and (6.46), we can assume that

$$1-2\eta \leqslant \mu+\nu \leqslant 1+2\eta, \tag{6.47}$$

otherwise we get $\mathcal{C}_2^{\pm}(f,N,M;q) = O(q^{-\eta+\varepsilon})$. We now let $\delta_1,\delta_2,\delta_3 > 0$ be sufficiently small auxiliary parameters and we distinguish four cases :

(a) Assume that $\mu \leqslant \delta_1$. In this case we apply the fourth estimate of Proposition 6.4 and we get by (6.47)

$$\mathcal{C}_2^{\pm}(f,N,M;q) \ll_{\varepsilon,t_f} q^{\varepsilon}\left(q^{\eta-\frac{1}{16}}+q^{\frac{\delta_1}{2}-\frac{1}{16}}\right) \leqslant q^{-\eta+\varepsilon},$$

provided

$$\eta \leqslant \frac{1}{32} \text{ and } \delta_1 \leqslant \frac{1}{8}-2\eta. \tag{6.48}$$

(b) If $\delta_1 < \mu \leqslant \frac{1}{2} - \delta_2$, the first bound of Proposition 6.4 yields

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll_\varepsilon q^\varepsilon \left( q^{\eta - \frac{1}{4}} + q^{\frac{1}{2}(\nu - 1)} + q^{\frac{1}{2}(\mu - \frac{1}{2})} \right).$$

The first term is less than $q^{\varepsilon - \eta}$ since $\eta \leqslant \frac{1}{32}$. For the second, we have $\nu - 1 \leqslant 2\eta - \delta_1$ (use (6.47) and $\mu \geqslant \delta_1$). Thus it is less than $q^{\varepsilon - \eta}$ under the assumption that

$$\delta_1 \geqslant 4\eta. \tag{6.49}$$

The third term is at most $q^{-\delta_2/2} \leqslant q^{-\eta}$ if

$$\delta_2 \geqslant 2\eta. \tag{6.50}$$

(c) Suppose that $\frac{1}{2} - \delta_2 < \mu \leqslant \frac{1}{2} + \delta_3$. In this configuration, we apply the third bound and we obtain

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll_\varepsilon q^{\varepsilon - \frac{1}{4} + \frac{\nu}{3} + \frac{\mu}{12}} = q^{\varepsilon - \frac{1}{4} + \frac{1}{12}(\mu + \nu) + \frac{\nu}{4}}.$$

Using (6.47) and $\nu \leqslant 1 + 2\eta - \mu \leqslant \frac{1}{2} + 2\eta + \delta_2$ allows us to bound the above expression by

$$q^{\varepsilon - \frac{1}{4} + \frac{1}{12}(1 + 2\eta) + \frac{1}{4}(\frac{1}{2} + 2\eta + \delta_2)} = q^{\varepsilon - \frac{1}{12}(\frac{1}{2} - 8\eta - 3\delta_2)} \leqslant q^{\varepsilon - \eta},$$

provided

$$3\delta_2 \leqslant \frac{1}{2} - 20\eta. \tag{6.51}$$

(d) Assume that $\mu > \frac{1}{2} + \delta_3$ the second bound gives

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll_\varepsilon q^\varepsilon \left( q^{\eta - \frac{1}{2}} + q^{\eta + \frac{1}{2} - \mu} \right) \leqslant q^{\varepsilon - \eta} + q^{\varepsilon + \eta - \delta_3} \ll q^{\varepsilon - \eta},$$

if we assume that

$$\delta_3 \geqslant 2\eta.$$

Finally, the combination of conditions (6.48) and (6.49) forces $\eta \leqslant \frac{1}{48}$ and (6.50)-(6.51) are simultaneously satisfied as long as $\eta \leqslant \frac{1}{52}$, which gives the correct exponent of the error term in Theorem 4 (0.11).

**Remark 6.5.** The treatment carried out in Section 5.1 remains almost identical if $f$ is level 1 Hecke cusp form. The only change we have to make is to replace the exponent $1/16$ by $1/8$ in the fourth bound of Proposition 6.4, which is due to the original Theorem [24, Theorem 1.2] for small level compared with $q$. However, it does not improve the final exponent $\frac{1}{52}$ since (6.50)-(6.51) is more restrictive and independent of (6.48)-(6.49).

# 7 Proof of Theorems 1 and 2

## 7.1 The mollification method

We show here how to derive Theorems 1 and 2 from Theorems 3 and 5. Let $1 < L < q$ be a real number such that $\log L \asymp \log q$. For any multiplicative character $\chi \pmod q$, we define the short linear form

$$\mathrm{M}(\chi; L) := \sum_{\ell \leqslant L} \frac{\chi(\ell)\mu(\ell)}{\ell^{1/2}} \left(\frac{\log L/\ell}{\log L}\right)^2, \tag{7.1}$$

where $\mu$ is the Möbius function. Let $\{\lambda_f(n)\}_{n \geqslant 1}$ denotes the sequence of Hecke eigenvalues of a Hecke cusp form of level one and $\mu_f(n)$ be the convolution inverse of $\lambda_f(n)$ given by

$$L(f,s)^{-1} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right) = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s}, \ \Re e(s) > 1.$$

For $1 < L' < q$ with $\log L' \asymp \log q$, we also define

$$\mathrm{M}(f \otimes \chi; L') := \sum_{\ell \leqslant L'} \frac{\chi(\ell)\mu_f(\ell)}{\ell^{1/2}} \left(\frac{\log L'/\ell}{\log L'}\right). \tag{7.2}$$

We finally consider the two mollified cubic moments

$$\mathscr{M}^3(\chi_1, \chi_2; q) := \frac{1}{q-1} \sum_{\chi \in \mathcal{D}_{\chi_1,\chi_2}(q)} \prod_{i=0}^{2} L\left(\chi\chi_i, \tfrac{1}{2}\right) \mathrm{M}(\chi\chi_i; L), \tag{7.3}$$

where $\chi_0$ is the trivial character, and

$$\mathscr{M}^3(f; q) := \frac{1}{q-1} \sum_{\substack{\chi \pmod q \\ \chi \neq 1}} L\left(f \otimes \chi, \tfrac{1}{2}\right) \mathrm{M}(f \otimes \chi; L') L\left(\chi, \tfrac{1}{2}\right) \mathrm{M}(\chi; L). \tag{7.4}$$

Note that (7.3) and (7.4) can be written in the form

$$\mathscr{M}^3(\chi_1,\chi_2;q)=\sum_{\ell_1,\ell_2,\ell_3\leqslant L}\frac{\boldsymbol{x}(\ell_1)\boldsymbol{x}(\ell_2)\boldsymbol{x}(\ell_3)}{(\ell_1\ell_2\ell_3)^{1/2}}\mathscr{T}^3(\chi_1,\chi_2,\ell_1\ell_2\ell_3;q),$$

$$\mathscr{M}^3(f;q)=\sum_{\ell\leqslant L,\ell'\leqslant L'}\frac{\boldsymbol{x}_f(\ell)\boldsymbol{x}(\ell')}{(\ell\ell')^{1/2}}\mathscr{T}^3(f,\ell\ell';q),$$

where $\mathscr{T}^3(\chi_1,\chi_2,\ell_1\ell_2\ell_3;q)$ are $\mathscr{T}^3(f,\ell\ell';q)$ are the twisted cubic moments defined in (0.6) and (0.7) respectively and with

$$\boldsymbol{x}(\ell):=\mu(\ell)\left(\frac{\log L/\ell}{\log L}\right)^2 \quad\text{and}\quad \boldsymbol{x}_f(\ell'):=\mu_f(\ell')\left(\frac{\log L/\ell'}{\log L}\right).$$

Since $f$ satisfies the Ramanujan-Petersson conjecture, we have for $1\leqslant\ell\leqslant L$ and $1\leqslant\ell'\leqslant L'$,

$$|\boldsymbol{x}(\ell)|\leqslant 1 \quad\text{and}\quad |\boldsymbol{x}_f(\ell')|\leqslant\tau(\ell'),$$

Hence an immediate consequence of Theorem 4 is the following corollary :

**Corollary 7.1.** *For any $\varepsilon>0$, the mollified cubic moments* (7.3) *and* (7.4) *satisfy the asymptotic formula*

$$\mathscr{M}^3(\chi_1,\chi_2;q)=1+O\left(L^{3/2}q^{-\frac{1}{64}+\varepsilon}\right),\ \mathscr{M}^3(f;q)=1+O\left((LL')^{1/2}q^{-\frac{1}{52}+\varepsilon}\right),$$

*where the implied constant only depends on $\varepsilon>0$ and polynomially on $t_f$ in the second expression.*

## 7.2 Proof of Theorem 1

We first present the proof of Theorem 1. For any $\chi$ (mod $q$), we define the following characteristic function

$$\mathbf{1}(\chi):=\delta_{|L(\chi,1/2)|\geqslant\frac{1}{\log q}}\delta_{|L(\chi\chi_1,1/2)|\geqslant\frac{1}{\log q}}\delta_{|L(\chi\chi_2,1/2)|\geqslant\frac{1}{\log q}}.$$

Using the generalized Hölder's inequality, we infer

$$\left|\frac{1}{q-1}\sum_{\chi\in\mathfrak{D}_{\chi_1,\chi_2}(q)}\mathbf{1}(\chi)\prod_{i=0}^{2}L\big(\chi\chi_i,\tfrac12\big)\mathrm{M}(\chi\chi_i;L)\right|\leqslant\left(\frac{1}{q-1}\sum_{\chi\in\mathfrak{D}_{\chi_1,\chi_2}(q)}\mathbf{1}(\chi)\right)^{1/4}\big(\mathscr{M}^4(q)\big)^{3/4},$$

where $\mathscr{M}^4(q)$ is the mollified fourth moment of Dirichlet $L$-functions defined in (0.5). On the other hand, we have the lower bound for the left handside

$$\left|\frac{1}{q-1}\sum_{\chi\in\mathfrak{D}_{\chi_1,\chi_2}(q)}\mathbf{1}(\chi)\prod_{i=0}^{2}L\big(\chi\chi_i,\tfrac12\big)\mathrm{M}(\chi\chi_i;L)\right|\geqslant\big|\mathscr{M}^3(\chi_1,\chi_2;q)\big|-\mathscr{D},$$

where $\mathscr{M}^3(\chi_1,\chi_2;q)$ is defined in (7.3) and

$$\mathscr{D} := \frac{1}{q-1}\sum_{\substack{\chi\in\mathcal{D}_{\chi_1,\chi_2}(q)\\ \mathbf{1}(\chi)=0}}\left|\prod_{i=0}^{2} L\big(\chi\chi_i,\tfrac12\big)\mathrm{M}(\chi\chi_i;L)\right|.$$

To estimate $\mathscr{D}$, note that the condition $\mathbf{1}(\chi)=0$ means that one of the central values is less than $\log(q)^{-1}$. Therefore, if for $i=0,1,2$, $\mathscr{D}_i$ is the subsum of $\mathscr{D}$ restricted to $\chi$ such that $|L(\chi\chi_i,\frac12)|\leqslant\log(q)^{-1}$, we obtain, by positivity, $\mathscr{D}\leqslant\mathscr{D}_0+\mathscr{D}_1+\mathscr{D}_2$ with for each $i=0,1,2$,

$$\mathscr{D}_i \leqslant \frac{1}{\log(q)}\left(\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}\big|\mathrm{M}(\chi;L)\big|^2\right)^{1/2}\left(\frac{1}{q-1}\sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\big|L\big(\chi,\tfrac12\big)\mathrm{M}(\chi;L)\big|^4\right)^{1/2}$$

using again twice Cauchy-Schwarz inequality. Assuming that $L=q^\lambda$ with $0<\lambda<\frac{11}{8064}$, we can use the asymptotic formula provided by Theorem 5 to obtain

$$\mathscr{D}_i \ll_\lambda \frac{1}{\log(q)}\left(\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}\big|\mathrm{M}(\chi;L)\big|^2\right)^{1/2}.$$

Moreover, opening the square in $|\mathrm{M}(\chi;L)|^2$ and applying the orthogonality relation yields

$$\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}|\mathrm{M}(\chi;L)|^2 \leqslant \sum_{\substack{\ell\equiv\ell'\ (\mathrm{mod}\ q)\\ \ell,\ell'\leqslant L}}\frac{|\boldsymbol{x}(\ell)\boldsymbol{x}(\ell')|}{(\ell\ell')^{1/2}} \leqslant \sum_{\ell\leqslant L}\frac{1}{\ell} \ll \log L, \tag{7.5}$$

since $L<q$. Hence we get

$$\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}\mathbf{1}(\chi) \geqslant \frac{\big|\mathscr{M}^3(\chi_1,\chi_2;q)\big|^4}{\mathscr{M}^4(q)^3} + O_\lambda\left(\frac{1}{\log(q)^{1/2}}\right) = \frac{1}{P(\lambda^{-1})^3}+o_\lambda(1).$$

If

$$c_1 := \max_{0<\lambda\leqslant\frac{11}{8064}} P(\lambda^{-1})^{-3},$$

then for any $\varepsilon>0$, there exists $0<\tilde\lambda<\frac{11}{8064}$ depending on $\varepsilon$ satisfying $|P(\tilde\lambda^{-1})^{-3}-c_1|\leqslant\varepsilon/2$. Finally, choosing $Q=Q(\varepsilon)$ large enough such that $|o_{\tilde\lambda}(1)|\leqslant\varepsilon/2$ for $q\geqslant Q$ and the result follows.

## 7.3 Proof of Theorem 2

We proceed in a similar way. Setting

$$\mathbf{1}(\chi,f) := \delta_{|L(\chi,1/2)|\geqslant\frac{1}{\log q}}\,\delta_{|L(f\otimes\chi,1/2)|\geqslant\frac{1}{\log^2 q}},$$

we obtain

$$\left|\frac{1}{q-1}\sum_{\chi\neq 1}\mathbf{1}(\chi,f)L\big(f\otimes\chi,\tfrac{1}{2}\big)\mathrm{M}(f\otimes\chi;L')L\big(\chi,\tfrac{1}{2}\big)\mathrm{M}(\chi;L)\right| \leqslant \left(\frac{1}{q-1}\sum_{\chi\neq 1}\mathbf{1}(\chi,f)\right)^{1/4} \times \big(\mathcal{M}^4(q)\big)^{1/4}\big(\mathcal{M}^4(f;q)\big)^{1/2} \tag{7.6}$$

where $\mathcal{M}^4(f;q)$ is the mollified twisted second moment

$$\mathcal{M}^2(f;q) := \frac{1}{q-1}\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq 1}}\big|L\big(f\otimes\chi,\tfrac{1}{2}\big)\mathrm{M}(f\otimes\chi;L')\big|^2.$$

In [6], the authors proved establishes the following asymptotic formula

$$\mathcal{M}^4(f;q) = 2\eta(1+\frac{2}{\lambda'}) + o_{\lambda',t_f}(1), \tag{7.7}$$

for $L' = q^{\lambda'}$ with $0<\lambda'<\frac{1}{360}$ and $\eta$ is an absolute constant satisfying $\eta\leqslant\zeta(3/2)$. As in the previous part, the left handside of (7.6) admits the lower bound

$$\geqslant \big|\mathcal{M}^3(f;q)\big| - \mathcal{C},$$

where $\mathcal{C}$ is the same as $\mathcal{M}^3(f;q)$, but with the absolute values inside and with the restriction in the summation to $\chi$ such that $\mathbf{1}(\chi,f)=0$. Writing $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) for the contribution of $|L(\chi,1/2)|\leqslant\frac{1}{\log q}$ (resp. $|L(f\otimes\chi,1/2)|\leqslant\frac{1}{\log^2 q}$), we get $\mathcal{C}\leqslant\mathcal{C}_1+\mathcal{C}_2$ with

$$\mathcal{C}_1 \leqslant \frac{1}{\log q}(\mathcal{M}^4(f;q))^{1/2}\left(\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}|\mathrm{M}(\chi;L)|^2\right)^{1/2} \ll_{\lambda,\lambda'} \frac{1}{(\log q)^{1/2}}$$

by (7.7) and (7.5). Finally, using again Theorem 5, we have

$$\begin{aligned}
\mathcal{C}_2 &\leqslant \frac{1}{\log^2(q)}\left(\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}|\mathrm{M}(f\otimes\chi;L')|^2\right)^{1/2}\left(\frac{1}{q-1}\sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\big|L\big(\chi,\tfrac{1}{2}\big)\mathrm{M}(\chi;L)\big|^2\right)^{1/2}\\
&\leqslant \frac{1}{\log^2 q}\left(\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}|\mathrm{M}(f\otimes\chi;L')|^2\right)^{1/2}\mathcal{M}^4(q)^{1/4}\\
&\ll_\lambda \frac{1}{\log^2(q)}\left(\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}|\mathrm{M}(f\otimes\chi;L')|^2\right)^{1/2},
\end{aligned}$$

with

$$\frac{1}{q-1}\sum_{\chi\ (\mathrm{mod}\ q)}|\mathrm{M}(f\otimes\chi;L')|^2 \leqslant \sum_{\substack{\ell\equiv\ell'\ (\mathrm{mod}\ q)\\ \ell,\ell'\leqslant L'}}\frac{\tau(\ell)\tau(\ell')}{(\ell\ell')^{1/2}} = \sum_{\ell\leqslant L'}\frac{\tau(\ell)^2}{\ell} \ll \log^3 L'.$$

Hence,

$$\mathcal{C}_2 = O_{\lambda',\lambda}\left(\frac{1}{\log(q)^{1/2}}\right),$$

and the rest of the proof is exactly the same as in the previous case.

**Remark 7.2.** Let $f$ be a primitive Hecke cusp form of prime level $q$ satisfying the Ramanujan-Petersson conjecture. The formula (0.11) could be used to prove simultaneous non-vanishing for $L(f\otimes\chi,\frac{1}{2})L(\chi,\frac{1}{2})$ as $\chi$ runs over non trivial Dirichlet characters modulo $q$ provided that it is possible to evaluate a second twisted moment of the form

$$\frac{1}{q-1}\sum_{\substack{\chi \pmod q\\ \chi\neq 1}}\left|L\left(f\otimes\chi,\tfrac{1}{2}\right)\right|^2\chi(\ell_1)\overline{\chi}(\ell_2),$$

where $(\ell_1,\ell_2)=1$ and are coprime with $q$. An asymptotic formula for this moment is given in [5] in the special case where the level is 1 and $\ell_1=\ell_2=1$ and for general $(\ell_1,\ell_2)=1$ in [6] (also for level 1). The principal difficulty here is that since the level is $q$, we have the solve a shifted convolution problem of the shape

$$\sum\sum_{\ell_1 n-\ell_2 m=hq}\lambda_f(n)\lambda_f(m)W_1\left(\frac{n}{N}\right)W_2\left(\frac{m}{M}\right),$$

for Hecke eigenvalues $\lambda_f(n)$ of level $q$.

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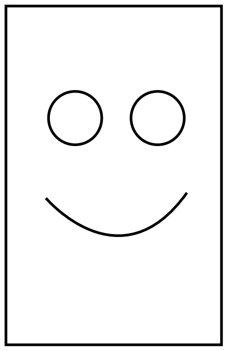
## Personal details:



| | |
|---|---|
| Name : | Mr. Sample CV |
| Address : | Samplestreet<br>70<br>6005 Luzern<br>Switzerland |
| Date of Birth : | 2nd of October 1981 |
| Nationality : | Swiss |
| Legally work : | legally work in EU |
| Marital status : | with partner |
| Children : | none |
| | |
| Languages : | Chinese/Mandarin, English, French, German |
| Education level : | Bachelors degree |
| Hospitality work experience : | 3-5 years |
| Special experience : | Europe work experience |
| | |
| Date of availability : | September 2009 |
| Current location : | Africa |
| Travelling Status : | will be travelling single status |
| Telephone : | 0041 41 370 6759 |
| Email address : | jeff@h-g-r.com |
| Position(s) sought : | Permanent position for graduates |
| Department(s) sought : | Food & Beverage Bar/Sommelier |

## Personal profile:

As a Bachelor of Business Administration and after obtaining first relevant international work experience within the hospitality industry, I am now ready to take on new responsibilities to further my professional career.
My key strengths include strong analytical and logical skills, an eye for detail, communication and interpersonal skills.
I enjoy working in a team and help others progress. At the same time I work well independently.
As a highly motivated and driven individual I strive on taking up challenges.

## Interests:

Travelling
Foreign Cultures
Photography
Sports

## Educational qualifications:

| | |
|---|---|
| Oct 99 - Feb 02 | Higher Diploma (Hotel Management)<br>Swiss Hotelmanagement School, SHL |

## Employment history:

| | |
|---|---|
| Mar 04 - Ongoing | Assistant Manager (Rooms Division/Food & Beverage)<br>Hotel Atlantic Kempinski Hamburg www.kempinski.com 5 star business hotel, part of Leading Hotels of the World 412 guest rooms, large function facilities, 3 food & beverage outlets<br>Optimization of bar procedures, reinforcing SOPs<br>Developing & implementing promotions<br>Responsible for day-to-day operations<br>Optimization and streamlining of housekeeping and laundry procedures<br>Implementation of new SOPs<br>Analyzing monthly reports for rooms division performance and sub departments |
| Mar 03 - Mar 04 | Management Trainee<br>Hospitality Graduate Recruitment www.h-g-r.com Leading company for placements within the Hospitality industry.<br>Traineeship covering all aspects of an online recruitment agency. |
| Mar 02 - Mar 03 | Management Trainee (Rooms Division)<br>Hyatt Regency Xian, China www.hyatt.com 5 star business hotel 404 guest rooms, 4 food & beverage outlets<br>Traineeship covering all rooms division departments on operational as well as supervisory level. |

## Training courses attended:

| | |
|---|---|
| Mar 02 - Ongoing | OpenOffice - IT Courses |
| May 01 - Jan 03 | Language Course - Chinese |

## References:

Hyatt Regency Xian
Patrick Sawiri, Phone: 86 22 2330 7654

Hospitality Graduate Recruitment
Jeff Ross, Phone: 41 41 370 99 88

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