Ising Model and Field Theory: Lattice Local Fields and Massive Scaling Limit

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Abstract

This thesis is devoted to the study of the local fields in the Ising model. The scaling limit of the critical Ising model is conjecturally described by Conformal Field Theory. The explicit predictions for the building blocks of the continuum theory (spin and energy density) have been rigorously established [HoSm13, CHI15]. We study how the field-theoretic description of these random fields extends beyond the critical regime of the model. Concretely, the thesis consists of two parts:

The first part studies the behaviour of lattice local fields in the critical Ising model. A lattice local field is a function of a finite number of spins at microscopic distances from a given point. We study one-point functions of these fields (in particular, their asymptotics under scaling limit and conformal invariance). Our analysis, based on discrete complex analysis methods, results in explicit computations which are of interest in applications (e.g. [HKV17]).

The second part considers the behaviour of the massive spin field. In the subcritical massive scaling limit regime first considered by Wu, McCoy, Tracy, and Barouch [WMTB76], we show that the correlations of the massive spin field in a bounded domain have a scaling limit. Furthermore, to this end we generalise the notions and methods of discrete complex analysis in the critical case to the massive regime, and give a new derivation of the formula for the two-point correlation in the full plane in terms of a Painlevé III transcendent.

Keywords: Ising Model, Statistical Mechanics, Probability Theory, Conformal Field Theory, Isomonodromy, Discrete Complex Analysis.

Résumé

Cette thèse est consacrée à l'étude des champs locaux du modèle d'Ising. Il est conjecturé que la limite d'échelle du modèle d'Ising critique est décrite par une théorie conforme des champs. Les prédictions explicites pour les éléments de base (le spin et la densité d'énergie) de la théorie continue ont été démontrées rigoureusement dans [HoSm13, CHI15]. On étudie comment la description du point de vue de la théorie des champs s'étend au-delà du comportement critique de ces champs aléatoires. Concrètement, cette thèse se compose de deux parties :

La première partie étudie le comportement des champs locaux de réseau. Un champ local au niveau du réseau est une fonction d'un nombre fini des spins à distances microscopiques d'un point donné. On étudie les fonctions de corrélation à 1 point de ces champs (en particulier, comment ils se comportent dans la limite d'échelle). L'analyse, basée sur des méthodes d'analyse complexe discrète, a pour résultat des calculs explicits d'intérêt pour applications (e.g. [HKV17]).

La seconde partie examine le comportement du champ de spin. Dans le régime de limite d'échelle sous-critique considéré pour la première fois par Wu, McCoy, Tracy et Barouch [WMTB76], on démontre que les corrélations du champ de spin massif dans un domaine à borne possèdent une limite d'échelle. Dans ce but, on généralise les notions et méthodes d'analyse complexe discrète du cas critique au régime massif, et on donne une nouvelle dérivation la formule pour la corrélation à 2 points dans le plan en termes de la fonction transcendante de Painlevé III.

Mots-clés : modèle d'Ising, mécanique statistique, théorie des probabilités, théorie conforme des champs, déformation isomonodromique, analyse complexe discrète.

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To Jackson and Liam

Contents

1 Introduction

1.1 The Ising Model

1.1.1 Definition

The *Ising model* was introduced as a mathematical model of ferromagnetism by Lenz [Len20] and has been applied to a wide range of fields (see e.g. [DIK13]) and has seen a tremendous mathematical development in the field of equilibrium statistical mechanics. Given a finite graph Λ and *inverse temperature* $\beta > 0$, one defines the probability measure $\mathbb{P}_{\beta,\Lambda}$ on a configuration of ± 1 *spins* σ_i on the sites (or vertices) i in Λ by:

$$\mathbb{P}_{\beta,\Lambda}[\sigma] \propto \exp\left[\beta \sum_{i \sim j} \sigma_i \sigma_j\right],$$

where the sum is over pairs of adjacent sites i, j. As the inverse temperature β increases, configurations with fewer disagreements between neighbouring σ_i, σ_j become more likely.

1.1.2 Phase Transition

In the case where the graph Λ is taken to be a finite subdomain of a lattice (such as $\mathbb{Z}^2$) the Ising model is an emblematic example of a *lattice model*. One is typically interested in a large scale behaviour which emerges as Λ becomes progressively bigger (*thermodynamic limit*). A typical question one may ask is:

- In a large $\Lambda \subset \mathbb{Z}^2$, how strongly do spins at far-apart sites i, j interact? In other words: how does the correlation $\mathbb{E}_{\beta,\Lambda}\left[\sigma_i \sigma_j\right]$ behave as $|i - j|$ grows?

The Ising model goes through a *phase transition* at $\beta_c = \frac{1}{2}\ln\left(1 + \sqrt{2}\right)$: at $\beta < \beta_c$, the model is disordered, and at $\beta > \beta_c$, there is a long range order. In fact, the Ising model on $\mathbb{Z}^2$ undergoes a continuous phase transition: as $|i - j|$ grows, the correlation $\mathbb{E}_{\beta,\mathbb{Z}^2}\left[\sigma_i \sigma_j\right]$ tends to a continuous function of β which is zero if $\beta \le \beta_c$, and strictly positive if $\beta > \beta_c$. Its explicit formula was given by Onsager and Yang [Yan52]. It was Onsager [Ons44] who had given a celebrated exact solution of the model in $\mathbb{Z}^2$ which allowed for the demonstration that the model admits a continuous phase transition at β_c. Previously, Peierls [Pei36] had shown the existence of a phase transition, and Kramers and Wannier [KrWa41] had determined β_c to be the self-dual point.

While the correlation does not decay to zero when $\beta > \beta_c$, the *truncated correlation* $\mathbb{E}_{\beta,\mathbb{Z}^2}\left[\sigma_i\sigma_j\right] - \mathbb{E}_{\beta,\mathbb{Z}^2}\left[\sigma_i\right]\mathbb{E}_{\beta,\mathbb{Z}^2}\left[\sigma_j\right]$ decays at an exponential rate ξ, known as the *correlation length*, for any $\beta \neq \beta_c$ [McWu73].

1.1.3 Relation to Other Models

The Ising model has a rich array of equivalent formulations and generalisations based on them. Many *spin models* can be considered to be natural generalisations of the model: among them are the *q-Potts*, $\mathbb{Z}_n$, and $O(N)$*-models*, all of which allow the set of possible spins to be bigger than $\{\pm 1\}$ (see e.g. [Mus10]). The family of q-Potts models, including the Ising ($q = 2$) case, also has a fundamental connection to models called the *Random Cluster* or *Fortuin-Kasteleyn (FK) models*; under the *Edwards-Sokal coupling* (see e.g. [Gri06]), one may sample a Potts model from the corresponding FK model, and vice versa.

Another generalisation is based on the representation of the two-dimensional Ising model in terms of loops. For example, with a + boundary condition, i.e. conditioning the boundary spins to be $+1$, the interfaces separating plus and minus spins are precisely loops made of dual edges, leading to the *low-temperature expansion*, where we describe a spin configuration (up to $\sigma \to -\sigma$ symmetry) in terms of them. This loop model was then generalised to the *loop* $O(n)$ *model*, where the Ising case corresponds to $n = 1$. The loop $O(n)$ models are themselves a form of resummation known as *high-temperature expansion* of $O(N)$ models with $N = n - 1$ (see e.g. [Smi06]).

In both cases, the Ising model serves as the central example of the above families. As we will see below, the theory is incredibly rich; the Ising model is arguably the best known example of *exactly solvable models* in two dimensions (see e.g. [Bax89]). In the recent years, significant progress was made possible towards understanding these models rigorously (e.g. [DGHMT16]), thanks to Smirnov's introduction of discrete holomorphic parafermionic observables for these families [Smi06]. The exact analysis possible for the Ising model has not only enriched our understanding of the model itself, but also shows potential for far-reaching generalisation in analysis of various statistical mechanics models.

1.2 Scaling Limit and Conformal Invariance

1.2.1 Scaling Limit and Universality

At the critical temperature β_c of the 2D Ising model, the one-point functions $\mathbb{E}_{\beta,\Lambda}\left[\sigma_{i,j}\right]$ decay to zero proportionally to $\mathrm{dist}(i,\partial\Lambda)^{-1/8}$ as Λ grows [CHI15]. In the thermodynamic limit, the correlation $\mathbb{E}_{\beta,\mathbb{Z}^2}\left[\sigma_i\sigma_j\right]$ decays proportionally to $|i-j|^{-1/4}$, and the correlation length is infinite, indicating there are fluctuations at all scales [CHI15]. To extract long-range information from the decaying correlations, it is natural to consider the (renormalised) *scaling limit*: while looking at progressively larger scales, offset the decay by multiplying a factor growing with the scale. Renormalisation group arguments (see e.g. [Hen99]) suggest the emergence of a continuous regime, which is scale invariant and is independent of the underlying square lattice structure, a phenomenon known as *universality*.

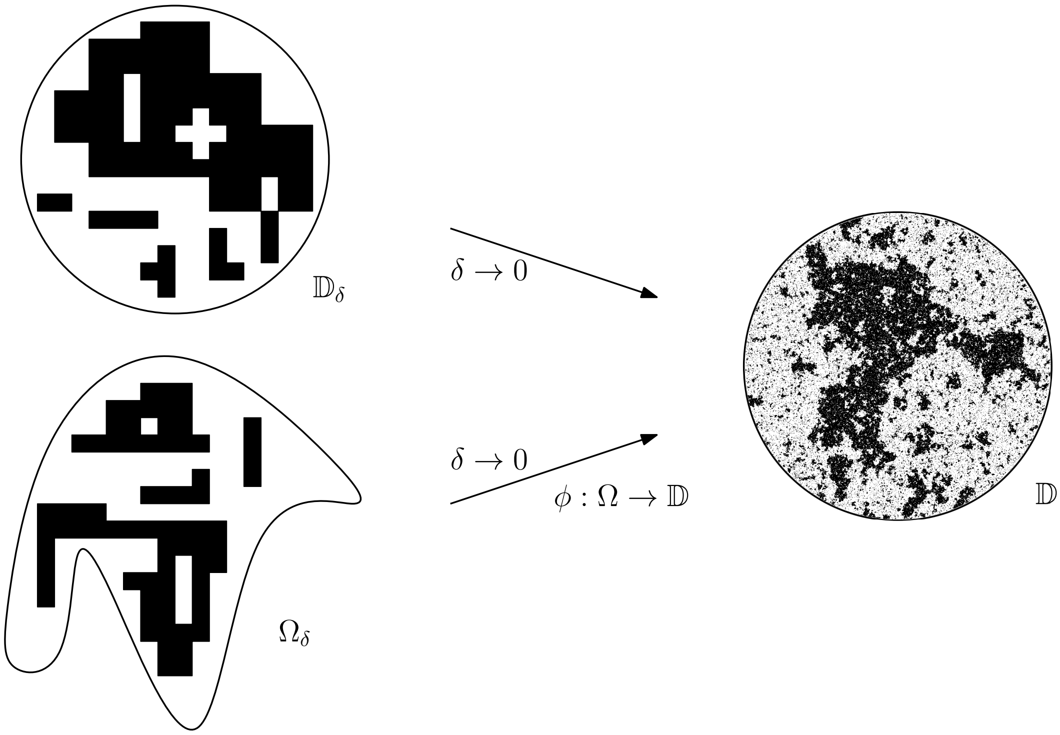

Figure 1.2.1 – Convergence of the spin interface to conformally invariant random curves.

1.2.2 Conformal Invariance

Concretely, instead of taking domains Λ approaching $\mathbb{Z}^2$, consider a domain $\Omega \subset \mathbb{C}$ and set $\Omega_\delta := \Omega \cap \delta\mathbb{Z}^2$ with $\delta \downarrow 0$, fixing appropriate boundary conditions (say, + boundary conditions). While Ω_δ still approximates $\delta\mathbb{Z}^2 \cong \mathbb{Z}^2$ as a graph as $\delta \downarrow 0$, it becomes an approximation of the continuous domain Ω by considering a sequence of sites $z^\delta \to z \in \Omega$ in Ω_δ. This setup is particularly useful to understand the fine structures of the critical model. The scaling limit is conjectured to be universal and connected to two fundamental theories: Schramm-Loewner

Evolutions (SLE), a family of conformally invariant random curves, and Conformal Field Theory, (CFT), a framework providing a conjectural description of the local fields of the model, to be elaborated below.

At the heart of both SLE and CFT is a symmetry called *conformal symmetry* or *conformal invariance*. Loosely speaking, this symmetry may be formulated as follows: for any conformal mapping (i.e. any angle-preserving diffeomorphism) $\phi : \Omega_1 \to \Omega_2$, we have

$$\phi\left(\text{Scaling limit of Ising Model on } \Omega_1\right) \overset{\text{(law)}}{=} \text{Scaling limit of Ising Model on } \Omega_2.$$

Conformal invariance is an especially powerful symmetry in two dimensions thanks to the size of the family of conformal transformations, as exemplified by the Riemann Mapping Theorem, which states that such a map always exists between two simply connected proper subdomains in the complex plane. As a result, the data of the critical Ising scaling limit on a simply connected domain different from $\mathbb{C}$ can be reduced to that on, e.g., the disc.

1.2.3 Schramm-Loewner Evolutions

A particularly natural framework to understand the conformally invariant scaling limit of the Ising model is that of Schramm's [Sch00] SLE curves (see Figure 1.2.1): at criticality, the macroscopic interfaces which separate plus and minus spins converge in law to continuous random curves which are conformally invariant (in law). The SLE curves and their variants form a one-parameter family indexed by a parameter $\kappa > 0$ corresponding to various universality classes. The universality class of the Ising model corresponds to $\kappa = 3$. The convergence to SLE_3 and variants has been shown in a number of cases, in particular for the interfaces generated by boundary condition changes [CDHKS14, HoKy10, Izy17] and for the so-called *full scaling limit* (the set of all loops arising with + boundary conditions) [BeHo16].

1.2.4 Conformal Field Theory

A manifestation of conformal invariance in the Ising model at a more local level is given by the fact that Conformal Field Theory (CFT) accurately predicts various local statistics. For example, if z^δ is an Ω_δ-approximation to $z \in \Omega$, Chelkak, Hongler, and Izyurov [CHI15] have shown that

$$\langle \sigma_z \rangle_\Omega := \delta^{-1/8} \mathbb{E}^+_{\beta_c, \Omega_\delta}[\sigma_{z^\delta}] \xrightarrow{\delta \downarrow 0} C \left| r'_\Omega(z) \right|^{-1/8},$$

where the boundary spins on $\partial\Omega_\delta$ are set to $+1$ and the *conformal radius* $r_\Omega(z)$ of Ω seen from $z = x + iy$ is defined as $r_\Omega(z) := |\varphi'_z(0)|$ where $\varphi_z : \mathbb{D} \to \Omega$ as a conformal map mapping $0 \in \mathbb{D}$ to $z \in \Omega$. The expectation decays with a factor of $\delta^{1/8}$, and both the shape of the domain Ω and the + boundary condition have a nontrivial effect as we renormalise; furthermore, their effect reflects the overall continuous conformal geometry of the domain independent of the square lattices structure, as suggested by universality.

Conformal Field Theory was initiated in the 1980s by physicists Belavin, Polyakov, and Zamolodchikov [BPZ84]. Based on these ideas, [BuGu93] provided precise predictions for the limit of $\delta^{-n/8}\mathbb{E}^{+}_{\beta_c,\Omega_\delta}[\sigma_{z_1^\delta}\cdots\sigma_{z_n^\delta}]$, later proven by [CHI15]. It is a theory which conjecturally describes conformally covariant *fields* in the continuum, such as the spin σ_z which may be understood as the continuous limit of the discrete spin σ_{z_δ}. The spin field σ_z is treated as an abstract object of the correlation operation which transforms well under conformal maps: for example, it scales as $\sigma \to a^{-1/8}\sigma$ when the underlying domain is dilated by a factor of a. First developed in terms of physical arguments, CFT has attracted diverse attempts to formalise it (e.g. [Seg88]); a surprising facet to a result of the above type is that it may be also used to make sense of the field σ_z itself in the continuum as a random distribution [CGN15]. In other words, proving that the predictions of a CFT hold true may be the key to rigorously establish its mathematical foundation.

Recently, a new approach to understand the relationship between the Ising model and CFT in the continuum has been proposed. The idea is to define *lattice local fields* and try to realise CFT directly on the lattice (cf. [GHP19, HKV17]). In this setup, a lattice local field $\phi(z^\delta)$ is a function $F(\sigma_{z^\delta+\delta v} : v \in V)$ of the spins applied to a finite neighbourhood $z^\delta + \delta v$ of a given point z^δ, and serves as the building blocks of the discrete field theory.

1.3 Massive Regimes

The vicinity of the critical point may also be studied by field theories, which are known as *massive* field theories (as opposed to *massless*, i.e. scale invariant, field theories). In the physical terminology, the mass of a field theory refers to the reciprocal of its correlation length ξ. Renormalisation group arguments suggest that there are two ways of perturbing the Ising model around criticality that leads to nontrivial perturbation of the Ising CFT (see e.g. [Mus10]).

1.3.1 Magnetic Perturbation

A *magnetic* perturbation introduces a bias between ± 1 spins by introducing a *magnetic field* parameter h in the measure $\exp\left[\beta_c \sum_{i\sim j}\sigma_i\sigma_j + h\sum_i \sigma_i\right]$. To yield a massive theory in the scaling limit $\delta \to 0$, h should be simultaneously scaled like $\delta^{15/8}$, and this setup has yielded some preliminary results [CGN16] based on the convergence of the critical spin. In general, the physical analysis of this regime has proved deeply complex and is still an active area of research (see e.g. [Zam87]).

1.3.2 Thermal Perturbation

In this thesis, we are interested in the perturbation by the other relevant perturbation of the Ising model: the *thermal* perturbation to the critical model, which corresponds to $\beta \neq \beta_c$. To keep the correlation length finite as $\delta \to 0$, we simultaneously take $\beta \to \beta_c$ so that $\xi \to \infty$.

Below β_c, the two-spin correlation $\mathbb{E}^+_{\beta,\mathbb{Z}^2}\left[\sigma_i\sigma_j\right]$ decays exponentially; in this *supercritical* regime, the correlation length ξ scales like $\left(\beta_c - \beta\right)^{-1}$, meaning that δ should be scaled like $\beta_c - \beta$ to get nontrivial correlation length within the scaling limit. In the *subcritical* regime ($\beta > \beta_c$), instead of the correlation, the truncated correlation $\mathbb{E}^+_{\beta,\mathbb{Z}^2}\left[\sigma_i\sigma_j\right] - \mathbb{E}^+_{\beta,\mathbb{Z}^2}\left[\sigma_i\right]\mathbb{E}^+_{\beta,\mathbb{Z}^2}\left[\sigma_j\right]$ decays exponentially with correlation length ξ scaling like $(\beta - \beta_c)^{-1}$ [McWu73]. In other words, one should scale the temperature $\beta(\delta)$ such that $\delta \propto |\beta(\delta) - \beta_c|$ in both cases.

1.3.3 Massive Ising Correlations

Unlike in the critical regime, the massive regime lacks conformal symmetry in the continuum, and as such a theory analogous to CFT. Nonetheless, the scaling limit in the full plane $\mathbb{C}_\delta \to \mathbb{C}$ has been studied by Wu, McCoy, Tracy, and Barouch [WMTB76] and subsequently Sato, Miwa, and Jimbo and others [SMJ77, KaKo80]. It has been shown [PaTr83] that the *scaling functions*

$$z_1, \ldots, z_n \mapsto \lim_{\delta \to 0} \delta^{-n/8} \mathbb{E}^+_{\beta(\delta),\mathbb{C}_\delta}[\sigma_{z_1^\delta}, \ldots, \sigma_{z_n^\delta}]$$

exist in both super- and subcritical regimes, and describe a highly nontrivial integrable system: for example, the two-point function, as a function of the distance $r := |z_1 - z_2|$, may be written exactly in terms of a *Painlevé transcendent of the third kind.* The Painlevé equations have a central role in the *isomonodromic deformation* of holomorphic functions (see e.g. [FIKN06]), or infinitesimal movement of their singularities and monodromies; indeed [SMJ77] derives the formula by considering a similar deformation for certain massive correlation functions. Their continuous setup, a general quantum field theoretic analysis known as *holonomic field theory*, is related to but independent of CFT.

1.4 Methods of Analysis

1.4.1 Field Theory and Combinatorics

The two-dimensional Ising model has been analysed in a variety of ways. The *Transfer matrix* method takes a view akin to a discretisation of a quantum mechanical setup, in that it considers the evolution of a one-dimensional state (for example, the plus boundary condition on the x-axis) under a suitable (possibly infinite dimensional) matrix, one lattice spacing at a time, with the y-axis playing the role of (pure imaginary) time. Taking the scaling limit is equivalent to finding the operator theoretical limit of the discrete transfer matrix as mesh size tends to zero.

For example, this is the view taken by [SMJ77], subsequently used in a proof of convergence by [PaTr83]; note that the method is inherently suited for rectangular domains, rendering modification of their methods into the case of general domains unfeasible.

As is common with other statistical mechanics models, combinatorial bijections have also been fruitful in the study of the Ising model. A well-known example is a mapping to the dimer model (perfect matching) through a modification of the underlying lattice, such as the *Fisher lattice* [Fis66]. The investigation of such bijections enjoys continued attention and progress to this day, being associated to other combinatorial representations of the model (e.g. Kac-Ward determinants) [CCK17] or involving also the six-vertex and the eight-vertex models [Dub11].

1.4.2 Fermionic Observables

Throughout this thesis, we use *lattice fermionic observables* to analyse the model, which combines combinatorics with the field-theoretic approach. These observables, which are deterministic functions, are correlations of a discrete field ψ called *fermion* with other fields (possibly including ψ itself) in the model, and converge to continuous functions which are correlations of the continuous fermion, described at criticality by CFT. The fermion is not a lattice local field in that its definition involves non-local quantities; it is designed precisely to exploit the Kramers-Wannier duality inherent in the model [KrWa41, KaCe71].

A concrete definition for fermionic observables is given combinatorially, for example [HoSm13]:

$$F_{\Omega_\delta}(a,z) := \frac{1}{\mathcal{Z}_{\Omega_\delta}} \sum_{\gamma \in \mathscr{C}(a,z)} \exp\left[-\frac{i}{2} W(\gamma)\right] \exp\left[-2\beta|\gamma|\right],$$

where the sum is over sets γ of edges of Ω_δ which form a path from a to z with prescribed orientation at a and closed loops, and $W(\gamma)$ is the total turning of the tangent vector of the path (see Figure 1.4.1). The normalisation factor $\mathcal{Z}_{\Omega_\delta}$ is the partition function of the aforementioned low-temperature expansion of the Ising model (loop $O(n=1)$ model), defined by sum over loops γ with weight $\exp\left[-2\beta|\gamma|\right]$. F_{Ω_δ} is a natural modification of the low-temperature expansion obtained by adding a path from a to z and complexify the weights. As z approaches a, or both a, z approach the boundary, the sum reduces to recognisable combinatorial representations of spin or energy density (product of two adjacent spins) correlations.

1.4.3 Criticality and Discrete Holomorphicity

The feature that makes fermionic observables particularly suitable for analysing the scaling limit is that they are *discrete holomorphic* at the critical point: they satisfy discrete analogues of the Cauchy-Riemann equations. Discrete holomorphic functions, being discrete harmonic functions, are suitable for analysis using probabilistic methods; this follows from the connection between discrete harmonic functions and simple random walk. There is a rich theory

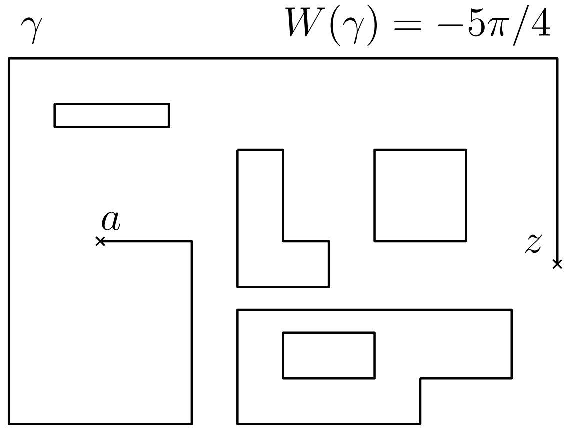

Figure 1.4.1 – An example loop-path configuration γ.

of discrete complex analysis mirroring that of the continuous analysis developed based on assumptions on the behaviour of random walk on a general lattice [Che16].

While methods involving holomorphic fermionic observables were among the many early ones applied to the Ising model [SML64], there was a groundbreaking progress made in the last 15 years with Smirnov's identification of the notion of *s-holomorphicity* [Smi10]. The observable F_{Ω_δ} satisfies this strong notion of discrete holomorphicity, which allows for an effective *integration of the square* of the observable: concretely, one may define the line integral $\mathrm{Re} \int F_{\Omega_\delta}^2 dz$. The existence of such integral is remarkable, since the square $F_{\Omega_\delta}^2$ is not discrete holomorphic. S-holomorphicity guarantees just enough regularity for the real part of the integral to be well-defined and *approximately* harmonic. It has consequences for universality in the aforementioned sense as well, being suited to a large class of lattices called *isoradial* lattices (where each face is inscribed in circles of identical radius) [ChSm11, ChSm12], and even larger families called *s-embeddings* [Che18].

A notable consequence of integrability of the square is that it allows for a unique characterisation of the fermionic observables in general simply connected domains through a boundary value problem. One sees from the complex weight in the definition of F_{Ω_δ} that it has a prescribed phase at any boundary point, which can be normalised as $\nu_{out}^{-1/2}$ where ν_{out} is the unit complex number in the direction exiting the domain. This condition translates to a constant Dirichlet condition for the (approximate) discrete Laplace equation which $\mathrm{Re} \int F_{\Omega_\delta}^2 dz$ solves.

Given s-holomorphicity and integrability of the square, the strategy for the analysis of the scaling limit of the Ising model is a natural one in the field-theoretic viewpoint: one constructs a discrete fermion correlation related to the desired physical quantity, then shows that it converges to a continuous 'correlation' uniquely identified by an analogous continuous boundary value problem, suggesting some sort of convergence of the discrete fermion to a continuous field. S-holomorphicity, the integral of the square in particular, is used to show precompactness of the discrete correlations (in a suitable function space). Convergence then follows once one has uniquely identified the limit.

Discrete complex analysis is useful then to analyse the model in the scaling limit, but it also provides the appropriate setting to study CFT symmetries at the lattice level. In general, the symmetries of a CFT include actions of the Virasoro algebra on the space of local fields. Surprisingly, one may implement an analogous representation of the Virasoro algebra at the lattice level as well [HKV17]; it relies heavily on the discrete holomorphicity of the fermion and construction of a basis of discrete holomorphic functions.

1.4.4 Massive Discrete Complex Analysis

As mentioned above, the thermal massive regime corresponds to perturbing the inverse temperature β away from the critical point β_c. The discrete notion of s-holomorphicity survives in a perturbed form [BeDC12, HKZ15], facilitating analysis of the thermal massive regime: instead of converging to continuous holomorphic functions satisfying the Cauchy-Riemann PDE ($\partial_{\bar{z}} f := \frac{1}{2}(\partial_x + i\partial_y) f = 0$), a *massive s-holomorphic* function should converge to continuous functions satisfying the Vekua equation $\partial_{\bar{z}} f = m\bar{f}$, where m is a real constant. Many of the useful features in discrete complex analysis of s-holomorphic objects, such as the integral of the square and harmonicity, survive in a perturbed form: the discrete *massive harmonic* functions should converge to functions satisfying $\Delta f = m^2 f$.

1.5 Objectives

The precise correspondence between discrete quantities and continuous fields is a matter of considerable interest in the analysis of the Ising model in terms of Conformal Field Theory. As mentioned in preceding sections, many foundational conjectures on the description of the scaling limit of the critical Ising model in terms of Conformal Field Theory have been verified explicitly; the pioneering results of Smirnov and Hongler [HoSm13], and Chelkak, Hongler, and Izyurov [CHI15] have yielded convergence of the most basic field correlations, those of the energy density and the spin. Smirnov's introduction of s-holomorphic fermions was instrumental in both formulating the physics of the model in terms of discrete complex analysis and the subsequent analysis.

Armed by these results and their methods, we go on to give a more detailed field-theoretical description of the model. The building blocks of the discrete field theory are the random spin configurations, and any local functions of the configurations constitute a meaningful quantity in the discrete field theory. Given the fact that the continuous theory primarily revolves around the two aforementioned local fields [DMS97], how would a general local statistic fit into the continuous CFT? Concretely, how would it scale and what would the rescaled limit be? The second chapter of this thesis studies this question.

In the full plane, we know that spin correlations converge to massive limit (see e.g. [Pal07]); does the bounded domain field theory generalise to the massive setup as well? The third chapter of this thesis studies this question.

All in all, the questions treated in this thesis arise naturally in the discrete Ising field theory but have not been treated before. In the process of answering them, we situate the analysis in the context of precursors and identify the relevant analytical ingredients and objects.

1.6 Main Theorems

In this section, we outline the main theorems of this thesis. For technical reasons, we consider the Ising model on the *dual rotated square lattice*: consider the *faces* $\mathbb{C}_\delta := (1+i)\delta\mathbb{Z}^2$. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and define $\Omega_\delta := \mathbb{C}_\delta \cap \Omega$. We consider the model at $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$.

Theorem A (Theorems 2.1.1, 2.1.2; page 24)**.** *Assume* $0 \in \Omega$. *Let* $\mathscr{B} := \{e_1, \dots, e_n\}$ *be a set of edges in* $\mathbb{C}_{\delta=1}$. *Then* $\delta e_1, \dots, \delta e_n$ *are edges in* Ω_δ *for small enough* $\delta > 0$. *Define the* energy density *at an edge* $e = \{f_1, f_2\} \subset (\Omega_\delta)$ *by* $\epsilon(e) := \frac{\sqrt{2}}{2} - \sigma_{f_1}\sigma_{f_2}$ *and its correlation* $\epsilon(\delta\mathscr{B}) := \prod_j \epsilon(\delta e_j)$. *The* spin-weighted energy density $\epsilon_{[0]}(e)$ *and its correlation* $\epsilon_{[0]}(\delta\mathscr{B}) := \prod_j \epsilon_{[0]}(\delta e_j)$ *are defined analogously, replacing* $\frac{\sqrt{2}}{2}$ *by the spin-weighted volume limit* (2.1.1).

Then there exist explicit, real-valued constants $\mathfrak{F}^{\mathscr{B}}, \mathfrak{E}^{\mathscr{B}}, \mathfrak{F}^{\mathscr{B}}_{[0]}, \mathfrak{E}^{\mathscr{B}}_{x,[0]}, \mathfrak{E}^{\mathscr{B}}_{y,[0]}$ *such that*

$$\mathbb{E}^+_{\beta_c,\Omega_\delta}[\epsilon(\delta\mathscr{B})] = \mathfrak{F}^{\mathscr{B}} + \delta r_\Omega^{-1}(0)\mathfrak{E}^{\mathscr{B}} + o(\delta);$$
$$\frac{\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_0\epsilon_{[0]}(\delta\mathscr{B})]}{\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_0]} = \mathfrak{F}^{\mathscr{B}}_{[0]} + \delta\left(\partial_x \log r_\Omega(0)\mathfrak{E}^{\mathscr{B}}_{x,[0]} + \partial_y \log r_\Omega(0)\mathfrak{E}^{\mathscr{B}}_{y,[0]}\right) + o(\delta),$$

in the scaling limit $\delta \downarrow 0$, *where* r_Ω *is the conformal radius (defined below* (1.2.4)*).*

Theorem A yields an analogous scaling result on the probability of a specific *spin pattern* occuring at a point $z \in \Omega$: see Corollary 2.1.1.

Outline of Proof. We outline the different steps in the proof.

Combinatorial Representation. As noted above, we use discrete complex analytic methods to analyse the quantities $\mathbb{E}^+_{\beta_c,\Omega_\delta}[\epsilon(\delta\mathscr{B})]$, $\frac{\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_0\epsilon(\delta\mathscr{B})]}{\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_0]}$. Recall that the notion of discrete holomorphicity we use is the *s-holomorphicity* (Definition 2.2.1). The connection to discrete s-holomorphic functions is made in Propositions 2.4.9 and 2.4.10:

$$\mathbb{E}^+_{\beta_c,\Omega_\delta}[\epsilon(\delta\mathscr{B})] = (-2)^n \mathrm{Pf}\, \mathbf{F}^{[\{e_k\}]}_{\Omega_\delta}; \quad \frac{\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_0\epsilon_{[0]}(\delta\mathscr{B})]}{\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_0]} = (-2)^n \mathrm{Pf}\, \mathbf{F}^{[\{e_k\}]}_{[\Omega_\delta,0]},$$

where $\mathbf{F}^{[\{e_k\}]}_{\Omega_\delta}, \mathbf{F}^{[\{e_k\}]}_{[\Omega_\delta,0]}$ are $2n \times 2n$ antisymmetric matrices whose entries are values of s-holomorphic *two-point fermions* $F_{\Omega_\delta}, F_{[\Omega_\delta,0]}$ (Definitions 2.3.2, 2.3.3) evaluated at the edges in $\mathscr{B}$. This decomposition of the energy multipoint function is based on the combinatorial definition of the fermions as in Figure 1.4.1 and requires a decomposition of the $2n$*-point fermion*, which are discrete s-holomorphic functions identified by their singularities and a boundary condition, in terms of the two-point fermions. In the case of $\mathbb{E}^{+}_{\beta_c,\Omega_\delta}[\epsilon(\delta\mathscr{B})]$, the formula is essentially identical to [Hon10, Proposition 85], but we have to verify that the formula applies to adjacent correlations as well: this is based on a refinement (2.2.1) of the definition of s-holomorphicity on *corners*.

Boundary Value Problem. It suffices to study the asymptotics of the two-point observables, concretely the complexified ones $H^{a}_{\Omega_\delta}, H^{a}_{[\Omega_\delta,0]}$ (Definitions 2.3.4, 2.3.5). These are discrete s-holomorphic functions in the variable z defined in terms of contours as in Figure 1.4.1, with defining properties:

- Singularity: s-holomorphicity fails when z approaches a, with explicit *discrete residue* (Lemma 2.4.3);
- Monodromy: in the case of the *spin-fermion* observable $\mathbb{H}^{a}_{[\Omega_\delta,0]}$, the natural domain of definition is the *double cover* of Ω_δ ramified at 0, i.e. the covering space of $\Omega_\delta \setminus \{0\}$ with two sheets, and the observable might have a singular behaviour (but *not* an s-holomorphic pole) at 0 (Lemma 2.2.4);
- Boundary Condition: when z is taken on a boundary outer edge of the direction ν_{out}, $\sqrt{\nu_{out}}H \in \mathbb{R}$. (Lemma 2.4.4).

These properties uniquely determine the observables by an argument based on the *integral of the square* (Propositions 2.5.1-2.5.1), and their natural continuous counterparts similarly constitute a boundary value problem (Remark 2.5.3).

Bulk Convergence. We show that the *renormalised* discrete two-point observables converge as $\delta \to 0$ to continuous functions in the sense that the discrete functions, suitably interpolated to the continuous domain, converge to continuous functions, uniformly in compact subsets away from a and the monodromy at 0, if present. The case of $\delta^{-1}H^{a}_{\Omega_\delta}$ is a known result [Hon10, Theorem 91], so we have to treat the observables $H^{a}_{[\Omega_\delta,0]}$ with monodromy. In our scheme, since we study local correlations near 0, a is in fact taken to be a lattice point δa at a microscopic distance from 0.

We use a precompactness argument: the functions $\{\delta^{-1/2}H^{\delta a}_{[\Omega_\delta,0]}\}_{\delta>0}$, are precompact and thus admit a subsequential limit, but then we show that there can be at most one continuous limit, proving convergence.

By the Arzelà-Ascoli theorem, it suffices to show uniform boundedness and equicontinuity on any compact subset. The aforementioned integration of the square (Proposition 2.5.3) is useful in this setup. The (approximately) discrete holomorphic functions $\delta^{-1}\left(H^{\delta a}_{[\Omega_\delta,0]}\right)^2$ are derivatives of the (approximately) discrete harmonic functions $\mathbb{I}\left[\delta^{-1/2}H^{\delta a}_{[\Omega_\delta,0]}\right] := \operatorname{Re}\int \delta^{-1}\left(H^{\delta a}_{[\Omega_\delta,0]}\right)^2 dz$. As long as $\mathbb{I}\left[\delta^{-1/2}H^{\delta a}_{[\Omega_\delta,0]}\right]$ is locally uniformly bounded, the derivatives $\delta^{-1}\left(H^{\delta a}_{[\Omega_\delta,0]}\right)^2$ are locally bounded and equicontinuous. To show that the integral $\mathbb{I}$ is bounded, we use a *full-plane* observable $H^{\delta a}_{[\mathbb{C}_\delta,0]}$, constructed by an infinite volume limit procedure in the discrete (Theorem 2.3.2). This observable has the same discrete singularity as the domain observable, and can be explicitly shown to scale at power $\delta^{-1/2}$ with a limit of the form $\frac{C_\alpha}{\sqrt{z}}$ (Lemma 2.5.4).

We finally argue that the integral of the square of the domain observable $H^{\delta a}_{[\Omega_\delta,0]}$ cannot scale differently, yielding $\delta^{-1/2}$ scaling. Comparison with the full-plane observable uniquely identifies the limit as well: we expect the continuous limit of $\delta^{-1/2} H^{\delta a}_{[\Omega_\delta,0]}$ to have a singularity and a monodromy at 0 behaving like $\frac{C_\alpha}{\sqrt{z}}$, since that is the limit of the full-plane observable sharing the same discrete singularity. Such comparison arguments are done by cancelling out the discrete singularity using the full-plane observable and showing that the remainder needs to be well-behaved (Proposition 2.5.7); an added complication is that we need to control potential singular behaviour directly near the monodromy, which is done by first estimating the value of the function near the monodromy (Lemma 2.5.6).

The boundary condition $\sqrt{\nu_{out}} H \in \mathbb{R}$ of the continuous limit is verified in terms of its square integral: this boundary condition is equivalent to having a square integral (the real part thereof) which is constant and has positive outer normal derivative on the boundary. While we use familiar arguments based on uniform bounds (Beurling estimate) on discrete harmonic functions here, we note that this part poses considerable challenge in the massive case in the next chapter.

Analysis near the Singularity. In view of the statement of the Theorem A, we would like a convergence result of the following type (Theorem 2.5.2): with both $\delta a, \delta b$ scaling to zero,

$$H^{\delta a}_{[\Omega_\delta,0]}(\delta z) = F_{a,b} + \delta \cdot E_{a,b} + o(\delta),$$

where $F_{a,b}, E_{a,b}$ are explicitly identifiable constants, the latter in terms of the conformal map φ_z.

The bulk convergence gives that the discrete fermion $\delta^{-1} H^{\delta a}_{[\Omega_\delta,0]}$ is uniformly close on any small but macroscopic domain around 0 to its continuous limit $h^{da}_{[\Omega,0]}$, which indeed has a series expansion near z (Definition 2.5.4, Theorem 2.5.2) which is

$$C_\alpha^{-1} h^{da}_{[\Omega,0]} = \frac{1}{\sqrt{z}} + 2\mathscr{A}_\Omega \sqrt{z} + o(|z|^{1/2}),$$

where the coefficient $\mathscr{A}_\Omega$ is determined by φ_z (Remark 2.5.4).

But we are not allowed yet to compare the discrete observable to the continuous at the point $\delta\beta$ scaling to zero. We need to find a way to say that the discrete fermion mimics the above expansion at microscopic scales near 0. The strategy for the proof of Theorem 2.5.2 is to model the series expansion above using discrete counterparts. The singularity $C_\alpha/\sqrt{z}$ already had to be cancelled out with the full-plane fermion, and the next is the square root behaviour. We construct a discrete s-holomorphic square root $G_{[\mathbb{C}_\delta,0]}$ (Definition 2.3.7) for this purpose using a discrete integration procedure.

Then we subtract a suitable multiple of $G_{[\mathbb{C}_\delta,0]}$ from the discrete fermion to get $o(\delta)$ behaviour near 0; by bulk convergence, the difference can be controlled in any small but macroscopic circle. While a holomorphic (thus harmonic) function which is small on the boundary should be small everywhere, there is a potential problem here because of the monodromy at 0. To use uniform bounds on discrete harmonic functions (Beurling estimate) we additionally use an inherent symmetry (Remark 2.3.4) in the discrete fermion to produce a discrete harmonic function in the slit plane and vanishing on the slit.

Explicit Formulae and Spin Patterns. As mentioned above, we give explicit formulae for all coefficients appearing in the statements of Theorem A. To this end, we introduce a new explicit formula for the *discrete harmonic measure of the tip of the slit plane* (Proposition 2.7.1, Figure 2.3.2): the probability of a random walk on $\delta(1+i)\mathbb{Z}^2$ hitting 0 before any point on the negative real line. All of our auxiliary functions (full-plane observables and discrete square root) may be expressed in terms of this harmonic function (Proposition 2.3.9, Corollary 2.3.11, Definition 2.3.7). The formula is based on Fourier analysis, and has been instrumental in fixing the scaling factor $\vartheta(\delta)$ (Section 2.5.4) implicitly used in [CHI15].

Spin patterns are treated by noting that the correlations of the type treated in this chapter form a basis for any local function of the spin. We give an explicit matrix to translate correlations into probabilities of specific patterns in Section 2.6.3.

□

Theorem B (Theorem 3.1.1; page 90)**.** *Assume the boundary of* Ω *is smooth. Let* $\beta(\delta) = \beta_c - \frac{m\delta}{2}$ *for a fixed* $m < 0$*. Then for* $a_1, \ldots, a_n \in \Omega$*, there is a continuous function* $\langle a_1, \ldots, a_n \rangle^+_{\Omega,m}$ *such that*

$$\delta^{-n/8}\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}] \xrightarrow{\delta\downarrow 0} \langle a_1, \ldots, a_n \rangle^+_{\Omega,m},$$

where $a_1^\delta, \ldots a_n^\delta \in \Omega_\delta$ *are the closest faces to the respective points. In other words, the subcritical massive spin correlations converge to continuous scaling functions when properly normalised. The scaling functions* $< \cdot >_{\Omega,m}$ *in Theorem B are identified by their logarithmic derivatives and asymptotics near the diagonal and* $\partial\Omega$*; for example the one-point scaling functions approach their critical counterparts near* $\partial\Omega$*.*

This theorem allows us to give a new derivation of the formula [WMTB76] of the spin two-point function in the full plane in terms of Painlevé III transcendent (Corollary 3.1.2).

Outline of Proof. We again outline the different steps in the proof.

Logarithmic Derivatives. As in the previous chapter, we first need to connect physical quantities to discrete complex analytic objects. Concretely, if $a_1, \ldots, a_n \in \Omega$ are identified with the locations $a_1^\delta, \ldots, a_n^\delta$ of Ising spins, we will show the convergence of:

$$\frac{1}{\delta}\left(\frac{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta+2\delta}\cdots\sigma_{a_n^\delta}]}{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}]}-1\right); \quad \frac{1}{\delta}\left(\frac{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta+2i\delta}\cdots\sigma_{a_n^\delta}]}{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}]}-1\right),$$

i.e. the discrete *logarithmic derivatives* of the spin correlation $\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}]$ in the x, y directions. Note that above convergence, by a discrete integration procedure, implies the convergence of

$$\frac{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}]}{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{b_1^\delta}\cdots\sigma_{b_n^\delta}]},$$

for any other $b_1, \ldots, b_n \in \Omega$. The connection to discrete fermions is made in Proposition 3.2.3, which says that

$$F_{[\Omega_\delta,a_1,\ldots,a_n]}(a_1+\frac{3\delta}{2}) = \frac{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta+2\delta}\cdots\sigma_{a_n^\delta}]}{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}]},$$

where $F_{[\Omega_\delta,a_1,\ldots,a_n]}$ is a massive generalisation (Definition 3.2.1) of the observable with monodromy defined in the previous chapter; in this case, it is defined on the double cover ramified at points $a_1, \ldots, a_n$. Our desired convergence result (Theorem 3.3.5) is of the form

$$F_{[\Omega_\delta,a_1,\ldots,a_n]}(a_1+\frac{3\delta}{2}) = 1 + 2\mathscr{A}_\Omega(a_1,\ldots,a_n)\delta + o(\delta),$$

where $\mathscr{A}_\Omega(a_1,\ldots,a_n)$, the logarithmic derivative, is a quantity uniquely determined by the continuous domain Ω and the positions of $a_1, \ldots, a_n$ (Definition 3.2.17).

Massive Holomorphicity. To use a precompactness argument, we need analytic means to both analyse and identify the discrete and continuous fermions. This is given by the notion of *massive s-holomorphicity*, which the massive fermions $F_{[\Omega_\delta,a_1,\ldots,a_n]}$ satisfy (Proposition 3.2.2). It is a discrete version of the continuous equation $\partial_{\bar z} f = m\bar f$ (Proposition 3.5.1, Remark 3.5.2) which we call *massive holomorphicity*.

Then we prove that many of the helpful notions from the massless analysis also exist in this regime, namely:

- Integration of the square: the real part of the integral of the square $H_{[\Omega_\delta,a_1,\ldots,a_n]} := \mathbb{I}\left[F^2_{[\Omega_\delta,a_1,\ldots,a_n]}\right]$ exists (Proposition 3.2.4). Instead of being approximately harmonic, it satisfies a discretisation of the continuous notion $\Delta\mathbb{H}_{[\Omega_\delta,a_1,\ldots,a_n]} = 4m\left|\nabla H_{[\Omega_\delta,a_1,\ldots,a_n]}\right|^2 = 4m\left|F_{[\Omega_\delta,a_1,\ldots,a_n]}\right|^2$ (Proposition 3.2.7).
- Identification via a boundary value problem: there is a boundary value problem for a continuous massive holomorphic functions that admits at most one solution (Proposition 3.2.15).

Massive holomorphic functions belong to a more general class of functions called *generalised analytic functions*, and we use some parts of the continuous theory, such as the fact that they may be expanded in a series resembling the power series for holomorphic functions (Corollary 3.2.13). In addition, in both discrete and continuous senses, massive holomorphic functions are *massive harmonic*, i.e. solves a version of the equation $\Delta f = m^2 f$. Massive harmonic functions share many regularity properties with harmonic functions; in particular, L^2 bounded massive harmonic functions are locally bounded and equicontinuous (Proposition 3.5.4).

Convergence. Instead of the uniform boundedness of the integral of the square $\mathbb{I}\left[\delta^{-1}F^2_{[\Omega_\delta,a_1,\ldots,a_n]}\right]$ as in the previous chapter, we rely on an L^2 estimate and massive harmonicity of the renormalised fermion $\delta^{-1/2}F_{[\Omega_\delta,a_1,\ldots,a_n]}$ for the bulk convergence.

- Precompactness: since $m<0$, the integral of the square is nearly superharmonic with approximate laplacian $4m\left|F_{[\Omega_\delta,a_1,\ldots,a_n]}\right|^2<0$. Superharmonicity fails at the singularity a_1, where $F_{[\Omega_\delta,a_1,\ldots,a_n]}$ is singular but has very explicit bounds (Proposition 3.2.3). We may bound the singular laplacian at a_1, and this gives an L^2 bound for the fermion (Proposition 3.3.1).
- Identification of the limit: in our subcritical massive regime where the integral of the square $H_{[\Omega_\delta,a_1,\ldots,a_n]}$ is approximately superharmonic, many of the two-sided estimates from the massless case, where the integral is approximately harmonic, become only one-sided inequalities. As mentioned before, the boundary condition $\sqrt{\nu_{out}}F_{[\Omega_\delta,a_1,\ldots,a_n]}\in\mathbb{R}$ translates to the integral of the square being constant and of positive outer normal on the boundary (Remark 3.2.16). Even given boundedness in the bulk, this condition might not be preserved for a continuous bulk limit of general discrete superharmonic functions satisfying it.

  Our strategy in the proof of unique identification is to show uniform boundedness of $\delta^{-1/2}F_{[\Omega_\delta,a_1,\ldots,a_n]}$ away from the singularities at $a_1,\ldots,a_n$, and thus uniform convergence of $\delta^{-1}H_{[\Omega_\delta,a_1,\ldots,a_n]}$ near the boundary. Since $\delta^{-1/2}F_{[\Omega_\delta,a_1,\ldots,a_n]}$ is bounded in the bulk, it suffices to show that it is bounded at the boundary by massive harmonicity. We have that the superharmonic $\delta^{-1}H_{[\Omega_\delta,a_1,\ldots,a_n]}$ is bounded below by its harmonic minorant (a constant multiple of the domain Green's function), whose values are $O(\delta)$ on points adjacent to the boundary (Lemma 3.5.3). But by positivity of outer derivatives, this is the only derivative bound one needs at the boundary; thus the derivative $\delta^{-1/2}F_{[\Omega_\delta,a_1,\ldots,a_n]}$ is bounded at the boundary (Proposition 3.3.3).
- Analysis near the Singularity: near the singularity, we rely on short-scale behaviour of massive harmonic functions, which asymptotically approaches that of harmonic functions. For example, the massive counterpart of the slit plane discrete harmonic measure of the previous chapter is the hitting probability of a random walk extinguished at each step with a given probability (Proposition 3.2.11); full-plane observable converges to exponentially weighted $\frac{e^{2m|z|}}{\sqrt{z}}$ (Lemma 3.3.2). Therefore, analysis near the singularity proceeds as in the previous chapter, with discrete square root constructed using a discrete massive integration procedure (Proposition 3.3.4).

Integration of the Coefficients. Given convergence of the logarithmic derivatives, and therefore convergence of the ratios $\frac{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}]}{\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{b_1^\delta}\cdots\sigma_{b_n^\delta}]}$, we simply need to show the convergence of *one* renormalised correlation $\delta^{-n/8}\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}]$ for any domain Ω and n.

The convergence of the two-point fermion (3.3.7) in fact gives additional information, in that $\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\sigma_{a_2^\delta}] \sim \mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_{a_1^\delta}\sigma_{a_2^\delta}]$ (3.1.1) as a_1, a_2 approach themselves in the bulk. Given the convergence result at β_c of [CHI15], this yields convergence of the two-point functions.

For the convergence of other n, we use decorrelation near the boundary: as $a_j \to \partial\Omega$, the correlation of the spins is primarily determined by interaction with the boundary, and not so much the interaction within themselves. In fact, by [CHI15, (1.3)], $\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_{a_1^\delta}\sigma_{a_2^\delta}] \sim \mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_{a_1^\delta}]\mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_{a_2^\delta}]$ as a_1, a_2 approach the boundary away from each other. To use this information, we bound the difference of logarithmic derivatives $\mathscr{A}_\Omega(a_1, a_2)$ in the massive and massless cases (Lemma 3.4.9) to show that (3.1.2) one may make $a_1, a_2 \to \partial\Omega$ such that we still have $\mathbb{E}^+_{\beta(\delta),\Omega_\delta}[\sigma_{a_1^\delta}\sigma_{a_2^\delta}] \sim \mathbb{E}^+_{\beta_c,\Omega_\delta}[\sigma_{a_1^\delta}\sigma_{a_2^\delta}]$ while at the same time there is a decorrelation of the two-point functions. This yields the fact that massive one-point functions approach the critical one-point functions near the boundary (3.1.3), and therefore convergence of the massive one-point functions. For $n \geq 3$, we show that the n-point correlation near the boundary should decorrelate into products of n one-point functions, which results in a uniquely identified n-point function scaling at n-th power of the 1-point function.

Painlevé Transcendent. The formula [WMTB76] for the two-point spin correlation in the full plane in terms of the Painlevé III transcendent has been obtained by an isomonodromic deformation procedure [SMJ77]. In Section 3.4.2, we follow this strategy (as presented in [KaKo80]) with our fermionic observables. The idea is simple: the change of the observable under infinitesimal movements of the points a_1, a_2 may be expressed in terms of the observables and their derivatives, since they form a basis of the space of solutions to the massive boundary value problem (Proposition 3.2.15). To justify differentiation in a_j, we verify that the continuous observables are indeed differentiable in the positions of the monodromies (Proposition 3.4.10).

□

2 Local Correlations and Conformal Invariance

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Contribution of the author of the thesis

Drafting of Sections 2 and later; strategies for the bulk and near-singularity convergence in the scaling limit; explicit formula for the harmonic measure and the discrete square root; setup for restatement in terms of spin patterns.

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Abstract.

We study the 2-dimensional Ising model at critical temperature on a simply connected subset Ω_δ of the square grid $\delta\mathbb{Z}^2$. The scaling limit of the critical Ising model is conjectured to be described by Conformal Field Theory; in particular, there is expected to be a precise correspondence between local lattice fields of the Ising model and the local fields of Conformal Field Theory.

Towards the proof of this correspondence, we analyze arbitrary spin pattern probabilities (probabilities of finite spin configurations occurring at the origin), explicitly obtain their infinite-volume limits, and prove their conformal covariance at the first (non-trivial) order. We formulate these probabilities in terms of discrete fermionic observables, enabling the study of their scaling limits. This generalizes results of [Hon10, HoSm13] and [CHI15] to one-point functions of any local spin correlations.

We introduce a collection of tools which allow one to exactly and explicitly translate any spin pattern probability (and hence any lattice local field correlation) in terms of discrete complex analysis quantities. The proof requires working with multipoint lattice spinors with monodromy (including construction of explicit formulae in the full plane), and refined analysis near their source points to prove convergence to the appropriate continuous conformally covariant functions.

2.1 Introduction

The 2D Ising model is one of the most studied models of statistical mechanics. In its simplest formulation it consists of a random assignment of ± 1 spins σ_x to the faces of (subgraphs of) the square grid $\mathbb{Z}^2$; the spins tend to align with their neighbors; the probability of a configuration is proportional to $e^{-\beta H(\sigma)}$ where the energy $H(\sigma) = -\sum_{i\sim j}\sigma_i\sigma_j$ sums over pairs of adjacent faces; alignment strength is controlled by the parameter $\beta > 0$, usually identified with the inverse temperature.

The 2D Ising model has found applications in many areas of science, from description of magnets to ecology and image processing. Due to its simplicity and emergent features, it is interesting both as a discrete probability and statistical field theory model. Of particular physical interest is the phase transition at the critical point β_c: for $\beta < \beta_c$ the system is disordered at large scales while for $\beta > \beta_c$ a long-range ferromagnetic order arises. Classically, the phase transition can be described in terms of the infinite-volume limit: in the disordered phase $\beta < \beta_c$ there is a unique Gibbs measure, while in the ordered phase $\beta > \beta_c$ infinite-volume measures are convex combinations of two extremal measures. It has a continuous phase transition: only one Gibbs measure exists at $\beta = \beta_c$.

Critical lattice models at continuous phase transition points are widely expected to have universal scaling limits (independent of the choice of lattice and other details). In 2D, such scaling limits are expected to exhibit conformal symmetry. This can be loosely formulated as follows: for a conformal mapping $\varphi : \Omega \to \tilde{\Omega}$,

$$\varphi\left(\text{scaling limit on } \Omega\right) = \text{scaling limit on } \tilde{\Omega}.$$

There are two main tools used to describe the scaling limits of planar lattice models: curves and fields. Schramm-Loewner Evolution (SLE) curves naturally arise in conformally invariant setups: for the Ising model, they describe the scaling limit of interfaces between opposite spins ([CDHKS14, BeHo16]). The fields on a discrete level, such as the ± 1-valued *spin field* σ_i, can be described by Conformal Field Theory (CFT): their correlations, in principle, are conjecturally described using representation-theoretic methods. Such conjectures have been proved for a number of natural fields ([Hon10, HoSm13, CHI15]); the present paper is part of this program.

What makes it possible to mathematically analyze the 2D Ising model with great precision is its exactly solvable structure, first revealed by Onsager [Ons44]. The exact solvability can be formulated in many different ways; in recent years, the formulation in terms of discrete complex analysis has emerged as one the most powerful ways to understand the scaling limit of the model rigorously. In particular, the model's conformal symmetry becomes much more transparent in this context.

The results of [Hon10] and [CHI15] on (asymptotic) conformal invariance of spin and energy fields can be formulated, in their simplest cases, as follows: consider the critical Ising model with plus boundary conditions on the discretization Ω_δ by a square grid of mesh size $\delta > 0$ of a simply-connected domain Ω around the origin. Take the origin $0 \in \Omega$, identify it with the closest face of Ω_δ and let δ be the face to the right of 0. Then, as $\delta \to 0$, we have the asymptotic expansions,

$$\text{[spin field]} \qquad \mathbb{E}_{\Omega_\delta}[\sigma_0] = 0 + C_\sigma \left|\varphi'(0)\right|^{-\frac{1}{8}} \delta^{\frac{1}{8}} + o\left(\delta^{\frac{1}{8}}\right),$$

$$\text{[energy density field]} \qquad \mathbb{E}_{\Omega_\delta}[\sigma_0 \sigma_\delta] = \frac{\sqrt{2}}{2} + C_\epsilon \left|\varphi'(0)\right|^{-1} \delta + o(\delta),$$

where $C_\sigma, C_\epsilon > 0$ are explicit (lattice-dependent) constants and φ is any conformal map from the unit disk $\mathbb{D}$ to Ω fixing the origin. The first terms in the respective expansions are the infinite-volume limits of the left-hand side quantities. These results illustrate the following: for any local field one-point function, the correction to its infinite-volume expectation is described by Conformal Field Theory (CFT) quantities.

The Ising Model is conjectured to be described by the unitary CFT minimal model $\mathcal{M}_{3,4}$ (see e.g. [BPZ84], [DMS97]), also known as the Ising CFT. The Ising CFT consists of three primary fields: those of respective scaling dimensions 0 (the identity field – a constant field identically

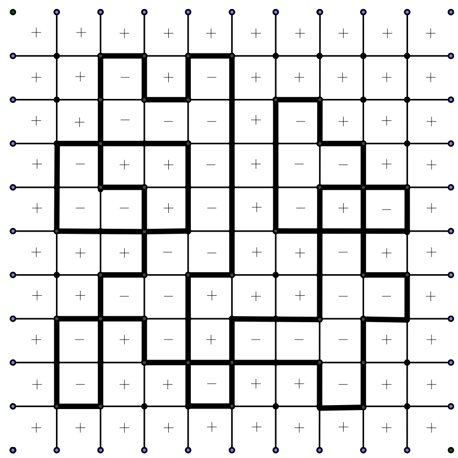

Figure 2.1.1 – An Ising model configuration on the faces of a square subset of $\mathbb{Z}^2$ with all-plus boundary conditions, along with its "low-temperature expansion", indicating interfaces separating plus and minus spins.

equal to one), $\frac{1}{8}$ (the spin field) and 1 (the energy field). Each of these primary fields generates an infinite-dimensional tower of fields called its *descendants*.

We conjecture that the space of the Ising CFT fields describes the limits of Ising *lattice local fields*:

Definition. Let $\mathscr{F}$ be a finite connected collection of faces of $\mathbb{Z}^2$ including 0. For any $F : \{\pm 1\}^{\mathscr{F}} \to \mathbb{C}$, a *lattice local field* Φ_δ^F is a random field on the faces of Ω_δ whose values are given by $\Phi_\delta^F(x) = F(\sigma|_{x+\delta\mathscr{F}})$. We call a local field *spin-antisymmetric* if $F(-\sigma) = -F(\sigma)$ and *spin-symmetric* if $F(-\sigma) = F(\sigma)$.

Conjecture. *For any nonzero lattice field (i.e. whose correlations do not vanish generically; see [HKV17]) Φ_δ^F, there exists $D \in \mathbb{N} \cup \left(\mathbb{N} + \frac{1}{8}\right)$ such that*

$$\delta^{-D}\Phi_\delta^F \to \Phi$$

in the sense of correlations (meaning that the n point functions converge), where Φ is a nonzero primary or descendant CFT field. If Φ is spin-antisymmetric then $D \in \mathbb{N} + \frac{1}{8}$; if Φ is spin-symmetric then $D \in \mathbb{N}$. Moreover every Ising CFT field can be obtained in such a manner.

Spin Pattern Probabilities

Correlations of any lattice local field at a point x can be rewritten in terms of probabilities of observing certain *spin patterns* centered at x, i.e. probabilities of spin configurations in a microscopic neighborhood of x. The main objective of this paper is to obtain explicit representations for probabilities of spin pattern events, which are the most general local quantities describing the model: we obtain infinite-volume limits for arbitrary local pattern probabilities and give their first-order corrections corresponding to what can be expected from the Ising-CFT correspondence (see Theorems 2.1.1–2.1.2 and Corollary 2.1.1).

More precisely, Let $\mathscr{F}$ be a finite connected set of faces in $\mathbb{Z}^2$ and fix a configuration $\rho \in \{\pm 1\}^{\mathscr{F}}$. We look at two types of lattice local fields:

- *spin-antisymmetric pattern fields* $\Phi_\delta^{F_\rho}(x)$ where $F_\rho = \mathbf{1}\{\sigma|_{\mathscr{F}} = \rho\}$, whose expectation gives the probability of the *spin-antisymmetric pattern* ρ on $\mathscr{F}$,
- *spin-symmetric pattern fields* $\Phi_\delta^{F_\rho^\pm}(x)$ where $F_\rho^\pm = \mathbf{1}\{\sigma|_{\mathscr{F}} \in \{\pm\rho\}\}$, whose expectation gives the probability of the *spin-symmetric pattern* $\pm\rho$ on $\mathscr{F}$.

Every Ising lattice local field can easily be seen to be a finite linear combination of such fields. The main result of the paper is the following.

Theorem (see Corollary 2.1.1)**.** *Let $\mathscr{F}$ and ρ be as above and let $\mathbb{P}_{\mathbb{Z}^2}$ be the infinite-volume measure of the critical Ising model. Consider the critical Ising model on Ω_δ with plus boundary conditions, and denote it by $\mathbb{P}_{\Omega_\delta}$. Then as $\delta \to 0$, we have*

$$\begin{aligned}\mathbb{P}_{\Omega_\delta}\left[\sigma|_{\delta\mathscr{F}} = \rho\right] &= \mathbb{P}_{\mathbb{Z}^2}\left[\sigma|_{\mathscr{F}} = \rho\right] + \delta^{\frac{1}{8}} \cdot \text{geometric effect}(\rho, \Omega) + o\left(\delta^{\frac{1}{8}}\right),\\ \mathbb{P}_{\Omega_\delta}\left[\sigma|_{\delta\mathscr{F}} \in \{\pm\rho\}\right] &= \mathbb{P}_{\mathbb{Z}^2}\left[\sigma|_{\mathscr{F}} \in \{\pm\rho\}\right] + \delta \cdot \text{geometric effect}(\pm\rho, \Omega) + o(\delta).\end{aligned}$$

The infinite-volume probability and the geometric effects are given in terms of explicit Pfaffian formulae.

The distinction between spin-symmetric and spin-antisymmetric pattern fields is both natural and important in the CFT framework: the spin and energy fields are the most elementary instances of spin-antisymmetric and spin-symmetric local fields respectively. The space of lattice local fields is a vector space that can be decomposed into the direct sum of fields that are symmetric and antisymmetric under spin flip.

The above theorem proves the aforementioned conjecture in the following specific case: it allows one to study the scaling limit of the one-point function of $\delta^{-D}\Phi_\delta^F$ for $D \leq 1$ in simply-connected domains. Following [Hon10, HoSm13, CHI15], we expect the proof naturally extends to multi-field correlations, in order to prove—in full—the conjecture for fields of scaling dimension $D \in \{0, 1/8, 1\}$. Beyond that, the method we use provides a general toolbox to express multipoint correlations of any local lattice field in terms of discrete fermionic observables in discrete domains, and hence to give explicit infinite-volume limits and first order corrections, and reduce the calculation of all subsequent CFT terms to questions in discrete complex analysis (see Applications 2.1.3).

Moreover, the results allow one to study new interesting quantities: for instance, one can estimate spin flip rates for critical Ising Glauber dynamics, including the geometric effects on them up to first order in the mesh size (see Applications 2.1.3).

The results and approach of this paper, as well as the conjectured connection between Ising lattice local fields and CFT, are expected to straightforwardly generalize in two directions. First, the approach can be generalized to arbitrary combinations of $+, -$ and free boundary conditions (the three conformally invariant boundary conditions according to CFT [Car84]). Second, the results should extend to more general planar graphs, including, in particular, isoradial graphs.

Let us also point out similar connections between pattern probabilities and conformal invariance obtained by Boutillier in [Bo07] in the context of the dimer model.

The proof relies mainly on discrete complex analytic methods: we use lattice observables, modifying the objects introduced in [Hon10] and [CHI15], to connect pattern probabilities with solutions of discrete boundary value problems. This requires precise treatment of multipoint observables on a topological double cover of the lattice with microscopically separated source points at their singularities. We then study the scaling limits of such solutions using discrete complex analysis technique, where, in particular, the neighborhood of the monodromy of the double cover needs to be analyzed delicately. The new techniques introduced for this purpose are: refined analysis of convergence of observables and constructions and characterizations of lattice spinor observables on the slit plane $(\mathbb{C}\setminus\mathbb{R}_{>0})_{\delta>0}$, both as limits of finite-volume ones and in terms of discrete harmonic measures (explicitly computed with Fourier techniques).

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2.1.1 Notation

We begin by defining the most important notation, that will be necessary for the statements of the main theorems. We defer a more extensive discussion of the notation used in the proofs to §2.1.5.

In this paper, we consider the Ising model with spins on the faces of the graph Ω_δ, a discretization of Ω of mesh size $\delta > 0$. More precisely:

Identify $\mathbb{Z}^2$ with the square lattice with vertex set at $\mathbb{Z} + i\mathbb{Z} \subset \mathbb{C}$ and nearest-neighbor edges. Let

$$\mathbb{C}_1 := (1+i)\mathbb{Z}^2 + 1 \qquad \text{and} \qquad \mathbb{C}_\delta = \delta\mathbb{C}_1$$

be the rescaled, rotated and shifted lattice, and its rescaling by a *mesh size* $\delta > 0$, respectively.

For a simply connected open domain $\Omega \subset \mathbb{C}$ bounded by a smooth curve containing 0 (this is easily relaxed to arbitrary simply connected domains [CHI15, Remark 2.10]), define Ω_δ as the largest connected component of the graph $\Omega \cap \mathbb{C}_\delta$. Denote by $\mathcal{V}_{\Omega_\delta}$ the set of vertices of Ω_δ, and by $\mathcal{E}_{\Omega_\delta}$ the set of edges in Ω_δ. We denote the set of faces of the graph by $\mathcal{F}_{\Omega_\delta}$. Whenever needed, we identify the edges in $\mathcal{E}_{\Omega_\delta}$ with their midpoints, and the faces in $\mathcal{F}_{\Omega_\delta}$ with their centers, such that the origin is identified with a face.

Ising Model

An Ising *configuration* σ is an assignment of ± 1 spins to the faces in $\mathcal{F}_{\Omega_\delta}$. We consider the *critical Ising model* on $\mathcal{F}_{\Omega_\delta}$ with *plus boundary conditions*, given by,

$$\mathbb{P}_{\Omega_\delta}(\sigma) = \mathbb{P}^+_{\Omega_\delta}(\sigma) \propto e^{-\beta_c H(\sigma)},$$

where $\beta_c = \frac{1}{2}\ln(\sqrt{2}+1)$ is the critical inverse temperature and $H(\sigma) = -\sum_{x\sim y}\sigma_x\sigma_y$ with boundary faces fixed to have $+1$ spin. Let $\mathbb{E}_{\Omega_\delta} = \mathbb{E}^+_{\Omega_\delta}$ be its corresponding expectation.

Define the *energy density field* $(\epsilon(\delta e))_{e\in\mathbb{C}_1}$ as follows: for $\delta e \in \mathcal{E}_{\Omega_\delta}$ separating faces $\delta f_1 \sim \delta f_2$,

$$\epsilon(\delta e) = \mu - \sigma_{\delta f_1}\sigma_{\delta f_2}, \qquad \text{where} \qquad \mu := \frac{\sqrt{2}}{2} = \mathbb{E}_{\mathbb{C}_\delta}[\sigma_{\delta f_1}\sigma_{\delta f_2}] = \mathbb{E}_{\mathbb{C}_1}[\sigma_{f_1}\sigma_{f_2}].$$

Define the *spin-weighted energy density field* $(\epsilon_{[0]}(\delta e))_{e\in\mathbb{C}_1}$ on $\mathcal{E}_{\Omega_\delta}$ by

$$\epsilon_{[0]}(\delta e) = \mu_e - \sigma_{\delta f_1}\sigma_{\delta f_2}, \qquad \text{where} \qquad \mu_e := \lim_{\Omega\to\mathbb{C}} \frac{\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{\delta f_1}\sigma_{\delta f_2}]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} = \lim_{\Omega\to\mathbb{C}} \frac{\mathbb{E}_{\Omega_1}[\sigma_0\sigma_{f_1}\sigma_{f_2}]}{\mathbb{E}_{\Omega_1}[\sigma_0]} \tag{2.1.1}$$

where the limit is trivially independent of δ and exists for every $e \in \mathbb{C}_1$ by Theorem 2.1.2 (see

Figure 2.4.2 for some exact values). Given a set of edges $\mathscr{B} \subset \mathscr{E}_{\mathbb{C}_1}$, we write

$$\epsilon(\delta\mathscr{B}) := \prod_{e\in\mathscr{B}} \epsilon(\delta e) \qquad \text{and analogously} \qquad \epsilon_{[0]}(\delta\mathscr{B}) := \prod_{e\in\mathscr{B}} \epsilon_{[0]}(\delta e).$$

2.1.2 Main Results

In this section we present the main results. By translation, it suffices to consider the statistics of fields centered at $x = 0$. For a collection $\mathscr{B} = \{e_1, \ldots, e_n\} \subset \mathscr{E}_{\mathbb{C}_1}$, consider the *spin-symmetric field* $\epsilon(\delta\mathscr{B}) = \epsilon(\delta e_1)\cdots\epsilon(\delta e_n)$, i.e., the product of energy densities on a collection of edges around x, and the *spin-antisymmetric field* $\sigma_0\epsilon_{[0]}(\delta\mathscr{B}) = \sigma_0\epsilon_{[0]}(\delta e_1)\cdots\epsilon_{[0]}(\delta e_n)$. If $\mathbf{A}, \mathbf{B}$ are anti-symmetric square matrices of the same dimensions, define the directional derivative of the Pfaffian $\mathrm{Pf}(\mathbf{B})$ (defined in (2.4.6)) by

$$D_{\mathbf{A}}\mathrm{Pf}(\mathbf{B}) = \lim_{t\downarrow 0} \frac{\mathrm{Pf}(\mathbf{B} + t\mathbf{A}) - \mathrm{Pf}(\mathbf{B})}{t}.$$

For spin-symmetric and spin-antisymmetric fields, we obtain the following two convergence results:

Theorem 2.1.1 (Spin-symmetric correlations)**.** *Let $\mathscr{B} = \{e_1, ..., e_n\} \subset \mathscr{E}_{\mathbb{C}_1}$. There exist explicit, real-valued anti-symmetric $2n \times 2n$ matrices $\mathbf{F}^{\mathscr{B}}$ and $\mathbf{E}^{\mathscr{B}}$, such that as $\delta \to 0$,*

$$\mathbb{E}_{\Omega_\delta}[\epsilon(\delta\mathscr{B})] = (-2)^n \cdot \mathrm{Pf}(\mathbf{F}^{\mathscr{B}}) + (-2)^n \cdot \delta \cdot r_\Omega^{-1}(0) \cdot D_{\mathbf{E}^{\mathscr{B}}}\mathrm{Pf}(\mathbf{F}^{\mathscr{B}}) + o(\delta),$$

where $r_\Omega(z)$ is the conformal radius of Ω seen from $z \in \Omega$ (i.e. $r_\Omega(z) = |\varphi'(0)|$ where $\varphi : \mathbb{D} \to \Omega$ is the conformal map such that $\varphi(0) = z$).

Theorem 2.1.2 (Spin-antisymmetric correlations)**.** *Let $\mathscr{B} = \{e_1, ..., e_n\} \subset \mathscr{E}_{\mathbb{C}_1}$. The limits μ_e defined in* (2.1.1) *exist for every $e \in \mathscr{E}_{\mathbb{C}_1}$ and are given explicitly. There exist explicit anti-symmetric $2n \times 2n$ matrices $\mathbf{F}_{[0]}^{\mathscr{B}}$ and $\mathbf{E}_{[0]}^{\mathscr{B}}$, the former being real-valued, such that as $\delta \to 0$,*

$$\frac{\mathbb{E}_{\Omega_\delta}\left[\sigma_0\epsilon_{[0]}(\delta\mathscr{B})\right]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} = (-2)^n \cdot \mathrm{Pf}(\mathbf{F}_{[0]}^{\mathscr{B}}) + (-2)^n \cdot \delta \cdot Re\left[-\frac{1}{4}\partial_z \log r_\Omega(z)\Big|_{z=0} \cdot D_{\mathbf{E}_{[0]}^{\mathscr{B}}}\mathrm{Pf}\big(\mathbf{F}_{[0]}^{B}\big)\right] + o(\delta),$$

where $z = x + iy$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

Remark. Theorems 2.1.1 and 2.1.2 yield that the infinite-volume limits of $\mathbb{E}_{\Omega_\delta}\left[\sigma_0\epsilon_{[0]}(\delta\mathscr{B})\right]/\mathbb{E}_{\Omega_\delta}[\sigma_0]$ and $\mathbb{E}_{\Omega_\delta}[\epsilon(\delta\mathscr{B})]$ exist and are given explicitly by

$$\lim_{\Omega\to\mathbb{C}} \mathbb{E}_{\Omega_\delta}[\epsilon(\delta\mathscr{B})] = (-2)^n \cdot \mathrm{Pf}(\mathbf{F}^{\mathscr{B}}), \qquad \text{and} \qquad \lim_{\Omega\to\mathbb{C}} \frac{\mathbb{E}_{\Omega_\delta}\left[\sigma_0\epsilon_{[0]}(\delta\mathscr{B})\right]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} = (-2)^n \cdot \mathrm{Pf}(\mathbf{F}_{[0]}^{\mathscr{B}}).$$

For spin pattern fields, our results translate to the following:

Corollary 2.1.1 (Conformal invariance of pattern probabilities)**.** *Let $\mathscr{F}$ be a finite connected collection of faces of $\mathbb{C}_1$ including 0. For any $\rho \in \{\pm 1\}^{\mathscr{F}}$ we have:*

$$\delta^{-1}\left(\mathbb{P}_{\Omega_\delta}\left[\sigma|_{\delta\mathscr{F}} \in \{\pm\rho\}\right] - \mathbb{P}_{\mathbb{C}_1}\left[\sigma|_{\mathscr{F}} \in \{\pm\rho\}\right]\right) \xrightarrow[\delta\to 0]{} \langle\langle \mathscr{F}, \{\pm\rho\}\rangle\rangle_\Omega,$$
$$\delta^{-1/8}\left(\mathbb{P}_{\Omega_\delta}[\sigma|_{\delta\mathscr{F}} = \rho] - \mathbb{P}_{\mathbb{C}_1}[\sigma|_{\mathscr{F}} = \rho]\right) \xrightarrow[\delta\to 0]{} \langle\langle \mathscr{F}, \rho\rangle\rangle'_\Omega,$$

where the functions $\langle\langle\cdot\rangle\rangle_\Omega$ and $\langle\langle\cdot\rangle\rangle'_\Omega$ depend only on Ω, and where:

- *infinite-volume limits $\mathbb{P}_{\mathbb{C}_1}\left[\sigma|_{\mathscr{F}} \in \{\pm\rho\}\right] = \lim_{\Omega\to\mathbb{C}} \mathbb{P}_{\Omega_1}[\sigma|_{\mathscr{F}} \in \{\pm\rho\}]$ and $\mathbb{P}_{\mathbb{C}_1}[\sigma|_{\mathscr{F}} = \rho]$ are explicit.*
- *$\langle\langle \mathscr{F}, \{\pm\rho\}\rangle\rangle_\Omega$ and $\langle\langle \mathscr{F}, \rho\rangle\rangle'_\Omega$ are explicit and are such that for the map $\varphi : \mathbb{D} \to \Omega$ as in Theorem 2.1.1,*

$$\langle\langle \mathscr{F}, \{\pm\rho\}\rangle\rangle_\Omega = r_\Omega^{-1}(0)\langle\langle \mathscr{F}, \{\pm\rho\}\rangle\rangle_{\mathbb{D}}, \qquad \textit{and} \qquad \langle\langle \mathscr{F}, \rho\rangle\rangle'_\Omega = r_\Omega^{-\frac{1}{8}}(0)\langle\langle \mathscr{F}, \rho\rangle\rangle'_{\mathbb{D}}.$$

As a result of Corollary 2.1.1, our results include explicit expressions for all the finite-dimensional distributions of $\mathbb{P}_{\mathbb{Z}^2}$ as finite linear combinations of a certain Fourier integral given in Theorem 2.3.3.

2.1.3 Applications

In this subsection, we briefly detail three applications of our results: the lattice local field conjecture of the introduction, relations between Markov chain dynamics flip rates and the geometry of the domain where the dynamics live, and explicit computations of pattern probabilities under the Gibbs measure.

Lattice Local Fields and CFT

Returning to the conjectured Ising-CFT correspondence in terms of lattice fields, we observe that any lattice local field $\Phi_\delta^F(x)$ is such that F can be expressed as a linear combination of indicator functions of spin-pattern events in a microscopic neighborhood of x; a spin-symmetric lattice local field can in particular be written in terms of indicators of spin-symmetric pattern events.

Then Theorems 2.1.1–2.1.2 give the infinite-volume limits, and first-order CFT corrections of the one point function of any lattice local field $\Phi_\delta^F(x)$ in terms of those of spin-symmetric and spin-antisymmetric pattern fields, whose one-point functions can be obtained explicitly. We believe extending this to multi-field correlations of fields with scaling dimension $D \leq 1$ should carry over from [Hon10, CHI15].

In the other direction, though our main statements only go up to first-order corrections ($\delta^{1/8}$ or δ), the methods of this paper can, in principle, be employed to reduce the computation of higher order CFT corrections to correlations of any local lattice field to questions of discrete complex analysis. Of course, then obtaining the necessary sharper discrete complex analytic expansions is itself a major obstacle to the extension of such results. All the same, using the present framework along with, hypothetically, improved discrete complex analysis asymptotics, should yield that all Ising lattice local fields are, as conjectured in the Introduction, either zero in correlations, or have scaling dimensions $\Delta \in \mathbb{N} \cup (\mathbb{N} + \frac{1}{8})$, as predicted from the Ising CFT.

Local Markov Chain Dynamics

There are a number of Markov chain dynamics for which the Ising measure is the stationary measure; as a result, an efficient way to sample the Ising model is to run such a Markov chain for long times.

Of particular importance are local dynamics, such as the Glauber dynamics, where one picks a spin at random, and flips it with a probability given by the state of spins in a microscopic neighborhood of it (the simplest one using only the four neighbors). At critical temperature interesting dynamical behavior arises (see [LuSl12]). In particular, as our results explain, the geometry of the domain Ω has a measurable (i.e. inverse polynomial-sized) effect on the local dynamics of the Markov chain.

For such a dynamics, our results allow one to describe, once we are at equilibrium, the relevant observables to compute the average flip rates of the system: those are indeed given in terms of the occurence probabilities of various spin patterns (typically spin-symmetric patterns).

In particular Corollary 2.1.1 gives us the following: at criticality, for any Glauber dynamics (see e.g., [LuSl12, Section 2.1]), we can derive exact information about spin-symmetric pattern probabilities, how they behave at constant order, and how the first-order correction depends on a geometric quantity.

Knowing the long-term history of a Glauber dynamics in a microscopic neighborhood of a point enables the computation of various spin pattern probabilities and hence lattice local field one-point functions. Higher order corrections of these terms in turn give geometric information beyond the conformal radius of the domain. A particularly interesting question, for which our results provide relevant tools, is the following one, due to Benjamini (private communication to the second author): *does the complete (i.e., unbounded in time) knowledge of the flip history of a single spin allow one to recover the shape of the domain Ω, up to isometry?*

Explicit Computations

Explicit calculation of infinite-volume limits and finite-size corrections of pattern probabilities in the critical Ising model is of general interest, and may be particularly useful for the program of Application 1.3.2. Such computation requires the explicit matrices of Theorems 2.1.1–2.1.2, which are expressed in terms of the full-plane fermion and spin-fermion observables: some values of the former are given in e.g., [Ken00]; we characterize the latter as a Fourier integral (see Theorem 2.3.3) and give some of its values in Figure 2.3.2. In particular, the entries of the matrices in Theorems 2.1.1–2.1.2 are given as finite linear combinations of a slit-plane harmonic measure, whose values are explicitly computable as a Fourier integral.

We present an example computation of the infinite-volume limit and first-order conformal correction of $\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{2\delta}]$ in Corollary 2.9.1 of Appendix 2.9, where, since the spins live on the rotated square lattice, this is a pair of diagonally "adjacent" spins. The first and second order corrections to this term, and their representation in terms of discrete complex analysis will be used in the Ising stress tensor on the lattice level (see [BeHo18] and, for an alternative approach to the stress tensor, [CGS17]).

As a computation of spin-antisymmetric fields, Corollary 2.9.2 gives the values of the infinite-volume limit and conformal correction to the spin weighted "L"-shaped correlation $\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{(1+i)\delta}\sigma_{2\delta}]/\mathbb{E}_{\Omega_\delta}[\sigma_0]$.

2.1.4 Proof Outline

In this subsection we outline the strategy for proving our main results: Theorems 2.1.1–2.1.2. The proof combines ideas from [Hon10, CHI15], and we try to focus the outline on the places where substantial new ingredients are needed. The steps in proving the main theorems broadly consist of the following.

Section 2.2 defines standard concepts in discrete complex analysis as well as the discrete Riemann boundary value problems solved by certain discrete observables. Section 2.3 begins by defining the two-point discrete *fermion* and *spin-fermion* (given by $(\alpha,\zeta)\mapsto F^{\alpha,\zeta}_{\Omega_\delta}$ and $F^{\alpha,\zeta}_{[\Omega_\delta,0]}$ respectively) via the low-temperature expansion of the Ising model and disorder lines, as well as their full-plane analogues.

- In §2.3.1, we define the bounded domain observables on Ω_δ, as previously defined in [HoSm13, CHI15].
- In §2.3.2, we introduce their full-plane analogues: the full plane fermion $H^\alpha_{\mathbb{C}_1}(z)$ is given explicitly by a formula due to Kenyon. For the special value of $\alpha_0=\frac{1}{2}$, the full plane spin-fermion $H^{\alpha_0}_{[\mathbb{C}_1,0]}(z)$ was given by [CHI15]. Here, we prove existence of the infinite-volume limit of the spin-fermion $H^{\alpha}_{[\mathbb{C}_1,0]}$ for every α, and express it as a finite linear combination of discrete harmonic measures on $\mathbb{C}_1\setminus\mathbb{R}_{>0}$. Moreover, we give an explicit representation formula using Fourier techniques for this discrete harmonic measure

(see Theorem 2.3.3), allowing computation of $H^{\alpha}_{[\mathbb{C}_1,0]}(z)$ for arbitrary α.

Section 2.4 defines and analyzes n-point analogues of the two-point fermion and spin-fermion. This section is notationally heavy, but many of its proofs are straightforward adaptations of proofs in [Hon10].

- We first recall the multipoint fermion defined in [Hon10] and slightly, but crucially, generalize its properties to the setting where its arguments are permitted to be adjacent edges.
- Motivated by this definition, we consider a multipoint version of the spin-fermion observable in Definition 2.4.2 and prove that the same properties hold after minor modifications.
- In Proposition 2.4.6 we relate specific values taken by the multipoint fermion and spin-fermion to the n-point spin-symmetric and spin-antisymmetric correlations of Theorems 2.1.1–2.1.2.
- These results allow us to arrive at the spin-antisymmetric analogues of the Pfaffian formulae of [Hon10], connecting spin-antisymmetric n-point Ising correlations to the Pfaffian of a matrix with entries consisting only of the two-point spin-fermion (see Proposition 2.4.10).

Section 2.5 defines two-point continuous observables, and proves that they are the renormalized scaling limits (as $\delta \to 0$) of the discrete two-point observables.

- In §2.5.2 we introduce the continuous analogues of the discrete Riemann boundary value problem defined in Section 2.2 and give their full-plane solutions and bounded domain solutions h^{α}_{Ω} and $h^{\alpha}_{[\Omega,0]}$. Again this was already done for the fermion in [Hon10, HoSm13] and for the spin-fermion in the particular case of $h^{\alpha=\frac{1}{2}}_{[\Omega,0]}$ [CHI15]; extra care is needed in constructing the continuous bounded domain and full-plane spin-fermions for arbitrary source point α.
- The heart of Section 2.5, §2.5.4 proves convergence of a rescaled, renormalized discrete spin-fermion to a conformally covariant quantity obtained by Taylor expanding $h^{\alpha}_{[\Omega,0]}$, for arbitrary α. In adapting the proof of convergence in [CHI15] to the case where $\alpha \neq \frac{1}{2}$, we require refined analysis of the observables near their branch points and singularities. Here we encounter some discrete complex analytic peculiarities regarding discretizations of the function $i\sqrt{z}$ which are independently interesting.

Section 2.6 combines the Pfaffian formulae of §2.4.4 expressing n-point correlations in terms of two-point discrete observables, with the convergence results of §2.5, to prove Theorems 2.1.1–2.1.2.

In Appendix 2.7, we prove the validity of our explicit construction of the discrete harmonic measure on $\mathbb{C}_1 \backslash \mathbb{R}_{>0}$ and provide a recursive formula to obtain its value at any lattice point as a finite linear combination of Fourier integrals. In Appendix 2.8, Proposition 2.8.1, we provide a combinatorial proof of the well-definedness of the discrete multipoint spin-fermion we introduce in Definition 2.4.2. As mentioned earlier, Appendix 2.9 consists of explicit computations of the infinite-volume limit and first-order correction of the correlations of two diagonally adjacent spins and three spins in an "L" shape.

2.1.5 Extra Notation and Glossary

We now introduce extended notation that will be used globally throughout the paper. This notation mimics very closely that of [CHI15], and we try to make note of places where our conventions differ.

Relevant Constants

The following constants will recur throughout the paper.

$\beta_c = \frac{1}{2}\log(1+\sqrt{2})$	$\mu = \frac{\sqrt{2}}{2}$	$\lambda = e^{i\pi/4}$

Graph Notation

We list below the additional graph notation that will be used throughout the paper.

- For two adjacent vertices $a, b \in \mathcal{V}_{\Omega_\delta}$ the edge $e = \{a, b\}$ is identified with the line segment in Ω connecting a and b; we define the set of *medial vertices* $\mathcal{V}^m_{\Omega_\delta}$ as the set of edge midpoints; given an edge $e \in \mathcal{E}_{\Omega_\delta}$, we denote its midpoint by $m(e) \in \mathcal{V}^m_{\Omega_\delta}$, and conversely for $m \in \mathcal{V}^m_{\Omega_\delta}$ the corresponding edge $e(m) \in \mathcal{E}_{\Omega_\delta}$.

- We call *corners* the points that are at distance $\delta/2$ from the vertices in one of the four $\pm 1, \pm i$ directions. Following [CHI15], we set

$$\mathcal{V}^1_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} + \frac{\delta}{2}, \quad \mathcal{V}^i_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} - \frac{\delta}{2}, \quad \mathcal{V}^\lambda_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} - \frac{i\delta}{2}, \quad \text{and } \mathcal{V}^{\bar{\lambda}}_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} + \frac{i\delta}{2}.$$

  The set of corners $\mathcal{V}^c_{\Omega_\delta}$ is the union $\mathcal{V}^1_{\Omega_\delta} \cup \mathcal{V}^i_{\Omega_\delta} \cup \mathcal{V}^\lambda_{\Omega_\delta} \cup \mathcal{V}^{\bar{\lambda}}_{\Omega_\delta}$.

- The domain of definitions for most discrete functions in the following sections is the set of both corners and medial vertices, or $\mathcal{V}^{cm}_{\Omega_\delta} := \mathcal{V}^c_{\Omega_\delta} \cup \mathcal{V}^m_{\Omega_\delta}$. We declare a medial vertex and a corner *adjacent* if they are $\frac{\delta}{2}$ apart from each other.

- The *boundary faces* $\partial\mathcal{F}_{\Omega_\delta}$, *boundary medial vertices* $\partial\mathcal{V}^m_{\Omega_\delta}$, *boundary edges* in $\partial\mathcal{E}_{\Omega_\delta}$ are

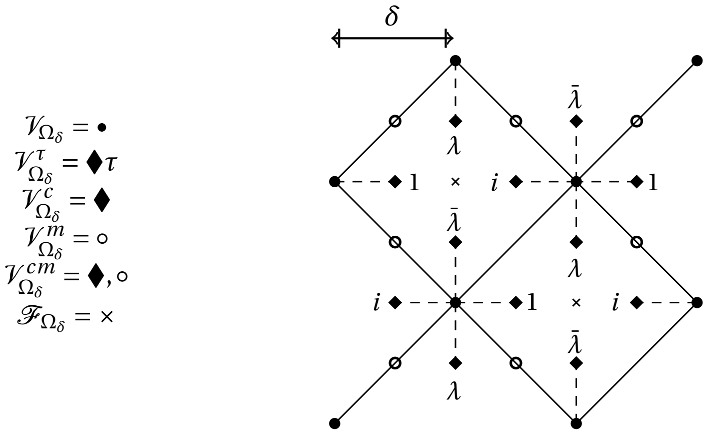

Figure 2.1.2 – The graph notation on discretizations of Ω where $\tau \in \{1, i, \lambda, \bar{\lambda}\}$; the notation on $[\Omega_\delta, 0]$ is analogously defined.

those faces, medial vertices, and edges in $\mathbb{C}_\delta$ that are incident to but not contained in $\mathcal{F}_{\Omega_\delta}$, $\mathcal{V}^m_{\Omega_\delta}$, and $\mathcal{E}_{\Omega_\delta}$.

- Given a boundary edge z (resp., boundary medial vertex), we define the *unit normal outward vector* ν_z as the unit vector in the direction of the vertex in $\mathbb{C} \setminus \Omega$ viewed from the vertex inside Ω.

Graph lifts to the double cover

For the discrete functions with monodromy which will be introduced in Section 3, we work with graphs lifted to the double cover $[\Omega, 0]$ of $\Omega \setminus \{0\}$.

- We denote by $[\mathbb{C}, 0]$ the double cover of the plane $\mathbb{C}$ ramified at 0, i.e. the surface on which the function $z \mapsto \sqrt{z} \in \mathbb{C} \setminus \{0\}$ is naturally defined; above each point of $\mathbb{C} \setminus \{0\}$ lie exactly two points of $[\mathbb{C}, 0]$. We will sometimes use $z \in [\mathbb{C}, 0]$ to refer to its projection on $\mathbb{C}$ in unambiguous cases.

- If $z_1, z_2 \in [\mathbb{C}, 0]$ are two points above $w_1, w_2 \in \mathbb{C} \setminus \{0\}$ such that $\mathrm{Re}\left(\frac{w_1}{w_2}\right) > 0$, we say that they are *on the same sheet* if $\mathrm{Re}\left(\frac{\sqrt{z_1}}{\sqrt{z_2}}\right) > 0$ and that they are *on opposite sheets* if $\mathrm{Re}\left(\frac{\sqrt{z_1}}{\sqrt{z_2}}\right) < 0$.

- If $z \in [\mathbb{C}, 0]$ lies above $w \in \mathbb{C} \setminus \{0\}$, we define $z + x \in [\mathbb{C}, 0]$, for $x \in \mathbb{C}$ small enough, as the point above $w + x$ that is on the same sheet as z.

- We define complex conjugation on the double cover by conjugating the square root. In other words, the complex conjugate $\bar{z}$ of $z \in [\mathbb{C}, 0]$ is defined by the condition that $\sqrt{\bar{z}} = \overline{\sqrt{z}}$.

- We call functions with monodromy -1 around 0 *spinors*; these are naturally defined on $[\mathbb{C},0]$.
- We denote by $[\Omega_\delta,0]$ the double cover of Ω_δ ramified at 0; in other words, the vertices, medial vertices, and corners get lifted from Ω_δ to yield the lifted vertex, edge and corner sets. We use similar notations for the lifted vertex, edge, and corner sets as above by replacing Ω_δ with $[\Omega_\delta,0]$. Moreover, $[\Omega_\delta,0]$ can be naturally viewed as a subgraph of $[\mathbb{C}_\delta,0]$ via the natural inclusion $[\Omega,0]\subset[\mathbb{C},0]$.
- Identify the branches of the double cover $[\mathbb{C},0]$ using the function $\sqrt{z}$ as follows:

$\mathbb{X} := \mathbb{C}\setminus\mathbb{R}_{<0}$	with $\mathbb{X}^+ = \{z\in[\mathbb{C},0] : \mathrm{Re}(\sqrt{z})>0\}$ and $\mathbb{X}^- = \{z\in[\mathbb{C},0] : \mathrm{Re}(\sqrt{z})<0\}$
$\mathbb{Y} := \mathbb{C}\setminus\mathbb{R}_{>0}$	with $\mathbb{Y}^+ = \{z\in[\mathbb{C},0] : \mathrm{Im}(\sqrt{z})>0\}$ and $\mathbb{Y}^- = \{z\in[\mathbb{C},0] : \mathrm{Im}(\sqrt{z})<0\}$

On the discrete level, define the lift of $\mathcal{V}^1_{\Omega_\delta}$ to $\mathbb{X}^\pm$ as $\mathbb{X}^\pm_\delta$, and the lift of $\mathcal{V}^i_{\Omega_\delta}$ to $\mathbb{Y}^\pm$ as $\mathbb{Y}^\pm_\delta$.

Orientations

We define orientations and s-orientations for corners and medial vertices.

- Given an edge $e=\{a,b\}\in\mathcal{E}_{\Omega_\delta}$, we denote the two orientations of e by the complex numbers $(a-b)/|a-b|$ and $(b-a)/|b-a|$. We can subdivide e into two *half edges* $\{a,m(e)\}$ and $\{m(e),b\}$; their union is identified with the whole edge e. An orientation $o=o(e)$ is *compatible* with a half edge $\{a,m(e)\}$ if $a-m(e)$ points to the same direction: i.e., $o=(a-m(e))/|a-m(e)|$.
- We call an *oriented medial vertex* and denote by m^o an edge midpoint $m(e)$ together with an orientation of the edge e. We denote by $\mathcal{V}^o_{\Omega_\delta}$ the set of oriented medial vertices.
- For a corner c, we define its orientation $o=o(c)$ as the complex number $(v-c)/|v-c|$, where v is the nearest vertex to c.
- To an orientation o we further associate two *s-orientations* corresponding to the two choices of square root for o; we often denote an s-orientation by $(\sqrt{o})^2$, indicating this choice.

Glossary

For the reader's convenience, we compile some of the most important terminology and quantities used across the paper (see also Fig. 2.1.2 for the graph notation). We first recall the various graphs we work with: if Ω is a simply-connected smooth domain containing the origin and $\overline{\Omega}$ is its complex conjugate,

$\mathbb{C}_1=(1+i)\mathbb{Z}^2+1$	$\mathbb{C}_\delta=\delta\mathbb{C}_1$	$\Omega_\delta=\Omega\cap\mathbb{C}_\delta$	$\Lambda_\delta=\Omega_\delta\cap(\overline{\Omega})_\delta$	$a_0=\frac{1}{2}$

and the quantities $[\mathbb{C}_1, 0]$, $[\Omega_\delta, 0]$, etc. are analogously defined on the ramified plane $[\mathbb{C}, 0]$. When proofs are independent of the choice, we let D_δ denote *either of* Ω_δ *or* $[\Omega_\delta, 0]$.

The domain-dependent quantities of interest match with [CHI15] and read as follows for fixed $z \in \Omega$.

Geometric quantities		
$\varphi(\omega)$	conformal map $\varphi : \mathbb{D} \to \Omega$ with $\varphi(0) = z$	
$r_\Omega(z)$	$r_\Omega(z) = \lvert\varphi'(0)\rvert$	conformal radius of z
$\mathscr{A}_\Omega$	$\mathscr{A}_\Omega = -\frac{1}{4}\partial_z \log r_\Omega(z)\big\rvert_{z=0}$	Remark 2.5.4

In what follows, we present the notation for the fundamental observables we deal with; in order to reduce multipoint observables to such two-point functions via Pfaffian relations we need much heavier notation, that is restricted to §2.4. In the sequel, a, z will be medial vertices or corners and α and ζ will be their s-oriented counterparts, e.g., $\alpha = a^{(\sqrt{o})^2}$.

Fermion observables		
$F^{\alpha,\zeta}_{\Omega_\delta}$	discrete real fermion	Definition 2.3.2
$H^{\alpha}_{\Omega_\delta}(z)$	discrete complexified fermion	Definition 2.3.4
$F^{\alpha,\zeta}_{\mathbb{C}_1}$	discrete full-plane fermion	Theorem 2.3.1
$H^{\alpha}_{\mathbb{C}_1}(z)$	discrete full-plane complexified fermion	Proposition 2.3.5
$f_\Omega, h_\Omega, f_\mathbb{C}, h_\mathbb{C}$	continuous counterparts of the above	Definitions 2.5.1, 2.5.3
$F^\dagger_{\Omega_\delta}, H^\dagger_{\Omega_\delta}, f^\dagger_\Omega, h^\dagger_\Omega$	e.g., $F^{\dagger,\alpha,\zeta}_{\Omega_\delta} = F^{\alpha,\zeta}_{\Omega_\delta} - F^{\alpha,\zeta}_{\mathbb{C}_\delta}$	

The notation for the spin-fermions is analogous to the above, but on the respective double covers.

Spinor observables		
$F^{\alpha,\zeta}_{[\Omega_\delta,0]}$	discrete real spin-fermion	Definition 2.3.3
$H^{\alpha}_{[\Omega_\delta,0]}(z)$	discrete complexified spin-fermion	Definition 2.3.5
$F^{\alpha,\zeta}_{[\mathbb{C}_1,0]}$	discrete full-plane spin-fermion	Theorem 2.3.2, item (E)
$H^{\alpha}_{[\mathbb{C}_1,0]}(z)$	discrete full-plane complexified spin-fermion	Proposition 2.3.9
$S^{\alpha}_{[\Omega_\delta,0]}(z), A^{\alpha}_{[\Omega_\delta,0]}(z)$	symmetrized and anti-symmetrized observables	Definition 2.3.6
$G_{[\mathbb{C}_1,0]}(z), \tilde{G}^{\pm}_{[\mathbb{C}_1,0]}(z)$	auxiliary functions	Definitions 2.3.7
$C_\alpha = C_{a^o}$	$C_\alpha = -\mathrm{Re}\left[i\sqrt{o}(\tilde{G}^{+}_{[\mathbb{C}_1,0]} - \tilde{G}^{-}_{[\mathbb{C}_1,0]})(a)\right]$	Corollary 2.5.8

The lower-case versions of the spin-fermions above again are their continuous counterparts, and when there is a † superscript, that denotes the difference of the bounded-domain and full-plane spin-fermions.

2.2 Discrete Complex Analysis

In this section, we review basic notions of discrete complex analysis that will be useful in this paper. We use discrete complex analysis for the following:

- To relate the Ising correlations to Pfaffians of fermion and spin-fermion observables.
- To obtain explicit formulae for the full-plane observables.
- To establish the convergence of the two-point observables and study their local behavior.

2.2.1 S-holomorphicity

Definition 2.2.1. Associate to each corner $c \in \mathcal{V}^{\tau}_{\Omega_\delta}$ with $\tau \in \{1, i, \lambda, \bar{\lambda}\}$, the line $l(c) := \tau\mathbb{R}$ in the complex plane. A function H_δ defined on corners and medial vertices of a discrete domain Ω_δ is said to be *s-holomorphic* at a corner $c \in \mathcal{V}^{\tau}_{\Omega_\delta}$ if for any adjacent medial vertex $a \in \mathcal{V}^{m}_{\Omega_\delta}$ we have

$$H_\delta(c) = \mathsf{P}_{l(c)}[H_\delta(a)] := \frac{1}{2}\left(H_\delta(a) + \tau^2 \bar{H}_\delta(a)\right), \tag{2.2.1}$$

where $\mathsf{P}_{l(c)}$ denotes orthogonal projection in the complex plane onto the line $l(c)$. The function H_δ is said to be s-holomorphic at a medial vertex $a \in \mathcal{V}^{m}_{\Omega_\delta}$ if Eq. (2.2.1) holds for all corners c adjacent to a. A function is said to be s-holomorphic on Ω_δ if it is s-holomorphic at every $c \in \mathcal{V}^{c}_{\Omega_\delta}$.

Remark 2.2.1. If a function H_δ is s-holomorphic on a discrete domain then it is purely real on the corners of type 1 and purely imaginary on the corners of type i. We call respective restrictions to those corners the *real part* and the *imaginary part* of H_δ.

Remark 2.2.2. S-holomorphicity implies usual *discrete holomorphicity* of the real and imaginary parts, defined by a lattice version of Cauchy-Riemann equations ([Smi10]). If H_δ is s-holomorphic, then the following discrete derivative vanishes:

$$\bar{\partial}_\delta H_\delta(x) := H_\delta\left(x + \frac{\lambda\delta}{\sqrt{2}}\right) - H_\delta\left(x - \frac{\lambda\delta}{\sqrt{2}}\right) + i\left(H_\delta\left(x - \frac{\bar{\lambda}\delta}{\sqrt{2}}\right) - H_\delta\left(x + \frac{\bar{\lambda}\delta}{\sqrt{2}}\right)\right) = 0$$

for $x \in \mathcal{V}_{\Omega_\delta}^{\lambda} \cup \mathcal{V}_{\Omega_\delta}^{\bar{\lambda}}$. One similarly defines ∂_δ by taking a negative sign in front of i. We extend the definition to $x \in \mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta}$ by setting $\partial_\delta H_\delta(x) := \partial_\delta H_\delta(x - \frac{i\delta}{2}) + \partial_\delta H_\delta(x + \frac{i\delta}{2})$. Note the differences in our definitions compared to their continuous counterparts, as the discrete derivatives are taken in rotated directions (thus differing by a phase factor); however, we will not take direct scaling limits of the operator and this poses no problem.

The information defined on corners of type $1, i$ is enough to recover an s-holomorphic function on $\mathcal{V}_{\Omega_\delta}^{cm}$: one can start from a discrete holomorphic function defined on corners of type 1 and i, reconstruct values on medial vertices based on their projections onto $\mathbb{R}$ and $i\mathbb{R}$, then project to corners of type λ and $\bar{\lambda}$ (discrete holomorphicity guarantees well-definedness at those corners); see [CHI15, Remark 3.1].

Definition 2.2.2. We define the *discrete Laplacian* Δ_δ by

$$\Delta_\delta H_\delta(x) = H_\delta(x + \delta + i\delta) + H_\delta(x - \delta + i\delta) + H_\delta(x - \delta - i\delta) + H_\delta(x + \delta - i\delta) - 4H_\delta(x).$$

This quantity makes sense on any discrete domain of rotated square type, for example $\mathcal{V}_{\Omega_\delta}^1$ or $\mathcal{V}_{\Omega_\delta}^i$. A function H_δ on such a lattice is said to be *discrete harmonic* if $\Delta_\delta H_\delta(x) = 0$ for all x at which $\Delta_\delta H_\delta$ is defined. Analogously, it is *discrete sub-harmonic* if $\Delta_\delta H_\delta \geq 0$, and *discrete super-harmonic* if $\Delta_\delta H_\delta \leq 0$.

Remark 2.2.3. The real and imaginary parts of an s-holomorphic function on a planar domain are discrete harmonic on their respective lattices, $\mathcal{V}_{\Omega_\delta}^1$ and $\mathcal{V}_{\Omega_\delta}^i$. This is a direct consequence of Remark 2.2.2: discrete holomorphicity converts discrete outward derivatives from the center point in the Laplacian into discrete derivatives in the angular direction, and going in a closed loop around the center point gives zero.

Remark 2.2.4. The notions of discrete complex analysis thus far introduced have been defined on the planar domain Ω_δ, but they generalize to $[\Omega_\delta, 0]$ in a straightforward manner since the double cover is locally isomorphic to a planar domain (cf. Section 2.1.5). However, great care is needed in applying Remark 2.2.3 because, if the center point of the Laplacian is one of the corners on the monodromy face labeled by 0, the loop around the center point must enclose the monodromy; as a result its lift to $[\Omega_\delta, 0]$ is not closed, and thus discrete holomorphicity *does not* imply harmonicity at the real and imaginary corners on the face 0. We may still obtain harmonicity of a discrete holomorphic spinor at one of those two types of corners if we assume in addition that it vanishes at the other, since the sum of discrete derivatives will vanish as though the spinor does not branch at 0.

2.2.2 Discrete Singularities

Discrete singularities appear as violations of the s-holomorphicity projection relations relations. To study these, we define front and back values at a singularity in order to introduce the notion of *discrete residue* of a function H_δ at an oriented medial vertex.

Definition 2.2.3. Let H_δ be a function defined on an oriented medial vertex $\alpha = a^o$ and its adjacent four corners. Let c_1, c_2 be the two corners adjacent to a in the direction of o (i.e., $c_1 = a + \frac{\sqrt{2}\mathrm{Re}(o)\delta}{2}$ and $c_2 = a + \frac{\sqrt{2}\mathrm{Im}(o)\delta}{2}$). We define the *front* value $H_\delta(\alpha_+)$ as the unique value such that

$$H_\delta(c_1) = \mathsf{P}_{l(c_1)}\left[H_\delta(\alpha_+)\right], \qquad \text{and} \qquad H_\delta(c_2) = \mathsf{P}_{l(c_2)}\left[H_\delta(\alpha_+)\right].$$

Likewise if c_3, c_4 are the two corners adjacent to a in the direction of $-o$ (i.e., $c_3 = a - \frac{\sqrt{2}\mathrm{Re}(o)\delta}{2}$ and $c_4 = a - \frac{\sqrt{2}\mathrm{Im}(o)\delta}{2}$), we set the *back* value $H_\delta(\alpha_-)$ as the unique value such that

$$H_\delta(c_3) = \mathsf{P}_{l(c_3)}\left[H_\delta(\alpha_-)\right], \qquad \text{and} \qquad H_\delta(c_4) = \mathsf{P}_{l(c_4)}\left[H_\delta(\alpha_-)\right].$$

Definition 2.2.4. The *discrete residue* of H_δ at α is the difference $\mathrm{Res}_\alpha(H_\delta) := H_\delta(\alpha_+) - H_\delta(\alpha_-)$.

By definition H_δ has an s-holomorphic extension to a if and only if the discrete residue is zero at α. It is an analog of the residue in the continuous setting in that doing a closed contour sum around a along the edges of the lattice (i.e. summing $H_\delta(e)\vec{e}$ where $\vec{e}$ is the vector pointing from the start to the end of the edge e on the closed counterclockwise path) will yield $\sqrt{2}oi\delta\,\mathrm{Res}_\alpha(H_\delta)$, for any choice of o.

2.2.3 Discrete Riemann Boundary Value Problems

The key tool for our analysis is the study of discrete Riemann boundary value problems. To prove the convergence of the Ising model observables as the mesh size goes to zero, we will formulate them as the unique solutions to such problems.

Recall that we denote by $\partial\mathcal{V}^m_{\Omega_\delta}$ the set of boundary medial vertices, by ν_z the outer normal at any $z \in \partial\mathcal{V}^m_{\Omega_\delta}$, i.e. the orientation at z which points outward from Ω_δ, and ∂_{ν_z} the outer normal difference, i.e. the value on the outer adjacent vertex minus the value on the inner adjacent vertex.

Definition 2.2.5. We say that a function $H_\delta : \mathcal{V}^{cm}_{\Omega_\delta} \to \mathbb{C}$ defined on corners and medial vertices of a discrete domain Ω_δ is the solution to the *discrete Riemann boundary value problem* on Ω_δ with boundary data $f : \partial\mathcal{V}^m_{\Omega_\delta} \to \mathbb{C}$ if it is s-holomorphic and $H_\delta(z) - f_\delta(z) \in \nu_z^{-\frac{1}{2}}\mathbb{R}$ for any boundary medial vertex $z \in \partial\mathcal{V}^m_{\Omega_\delta}$; note that the definition is independent of the branch of the square root $\nu_z^{-\frac{1}{2}}$. This notion is straightforwardly generalized to a function on the double cover $H'_\delta : \mathcal{V}^{cm}_{[\Omega_\delta,0]} \to \mathbb{C}$ and the boundary data $q : \partial\mathcal{V}^m_{[\Omega_\delta,0]} \to \mathbb{C}$ by adding the assumption that both have monodromy -1 around the origin.

Before proving a useful uniqueness result for the discrete Riemann boundary value problems, we introduce the crucial notion of *integration of the square* of an s-holomorphic function, defined on the vertices and faces (see also [Smi10, Hon10, CHI15]). Although the square of an s-holomorphic function is not s-holomorphic, we can "line-integrate" the square of its magnitude to obtain a single-valued function without monodromy. Its restrictions to the two rotated square lattices respectively of faces and vertices are not harmonic, but they are respectively super-harmonic and sub-harmonic, which will allow us to derive estimates crucial for proofs of the convergence.

$\mathbb{I}_\delta(H_\delta)$ is a discrete analogue of the line integral $\operatorname{Re}\int [H_\delta]^2\, dz$, defined as follows.

Proposition 2.2.1 ([Smi10, Lemma 3.8])**.** *Let H_δ be an s-holomorphic function on Ω_δ. There exists a function $\mathbb{I}_\delta[H_\delta] : \mathscr{F}_{\Omega_\delta} \cup \mathcal{V}_{\Omega_\delta} \to \mathbb{R}$ uniquely constructed (up to an additive constant) with the rule*

$$\mathbb{I}_\delta[H_\delta](w) - \mathbb{I}_\delta[H_\delta](v) = 2\delta \left| H_\delta\left(\frac{1}{2}(w+v)\right)\right|^2,$$

where w is a face, v is a vertex incident to the face, so that $\frac{1}{2}(w+v)$ is the corner between them.

It has $\Delta_\delta \mathbb{I}_\delta[H_\delta] = 2\delta\,|\partial_\delta H_\delta|^2$ on $\mathscr{F}_{\Omega_\delta}$, $\Delta_\delta \mathbb{I}_\delta[H_\delta] = -2\delta\,|\partial_\delta H_\delta|^2$ on $\mathcal{V}_{\Omega_\delta}$.

The following uniqueness statement for both types of the discrete Riemann boundary value problems then allows one to characterize s-holomorphic functions in terms of their boundary values.

Lemma 2.2.2. *If H_δ is a solution of the discrete Riemann boundary value problem on Ω_δ with boundary data 0, it is identically zero. Similarly, if H'_δ is a solution of the discrete Riemann boundary value problem on $[\Omega_\delta, 0]$ with boundary data 0, it is identically zero.*

Proof. The case of H_δ is treated in [Hon10, Proposition 28], but we summarize it here. Given s-holomorphicity and $\mathsf{P}_{\nu_z^{-\frac{1}{2}}} H_\delta = 0$, we can calculate $\partial_{\nu_z}\mathbb{I}_\delta(H_\delta)(z) = \sqrt{2}\delta \left|\mathsf{P}_{i\nu_z^{-\frac{1}{2}}} H_\delta(z)\right|^2$. Then by using the discrete divergence formula ([Hon10, Lemma 6]) and the Laplacian, we can bound from above the orthogonal component of H_δ on the boundary:

$$0 \le \sum_{z\in\partial\mathcal{V}^m_{\Omega_\delta}} \partial_{\nu_z}\mathbb{I}_\delta(H_\delta)(z) = \sum_{v\in\mathcal{V}_{\Omega_\delta}} \Delta_\delta \mathbb{I}_\delta(H_\delta)(v) \le 0,$$

which implies that $H_\delta \equiv 0$ on $\partial\mathcal{V}^m_{\Omega_\delta}$ and that $\Delta_\delta\mathbb{I}_\delta[H_\delta] = -2\delta\,|\partial_\delta H_\delta|^2 \equiv 0$ in $\mathcal{V}_{\Omega_\delta}$, so $H_\delta \equiv 0$ in $\mathcal{V}_{\Omega_\delta}$.

For H'_δ, note that (see [ChIz13, Proposition 4.1]) we can similarly define the single-valued square integral $\mathbb{I}_\delta(H'_\delta)$ with single valued increments $\mathbb{I}_\delta\left[H'_\delta\right](w) - \mathbb{I}_\delta\left[H'_\delta\right](v) = 2\delta\left|H'_\delta\left(\frac{1}{2}(w+v)\right)\right|^2$. While its restriction to faces fails to be sub-harmonic at the monodromy face in general, $\mathbb{I}_\delta(H'_\delta)$

on vertices is nonetheless super-harmonic everywhere with positive outer difference, and we can apply the same argument as in the H_δ case. □

2.3 Discrete Two-point Observables

In this section, we introduce discrete observables, which connect Ising model correlations to discrete complex analysis. Bounded domain observables are defined by summing Boltzmann weights over the set of contours made of the edges in the lattice, alluding to a path integral formulation.

In this section, we define the two-point functions, in terms of which the correlations will be formulated in Propositions 2.4.9 and 2.4.10. In §2.4, multipoint versions of these observables are introduced to prove these statements.

Low-temperature Expansion

In this paper, we will use the *low-temperature expansion* of the Ising model: we represent a spin configuration by the set of edges separating faces of opposite spins. Through this representation, the probability of a set of edges ω is proportional to $e^{-2\beta|\omega|}$. When considering + boundary conditions, the relevant set of edges form a collection of loops (sets of distinct edges $\{e_1,...,e_k\}$ such that e_i is incident to e_{i+1} for every $1 \le i \le k$ with $e_{k+1} = e_1$). In identifying an edge configuration ω in which every vertex is incident to an even number of edges, with a *collection of loops*, there is a possible ambiguity at vertices incident to four edges. For concreteness, we fix the convention that at such ambiguous vertices, loops proceed by joining northwest edges to northeast edges and southwest edges to southeast edges. We further prove that all quantities we consider are independent of choice of convention at ambiguous vertices. Denote by $\mathscr{C}_{\Omega_\delta}$ the set of all such ω (subsets of edges of $\mathscr{E}_{\Omega_\delta}$ with every vertex incident an even number of edges), corresponding to collections of closed loops in Ω_δ.

As a result, for the critical Ising model with + boundary conditions, the low-temperature expansion of the partition function is thus obtained by summing over the set $\mathscr{C}_{\Omega_\delta}$:

$$\mathscr{Z}_{\Omega_\delta} := \sum_{\omega\in\mathscr{C}_{\Omega_\delta}} e^{-2\beta_c|\omega|}.$$

We also note that in this representation, the value of a spin is determined by the parity of the number of loops around it (independently of the choice of convention above), and it is easy to see that

$$\mathbb{E}_{\Omega_\delta}[\sigma_0] = \frac{\sum_{\omega\in\mathscr{C}_{\Omega_\delta}} e^{-2\beta_c|\omega|}(-1)^{\ell(\omega)}}{\mathscr{Z}_{\Omega_\delta}},$$

where $\ell(\omega)$ counts the number of loops in ω that surround 0.

Disorder Lines

The main tool in the study of the 2D Ising model is its fermionic formulation. In this paper, we use the low-temperature representation of the Ising fermion. The relevant sets of contours are deformed versions of $\mathscr{C}_{\Omega_\delta}$ above: in addition to the collection of closed loops in Ω_δ, there are paths linking a pair of marked points of the lattice. In the language of Kadanoff and Ceva (see [KaCe71]), these correspond to the results of the insertion of disorder operators next to the spin.

A medial vertex divides the corresponding edge into two *half-edges*. A *walk* between two medial vertices a, z is a sequence that consists of a half-edge of a, then continues on successively adjacent, all distinct edges, before reaching a half-edge of z. If $\gamma^{a,z}$ is any such a walk, $\mathscr{C}_{\Omega_\delta}^{a,z} := \{\omega \oplus \gamma^{a,z} : \omega \in \mathscr{C}_{\Omega_\delta}\}$, where $\oplus$ denotes the symmetric difference operation (where the symmetric difference of a half-edge and its edge is defined in the natural way), clearly does not depend on the choice of $\gamma^{a,z}$. Each $\gamma \in \mathscr{C}_{\Omega_\delta}^{a,z}$ corresponds to a walk from a and to z and possibly some collection of loops.

For an element $\gamma \in \mathscr{C}_{\Omega_\delta}^{a,z}$, we say a walk $\pi(\gamma) \subset \gamma$ from a to z is an *admissible choice of walk* if whenever it arrives at an ambiguous vertex, i.e. incident to four edges in γ, it chooses to connect northeast with northwest edges and southeast with southwest edges (in accordance with the aforementioned convention for loops). Again, we will prove well-definedness of relevant quantities so that the choice of convention here is irrelevant. When one or both of a, z are instead corners, the above is defined analogously, where "half-edge" is understood to mean the segment joining the corner to its nearest vertex.

Recall that any choice of orientation on a medial vertex is in the direction of exactly one of its two incident half-edges (see Section 2.1.5). If $\alpha = a^o$ and $\zeta = z^p$ are oriented medial vertices, set $\mathscr{C}_{\Omega_\delta}^{\alpha,\zeta}$ to be the subset of $\gamma \in \mathscr{C}_{\Omega_\delta}^{a,z}$ including the particular half-edges given by the respective orientations at a and z.

2.3.1 Bounded Domain Observables

In this subsection, we define the fermion and the spin-fermion observables. The former is a function defined on the discrete domain Ω_δ, whereas the latter is defined on the double cover, $[\Omega_\delta, 0]$. Using the above definitions of loops, walks, $\mathscr{C}_{\Omega_\delta}^{\alpha,\zeta}$, and admissible choices of walks, we define some quantities central to the presentation of the (two-point) fermion and spin-fermion observables.

Definition 2.3.1. If $\alpha = a^o$ and $\zeta = z^p$ are s-oriented medial vertices or corners, define the constants,

$$\mathrm{c}_v := \begin{cases} 1 & \text{if } v \in \mathcal{V}_{\Omega_\delta}^m \\ \cos\frac{\pi}{8} & \text{if } v \in \mathcal{V}_{\Omega_\delta}^c \end{cases} \qquad \text{and} \qquad \lambda_{\alpha,\zeta} := \frac{\sqrt{p}}{\sqrt{o}}\mathrm{c}_a\mathrm{c}_z .$$

For a walk and loops $\gamma \in \mathscr{C}_{\Omega_\delta}^{\alpha,\zeta}$, define its length $|\gamma|$ as the number of full edges (where the two half-edges at the ends together count as one) in γ and for an admissible choice of walk $\pi(\gamma)$ in γ from a to z, denote its winding (more accurately, turning) angle by $\mathbf{W}(\pi(\gamma))$ as the total change in argument of the velocity vector of the walk $\pi(\gamma)$ from a to z (see §5.2.1 of [Hon10]). The choice of counting two half-edges together as one full-edge, is different from the counting in [CHI15]; this leads to an appearance of a normalizing factor of $(\cos\frac{\pi}{8})^2 e^{-2\beta_c} = \frac{1}{2\sqrt{2}}$, whenever considering observables with both arguments in corners.

The following is a real-valued weight on $\gamma \in \mathscr{C}_{\Omega_\delta}^{\alpha,\zeta}$:

$$\phi_{\alpha,\zeta}(\gamma) := i\lambda_{\alpha,\zeta}\, e^{-2\beta_c|\gamma|}\, e^{-\frac{i}{2}\mathbf{W}(\pi(\gamma))}.$$

We also recall for a collection of loops $\omega \in \mathscr{C}_{\Omega_\delta}$ the definition of $\ell(\omega)$ as the number of loops of ω around 0 (whose parity is independent of our convention for loops). See [Hon10, Proposition 67] for the well-definedness (i.e., independence of the choice of convention for the admissible path $\pi(\gamma)$) of $\phi_{\alpha,\zeta}$.

When α,ζ are s-oriented medial vertices or corners on $[\Omega_\delta,0]$, we define the spin-fermion weight as

$$\phi^{\Sigma}_{\alpha,\zeta}(\gamma) := \phi_{\alpha,\zeta}(\gamma)\,(-1)^{\ell(\gamma\setminus\pi(\gamma))}\,\mathrm{s}_{\alpha,\zeta}(\pi(\gamma)),$$

where $\pi(\gamma)$ is any admissible choice of walk, and $\mathrm{s}_{\alpha,\zeta}$ is the *sheet number* defined by

$$\mathrm{s}_{\alpha,\zeta}(\pi(\gamma)) = \begin{cases} +1 & \text{if } \pi(\gamma) \text{ lifted to } [\Omega_\delta,0] \text{ connects } \alpha \text{ to } \zeta \\ -1 & \text{if } \pi(\gamma) \text{ lifted to } [\Omega_\delta,0] \text{ connects } \alpha \text{ to } \zeta^* \end{cases},$$

where ζ^* is the point on $[\Omega_\delta,0]$ that is distinct from ζ but shares its projection with ζ. Here, the real-valued weight $\phi_{\alpha,\zeta}(\gamma)$ is still computed by identifying α,ζ with their projections to Ω_δ.

See Remark 2.2(ii) of [CHI15] for the well-definedness of the spin-fermion weight $\phi^{\Sigma}_{\alpha,\zeta}$.

We are now in position to define the real fermion $F_{\Omega_\delta}^{\alpha,\zeta}$ and the real spin-fermion $F_{[\Omega_\delta,0]}^{\alpha,\zeta}$.

Definition 2.3.2. The (real) *fermion observable* F_{Ω_δ} is a function of two variables $(\alpha,\zeta) \mapsto F_{\Omega_\delta}^{\alpha,\zeta}$, where $\alpha := a^o$ and $\zeta := z^p$ are s-oriented corners or medial vertices of Ω_δ given by

$$F_{\Omega_\delta}^{\alpha,\zeta} := \frac{1}{\mathcal{Z}_{\Omega_\delta}} \sum_{\gamma \in \mathscr{C}_{\Omega_\delta}^{\alpha,\zeta}} \phi_{\alpha,\zeta}(\gamma).$$

Definition 2.3.3. The (real) *spin-fermion observable* $F_{[\Omega_\delta,0]}$ is a function of two variables $(\alpha,\zeta) \mapsto F_{[\Omega_\delta,0]}^{\alpha,\zeta}$, where $\alpha := a^o$ and $\zeta := z^p$ live on the double cover $[\Omega_\delta,0]$ of domain Ω_δ

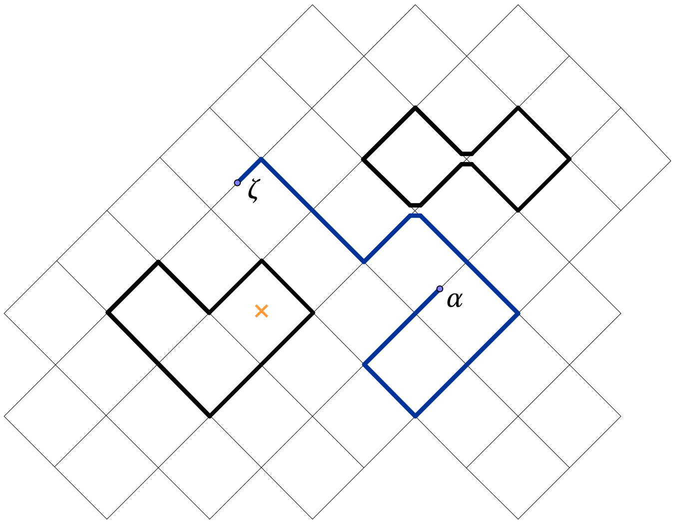

Figure 2.3.1 – An configuration $\gamma \in \mathscr{C}^{\alpha,\zeta}_{[\Omega_\delta,0]}$, with an admissible choice of walk $\pi(\gamma)$ in blue, from α to ζ. The winding of the walk has $\mathbf{W}(\pi(\gamma)) = 2\pi$. The loop number $\ell(\gamma)$ is 1 since there is precisely one loop with 0 in its interior and $s_{\alpha,\zeta}(\pi(\gamma)) = 1$.

ramified at 0. Define $F^{\alpha,\zeta}_{[\Omega_\delta,0]}$ by

$$F^{\alpha,\zeta}_{[\Omega_\delta,0]} := \frac{1}{\mathcal{Z}_{\Omega_\delta}\mathbb{E}_{\Omega_\delta}[\sigma_0]} \sum_{\gamma \in \mathscr{C}^{\alpha,\zeta}_{\Omega_\delta}} \phi^{\Sigma}_{\alpha,\zeta}(\gamma).$$

Again, when computing $\mathcal{Z}_{\Omega_\delta}$, and $\mathscr{C}^{\alpha,\zeta}_{\Omega_\delta}$ and $\phi_{\alpha,\zeta}$, identify α, ζ with their projections to Ω_δ.

The well-definedness of $F^{\alpha,\zeta}_{\Omega_\delta}$ and $F^{\alpha,\zeta}_{[\Omega_\delta,0]}$ are implied by well-definedness of $\phi_{\alpha,\zeta}$ and $\phi^{\Sigma}_{\alpha,\zeta}$ respectively.

Remark 2.3.1. Informally, one can think of $F_{[\Omega_\delta,0]}$ as the natural modification to F_{Ω_δ} when one tries to reweight it by the value of the spin at 0. Since the spin and the disorders are not mutually local (but quasi-local instead), this gives rise to a multivalued function (with monodromy -1 around 0).

Antisymmetry of the Observables

An important elementary feature of the observables $F_{\Omega_\delta}, F_{[\Omega_\delta,0]}$ is their antisymmetry properties, immediate from their definitions. The fermion observable satisfies the following:

Lemma 2.3.1. *If $\alpha := a^o, \zeta := z^p$ are s-oriented corners or medial vertices of Ω_δ and $\alpha' := a^{o'}, \zeta' := z^{p'}$ where $o' := e^{2\pi i} o$ and $p' := e^{2\pi i} p$, then we have the following antisymmetry prop-*

erties:

$$F_{\Omega_\delta}^{\alpha,\zeta} = -F_{\Omega_\delta}^{\zeta,\alpha} = -F_{\Omega_\delta}^{\alpha',\zeta} = -F_{\Omega_\delta}^{\alpha,\zeta'}.$$

Similarly, the spin-fermion observable satisfies the following antisymmetry properties.

Lemma 2.3.2. *If $\alpha := a^o, \zeta := z^p$ are s-oriented corners or medial vertices of $[\Omega_\delta, 0]$, and α', ζ' are as in the previous lemma and $\alpha^* := (a^*)^o, \zeta^* := (z^*)^p$ where a, a^* and z, z^* are respectively distinct lifts of the same points in $\Omega \setminus \{0\}$, we have*

$$F_{[\Omega_\delta,0]}^{\alpha,\zeta} = -F_{[\Omega_\delta,0]}^{\zeta,\alpha} = -F_{[\Omega_\delta,0]}^{\alpha',\zeta} = -F_{[\Omega_\delta,0]}^{\alpha,\zeta'} = -F_{[\Omega_\delta,0]}^{\alpha^*,\zeta} = -F_{[\Omega_\delta,0]}^{\alpha,\zeta^*}.$$

Recall that we define complex conjugation on the double cover by letting the square root be conjugated. We similarly conjugate the s-orientations, and define $\bar{\alpha} := \bar{a}^{\bar{o}}$ if $\alpha = a^o$. The fact that the contour set of Ω_δ and its mirror image $\overline{\Omega_\delta}$ have a natural bijection arising from the complex conjugation immediately yields the following.

Lemma 2.3.3. *If α, ζ are s-oriented corners or medial vertices in $[\Omega_\delta, 0]$, we have $F_{[\Omega_\delta,0]}^{\alpha,\zeta} = -F_{\left[\overline{\Omega_\delta},0\right]}^{\bar{\alpha},\bar{\zeta}}$.*

Complexified Observables

As explained in the previous subsections, the observables introduced in the previous subsections are real quantities antisymmetric in their two variables; exploiting those properties we can define the following complexified versions, which can be analyzed with discrete complex analysis.

Definition 2.3.4. Let α be an s-oriented corner or medial vertex of Ω_δ. For an (unoriented) medial vertex $z \in \mathcal{V}_{\Omega_\delta}^m$, we define the *complex fermion-fermion observable* $H_{\Omega_\delta}^{\alpha}$ by

$$H_{\Omega_\delta}^{\alpha}(z) := \frac{1}{i\sqrt{p_1}} F_{\Omega_\delta}^{\alpha,\zeta^1} + \frac{1}{i\sqrt{p_2}} F_{\Omega_\delta}^{\alpha,\zeta^2},$$

where $\zeta^1 := z^{p_1}$ and $\zeta^2 := z^{p_2}$ are arbitrary s-orientations of z with opposite orientations, i.e. $p_2 = e^{\pm\pi i} p_1$. The resulting quantity is easily seen to be well-defined regardless of the choice of s-orientations. Similarly for a corner $c \in \mathcal{V}_{\Omega_\delta}^c$ with s-orientation $\kappa = c^q$, define

$$H_{\Omega_\delta}^{\alpha}(c) := \frac{1}{i\sqrt{q}} F_{\Omega_\delta}^{\alpha,\kappa}.$$

Define the complexified spin-fermion observable in the same way:

Definition 2.3.5. Let α be an s-oriented corner or medial vertex of $[\Omega_\delta, 0]$, let z be a medial vertex of $[\Omega_\delta, 0]$, and let c be a corner of $[\Omega_\delta, 0]$. Using the same notation as in Definition 2.3.4, we define the *complex spin-fermion observable* $H_{[\Omega_\delta,0]}$ by

$$\begin{aligned} H^{\alpha}_{[\Omega_\delta,0]}(z) &:= \frac{1}{i\sqrt{p_1}} F^{\alpha,\zeta^1}_{[\Omega_\delta,0]} + \frac{1}{i\sqrt{p_2}} F^{\alpha,\zeta^2}_{[\Omega_\delta,0]}, \\ H^{\alpha}_{[\Omega_\delta,0]}(c) &:= \frac{1}{i\sqrt{q}} F^{\alpha,\kappa}_{[\Omega_\delta,0]}. \end{aligned}$$

Remark 2.3.2. Note that, given a complexified observable, the real observables can be recovered by $F^{\alpha,\zeta}_{\Omega_\delta} = i\sqrt{p}\mathsf{P}_{\frac{1}{i\sqrt{p}}\mathbb{R}}\left[H^{\alpha}_{\Omega_\delta}(z)\right] = \mathrm{Re}\left[i\sqrt{p}H^{\alpha}_{\Omega_\delta}(z)\right]$ for medial vertices $\zeta = z^p$, and obviously at corners $\kappa = c^q$, we have $F^{\alpha,\kappa}_{\Omega_\delta} = \mathrm{Re}\,[i\sqrt{q}H^{\alpha}_{\Omega_\delta}(c)]$.

Definition 2.3.6. Let $\Lambda_\delta = \Omega_\delta \cap \overline{\Omega_\delta}$. Then define the symmetrized and antisymmetrized observables:

$$S^{\alpha}_{[\Omega_\delta,0]} := \frac{1}{2}\left[H^{\alpha}_{[\Omega_\delta,0]} + \overline{H^{\bar{\alpha}}_{[\Omega_\delta,0]}}\right], \qquad A^{\alpha}_{[\Omega_\delta,0]} := \frac{1}{2}\left[H^{\alpha}_{[\Omega_\delta,0]} - \overline{H^{\bar{\alpha}}_{[\Omega_\delta,0]}}\right].$$

The following lemma then follows immediately from Lemma 2.3.1 and the definition of $H_{[\Omega_\delta,0]}$.

Lemma 2.3.4. *Let α be an s-oriented corner or medial vertex in $[\Omega_\delta,0]$, and z be a corner or medial vertex in $[\Lambda_\delta,0]$. Then,*

$$\begin{aligned} &\text{on } \mathcal{V}^1_{[\Omega_\delta,0]} \cap \mathbb{R}_{>0}, \mathcal{V}^i_{[\Omega_\delta,0]} \cap \mathbb{R}_{<0}, && \text{we have} && S^{\alpha}_{[\Omega_\delta,0]} = H^{\alpha}_{[\Omega_\delta,0]} \text{ and } A^{\alpha}_{[\Omega_\delta,0]} = 0; \\ &\text{on } \mathcal{V}^i_{[\Omega_\delta,0]} \cap \mathbb{R}_{>0}, \mathcal{V}^1_{[\Omega_\delta,0]} \cap \mathbb{R}_{<0}, && \text{we have} && A^{\alpha}_{[\Omega_\delta,0]} = H^{\alpha}_{[\Omega_\delta,0]} \text{ and } S^{\alpha}_{[\Omega_\delta,0]} = 0. \end{aligned}$$

2.3.2 Full Plane Observables

In this section, we study the infinite-volume limits of the fermion and spin-fermion observables. By scale invariance, it is enough to give a characterization on the rotated unit grid $\mathbb{C}_1 = (1+i)\mathbb{Z}^2 + 1$ placed on increasing domains. On $\mathbb{C}_\delta$ we can define $H_{\mathbb{C}_\delta}(a\delta) := H_{\mathbb{C}_1}(a)$ and $H_{[\mathbb{C}_\delta,0]}(a\delta) := H_{[\mathbb{C}_1,0]}(a)$. In the $\delta \to 0$ scaling limit, these converge to meromorphic functions with a singularity at zero.

We first give a unique characterization for the full-plane limits and establish their existence, and then we give an explicit construction. Using those explicit formulae, we define auxiliary s-holomorphic functions on the double cover which are discrete forms of $\sqrt{z}$ and $i\sqrt{z}$.

We take the limit $\Omega_1 \to \mathbb{C}_1$ using an increasing sequence of bounded domains $\Omega_1^1 \subset \Omega_1^2 \subset \cdots \subset \Omega_1^n \subset \cdots$ such that $\bigcup_n \Omega_1^n = \mathbb{C}_1$. The limiting functions will be seen to be unique, so that they do not depend on the particular sequence.

Full Plane Fermion Observable

The following are straightforward modifications to our setting, of the construction [Hon10] of the full-plane fermion.

Theorem 2.3.1. *As $\Omega_1 \to \mathbb{C}_1$, the complexified fermion observable $H_{\mathbb{C}_1} := \lim_{\Omega\to\mathbb{C}} H_{\Omega_1}$ exists and is uniquely characterized by the following properties:*

- *if $\alpha = a^o$ is an s-oriented medial vertex,*
  - *$H^{\alpha}_{\mathbb{C}_1}$ is s-holomorphic on $\mathbb{C}_1 \setminus \{a\}$;*
  - *At α we have the discrete residue $H^{\alpha}_{\mathbb{C}_1}(\alpha_+) - H^{\alpha}_{\mathbb{C}_1}(\alpha_-) = \frac{1}{\sqrt{o}}$;*
  - *$H^{\alpha}_{\mathbb{C}_1}(l) \to 0$ as $|l| \to \infty$.*
- *For $\zeta = z^p$, the full-plane fermion*

$$F^{\alpha,\zeta}_{\mathbb{C}_1} := i\sqrt{p}\cdot \mathsf{P}_{\frac{1}{i\sqrt{p}}\mathbb{R}}\left[H^{\alpha}_{\mathbb{C}_1}(z)\right]$$

  satisfies the antisymmetry properties of Lemma 2.3.1.

Proof. For a given s-oriented medial vertex α, such an $H^{\alpha}_{\mathbb{C}_1}$ must be uniquely determined: if any two satisfy the above properties, their difference will have an s-holomorphic extension to a, but any entire s-holomorphic function which decays to 0 at infinity must be zero, since its real and imaginary parts will be discrete harmonic functions.

To use the same reasoning when α is an s-oriented corner with adjacent medial vertices z, z', it suffices to show that the s-holomorphic singularity at α, i.e. concretely the value of nonzero $\mathsf{P}_{l(\alpha)}\left[H^{\alpha}_{\mathbb{C}_1}(z')\right] - \mathsf{P}_{l(\alpha)}\left[H^{\alpha}_{\mathbb{C}_1}(z)\right]$, is fixed by the medial vertex case above. But the antisymmetry relation of Lemma 2.3.1 gives $H^{\alpha}_{\mathbb{C}_1}(z) = \frac{1}{i\sqrt{p_1}}F^{\alpha,\zeta^1}_{\mathbb{C}_1} + \frac{1}{i\sqrt{p_2}}F^{\alpha,\zeta^2}_{\mathbb{C}_1}$ for $\zeta^1 = z^{p_1}$ and $\zeta^2 = z^{p_2}$, where p_1 and p_2 are s-orientations of the two opposite orientations of z. Since $F^{\alpha,\zeta^1}_{\mathbb{C}_1} = -F^{\zeta^1,a}_{\mathbb{C}_1}$ and $F^{\alpha,\zeta^2}_{\mathbb{C}_1} = -F^{\zeta^2,a}_{\mathbb{C}_1}$, both terms are determined by their values on medial vertices, and similarly for $H^{\alpha}_{\mathbb{C}_1}(z')$.

An explicit formula, Eq. (2.3.1), for this full-plane observable, and thus its existence, is given by Proposition 2.3.5. Then the fact that the given explicit function is the infinite-volume limit is immediate from Theorem 2.5.1. □

Proposition 2.3.5. *Let $a, z \in \mathcal{V}^m_{\mathbb{C}_1}$ and for an s-orientation $(\sqrt{o})^2$ on a, write $\alpha = a^o$ for the s-oriented medial vertex. The function*

$$\begin{aligned} H^{\alpha}_{\mathbb{C}_1}(z) = {} & \frac{e^{\pi i/8}}{\sqrt{o}}\cos\frac{\pi}{8}\left(C_0\left(\frac{\sqrt{2}a}{o}+1, \frac{\sqrt{2}z}{o}\right) + C_0\left(\frac{\sqrt{2}a}{o}-i, \frac{\sqrt{2}z}{o}\right)\right) \\ & + \frac{e^{-3\pi i/8}}{\sqrt{o}}\sin\frac{\pi}{8}\left(C_0\left(\frac{\sqrt{2}a}{o}-1, \frac{\sqrt{2}z}{o}\right) + C_0\left(\frac{\sqrt{2}a}{o}+i, \frac{\sqrt{2}z}{o}\right)\right) \end{aligned} \tag{2.3.1}$$

for $z \neq a$ satisfies the properties of Theorem 2.3.1, where the translation invariant function C_0 is *the dimer coupling function defined in [Ken00]:*

$$C_0(z_1,z_2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\exp(i(xs-yt))}{2i\sin s+2\sin t}dsdt, \quad \text{if } z_2-z_1 = x+iy.$$

Moreover, at α, the front and back values of the s-holomorphic singularity are given by $H^{\alpha}_{\mathbb{C}_1}(\alpha_\pm) = \frac{\mu\pm 1}{2\sqrt{o}}$.

Proof. These properties were verified in [Hon10, Proposition 22] for a version on the non-rotated lattice, which we will call $H_{\mathbb{Z}^2}$. We note that for any s-oriented medial vertex $\alpha = a^o$, we have

$$H^{\alpha}_{\mathbb{C}_1}(z) = e^{-\frac{\pi i}{8}}H^{\alpha'}_{\mathbb{Z}^2}(z')$$

if $\alpha' = (a')^{o'}$ is the rotated medial vertex $a' = \frac{a}{1+i}\in\mathcal{V}^m_{\mathbb{Z}^2}$ oriented to $\sqrt{o'} = e^{-\frac{\pi i}{8}}\sqrt{o}$, and $z' = \frac{z}{1+i}$. Given that the projection lines in the s-holomorphicity relation Eq. (2.2.1) are also rotated by $e^{-\frac{\pi i}{8}}$ from the definition in [Hon10], the results are easily seen to carry over. The explicit front and back values $H^{\alpha}_{\mathbb{C}_1}(\alpha_\pm)$ follow from straightforward computation. □

Full Plane Spin-fermion Observable

First we recall some notation. Define the left slit plane $\mathbb{X} = \mathbb{C}\setminus\mathbb{R}_{<0}$ and the right slit plane $\mathbb{Y} = \mathbb{C}\setminus\mathbb{R}_{>0}$. The double cover $[\mathbb{C},0]$ contains two lifts $\mathbb{X}^\pm$ of $\mathbb{X}$ and two lifts $\mathbb{Y}^\pm$ of $\mathbb{Y}$; define $\sqrt{z}$ on the double cover such that the superscripts of $\mathbb{X}^\pm$ denote the sign of the real part of $\sqrt{z}$ and those of $\mathbb{Y}^\pm$ denote the sign of the imaginary part of $\sqrt{z}$. In other words, $\mathbb{X}^+\cap\mathbb{Y}^+, \mathbb{X}^-\cap\mathbb{Y}^-$ are lifts of the upper half plane, and $\mathbb{X}^+\cap\mathbb{Y}^-, \mathbb{X}^-\cap\mathbb{Y}^+$ are lifts of the lower half plane. We use the process outlined in Remark 2.2.2 to define an s-holomorphic extension from a discrete holomorphic function defined on type 1 and i corners, so let us define the slit discrete domains $\mathbb{X}^1_1 := \mathcal{V}^1_{\mathbb{C}_1}\cap\mathbb{X}\cong\mathcal{V}^1_{[\mathbb{C}_1,0]}\cap\mathbb{X}^\pm$ and $\mathbb{Y}^i_1 := \mathcal{V}^i_{\mathbb{C}_1}\cap\mathbb{Y}\cong\mathcal{V}^i_{[\mathbb{C}_1,0]}\cap\mathbb{Y}^\pm$.

Theorem 2.3.2. *As $\Omega_1\to\mathbb{C}_1$, the complexified spin-fermion observable $H_{[\mathbb{C}_1,0]} := \lim_{\Omega\to\mathbb{C}} H_{[\Omega_1,0]}$ exists and is uniquely characterized by the following properties: for every $\alpha = a^o\in\mathcal{V}^{cm}_{[\mathbb{C}_1,0]}$,*

A $H^{\alpha}_{[\mathbb{C}_1,0]}$ *has monodromy -1 around 0.*

B $H^{\alpha}_{[\mathbb{C}_1,0]}$ *is s-holomorphic on $[\mathbb{C}_1,0]\setminus\{a,a^*\}$, where a^* is the point in $[\mathbb{C}_1,0]$ distinct from a which shares its projection onto $\mathbb{C}_1$ with the projection of a.*

C *if $\alpha = a^o$ is an s-oriented medial vertex, we have discrete residue $H^{\alpha}_{[\mathbb{C}_1,0]}(\alpha_+) - H^{\alpha}_{[\mathbb{C}_1,0]}(\alpha_-) = \frac{1}{\sqrt{o}}$.*

D *If $\alpha = a^o$ is an s-oriented real or imaginary corner, $P_{l(a)}H^{\alpha}_{[\Omega_1,0]}(a\pm\frac{i}{2}) = \mp\frac{i}{2\sqrt{2}}\sqrt{o}$.*

E *For $\zeta = z^p$, the full-plane spin-fermion,*

$$F^{\alpha,\zeta}_{[\mathbb{C}_1,0]} := i\sqrt{p}\cdot\mathsf{P}_{\frac{1}{i\sqrt{p}}\mathbb{R}}\left[H^{\alpha}_{[\mathbb{C}_1,0]}(z)\right]$$

satisfies the antisymmetry properties laid out in Lemmas 2.3.2–2.3.3.

F $H^{\alpha}_{[\mathbb{C}_1,0]}(l) \to 0$ *as* $|l| \to \infty$.

We first present the following three lemmas, then conclude the proof using them.

Lemma 2.3.6. *There exists a uniform constant* $M > 0$ *such that* $\left|H^{\alpha}_{[\Omega_1^n,0]}(z)\right| \leq M$ *for all* $n \geq 0$ *and any s-oriented corner* α *and any corner* z.

Proof. The strategy will be to progressively extend the validity of the result to more and more points of the domain. Below we will denote by a and z both corners and medial vertices interchangeably:

1. When $\alpha := \alpha_0$ is the imaginary corner on the monodromy face 0 (specifically, the lift of $\frac{1}{2}$ to $\mathbb{X}_\delta$) and $z = a+1$, $H^{\alpha}_{[\Omega_1^n,0]}(z)$ has a probabilistic interpretation as a ratio of magnetizations $\mathbb{E}_{\Omega_1^n}[\sigma_2] / \mathbb{E}_{\Omega_1^n}[\sigma_0]$, which is bounded from above by the finite-energy property of the model.

2. When $\alpha = \alpha_0$ and z is on the boundary, we claim that $\sum_{z \in \partial \mathcal{V}^m_{[\Omega_1^n,0]}} \left|H^{\alpha}_{[\Omega_1^n,0]}(z)\right|^2 \leq \mathrm{Cst} \cdot H^{\alpha}_{[\Omega_1^n,0]}(a+1)$ (the right hand side of which is bounded by step 1). This inequality follows by considering the discrete analogue $Q_1 := \mathbb{I}_\delta(H^{\alpha}_{[\Omega_1^n,0]})$ of $\mathrm{Re}\left(\int (H^{n,\alpha_0})^2\right)$ analyzed in Section 2.5.1. By Proposition 2.5.3, the restriction of Q_1 to the vertices is super-harmonic (except perhaps at $a+1$), the sum of the Laplacians is hence bounded from above by $\mathrm{Cst} \cdot H^{\alpha}_{[\Omega_1^n,0]}(a+1)$. At the same time the sum of the Laplacians equals the sum of the outer normal derivatives $\partial_{\nu_z} Q_1$ on the boundary of $[\Omega_1^n, 0]$, and these normal derivatives $\partial_{\nu_z} Q_1$ equal $\sqrt{2}\left|H^{\alpha}_{[\Omega_1^n,0]}(z)\right|^2$. Hence we deduce the inequality.

3. When $\alpha = \alpha_0$ and z (corner or medial vertex) is in the interior, we extend the bound of step 2 by the maximum principle.

4. When α is on the boundary and z is the imaginary corner adjacent to the monodromy ($z = \frac{1}{2}$), the bound follows from the antisymmetry of H and step 2.

5. When α and z are on the boundary, we have that $\left|H^{\alpha}_{[\Omega_1^n,0]}(z)\right|$ acquires a probabilistic interpretation: the winding factors out from the sum in the definition of H, and we sum over contours that represent the low-temperature expansion of an Ising model with $+/-$ boundary conditions switching at a and z. As a result, it is easy to show that $\left|H^{\alpha}_{[\Omega_1^n,0]}(z)\right|$ is the ratio $\mathbb{E}^{\pm}_{\Omega_1^n}[\sigma_0] / \mathbb{E}^{+}_{\Omega_1^n}[\sigma_0]$, where $\pm$ and $+$ indicate the boundary conditions. By monotonocity of the Ising magnetization in boundary conditions, this ratio is less than one, which gives us the desired bound.

6. When α is on the boundary and z in the interior, the result follows from steps 4 and 5 and the maximum principle.

7. When α is in the interior and z is next to the monodromy or on the boundary, the result follows from steps 3 and 6 by antisymmetry.

8. When α and z are in the interior, the result follows from the maximum principle. □

Lemma 2.3.7. *Any bounded function $H := H_{[\mathbb{C}_1,0]}$ that satisfies the properties A-E decays at infinity.*

Proof. We exploit the antisymmetry properties E, as specified in Lemmas 2.3.2–2.3.3. The idea is to symmetrize-antisymmetrize H as in Definition 2.3.6 by writing it as $S + A$, where $S^\alpha = \frac{1}{2}\left(H^\alpha + H^{\bar{\alpha}}\right)$ and $A^\alpha = \frac{1}{2}\left(H^\alpha - H^{\bar{\alpha}}\right)$. Let us now show that S and A both decay at infinity.

We have that the restriction of S to real corners vanishes of the positive half-line. We can make a branch cut where it vanishes and study the function on both slit-plane sheets separated by the cut. Since it is uniformly bounded and harmonic except near 0 and $a, \bar{a}$, one can use planar random walk arguments (Beurling estimate) to show that the function vanishes at infinity.

Similarly, the restriction to imaginary corners and analogous restrictions of A vanish on either the positive or the negative half-line. By the same arguments as above, we can conclude the proof. □

Lemma 2.3.8. *There is at most one function satisfying the properties A-F.*

Proof. To prove the uniqueness, it suffices that if we have two such functions, their difference is zero. Denote by $D^\alpha(z)$ this difference, which will be everywhere s-holomorphic and decay at infinity. However, as noted in Remark 2.2.4, the absence of s-holomorphic singularities does not guarantee harmonicity on the monodromy face, and some care is needed there (note that below, we abuse notation to refer to points of $[\mathbb{C}, 0]$ by their projections on $\mathbb{C}$).

For $\alpha_0 = (\frac{1}{2})^o$, D^{α_0} extends s-holomorphically to a_0 by zero by property D, and we have that the real part of D^{α_0} is everywhere harmonic by Remark 2.2.4. As a result, by the maximum principle and discrete holomorphicity, the real part of D^{α_0} vanishes and $D^{\alpha_0} \equiv 0$.

For an arbitrary corner a, by antisymmetry and the previous step, we have $D^\alpha(a_0) = 0$. As a result the real part of $D^\alpha(z)$ is harmonic. Thus, as in the previous step, $D^\alpha(z)$ vanishes everywhere. □

Proof of Theorem 2.3.2. By Lemma 2.3.6, we have that for each s-oriented corner α of $[\mathbb{C}_1, 0]$, the sequence of harmonic functions $H^\alpha_{[\Omega_1^n,0]}$ is uniformly bounded and hence by standard arguments, it admits convergent subsequences as $n \to \infty$ on any finite graph. Any limit along such subsequences satisfies properties A-E and as a result tends to 0 at infinity by Lemma 2.3.7. By Lemma 2.3.8, it is uniquely determined. This shows the convergence of the sequence itself to a limit which satisfies the conditions of the theorem, which we call $H_{[\mathbb{C}_1,0]}$. □

Analytical Expressions

In this subsection, we give characterizations of $H_{[\mathbb{C}_1,0]}$ in a few special cases, then outline an inductive process to construct it explicitly in general.

For the observables with monodromy, we have the following characterization near 0 from [CHI15]. Recall $\alpha_0 = a_0^o$ where $a_0 \in \mathbb{X}^+$ is the lift of $\frac{1}{2}$ to $\mathbb{X}^+$ and $o = (e^{2\pi i})^2$.

Proposition 2.3.9. *For $z \in \mathcal{V}^1_{[\mathbb{C}_1,0]} \cup \mathcal{V}^i_{[\mathbb{C}_1,0]} \setminus \{\frac{1}{2}\}$ we have the characterization*

$$H^{\alpha_0}_{[\mathbb{C}_1,0]}(z) = \begin{cases} \pm\frac{1}{2\sqrt{2}} hm^{\mathbb{X}^1_1}_{3/2}(z) & \text{if } z \in \mathcal{V}^1_{[\mathbb{C}_1,0]} \cap \mathbb{X}^\pm \\ \mp\frac{i}{2\sqrt{2}} hm^{\mathbb{Y}^i_1}_{1/2}(z) & \text{if } z \in \mathcal{V}^i_{[\mathbb{C}_1,0]} \cap \mathbb{Y}^\pm \\ 0 & \text{otherwise} \end{cases},$$

where $hm^{\mathbb{D}_\delta}_a(z)$, for a discrete domain $\mathbb{D}_\delta$ and $a \in \mathbb{D}_\delta \cup \partial\mathbb{D}_\delta$, denotes the harmonic measure of a as seen from z, i.e., the probability that a simple symmetric random walk on $\mathbb{D}_\delta$ started at z will first hit a when or before exiting $\mathbb{D}_\delta$.

Proof. In [CHI15, Lemma 2.14] the function defined above (without the additional normalization factor $\cos^2\frac{\pi}{8}\cdot e^{-2\beta_c} = \frac{1}{2\sqrt{2}}$) is proved to be the only function on $[\mathbb{C}_1, 0]$ which decays at infinity and is s-holomorphic everywhere away from the singularity at $a_0 = \frac{1}{2}$ given by $\mathsf{P}_{l(a)} H^{\alpha}_{[\mathbb{C}_1,0]}(\frac{1\pm i}{2}) = \mp i$. Thus we can identify it as the unique infinite-volume limit introduced in Theorem 2.3.2 for $\alpha = \alpha_0$. □

Remark 2.3.3. The zeros in the definition reflect the fact that a slit plane harmonic measure vanishes everywhere on the slit except at the tip, e.g. $\frac{1}{2}$ in case of $\mathrm{hm}^{\mathbb{Y}^i_1}_{1/2}(z)$. This function is harmonic on all points of $\mathbb{Y}^i_1$, but harmonicity fails on the slit (positive real axis), the boundary of the domain.

The following explicit characterization of the discrete harmonic measure of the slit plane may be of independent interest. Using this, we provide the values of $2\sqrt{2}H^{\alpha_0}_{[\mathbb{C}_1,0]}(z)$ near the origin in Figure 2.3.2.

Theorem 2.3.3. *We have the following expression for the discrete harmonic measure:*

$$H_0(z) := hm^{\mathbb{Y}^i_1}_{1/2}(z = s + ik + \tfrac{1}{2}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{C^{|k|}(\theta)}{\sqrt{1-e^{-2i\theta}}} e^{-is\theta}\, d\theta,$$

where $C(\theta) := \frac{\cos\theta}{1+|\sin\theta|}$.

Proof. We defer the proof of this theorem to Appendix 2.7, Proposition 2.7.1. □

Now we inductively characterize $H^{\alpha}_{[\mathbb{C}_1,0]}$ in the cases where $\alpha \in \mathcal{V}^{1,i}_{[\mathbb{C}_1,0]} \cap \mathbb{R}_{>0}$.

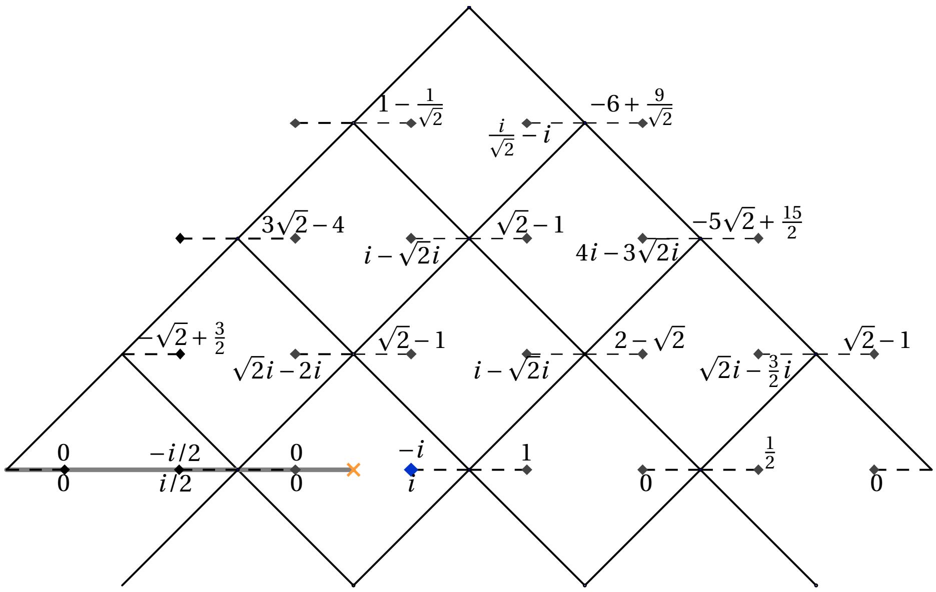

Figure 2.3.2 – Some explicit values for the full-plane spin-fermion observable $2\sqrt{2}H^{\alpha_0}_{[\mathbb{C}_1,0]}(z)$ for $z \in \mathcal{V}^{1,i}_{[\mathbb{C}_1,0]} \cap \mathbb{X}^+$, where $a_0 = \frac{1}{2}$. The function has a monodromy about the origin, marked by the orange ×, and singularity at α_0 marked by the blue ♦.

Proposition 2.3.10. *For an s-oriented corner $\alpha = a^o$ with $a \in \mathbb{X}^+ \cap \mathbb{R}_{\geq 0}$, we can recursively compute $H^{\alpha}_{[\mathbb{C}_1,0]}$ starting from the case $\alpha_0 = \frac{1}{2}^{o_0}, o_0 = (e^{2\pi i})^2$, as a finite linear combination of functions given explicitly in Theorem 2.3.3. Explicitly, for s-oriented corners $\alpha_0 + 2n := (a_0+2n)^{o_0}, \alpha_0 + 2n + 1 := (a_0 + 2n + 1)^{o'}, o' = \left(e^{\frac{\pi i}{2}}\right)^2$, we have on any of the four half-planes $\mathbb{X}^{\pm} \cap \mathbb{Y}^{\pm}$,*

$$H^{\alpha_0+2n}_{[\mathbb{C}_1,0]}(z) = H^{\alpha_0}_{[\mathbb{C}_1,0]}(z-2n) - \sum_{m=1}^{n} \frac{H_0(-2m+2)}{2m} H^{\alpha_0}_{[\mathbb{C}_1,0]}(z-2(n-m)), \text{ and}$$
$$H^{\alpha_0+2n+1}_{[\mathbb{C}_1,0]}(z) = i H^{\alpha_0+2n}_{[\mathbb{C}_1,0]}(z-1) - i\left(H_0(2n+2) + \frac{H_0(2n)}{2(n+1)}\right) H^{\alpha_0}_{[\mathbb{C}_1,0]}(z+1).$$

In fact, for any $a \in \mathcal{V}^1_{[\mathbb{C}_1,0]} \cup \mathcal{V}^i_{[\mathbb{C}_1,0]}$ on the real or imaginary axes, we can compute $H^{\alpha}_{[\mathbb{C}_1,0]}$ using rotational symmetry: if $\alpha = a^o$ is any s-oriented corner on the real or imaginary line, and $\alpha' = (a')^{o'}$ is the rotated corner $a' = e^{-\frac{\pi i}{2}} a \in \mathcal{V}^c_{[\mathbb{C}_1,0]}$ oriented to $o' = (e^{-\frac{\pi i}{4}}\sqrt{o})^2$, and $z' = e^{-\frac{\pi i}{2}} z$, we have,

$$H^{\alpha}_{[\mathbb{C}_1,0]}(z) = e^{-\frac{\pi i}{4}} H^{\alpha'}_{[\mathbb{C}_1,0]}(z').$$

Proof. Generalizing from the proof of Proposition 2.3.9, we construct a function on $[\mathbb{C}_1, 0]$ which decays at infinity and is s-holomorphic everywhere away from the specified singularity at a, and we argue that it is the unique function which can satisfy the properties specified in

Theorem 2.3.2. For convenience, we will take source points α on the sheet $\mathbb{X}^+$ unless otherwise specified; i.e. the positive real line approached from above on $\mathbb{Y}^+$.

Assume $a \in \mathcal{V}^i_{[\mathbb{C}_1,0]} \cap \mathbb{R}_{>0}$ and consider the restriction H^α of $H^\alpha_{[\mathbb{C}_1,0]}$ to the imaginary corners on the slit plane $\mathcal{V}^i_{[\mathbb{C}_1,0]} \cap \mathbb{Y}^+$. The function H^α can be characterized as the unique discrete harmonic function on the slit plane $\mathcal{V}^i_{[\mathbb{C}_1,0]} \cap \mathbb{Y}^+$ which has the single-valued boundary data $H^\alpha(a) := -\frac{i}{2\sqrt{2}}\sqrt{o}$ and zero elsewhere on the slit $\mathbb{R}_{>0}$, and decays at infinity. The harmonic function with these properties can be obtained by translating $H^{\alpha-2}$ for $\alpha - 2 := (a-2)^o$ (which takes its only nonzero boundary value at $a-2$) to the right, and subtracting off a multiple of H^{α_0} in order to cancel the nonzero value at $a_0 = \frac{1}{2}$. Specifically, letting

$$H^\alpha(z) = H^{\alpha-2}(z-2) - H^{\alpha-2}(-\tfrac{3}{2}) \cdot \mathrm{hm}\,^{\mathbb{Y}^i_1}_{1/2}(z) = H^{\alpha-2}(z-2) - \mathrm{hm}\,^{\mathbb{Y}^i_1}_{a-2}(-\tfrac{3}{2}) \cdot H^{\alpha_0}(z)\,,$$

then discrete holomorphicity relations imply that the real part is determined up to a an additive constant; however since the real part must vanish on $\mathbb{R}_{<0}$, the real part is uniquely determined. In fact, it is easy to see that, in order to maintain discrete holomorphicity, the above recursive relation should also hold for the entire $H^\alpha_{[\mathbb{C}_1,0]}$ as long as we are in the four real-translation invariant half-planes $\mathbb{X}^\pm \cap \mathbb{Y}^\pm$. For the explicit identification of the coefficients, we refer to Proposition 2.7.2.

For $a \in \mathcal{V}^1_{[\mathbb{C}_1,0]} \cap \mathbb{R}_{>0}$, we use a similar recursive process but now instead first construct the restriction H'^α of $H^\alpha_{[\mathbb{C}_1,0]}$ to the real corners on $\mathbb{Y}^+$, starting from the case $a = \frac{3}{2}$. Unlike the imaginary case, we need to consider $-\frac{1}{2}$ as well as $\mathcal{V}^1_{[\mathbb{C}_1,0]} \cap \mathbb{R}_{>0}$ as part of the slit boundary (where harmonicity fails): since $H^\alpha_{[\mathbb{C}_1,0]}(\frac{1}{2}) \neq 0$ in general, we cannot assume that the real part is harmonic at $-\frac{1}{2}$ (see Remark 2.2.4).

In other words, H'^α is the function harmonic on the slit plane $\mathcal{V}^1_{[\mathbb{C}_1,0]} \cap \mathbb{Y}^+ \setminus \{-\frac{1}{2}\}$ which takes nonzero boundary values only at a and $-\frac{1}{2} = -a_0$. As above $H'^\alpha(a) = -\frac{i}{2\sqrt{2}}\sqrt{o}$.

For the value at $-a_0 := e^{\pi i}a_0$ with oriented version $-\alpha_0 := (-a_0)^{o'}$, where $o' = (e^{\frac{\pi i}{2}})^2$ we use antisymmetry in the two inputs to write $H^\alpha_{[\mathbb{C}_1,0]}(-a_0) = -i\sqrt{o}H^{-\alpha_0}_{[\mathbb{C}_1,0]}(a)$. Since by rotation $H^{-\alpha_0}_{[\mathbb{C}_1,0]}(a) = -iH^{\alpha_0}_{[\mathbb{C}_1,0]}(e^{-\pi i}a)$, we conclude $H^\alpha_{[\mathbb{C}_1,0]}(-a_0) = -\sqrt{o}H^{\alpha_0}_{[\mathbb{C}_1,0]}(e^{-\pi i}a) = -\frac{i}{2\sqrt{2}}\sqrt{o} \cdot \mathrm{hm}\,^{\mathbb{Y}^i_1}_{1/2}(-a)$. We can match these boundary values with recursion as above.

The general rotation identity can be verified independently with the same strategy, i.e., identifying the restriction of the left-hand side to a specific type of corner as the unique harmonic function with suitable boundary values, which the right-hand side solves. □

Remark 2.3.4. The coefficients in Proposition 2.3.10 of various translated and rotated versions of $H^{\alpha_0}_{[\mathbb{C}_1,0]}$ (with the same scaling limit) become important when identifying the scaling limit of the observables in Lemma 2.5.4. In particular, by Proposition 2.7.2, the coefficients in the recursive expansion sum to zero in the case of $a \in \mathcal{V}^1_{[\mathbb{C}_1,0]} \cap \mathbb{R}_{>0}$, which will yield that $\tilde{C}_\alpha = 0 = C_\alpha$ in Corollary 2.5.8.

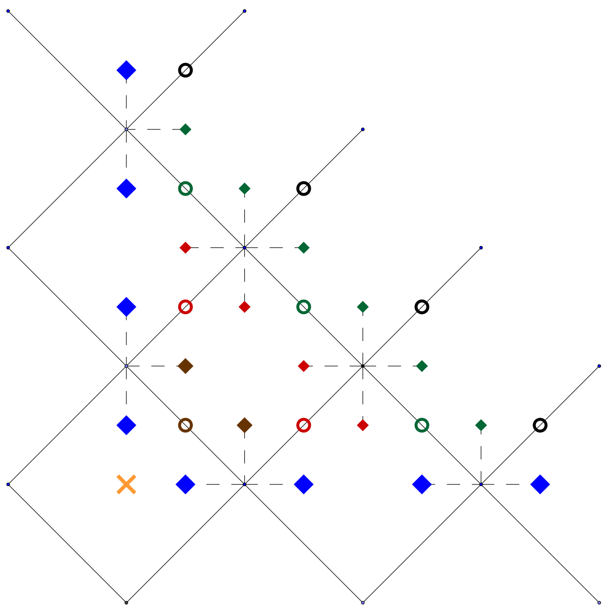

Figure 2.3.3 – The full plane spinor $H^{\alpha}_{[\mathbb{C}_1,0]}(z)$ is first defined at all blue corners ♦ by its antisymmetry relations and Proposition 2.3.10. By s-holomorphicity, we can then deduce its value at the brown medial vertex ∘, and then project from there onto the two brown corners ♦. We continue this process, next moving to the red ∘, then the red ♦, and then green etc.

Corollary 2.3.11. *For any $a \in \mathcal{V}^{cm}_{[\mathbb{C}_1,0]}$, we can recursively compute $H^{\alpha}_{[\mathbb{C}_1,0]}$ as a finite linear combination of functions given by Theorem 2.3.3.*

Proof. By Proposition 2.3.10, $H^{\alpha}_{[\mathbb{C}_1,0]}(z)$ is given for all $z \in \mathcal{V}^{cm}_{[\mathbb{C}_1,0]}$ whenever $\alpha \in \mathcal{V}^{1,i}_{[\mathbb{C}_1,0]} \cap (\mathbb{R}_{>0} \cup i\mathbb{R}_{>0})$. The antisymmetry relations it satisfies thus give $H^{\alpha}_{[\mathbb{C}_1,0]}(z)$ for every α, whenever $z \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}$ (as a finite linear combination of explicit harmonic measures).

Now observe that s-holomorphicity implies that the value, say, of $H^{\alpha}_{[\mathbb{C}_1,0]}(\frac{1+i}{2})$ can be recovered from its values at the $\frac{1}{2}, \frac{i}{2}$. From there, one can project the values of $H^{\alpha}_{[\mathbb{C}_1,0]}(z)$ when $z = 1+\frac{i}{2}$ and $\frac{1}{2}+i$ (see Figure 2.3.3). Continuing this process allows for a recursive construction of $H^{\alpha}_{[\mathbb{C}_1,0]}(z)$ for any $z \in \mathcal{V}^{cm}_{[\mathbb{C}_1,0]}$ as a finite linear combination of the explicit functions of Theorem 2.3.3. □

Auxiliary Functions

We introduce here full-plane auxiliary functions G and $\tilde{G}^{\pm}$, which are everywhere s-holomorphic functions on $[\mathbb{C}_1,0]$ which do not decay at infinity. The real part of G was defined in [CHI15] as a discrete version of the holomorphic function $\sqrt{z}$ on the double cover; we extend the result in order to give full s-holomorphic discrete representations of $\frac{1}{2\sqrt{2}}\sqrt{z}$ and $\frac{i}{2\sqrt{2}}\sqrt{z}$. Convergence results for these functions will be proved in Section 2.5.4.

As in previous subsections, we define the functions on the unit grid $[\mathbb{C}_1,0]$, and then scale them by $G_{[\mathbb{C}_\delta,0]}(z\delta) := \delta G_{[\mathbb{C}_1,0]}(z)$ and $\tilde{G}^{\pm}_{[\mathbb{C}_\delta,0]}(z\delta) := \delta \tilde{G}^{\pm}_{[\mathbb{C}_1,0]}(z)$. We define them first on real and imaginary corners by "integrating" the harmonic measures, then extend to other points by s-holomorphicity. The fact that there are two discrete versions of $\frac{i}{2\sqrt{2}}\sqrt{z}$ is a peculiarity that will be important in the proof of the main convergence result in Section 2.5.4 (see also [Dub15]).

Definition 2.3.7. Define for $z \in \mathcal{V}^1_{[\mathbb{C}_1,0]} \cup \mathcal{V}^i_{[\mathbb{C}_1,0]}$, the auxiliary functions

$$G_{[\mathbb{C}_1,0]}(z) := \begin{cases} \sum_{n=0}^{\infty} \pm \frac{1}{2\sqrt{2}} \mathrm{hm}\,{}^{\mathbb{X}^1_1}_{3/2}(z-2n) & \text{if } z \in \mathcal{V}^1_{[\mathbb{C}_1,0]} \cap \mathbb{X}^{\pm} \\ \sum_{n=0}^{\infty} \pm \frac{i}{2\sqrt{2}} \mathrm{hm}\,{}^{\mathbb{Y}^i_1}_{-3/2}(z+2n) & \text{if } z \in \mathcal{V}^i_{[\mathbb{C}_1,0]} \cap \mathbb{Y}^{\pm} \\ 0 & \text{otherwise} \end{cases},$$

and

$$\tilde{G}^{\pm}_{[\mathbb{C}_1,0]}(z) := i G_{[\mathbb{C}_1,0]}(z \pm 1),$$

where translation by 1 is well-defined at any point other than $\pm\frac{1}{2}$; $G_{[\mathbb{C}_1,0]}(\pm\frac{1}{2}) = 0$ on both sheets so there is no ambiguity in defining $\tilde{G}^{\pm}_{[\mathbb{C}_1,0]}(\mp\frac{1}{2})$.

Remark 2.3.5. In [CHI15, Lemma 2.17] well-definedness and harmonicity of the real part of $G_{[\mathbb{C}_1,0]}$ were proven. From symmetry, we see the same holds for the imaginary part. Discrete holomorphicity of $G_{[\mathbb{C}_1,0]}$, and thus of $\tilde{G}^{\pm}_{[\mathbb{C}_1,0]}$, is proved in Appendix 2.7 using the explicit formula of Theorem 2.3.3. Using that, we then extend these to s-holomorphic functions on the corners and medial vertices, again using the process of Remark 2.2.2.

Remark 2.3.6. Since $G_{[\mathbb{C}_1,0]}$ (and thus $\tilde{G}^{\pm}_{[\mathbb{C}_1,0]}$) is defined using infinite sums, one cannot a priori calculate them exactly. However, once its s-holomorphicity is exhibited in Appendix 2.7, we can use a propagation procedure similar to one shown in Fig. 2.3.3 and explained in Corollary 2.3.11 to recursively calculate its values from its values on the real and imaginary axes. Indeed, $G_{[\mathbb{C}_1,0]}$ is explicitly computable on the real line since the summands eventually become zero; then we use the rotation identity $e^{\pi i/4} G_{[\mathbb{C}_1,0]}(e^{\pi i/2} z) = \frac{1}{2}\left[\tilde{G}^{+}_{[\mathbb{C}_1,0]} + \tilde{G}^{-}_{[\mathbb{C}_1,0]}\right](z)$, proved in Proposition 2.7.4, to find the values on the imaginary axis.

Using this procedure, we provide some explicit values of $G_{[\mathbb{C}_1,0]}(z)$ near the origin in Figure 2.3.4

2.4 Discrete Multipoint Observables

In this section, we prove Pfaffian formulae expressing n-point energy correlations, with or without a spin weight, in terms of the real fermion and spin-fermion two-point functions.

For the n-point energy correlations, we formulate them in terms of s-holomorphic multipoint fermion observables introduced in [Hon10] and follow the strategy there to obtain their Pfaffian formulation with the two-point observables introduced in Section 2.3. In [Hon10], the arguments of the multipoint observable were required to be distinct, non-adjacent medial vertices; we slightly but crucially generalize this to allow for adjacent medial vertices in

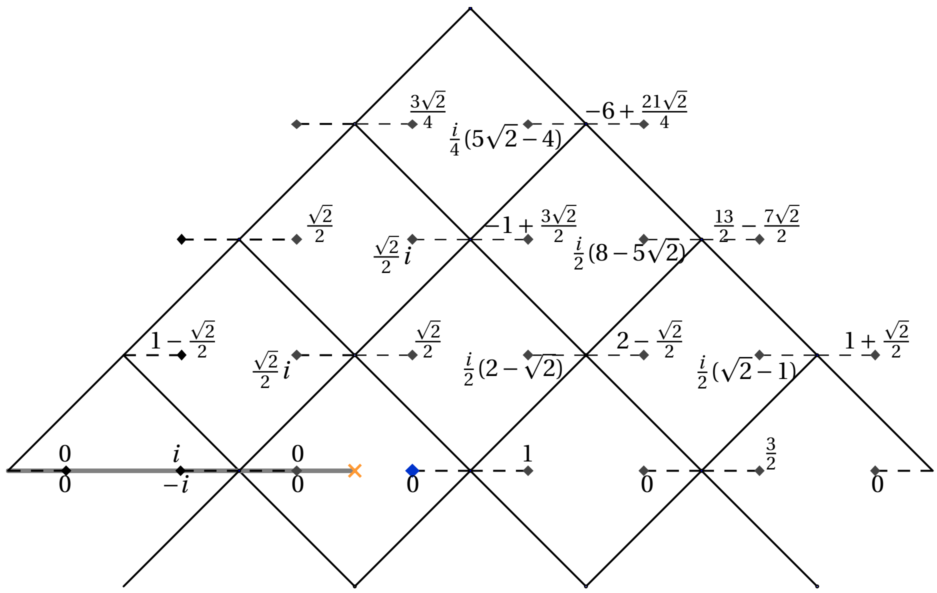

Figure 2.3.4 – Some explicit values for the auxiliary function $2\sqrt{2}G_{[\mathbb{C}_1,0]}(z)$ for $z \in \mathcal{V}^{1,i}_{[\mathbb{C}_1,0]} \cap \mathbb{X}^+$, as defined in Definition 2.3.7. The function has monodromy about the origin, represented by the orange ×.

Proposition 2.4.1 and Lemma 2.4.2, using a combinatorial correspondence between paths sourced at medial vertices and at corners.

When looking at n-point energy correlations weighted by a spin, the process is analogous, and we get fused multipoint spin-fermion observables, which then reduce to Pfaffians of the two-point spin-fermion observables of Section 2.3.1.

The proofs of these relations connecting the two-point fermion and spin-fermion observables to edge correlations (namely, Propositions 2.4.9–2.4.10) are quite notationally heavy, but all the extra notation is contained completely to this section. After generalizing the discrete complex analytic properties of the multipoint observables to adjacent medial vertices in Proposition 2.4.1, the rest of the proof is just a natural extension of the steps outlined in [Hon10] to prove the desired relations. For an alternative approach using mostly combinatorial arguments, see [CCK17].

2.4.1 Multipoint Observables

Recall that in Section 2.3 we denoted by $\mathcal{Z}_{\Omega_\delta}$ the low-temperature expansion of the partition function defined by summing $e^{-2\beta|\omega|}$ over all $\omega \in \mathscr{C}_{\Omega_\delta}$, where $\mathscr{C}_{\Omega_\delta}$ is the set of closed loops in Ω_δ. Furthermore, for oriented corners or medial vertices α, ζ, we defined the contour sets $\mathscr{C}^{\alpha,\zeta}_{\Omega_\delta}$ as well as admissible walks. We also chose the convention that and at ambiguous vertices in the walk, we connect northeast to northwest edges and southeast and southwest edges (see §2.3.1).

We now generalize the two-point observables defined in Section 2.3. First, let us define the generalized contour set $\mathscr{C}_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n}}$ for s-oriented corners or medial vertices $\alpha_j = a_j^{o_j}$ for $j = 1,2,\dots,2n$. For now we assume that the underlying points $a_1,\dots,a_{2n}$ are distinct (we will later generalize to the case where they can take the same value: see Remark 2.4.3). In [Hon10], these points were all medial vertices, and were required to be non-adjacent (the edges corresponding to them were not allowed to be adjacent). We do not impose this non-adjacency requirement, and this small extension is important to the proofs. As before, the half-edge of a corner is the line segment connecting it to its nearest vertex.

Each element $\gamma \in \mathscr{C}_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n}}$ is a set containing

- $2n$ half-edges of $a_1,\dots,a_{2n}$ selected by their respective orientations $o_1,\dots,o_{2n}$, and
- a (possibly empty) collection of edges distinct from the above-mentioned half-edges of $\{a_j\}_{j=1}^{2n}$,

such that any vertex of Ω_δ is incident to an even number of the edges and half-edges. Such a γ will be an edge-disjoint union of n walks connecting $\alpha_1,\dots\alpha_{2n}$ pairwise as well as a (possibly empty) collection of edge-disjoint loops. In particular, the definition of the 2-point set $\mathscr{C}_{\Omega_\delta}^{\alpha,\zeta}$ coincides with the one given in Section 2.3.

Generalizing from the 2-point case in Section 2.3, given γ we can pick n admissible walks which connect $\{\alpha_i\}$ pairwise. We denote by $\Gamma(\gamma) \subset \gamma$, a set of those n edge-disjoint admissible walks $\{\gamma^{\alpha_j,\alpha_k}\}$ chosen from the half-edges and edges constituting γ. We label them so that $j < k$ for each $\gamma^{\alpha_j,\alpha_k}$. Define the *crossing parity* $\mathbf{c}(\Gamma(\gamma))$ as the number of crossings, modulo 2 when linking $1,\dots,2n \in \mathbb{R}$ pairwise with generic simple curves in the upper half plane (i.e. connect j,k if there is a walk in $\Gamma(\gamma)$ connecting α_j,α_k). Moreover, recall the definitions of $\lambda_{\alpha_i,\alpha_j}$ and $\mathbf{W}(\gamma)$ from Definition 2.3.1.

Definition 2.4.1. For a collection of s-oriented corners or medial vertices $\{\alpha_j\}_{j=1}^{2n}$ in Ω_δ, define the *multipoint fermion observable* as

$$F_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n}} := \frac{1}{\mathcal{Z}_{\Omega_\delta}} \sum_{\gamma\in\mathscr{C}_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n}}} \phi_{\{\alpha_j\}}(\gamma),$$

$$\phi_{\{\alpha_j\}}(\gamma) := e^{-2\beta_c|\gamma|}(-1)^{\mathbf{c}(\Gamma(\gamma))} \prod_{\gamma^{\alpha_j,\alpha_k}\in\Gamma(\gamma)} i\lambda_{\alpha_i,\alpha_j} e^{-\frac{i}{2}\mathbf{W}(\gamma^{\alpha_j,\alpha_k})}.$$

In analogy with the two point case, if $\zeta^1 = z^{p_1}$ and $\zeta^2 = z^{p_2}$ are s-orientations of the two opposite orientations of $z := a_{2n}$, so that $p_2 = e^{\pm\pi i} p_1$, and $\kappa = c^q$ is an s-orientation of a corner

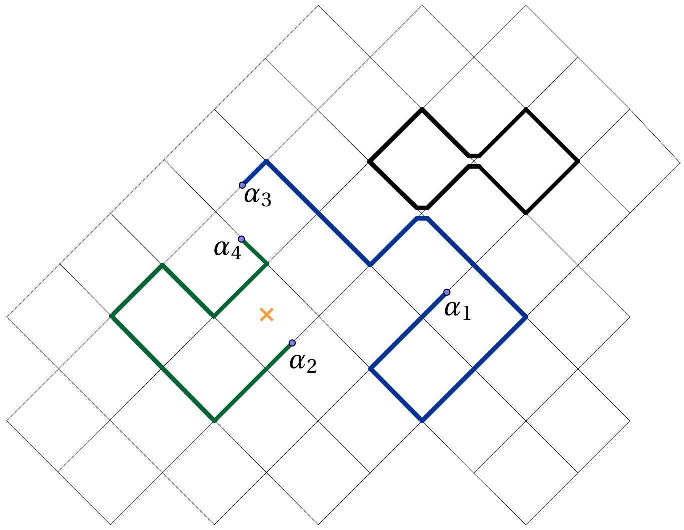

Figure 2.4.1 – An example of a choice of two admissible walks between α_1 and α_3 and α_2 and α_4 and a loop. The winding of $\gamma^{\alpha_1,\alpha_3}$ is 2π while the winding of $\gamma^{\alpha_2,\alpha_4}$ is $-\frac{\pi}{2}$. The crossing parity has $\mathbf{c}(\Gamma(\gamma)) = 1$.

c in Ω_δ, we define the complexification of the multipoint fermion observable,

$$H_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n-1}}(z) := \frac{1}{i\sqrt{p_1}} F_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n-1},\zeta^1} + \frac{1}{i\sqrt{p_2}} F_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n-1},\zeta^2}, \tag{2.4.1}$$

$$H_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n-1}}(c) := \frac{1}{i\sqrt{q}} F_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n-1},\kappa}.$$

Definition 2.4.2. For a collection of s-oriented corners or medial vertices $\{\alpha_j\}_{j=1}^{2n}$ in $[\Omega_\delta, 0]$, define the *multipoint spin-fermion observable,*

$$F_{[\Omega_\delta,0]}^{\alpha_1,\dots,\alpha_{2n}} := \frac{1}{\mathcal{Z}_{\Omega_\delta}\mathbb{E}_{\Omega_\delta}^{+}[\sigma_0]} \sum_{\gamma\in\mathscr{C}_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n}}} \phi_{\{\alpha_j\}}^{\Sigma}(\gamma),$$

$$\phi_{\{\alpha_j\}}^{\Sigma}(\gamma) := \phi_{\{\alpha_j\}}(\gamma)(-1)^{\ell(\gamma\setminus\cup\Gamma(\gamma))} \prod_{\gamma^{\alpha_j,\alpha_k}\in\Gamma(\gamma)} \mathrm{s}_{\alpha_j,\alpha_k}(\gamma^{\alpha_j,\alpha_k}),$$

where ℓ and $\mathrm{s}_{\alpha_j,\alpha_k}$ are defined as in Definition 2.3.1. As in the two-point spin fermion, the contour collection $\mathscr{C}_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n}}$ and the weight $\phi_{\{\alpha_j\}}$ are computed with respect to the projections of α_j onto Ω_δ, but the sheet choices come in the terms $\mathrm{s}_{\alpha,\zeta}$. Define the complexified spin-fermion $H_{[\Omega_\delta,0]}^{\alpha_1,\dots,\alpha_{2n-1}}$ on medial vertices and corners of $[\Omega_\delta, 0]$ analogously to Eq. (2.4.1).

Remark 2.4.1. [Hon10, Propositions 67, 68] proves well-definedness of $\phi_{\{\alpha_j\}}(\gamma)$. In Proposition 2.8.1 we prove the well-definedness of $(-1)^{\ell(\gamma\setminus\cup\Gamma(\gamma))}\prod \mathrm{s}_{\alpha_j,\alpha_k}(\gamma^{\alpha_j,\alpha_k})$, and thus of $\phi_{\{\alpha_j\}}^{\Sigma}$, so that it is independent of our convention for admissible walks and the choice of pairings of source points; in particular, it is independent of our choice of admissible $\Gamma(\gamma)$.

In addition to increasing the number of inputs, we can also define observables summing over a subset $\mathscr{C}^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta:\{e_k^{s_k}\}} \subset \mathscr{C}^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta}$ formed by specifying the inclusion or exclusion of given edges. Given a collection of edges $\{e_k\}_{k=1}^m$ in Ω_δ (disjoint to the half-edges given by $\{a_j\}_{i=1}^{2n}$) and corresponding *inclusion variables* $s_k \in \{\bullet, \circ\}$, let

$$\mathscr{C}^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}} = \left\{\gamma \in \mathscr{C}^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta} : e_k \in \gamma \text{ if } s_k = \bullet, \text{ and } e_k \notin \gamma \text{ if } s_k = \circ\right\}.$$

Definition 2.4.3. We define the *restricted fermion and spin-fermion observables*, denoted $F^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}$ and $F^{\alpha_1,\dots,\alpha_{2n}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}$, and their complexifications as in Definitions 2.4.1–2.4.2, replacing the contour set $\mathscr{C}^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta}$ by the restricted contour set $\mathscr{C}^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}$.

The following propositions will characterize the complexified fermion and spin-fermion observables in terms of discrete complex analysis, proving the connection to the discrete Riemann boundary value problem defined in Section 2.2 for possibly adjacent medial vertices $a_1, \dots, a_{2n}$.

We first modify the real weight $\phi_{\{\alpha_k\}}$ defined on $\mathscr{C}^{\alpha_1,\dots,\alpha_{2n}}_{\Omega_\delta}$ for $2n$ s-oriented vertices $\alpha_1, \dots, \alpha_{2n}$ into a *complex weight* χ defined on

$$\mathscr{C}^{\alpha_1,\dots,\alpha_{2n-1},a_{2n}}_{\Omega_\delta} := \mathscr{C}^{\alpha_1,\dots,\alpha_{2n-1},\alpha^1_{2n}}_{\Omega_\delta} \sqcup \mathscr{C}^{\alpha_1,\dots,\alpha_{2n-1},\alpha^2_{2n}}_{\Omega_\delta}$$

for $2n-1$ s-oriented medial vertices or corners $\alpha_1, \dots, \alpha_{2n-1}$ and another medial vertex a_{2n}. Choose two s-orientations $\alpha^1_{2n} = a^{p_1}_{2n}$ and $\alpha^2_{2n} = a^{p_2}_{2n}$ of a_{2n} such that the orientations p_1, p_2 are the two opposite permissible orientations on a_{2n}. Define for $\gamma \in \mathscr{C}^{\alpha_1,\dots,\alpha^j_{2n}}_{\Omega_\delta} \subset \mathscr{C}^{\alpha_1,\dots,\alpha_{2n-1},a_{2n}}_{\Omega_\delta}$ the complex weight (where dependence on the choice of $\sqrt{p_j}$ eventually cancels out),

$$\chi(\gamma) := \frac{1}{i\sqrt{p_j}} \phi_{\{a_1,\dots,\alpha_{2n-1},a^j_{2n}\}}(\gamma),$$

noting that the complexified observables can be defined in terms of sums of this weight. If $\alpha_{2n} = a^o_{2n}$ is an s-oriented corner, there is only one corresponding orientation and $\chi(\gamma) := \frac{1}{i\sqrt{o}} \phi_{\{\alpha_k\}}(\gamma)$.

Proposition 2.4.1. *$H^{\alpha_1,\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}$ and $H^{\alpha_1,\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}$ are s-holomorphic wherever defined.*

Proof. This was proven for the complexified fermion in [Hon10, Lemma 74] in the setting where the α_i are non-adjacent medial vertices. We extend this to all possible α_i via the extension of the complexified fermion to corners in addition to medial vertices. The idea is that there is a natural bijection between the set of paths to a medial vertex e and those to an adjacent corner c. Namely, if $e(c)$ is the shortest walk from a_{2n} to c consisting of two half-edges both incident to the common vertex v, the map $\gamma \mapsto \gamma \oplus e(c)$ is a bijection. One needs to show that the projection in s-holomorphicity relations (2.2.1) sends the winding weight $e^{-\frac{i}{2}\mathbf{W}(\gamma)}$ to the winding weight of the image, times a factor of $\cos(\pi/8)$.

Following this, the next lemma therefore proves s-holomorphicity in the case of $H^{\alpha_1,\ldots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\ldots,e_m^{s_m}\}}$. The summands in the definition of $H^{\alpha_1,\ldots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\ldots,e_m^{s_m}\}}$ only have additional real factors invariant under the $\cdot \oplus e(c)$ bijection, so the lemma implies the s-holomorphicity for $H^{\alpha_1,\ldots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\ldots,e_m^{s_m}\}}$ as well. ☐

Lemma 2.4.2. *Let $\{a_j\}_{j=1}^{2n-1}$, $\{e_k\}_{k=1}^{m}$ be distinct medial vertices of Ω_δ and denote the s-oriented versions of $\{a_j\}_{j=1}^{2n-1}$ by $\alpha_j = a_j^{o_j}$, and inclusion variables $s_1,\ldots,s_m \in \{\bullet,\circ\}$. Suppose c is an interior corner and a_{2n} is an adjacent interior medial vertex distinct from $\{a_j\}_{j=1}^{2n-1}$. Let $e(c)$ be the shortest walk from a_{2n} to c consisting of two half-edges both incident to a common vertex v. Then the bijection,*

$$\gamma \mapsto \gamma \oplus e(c),$$

from $\mathscr{C}^{\alpha_1,\ldots,\alpha_{2n-1},a_{2n}}_{\Omega_\delta:\left\{e_k^{s_k}\right\}}$ to $\mathscr{C}^{\alpha_1,\ldots,\alpha_{2n-1},c}_{\Omega_\delta:\left\{e_k^{s_k}\right\}}$ satisfies the projection relation,

$$\mathsf{P}_{l(c)}\chi(\gamma) = \chi(\gamma \oplus e(c)).$$

Proof. Suppose $\gamma_f \in \Gamma(\gamma)$ is the walk ending at a_{2n}, starting at some α_s. Suppose c is a corner of type τ adjacent to a_{2n}. Then $\Gamma(\gamma \oplus e(c))$, chosen using paths in $\Gamma(\gamma)$ with γ_f replaced by $\gamma_f \oplus e(c)$, is clearly an admissible choice of walks in $\gamma \oplus e(c)$. Note that if o_{2n} is any s-orientation of a_{2n} compatible with γ_f, $i\frac{\sqrt{o_{2n}}}{\sqrt{o_s}}e^{-\frac{i}{2}\mathbf{W}(\gamma_f)}$ is a real quantity. Thus it suffices to show that

$$\mathsf{P}_{l(c)}\frac{e^{-2\beta_c|\gamma|}}{\sqrt{o_s}}e^{-\frac{i}{2}\mathbf{W}(\gamma_f)} = \cos\frac{\pi}{8}\cdot\frac{e^{-2\beta_c|\gamma\oplus e(c)|}}{\sqrt{o_s}}e^{-\frac{i}{2}\mathbf{W}(\gamma_f\oplus e(c))}.$$

Commuting real quantities with projections, we may rewrite the left hand side as

$$i\frac{\sqrt{o_{2n}}}{\sqrt{o_s}}e^{-\frac{i}{2}\mathbf{W}(\gamma_f)}e^{-2\beta_c|\gamma|}\mathsf{P}_{l(c)}\frac{1}{i\sqrt{o_{2n}}}.$$

There are two cases to consider: the cases when the half-edge $\langle a_{2n}, v\rangle \in \gamma_f$ and when $\langle a_{2n}, v\rangle \notin \gamma_f$.

- In the first case, $\left|\gamma \oplus e(c)\right| = \left|\gamma\right|$. Subsequently $\tau^2 o_{2n} = -e^{\pm\frac{\pi}{4}i}$, where the sign depends on the winding change $\mathbf{W}(\gamma \oplus e(c)) = \mathbf{W}(\gamma) \mp \frac{\pi}{4}$. Now we have

  $$\mathsf{P}_{l(c)}\frac{1}{i\sqrt{o_{2n}}} = \frac{1-\tau^2 o_{2n}}{2i\sqrt{o_{2n}}} = \frac{e^{\pm\frac{\pi}{8}i}}{i\sqrt{o_{2n}}}\cos\frac{\pi}{8},$$

  and the result follows.

- In the second case, $\left|\gamma\oplus e(c)\right| = \left|\gamma\right|+1$. Then $\tau^2 o_{2n} = e^{\pm\frac{\pi}{4}i}$, where the sign depends on the winding change $\mathbf{W}(\gamma\oplus e(c)) = \mathbf{W}(\gamma)\pm\frac{3\pi}{4}$. Then

$$\mathsf{P}_{l(c)}\frac{1}{i\sqrt{o_{2n}}} = \frac{1-\tau^{-2}o_{2n}}{2i\sqrt{o_{2n}}} = \frac{e^{\mp\frac{3\pi}{8}i}}{i\sqrt{o_{2n}}}\sin\frac{\pi}{8}.$$

Upon noting that $\tan\frac{\pi}{8} = e^{-2\beta_c}$, the result follows. □

Lemma 2.4.3. *The discrete residues at an s-oriented interior oriented medial vertex α_j are*

$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}(\alpha_{j+}) - H^{\alpha_1,\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}(\alpha_{j-}) = \frac{(-1)^{j+1}}{\sqrt{o_j}}F^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}},$$
$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}(\alpha_{j+}) - H^{\alpha_1,\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}(\alpha_{j-}) = \frac{(-1)^{j+1}}{\sqrt{o_j}}F^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}.$$

where in the two cases $\alpha_1,\dots,\alpha_{2n-1}$ are all taken respectively on Ω_δ and $[\Omega_\delta,0]$.

Proof. We prove this for the fermion and spin-fermion simultaneously, letting D_δ represent Ω_δ or $[\Omega_\delta,0]$. It suffices to show that the front and back values of $H^{\alpha_1,\dots,\alpha_{2n-1}}_{D_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}$ at α_j are given by

$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{D_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}(\alpha_{j+}) = \frac{(-1)^{j+1}}{\sqrt{o_j}}F^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots\alpha_{2n-1}}_{D_\delta:\left\{e_1^{s_1},\dots,e_m^{s_m},a_j^\circ\right\}},$$
$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{D_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}(\alpha_{j-}) = \frac{(-1)^{j}}{\sqrt{o_j}}F^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots\alpha_{2n-1}}_{D_\delta:\left\{e_1^{s_1},\dots,e_m^{s_m},a_j^\bullet\right\}}, \tag{2.4.2}$$

where in the case $D_\delta = [\Omega_\delta,0]$ the projection of a_j onto Ω_δ is considered in $a_j^{\circ,\bullet}$.

Let c_1 denote one of the two corners adjacent to the end vertex of a_j in the direction o_j. We verify that $H^{\alpha_1,\dots,\alpha_{2n-1}}_{D_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}(\alpha_{j+})$ projects to $H^{\alpha_1,\dots,\alpha_{2n-1}}_{D_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}(c_1)$. Let c_1 be a corner of type τ.

We note that any $\gamma\in\mathscr{C}^{\alpha_1,\dots,\alpha_{2n-1},c_1}_{D_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}$ contains the walk $e(c_1) := \left\{\left\langle a_j, a_j+o_j\frac{\delta}{\sqrt{2}}\right\rangle,\left\langle a_j+o_j\frac{\delta}{\sqrt{2}},c_1\right\rangle\right\}$, so we can start from γ_f and complete $\Gamma(\gamma)$. In addition, $\gamma\oplus e(c_1)\in\mathscr{C}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}}_{D_\delta:\left\{e_1^{s_1},\dots,e_m^{s_m},a_j^\circ\right\}}$ and $\Gamma(\gamma)\setminus\{e(c_1)\}$ is naturally an admissible choice of walk for $2n-2$ points. Since the additional real factor in the case of $D_\delta = [\Omega_\delta,0]$ is easily seen to be invariant under the bijection, it now suffices to show the projection relation

$$\mathsf{P}_{l(c_1)}\frac{(-1)^{j+1}}{\sqrt{o_j}}\phi_{\{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}\}}(\gamma\oplus e(c_1)) = \chi(\gamma).$$

This relation can be rewritten as $\mathsf{P}_{l(c_1)}\frac{1}{\sqrt{o_j}} = e^{-2\beta_c}\cos\frac{\pi}{8}\cdot\frac{e^{-\frac{i}{2}\mathbf{W}(e(c_1))}}{\sqrt{o_j}} = \cos\frac{3\pi}{8}\cdot\frac{e^{-\frac{i}{2}\mathbf{W}(e(c_1))}}{\sqrt{o_j}}$ using explicit formulae and admissible choices $\Gamma(\gamma)$ and $\Gamma(\gamma\oplus e(c_1))$ (where $(-1)^{j+1}$ is precisely the ratio between the crossing parity factor of $\Gamma(\gamma\oplus e(c_1))$ and $\Gamma(\gamma)$; one sees this by drawing the pairing between j and $2n$ very close to $[j,2n] \subset \mathbb{R}$, so that it crosses exactly $2n-j-1$ other lines).

Note that $o_j e^{i\mathbf{W}(e(c_1))} = \tau^2$ and $\mathbf{W}(e(c_1)) = \pm\frac{3\pi}{4}$. Commuting real values with projection, we have,

$$\mathsf{P}_{l(c_1)}\left[\frac{1}{\sqrt{o_j}}\right] = \frac{1}{2}\left[\frac{1}{\sqrt{o_j}} + \frac{\sqrt{o_j}}{\tau^2}\right] = \frac{1}{\sqrt{o_j}}\frac{1+e^{-i\mathbf{W}(e(c_1))}}{2} = \cos\frac{3\pi}{8}\cdot\frac{e^{-\frac{i}{2}\mathbf{W}(e(c_1))}}{\sqrt{o_j}}. \qquad \square$$

Remark 2.4.2. The explicit front and back values (2.4.2) shown in the proof above give us a simple correspondence between the full-plane observables and the full-plane (normal and spin-weighted) nearest-pair correlations $\mu, \mu_{a'}$ for edges $a = \delta a' \in \mathscr{E}_{\Omega_\delta}$ defined in (2.1.1).

Notice that, by low-temperature expansion, the correlation of the nearest spin pair separated by an edge a can be written $\frac{1}{\mathcal{Z}_{\Omega_\delta}}\left[\sum_{\omega\in\mathscr{C}_{\Omega_\delta:\{a^\circ\}}} e^{-2|\omega|} - \sum_{\omega\in\mathscr{C}_{\Omega_\delta:\{a^\bullet\}}} e^{-2|\omega|}\right] = F_{\Omega_\delta:\{a^\circ\}} - F_{\Omega_\delta:\{a^\bullet\}}$. By the values given in (2.4.2), this is precisely $\sqrt{o}\left[H^\alpha_{\Omega_\delta}(\alpha_+) + H^\alpha_{\Omega_\delta}(\alpha_-)\right]$, for any s-oriented version α of a. Now taking the infinite-volume limit by sending $\Omega \uparrow \mathbb{C}$ (whose existence is known by Theorem 2.3.1), we have

$$\mu = \sqrt{o}\left[H^\alpha_{\mathbb{C}_\delta}(\alpha_+) + H^\alpha_{\mathbb{C}_\delta}(\alpha_-)\right],$$

matching the values $H^\alpha_{\mathbb{C}_\delta}(\alpha_\pm) = \frac{\mu\pm1}{2\sqrt{o}}$ given by Proposition 2.3.5.

By analogous reasoning, for every $a = \delta a' \in \mathscr{E}_{\Omega_\delta}$ we can calculate $\mu_{a'}$ as defined in (2.1.1) by

$$\mu_{a'} = \sqrt{o}\left[H^\alpha_{[\mathbb{C}_\delta,0]}(\alpha_+) + H^\alpha_{[\mathbb{C}_\delta,0]}(\alpha_-)\right],$$

or equivalently for every $a = \delta a' \in \mathscr{E}_{\Omega_\delta}$ we have $H^\alpha_{[\mathbb{C}_\delta,0]}(\alpha_\pm) = \frac{\mu_{a'}\pm1}{2\sqrt{o}}$, where we know the infinite-volume limit $\Omega \uparrow \mathbb{C}$ of $\sqrt{o}[H^\alpha_{[\Omega_\delta,0]}(\alpha_+) + H^\alpha_{[\Omega_\delta,0]}(\alpha_-)]$ exists by Theorem 2.3.2.

Recall the definition of ν_z as the outer normal at a boundary medial vertex z.

Lemma 2.4.4. *If $\{\alpha_i\}_{i=1}^{2n-1}$ are interior s-oriented medial vertices, $\{e_1^{s_1}, \dots e_m^{s_m}\}$ are edges with corresponding inclusion variables, and $z = a_{2n}$ is a boundary medial vertex, we have that*

$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}(z)\sqrt{\nu_z} \in \mathbb{R};$$
$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}(z)\sqrt{\nu_z} \in \mathbb{R}.$$

Proof. [Hon10, Proposition 79] proves the lemma for $H^{\alpha_1,\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}$ with non-adjacent α_i. The idea is that if z is on the boundary, a path can only reach z via the half-edge in the inner direction, which fixes the complex phase in the weight; this goes through unchanged for possibly adjacent α_i. For the spin-fermion observable, since the additional factors in $H^{\alpha_1,\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}$ are real, the result holds. $\square$

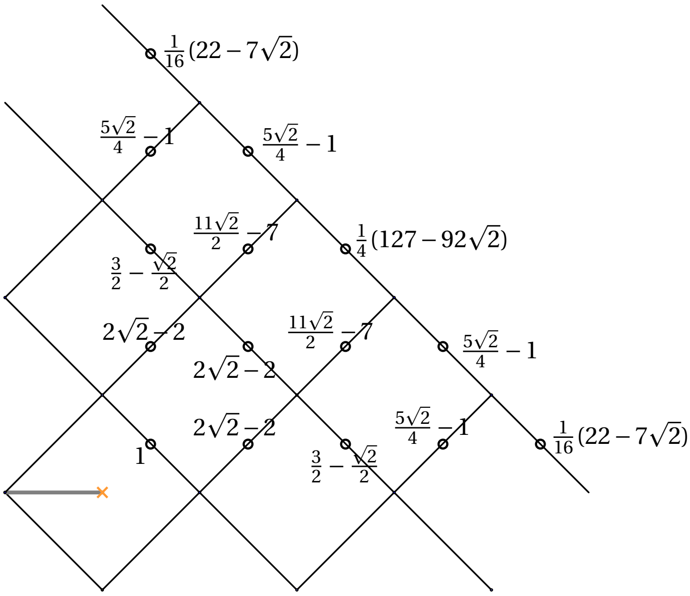

Figure 2.4.2 – Some explicit values of $(\mu_e)_{e\in\mathscr{E}_{\mathbb{C}_1}}$. This represents the mean of the physical quantity, the *spin-weighted energy field*. The orange × marks the identified origin face, whose spin μ_e is weighted by; as $|e|\to\infty$, we have $\mu_e\to\mu$.

2.4.2 Fused Observables and Ising Correlations

In this subsection, we formulate the Ising correlations in a bounded domain in terms of fusions of the observables introduced above. We again write $\alpha_1,\dots,\alpha_{2n}$ for distinct s-oriented medial vertices in Ω_δ or $[\Omega_\delta,0]$ and $e_1,\dots,e_m$ distinct edges with inclusion variables $s_1,\dots,s_m\in\{\bullet,\circ\}$.

Definition 2.4.4. Suppose $b_1,\dots,b_N$ are distinct medial vertices in Ω_δ with $b_N=\delta b'_N$. Define the *fused fermion* and *fused spin-fermion observables* and their complexifications inductively by

$$F^{\alpha_1,\dots,\alpha_{2n}:[b_1,\dots,b_N]}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}:=F^{\alpha_1,\dots,\alpha_{2n}:[b_1,\dots,b_{N-1}]}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m},b_N^{\circ}\}}-\frac{\mu+1}{2}F^{\alpha_1,\dots,\alpha_{2n}:[b_1,\dots,b_{N-1}]}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}},\tag{2.4.3}$$

$$F^{\alpha_1,\dots,\alpha_{2n}:[b_1,\dots,b_N]}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}:=F^{\alpha_1,\dots,\alpha_{2n}:[b_1,\dots,b_{N-1}]}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m},b_N^{\circ}\}}-\frac{\mu_{b'_N}+1}{2}F^{\alpha_1,\dots,\alpha_{2n}:[b_1,\dots,b_{N-1}]}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}},$$

and the usual complexification scheme on α_{2n} (see e.g. (2.4.1)).

These fusions arise naturally from the process of removing singularities. Suppose α_j is an s-oriented interior medial vertex. Note that, by Lemma 2.4.3 and Theorems 2.3.1–2.3.2, the functions

$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}+(-1)^jF^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}H^{\alpha_j}_{\mathbb{C}_\delta},\tag{2.4.4}$$

$$H^{\alpha_1,\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}+(-1)^jF^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}}_{[\Omega_\delta,0]:\{e_1^{s_1},\dots,e_m^{s_m}\}}H^{\alpha_j}_{[\mathbb{C}_\delta,0]},\tag{2.4.5}$$

have discrete residue 0 at α_j; thus they extend to a_j s-holomorphically. In fact, we have the following extension result for them, which also applies to the corresponding fused observables.

Lemma 2.4.5. *The following fused versions of the functions defined in* (2.4.4)–(2.4.5) *have s-holomorphic extensions to a_j given by the values*

$$\left[H_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{2n-1}:[e_1,\dots,e_m]} + (-1)^j F_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}:[e_1,\dots,e_m]} H_{\mathbb{C}_\delta}^{\alpha_j}\right](a_j)$$
$$:= \frac{(-1)^{j+1}}{\sqrt{o_j}} F_{\Omega_\delta}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots\alpha_{2n-1}:[e_1,\dots,e_m,a_j]},$$
$$\left[H_{[\Omega_\delta,0]}^{\alpha_1,\dots,\alpha_{2n-1}:[e_1,\dots,e_m]} + (-1)^j F_{[\Omega_\delta,0]}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}:[e_1,\dots,e_m]} H_{[\mathbb{C}_\delta,0]}^{\alpha_j}\right](a_j)$$
$$:= \frac{(-1)^{j+1}}{\sqrt{o_j}} F_{[\Omega_\delta,0]}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots\alpha_{2n-1}:[e_1,\dots,e_m,a_j]},$$

where in the second case, a_j in $[\Omega_\delta,0]$ is identified with its projection on Ω_δ in the expression $[e_1,\dots,e_m,a_j]$.

Proof. By $\mathbb{R}$-linearity of s-holomorphicity and the definition of fused observables, it suffices to show that the unfused observables given by (2.4.4)–(2.4.5) extend s-holomorphically to a_j with above right hand side values (without $e_1,\dots,e_m$).

Since the function (2.4.4) has discrete residue 0 at a_j, it has an s-holomorphic extension to a_j given by

$$\left[H_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}^{\alpha_1,\dots,\alpha_{2n-1}} + (-1)^j F_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}} H_{\mathbb{C}_\delta}^{\alpha_j}\right](a_j)$$
$$= H_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}^{\alpha_1,\dots,\alpha_{2n-1}}(\alpha_{j+}) + (-1)^j F_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}} H_{\mathbb{C}_\delta}^{\alpha_j}(\alpha_{j+}).$$

In turn, by the explicit values in the proof of Lemma 2.4.3, this is given by

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} F_{\Omega_\delta:\left\{e_1^{s_1},\dots,e_m^{s_m},a_j^\circ\right\}}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots\alpha_{2n-1}} + (-1)^j F_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{2n-1}} H_{\mathbb{C}_\delta}^{\alpha_j}(\alpha_{j+})$$
$$= \frac{(-1)^{j+1}}{\sqrt{o_j}} F_{\Omega_\delta:\{e_1^{s_1},\dots,e_m^{s_m}\}}^{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots\alpha_{2n-1}:[a_j]},$$

using that $H_{\mathbb{C}_\delta}^{\alpha_j}(a_{j+}) = \frac{\mu+1}{2\sqrt{o_j}}$ (see Remark 2.4.2). Identical computation gives the $[\Omega_\delta,0]$ case, where now we use the fact that for every $a_j = \delta a_j'$, we have $H_{[\mathbb{C}_\delta,0]}^{\alpha_j}(\alpha_{j+}) = \frac{\mu_{a_j'}+1}{2\sqrt{o_j}}$. □

Remark 2.4.3. So far, we have assumed that the $2n$ inputs $\alpha_1,\dots,\alpha_{2n}$ of the real observable are s-orientations of distinct $a_1,\dots,a_{2n}$. An important observation to be made is that the combinatorial definition of the real observable is robust enough for the *pairwise-fused* case, where a medial vertex e appears twice among the $2n$ inputs (say $a_j = a_k = e$), as long as their respective orientations o_j, o_k point to opposite directions.

In the complexified case, a medial vertex can appear twice, again oppositely oriented, among the first $2n-1$ s-oriented medial vertices; if $a_j = a_k$ on Ω_δ with opposite orientations o_j and o_k, one can verify that the residue at α_j is given by

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} F^{\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1},\ldots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\ldots,e_m^{s_m}\}} - \frac{(-1)^{k+1}}{\sqrt{o_k}} F^{\alpha_1,\ldots,\alpha_{k-1},\alpha_{k+1},\ldots,\alpha_{2n-1}}_{\Omega_\delta:\{e_1^{s_1},\ldots,e_m^{s_m}\}}.$$

This is precisely a directed superposition of the two residues derived in Lemma 2.4.3 for s-oriented medial vertices α_j, α_k; similarly Lemma 2.4.5 and the resulting Propositions 2.4.7–2.4.8, introduced in the next subsection, easily generalize to the pairwise-fused case.

We are now in position to connect the fused multipoint observables to multipoint Ising correlations.

Proposition 2.4.6. *Suppose* $\{b_k\}_{k=1}^N$ *is a set of N distinct interior edges in* Ω_δ. *We have*

$$\mathbb{E}_{\Omega_\delta}\left[\prod_{b\in\{b_k\}} \epsilon(b)\right] = (-1)^N 2^N F^{[b_1,\ldots,b_N]}_{\Omega_\delta},$$

$$\frac{\mathbb{E}_{\Omega_\delta}\left[\sigma_0 \prod_{b\in\{b_k\}} \epsilon_{[0]}(b)\right]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} = (-1)^N 2^N F^{[b_1,\ldots,b_N]}_{[\Omega_\delta,0]}.$$

Proof. The first identity was proved in [Hon10, Proposition 72] inductively, where our above extensions of the projection relations to adjacent edges now allow for the b_k to be adjacent. Explicitly, denoting $\left\{e_j^{s_j}\right\} = \{e_1^{s_1},\ldots,e_m^{s_m}\}$ for a set of edges (distinct from b_k) with inclusion variables $s_j \in \{\bullet,\circ\}$ and recalling $\mathscr{C}_{\Omega_\delta:\left\{e_j^{s_j}\right\}}$, it is straightforward to show by induction on N (the base case is trivial),

$$\mathbb{E}_{\Omega_\delta}\left[\mathbf{1}_{\left\{e_j^{s_j}\right\}} \prod_{b\in\{b_k\}} \epsilon(b)\right] = (-1)^N 2^N F^{[b_1,\ldots,b_N]}_{\Omega_\delta:\left\{e_j^{s_j}\right\}},$$

(where for an edge e separating faces x, y with inclusion variable $s \in \{\bullet,\circ\}$, the indicator $\mathbf{1}_{\{e^s\}}$ denotes an indicator on the event that $\sigma_x = \sigma_y$ if $s = \circ$ and $\sigma_x \neq \sigma_y$ if $s = \bullet$) using the expansion

$$\begin{aligned}\mathbb{E}_{\Omega_\delta}\left[\mathbf{1}_{\left\{e_j^{s_j}\right\}}\epsilon(b_1)\cdots\epsilon(b_{N+1})\right] &= (\mu-1)\mathbb{E}_{\Omega_\delta}\left[\mathbf{1}_{\left\{e_j^{s_j},b_{N+1}^\circ\right\}}\epsilon(b_1)\cdots\epsilon(b_N)\right] \\ &\quad + (\mu+1)\mathbb{E}_{\Omega_\delta}\left[\mathbf{1}_{\left\{e_j^{s_j},b_{N+1}^\bullet\right\}}\epsilon(b_1)\cdots\epsilon(b_N)\right],\end{aligned}$$

and that by definition $F^{[b_1,\ldots,b_N]}_{\Omega_\delta:\left\{e_j^{s_j}\right\}} = F^{[b_1,\ldots,b_N]}_{\Omega_\delta:\left\{e_j^{s_j},b_{N+1}^\circ\right\}} + F^{[b_1,\ldots,b_N]}_{\Omega_\delta:\left\{e_j^{s_j},b_{N+1}^\bullet\right\}}$.

The second identity follows from an analogous process, where we note that for a collection of loops and walks ω, we have $\sigma_0(\omega) = (-1)^{\ell(\omega)}$ due to the plus boundary condition. □

2.4.3 Pfaffian Formulae

Having related the Ising correlations to the fused observables, in this subsection we will elucidate how the recursive relation (2.4.3) gives rise to the Pfaffian relation in Section 2.4.4. The argument is identical to the one presented in Chapter 6 of [Hon10], albeit with the stronger lemmas introduced in Section 2.4.1 allowing the α_i's to be adjacent.

We first prove that a $2n$-point observable is in fact a Pfaffian of a matrix of two-point observables. Recall that for a $2n \times 2n$ antisymmetric matrix $A = (A_{jk})_{j,k=1,\ldots,2n}$,

$$\mathrm{Pf}\, A := \frac{1}{2^n n!} \sum_{\sigma \in S^{2n}} \mathrm{sgn}(\sigma) A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(2n-1)\sigma(2n)}\,, \tag{2.4.6}$$

and we have the recursive expansion formula,

$$\mathrm{Pf}\, A = \sum_{j=1}^{2n-1} (-1)^j A_{j,2n} \mathrm{Pf}\, A_{\widehat{j;2n}}\,, \tag{2.4.7}$$

where $A_{\widehat{j;2n}}$ is the matrix where the j and $2n$-th rows and columns are removed.

Proposition 2.4.7. *Suppose $\alpha_j = a_j^{o_j}, j = 1, \ldots, 2n$ are distinct (possibly pairwise-fused) s-oriented interior medial vertices of $D_\delta = \Omega_\delta$ or $[\Omega_\delta, 0]$. Define the $2n \times 2n$ antisymmetric matrix*

$$\mathbf{F}_{D_\delta}^{\{\alpha_j\}} = \begin{pmatrix} 0 & F_{D_\delta}^{\alpha_1,\alpha_2} & \cdots & F_{D_\delta}^{\alpha_1,\alpha_{2n-1}} & F_{D_\delta}^{\alpha_1,\alpha_{2n}} \\ & 0 & \cdots & F_{D_\delta}^{\alpha_2,\alpha_{2n-1}} & F_{D_\delta}^{\alpha_2,\alpha_{2n}} \\ & & 0 & \vdots & \vdots \\ & & & \ddots & F_{D_\delta}^{\alpha_{2n-1},\alpha_{2n}} \\ & & & & 0 \end{pmatrix}.$$

Then

$$F_{D_\delta}^{\alpha_1,\ldots,\alpha_{2n}} = Pf\mathbf{F}_{D_\delta}^{\{\alpha_j\}}\,.$$

Proof. The 2×2 case is trivial. Given the recursive formula (2.4.7) for the Pfaffian, inductively it suffices to show

$$F_{D_\delta}^{\alpha_1,\ldots,\alpha_{2n}} = \sum_{j=1}^{2n-1} (-1)^j F_{D_\delta}^{\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1}\ldots,\alpha_{2n-1}} F_{D_\delta}^{\alpha_j,\alpha_{2n}}\,.$$

The strategy is to use the boundary value problem uniqueness result (Lemma 2.2.2) to show that the function $H_{D_\delta}^{\alpha_1,\ldots,\alpha_{2n-1}}(a_{2n}) - \sum_{j=1}^{2n-1} (-1)^j F_{D_\delta}^{\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1}\ldots,\alpha_{2n-1}} H_{D_\delta}^{\alpha_j}(a_{2n})$ is identically zero. The boundary condition is obviously satisfied; the fact that all singularities at $\alpha_1, \ldots, \alpha_{2n-1}$ are removable, i.e. have residue zero, is immediate from Lemma 2.4.3. □

We now extend the Pfaffian representation to the fused observables. For any edge e, write $e^+ = e^{o^+}$, $e^- = e^{o^-}$ for a pair of s-orientations $o^\pm$ of e, so that $o^+ = e^{\pi i} o^-$. Recall the use of the † to denote

$$F^{\dagger}_{\Omega_\delta} = F_{\Omega_\delta} - F_{\mathbb{C}_\delta} \qquad \text{and} \qquad F^{\dagger}_{[\Omega_\delta,0]} = F_{[\Omega_\delta,0]} - F_{[\mathbb{C}_\delta,0]}.$$

For edges $e_1, \dots, e_m$ define the $2m \times 2m$ antisymmetric matrix with $F^{\dagger e_i^+, e_i^-}_{D_\delta}$ on the anti-diagonal $i + j = 2m + 1$, $i \le m$, and more generally, entries,

$$\mathbf{F}^{[\{e_k\}]}_{D_\delta} := \begin{pmatrix} 0 & F^{e_1^+,e_2^+}_{D_\delta} & \cdots & F^{e_1^+,e_m^+}_{D_\delta} & F^{e_1^+,e_m^-}_{D_\delta} & \cdots & F^{e_1^+,e_2^-}_{D_\delta} & F^{\dagger e_1^+,e_1^-}_{D_\delta} \\ & 0 & \cdots & F^{e_2^+,e_m^+}_{D_\delta} & F^{e_2^+,e_m^-}_{D_\delta} & \cdots & F^{\dagger e_2^+,e_2^-}_{D_\delta} & F^{e_2^+,e_1^-}_{D_\delta} \\ & & 0 & \vdots & & \iddots & \vdots & \vdots \\ & & & \ddots & F^{\dagger e_m^+,e_m^-}_{D_\delta} & & F^{e_m^+,e_2^-}_{D_\delta} & F^{e_m^+,e_1^-}_{D_\delta} \\ & & & & 0 & \cdots & F^{e_m^-,e_2^-}_{D_\delta} & F^{e_m^-,e_1^-}_{D_\delta} \\ & & & & & \ddots & \vdots & \vdots \\ & & & & & & 0 & F^{e_2^-,e_1^-}_{D_\delta} \\ & & & & & & & 0 \end{pmatrix}.$$

Proposition 2.4.8. *Suppose $\alpha_j = a_j^{o_j}$ for $j = 1, \dots, 2n$ and e_k for $k = 1, \dots, m$ are distinct (possibly pairwise-fused) interior medial vertices of $D_\delta = \Omega_\delta$ or $[\Omega_\delta, 0]$. Define the block antisymmetric $2(m+n) \times 2(m+n)$ matrix $F^{\{a_j\}:[\{e_k\}]}_{D_\delta}$ by*

$$\mathbf{F}^{\{a_j\}:[\{e_k\}]}_{D_\delta} := \begin{pmatrix} \mathbf{F}^{[\{e_k\}]}_{D_\delta} & -\left[\mathbf{F}^{\{a_j\}\times[\{e_k\}]}_{D_\delta}\right]^T \\ \mathbf{F}^{\{a_j\}\times[\{e_k\}]}_{D_\delta} & \mathbf{F}^{\{a_j\}}_{D_\delta} \end{pmatrix}, \textit{ where}$$

$$\mathbf{F}^{\{a_j\}\times[\{e_k\}]}_{D_\delta} := \begin{pmatrix} F^{\alpha_1,e_1^+}_{D_\delta} & \cdots & F^{\alpha_1,e_m^+}_{D_\delta} & F^{\alpha_1,e_m^-}_{D_\delta} & \cdots & F^{\alpha_1,e_1^-}_{D_\delta} \\ \vdots & & \vdots & \vdots & & \vdots \\ F^{\alpha_{2n},e_1^+}_{D_\delta} & \cdots & F^{\alpha_{2n},e_m^+}_{D_\delta} & F^{\alpha_{2n},e_m^-}_{D_\delta} & \cdots & F^{\alpha_{2n},e_1^-}_{D_\delta} \end{pmatrix}.$$

Then

$$F^{\alpha_1,\dots,\alpha_{2n}:[e_1,\dots,e_m]}_{D_\delta} = Pf\,\mathbf{F}^{\{a_j\}:[\{e_k\}]}_{D_\delta}.$$

In particular, since $F^{\alpha_1,\dots,\alpha_{2n}:[e_1,\dots,e_m]}_{D_\delta}$ does not depend on the choice of s-orientations o_j^+, o_j^- on e_k, the Pfaffian does not.

Proof. Without loss of generality, we will assume $D_\delta = \Omega_\delta$; the case $D_\delta = [\Omega_\delta, 0]$ can be treated identically. We use induction on m. The case $m = 0$ is given by Proposition 2.4.7. Now we assume the result holds for m, and consider the case $m+1$. By Lemma 2.4.5, if $e_{m+1}^+ = e_{m+1}^o$ is an s-orientation of e_{m+1}, we can extend to the removed singularity

$$\left[H^{\alpha_1,\dots,\alpha_{2n},e_{m+1}^+:[e_1,\dots,e_m]}_{\Omega_\delta} - F^{\alpha_1,\dots,\alpha_{2n}:[e_1,\dots,e_m]}_{\Omega_\delta} H^{e_{m+1}^+}_{\mathbb{C}_\delta}\right](e_{m+1}) := \frac{1}{\sqrt{o}} F^{\alpha_1,\dots\alpha_{2n}:[e_1,\dots,e_{m+1}]}_{\Omega_\delta}.$$

Then by the projection relations given by s-holomorphicity, we can deduce

$$F_{\Omega_\delta}^{\alpha_1,\ldots,\alpha_{2n},e_{m+1}^+,e_{m+1}^-:[e_1,\ldots,e_m]} - F_{\Omega_\delta}^{\alpha_1,\ldots,\alpha_{2n}:[e_1,\ldots,e_m]} F_{\mathbb{C}_\delta}^{e_{m+1}^+,e_{m+1}^-} := F_{\Omega_\delta}^{\alpha_1,\ldots\alpha_{2n}:[e_1,\ldots,e_{m+1}]}.$$

By the inductive hypothesis, we have the desired Pfaffian formulations of $F_{\Omega_\delta}^{\alpha_1,\ldots,\alpha_{2n},e_{m+1}^+,e_{m+1}^-:[e_1,\ldots,e_m]}$ and $F_{\Omega_\delta}^{\alpha_1,\ldots,\alpha_{2n}:[e_1,\ldots,e_m]}$. Expanding the Pfaffian along the last column as in (2.4.7) the result easily follows. □

2.4.4 Observables and Ising Correlations

In this subsection we present the connection between Ising model correlations and spin-weighted correlations and the two-point observables defined in Section 2.3.1. These formulae follow immediately from the results of Sections 2.4.2 and 2.4.3.

Spin-symmetric correlations

Recall that we have defined a spin-symmetric correlation as the expectation of a product of energy densities which scale with the mesh size δ. Our characterization of the correlation consists of a Pfaffian involving the real observables introduced in Section 2.3. For an edge e, we will denote by $e^+ := e^{o^+}, e^- := e^{o^-}$ for any pair of s-orientations o^+, o^- such that $o^+ = e^{\pi i} o^-$; the following characterization shows in particular that the Pfaffian does not depend on such choice.

Proposition 2.4.9. *For any collection of distinct (possibly adjacent) interior edges $e_1, \ldots, e_m$ in Ω_δ,*

$$\mathbb{E}_{\Omega_\delta}\left[\prod_{e\in\{e_k\}} \epsilon(e)\right] = (-1)^m 2^m \mathrm{Pf}\left(\mathbf{F}_{\Omega_\delta}^{[\{e_k\}]}\right),$$

where for any admissible s-orientations, $o_1^\pm, \ldots, o_m^\pm$, the antisymmetric matrix $\mathbf{F}_{\Omega_\delta}^{[\{e_k\}]}$ is given by

$$\begin{pmatrix}
0 & F_{\Omega_\delta}^{e_1^+,e_2^+} & \cdots & F_{\Omega_\delta}^{e_1^+,e_m^+} & F_{\Omega_\delta}^{e_1^+,e_m^-} & \cdots & F_{\Omega_\delta}^{e_1^+,e_2^-} & F_{\Omega_\delta}^{\dagger e_1^+,e_1^-} \\
 & 0 & \cdots & F_{\Omega_\delta}^{e_2^+,e_m^+} & F_{\Omega_\delta}^{e_2^+,e_m^-} & \cdots & F_{\Omega_\delta}^{\dagger e_2^+,e_2^-} & F_{\Omega_\delta}^{e_2^+,e_1^-} \\
 & & 0 & \vdots & & \iddots & \vdots & \vdots \\
 & & & \ddots & F_{\Omega_\delta}^{\dagger e_m^+,e_m^-} & & F_{\Omega_\delta}^{e_m^+,e_2^-} & F_{\Omega_\delta}^{e_m^+,e_1^-} \\
 & & & & 0 & \cdots & F_{\Omega_\delta}^{e_m^-,e_2^-} & F_{\Omega_\delta}^{e_m^-,e_1^-} \\
 & & & & & \ddots & \vdots & \vdots \\
 & & & & & & 0 & F_{\Omega_\delta}^{e_2^-,e_1^-} \\
 & & & & & & & 0
\end{pmatrix},$$

where $F_{\Omega_\delta}^{\dagger\alpha,\zeta} := F_{\Omega_\delta}^{\alpha,\zeta} - F_{\mathbb{C}_\delta}^{\alpha,\zeta}$.

Proof. Starting from the characterization of Proposition 2.4.6, we can calculate the fused observable via an application of Proposition 2.4.8. □

Spin-antisymmetric Correlations

Recall that a spin-weighted correlation was defined as the expectation of a product of the spin at 0 and energy densities on adjacent sites. We characterize it as a Pfaffian analogously to the previous subsection, but with Ω_δ replaced by its double cover and the two-point fermion replaced with the spin-fermion observable. Accordingly, we again fix orientations e^+, e^- as well as a choice of lift in $[\Omega_\delta, 0]$ for an edge e in Ω_δ, on which the value of the Pfaffian does not depend.

Proposition 2.4.10. *For any collection of distinct (possibly adjacent) interior edges $e_1, \ldots, e_m$ in Ω_δ,*

$$\frac{1}{\mathbb{E}_{\Omega_\delta}[\sigma_0]}\mathbb{E}_{\Omega_\delta}\left[\sigma_0 \prod_{e\in\{e_k\}} \epsilon_{[0]}(e)\right] = (-1)^m 2^m \mathrm{Pf}\left(\mathbf{F}^{[\{e_k\}]}_{[\Omega_\delta,0]}\right),$$

where for any admissible s-orientations $o_1^\pm, \ldots, o_m^\pm$, the antisymmetric matrix $\mathbf{F}^{[\{e_k\}]}_{[\Omega_\delta,0]}$ is given by

$$\begin{pmatrix} 0 & F^{e_1^+,e_2^+}_{[\Omega_\delta,0]} & \cdots & F^{e_1^+,e_m^+}_{[\Omega_\delta,0]} & F^{e_1^+,e_m^-}_{[\Omega_\delta,0]} & \cdots & F^{e_1^+,e_2^-}_{[\Omega_\delta,0]} & F^{\dagger e_1^+,e_1^-}_{[\Omega_\delta,0]} \\ & 0 & \cdots & F^{e_2^+,e_m^+}_{[\Omega_\delta,0]} & F^{e_2^+,e_m^-}_{[\Omega_\delta,0]} & \cdots & F^{\dagger e_2^+,e_2^-}_{[\Omega_\delta,0]} & F^{e_2^+,e_1^-}_{[\Omega_\delta,0]} \\ & & 0 & \vdots & & \iddots & \vdots & \vdots \\ & & & \ddots & F^{\dagger e_m^+,e_m^-}_{[\Omega_\delta,0]} & \cdots & F^{e_m^+,e_2^-}_{[\Omega_\delta,0]} & F^{e_m^+,e_1^-}_{[\Omega_\delta,0]} \\ & & & & 0 & \cdots & F^{e_m^-,e_2^-}_{[\Omega_\delta,0]} & F^{e_m^-,e_1^-}_{[\Omega_\delta,0]} \\ & & & & & \ddots & \vdots & \vdots \\ & & & & & & 0 & F^{e_2^-,e_1^-}_{[\Omega_\delta,0]} \\ & & & & & & & 0 \end{pmatrix},$$

where $F^{\dagger\alpha,\zeta}_{[\Omega_\delta,0]} := F^{\alpha,\zeta}_{[\Omega_\delta,0]} - F^{\alpha,\zeta}_{[\mathbb{C}_\delta,0]}$.

Proof. Again starting from the characterization of Proposition 2.4.6, we can calculate the fused observable via Proposition 2.4.8, using now the equations corresponding to the spin-fermion. □

2.5 Scaling Limits of Observables

We have thus far defined the discrete observables which encode probabilistic information in the form of n-point correlations. As a result of the Pfaffian formulae of Section 2.4.4, it suffices to consider two-point discrete observables since all multipoint correlations can now be written in terms of only two-point observables. In this section, we introduce the continuous observables, which are precisely defined to have continuous analogues of the properties satisfied by the discrete observables; see Section 2.2. With the appropriate scaling, the discrete observables will be shown to converge to these continuous observables.

As the heart of convergence proofs is the extension of the convergence results of [CHI15] for $H^{\alpha}_{[\Omega_\delta,0]}$ to the case where the singularity point α is no longer at α_0, familiarity with the proofs of convergence in [CHI15] for the case $\alpha = \alpha_0$ is very helpful to understanding the sequel.

2.5.1 Integration of the Square

Write ν_z for the unit outward normal vector at $z \in \partial\Omega$. We have the following characterization of the complex fermion observable H_{Ω_δ} in terms of discrete complex analysis:

Proposition 2.5.1 (see [Hon10])**.** *Let $\alpha = a^o$ be an s-oriented medial vertex of Ω_δ that is not on the boundary. The function $H^{\alpha}_{\Omega_\delta}$ is the unique function such that:*

- $H^{\alpha}_{\Omega_\delta}$ *is s-holomorphic on* $\Omega_\delta \setminus \{a\}$;
- $H^{\alpha}_{\Omega_\delta}$ *has discrete residue* 1 *at* α: $H^{\alpha}_{\Omega_\delta}(\alpha_+) - H^{\alpha}_{\Omega_\delta}(\alpha_-) = \frac{1}{\sqrt{o}}$;
- $H^{\alpha}_{\Omega_\delta}(z)\sqrt{\nu_z} \in \mathbb{R}$ *for any boundary medial vertex* z.

Similarly, we have the following characterization of the complex spin-fermion observable $H_{[\Omega_\delta 0]}$.

Proposition 2.5.2. *Let $\alpha = a^o$ be an s-oriented medial vertex of $[\Omega_\delta, 0]$ that is not on the boundary. The function $H^{\alpha}_{[\Omega_\delta,0]}$ is the unique function such that:*

- $H^{\alpha}_{[\Omega_\delta,0]}$ *has monodromy* -1 *around* 0;
- $H^{\alpha}_{[\Omega_\delta,0]}$ *is s-holomorphic on* $[\Omega_\delta, 0] \setminus \{a, a^*\}$, *where* a, a^* *are on opposite sheets of* $[\Omega_\delta, 0]$;
- $H^{\alpha}_{[\Omega_\delta,0]}$ *has discrete residue* 1 *at* α: $H^{\alpha}_{[\Omega_\delta,0]}(\alpha_+) - H^{\alpha}_{[\Omega_\delta,0]}(\alpha_-) = \frac{1}{\sqrt{o}}$;
- $H^{\alpha}_{[\Omega_\delta,0]}(z)\sqrt{\nu_z} \in \mathbb{R}$ *for any boundary medial vertex* z.

Proof of Propositions 2.5.1–2.5.2. Monodromy for the latter is clear from the sheet factor $\mathrm{s}_{\alpha,\zeta}$. Proposition 2.4.1 and Lemmas 2.4.3 and 2.4.4 provide the remaining properties. Uniqueness follows from Lemma 2.2.2. □

We now consider the square integral Q^{α}_{δ} of the observables. These are the same discrete square integral analogues $\mathbb{I}_\delta(H_\delta) = \mathrm{Re} \int (H_\delta)^2$ introduced in Section 2.2.3 in the case where H_δ is one of $H^{\alpha}_{\Omega_\delta}, H^{\alpha}_{[\Omega_\delta,0]}$, but we need to analyze their properties near the singularity at α and the monodromy. These square integrals will be our primary means of estimating both observables, but for conciseness, we will assume that we are working with $H^{\alpha}_{[\Omega_\delta,0]}$. Neither the construction nor the properties are changed in the $H^{\alpha}_{\Omega_\delta}$ case, except for the fact that there are no longer complications arising from the branching at 0.

Proposition 2.5.3. *Suppose $\alpha = a^o$ is an s-oriented corner in $[\Omega_\delta, 0]$. Consider the single-valued integral of the square $Q^{\alpha}_{\delta} := \mathbb{I}_\delta \left[H^{\alpha}_{[\Omega_\delta,0]} \right] : \mathscr{F}_{\Omega_\delta} \cup \mathscr{V}_{\Omega_\delta} \to \mathbb{R}$ constructed with the usual rule*

$$Q^{\alpha}_{\delta}(w) - Q^{\alpha}_{\delta}(v) = 2\delta \left| H^{\alpha}_{[\Omega_\delta,0]} \left(\frac{1}{2}(w+v) \right) \right|^2,$$

where w is a face, v is a vertex incident to the face, so that $\frac{1}{2}(w+v)$ is the corner between them (note that at the singularity a, $\left| H^{\alpha}_{[\Omega_\delta,0]}(a) \right|^2 = \frac{1}{8}$), and the Dirichlet boundary condition

$$Q^{\alpha}_{\delta}(w) = 0 \text{ for } w \in \partial \mathscr{F}_{\Omega_\delta}.$$

It has

- $\Delta_\delta Q^{\alpha}_{\delta} = 2\delta \left| \partial_\delta H^{\alpha}_{[\Omega_\delta,0]} \right|^2$ *on* $\mathscr{F}_{\Omega_\delta} \setminus \left\{ 0, a - o\frac{\delta}{2} \right\}$, $\Delta_\delta Q^{\alpha}_{\delta} = -2\delta \left| \partial_\delta H^{\alpha}_{[\Omega_\delta,0]} \right|^2$ *on* $\mathscr{V}_{\Omega_\delta} \setminus \left\{ a + o\frac{\delta}{2} \right\}$;
- *The outer normal derivative* $\partial_{out} Q^{\alpha}_{\Omega} = \sqrt{2} \left| H^{\alpha}_{[\Omega_\delta,0]} \right|^2$ *on* $\partial \mathscr{V}^{m}_{\Omega_\delta}$.

Proof. The proof follows from direct computations in Chapter 2 of [Hon10] and Section 3.3 of [ChSm12]. Note that as in [CHI15, Proposition 3.6], the singularity at a (two projections from neighboring medial vertices differing only by a sign) and branching at 0 does not affect well-definedness of Q^{α}_{δ}, but does affect the Laplacian at $0, a \pm o\frac{\delta}{2}$. □

Remark 2.5.1. In keeping with the Dirichlet boundary condition, we can define $\tilde{Q}^{\alpha}_{\delta}$ which simply modifies the value of Q^{α}_{δ} on $\partial \mathscr{V}_{\Omega_\delta}$ to be zero. This affects the Laplacian and normal derivative, but one can define an alternate Laplacian $\tilde{\Delta}_\delta$ modified at the boundary which gives $\tilde{\Delta}_\delta \tilde{Q}^{\alpha}_{\delta} = \Delta_\delta Q^{\alpha}_{\delta}$; see [CHI15, Preposition 3.6].

Remark 2.5.2. Similarly, one can define the integral $Q^{\dagger\alpha}_{\delta} := \mathbb{I}_\delta \left[H^{\alpha}_{[\Omega_\delta,0]} - H^{\alpha}_{[\mathbb{C}_\delta,0]} \right]$; this has the advantage of removing the singularity at α, so that we have sub- and super-harmonicity at all points except for 0. [CHI15, Remark 3.8] notes that in the special case $a = \frac{\delta}{2}$ we in fact do have sub-harmonicity at a, owing to the fact that $\left[H^{\alpha}_{[\Omega_\delta,0]} - H^{\alpha}_{[\mathbb{C}_\delta,0]} \right](a) := 0$.

2.5.2 Continuous Observables

The continuous observables are the solutions of the continuous Riemann boundary value problem corresponding to the discrete b.v.p. of Lemma 2.2.2. We prove that they are in fact the scaling limits of their discrete counterparts in Section 2.5.3.

Continuous Full-Plane Observables

We begin by defining the continuous full-plane observables. In analogy with the discrete case, we define an s-orientation o of a point a in a continuous domain Ω as a choice of any unit complex number o with a specified square root.

Definition 2.5.1. Let $\alpha := a^o$ and $\zeta := z^p$ be two s-oriented points of $\mathbb{C}$ with $a \neq z$. We define the continuous fermion observable $f_{\mathbb{C}}$ by

$$f_{\mathbb{C}}^{\alpha}(\zeta) := i\sqrt{p} \cdot \mathsf{P}_{\frac{1}{i\sqrt{p}}\mathbb{R}}\left[h_{\mathbb{C}}^{\alpha}(z)\right] = \mathrm{Re}\,\left[i\sqrt{p} \cdot h_{\mathbb{C}}^{\alpha}(z)\right],$$

where the complex observable $h_{\mathbb{C}}$ is defined by

$$h_{\mathbb{C}}^{\alpha}(z) := \frac{1}{\sqrt{2}\pi} \frac{\sqrt{o}}{z-a}.$$

Analogously to the previous definition, we proceed to define the continuous full-plane spin-fermion observable. Since the source point $\delta\alpha$ of the discrete two-point spin-fermion tends to the monodromy point 0 as $\delta \to 0$, we fix it as $\alpha \in \mathcal{V}^{cm}_{[\mathbb{C}_1,0]}$, and formally denote the dependence of the continuous spin-fermion on α by $d\alpha$ (not to be confused with $\delta\alpha \in \mathcal{V}^{cm}_{[\mathbb{C}_\delta,0]}$, which is the scaling of α by $\delta > 0$).

Definition 2.5.2. Let $\alpha := a^o$ be an s-oriented corner or medial vertex of $[\mathbb{C}_1, 0]$. Let $\zeta := z^p$ be an s-oriented point of $[\mathbb{C}, 0]$. We define the continuous spin-fermion observable $f_{[\mathbb{C},0]}$ by

$$f_{[\mathbb{C},0]}^{d\alpha}(\zeta) := \mathrm{Re}\,[i\sqrt{p} \cdot h_{[\mathbb{C},0]}^{d\alpha}(z)],$$

where the complex observable $h_{[\mathbb{C},0]}^{d\alpha}$ is defined by

$$h_{[\mathbb{C},0]}^{d\alpha}(z) := \frac{C_\alpha}{\sqrt{z}},$$

and the scaling limit factor $C_\alpha \in \mathbb{R}$ is given by

$$C_\alpha = -\mathrm{Re}\,\left[i\sqrt{o}\left(\tilde{G}^{+}_{[\mathbb{C}_1,0]} - \tilde{G}^{-}_{[\mathbb{C}_1,0]}\right)(a)\right]. \tag{2.5.1}$$

Recall, here, that $\tilde{G}^{\pm}$ are the discrete analogs of $\frac{i}{2\sqrt{2}}\sqrt{z}$ given by Definition 2.3.7. Note that when α is an s-oriented corner, we do not need to take the real-part in the definition above as $i\sqrt{o}\tilde{G}^{\pm}(a)$ are already real, due to s-holomorphicity. In particular, if α is an s-oriented real or imaginary corner we have that

$$C_\alpha = \begin{cases} \pm\frac{1}{2\sqrt{2}}\sqrt{o} \cdot \mathrm{hm}\,{}^{\mathbb{X}^i_1}_{1/2}(a) & \text{if } a \in \mathcal{V}^{i}_{[\mathbb{C}_1,0]} \cap \mathbb{X}^{\pm} \\ \mp\frac{i}{2\sqrt{2}}\sqrt{o} \cdot \mathrm{hm}\,{}^{\mathbb{Y}^1_1}_{-1/2}(a) & \text{if } a \in \mathcal{V}^{1}_{[\mathbb{C}_1,0]} \cap \mathbb{Y}^{\pm} \end{cases}.$$

Continuous Bounded Domain Observables

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain containing 0. Recall that we denote by φ the conformal mapping from Ω to the open unit disk $\mathbb{D}$ with $\varphi(0) = z$ and $\varphi'(0) > 0$. Here we set $z = 0$. We transform $\alpha = a^o$ as $\varphi(\alpha) = \varphi(a)^{o'}, o' = \left(\sqrt{\varphi'(a)}\sqrt{o}\right)^2 / \left|\varphi'(a)\right|$ under φ (with a continuous square root branch choice for φ' such that $\sqrt{\varphi'(0)} > 0$).

Definition 2.5.3. Let $\alpha = a^o, \zeta = z^p$ be two s-oriented points of Ω with $a \neq z$. We define the fermion observable f_Ω by

$$f_\Omega^\alpha(\zeta) := i\sqrt{p}\mathsf{P}_{\frac{1}{i\sqrt{p}}\mathbb{R}}\left[h_\Omega^\alpha(z)\right] = \mathrm{Re}\left[i\sqrt{p}\cdot h_\Omega^\alpha(z)\right],$$

where the complexified fermion observable h_Ω is defined by

$$h_\Omega^\alpha(z) := \sqrt{\left|\varphi'(a)\right|}\sqrt{\varphi'(z)}h_{\mathbb{D}}^{\varphi(\alpha)}(\varphi(z)), \quad \text{where} \quad h_{\mathbb{D}}^\alpha(z) := \frac{1}{\sqrt{2}\pi}\left(\frac{1}{\sqrt{o}}\cdot\frac{1}{1-\bar{a}z} + \frac{\sqrt{o}}{z-a}\right).$$

Definition 2.5.4. Let $\alpha = a^o$ be an s-oriented corner or medial vertex on $[\mathbb{C}_1, 0]$, and $\zeta = z^p$ be an s-oriented point of Ω with $a \neq z$. We define the spin-fermion observable $f_{[\Omega,0]}$ by

$$f_{[\Omega,0]}^{d\alpha}(\zeta) := i\sqrt{p}\mathsf{P}_{\frac{1}{i\sqrt{p}}\mathbb{R}}\left[h_{[\Omega,0]}^{d\alpha}(z)\right] = \mathrm{Re}\left[i\sqrt{p}\cdot h_{[\Omega,0]}^{d\alpha}(z)\right],$$

where the complexified spin-fermion observable $h_{[\Omega,0]}$ is defined by

$$h_{[\Omega,0]}^{d\alpha}(z) := \sqrt{\varphi'(z)}h_{\mathbb{D}}^{d\alpha}(\varphi(z)), \quad \text{where} \quad h_{[\mathbb{D},0]}^{d\alpha}(z) = \frac{C_\alpha}{\sqrt{z}}.$$

Remark 2.5.3. As explained above, these definitions imply continuous versions of the properties satisfied by the discrete observables; they share the singularities of the full-plane observables, and satisfy the boundary condition $h_\Omega^\alpha(z), h_{[\Omega,0]}^{d\alpha}(z) \in \nu_z{}^{-1/2}\mathbb{R}$, if ν_z is the unit outward normal vector at $z \in \partial\Omega$. These also uniquely characterize a holomorphic function: see [Hon10, Proposition 48] and [CHI15, Lemma 2.9].

2.5.3 Observable Convergence: Statements

We now state the two observable convergence results for the fermion and spin-fermion that are needed for the proof of the main theorem in Section 2.6. The bulk of the remaining work is in the proof of the spin-fermion observable convergence; that proof is deferred until the next subsection.

In what follows, we say an s-holomorphic function $H_\delta : \mathcal{V}_{D_\delta}^{cm} \to \mathbb{C}$ *converges* to a continuous function $h : D \to \mathbb{C}$ if for any sequence $a_\delta \in \mathcal{V}_{D_\delta}^m$ such that $a_\delta \to a \in D$ we have $H_\delta(a_\delta) \to h(a)$. Equivalently, the values of H_δ on type 1 and i corners respectively converge to the real and imaginary parts of h.

For notational convenience and concreteness, when we take $z \in D$ to be the argument of an s-holomorphic function H_δ defined on $\mathcal{V}^{cm}_{D_\delta}$, we will take a closest medial vertex $z_\delta \in \mathcal{V}^m_{D_\delta}$ to z and evaluate it at z_δ. Then the convergence is often *uniform* on a compact set $K \subset D$, i.e. $|H_\delta(z_\delta) - h(z)|$ is small uniformly in $z \in K$.

By a slight abuse of notation, we will then use the notation $\alpha = a^o$ and $\zeta = z^p$ both for the the s-orientations of a or z, and those of a_δ and z_δ.

Theorem 2.5.1 ([Hon10, Theorem 91]). *As $\delta \to 0$, we have*

$$\frac{1}{\delta}\left(F^{\alpha}_{\Omega_\delta} - F^{\alpha}_{\mathbb{C}_\delta}\right)(\zeta) \to \left(f^{\alpha}_{\Omega} - f^{\alpha}_{\mathbb{C}}\right)(\zeta)\,,$$

uniformly for a, z away from $\partial\Omega$.

Now let $\alpha := a^o$ and $\zeta := z^p$ be s-oriented corners on $[\mathbb{C}_1, 0]$. For $\delta > 0$, denote by $\delta\alpha := (\delta a)^o$ and $\delta\zeta := (\delta z)^p$ their corresponding scaled versions on $[\mathbb{C}_\delta, 0]$ and, for sufficiently small δ, on $[\Omega_\delta, 0]$. Recall that $G, \tilde{G}^{\pm}$ are the discrete analogues of $\frac{1}{2\sqrt{2}}\sqrt{z}$, $\frac{i}{2\sqrt{2}}\sqrt{z}$, respectively, introduced in Definition 2.3.7.

Theorem 2.5.2. *For α, ζ any s-oriented corners or medial vertices in $[\mathbb{C}_1, 0]$, we have as $\delta \to 0$,*

$$\begin{aligned}\frac{1}{\delta}\left(F^{\delta\alpha}_{[\Omega_\delta,0]} - F^{\delta\alpha}_{[\mathbb{C}_\delta,0]}\right)(\delta\zeta) \to & \, 2\cdot 2\sqrt{2}Re\,\mathscr{A}_\Omega \cdot \left(C_\alpha Re\left[i\sqrt{p}G_{[\mathbb{C}_1,0]}(z)\right] - C_\zeta Re\left[i\sqrt{o}G_{[\mathbb{C}_1,0]}(a)\right]\right) \\ & + 2\cdot 2\sqrt{2}Im\,\mathscr{A}_\Omega \cdot \left(C_\alpha Re\left[i\sqrt{p}\tilde{G}^{-}_{[\mathbb{C}_1,0]}(z)\right] - C_\zeta Re\left[i\sqrt{o}\tilde{G}^{-}_{[\mathbb{C}_1,0]}(a)\right]\right),\end{aligned}$$

where, for $\alpha_0 = \frac{1}{2}^o$ as before, $\mathscr{A}_\Omega$ is the coefficient in the expansion

$$h^{d\alpha_0}_{\Omega}(z) = \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{z}} + 2\mathscr{A}_\Omega\sqrt{z} + O(|z|^{3/2})\right).$$

Proof. This statement is the consequence of results proved in §2.5.4. Theorem 2.5.3 shows the following in the case where α, ζ are both s-oriented corners: $\frac{1}{\delta}[H^{\delta\alpha}_{[\Omega_\delta,0]}(\delta z) - H^{\delta\alpha}_{[\mathbb{C}_\delta,0]}(\delta z)]$ converges to

$$\begin{aligned}2\cdot 2\sqrt{2}\cdot & \left[\mathrm{Re}\,\mathscr{A}_\Omega \cdot \left(C_\alpha G_{[\mathbb{C}_1,0]}(z) + i\sqrt{o}\left[\tilde{G}^{+}_{[\mathbb{C}_1,0]} - \tilde{G}^{-}_{[\mathbb{C}_1,0]}\right](z) G_{[\mathbb{C}_1,0]}(a)\right)\right. \\ & \left. + \mathrm{Im}\,\mathscr{A}_\Omega \cdot \left(C_\alpha \tilde{G}^{-}_{[\mathbb{C}_1,0]}(z) + i\sqrt{o}\left[\tilde{G}^{+}_{[\mathbb{C}_1,0]} - \tilde{G}^{-}_{[\mathbb{C}_1,0]}\right](z)\tilde{G}^{-}_{[\mathbb{C}_1,0]}(a)\right)\right],\end{aligned}$$

where we used Corollary 2.5.8 in order to identify $\tilde{C}_\alpha$ with C_α defined by (2.5.1). This formula also applies in the case where z is a medial vertex by linearity. Reformulation in the form of the theorem statement follows from the definition of the real observable; we can then swap α, ζ from the antisymmetry of the real observable and proceed to prove the result for the case where both are medial vertices. □

Remark 2.5.4. It was shown in [CHI15, Lemma 2.21] that $\mathscr{A}_\Omega = \mathscr{A}_{[\Omega,0]} = -\frac{1}{4}\partial_z \log r_\Omega(z)\big|_{z=0}$ which is the logarithmic derivative of the conformal radius as viewed from $0 \in \Omega$. This is used in Theorem 2.6.2 to identify the coefficient in the first-order correction to spin weighted correlations with $-\frac{1}{4}\partial_z \log r_\Omega(z)|_{z=0}$.

2.5.4 Observable Convergence: Proofs

We now prove the aforementioned observable convergence theorems, in particular, the convergence of the full-plane observables and Theorem 2.5.2.

A tricky point of this subsection is that the explicit coefficient C_α of Definition 2.5.2, in the claimed limit $h^{d\alpha}_{[\mathbb{C},0]}(z) = C_\alpha/\sqrt{z}$ of the full-plane spinor $H^{\delta\alpha}_{[\mathbb{C}_\delta,0]}$, can only be identified as such once all convergence results are proven; until then we introduce a recursively defined stand-in $\tilde{C}_\alpha$ and $\tilde{h}^{d\alpha}_{[\mathbb{C},0]}(z) := \tilde{C}_\alpha/\sqrt{z}$. We then show the equality $\tilde{C}_\alpha = C_\alpha$ at the end of this subsection in Corollary 2.5.8.

Convergence of the Full-plane Observables

We first state direct extensions of some results in [CHI15] regarding convergence of the full-plane observables and functions. In [CHI15], the renormalization factor $\vartheta(\delta) := \mathrm{hm}_0^{\mathbb{C}_\delta \setminus \mathbb{R}_{<0}}(2\delta\lfloor(2\delta)^{-1}\rfloor)$ is used; we are able to calculate the constant explicitly thanks to Proposition 2.7.1. Specifically, writing $N := \lfloor(2\delta)^{-1}\rfloor$, $\vartheta(\delta) = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2N-1}{2N} = \frac{(2N)!}{4^N N!^2}$ and Stirling's approximation shows $\vartheta(\delta) \sim \frac{1}{\sqrt{\pi N}} \sim \sqrt{\frac{2\delta}{\pi}}$.

Lemma 2.5.4. *For $\alpha \in \mathcal{V}^{cm}_{[\mathbb{C}_1,0]}$, we have $\sqrt{\frac{\pi}{2\delta}} H^{\delta\alpha}_{[\mathbb{C}_\delta,0]}(z) \xrightarrow{\delta\downarrow 0} \tilde{h}^{d\alpha}_{[\mathbb{C},0]}(z) := \frac{\tilde{C}_\alpha}{\sqrt{z}}$ uniformly on compact subsets in $\mathbb{C}\setminus\{0\}$ for some $\tilde{C}_\alpha \in \mathbb{R}$, which can be computed recursively.*

Proof. [CHI15, Lemma 2.14] provides the case where $\alpha = \alpha_0 = \frac{1}{2}^o, o = (e^{2\cdot\pi i})^2$ ($\tilde{C}_{\alpha_0} = \frac{1}{2\sqrt{2}}$ in our normalization). The other cases easily follow from the recursive constructions given in Proposition 2.3.10 and Corollary 2.3.11. The fact that $\tilde{C}_\alpha \in \mathbb{R}$ is inductively apparent from the fact that for any source point α, the function $H^{\delta\alpha}_{[\mathbb{C}_\delta,0]}(z)$ vanishes if z is an imaginary corner of the positive real line, or a real corner of the negative real line. □

Lemma 2.5.5. *We have that $\sqrt{\frac{\pi}{2\delta}} G_{[\mathbb{C}_\delta,0]}(z) \xrightarrow{\delta\downarrow 0} \frac{1}{2\sqrt{2}}\sqrt{z}$ and $\sqrt{\frac{\pi}{2\delta}} \tilde{G}^{\pm}_{[\mathbb{C}_\delta,0]}(z) \xrightarrow{\delta\downarrow 0} \frac{i}{2\sqrt{2}}\sqrt{z}$ uniformly on compact subsets in $\mathbb{C}\setminus\{0\}$.*

Proof. By [CHI15, Lemma 2.17], we have convergence of the real part of $G_{[\mathbb{C}_\delta,0]}$. The lemma follows by rotation and multiplication by i. □

Convergence of the Bounded Domain Observables

We begin by proving the following convergence result, which is a simple generalization of [CHI15, Theorem 2.18]. It provides a local estimate near the monodromy point 0 which will be crucial to the proof of the general global convergence.

As we have introduced $\tilde{h}^{d\alpha}_{[\mathbb{C},0]}$ as the counterpart of $h^{d\alpha}_{[\mathbb{C},0]}$ where C_α was replaced by $\tilde{C}_\alpha$ in Lemma 2.5.4, we define $\tilde{h}^{d\alpha_0}_{[\Omega,0]}$ using Definition 2.5.4 where C_α in $h^{d\alpha}_{[\Omega,0]}$ is replaced by $\tilde{C}_\alpha$ (i.e. $\tilde{h}^{d\alpha_0}_{[\Omega,0]}(z) = \sqrt{\varphi'(z)}\frac{\tilde{C}_\alpha}{\sqrt{\varphi(z)}}$). Note $C_{\alpha_0} = \tilde{C}_{\alpha_0} = \frac{1}{2\sqrt{2}}$. Recall also that we defined $\mathscr{A}_\Omega$ as the coefficient in the expansion,

$$h^{d\alpha_0}_{[\Omega,0]}(z) = \tilde{h}^{d\alpha_0}_{[\Omega,0]}(z) = \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{z}} + 2\mathscr{A}_\Omega\sqrt{z} + O(|z|^{3/2})\right).$$

Lemma 2.5.6. *For $\alpha_0 = \frac{1}{2}^o, \sqrt{o} = 1$, and a corner or medial vertex a on $[\mathbb{C}_1, 0]$,*

$$H^{\dagger\delta\alpha_0}_{[\Omega_\delta,0]}(\delta a) = H^{\delta\alpha_0}_{[\Omega_\delta,0]}(\delta a) - H^{\delta\alpha_0}_{[\mathbb{C}_\delta,0]}(\delta a) = \left(2Re\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_\delta,0]} + 2Im\,\mathscr{A}_\Omega \cdot \tilde{G}^{-}_{[\mathbb{C}_\delta,0]}\right)(\delta a) + o(\delta).$$

Proof. We closely follow the strategy in Section 3.5 of [CHI15]. Note that

$$H^{\dagger\delta\alpha_0}_{[\Omega_\delta,0]} - \left(2\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_\delta,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}^{-}_{[\mathbb{C}_\delta,0]}\right)$$

is s-holomorphic, so it suffices to show that it is $o(\delta)$ on real and imaginary corners and propagate.

Recalling the symmetrized and antisymmetrized observables $S^{\alpha}_{[\Omega_\delta,0]} := \frac{1}{2}\left[H^{\alpha}_{[\Omega_\delta,0]} + H^{\bar{\alpha}}_{\overline{[\Omega_\delta,0]}}\right], A^{\alpha}_{[\Omega_\delta,0]} := \frac{1}{2}\left[H^{\alpha}_{[\Omega_\delta,0]} - H^{\bar{\alpha}}_{\overline{[\Omega_\delta,0]}}\right]$ and $\Lambda_\delta = \Omega_\delta \cap \overline{\Omega_\delta}$ from Definition 2.3.6, define the following functions, s-holomorphic everywhere on $[\Lambda_\delta, 0]$:

$$S_\delta := S^{\delta\alpha_0}_{[\Omega_\delta,0]} - S^{\delta\alpha_0}_{[\mathbb{C}_\delta,0]} - 2\,\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_\delta,0]},$$
$$A_\delta := A^{\delta\alpha_0}_{[\Omega_\delta,0]} - A^{\delta\alpha_0}_{[\mathbb{C}_\delta,0]} - 2\,\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}^{-}_{[\mathbb{C}_\delta,0]}.$$

Given that $S_\delta + A_\delta = H^{\dagger\delta\alpha_0}_{[\Omega_\delta,0]} - \left(2\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_\delta,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}_{[\mathbb{C}_\delta,0]}\right)$, it remains to estimate the real and imaginary parts $S^1_\delta := S_\delta|_{\mathcal{V}^1_{[\Lambda_\delta,0]}\cap\mathbb{X}^+}, S^i_\delta := S_\delta|_{\mathcal{V}^i_{[\Lambda_\delta,0]}\cap\mathbb{Y}^+}$ and $A^1_\delta := A_\delta|_{\mathcal{V}^1_{[\Lambda_\delta,0]}\cap\mathbb{Y}^+}, A^i_\delta := A_\delta|_{\mathcal{V}^i_{[\Lambda_\delta,0]}\cap\mathbb{X}^+}$. Without loss of generality we show the $o(\delta)$ estimate for $S^1_\delta(\delta a)$, where S^1_δ is harmonic in the slit domain $\mathcal{V}^1_{[\Lambda_\delta,0]} \cap \mathbb{X}^+$ and vanishes on $\mathcal{V}^1_{[\Lambda_\delta,0]} \cap \mathbb{R}_{<0}$.

Define the discrete circle $w(r) := \{z \in \mathrm{Dom}(S^1_\delta) : r < |z| < r + 5\delta\}$ for small $r > 0$. The same twist of the discrete Beurling estimate ([LaLi04, Theorem 1]) as in [CHI15, Lemma 3.3] or our proof of Theorem 2.3.2 gives $\mathrm{hm}^{\mathbb{X}^+_\delta}_{\{\delta a\}}(z) \le C\delta^{1/2}|z|^{-1/2}$. By reversibility of the simple random walk we have $\mathrm{hm}^{\mathbb{X}^+_\delta}_{w(r)}(\delta a) \le C\delta^{1/2}r^{-1/2}$, which gives an estimate of a harmonic function identically 1 on $w(r)$ and vanishing on the slit. Comparing this with S^1_δ on $w(r)$ and applying the maximum principle in the interior gives

$$\left|S^1_\delta(\delta a)\right| \le C\delta^{\frac{1}{2}}r^{-\frac{1}{2}}\sup_{w(r)}\left|S^1_\delta\right|.$$

Now by convergence of $\sqrt{\frac{\pi}{2\delta}}H^{\delta\alpha_0}_{[\Omega_\delta,0]}$ to $h^{d\alpha_0}_{[\Omega,0]}$, away from the singularity, as $\delta \to 0$ (see Theorem 2.16 of [CHI15]), we have

$$\begin{aligned}\sqrt{\frac{\pi}{2\delta}}S^1_\delta(z) \to &\mathrm{Re}\left[\frac{1}{2}\left(\tilde{h}^{d\alpha_0}_{[\Omega,0]}(z)+\overline{\tilde{h}^{d\alpha_0}_{[\Omega,0]}(\bar{z})}\right)-\tilde{h}^{d\alpha_0}_{[\mathbb{C},0]}(z)-2\mathscr{A}_\Omega \mathrm{Re}\sqrt{z}\right]\\ &= O(|z|^{3/2}).\end{aligned}$$

Here we used the fact that $\tilde{h}^{\overline{d\alpha_0}}_{[\Omega,0]}(\cdot) = \overline{\tilde{h}^{d\alpha_0}_{[\Omega,0]}(\bar{\cdot})}$ since the right hand side is the unique solution to the boundary value problem in Remark 2.5.3. Thus, we have $\left|S^1_\delta(\delta a)\right| \le C'\delta r$ and since r is arbitrary we have $S^1_\delta(\delta a) = o(\delta)$ as $\delta \to 0$.

The estimate follows analogously for $S^i_\delta, A^1_\delta, A^i_\delta$, since they share the following properties which were the two properties needed to deduce that $S^1_\delta(\delta a) = o(\delta)$ above:

1. they are harmonic functions on their respective slit domains and vanish on the slits; we might extend the slit by a point, specifically the slit for A^i_δ includes $\frac{\delta}{2} \in \mathcal{V}^i_{[\Lambda_\delta,0]} \cap \mathbb{X}^+$ given that $A_\delta(\frac{\delta}{2}) = A^i_\delta(\frac{\delta}{2}) = 0$ (which also implies that A^1_δ is harmonic at $-\frac{\delta}{2}$).

2. they are $O(|z|^{3/2})$ on the discrete circle $w(r)$ defined above. □

Now we prove global convergence for general source point $\alpha \neq \alpha_0$, away from 0.

Proposition 2.5.7. *If $\alpha = a^o$ is an s-oriented corner or medial vertex on $[\mathbb{C}_1,0]$; then as $\delta \to 0$, we have $\sqrt{\frac{\pi}{2\delta}}H^{\delta\alpha}_{[\Omega_\delta,0]}(z) \xrightarrow{\delta\downarrow 0} \tilde{h}^{d\alpha}_{[\Omega,0]}(z)$ uniformly on compact subsets of $\Omega \setminus \{0\}$.*

Proof. [CHI15, Theorem 2.16] proves the case where $\alpha = \alpha_0 = \frac{1}{2}^o, \sqrt{o} = 1$; they consider the square integrals $\tilde{Q}^{\delta\alpha_0}_\delta$ and $Q^{\dagger\delta\alpha_0}_\delta$ as introduced in Section 2.5.1 (in their notation, $H_\delta, H^\dagger_\delta$), and show that they converge to the continuous functions Re $\int\left(\tilde{h}^{\delta\alpha_0}_{[\Omega,0]}\right)^2$ and Re $\int\left(\tilde{h}^{\delta\alpha_0}_{[\Omega,0]}-\tilde{h}^{\delta\alpha_0}_{[\mathbb{C},0]}\right)^2$, which implies convergence of the integrand (see Section 3.4 of [CHI15]). In our notation, they show uniform boundedness, and thus equicontinuity, of $\tilde{Q}^{\delta\alpha_0}_\delta, Q^{\dagger\delta\alpha_0}_\delta$ in δ in each subdomain of Ω_δ, away from the boundary and 0. Their subsequential limits are then identified with the continuous square integrals above.

We argue that a similar strategy works for all α. In fact, the only difference here is that the sub-harmonicity of $Q^{\dagger\delta\alpha}_\delta$ at 0 fails, since in general $H^{\dagger\delta\alpha}_{[\Omega_\delta,0]}(\frac{\delta}{2}) \neq 0$. Sub-harmonicity of the square integral is used twice in the proof of [CHI15, Theorem 2.16]: it shows that their H_δ (which corresponds to our $Q^{\dagger\delta\alpha}_\delta$) is uniformly bounded near 0, which is needed to identify the limit, and it is needed to apply the maximum principle near 0 and obtain [CHI15, Lemma 3.10]. We will thus reproduce these two bounds, except that we replace $Q^{\dagger\delta\alpha}_\delta$ in their argument by a modified version $Q^{\dagger\dagger\delta\alpha}_\delta$, which we now introduce.

By Lemma 2.5.6 and rescaling,

$$H^{\dagger\delta\alpha_0}_{[\Omega_\delta,0]}(\delta a) = \delta\left(2\mathrm{Re}\,\mathscr{A}_\Omega\cdot G_{[\mathbb{C}_1,0]}(a) + 2\mathrm{Im}\,\mathscr{A}_\Omega\cdot\tilde{G}^-_{[\mathbb{C}_1,0]}(a) + o(1)\right),$$

and by antisymmetry between the two arguments, it is easy to see that $H^{\dagger\delta\alpha}_{[\Omega_\delta,0]}(\delta a_0) = \frac{1}{2\sqrt{2}}(A_\alpha + o(1))i\delta$ as $\delta\to 0$ for some constant A_α. Then the modified observable

$$H^{\dagger\dagger\delta\alpha}_{[\Omega_\delta,0]} := H^{\dagger\delta\alpha}_{[\Omega_\delta,0]} - 2\sqrt{2}\frac{H^{\dagger\delta\alpha}_{[\Omega_\delta,0]}(\delta a_0)}{\delta i}\tilde{G}^+_{[\mathbb{C}_\delta,0]} = H^{\dagger\delta\alpha}_{[\Omega_\delta,0]} - (A_\alpha + o(1))\tilde{G}^+_{[\mathbb{C}_\delta,0]}$$

is everywhere s-holomorphic and satisfies $H^{\dagger\dagger\delta\alpha}_{[\Omega_\delta,0]}(\delta a_0) = 0$, so its integral $Q^{\dagger\dagger\delta\alpha}_\delta := \mathbb{I}_\delta\left[H^{\dagger\dagger\delta\alpha}_{[\Omega_\delta,0]}\right]$ is sub-harmonic on faces and super-harmonic on vertices. It converges to

$$\mathrm{Re}\int\left(\tilde{h}^{d\alpha}_{[\Omega,0]}(z) - \tilde{h}^{d\alpha}_{[\mathbb{C},0]}(z) - \frac{i}{2\sqrt{2}}A_\alpha\sqrt{z}\right)^2 dz$$

uniformly away from 0, so both the discrete observable and the continuous function are single-valued and bounded near 0. This fact, alternatively to its analogue for $Q^{\dagger\delta\alpha}_\delta$, also implies [CHI15, (2.8)], which identifies the singularity at 0. The analog of [CHI15, Lemma 3.10] also easily follows by replacing $H^\dagger_\delta$ in their proof by $Q^{\dagger\dagger\delta\alpha}_\delta$ as defined above. □

We now analyze the observable near the singularity at 0, finally giving the proof of the main convergence theorem. Balancing the two discrete analogues $\tilde{G}^\pm_{[\mathbb{C}_\delta,0]}$ of $\frac{i}{2\sqrt{2}}\sqrt{z}$ to create a harmonic function which is amenable to the methods of analysis used thus far is crucial to the proof.

Theorem 2.5.3 (Convergence Content of Theorem 2.5.2)**.** If $z\in\mathcal{V}^c_{[\mathbb{C}_1,0]}$ and $\alpha = a^o$ is an s-oriented corner on $[\mathbb{C}_1,0]$, then as $\delta\to 0$,

$$\begin{aligned}H^{\dagger\delta\alpha}_{[\Omega_\delta,0]}(\delta z) =& 2\sqrt{2}\tilde{C}_\alpha\left(2\mathrm{Re}\,\mathscr{A}_\Omega\cdot G_{[\mathbb{C}_\delta,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega\cdot\tilde{G}^-_{[\mathbb{C}_\delta,0]}\right)(\delta z)\\ &+ i2\sqrt{2}\sqrt{o}\left(2\mathrm{Re}\,\mathscr{A}_\Omega\cdot G_{[\mathbb{C}_1,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega\cdot\tilde{G}^-_{[\mathbb{C}_1,0]}\right)(a)\left[\tilde{G}^+_{[\mathbb{C}_\delta,0]} - \tilde{G}^-_{[\mathbb{C}_\delta,0]}\right](\delta z)\\ &+ o(\delta).\end{aligned}$$

Proof. We argue that the same strategy as in the proof of Lemma 2.5.6 works here. Indeed, after defining

$$\begin{aligned}S_\delta &:= S^{\delta\alpha}_{[\Omega_\delta,0]} - S^{\delta\alpha}_{[\mathbb{C}_\delta,0]} - 2\cdot 2\sqrt{2}\tilde{C}_\alpha\,\mathrm{Re}\mathscr{A}_\Omega\cdot G_{[\mathbb{C}_\delta,0]}\\ A_\delta &:= A^{\delta\alpha}_{[\Omega_\delta,0]} - A^{\delta\alpha}_{[\mathbb{C}_\delta,0]} - 2\cdot 2\sqrt{2}\tilde{C}_\alpha\,\mathrm{Im}\mathscr{A}_\Omega\cdot\tilde{G}^-_{[\mathbb{C}_\delta,0]}\\ &\quad - i2\sqrt{2}\sqrt{o}\left(2\mathrm{Re}\,\mathscr{A}_\Omega\cdot G_{[\mathbb{C}_1,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega\cdot\tilde{G}^-_{[\mathbb{C}_1,0]}\right)(a)\left[\tilde{G}^+_{[\mathbb{C}_\delta,0]} - \tilde{G}^-_{[\mathbb{C}_\delta,0]}\right],\end{aligned}$$

one sees that the real and imaginary parts of these two functions satisfy properties (1) and (2) at the end of the proof of Lemma 2.5.6, sufficient to conclude that $S^1_\delta, S^i_\delta, A^1_\delta, A^i_\delta$ evaluated at

(δz) are $o(\delta)$. The additional term in A_δ above is needed because we require $A_\delta(\delta a_0) = 0$;

$$\begin{aligned}\left[A^{\delta\alpha}_{[\Omega_\delta,0]} - A^{\delta\alpha}_{[\mathbb{C}_\delta,0]}\right](\delta a_0) = H^{\dagger\delta\alpha}_{[\Omega_\delta,0]}(\delta a_0) = -\sqrt{o}H^{\dagger\delta\alpha_0}_{[\Omega_\delta,0]}(\delta a) \\ = -\delta\sqrt{o}\left(2\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_1,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}^-_{[\mathbb{C}_1,0]}\right)(a) + o(\delta)\,,\end{aligned}$$

so we insert $\tilde{G}^+$ to cancel out this nonzero value, then adjust the coefficient in front of $\tilde{G}^-$ to match the global limit. □

We are now in position to explicitly characterize $\tilde{C}_\alpha$. The following is a consequence of Theorem 2.5.3.

Corollary 2.5.8. *For every $\alpha \in \mathcal{V}^{cm}_{[\mathbb{C}_1,0]}$, the constant $\tilde{C}_\alpha$ defined in Lemma 2.5.4 is given explicitly by*

$$\tilde{C}_\alpha = -Re\left[i\sqrt{o}\left(\tilde{G}^+_{[\mathbb{C}_1,0]} - \tilde{G}^-_{[\mathbb{C}_1,0]}\right)(a)\right] =: C_\alpha\,,$$

and therefore $\tilde{h}^{d\alpha}_{[\mathbb{C},0]} = h^{d\alpha}_{[\mathbb{C},0]}$.

Proof. First we suppose a is a real or imaginary corner; note that in this case the real part operator in the definition of C_α or real observables is superfluous. In Theorem 2.5.3, let $z = \frac{3}{2}$, say on $\mathbb{X}^+$, and let $\zeta = z^p$ with $\sqrt{p} = i$. Since $F^{\dagger\delta\alpha,\delta\zeta}_{[\Omega_\delta,0]} = \mathrm{Re}\, i\sqrt{p}H^{\dagger\delta\alpha}_{[\Omega_\delta,0]}(\delta z) = i\sqrt{p}H^{\dagger\delta\alpha}_{[\Omega_\delta,0]}(\delta z)$,

$$\begin{aligned}(2\sqrt{2}\delta)^{-1}F^{\dagger\delta\alpha,\delta\zeta}_{[\Omega_\delta,0]} \to &\tilde{C}_\alpha\left[i\sqrt{p}\left(2\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_1,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}^-_{[\mathbb{C}_1,0]}\right)(z)\right] \\ &- C_\zeta\left[i\sqrt{o}\left(2\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_1,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}^-_{[\mathbb{C}_1,0]}\right)(a)\right]\,, \\ (2\sqrt{2}\delta)^{-1}F^{\dagger\delta\zeta,\delta\alpha}_{[\Omega_\delta,0]} \to &\tilde{C}_\zeta\left[i\sqrt{o}\left(2\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_1,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}^-_{[\mathbb{C}_1,0]}\right)(a)\right] \\ &- C_\alpha\left[i\sqrt{p}\left(2\mathrm{Re}\,\mathscr{A}_\Omega \cdot G_{[\mathbb{C}_1,0]} + 2\mathrm{Im}\,\mathscr{A}_\Omega \cdot \tilde{G}^-_{[\mathbb{C}_1,0]}\right)(z)\right]\,.\end{aligned}$$

Since the two limits should differ only by sign, the result follows by using that $G_{[\mathbb{C}_1,0]}(z) \neq 0, \tilde{G}^\pm_{[\mathbb{C}_1,0]}(z) = C_\zeta = 0$, and the fact that, by our recursive construction, $\tilde{C}_\zeta = 0$ (see Proposition 2.3.10, and in particular, Proposition 2.7.2).

In the case where a is a medial vertex, note that by s-holomorphicity of $H^{\delta\alpha}_{[\mathbb{C}_\delta,0]}$ and antisymmetry of its real counterpart, in their arguments, we can express $\tilde{C}_\alpha$ as a linear combination of $\tilde{C}_{\beta^\pm}$, where $\beta^\pm$ are adjacent real and imaginary (s-oriented) corners. Notice that this linear combination is exactly mirrored in the case of C_α, since $\tilde{G}^+_{[\mathbb{C}_1,0]} - \tilde{G}^-_{[\mathbb{C}_1,0]}$ is s-holomorphic; thus the desired equality holds. □

2.6 Proofs of Theorems

In this section, we complete the proofs of Theorems 2.1.1–2.1.2 and Corollary 2.1.1.

2.6.1 Spin-symmetric Fields

We first prove Theorem (2.1.1), establishing the conformal invariance of spin-symmetric fields.

Definition 2.6.1. Suppose $a \neq z$ are medial vertices on $\mathbb{C}_1$. For s-orientations o, p on a, z respectively, write $\alpha = a^o, \zeta = z^p$. Define

$$F_{\mathbb{C}_1}^{\alpha,\zeta} = \text{Re}\left[i\sqrt{p}H_{\mathbb{C}_1}^{\alpha}(z)\right],$$
$$E_{\mathbb{C}_1}^{\alpha,\zeta} = \text{Re}\left[\frac{i\sqrt{p}\overline{\sqrt{o}}}{\sqrt{2}\pi}\right],$$

where $H_{\mathbb{C}_1}^{\alpha}(z)$ was explicitly defined in (2.3.1).

Now let $\{e_k\} = \{e_1, \ldots, e_n\}$ be a collection of distinct edges of $\mathbb{C}_1$. Set

$$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) := \left(e_1^+, \ldots e_n^+, e_n^-, \ldots, e_1^-\right),$$

where $e_j^+ := e_j^{o^+}$ and $e_j^- := e_j^{o^-}$ denote a choice of opposite s-orientations of e_j such that $o^+ = e^{\pi i} o^-$.

We write $\mathbf{F}^{\{e_k\}}$ to denote the $2n \times 2n$ antisymmetric matrix with entries $\left(\mathbf{F}^{\{e_k\}}\right)_{jk} := F_{\mathbb{C}_1}^{x_j,x_k}$ for $j + k \neq 2n+1$, and $\left(\mathbf{F}^{\{e_k\}}\right)_{jk} := 0$ on the anti-diagonal $j + k = 2n+1$; we also set $\mathbf{E}^{\{e_k\}}$ to be the matrix taking values $\left(\mathbf{E}^{\{e_k\}}\right)_{jk} := E_{\mathbb{C}_1}^{x_j,x_k}$. Now define

$$\mathscr{P}^{\{e_k\}} := (-2)^n \text{Pf}\left(\mathbf{F}^{\{e_k\}}\right),$$
$$\mathscr{Q}^{\{e_k\}} := (-2)^n D_{\mathbf{E}^{\{e_k\}}} \text{Pf}(\mathbf{F}^{\{e_k\}}),$$

where $D_{\mathbf{E}^{\{e_k\}}}\text{Pf}$ denotes the directional derivative of the Pfaffian function in the direction of $\mathbf{E}^{\{e_k\}}$.

Remark 2.6.1. The values $\mathscr{P}^{\{e_k\}}$ and $\mathscr{Q}^{\{e_k\}}$ only depend on the unordered collection $\{e_k\}$ since they arise as limits of the Pfaffian from Proposition 2.4.9, which has an interpretation as a physical quantity only depending on $\{e_k\}$.

Theorem 2.6.1 (Restatement of Theorem 2.1.1). *Let $\{e_k\}_{k=1}^n$ be a collection of n distinct edges of $\mathbb{C}_1$. As $\delta \to 0$, we have*

$$\mathbb{E}_{\Omega_\delta}\left[\prod_{e \in \{e_k\}} \epsilon(\delta e)\right] = \mathscr{P}^{\{e_k\}} + \delta \cdot r_\Omega^{-1}(0) \cdot \mathscr{Q}^{\{e_k\}} + o(\delta),$$

where $r_\Omega(z)$ is the conformal radius of Ω at $z \in \Omega$, defined $r_\Omega(z) := \left|\varphi'(0)\right|$ where $\varphi : \mathbb{D} \to \Omega$ is a conformal map with $\varphi(0) = z$.

Proof of Theorem 2.6.1. By Proposition 2.4.9, we have that

$$\mathbb{E}_{\Omega_\delta}\left[\prod_{e\in\{e_k\}}\epsilon(\delta e)\right]=(-1)^n 2^n \mathrm{Pf}\left(\mathbf{F}_{\Omega_\delta}^{[\{\delta e_k\}]}\right).$$

Write $\mathbf{F}_{\Omega_\delta}^{[\{\delta e_k\}]}=\mathbf{F}_{\mathbb{C}_\delta}^{[\{\delta e_k\}]}+\left[\mathbf{F}_{\Omega_\delta}^{[\{\delta e_k\}]}-\mathbf{F}_{\mathbb{C}_\delta}^{[\{\delta e_k\}]}\right]$. By scale invariance, $\mathbf{F}_{\mathbb{C}_\delta}^{[\{\delta e_k\}]}=\mathbf{F}_{\mathbb{C}_1}^{[\{e_k\}]}$, which, by definition, satisfies $\mathbf{F}_{\mathbb{C}_1}^{[\{e_k\}]}=\mathbf{F}^{\{e_k\}}$.

By Theorem 2.5.1, for any $\alpha=a^o,\zeta=z^p$, if we set $\delta\alpha:=(\delta a)^o$, $\delta\zeta:=(\delta z)^p$ we can calculate

$$\lim_{\delta\to 0}\frac{1}{\delta}\left[F_{\Omega_\delta}^{\delta\alpha,\delta\zeta}-F_{\mathbb{C}_\delta}^{\delta\alpha,\delta\zeta}\right]=\left[f_\Omega^{0^o}-f_{\mathbb{C}}^{0^o}\right](0^p)=E_{\mathbb{C}_1}^{\alpha,\zeta}r_\Omega^{-1}(0),$$

and the result follows from Taylor expansion of $\mathrm{Pf}\left(\mathbf{F}_{\Omega_\delta}^{[\{\delta e_k\}]}\right)=\mathrm{Pf}\left(\mathbf{F}^{\{e_k\}}+\delta\mathbf{E}^{\{e_k\}}r_\Omega^{-1}(0)+o(\delta)\right)$. □

2.6.2 Spin-antisymmetric Fields

We now generalize the above proof to spin-antisymmetric fields, proving Theorem 2.1.2.

For any s-oriented medial vertex or corner $\zeta=z^p$, we introduce the real quantity

$$G_{[\mathbb{C}_1,0]}(\zeta)=\mathrm{Re}\,[i\sqrt{p}G_{[\mathbb{C}_1,0]}(z)],$$

and define the real quantities $\tilde{G}^{\pm}_{[\mathbb{C}_1,0]}(\zeta)$ analogously; $G_{[\mathbb{C}_1,0]}(z),\tilde{G}^{\pm}_{[\mathbb{C}_1,0]}(z)$ were defined in Section 2.3.2.

Definition 2.6.2. Suppose $a\neq z$ are medial vertices on $[\mathbb{C}_1,0]$. For s-orientations o,p respectively on a,z, write $\alpha=a^o,\zeta=z^p$ for the s-oriented medial vertices. Set

$$\begin{aligned}F^{\alpha,\zeta}_{[\mathbb{C}_1,0]}&=\mathrm{Re}\left[i\sqrt{p}H^{\alpha}_{[\mathbb{C}_1,0]}(z)\right],\\ E^{\alpha,\zeta}_{[\mathbb{C}_1,0]}&:=2\cdot 2\sqrt{2}\Big([G_{[\mathbb{C}_1,0]}-i\tilde{G}^-_{[\mathbb{C}_1,0]}](\alpha)[\tilde{G}^+_{[\mathbb{C}_1,0]}-\tilde{G}^-_{[\mathbb{C}_1,0]}](\zeta)\\ &\qquad-[G_{[\mathbb{C}_1,0]}-i\tilde{G}^-_{[\mathbb{C}_1,0]}](\zeta)[\tilde{G}^+_{[\mathbb{C}_1,0]}-\tilde{G}^-_{[\mathbb{C}_1,0]}](\alpha)\Big).\end{aligned}$$

Let $\{e_k\}=\{e_1,\ldots,e_n\}$ be a collection of distinct edges of $\mathbb{C}_1$. Let $\tilde{e}_1,\ldots,\tilde{e}_n$ be a choice of lifts of $e_1,\ldots,e_n$ to $[\mathbb{C}_1,0]$. Set

$$(x_1,\ldots,x_{2n}):=\left(\tilde{e}_1^+,\ldots,\tilde{e}_n^+,\tilde{e}_n^-,\ldots,\tilde{e}_1^-\right),$$

where $\tilde{e}_j^+:=\tilde{e}_j^{o^+}$ and $\tilde{e}_j^-:=\tilde{e}_j^{o^-}$ denote a choice of opposite s-orientations of $\tilde{e}_j$ such that $o^+=e^{\pi i}o^-$.

Define $\mathbf{F}_{[0]}^{\{e_k\}}$ as the $2n\times 2n$ antisymmetric matrix with entries $\left(\mathbf{F}_{[0]}^{\{e_k\}}\right)_{jk}:=F^{x_jx_k}_{[\mathbb{C}_\delta,0]}$ for $j+k\neq 2n+1$, and $\left(\mathbf{F}_{[0]}^{\{e_k\}}\right)_{jk}:=0$ on the anti-diagonal, $j+k=2n+1$. Define also $\mathbf{E}_{[0]}^{\{e_k\}}$ the $2n\times 2n$

antisymmetric matrix given by $\left(\mathbf{E}_{[0]}^{\{e_k\}}\right)_{jk} := E_{[\mathbb{C}_\delta,0]}^{x_j x_k}$. Let

$$\begin{aligned}\mathscr{P}_{[0]}^{\{e_k\}} &:= (-2)^n \mathrm{Pf}\left(\mathbf{F}_{[0]}^{\{e_k\}}\right),\\ \mathscr{Q}_{[0]}^{\{e_k\}} &:= (-2)^n D_{\mathbf{E}_{[0]}^{\{e_k\}}} \mathrm{Pf}\left(\mathbf{F}_{[0]}^{\{e_k\}}\right).\end{aligned}$$

Remark 2.6.2. By the same reasoning as in the spin-symmetric case, the values $\mathscr{P}^{\{e_k\}}$ and $\mathscr{Q}^{\{e_k\}}$ only depend on the unordered collection $\{e_k\}$.

Theorem 2.6.2 (Restatement of Theorem 2.1.2)**.** *Let $\{e_k\}_{k=1}^n$ be a set of n edges of $\mathbb{C}_1$. For every $1 \le k \le n$, the quantity μ_{e_k} defined in* (2.1.1) *exists and, independently of s-orientation o_k on e_k, is*

$$\mu_{e_k} = \sqrt{o_k}\left[H_{[\mathbb{C}_1,0]}^{e_k^{o_k}}(e_{k+}^{o_k}) + H_{[\mathbb{C}_1,0]}^{e_k^{o_k}}(e_{k-}^{o_k})\right], \tag{2.6.1}$$

so that $\epsilon_{[0]}(\delta e_k)$ is a well-defined random variable for every k, and as $\delta \to 0$,

$$\frac{\mathbb{E}_{\Omega_\delta}\left[\sigma_0 \prod_{e\in\{e_k\}} \epsilon_{[0]}(\delta e)\right]}{\mathbb{E}_{\Omega_\delta}\left[\sigma_0\right]} = \mathscr{P}_{[0]}^{\{e_k\}} + \delta \cdot Re\left[-\frac{1}{4}\partial_z \log r_\Omega(z)\Big|_{z=0} \cdot \mathscr{Q}_{[0]}^{\{e_k\}}\right] + o(\delta),$$

where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ if $z = x + iy$. In particular, it follows from the results of [CHI15] that

$$\mathbb{E}_{\Omega_\delta}\left[\sigma_0 \prod_{a\in\{e_k\}} \epsilon_{[0]}(\delta e)\right] = 0 + \mathscr{C} \cdot \delta^{\frac{1}{8}} \cdot \mathscr{P}_{[0]}^{\{e_k\}} \cdot 2^{\frac{1}{4}} \cdot r_\Omega(0)^{-\frac{1}{8}} + o\left(\delta^{\frac{1}{8}}\right),$$

where $\mathscr{C}$ is a constant given explicitly by Eq. (1.1) of [CHI15].

Proof of Theorem 2.6.2. The expression for μ_e for every $e \in \mathscr{E}_{\mathbb{C}_1}$ was given by Remark 2.4.2. Now by Proposition 2.4.10, we have that

$$\frac{\mathbb{E}_{\Omega_\delta}\left[\sigma_0 \prod_{e\in\{e_k\}} \epsilon_{[0]}(\delta e)\right]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} = (-1)^n 2^n \mathrm{Pf}\left(\mathbf{F}_{[\Omega_\delta,0]}^{\{\delta e_k\}}\right),$$

Write $\mathbf{F}_{[\Omega_\delta,0]}^{\{\delta e_k\}} = \mathbf{F}_{[\mathbb{C}_\delta,0]}^{\{\delta e_k\}} + \left[\mathbf{F}_{[\Omega_\delta,0]}^{\{\delta e_k\}} - \mathbf{F}_{[\mathbb{C}_\delta,0]}^{\{\delta e_k\}}\right]$.

As before, by scale invariance, $\mathbf{F}_{[\mathbb{C}_\delta,0]}^{\{\delta e_k\}} = \mathbf{F}_{[\mathbb{C}_1,0]}^{\{e_k\}}$, and by definition, $\mathbf{F}_{[\mathbb{C}_1,0]}^{\{e_k\}} = \mathbf{F}_{[0]}^{\{e_k\}}$. By Theorem 2.5.2, for any $\alpha = a^o, \zeta = z^p$, if we set $\delta\alpha := (\delta a)^o$, $\delta\zeta := (\delta z)^p$ we have

$$\lim_{\delta\to 0}\frac{1}{\delta} F_{[\Omega_\delta,0]}^{\dagger\delta\alpha,\delta\zeta} = -\mathrm{Re}\left[\frac{1}{4}\partial_z \log r_\Omega(z)\Big|_{z=0} \cdot E_{[0]}^{\alpha,\zeta}\right],$$

and hence we get

$$\lim_{\delta\to 0}\frac{1}{\delta}\left[\mathbf{F}_{[\Omega_\delta,0]}^{\{\delta a_k\}} - \mathbf{F}_{[\mathbb{C}_\delta,0]}^{\{\delta a_k\}}\right] = -\mathrm{Re}\left[\frac{1}{4}\partial_z \log r_\Omega(z)\Big|_{z=0} \cdot \mathbf{E}_{[0]}^{\{a_k\}}\right].$$

The first result then follows from Taylor expansion as in the proof of the previous theorem, and the second follows by multiplying through by the conformally covariant expansion of $\mathbb{E}_{\Omega_\delta}[\sigma_0] = \mathscr{C} \cdot 2^{\frac{1}{4}} \cdot r_\Omega(0)^{-\frac{1}{8}} \cdot \delta^{\frac{1}{8}} + o\left(\delta^{\frac{1}{8}}\right)$ given by [CHI15]. □

2.6.3 Spin Pattern Probabilities

Finally, we prove Corollary 2.1.1 as a consequence of the above two proofs.

Proof of Corollary 2.1.1. We begin by proving the corollary for the spin-symmetric pattern fields. For a subset $\mathscr{F} \subset \mathscr{F}_{\mathbb{C}_1}$, let $\mathscr{B}$ be the set of all edges separating two adjacent faces in $\mathscr{F}$. To any spin-symmetric pattern $\pm\rho$ on $\mathscr{F}$, we can associate an edge subset $B \subset \mathscr{B}$ via the usual low-temperature expansion (see Figure 2.1.1 and Section 2.3). Denote the collection of such edge subsets in $\mathscr{B}$ that are associated to a spin-symmetric pattern on $\mathscr{F}$ by $P_{\mathscr{F}}(\mathscr{B}) \subset P(\mathscr{B})$. We will index the $2^{|\mathscr{F}|-1}$-dimensional vector $P^{\mathscr{F}} = (P_B)_{B \in P_{\mathscr{F}}(\mathscr{B})}$ of the probabilities of all spin-symmetric patterns by corresponding edge subsets.

Given any such an edge subset B, we can calculate a spin-symmetric correlation $E_B = \mathbb{E}_{\Omega_\delta}[\epsilon(\delta B)] = \mathbb{E}_{\Omega_\delta}\left[\prod_{e \in B} \epsilon(\delta e)\right]$ and also form another vector $E^{\mathscr{F}} = (E_B)_{B \in P_{\mathscr{F}}(\mathscr{B})}$ of dimension $2^{|\mathscr{F}|-1}$.

Clearly, every correlation function $\mathbb{E}_{\Omega_\delta}[\epsilon(B)]$ can be expressed as a linear combination of probabilities of the $2^{|\mathscr{F}|-1}$ spin-symmetric patterns on $\mathscr{B}$.

Thus we have a $2^{|\mathscr{F}|-1} \times 2^{|\mathscr{F}|-1}$ matrix $(EP^{\mathscr{F}})_{BB'} = \prod_{e \in B}(\mu - (2\mathbf{1}_{\{e \in B'\}} - 1))$, such that $E^{\mathscr{F}} = (EP^{\mathscr{F}})P^{\mathscr{F}}$. One can check by hand that the matrix has inverse given by:

$$(PE^{\mathscr{F}})_{B'B} = \frac{1}{2^{(2^{|\mathscr{F}|-1})}}(-1)^{\sum_{e \in \mathscr{B}} \mathbf{1}_{\{e \in B\}} \oplus \mathbf{1}_{\{e \in B'\}}} \prod_{e \in B} (\mu + (2\mathbf{1}_{\{e \in B'\}} - 1)).$$

Applying the inverse to $E^{\mathscr{F}}$, consisting of conformally covariant spin-symmetric correlations from Theorem 2.1.1, yields the desired result for spin-symmetric patterns.

For the spin-antisymmetric patterns, an analogous approach but conditioning on $\sigma_0 = \pm 1$ and replacing μ by μ_e, combined with the conformally covariant expansion $\mathbb{E}_{\Omega_\delta}[\sigma_0] = \mathscr{C} \cdot 2^{\frac{1}{4}} \cdot r_\Omega(0)^{-\frac{1}{8}} \cdot \delta^{\frac{1}{8}} + o\left(\delta^{1/8}\right)$ from [CHI15], gives the desired result. □

2.7 Appendix: The Harmonic Measure on $\mathbb{C}_1 \setminus \mathbb{R}_{>0}$

We start this section by giving an analytic formula for the harmonic measure of the tip of the slit plane. This uses Fourier series techniques, which was inspired by [ChHo16]. Using the formula, we prove that the auxiliary functions $G_{[\mathbb{C}_1,0]}, \tilde{G}^{\pm}_{[\mathbb{C}_1,0]}$ are discrete holomorphic.

Since the slit-plane harmonic measures appear as different translations of the same function, we present the following prototype:

Proposition 2.7.1. *For $s + ik \in (1+i)\mathbb{Z}^2$, the function H_0 given by,*

$$H_0(z) := hm_0^{(1+i)\mathbb{Z}^2\setminus\mathbb{Z}_{\geq 0}}(z = s+ik) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{C^{|k|}(\theta)}{\sqrt{1-e^{-2i\theta}}} e^{-is\theta} d\theta, \tag{2.7.1}$$

where $C(\theta) := \dfrac{\cos\theta}{1+|\sin\theta|}$ and the square root is evaluated on the principal branch, is the unique discrete harmonic function on the diagonal slit plane $(1+i)\mathbb{Z}^2 \setminus \mathbb{Z}_{\geq 0}$ with boundary values 1 at the origin and 0 elsewhere on $\mathbb{Z}_{\geq 0}$ and as $\to \infty$.

Proof. We first state two Fourier expansions, thanks to the generalized binomial theorem:

$$\frac{1}{\sqrt{1-e^{-2i\theta}}} = \sum_{n=0}^{\infty}(-1)^n \binom{-\frac{1}{2}}{n} e^{-2ni\theta} = 1 + \frac{1}{2}e^{-2i\theta} + \frac{3}{8}e^{-4i\theta} + \frac{5}{16}e^{-6i\theta} + \dots, \tag{2.7.2}$$

$$\frac{|\sin\theta|}{\sqrt{1-e^{-2i\theta}}} = \sqrt{\frac{\sin^2\theta}{1-e^{-2i\theta}}} = \frac{1}{2}\sqrt{1-e^{2i\theta}} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n} e^{2ni\theta}.$$

The first identity immediately gives the boundary values on $\mathbb{Z}_{\geq 0}$. Discrete harmonicity when $k \neq 0$ follows directly from the structure of the integrand, and when $k = 0$ values of the discrete Laplacian correspond to the Fourier coefficients in the second identity of Eq. 2.7.2, so vanishes on $\mathbb{Z}_{<0}$ since there are no negative Fourier modes.

For the decay at infinity, it suffices to show $|H_0(s+ik)| \to 0$ as $|k| \to \infty$ uniformly in s and as $|s| \to \infty$ for fixed k. Note the latter is just the Riemann-Lebesgue Lemma. For the former, we can use dominated convergence since $\frac{|C^{|k|}(\theta)|}{\sqrt{|1-e^{-2i\theta}|}} \downarrow 0$ pointwise a.e. as $|k| \to \infty$ and $\frac{|C^{|k|}(\theta)|}{\sqrt{|1-e^{-2i\theta}|}} \leq \frac{1}{\sqrt{|1-e^{-2i\theta}|}} = \frac{1}{\sqrt{2|\sin\theta|}}$, which is integrable. □

Now, the above characterization of the harmonic measure of the tip of the slit plane leads to a recursive construction of harmonic measures of other points on the slit, as discussed in Proposition 2.3.10. Suppose $H_n := \mathrm{hm}_{2n}^{(1+i)\mathbb{Z}^2}$ denotes the harmonic measure of the point $2n$ on the slit plane $(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{\geq 0}$. We have the following recursion formula:

$$\begin{aligned} H_n(m) &= H_{n-1}(m-2) - H_{n-1}(-2)H_0(m) \\ &=: H_0(m-2n) - X_1H_0(m-2n+2) - X_2H_0(m-2n+4) - \dots - X_nH_0(m). \end{aligned} \tag{2.7.3}$$

The coefficients $X_n := H_{n-1}(-2)$ can be used to calculate the recursive coefficients in Proposition 2.3.10 and the scaling limit coefficients C_α in Section 2.5.2 explicitly. We now give a simple formula for X_i.

Proposition 2.7.2. *For all $i \geq 1$, we have that $X_n = \frac{H_0(-2n+2)}{2n}$. Consequently,*

$$1 - X_1 - X_2 - \dots - X_n = H_0(-2n).$$

Proof. Define the generating functions

$$X(z) := \sum_{n=1}^{\infty} X_n z^n = \frac{1}{2}z + \frac{1}{8}z^2 + \frac{1}{16}z^3 + \cdots,$$
$$F(z) := \sum_{n=0}^{\infty} H_0(-2i)z^n = 1 + \frac{1}{2}z + \frac{3}{8}z^2 + \cdots = \frac{1}{\sqrt{1-z}}.$$

Note that Eq. (2.7.3) implies a convolution identity by taking $m = 0$,

$$H_0(-2m) = \sum_{n=0}^{m} X_n H_0(-2m+2n),$$

and setting $X_0 = 0$. Thus $XF = F - 1$, and $X = 1 - \sqrt{1-z} = \sum_{n=1}^{\infty}(-1)^{n+1}\binom{1/2}{n}z^n$. Given that $H_0(-2n) = (-1)^n\binom{-1/2}{n}$ by Eq. (2.7.2), both results are straightforward. □

Corollary 2.7.3. *The auxiliary functions, $G_{[\mathbb{C}_1,0]}, \tilde{G}^{\pm}_{[\mathbb{C}_1,0]}$, defined in Definition 2.3.7 are discrete holomorphic on* $[\mathbb{C}_1, 0]$.

Proof. Without loss of generality, we will show discrete holomorphicity of $G_{[\mathbb{C}_1,0]}$ at a type-λ corner $z = s + i\left(k + \frac{1}{2}\right) \in \mathbb{X}^+ \cap \mathbb{Y}^+$ $(k \geq 0)$. By the fact that

$$\sum_{n=-\infty}^{\infty}\left[\operatorname{hm}{}^{\mathbb{X}_1^1}_{3/2}(z + \frac{1+i}{2} + 2n) - \operatorname{hm}{}^{\mathbb{X}_1^1}_{3/2}(z - \frac{1+i}{2} + 2n)\right] = \lim_{\theta\to 0} C^k(\theta)\frac{C(\theta)-1}{\sqrt{1-e^{-2i\theta}}} = 0,$$

and discrete holomorphicity of $H^{\alpha_0}_{[\mathbb{C}_1,0]}$, we have

$$\begin{aligned}
&G_{[\mathbb{C}_1,0]}(z - \frac{1-i}{2}) - G_{[\mathbb{C}_1,0]}(z + \frac{1-i}{2})\\
&= \frac{i}{2\sqrt{2}}\sum_{n=1}^{\infty}\left[\operatorname{hm}{}^{\mathbb{Y}_1^i}_{1/2}(z - \frac{1-i}{2} + 2n) - \operatorname{hm}{}^{\mathbb{Y}_1^i}_{1/2}(z + \frac{1-i}{2} + 2n)\right]\\
&= -\frac{i}{2\sqrt{2}}\sum_{n=1}^{\infty}\left[\operatorname{hm}{}^{\mathbb{X}_1^1}_{3/2}(z + \frac{1+i}{2} + 2n) - \operatorname{hm}{}^{\mathbb{X}_1^1}_{3/2}(z - \frac{1+i}{2} + 2n)\right]\\
&= \frac{i}{2\sqrt{2}}\sum_{n=0}^{\infty}\left[\operatorname{hm}{}^{\mathbb{X}_1^1}_{3/2}(z + \frac{1+i}{2} - 2n) - \operatorname{hm}{}^{\mathbb{X}_1^1}_{3/2}(z - \frac{1+i}{2} - 2n)\right]\\
&= i\left[G_{[\mathbb{C}_1,0]}(z + \frac{1+i}{2}) - G_{[\mathbb{C}_1,0]}(z - \frac{1+i}{2})\right],
\end{aligned}$$

as desired. □

Now we show that G in fact has some rotation symmetry, which can be exploited to recursively compute its values as outlined in Remark 2.3.6. The proof relies on the same kind of analysis of discrete harmonic functions as in the proof of Lemma 2.5.6 and therefore we omit some of the details.

Proposition 2.7.4. *On $\mathbb{C}_\delta$, the following holds:* $e^{\pi i/4} \cdot G_{[\mathbb{C}_\delta,0]}(e^{\pi i/2}z) = \frac{1}{2}\left[\tilde{G}^{+}_{[\mathbb{C}_\delta,0]} + \tilde{G}^{-}_{[\mathbb{C}_\delta,0]}\right](z)$.

Proof. We will write $L_\delta(z)$ for the left hand side, and $R_\delta(z) = \left[k\tilde{G}^{+}_{[\mathbb{C}_\delta,0]} + (1-k)\tilde{G}^{-}_{[\mathbb{C}_\delta,0]}\right](z)$ for an as yet undefined real number k. Both are s-holomorphic functions, and by Lemma 2.5.5, both $\sqrt{\frac{\pi}{2\delta}}L_\delta(z), \sqrt{\frac{\pi}{2\delta}}R_\delta(z)$ converge to $\frac{i}{2\sqrt{2}}\sqrt{z}$ on compact subsets away from 0 as $\delta \to 0$.

First, it is straightforward to check that $G_{[\mathbb{C}_\delta,0]}$ vanishes on $\bar{\lambda}$-corners on the upper half of the imaginary axis, and on λ-corners on the lower half by the symmetry between its real and imaginary parts (in addition, $G_{[\mathbb{C}_\delta,0]}(\bar{z}) = \overline{G_{[\mathbb{C}_\delta,0]}(z)}$). So $L_\delta(z)$ has zero real part on the positive real line, and zero imaginary part on the negative real line. This is also true for $R_\delta(z)$. Now, $L_\delta\left(\pm\frac{\delta}{2}\right)$ is not necessarily zero, so we will choose (taking $\frac{\delta}{2}$ on $\mathbb{X}^+$) $k = -iL_\delta\left(\frac{\delta}{2}\right)$. Then $L_\delta(z) - R_\delta(z)$ is zero at $\frac{\delta}{2}$, so we have harmonicity at $-\frac{\delta}{2}$ by Remark 2.2.4.

Without loss of generality, consider the restriction of $\sqrt{\frac{\pi}{2\delta}}\left[L_\delta - R_\delta\right]$ to $\mathcal{V}^{i}_{[\mathbb{C}_\delta,0]} \cap \mathbb{X}^+$. It vanishes on the boundary $\mathbb{R}_{<0}$ and $\frac{\delta}{2}$, and as $\delta \to 0$ the values on the boundary $w(1)$ of the discrete ball $B_1(0) \cap \mathcal{V}^{i}_{[\mathbb{C}_\delta,0]}$ decays as $o(1)$. By the discrete Beurling estimate (see proof of Lemma 2.5.6) we can bound $\left|\sqrt{\frac{\pi}{2\delta}}\left[L_\delta - R_\delta\right](z\delta)\right|$ for any $z \in \mathcal{V}^{i}_{[\mathbb{C}_1,0]} \cap \mathbb{X}^+$ from above by $C\delta^{1/2}o(1)$. Since by definition $\sqrt{\frac{\pi}{2\delta}}\left[L_\delta - R_\delta\right](z\delta) = \sqrt{\frac{\pi\delta}{2}}\left[L_1 - R_1\right](z)$, $\left[L_1 - R_1\right](z) = 0$, and thus $L_\delta = R_\delta$.

Then we conclude $k = \frac{1}{2}$ since $(1-k)\tilde{G}^{-}_{[\mathbb{C}_\delta,0]}\left(e^{\pi i}\frac{\delta}{2}\right) = L_\delta\left(e^{\pi i}\frac{\delta}{2}\right) = iL_\delta\left(\frac{\delta}{2}\right) = ik\tilde{G}^{+}_{[\mathbb{C}_\delta,0]}\left(\frac{\delta}{2}\right)$ and $-\tilde{G}^{-}_{[\mathbb{C}_\delta,0]}\left(e^{\pi i}\frac{\delta}{2}\right) = \tilde{G}^{+}_{[\mathbb{C}_\delta,0]}\left(\frac{\delta}{2}\right) = G_{[\mathbb{C}_\delta,0]}\left(\frac{3\delta}{2}\right)$ again by the symmetry between real and imaginary parts of $G_{[\mathbb{C}_\delta,0]}$. □

2.8 Appendix: Contour Weights

Here we prove the well-definedness of the spin-fermionic contour weights introduced in full generality in Section 2.4.1.

Recall from Sections 2.3 and 2.4.1 the definition of $\gamma \in \mathscr{C}^{\alpha_1,\ldots,\alpha_{2n}}_{\Omega_\delta}$ and the admissible choices of walks $\{\Gamma(\gamma)\}$ associated to it. Moreover recall the definition of the multipoint observable $F_{[\Omega_\delta,a]}$ from Definition 2.4.2.

Proposition 2.8.1. *For any collection of distinct oriented medial vertices $\alpha_1,\ldots,\alpha_{2n}$ and any $\gamma \in \mathscr{C}^{\alpha_1,\ldots,\alpha_{2n}}_{\Omega_\delta}$, for every two admissible choices of walks, $\Gamma(\gamma), \Gamma'(\gamma)$, we have*

$$(-1)^{\ell(\gamma\setminus\cup\Gamma(\gamma))} \prod_{\gamma^{\alpha_j,\alpha_k}\in\Gamma(\gamma)} \mathrm{s}_{\alpha_j,\alpha_k}\left(\gamma^{\alpha_j,\alpha_k}\right) = (-1)^{\ell(\gamma\setminus\cup\Gamma'(\gamma))} \prod_{\gamma^{\alpha_{j'},\alpha_{k'}}\in\Gamma(\gamma)} \mathrm{s}_{\alpha_{j'},\alpha_{k'}}\left(\gamma^{\alpha_{j'},\alpha_{k'}}\right).$$

As a result, the function $F_{[\Omega_\delta,a]}$ is well-defined.

We will need the following two lemmas for the proof of the above proposition.

Lemma 2.8.2. *If A, B are unions of disjoint loops in Ω_δ, $(-1)^{\ell(A\oplus B)} = (-1)^{\ell(A)}(-1)^{\ell(B)}$.*

Proof. For each of A and B, fill in the faces of the lattice with spins, beginning with the plus boundary conditions, such that there is an edge between two faces if and only if they differ in sign. For each loop collection, we have the spin at zero $\sigma_0^A = (-1)^{\ell(A)}$ and $\sigma_0^B = (-1)^{\ell(B)}$.

The result follows after noting that $A \oplus B$ is identified with the spin configuration constructed by multiplying the spins of configurations A and B pointwise. □

Lemma 2.8.3. *Suppose l walks $w_1, \dots, w_l \in \{\gamma_1, \dots, \gamma_n, \gamma_1', \dots, \gamma_n'\}$ and l distinct oriented medial vertices $\alpha^1, \dots, \alpha^l \in \{\alpha_1, \dots, \alpha_{2n}\}$ form a cycle, i.e. w_1 connects (projections of) α^1 with α^2, w_2 connects α^2 with $\alpha^3, \dots$, w_l connects α^l with $\alpha^{l+1} := \alpha^1$. Then we have,*

$$(-1)^{\ell(w_1 \oplus \cdots \oplus w_l)} = s_{\alpha^1, \alpha^2}(w_1) \cdots s_{\alpha^l, \alpha^1}(w_l).$$

Proof. Fix a half-line $\Lambda = e^{i\theta}\mathbb{R}_{\geq 0}$ such that it is not parallel to any edges and is disjoint from all $\alpha^1, \dots, \alpha^l$. Note that $\Omega \setminus \Lambda$ lifts to $[\Omega, 0]$ as two sheets. We now define two quantities: given a piecewise C^1 path in $\Omega \setminus \{0\}$, we can count the number of times N_Λ that the path crosses Λ; given two points $a, z \in [\Omega, 0] \setminus \Lambda$, we define $S_\Lambda^{\alpha,\zeta} := 1$ if they belong to the same sheet in $[\Omega, 0]$ of $\Omega \setminus \Lambda$ and $S_\Lambda^{\alpha,\zeta} := -1$ otherwise.

Concatenate the (possibly reversed) walks such that $w := w_1 \oplus w_2 \oplus \cdots \oplus w_l$ is a continuous loop on $\Omega \setminus \{0\}$ starting from α^1. Then clearly $\ell(w_1 \oplus \cdots \oplus w_l) \equiv N_\Lambda(w) \mod 2$. Now it suffices to note that for any j, $(-1)^{N_\Lambda(w_j)} s_{\alpha^j, \alpha^{j+1}}(w_j) = S_\Lambda^{\alpha^j, \alpha^{j+1}}$ so that

$$\begin{aligned}(-1)^{N_\Lambda(w)} \prod_{j=1}^{l} s_{\alpha^j, \alpha^{j+1}}(w_j) &= \prod_{j=1}^{l} (-1)^{N_\Lambda(w_j)} s_{\alpha^j, \alpha^{j+1}}(w_j) \\ &= \prod_{j=1}^{l} S_\Lambda^{\alpha^j, \alpha^{j+1}} = S_\Lambda^{\alpha^1, \alpha^1} = 1,\end{aligned}$$

from which the lemma follows. □

Proof of Proposition 2.8.1. First observe that $\gamma_1, \dots, \gamma_n, \gamma_1', \dots, \gamma_n'$ and $a_1, \dots, a_{2n}$ are partitioned into disjoint cycles $P_1, \dots, P_{l'}$ in the sense of Lemma 2.8.3 (suppose each P_j is the resulting collection of loops of the form $w_1 \oplus \cdots \oplus w_l$). Note that $P_1 \oplus \cdots \oplus P_{l'} = \cup\Gamma(\gamma) \oplus \cup\Gamma'(\gamma)$, and thus $\big(\gamma \setminus \cup\Gamma(\gamma)\big) \oplus \big(\gamma \setminus \cup\Gamma'(\gamma)\big) \oplus P_1 \oplus \cdots \oplus P_{l'} = \emptyset$. By Lemma 2.8.2,

$$\begin{aligned}&(-1)^{\ell(\gamma \setminus \cup\Gamma(\gamma))} \prod_{\gamma^{\alpha_j, \alpha_k} \in \Gamma(\gamma)} s_{\alpha_j, \alpha_k}\left(\gamma^{\alpha_j, \alpha_k}\right) \cdot (-1)^{\ell(\gamma \setminus \cup\Gamma'(\gamma))} \prod_{\gamma^{\alpha_{j'}, \alpha_{k'}} \in \Gamma(\gamma)} s_{\alpha_{j'}, \alpha_{k'}}\left(\gamma^{\alpha_{j'}, \alpha_{k'}}\right) \\ =&(-1)^{\ell(\gamma \setminus \cup\Gamma(\gamma))} (-1)^{\ell(\gamma \setminus \cup\Gamma'(\gamma))} \prod_{j=1}^{l'} (-1)^{\ell(P_j)} = (-1)^{\ell(\emptyset)} = 1.\end{aligned}$$

concluding the proof. □

2.9 Appendix: Explicit Pattern Probabilities

In this section we give an example using Theorem 2.1.1 by computing explicitly the infinite-volume limit of and first-order conformal correction to a diagonal spin-spin correlation. On rotated lattices, $\Omega_\delta \subset \mathbb{C}_\delta$ with plus boundary, this corresponds to $\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{2\delta}]$ and is a quantity that appears in the study of the lattice level Ising stress tensor ([BeHo18]). We then explain how the similar computation would be done for an “L" shaped spin-weighted correlation $\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{(1+i)\delta}\sigma_{2\delta}]/\mathbb{E}_{\Omega_\delta}[\sigma_0]$ and give the explicit values one gets from the explicit recursion process outlined to get values of the infinite-volume spin-fermion.

Corollary 2.9.1. *Consider the Ising model on Ω_δ with plus boundary conditions and $0 \in \mathscr{F}_{\Omega_\delta}$; then, as $\delta \to 0$,*

$$\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{2\delta}] = \frac{2}{\pi} + \delta \cdot \frac{2}{\pi} \cdot r_\Omega^{-1}(0) + o(\delta).$$

Proof. We first observe that $\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{2\delta}] = \mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{(1+i)\delta}\sigma_{(1+i)\delta}\sigma_{2\delta}]$ and

$$\mathbb{E}_{\Omega_\delta}[\epsilon(a_1)\epsilon(a_2)] = \frac{1}{2} + \mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{(1+i)\delta}\sigma_{(1+i)\delta}\sigma_{2\delta}] - \frac{\sqrt{2}}{2}(\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{(1+i)\delta}] + \mathbb{E}_{\Omega_\delta}[\sigma_{(1+i)\delta}\sigma_{2\delta}]),$$

where $a_1 = \frac{1+i}{2}$, $a_2 = \frac{3+i}{2}$. By the first-order Taylor expansion of [HoSm13] (after rescaling) for the energy density ($\mathbb{E}_{\Omega_\delta}[\epsilon(a_1)] = -\delta \cdot \frac{\sqrt{2}}{\pi} \cdot r_\Omega^{-1}(0) + o(\delta)$), this implies that as $\delta \to 0$,

$$\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{2\delta}] = \frac{1}{2} + \mathbb{E}_{\Omega_\delta}[\epsilon(a_1)\epsilon(a_2)] + \delta \cdot \frac{2}{\pi} \cdot r_\Omega^{-1}(0) + o(\delta).$$

From this it suffices to compute the first-order asymptotics of $\lim_{\delta\to 0}\mathbb{E}_{\Omega_\delta}[\epsilon(a_1)\epsilon(a_2)]$. In order to do so, we consider the antisymmetric matrix

$$\mathbf{F}^{\{e_k\}} = \begin{pmatrix} 0 & F_{\mathbb{C}_1}^{a_1^+ a_2^+} & F_{\mathbb{C}_1}^{a_1^+ a_2^-} & F_{\mathbb{C}_1}^{\dagger a_1^+ a_1^-} \\ & 0 & F_{\mathbb{C}_1}^{\dagger a_2^+ a_2^-} & F_{\mathbb{C}_1}^{a_2^+ a_1^-} \\ & & 0 & F_{\mathbb{C}_1}^{a_2^- a_1^-} \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b & c \\ & 0 & d & e \\ & & 0 & f \\ & & & 0 \end{pmatrix}$$

where we choose orientations $o_1^+ = e^{7\pi i/4}$ and $o_2^+ = e^{5\pi i/4}$ on $\mathbb{C}_1$. Plugging in the explicit values from the observable defined in Eq. (2.3.1) and using the definition $F_{\mathbb{C}_1}^{\alpha,\zeta} = \mathrm{Re}[i\sqrt{p}H_{\mathbb{C}_1}^{\alpha,z}]$, where we choose the principal branch of the square root, observe that

$$a = -\frac{1}{2\pi} + \frac{\sqrt{2}}{2\pi} + \frac{1}{4}, \qquad b = -\frac{1}{4} + \frac{1}{2\pi},$$
$$c = 0, \qquad d = 0,$$
$$e = -\frac{1}{4} + \frac{1}{2\pi}, \qquad f = \frac{1}{2\pi} + \frac{\sqrt{2}}{2\pi} - \frac{1}{4}.$$

Combined with $\text{Pf}(\mathbf{F}^{\{e_k\}}) = af - be + cd$ and $\lim_{\delta\to 0}\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{2\delta}] = \frac{1}{2} + 4\text{Pf}(\mathbf{F}^{\{e_k\}})$ (where we applied Theorem 2.1.1 to $\mathbb{E}_{\Omega_\delta}[\epsilon(a_1)\epsilon(a_2)]$),

$$\lim_{\delta\to 0}\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{2\delta}] = \mathbb{E}_{\mathbb{C}_1}[\sigma_0\sigma_{2\delta}] = \frac{2}{\pi}.$$

In order to compute the constant in the conformal correction, note that the matrix $\mathbf{E}^{\{e_k\}}$ is given by $E^{\alpha,\zeta}_{\mathbb{C}_1} = \text{Re}\left[\frac{i\sqrt{p}\sqrt{o}}{\sqrt{2}\pi}\right]$ so that, here,

$$\mathbf{E}^{\{e_k\}} = \begin{pmatrix} 0 & E^{a_1^+a_2^+}_{\mathbb{C}_1} & E^{a_1^+a_2^-}_{\mathbb{C}_1} & E^{a_1^+a_1^-}_{\mathbb{C}_1} \\ & 0 & E^{a_2^+a_2^-}_{\mathbb{C}_1} & E^{a_2^+a_1^-}_{\mathbb{C}_1} \\ & & 0 & E^{a_2^-a_1^-}_{\mathbb{C}_1} \\ & & & 0 \end{pmatrix} = \frac{1}{\sqrt{2}\pi}\begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ & 0 & 1 & \frac{\sqrt{2}}{2} \\ & & 0 & -\frac{\sqrt{2}}{2} \\ & & & 0 \end{pmatrix}.$$

Inverting $\mathbf{F}^{\{e_k\}}$ by hand, and using the Pfaffian expansion formula $\text{Pf}(A+\delta B) = \text{Pf}(A)+\delta\text{Pf}(A)\text{Tr}(A^{-1}B)$, along with the above expressions, we see that, in fact, $4D_{\mathbf{E}^{\{e_k\}}}\text{Pf}(\mathbf{F}^{\{e_k\}}) = 4\text{Pf}(\mathbf{F}^{\{e_k\}})\text{Tr}((\mathbf{F}^{\{e_k\}})^{-1}\mathbf{E}^{\{e_k\}}) =$ 0 which, combined with Theorem 2.1.1 implies the desired geometric correction. □

Corollary 2.9.2. *Consider the Ising model on Ω_δ with plus boundary conditions and $0 \in \mathscr{F}_{\Omega_\delta}$; then as $\delta \to 0$,*

$$\frac{\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{(1+i)\delta}\sigma_{2\delta}]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} = 2(\sqrt{2}-1) + \delta\cdot\frac{5\sqrt{2}-7}{2}\cdot Re\left[\partial_z \log r_\Omega(z)|_{z=0}\right] + o(\delta)$$

Proof. First observe that the edge e separating $\sigma_{(1+i)\delta}$ from $\sigma_{2\delta}$ has midpoint at δa where $a = \frac{3}{2} + \frac{i}{2}$.

$$\frac{\mathbb{E}_{\Omega_\delta}[\sigma_0\epsilon_{[0]}(\delta a)]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} = \frac{\mathbb{E}_{\Omega_\delta}[\sigma_0\sigma_{(1+i)\delta}\sigma_{2\delta}]}{\mathbb{E}_{\Omega_\delta}[\sigma_0]} - \mu_a.$$

They by Eq. (2.6.1), picking an s-orientation o on a,

$$\mu_a = \sqrt{o}[H^{a^o}_{[\mathbb{C}_1,0]}(a^o_+) + H^{a^o}_{[\mathbb{C}_1,0]}(a^o_-)]$$

To compute the front and back values of $H^{a^o}_{[\mathbb{C}_1,0]}(a_\pm)$ we have implemented the recursion procedure for H using Mathematica as outlined in Proposition 2.3.10 and Corollary 2.3.11 (see also Fig. 2.3.3). That yields that

$$\mu_{3/2+i/2} = 2\sqrt{2}-2.$$

Now by Theorem 2.1.2, we wish to compute $\mathscr{P}^a_{[0]} = (-2)\text{Pf}(\mathbf{F}^a_{[0]})$ but since $\mathbf{F}^a_{[0]}$ is a $2n\times 2n$ antisymmetric matrix that is zero on its anti-diagonal, $\mathscr{P}^a_{[0]} = 0$. On the other hand, that implies that $Q^a_{[0]} = \mathbf{E}^a_{[0]}$ whose entries are given by Definition 2.6.2 in terms of values of $G_{[\mathbb{C}_1,0]}$

and $\tilde{G}^{\pm}_{[\mathbb{C}_1,0]}$ evaluated on oppositely oriented $a^{o^{\pm}}$. Via the explicit construction of the slit-plane harmonic measure recursion procedure outlined in Remark 2.3.6, one can calculate

$$E^{a^{o^+},a^{o^-}}_{[\mathbb{C}_1,0]} = 5\sqrt{2}-7\,.$$

Putting these together and plugging in to the expansion given by Theorem 2.1.2 for the spin-weighted correlation, we obtain the desired geometric correction. □

3 Massive Scaling Limit

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Solely responsible.

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Abstract.

We study the spin n-point functions of the planar Ising model on a simply connected domain Ω discretised by the square lattice $\delta\mathbb{Z}^2$ under near-critical scaling limit. While the scaling limit on the full-plane $\mathbb{C}$ has been analysed in terms of a fermionic field theory, the limit in general Ω has not been studied. We will show that, in a massive scaling limit wherein the inverse temperature is scaled $\beta \sim \beta_c - m_0\delta$ for a constant $m_0 < 0$, the renormalised spin correlations converge to a continuous quantity determined by a boundary value problem set in Ω. In the case of $\Omega = \mathbb{C}$ and $n = 2$, this result reproduces the celebrated formula of [WMTB76] involving the Painlevé III transcendent. To this end, we generalise the comprehensive discrete complex analytic framework used in the critical setting to the massive setting, which results in a perturbation of the usual notions of analyticity and harmonicity.

3.1 Introduction

The Ising model is a classical model of ferromagnetism first introduced by Lenz [Len20], whose simplicity and rich emergent structure have allowed for applications in various areas of science. In two dimensions, tt famously exhibits a continuous phase transition [Ons44, Yan52], where the characteristic length of the model diverges and the model becomes scale invariant. Consequently the model at the critical temperature β_c is expected to exhibit conformal symmetry under scaling limit, a prediction which has been formalised in terms of the Conformal Field Theory [BPZ84].

Given the infinite dimensional array of 2D local conformal symmetry [DMS97], it is natural to study the scaling limit of the model not only on the full plane $\mathbb{C}$ but on an arbitrary simply connected domain Ω. Accordingly, recent research has focused on giving a rigorous description of the interaction between various physical quantities of the model under critical scaling limit and the conformal geometry of Ω, which results in explicit formulae for the limit of spin correlations at the microscopic scale [HoSm13, GHP19], at the macroscopic scale [CHI15], or in some mixture of the two [Hon10, CHI18] in terms of quantities such as the conformal radius and its derivatives. Convergence of correlations also proves to be useful in proving convergence in more general senses: [CGN15] proves that the discrete spin field converges to a continuous random distribution using the convergence of their correlations.

Central to such analyses of the critical regime are discrete fermion correlations, which manifest themselves as discrete functions capable of encoding relevant physical quantities. They are discrete counterparts of the massless free fermion correlations in the continuous CFT, which turn out to be explicit holomorphic functions thanks to conformal symmetry. Discrete fermions instead enjoy a strong notion of discrete analyticity [Smi10], which, unlike some of its weaker variants, readily lends itself to precompactness estimates that ultimately yield convergence to the continuous fermion.

However, the free fermion is not an object unique to the massless (conformal) field theory;

indeed, the general field theory of the free fermion specifies a mass parameter m, or equivalently a length scale $\xi \propto 1/|m|$. In general, the corresponding regime in the Ising model is the *near-critical* scaling limit, where the deviation from criticality $\beta_c - \beta$ scales proportionally to the lattice spacing δ. Such scaling keeps the physical correlation length ξ asymptotically constant, allowing the limit to be physically described by the massive fermion.

The discrete fermion survives in this near-critical setup, and discrete analyticity persists albeit in a perturbed sense [DGP14, HKZ15, dT18]. While the continuous massive fermion is usually described by the two-dimensional massive Dirac equation, our strong discrete analyticity in fact features twice as many relations, resulting in (the discrete counterpart of) a perturbation of the ordinary Cauchy-Riemann equations in 1D. Since there are as many lattice equations in the near-critical limit as in the critical limit, it is natural to attempt to carry out in the former the analogues of analyses from the latter. We note here that in addition to such a *thermal* perturbation, one may also consider a *magnetic* perturbation to introduce mass (e.g. [CGN16]). Another direction of research has recently focused on universality with respect to general lattice, see [Che18].

In this paper, we undertake the analysis of macroscopic Ising spin correlations on a simply connected domain Ω in the near-critical scaling limit where $\beta_c - \beta$ is held equal to $m_0\delta$ for a fixed $m_0 < 0$ with + boundary conditions. We establish the existence of scaling functions to which renormalised spin correlations converge, and show that their logarithmic derivatives are determined by an explicit boundary value problem set in Ω. This extends the results of [CHI15] to the massive regime (save for the conformal covariance, which should not hold), and our proof combines the strategies of that paper with a massive perturbation of analytic function theory, both in the discrete and the continuous settings. In the former, massive harmonic and holomorphic functions can be studied via their relation to massive (extinguished) random walk; in the latter, the perturbed Cauchy-Riemann equation is dubbed Vekua equation and extensively treated in a theory established by Carleman, Bers, and Vekua, among others [Ber56, Vek62].

In the full plane, the massive scaling limit of the spin correlations was revealed to exhibit a surprising integrability property. Wu, McCoy, Tracy, and Barouch [WMTB76] first demonstrated that the 2-point function on the plane can be described in terms of the Painlevé III transcendent. Subsequently, Sato, Miwa, and Jimbo [SMJ77] recast the continuous analysis in terms of isomonodromic deformation theory, where Painlevé equations are known to arise, and obtained a closed set of differential equations for the n-point function. Letting $\Omega = \mathbb{C}$, we reproduce the 2-point scaling limit in the case of full-plane (whose classical treatment is given in, e.g., [PaTr83, Pal07]), setting up the continuous analysis. We explicitly carry out the isomonodromic analysis following the formulation of [KaKo80].

3.1.1 Main Results

Let Ω be a bounded simply connected domain with smooth boundary. We will treat the unbounded cases $\Omega = \mathbb{C}, \mathbb{H}$ as well. Define the rotated square lattice $\Omega_\delta := \Omega \cap \delta(1+i)\mathbb{Z}^2 = \Omega \cap \mathbb{C}_\delta$. We define the *Ising probability measure* $\mathbb{P} = \mathbb{P}^+_{\Omega_\delta,\beta}$ with + boundary conditions at *inverse temperature* $\beta > 0$ on the space of *spin configurations* $\{\pm 1\}^{\Omega_\delta}$ by

$$\mathbb{P}^+_{\Omega_\delta,\beta}[\sigma : \Omega_\delta \to \{\pm 1\}] \propto \exp \sum_{i\sim j} \beta \sigma_i \sigma_j,$$

where the sum is over pairs $\{i,j\} \subset \mathbb{C}_\delta$, $i \in \Omega_\delta$ such that $|i-j| = \sqrt{2}\delta$ and we define $\sigma_j = 1$ for $j \notin \Omega_\delta$ ($j \in \mathscr{F}[\Omega_\delta]$ in terms of detailed notation in Section 3.1.2). If $a \in \Omega$, we understand by σ_a the spin at a closest point in Ω_δ to a.

The planar Ising model on the square lattice undergoes a phase transition at the critical temperature $\beta_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right)$. Henceforth we will fix a negative parameter m and set $\beta = \beta(\delta) = \beta_c - \frac{m\delta}{2}$. This is a *subcritical* massive limit, where the spins stay in the ordered phase while approaching criticality.

Theorem 3.1.1. *Let Ω be a bounded simply connected domain and suppose $a_1, \ldots, a_n \in \Omega$. Under $\delta \downarrow 0, \beta = \beta_c - \frac{m\delta}{2}$, the spin n-point function converges to a continuous function of $a_1, \ldots, a_n$,*

$$\delta^{-\frac{n}{8}} \mathbb{E}^+_{\Omega_\delta,\beta(\delta)}\left[\sigma_{a_1} \cdots \sigma_{a_n}\right] \to \langle a_1, \ldots, a_n \rangle^+_{\Omega,m},$$

and its logarithmic derivative $\partial_{a_1} \ln \langle a_1, \ldots, a_n \rangle^+_{\Omega,m} = \mathscr{A}^1_\Omega + i\mathscr{A}^i_\Omega$, where $\partial_{a_1} = \frac{1}{2}\left(\partial_{x_1} - i\partial_{y_1}\right)$, is determined by the solution to the boundary value problem of Proposition 3.2.15 set on the domain Ω. The functions $\langle a_1, \ldots, a_n \rangle^+_{\Omega,m}$ are uniquely determined by their diagonal and boundary behaviours, see Section 3.1.3. In particular, as $a_1 \to \partial\Omega$,

$$\langle a_1 \rangle^+_{\Omega,m} \sim \langle a_1 \rangle^+_{\Omega,0} := \mathscr{C} \cdot 2^{\frac{1}{4}} \, rad^{-\frac{1}{8}}(a_1, \Omega),$$

where $\mathscr{C} := 2^{\frac{1}{6}} e^{-\frac{3}{2}\zeta'(-1)}$, $rad(a_1,\Omega)$ is the conformal radius of Ω as seen from a_1, and ζ is the Riemann zeta function.

If $\phi : \mathbb{D} \to \Omega$ is a conformal map with $\phi(0) = a_1$, then $\mathrm{rad}(a_1, \Omega) = \left|\phi'(a_1)\right|$. The normalisation of our continuous functions $\langle\cdot\rangle^+_{\Omega,0}$ differ to that of [CHI15] by a factor of $\mathscr{C}^n$. The derivation of the following result in our analytical setup may be of independent interest.

Corollary 3.1.2 ([WMTB76, SMJ77, KaKo80])**.** *The 2-point function in the full-plane is given by*

$$\langle -a, a \rangle^+_{\mathbb{C},m} = cst \cdot \cosh h_0(am) \cdot \exp\left[\int_{-\infty}^{am} r\left[\left(h_0'(r)\right)^2 - 4\sinh^2 2h_0(r)\right] dr\right],$$

$$\langle -a, a \rangle^{free}_{\mathbb{C},-m} = cst \cdot \sinh h_0(am) \cdot \exp\left[\int_{-\infty}^{am} r\left[\left(h_0'(r)\right)^2 - 4\sinh^2 2h_0(r)\right] dr\right],$$

where $a > 0$ and $\eta_0 = -\frac{1}{2}\ln h_0$ is a solution to the Painlevé III equation

$$r\eta_0\eta_0'' = r\left(\eta_0'\right)^2 - \eta_0\eta_0' - 4r + 4r\eta_0^4.$$

The constants are fixed by the condition that as $a \to 0$

$$\langle -a, a\rangle^+_{\mathbb{C},m} \sim \langle -a, a\rangle^+_{\mathbb{C},0} := \frac{\mathscr{C}^2}{|2a|^{1/4}},$$

and $\mathscr{C} := 2^{\frac{1}{6}} e^{-\frac{3}{2}\zeta'(-1)}$ as above.

3.1.2 Notation

Following signs will be used throughout the paper:

$$\lambda := e^{\frac{i\pi}{4}}, \beta_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right), \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}, A \oplus B := (A \cup B) \setminus (A \cap B).$$

We will write partial derivatives in contracted form, i.e. $\partial_x = \frac{\partial}{\partial x}$, etc. And denote by $\partial_z, \partial_{\bar{z}}$ the Wirtinger derivatives: where $z = x + iy$,

$$\partial_z := \frac{\partial_x - i\partial_y}{2}, \partial_{\bar{z}} := \frac{\partial_x + i\partial_y}{2} = \frac{e^{i\theta}\left(\partial_r + ir^{-1}\partial_\theta\right)}{2}.$$

We will also use ∂ to denote directional derivatives; i.e. $\partial_1 = \partial_x$, $\partial_i = \partial_y$, and so on. If $z \in \partial\Omega$, denote by $\nu_{\mathrm{out}}(z) \in \mathbb{C}$ as the unit normal at x, i.e. the unit complex number which points to the direction of outer normal vector at z. Then $\partial_{\nu_{\mathrm{out}}}$ is the outer normal derivative in the direction of ν_{out}.

We denote by $\partial^\delta_\lambda, \Delta^\delta$ the following discrete operators:

$$\partial^\delta_\lambda f(z) := f(z + \sqrt{2}\lambda\delta) - f(z), \text{ etc.,}$$
$$\Delta^\delta f(z) := f(z + \sqrt{2}\lambda\delta) + f(z - \sqrt{2}\bar{\lambda}\delta) + f(z - \sqrt{2}\lambda\delta) + f(z + \sqrt{2}\bar{\lambda}\delta) - 4f(z),$$

wherever they make sense, if $z \notin \mathcal{V}[\Omega_\delta]$. On $\mathcal{V}[\Omega_\delta]$, we make a small modification in the coefficients in Δ^δ; see (3.2.10). Note that $(\sqrt{2}\delta)^{-2}\Delta^\delta \to \Delta$.

Mass Parametrisation.

There are various equivalent ways of parametrising the deviation $\beta_c - \beta$, and we summarise the relation amongst them here at once, which hold at all times. In this paper, M, m, Θ will be supposed to be **negative** unless otherwise specified. We also assume δ is small enough to, e.g., have $\beta \in (0, \infty)$.

- *Discrete mass* $M := \beta_c - \beta$ is scaled $M = \frac{m\delta}{2}$ with the continuous mass m being a constant.
- Pure phase factor $e^{2i\Theta} := \lambda^{-3}\frac{e^{-2\beta}+i}{e^{-2\beta}-i}$ with $\Theta \in \left]-\frac{\pi}{8}, \frac{\pi}{8}\right[$. Equivalently $e^{2\beta} = \cot\left(\frac{\pi}{8} + \Theta\right)$. Θ is scaled $\Theta \sim \frac{m\delta}{2}$. Also define $M_H := 2\sin 2\Theta\sqrt{\frac{2}{\cos 4\Theta}}$ which is the mass coefficient in massive harmonicity.

Graph Notation.

Recall that we work with the rotated square lattice $\mathbb{C}_\delta := \delta(1+i)\mathbb{Z}^2$. Our graph Ω_δ comprises the following components (see Figure 3.1.1):

- *faces* $\mathscr{F}[\Omega_\delta] := \Omega \cap \delta(1+i)\mathbb{Z}^2$,
- *vertices* $\mathcal{V}[\Omega_\delta] := \left\{f \pm \delta, f \pm i\delta : f \in \mathscr{F}[\Omega_\delta]\right\} \subset \mathbb{C}_\delta^*$,
- *edge* $\mathscr{E}[\Omega_\delta] := \left\{(ij) = (ji) : i, j \in \mathcal{V}[\Omega_\delta], \left|i - j\right| = \sqrt{2}\delta\right\}$, and
- *corners* $\mathscr{C}[\Omega_\delta] := \left\{(vf) : v \in \mathcal{V}[\Omega_\delta], f \in \mathscr{F}[\Omega_\delta], \left|v - f\right| = \delta\right\}$.

For consistency with the low-temperature expansion of the model, we prefer to visualise the lattice in its dual form. Note that just as the faces are represented by their midpoints above, an edge (ij) and a corner (vf) will be identified with their midpoints $\frac{i+j}{2}$ and $\frac{v+f}{2}$, respectively. Additionally, we draw a *half-edge* between either an edge midpoint or a corner to a nearest vertex.

For $\tau = 1, i, \lambda, \bar{\lambda}$, a corner $c \in \mathscr{C}[\Omega_\delta]$ is in $\mathscr{C}^\tau[\Omega_\delta]$ if the nearest vertex is in the direction $-\tau^{-2}$. The edges in $\mathscr{C}^1[\Omega_\delta], \mathscr{C}^i[\Omega_\delta]$ are respectively called *real* and *imaginary corners.*

We will frequently denote union of various sets by concatenation, e.g. $\mathscr{E}\mathscr{C}[\Omega_\delta] := \mathscr{E}[\Omega_\delta] \cup \mathscr{C}[\Omega_\delta]$.

Graph Boundary.

- $\overline{\mathscr{E}}[\Omega_\delta] := \left\{(ij) = (ji) : i \in \mathcal{V}[\Omega_\delta], j \in \mathcal{V}[\mathbb{C}_\delta], |i - j| = \sqrt{2}\delta\right\}$, $\partial\mathscr{E}[\Omega_\delta] := \overline{\mathscr{E}}[\Omega_\delta] \setminus \mathscr{E}[\Omega_\delta]$,
- boundary vertices $\partial\mathcal{V}[\Omega_\delta]$ are the endpoints of edges in $\partial\mathscr{E}[\Omega_\delta]$ not in $\mathcal{V}[\Omega_\delta]$,
- boundary faces $\partial\mathscr{F}[\Omega_\delta]$ are faces in $\mathscr{F}[\mathbb{C}_\delta] \setminus \mathscr{F}[\Omega_\delta]$ which are δ away from a vertex in $\mathcal{V}[\Omega_\delta]$, and
- boundary corners $\partial\mathscr{C}[\Omega_\delta] = \left\{(vf) : v \in \mathcal{V}[\Omega_\delta], f \in \partial\mathscr{F}[\Omega_\delta], \left|v - f\right| = \delta\right\}$
- $\nu_{\text{out}}(z)$ for $z \in \partial\mathscr{E}[\Omega_\delta]$ is the unit complex number corresponding to the orientation of z pointing outwards from Ω.

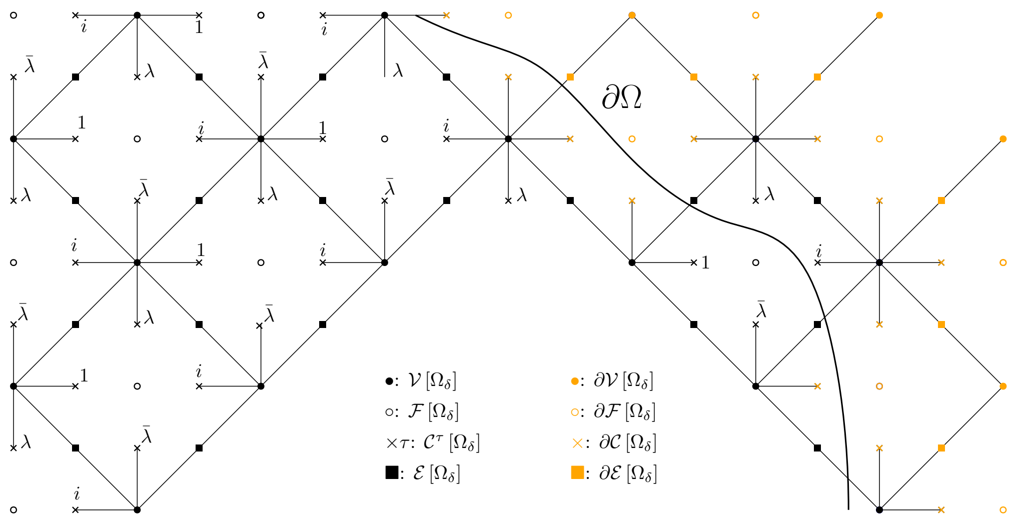

Figure 3.1.1 – The square lattice. Proposition 3.3.3: $F_{[\Omega_\delta,a_1,\dots,a_n]} = O(1)$ on boundary edges and corners in orange, since $H^\bullet_{[\Omega_\delta,a_1,\dots,a_n]} = O(\delta)$ on adjacent vertices in $\mathcal{V}[\Omega_\delta, a_1, \dots, a_n]$.

Double Cover.

The fermions we introduce in forthcoming sections are functions defined on the double cover of the continuous and discrete domains Ω, Ω_δ. Define $[\Omega, a_1, \dots, a_n]$ as the double cover of Ω ramified at distinct interior points $a_1, \dots, a_n \in \Omega$; in particular, it is a Riemann surface where $\sqrt{(z-a_1)\cdots(z-a_n)}$ is well-defined, smooth and single-valued. In the case where $\{\bar{a}_1, \dots, \bar{a}_n\} = \{a_1, \dots, a_n\}$, conjugation on the double cover is defined by requiring that $\sqrt{(\bar{z}-a_1)\cdots(\bar{z}-a_n)} = \overline{\sqrt{(z-a_1)\cdots(z-a_n)}}$. On $[\mathbb{C}, a_1]$, we will refer to the slit domains $\mathbb{X}^+ := \left\{\mathrm{Re}\sqrt{z} > 0\right\}$ and $\mathbb{Y}^+ := \left\{\mathrm{Im}\sqrt{z} > 0\right\}$.

When the choice of the lift of $z \in \Omega$ is clear, we will write z to denote the lift in $[\Omega, a_1, \dots, a_n]$. Conversely, if $z \in [\Omega, a_1, \dots, a_n]$, we will write z, or, for clarification, $\pi(z) \in \Omega \setminus \{a_1, \dots, a_n\}$ for the projection onto the planar domain; $z^{\cdot} \in [\Omega, a_1, \dots, a_n]$ is the lift of $\pi(z)$ which is not z. A function which switches sign under switching z and $z^{\cdot}$ is called a *spinor*.

We say that two points $z, w \in [\Omega, a_1, \dots, a_n]$ are *on the same sheet* if we can draw a straight line segment between them; i.e. the straight line segment on Ω which connects $\pi(z), \pi(w)$ can be lifted to connect z, w on $[\Omega, a_1, \dots, a_n]$.

For the discrete double cover $[\Omega_\delta, a_1, \dots, a_n]$, we will take closest faces in Ω_δ to $a_1, \dots, a_n \in \Omega$, and then lift components of Ω_δ minus those n faces. Clearly, $[\Omega_\delta, a_1, \dots, a_n]$ is a lattice which is locally isomorphic to the planar lattice Ω_δ. Given the first monodromy face a_1, we will fix a lift of $a_1 + \frac{\delta}{2}$ and refer to it throughout this paper.

3.1.3 Proof of the Main Theorems

Proof of Theorem 3.1.1. Given the fermion convergence of Theorem 3.3.5 and identification of Ising quantities in terms of the fermions of Proposition 3.2.3, we can integrate the discrete logarithmic derivative which converges in the scaling limit (see also [CHI15, Proposition 2.22, Remark 2.23])

$$\frac{1}{2\delta}\left[\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1+2\delta}\sigma_{a_2}\cdots\sigma_{a_n}\right]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\cdots\sigma_{a_n}\right]}-1\right]\to\mathscr{A}^1_\Omega(a_1,\dots,a_n),$$
$$\frac{1}{2\delta}\left[\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1+2i\delta}\sigma_{a_2}\cdots\sigma_{a_n}\right]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\cdots\sigma_{a_n}\right]}-1\right]\to-\mathscr{A}^i_\Omega(a_1,\dots,a_n),$$

to get that $\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}[\sigma_{b_1}\cdots\sigma_{b_n}]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}$ scales to a continuous limit $\frac{\langle b_1,\dots,b_n\rangle^+_{\Omega,m}}{\langle a_1,\dots,a_n\rangle^+_{\Omega,m}}$ for $a_1,\dots,a_n,b_1,\dots,b_n\in\Omega$. The convergence for $\mathscr{A}^i_\Omega$ follows by considering the result in a $-90°$ rotated domain (see also [CHI15, Proof of Theorems 1.5 and 1.7]).

Now it remains to uniquely relate the massive convergence rate to the massless convergence rate: $\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}\to\frac{\langle a_1,\dots,a_n\rangle^+_{\Omega,m}}{\langle a_1,\dots,a_n\rangle^+_{\Omega,0}}$ for some $a_1,\dots,a_n\in\Omega$. Given the convergence of $\delta^{-\frac{n}{8}}\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\cdots\sigma_{a_n}\right]$ to a continuous limit $\langle a_1,\dots,a_n\rangle^+_{\Omega,0}$, we have convergence of the massive correlation to unique $\langle a_1,\dots,a_n\rangle^+_{\Omega,m}$.

The procedure partially relies on the process in the massless case of relating the bounded domain correlations to full-plane correlations from [CHI15].

Note that $\beta>\beta_c$, and denote the dual temperature by $\beta^*<\beta_c$. We always assume a scaling $\beta_c-\beta=\frac{m\delta}{2}$ for $m<0$.

1. Relating two point functions: $\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}]}$ tends to a continuous limit, since

$$1\le\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}\le\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}{\mathbb{E}^{\beta^*,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}\le\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}{\mathbb{E}^{\beta^*,\mathrm{free}}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}\to|\mathscr{B}_\Omega(a_1,a_2|m)|^{-1},\qquad(3.1.1)$$

   where we successively used the monotonicity of spin correlation in inverse temperature (e.g. by coupling with FK-Ising) and in boundary condition (FKG inequality, [FrVe17, Theorem 3.21]). We also used the convergence of Theorem 3.3.5 of the Ising ratio to $|\mathscr{B}_\Omega(a_1,a_2|m)|$. We conclude by noting that $|\mathscr{B}_\Omega(a_1,a_2|m)|$ can be made arbitrarily close to 1 (Lemma 3.4.5) by merging a_1,a_2.

2. One point functions: note that by above, we may write

$$\frac{\langle a_1, a_2\rangle^+_{\Omega,m}}{\langle a_1, a_2\rangle^+_{\Omega,0}} = \exp\left[\int_{a_2}^{a_1} \left[\mathscr{A}^1_\Omega(z, a_2|m) - \mathscr{A}^1_\Omega(z, a_2|0)\right] dx - \left[\mathscr{A}^i_\Omega(z, a_2|m) - \mathscr{A}^i_\Omega(z, a_2|0)\right] dy\right], \tag{3.1.2}$$

along any line from a_2 to a_1 if the integral converges. Choose a_1 and a_2 as in Lemma 3.4.9 (see Figure 3.1.2), then we can bound the integral: $\frac{\langle a_1,a_2\rangle^+_{\Omega,m}}{\langle a_1,a_2\rangle^+_{\Omega,0}} \le e^{cst\cdot(\epsilon^{-\gamma}|a_1-a_2|+|a_1-a_2|^{1-\gamma})}$ for some fixed $\gamma \in (0,1)$. Now choose $|a_1 - a_2| = \epsilon^\kappa$ for some $\gamma < \kappa < 1$. Then $\frac{\langle a_1,a_2\rangle^+_{\Omega,m}}{\langle a_1,a_2\rangle^+_{\Omega,0}} \to 1$ as $\epsilon \to 0$, while the hyperbolic distance between a_1 and a_2 grows; indeed, the hyperbolic distance is comparable to $\ln \frac{|a_1-a_2|}{\mathrm{dist}(\{a_1,a_2\},\partial\Omega)} \propto (1-\kappa)\,|\ln\epsilon|$. Therefore, by [CHI15, (1.3)], $\frac{\langle a_1,a_2\rangle^+_{\Omega,0}}{\langle a_1\rangle^+_{\Omega,0}\langle a_2\rangle^+_{\Omega,0}} \to 1$. Now we may relate the massive one-point functions to the massless ones, so that the former are uniquely identified in terms of the latter. By the monotonicity of spin correlation, $\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\right] \le \mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]$, and by GKS inequality [FrVe17, Theorem 3.20], $\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_2}\right] \le \mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]$, so

$$1 \le \frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]} \le \frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_2}\right]} = \frac{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_2}\right]}\cdot\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}. \tag{3.1.3}$$

By the above discussion, both factors on the right is made arbitrarily close to 1 by taking small enough ϵ. Moreover, such choice of ϵ is uniform in the smoothness of the domain (concretely, the bound on the derivative of a conformal map to the disc).

3. More points: we again use the GKS inequality, in that

$$1 \le \frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\cdots\sigma_{a_n}\right]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]\cdots\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_n}\right]} \le \frac{\mathbb{E}^{\beta,+}_{\Omega^1_\delta}\left[\sigma_{a_1}\right]\cdots\mathbb{E}^{\beta,+}_{\Omega^n_\delta}\left[\sigma_{a_n}\right]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]\cdots\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_n}\right]}, \tag{3.1.4}$$

where $\Omega^1 \ni a_1, \ldots, \Omega^n \ni a_n$ are choices of disjoint smooth simply connected subdomains of Ω, each sharing a macroscopic boundary arc with Ω. By FKG inequality, requiring the spins in $\Omega \setminus \bigcup_{j=1}^n \Omega^j$ to be plus raises the correlation, which gives the second inequality above. By the uniform identification of the one point functions, there is $\epsilon > 0$ such that if each of $a_1, \ldots, a_n$ are ϵ-close to the boundary arc that $\Omega^1, \ldots, \Omega^n$ shares with Ω, both $\mathbb{E}^{\beta,+}_{\Omega^j_\delta}\left[\sigma_{a_j}\right], \mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_j}\right]$ become close to their massless counterparts. Since the massless correlations are Caratheodory stable ([CHI15, (1.3)]), as ϵ decreases, $\mathbb{E}^{\beta_c,+}_{\Omega^j_\delta}\left[\sigma_{a_j}\right] / \mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_j}\right]$ become close to 1. Clearly, any ratio between $\mathbb{E}^{\beta,+}_{\Omega^j_\delta}\left[\sigma_{a_j}\right]$, $\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_j}\right]$, $\mathbb{E}^{\beta_c,+}_{\Omega^j_\delta}\left[\sigma_{a_j}\right]$, $\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_{a_j}\right]$ may be made arbitrarily close to 1 by setting ϵ small enough, and thus $\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\cdots\sigma_{a_n}\right]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\right]\cdots\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_n}\right]}$ as well; this fixes the normalisation of an arbitrary n-point function.

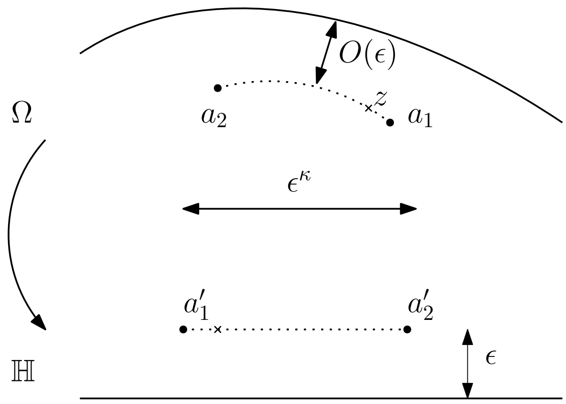

Figure 3.1.2 – Setup for applying Lemma 3.4.9 to have decorrelation between a_1, a_2.

□

Proof of Corollary 3.1.2. Note that the argument for the scaling limit of the two-point functions in the proof of Theorem 3.1.1 apply in the full plane as well. It also fixes the normalisation as $a \to 0$ by [CHI15, Remark 2.26]. From Theorem 3.3.5, we need to integrate $\mathscr{A}^1_{\mathbb{C}}(-a, a)$. Recall $r = am$, and define $\langle r\rangle^+_{\mathbb{C}} := \langle -a, a\rangle^+_{\mathbb{C},m}$.

By (3.4.22), $-m^{-1}\,\partial_a \ln\langle -a, b\rangle^+_{\mathbb{C},m}\Big|_{b=a} = -\frac{1}{2}(\ln\cosh h_0(r))' - r\left[\frac{1}{2}\left(h'_0(r)\right)^2 - 2\sinh^2 2h_0(r)\right]\Big|_{r=am}$, which can be rephrased as

$$\partial_r \ln\langle r\rangle^+_{\mathbb{C}} = m^{-1}\partial_a\langle -a, a\rangle^+_{\mathbb{C},m} = (\ln\cosh h_0(r))' + r\left[\left(h'_0(r)\right)^2 - 4\sinh^2 2h_0(r)\right].$$

Then

$$\langle -a, a\rangle^+_{\mathbb{C},m} = \langle r\rangle^+_{\mathbb{C}} = cst\cdot\cosh h_0(am)\cdot\exp\left[\int_{-\infty}^{am} r\left[\left(h'_0(r)\right)^2 - 4\sinh^2 2h_0(r)\right]dr\right].$$

Then the definition $\tanh h_0 := \mathscr{B}_0$ gives the other case. □

3.1.4 Structure of the Paper

This paper contains four sections and one appendix to which technical calculations and estimates are deferred. Section 3.2 defines our main analytical tool, the discrete fermions. The combinatorial definition in 3.2.1 involves contours on the discrete bounded domain and is seen to naturally encode the logarithmic derivative of the spin correlation. Its discrete complex analytic properties are then established, which are exploited in Section 3.2.2 to give a definition on the full plane by an infinite volume limit.

Since analysis of the continuous fermions is needed for the scaling limit process (for a unique characterisation of the continuous limit), we carry out the continuum analysis first in Section

3.2.3. We formulate the boundary value problem on Ω for our continuous fermions, which will be a massive perturbation of holomorphic functions treated in [Ber56, Vek62]. We verify various properties we will use: such as the expansion in terms of formal 'powers'. Analysis in the continuum is continued in Section 3.4.1.

Convergence of the discrete fermion under scaling limit is done in Section 3.3. The analysis is divided into two parts: bulk convergence (Section 3.3.1), where the discrete fermion evaluated on compact subsets of $[\Omega, a_1, \ldots, a_n]$ is shown to uniformly converge to the continuous fermion, and analysis near the singularity (Section 3.3.2), where the discrete fermion evaluated at a point in $[\Omega_\delta, a_1, \ldots, a_n]$ microscopically away from a monodromy face is identified from the coefficients of a massive analytic version of power series expansion of the continuous fermion. Bulk convergence is done in a standard manner, by first showing that the set of discrete fermion correlations are precompact and then uniquely identifying the limit. Analysis near the singularity mainly uses ideas from [CHI15], where the continuous power series expansion is modelled in the discrete setting then the coefficients carefully matched.

In Section 3.4, we collect the analysis of the massive fermions necessary for the integration of the logarithmic derivatives and isomonodromic analysis in Section 3.4.1, and finally in Section 3.4.2 we carry out the isomonodromic analysis and obtain the Painlevé III transcendent, which can be identified in the logarithmic derivative of spin correlations in $\mathbb{C}$ given the convergence results.

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3.2 Massive Fermions

In this section, we introduce the main discrete tool of our analysis, the massive discrete fermion correlations. While the object and the terminology hearkens back to the physical analysis of the Ising model, we shall give an explicit definition in Section 3.2.1 as a complex function which extrapolates the desired local physical quantity to the entire domain. In the same subsection we also show that, as a discrete function, it exhibits a notion in discrete complex analysis called (massive) s-holomorphicity. At first we only define the fermion in bounded discretised domains, i.e. finite sets; however we will define them in the complex plane in Section 3.2.2. Then, we carry out analysis of the continuous spinors, to which the discrete spinors presented in the previous section is shown to converge in Section 3.3. Since the proof of scaling limit requires unique identification of the continuous limit, we first give necessary analytic background in Section 3.2.3.

3.2.1 Bounded Domain Fermions and Discrete Analysis

We introduce here the main object of our analysis, the discrete fermion F. Note that this function is essentially the same object as in [CHI15, Definition 2.1] albeit at general β, and we try to keep the same normalisation and notation where appropriate. The contents of this subsection are valid for any $\beta > 0$.

In order to use the low-temperature expansion of the Ising model, we first define $\Gamma_{\Omega_\delta} \subset 2^{\mathscr{E}[\Omega_\delta]}$ as the collection of closed contours, i.e. set ω of edges in Ω_δ such that an even number of edges in ω meet at any given vertex. Given + boundary condition, any ω is clearly in one-to-one correspondence with a spin configuration σ (where ω delineates clusters of identical spins), and we can compute the partition function of the model and the correlation

$$\mathcal{Z}^{+,\beta}_{\Omega_\delta} := \sum_{\omega \in \mathscr{C}_{\Omega_\delta}} e^{-2\beta|\omega|}$$

$$\mathbb{E}^{+,\beta}_{\Omega_\delta}\left[\sigma_{a_1} \cdots \sigma_{a_n}\right] = \left|\mathcal{Z}^{+,\beta}_{\Omega_\delta}\right|^{-1} \sum_{\omega \in \Gamma_{\Omega_\delta}} e^{-2\beta|\omega|}(-1)^{\#\text{loops}_{a_1,\ldots,a_n}(\omega)},$$

where $\#\text{loops}_{a_1,\ldots,a_n}$ denotes the parity of loops in ω which separate the boundary spins from an odd number of $a_1, \ldots, a_n$. The unnormalised correlation $\mathcal{Z}^{+,\beta}_{\Omega_\delta}\left[\sigma_{a_1} \cdots \sigma_{a_n}\right] := \mathcal{Z}^{+,\beta}_{\Omega_\delta} \cdot \mathbb{E}^{+,\beta}_{\Omega_\delta}\left[\sigma_{a_1} \cdots \sigma_{a_n}\right]$ will be used for normalisation below.

Definition 3.2.1. For a bounded simply connected domain $\Omega \subset \mathbb{C}$ with n distinct interior points $a_1, \ldots a_n$ and inverse temperature $\beta > 0$, define for $z \in \mathscr{E}\mathscr{C}[\Omega_\delta, a_1, \ldots, a_n]$ which is not a lift of $a_1 + \frac{\delta}{2}$ the *discrete massive fermion correlation*, or simply the *discrete fermion*

$$F_{[\Omega_\delta, a_1, \ldots, a_n]}(z|\beta) = \frac{1}{\mathcal{Z}^{+,\beta}_{\Omega_\delta}\left[\sigma_{a_1} \cdots \sigma_{a_n}\right]} \sum_{\gamma \in \Gamma_{\Omega_\delta}(a_1 + \frac{\delta}{2}, z)} c_z e^{-2\beta|\gamma|} \cdot \phi_{a_1, \ldots, a_n}(\gamma, z)$$

where

- $\Gamma_{\Omega_\delta}(a_1 + \frac{\delta}{2}, z)$ is the collection of $\gamma = \omega \oplus \gamma_0$, where ω runs over elements over Γ_{Ω_δ}, γ_0 is a fixed simple lattice path from $a_1 + \frac{\delta}{2}$ to $\pi(z) \in \mathscr{E}\mathscr{C}[\Omega_\delta]$, and $\oplus$ refers to the XOR (symmetric difference) operation. $|\gamma|$ is the number of full edges in γ. $c_z := \cos\left(\frac{\pi}{8} + \Theta(\beta)\right)^{-1}$ if z is an edge, and 1 if it is a corner. Note that none of these definitions refer to the double cover.
- $\phi_{a_1, \ldots, a_n}(\gamma, z)$ is a pure phase factor, independent of β, defined by

$$\phi_{a_1, \ldots, a_n}(\gamma, z) = e^{-\frac{i}{2}\text{wind}(\text{p}(\gamma))}(-1)^{\#\text{loops}_{a_1,\ldots,a_n}(\gamma \setminus \text{p}(\gamma))}\text{sheet}_{a_1,\ldots,a_n}(\text{p}(\gamma)),$$

  where $\text{p}(\gamma)$ is a simple path (we allow for self-touching, as long as there is no self-crossing) from $a_1 + \frac{\delta}{2}$ to $\pi(z)$ chosen in γ, $\text{wind}(\text{p}(\gamma))$ is the total turning angle of the tangent of $\text{p}(\gamma)$, and $\text{sheet}_{a_1,\ldots,a_n}(\text{p}(\gamma)) \in \{\pm 1\}$ is defined to be $+1$ if the lift of $\text{p}(\gamma)$ to the double cover starting from the fixed lift of $a_1 + \frac{\delta}{2}$ (fixed once forever in Section 3.1.2) ends at z and -1 if it ends at $z^{\cdot}$. $\phi_{a_1,\ldots,a_n}$ is well-defined; see e.g. [ChIz13, CHI15].

Note that $F_{[\Omega_\delta,a_1,\ldots,a_n]}$ is naturally a spinor, i.e. $F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(z^{\cdot}|\beta\right) = -F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(z|\beta\right)$.

The massive fermion satisfies a perturbed notion of discrete holomorphicity, called *massive s-holomorphicity*. First, define the projection operator $\mathrm{Proj}_{e^{i\theta}\mathbb{R}}x := \frac{x+e^{2i\theta}\bar{x}}{2}$ to be the projection of the complex number x to the line $e^{i\theta}\mathbb{R}$.

Proposition 3.2.2. *The discrete massive fermion* $F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(\cdot|\beta\right)$ *is* massive s-holomorphic, *i.e. it satisfies*

$$e^{\mp i\Theta}\mathit{Proj}_{e^{\pm i\Theta}\tau(c)\mathbb{R}}F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(c\mp\frac{\tau(c)^{-2}i\delta}{2}|\beta\right) = F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(c|\beta\right), \tag{3.2.1}$$

between $c\mp\frac{\tau(c)^{-2}i\delta}{2}\in\mathscr{E}\left[\Omega_\delta,a_1,\ldots,a_n\right]$ *and* $c\in\mathscr{C}\left[\Omega_\delta,a_1,\ldots,a_n\right]$ *which is not a lift of* $a_1+\frac{\delta}{2}$*. At (the fixed lift of)* $a_1+\frac{\delta}{2}$*, we have instead*

$$e^{\mp i\Theta}\mathit{Proj}_{e^{\pm i\Theta}i\mathbb{R}}F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(a_1+\frac{\delta\pm\delta i}{2}|\beta\right) = \mp i. \tag{3.2.2}$$

Proof. The proof of massive s-holomorphicity, essentially identical to the massless case [CHI15, Subsection 3.1], uses the bijection between $\Gamma_{\Omega_\delta}(a_1+\frac{\delta}{2}, c\mp\frac{\tau(c)^{-2}i\delta}{2})$ and $\Gamma_{\Omega_\delta}(a_1+\frac{\delta}{2}, c)$ by the $\oplus$ (symmetric difference) operation. Without loss of generality, notice that $\gamma\mapsto\gamma\oplus t$ with the fixed path $t := \left\{\left(c-\frac{\tau(c)^{-2}i\delta}{2}, c-\frac{\tau(c)^{-2}\delta}{2}\right)\left(c-\frac{\tau(c)^{-2}\delta}{2}, c\right)\right\}$ maps $\Gamma_{\Omega_\delta}(a_1+\frac{\delta}{2}, c-\frac{\tau(c)^{-2}i\delta}{2})$ to $\Gamma_{\Omega_\delta}(a_1+\frac{\delta}{2}, c)$ bijectively (see Figure 3.1.1).

We now only need to show that the summand in the definition of $F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(c\mp\frac{\tau(c)^{2}i\delta}{2}|\beta\right)$ for a given γ transforms to the summand in $F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(c|\beta\right)$ for $\gamma\oplus t$. Given $\mathrm{p}(\gamma)\subset\gamma$, we may take $\mathrm{p}(\gamma\oplus t) := \mathrm{p}(\gamma)\oplus t$. Clearly $(-1)^{\#\mathrm{loops}}$, sheet remain the same for γ and $\gamma\oplus t$, while

$$\mathrm{wind}\left(\mathrm{p}(\gamma\oplus t)\right) = \begin{cases}\mathrm{wind}\left(\mathrm{p}(\gamma)\right)+\frac{\pi}{4} & \text{if } \gamma\cap t\neq\emptyset\\ \mathrm{wind}\left(\mathrm{p}(\gamma)\right)-\frac{3\pi}{4} & \text{if } \gamma\cap t=\emptyset\end{cases}\in\tau(c)\mathbb{R}.$$

In the case where $\gamma\cap t\neq\emptyset$, $\left|\gamma\oplus t\right| = \left|\gamma\right|$ and

$$\begin{aligned}e^{-i\Theta}\mathrm{Proj}_{e^{i\Theta}\tau(c)\mathbb{R}}e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma))} &= e^{-i\Theta}\mathrm{Proj}_{e^{i\Theta}\tau(c)\mathbb{R}}\left[e^{i\Theta}e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma\oplus t))}\right]e^{-\frac{\pi i}{8}-i\Theta}\\ &= e^{-i\Theta}\left[e^{i\Theta}e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma\oplus t))}\right]\mathrm{Re}e^{-\frac{\pi i}{8}-i\Theta}\\ &= e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma\oplus t))}\cos\left(\frac{\pi}{8}+\Theta\right),\end{aligned}$$

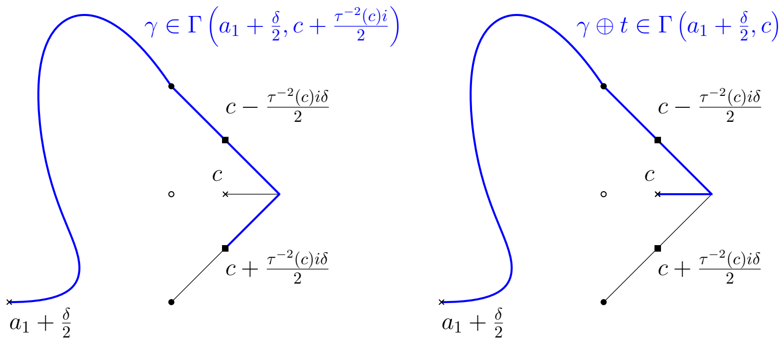

Figure 3.2.1 – The proof of massive s-holomorphicity by bijection.

while if $\gamma \cap t = \emptyset$, $|\gamma \oplus t| = |\gamma| + 1$ and similarly

$$
\begin{aligned}
e^{-i\Theta}\mathrm{Proj}_{e^{i\Theta}\tau(c)\mathbb{R}} e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma))} &= e^{-i\Theta}\left[e^{i\Theta}e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma\oplus t))}\right]\mathrm{Re}\, e^{\frac{3\pi i}{8}-i\Theta} \\
&= e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma\oplus t))}\sin\left(\frac{\pi}{8}+\Theta\right) \\
&= e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma\oplus t))}e^{2\beta}\cos\left(\frac{\pi}{8}+\Theta\right),
\end{aligned}
$$

and the result follows.

At $a_1 + \frac{\delta}{2}$, (say) $\gamma \in \Gamma_{\Omega_\delta}(a_1 + \frac{\delta}{2}, a_1 + \frac{\delta}{2} + \frac{i\delta}{2})$ is mapped bijectively to $\gamma \oplus t \in \Gamma_{\Omega_\delta}$. It is easy to see that

$$
\mathrm{wind}\left(\mathrm{p}(\gamma)\right) = \begin{cases} -\frac{5\pi}{4} \pm 2\pi & \text{if } t \subset \gamma, \\ -\frac{\pi}{4} \pm 2\pi & \text{if } \gamma \cap t = \emptyset \end{cases},
$$

and $(-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\gamma\setminus\mathrm{p}(\gamma))}\mathrm{sheet}_{a_1,\dots,a_n}(\mathrm{p}(\gamma)) = (-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\gamma\oplus t)}$. Now a simple computation similar to above yields the result. □

We will take (3.2.1) as the definition of *(M,β,Θ)-massive s-holomorphicity* (or just *s-holomorphicity* when the mass is clear) *at c* (or *between* $c \mp \frac{\tau(c)^{-2}i\delta}{2}$). We will see later that massive s-holomorphicity corresponds to the continuous notion of perturbed analyticity $\partial_z f = m\bar{f}$.

The fermion defined above is a deterministic function without explicit connection to the Ising model–we now record how it encodes probabilistic information.

Proposition 3.2.3. *For $v = -1, 0, 1$, we have*

$$
F_{[\Omega_\delta, a_1,\dots,a_n]}\left(a_1 + \delta + \frac{e^{iv\frac{\pi}{2}}}{2}\delta \middle| \beta\right) = e^{-iv\frac{\pi}{4}}\frac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1+\delta+e^{iv\frac{\pi}{2}}\delta}\sigma_{a_2}\cdots\sigma_{a_n}\right]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\cdots\sigma_{a_n}\right]}, \tag{3.2.3}
$$

where $a_1 + \delta + \frac{e^{iv\frac{\pi}{2}}}{2}\delta$ is lifted to the same sheet as the fixed lift of $a_1 + \frac{\delta}{2}$, and thus for any $\beta_1 < \beta_2$

there exist constants $C_1(\beta_1,\beta_2), C_2(\beta_1,\beta_2) > 0$*, uniform in* $\beta \in [\beta_1,\beta_2], \Omega, \delta$*, such that*

$$C_1 < \left| F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1+\delta+\frac{e^{i\nu\frac{\pi}{2}}}{2}\delta|\beta\right)\right| < C_2.$$

In the case $n=2$ *we have also*

$$\left|F_{[\Omega_\delta,a_1,a_2]}\left(a_2+\frac{\delta}{2}|\beta\right)\right| = \frac{\mathbb{E}^{\beta^*,free}_{\Omega^*_\delta}\left[\sigma_{a_1+\delta}\sigma_{a_2+\delta}\right]}{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\right]}. \tag{3.2.4}$$

Proof. (3.2.3) and (3.2.4) were proved for the massless case in [CHI15, Lemma 2.6] and the proof is easily seen to not depend on a specific value of β.

We carry out the $\nu=1$ case here: the path $t := \left\{(a_1+\frac{\delta}{2}, a_1+\delta), (a_1+\delta, a_1+\delta+\frac{i\delta}{2})\right\}$ is admissible (i.e. without a crossing) as $\mathrm{p}(\gamma)$ with $\mathrm{wind}(\mathrm{p}(\gamma)) = \frac{\pi}{2}, \mathrm{sheet}_{a_1,\dots,a_n}(\mathrm{p}(\gamma)) = 1$ in $\gamma \in \Gamma_{\Omega_\delta}(a_1+\frac{\delta}{2}, a_1+\delta+\frac{i\delta}{2})$ if and only if γ does not contain a loop separating a_1 from $a_1+\delta+i\delta$ and $(-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\gamma\setminus t)} = (-1)^{\#\mathrm{loops}_{a_1+\delta+i\delta,\dots,a_n}(\gamma\setminus t)}$. If γ does contain such a loop L, we need to choose $\mathrm{p}(\gamma) = t\cup L$ with $\mathrm{wind}(\mathrm{p}(\gamma)) = \frac{\pi}{2}\pm 2\pi$. In this case, $\mathrm{sheet}_{a_1,\dots,a_n}(\mathrm{p}(\gamma)) = 1$ if and only if L encloses an even number of a_i. This means that

$$\begin{aligned}(-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\gamma\setminus\mathrm{p}(\gamma))}\mathrm{sheet}_{a_1,\dots,a_n}(\mathrm{p}(\gamma)) &= (-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\gamma\setminus t)}\\ &= -(-1)^{\#\mathrm{loops}_{a_1+\delta+i\delta,\dots,a_n}(\gamma\setminus t)},\end{aligned}$$

so in all cases the summand in the definition of $F_{[\Omega_\delta,a_1,\dots,a_n]}$ is $\lambda^{-1}(-1)^{\#\mathrm{loops}_{a_1+\delta+i\delta,\dots,a_n}(\gamma\setminus t)}$ and the result follows. The uniform bound comes from the finite energy property of the Ising model (such expectations are uniformly bounded from 0 and ∞ in any finite distribution). □

A crucial feature of (discrete) massive s-holomorphicity, shared with its continuous counterpart, is that the line integral $\mathrm{Re}\int\left[F_{[\Omega_\delta,a_1,\dots,a_n]}(z|\beta)\right]^2 dz$ can be defined path-independently.

Proposition 3.2.4. *There is a single valued function* $H_{[\Omega_\delta,a_1,\dots,a_n]}(x|\beta)$ *up to a global constant on* $\mathcal{V}\mathcal{F}[\Omega_\delta]$ *constructed by*

$$H^\circ_{[\Omega_\delta,a_1,\dots,a_n]}(f|\beta) - H^\bullet_{[\Omega_\delta,a_1,\dots,a_n]}(v|\beta) := 2\delta\left|F_{[\Omega_\delta,a_1,\dots,a_n]}\left(\frac{v+f}{2}|\beta\right)\right|^2, \tag{3.2.5}$$

where $f\in\mathcal{F}[\Omega_\delta]\cup\partial\mathcal{F}[\Omega_\delta]$ *and* $v\in\mathcal{V}[\Omega_\delta]$ *are* δ *away from each other so that* $\frac{v+f}{2}$ *is the corner between them. Put* $\left|F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1+\frac{\delta}{2}|\beta\right)\right|^2 = 1$.

At the boundary faces H° *is constant, so we may put*

$$H^\circ_{[\Omega_\delta,a_1,\dots,a_n]}(f|\beta) := 0 \text{ if } f\in\partial\mathcal{F}[\Omega_\delta], \tag{3.2.6}$$

and further define $H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(v|\beta) := 0$ *if* $v \in \partial\mathcal{V}[\Omega_\delta]$*, then across a boundary edge* $e = (a_{int}a)$ *with* $a_{int} \in \mathcal{V}[\Omega_\delta]$*,* $a \in \partial\mathcal{V}[\Omega_\delta]$*, we have*

$$\begin{aligned}\partial^{\delta}_{\nu_{out}} H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(e|\beta) &:= -H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(a_{int}|\beta)\\ &= 2\cos^2\left(\frac{\pi}{8}+\Theta\right)\delta\left|F_{[\Omega_\delta,a_1,\dots,a_n]}(e|\beta)\right|^2 \\ &\geq 0.\end{aligned} \tag{3.2.7}$$

Proof. (3.2.5) gives rise to a single valued function because doing a loop around any edge will give zero: if $e = (v_1 v_2)$ is an edge which is incident to faces f_1, f_2, $H^{\circ}_{[\Omega_\delta,a_1,\dots,a_n]}(f_1) - H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(v_1) + H^{\circ}_{[\Omega_\delta,a_1,\dots,a_n]}(f_2) - H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(v_2)$ according to (3.2.5) is simply equal to $2\delta\left|F_{[\Omega_\delta,a_1,\dots,a_n]}\left(\frac{f_1+v_1}{2}\right)\right|^2 + 2\delta\left|F_{[\Omega_\delta,a_1,\dots,a_n]}\left(\frac{f_2+v_2}{2}\right)\right|^2 = 2\delta\left|F_{[\Omega_\delta,a_1,\dots,a_n]}(e)\right|^2$, since the two corner values at $\frac{f_{1,2}+v_{1,2}}{2}$ are simply projections of $F_{[\Omega_\delta,a_1,\dots,a_n]}(e)$ onto orthogonal lines by s-holomorphicity. $H^{\circ}_{[\Omega_\delta,a_1,\dots,a_n]}(f_2) - H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(v_1) + H^{\circ}_{[\Omega_\delta,a_1,\dots,a_n]}(f_1) - H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(v_2)$ is also equal to $2\delta\left|F_{[\Omega_\delta,a_1,\dots,a_n]}(e)\right|^2$, and increments along the loop $f_1 \sim v_1 \sim f_2 \sim v_2 \sim f_1$ sum to zero.

Then the boundary behaviour can be easily verified noting that $F_{[\Omega_\delta,a_1,\dots,a_n]}(e|\beta) \in \nu_{\text{out}}(e)^{-1/2}\mathbb{R}$ if e is a boundary edge. See [CHI15, Proposition 3.6] for a massless counterpart. □

Remark 3.2.5. Writing corner values in terms of edge projections, we obtain that $H_{[\Omega_\delta,a_1,\dots,a_n]}$ is a discrete version of the integral $\operatorname{Re}\int\left[F_{[\Omega_\delta,a_1,\dots,a_n]}(z|\beta)\right]^2 dz$ in that

$$H^{\circ}_{[\Omega_\delta,a_1,\dots,a_n]}(f_1|\beta) - H^{\circ}_{[\Omega_\delta,a_1,\dots,a_n]}(f_2|\beta) = \sqrt{2}\sin\left(\frac{\pi}{4}+2\Theta\right)\operatorname{Re}\left[F_{[\Omega_\delta,a_1,\dots,a_n]}(z|\beta)^2(f_1-f_2)\right], \tag{3.2.8}$$

for all f_1, f_2 of distance $\sqrt{2}\delta$ from each other, and

$$H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(v_1|\beta) - H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(v_2|\beta) = \sqrt{2}\sin\left(\frac{\pi}{4}-2\Theta\right)\operatorname{Re}\left[F_{[\Omega_\delta,a_1,\dots,a_n]}(z|\beta)^2(v_1-v_2)\right], \tag{3.2.9}$$

for v_1, v_2 of distance $\sqrt{2}\delta$ from each other.

The functions $H^{\circ,\bullet}$ constructed in the previous proposition satisfy a discrete version of a second-order partial differential equation. We first recall the standard discrete operators $\partial^{\delta}_{\bar{z}}, \Delta^{\delta}$; as alluded to in the notation section, we make a small modification to the conventional definition for the laplacian in $\mathcal{V}[\Omega_\delta]$. We make this boundary modification specifically for $\mathcal{V}[\Omega_\delta]$, which lets us define $H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]} = 0$ on boundary vertices; see [ChSm12] for a motivation.

Definition 3.2.6. Suppose A is a function defined on $\mathcal{E}[\Omega_\delta]$ and B on $\mathcal{V}[\Omega_\delta]$ (or $\mathcal{F}[\Omega_\delta]$, or any locally isomorphic graph). We define the discrete Wirtinger derivative and laplacian by

$$\partial_{\bar{z}}^{\delta} A(x) := \sum_{m=0}^{3} i^m e^{i\pi/4} A\left(x + i^m e^{i\pi/4} \frac{\delta}{\sqrt{2}}\right) \text{ if, e.g., } x \in \mathcal{V}\mathcal{F}[\Omega_\delta],$$
$$\Delta^{\delta} B(x) := \sum_{m=0}^{3} c_m \left[B\left(x + i^m e^{i\pi/4}\sqrt{2}\delta\right) - B(x)\right] \text{ if, e.g., } x \in \mathcal{V}[\Omega_\delta], \tag{3.2.10}$$

where $c_m = 1$ if $x + i^m e^{i\pi/4}\sqrt{2}\delta \in \mathcal{V}[\Omega_\delta]$, and $c_m = \frac{\sin(\frac{\pi}{4}-2\Theta)}{\cos^2(\frac{\pi}{8}+\Theta)}$ if $x + i^m e^{i\pi/4}\sqrt{2}\delta \in \partial\mathcal{V}[\Omega_\delta]$. For any other lattice, $c_m \equiv 1$. If $\Delta^\delta B(x) = M_H^2 B(x)$ for $M_H^2 = \frac{8\sin^2 2\Theta}{\cos 4\Theta}$, we call B *(Θ,M_H^2)-massive harmonic* at x.

The spinor $F_{[\Omega_\delta,a_1,\dots,a_n]}$ is Θ-massive harmonic [BeDC12, DGP14, HKZ15], at least away from monodromy; see Proposition 3.5.1, and also the proof of Proposition 3.2.11 for the behaviour near monodromy.

Note that in the scaling limit $\delta \downarrow 0$, $\frac{1}{2\sqrt{2}\delta}\partial_{\bar{z}}^{\delta} \to \partial_{\bar{z}} := \frac{1}{2}\left(\partial_x + i\partial_y\right)$ and $\frac{1}{(\sqrt{2}\delta)^2}\Delta^\delta \to \Delta$. Therefore, in the following result, the second terms become negligible as $\delta \to 0$ compared to the first, at least if F is in some sense regular; for an analogue in the scaling limit, see (3.3.3).

Proposition 3.2.7. *For $x \neq a_1 + \delta, a_2, \dots a_n$, we have*

$$\Delta^\delta H^{\circ}_{[\Omega_\delta,a_1,\dots,a_n]}(x) = 2\sin\left(\frac{\pi}{4} + 2\Theta\right)\delta\cdot$$
$$\left[A_\Theta \sum_{n=0}^{3} \left|F\left(x + i^n e^{i\pi/4}\frac{\delta}{\sqrt{2}}\right)\right|^2 + B_\Theta \left|\partial_{\bar{z}}\bar{F}\right|^2(x)\right] \tag{3.2.11}$$
$$\Delta^\delta H^{\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(x) = 2\sin\left(\frac{\pi}{4} - 2\Theta\right)\delta\cdot$$
$$\left[-A_{-\Theta} \sum_{n=0}^{3} \left|F\left(x + i^n e^{i\pi/4}\frac{\delta}{\sqrt{2}}\right)\right|^2 - B_{-\Theta} \left|\partial_{\bar{z}}\bar{F}\right|^2(x)\right]$$

where A_Θ, B_Θ are explicit numbers which depend on Θ. A is odd and B is even in Θ, $A_\Theta \sim 4\sqrt{2}\Theta$ and $B_\Theta \sim \frac{1}{2\sqrt{2}}$ as $\Theta \to 0$ (see (3.5.3)).

Proof. We calculate $\partial_{\bar{z}} F^2_{[\Omega_\delta,a_1,\dots,a_n]}$ in Proposition 3.5.1. The laplacians follow straightforwardly by noting that $\Delta^\delta H^{\circ,\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}(x) = \mathrm{Re}\left[2\sin\left(\frac{\pi}{4} \pm \Theta\right)\delta\partial_{\bar{z}} F^2_{[\Omega_\delta,a_1,\dots,a_n]}\right]$, which is true if $x \in \mathcal{V}[\Omega_\delta]$ is adjacent to $\partial\mathcal{V}[\Omega_\delta]$ as well (the boundary conductances are defined precisely to preserve this relation). See also Remark 3.2.8. □

Remark 3.2.8. Note that Proposition 3.2.7 applies for $x = a_1$ as well, since, thanks to the singularity (3.2.2), projections of $F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1 + e^{i\frac{\pi}{4}} \cdot \frac{\delta}{\sqrt{2}}\right), F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1 + e^{i\frac{3\pi}{2}} \cdot e^{i\frac{\pi}{4}} \cdot \frac{\delta}{\sqrt{2}}\right)$ (on the sheet which is cut along $\mathbb{R}_{\geq 0}$ and has the fixed lift of $a_1 + \frac{\delta}{2}$ on the upper side of $\mathbb{R}_{\geq 0}$) onto the line $i\mathbb{R}$ are equal to $-i$; on this sheet, $F_{[\Omega_\delta,a_1,\dots,a_n]}$ does show s-holomorphicity between 4 edges surrounding a_1. The singularity effectively transfers the monodromy at face a_1 to the vertex $a_1 + \delta$.

3.2.2 Full-Plane Fermions

We now define the fermion $F_{[\Omega_\delta,a_1,\dots,a_n]}$ for $\Omega = \mathbb{C}$ by taking increasingly bigger balls $B_R = B_R(0)$, allowing us to extract informations about the Ising measure on the corresponding discretised domains.

Lemma 3.2.9. *Fix $\delta > 0$ and $\Theta < 0$. There exists a constant $C = C(\delta,\Theta)$ such that $\left|F_{[(B_R)_\delta,a_1,\dots,a_n]}(c)\right| \leq C$ for any $c \in \mathscr{C}\left(\left[(B_R)_\delta, a_1,\dots,a_n\right]\right)$.*

Proof. Discrete Green's formula implies that

$$\sum_{v\in\mathcal{V}[(B_R)_\delta]} \Delta^\delta H^\bullet_{[(B_R)_\delta,a_1,\dots,a_n]}(v) = \frac{\sin\left(\frac{\pi}{4}-2\Theta\right)}{\cos^2\left(\frac{\pi}{8}+\Theta\right)} \sum_{e\in\partial\mathscr{E}[(B_R)_\delta]} \partial^\delta_{\nu_{\text{out}}} H^\bullet_{[(B_R)_\delta,a_1,\dots,a_n]}(e), \tag{3.2.12}$$

and we can use (3.2.11) on vertices other than $a_1+\delta$ for the laplacian and (3.2.7) for the boundary outer derivatives, so leaving just the laplacian at $(a_1+\delta)$ on the left hand side

$$\begin{aligned}
&\Delta^\delta H^\bullet_{[(B_R)_\delta,a_1,\dots,a_n]}(a_1+\delta) = \\
\sum_{\substack{v\in\mathcal{V}[(B_R)_\delta]\\ v\neq a_1+\delta}} &2\sin\left(\frac{\pi}{4}-\Theta\right)\delta\cdot\left[A_{-\Theta}\sum_{n=0}^{3}\left|F\left(v+i^n e^{i\pi/4}\frac{\delta}{\sqrt{2}}\right)\right|^2 + B_{-\Theta}\left|\partial_{\bar z}\bar F\right|^2(v)\right] \\
&+\sum_{e\in\partial\mathscr{E}[(B_R)_\delta]} 2\delta\sin\left(\frac{\pi}{4}-2\Theta\right)\left|F_{[(B_R)_\delta,a_1,\dots,a_n]}(e)\right|^2 \\
&\geq 2\sin\left(\frac{\pi}{4}-\Theta\right)A_{-\Theta}\delta \sum_{\substack{c\in\mathscr{C}[(B_R)_\delta]\\ c\neq a_1+\delta+\frac{i^m\delta}{2}}} |F(c)|^2
\end{aligned} \tag{3.2.13}$$

Note that $A_{-\Theta}, B_{-\Theta} > 0$, so we have the desired bound if $\Delta^\delta H^\bullet_{[(B_R)_\delta,a_1,\dots,a_n]}(a_1+\delta)$ is bounded. But by s-holomorphicity, clearly

$$\Delta^\delta H^\bullet_{[(B_R)_\delta,a_1,\dots,a_n]}(a_1+\delta) \leq 16\sqrt{2}\max_{0\leq m\leq 3}\delta\left|F_{[(B_R)_\delta,a_1,\dots,a_n]}\left(a_1+\delta+\frac{e^{i\nu\frac{\pi}{2}}}{2}\delta\right)\right|^2, \tag{3.2.14}$$

but the fermion value on the right hand side is bounded by a constant only depending on β by Proposition 3.2.3. □

Given uniform boundedness, we can use diagonalisation to get a subsequential limit $F_{[\mathbb{C}_\delta,a_1,\dots,a_n]}$ on the whole of $[\mathbb{C}_\delta, a_1,\dots,a_n]$. It suffices to show that such a limit must be unique.

Proposition 3.2.10. *Any subsequential limit $F_{[\mathbb{C}_\delta,a_1,\dots,a_n]}$ of $F_{[(B_R)_\delta,a_1,\dots,a_n]}$ as $R\to\infty$*

1. *shows massive s-holomorphicity and the singularity at $a_1+\frac{\delta}{2}$ as in Proposition 3.2.2,*

2. *at infinity:* $F_{[\mathbb{C}_\delta,a_1,\dots,a_n]} \to 0$ *uniformly and its square integral* $H_{[\mathbb{C}_\delta,a_1,\dots,a_n]}$ *tends to a finite constant,*

3. *has finite discrete 'L^2 norm':*

$$\sum_{\substack{v \in \mathcal{V}[\mathbb{C}_\delta] \\ v \neq a_1+\delta}} 2\sin\left(\frac{\pi}{4}-\Theta\right)\delta \cdot \left[A_{-\Theta}\sum_{n=0}^{3}\left|F\left(v+i^n e^{i\pi/4}\frac{\delta}{\sqrt{2}}\right)\right|^2\right] \tag{3.2.15}$$
$$\leq \sum_{\substack{v \in \mathcal{V}[\mathbb{C}_\delta] \\ v \neq a_1+\delta}} \left|\Delta^\delta H^\bullet_{[\mathbb{C}_\delta,a_1,\dots,a_n]}(v)\right| < cst\cdot\delta < \infty,$$

with constant independent of δ.

Moreover, there is only one such function $F_{[\mathbb{C}_\delta,a_1,\dots,a_n]}$.

Proof. The first entry is immediate from the corresponding properties of $F_{[(B_R)_\delta,a_1,\dots,a_n]}$.

The inequality (3.2.15) can be deduced from a uniform bound for the analogue for $\left|\Delta^\delta H^\bullet_{[(B_R)_\delta,a_1,\dots,a_n]}\right|$. It is bounded independently of R by (3.2.12) and (3.2.14), so monotone convergence gives the desired inequality for the limit as $R\to\infty$.

The infinity behaviour is then immediate from the fact that the sum of $\left|F_{[\mathbb{C}_\delta,a_1,\dots,a_n]}\right|^2$ along any line in $\mathbb{C}\setminus B_R$ vanishes uniformly as $R\to\infty$ by (3.2.15).

If there are two such $F_{[\mathbb{C}_\delta,a_1,\dots,a_n]}$, their difference $\hat{F}_{[\mathbb{C}_\delta,a_1,\dots,a_n]}$ is everywhere s-holomorphic, and since the sum of $\left|\hat{F}_{[\mathbb{C}_\delta,a_1,\dots,a_n]}\right|^2$ along any line in $\mathbb{C}\setminus B_R$ is finite and decays to zero, the square integral $\hat{H}_{[\mathbb{C}_\delta,a_1,\dots,a_n]}$ is finite and constant at infinity. $\hat{H}^\bullet_{[\Omega_\delta,a_1,\dots,a_n]}$ is everywhere superharmonic and is finite at infinity, so $H^{\dagger\bullet}_{[\Omega_\delta,a_1,\dots,a_n]}$ is constant and $F^\dagger_{[\Omega_\delta,a_1,\dots,a_n]}\equiv 0$. □

Fix $\Theta<0$. On the full plane, we have an explicit characterisation of the one point spinor in terms of the *massive harmonic measure* of the slit plane $\mathrm{hm}^0_{(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{<0}}$: $\mathrm{hm}^0_{(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{<0}}(z|\Theta)$ for $z\in(1+i)\mathbb{Z}^2$ is the probability of a simple random walk started at z extinguished at each step with probability $\left(1+\frac{2\sin^2 2\Theta}{\cos(4\Theta)}\right)^{-1}\frac{2\sin^2 2\Theta}{\cos(4\Theta)}$ to successfully reach 0 before hitting $(1+i)\mathbb{Z}^2\cap\mathbb{R}_{<0}$. $\mathrm{hm}^0_{(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{<0}}(\cdot|\Theta)$ is the unique Θ-massive harmonic function (in the sense of (3.5.2)) on $(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{\leq 0}$ which has the boundary values 1 at 0 and 0 on $(1+i)\mathbb{Z}^2\cap\mathbb{R}_{<0}$ and infinity.

Proposition 3.2.11. *Denote the slit planes* $\mathbb{X}^+ := \left\{z\in[\mathbb{C},0] : Re\sqrt{z}>0\right\}\cong\mathbb{C}\setminus\mathbb{R}_{>0}$ *and* $\mathbb{Y}^+ := \left\{z\in[\mathbb{C},0] : Im\sqrt{z}>0\right\}\cong\mathbb{C}\setminus\mathbb{R}_{>0}$. *Then*

$$F_{[\mathbb{C}_\delta,0]}(c\delta|\Theta) = \begin{cases} hm^0_{(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{<0}}\left(c-\frac{3}{2}|\Theta\right) & c\delta\in\mathbb{X}^+\cap\mathscr{C}^1[\mathbb{C}_\delta,0] \\ -i\,hm^0_{(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{<0}}\left(\frac{1}{2}-c|\Theta\right) & c\delta\in\mathbb{Y}^+\cap\mathscr{C}^i[\mathbb{C}_\delta,0] \end{cases}.$$

Proof. By Proposition 3.5.1, $F_{[\mathbb{C}_\delta,0]}(c\delta)$ restricted to $\mathscr{C}^1[\mathbb{C}_\delta,0]$ or $\mathscr{C}^i[\mathbb{C}_\delta,0]$ is Θ-massive harmonic, except possibly at the lifts of $\pm\frac{\delta}{2}$ (because there is no planar neighbourhood G_δ around these points in $[\mathbb{C}_\delta,0]$, see Figure 3.5.1) and $\frac{3\delta}{2}$ (because $F_{[\mathbb{C}_\delta,0]}(c\delta)$ is not s-holomorphic at the lifts of $\frac{\delta}{2}$).

From Definition 3.2.1, it is clear that $F_{[\mathbb{C}_\delta\cap B_R,0]}(c\delta)=0$ for any $c\in\mathscr{C}^1[\mathbb{C}_\delta,0]$ on the (lift of) negative real line and $c\in\mathscr{C}^i[\mathbb{C}_\delta,0]$ on the (lift of) positive real line, since the complex phase $\phi_{a_1,\ldots,a_n}(\gamma,c)$ for any $\gamma\in\Gamma_{\mathbb{C}_\delta\cap B_R}(\frac{\delta}{2},\pi(c))$ switches sign for the reflection across the real line $\gamma_r:=\{\bar{e}:e\in\gamma\}$, and $\Gamma_{\mathbb{C}_\delta\cap B_R}(\frac{\delta}{2},\pi(c))$ is invariant under the reflection. We conclude that $F_{[\mathbb{C}_\delta,0]}(c\delta)=0$ for $c\in\mathscr{C}^1[\mathbb{C}_\delta,0]$.

Also note that $F_{[\mathbb{C}_\delta,0]}(\frac{3\delta}{2})=1$, where $\frac{3\delta}{2}$ is evaluated at the lift which is on the same sheet as the fixed lift of $\frac{\delta}{2}$, since the corresponding Ising quantity in (3.2.3) is precisely the ratio of two adjacent magnetisations (spin 1-point functions); in the infinite volume limit with $\beta>\beta_c$, the ratio tends to 1.

All in all, $F_{[\mathbb{C}_\delta,0]}(\cdot|\Theta)$ restricted to $\mathbb{X}^+\cap\mathscr{C}^1[\mathbb{C}_\delta,0]$ is the massive harmonic function which takes the boundary values 1 on the lift of $\frac{3\delta}{2}$ in $\mathbb{X}^+\cap\mathscr{C}^1[\mathbb{C}_\delta,0]$ and 0 on the lift of the negative real line and infinity, whence the identification with $\mathrm{hm}^0_{(1+i)\mathbb{Z}^2\setminus\mathbb{R}_{<0}}$. The same argument applies for its restriction to $\mathbb{Y}^+\cap\mathscr{C}^i[\mathbb{C}_\delta,0]$, noting the boundary value at $\frac{\delta}{2}$ given by (3.2.2). □

3.2.3 Massive Complex Analysis: Continuous Fermions and their Uniqueness

We now present the boundary value problem which the limit of our discrete fermions solves. Since massive models by definition do not possess conformal invariance, we cannot expect the limit to be holomorphic as in the critical case; instead, it satisfies a particular perturbed notion of holomorphicity $\partial_{\bar{z}}f=m\bar{f}$, which belongs to various similar families of functions dubbed *generalised analytic* or *pseudoanalytic* in writings of, e.g., L. Bers and I. N. Vekua. Here we use a small excerpt of the established theory, saying that f is holomorphic up to a continuous factor. Although any nonzero smooth function is trivially pseudoanalytic (with the non-constant complex coefficient $\partial_{\bar{z}}/\bar{f}$), fixing the governing equation $\partial_{\bar{z}}f=m\bar{f}$ allows us to use known functions which satisfy the same equation to successively cancel out singularities; in other words, we have a generalised version of Laurent series. Imposing a real constant m in addition yields the possibility to define the line integral $\mathrm{Re}\int f^2dz$.

In analogy with the discrete terminology, we refer to our particular continuous condition *(m)-massive holomorphicity.* We will assume $m<0$ is fixed throughout this section.

Lemma 3.2.12 (Similarity Principle)**.** *Let D be a bounded domain in $\mathbb{C}$. If a continuously differentiable function $f:D\to\mathbb{C}$ is m-massive meromorphic, i.e. satisfies $\partial_{\bar{z}}f:=\frac{1}{2}\frac{\partial}{\partial x}f+\frac{i}{2}\frac{\partial}{\partial y}f=m\bar{f}$ except on a finite set, there exists a Hölder continuous function s on $\bar{U}$ such that $e^{-s}f$ is holomorphic in U.*

In particular, f only vanishes at isolated points.

Proof. See [Ber56, Section 10] or [Vek62, Section III.4]; we also explain the underlying idea and explicitly carry out specific cases in Section 3.4.1, e.g. the proof of Lemma 3.4.4. □

Corollary 3.2.13. *There exists a family of m-massive holomorphic functions Z_n^1, Z_n^i for each $n \in \mathbb{Z}$ such that as $z \to 0$*

$$Z_n^1(z) \sim z^n, Z_n^i(z) \sim iz^n,$$

and any function f that is m-massive holomorphic in a punctured neighbourhood of a can be expressed near a as formal power series in Z_n, i.e.

$$f(z) = \sum_n \left[A_n^1 Z_n^1(z-a) + A_n^i Z_n^i(z-a) \right] \tag{3.2.16}$$

for real numbers $A_n^{1,i}$. If $f(z)$ is a spinor defined on a double cover ramified at a, f admits formal power series in analogous functions indexed by half-integers $Z_{n+\frac{1}{2}}$. These expansions are uniformly convergent in a small disc or annulus around a, respectively if f has a singularity at a or not.

Proof. This corollary depends on the fact that s in Lemma 3.2.12 is continuous, so that if a massive holomorphic function is $o((z-a)^n)$ at a then it is automatically $O((z-a)^{n+1})$; see [Ber56, Section 5]. □

Remark 3.2.14. There is no canonical choice of the 'local formal powers' $Z_n^{1,i}$. We will need explicit functions to expand continuous fermions around their singularities and analyse them further to derive Painlevé III in Section 3.4; we hereby fix the following radially symmetric functions for half-integers ν:

$$Z_\nu^1(re^{i\theta}) := \frac{\Gamma(\nu+1)}{|m|^\nu} \left[e^{i\nu\theta} I_\nu(2|m|r) + (\mathrm{sgn} m) \cdot e^{-i(\nu+1)\phi} I_{\nu+1}(2|m|r) \right], \tag{3.2.17}$$
$$Z_\nu^i(re^{i\theta}) := \frac{\Gamma(\nu+1)}{|m|^\nu} \left[ie^{i\nu\theta} I_\nu(2|m|r) - i(\mathrm{sgn} m) \cdot e^{-i(\nu+1)\phi} I_{\nu+1}(2|m|r) \right],$$

where I_n is the modified Bessel function of the first kind. One can easily verify the desired asymptotics and massive holomorphicity from the corresponding facts about I_ν, namely that $I_\nu(r) \overset{r\to 0}{\sim} \Gamma(\nu+1)\left(\frac{r}{2}\right)^\nu$ and $I_\nu'(r) = I_{\nu\pm1}(r) \pm \frac{\nu}{r} I_\nu(r)$ [DLMF, Chapter 10] (See Section 3.4 for useful formulae, and note also that $\partial_{\bar{z}} = \frac{1}{2}e^{i\theta}\left(\partial_r + ir^{-1}\partial_\theta\right)$).

As a special case, we have $Z^1_{-\frac{1}{2}}(z) = \frac{e^{2m|z|}}{\sqrt{z}}$.

Proposition 3.2.15. *The following boundary value problem on a bounded smooth simply connected domain Ω has at most one solution:*

$f : [\Omega, a_1, \ldots, a_n] \to \mathbb{C}$ satisfies

1. *f is continuously differentiable, square integrable, and $\partial f = m\bar{f}$ in $[\Omega, a_1, \ldots, a_n]$,*
2. *$f(z) \in \sqrt{\nu_{out}^{-1}(z)}\mathbb{R}$ on the lift of $\partial\Omega$, where $\nu_{out}(z)$ is the outer normal at $\pi(z) \in \partial\Omega$,*

3. $(z-a_1)^{1/2} f(z) \to 1$ *as* $z \to a_1$, *and*

4. $\exists \mathscr{B}_j \in \mathbb{R}$ *such that* $(z-a_j)^{1/2} f(z) \to i\mathscr{B}_j$ *as* $z \to a_j$ *for* $j = 2, \dots, n$.

Proof. If there are any two such functions f_1, f_2, applying Green-Riemann's formula to their difference $\hat{f}$ on $\Omega_r = \Omega \setminus \cup_j B_r(a_j)$ for small $r > 0$ yields

$$\oint_{\partial\Omega_r} \hat{f}^2 dz = 2i \iint_{\Omega_r} \partial_{\bar{z}} \hat{f}^2 = 2i \iint_{\Omega_r} 2m|\hat{f}|^2 dz \in i\mathbb{R}. \tag{3.2.18}$$

$\hat{f} \sim cst \cdot (z-a_1)^{1/2}$ near a_1 in view of Corollary 3.2.13 so $\oint_{\partial B_r(a_1)} \hat{f}^2 dz$ tends to zero as $r \to 0$, and $\hat{f} \to i\hat{\mathscr{B}}_j (z-a_j)^{-1/2}$ for some $\hat{\mathscr{B}}_i \in \mathbb{R}$ as z tends to any other a_j. On the inner circles, we have

$$\oint_{\partial B_r(a_j)} \hat{f}^2 dz \xrightarrow{r\to 0} -2\pi i \hat{\mathscr{B}}_j^2 \in i\mathbb{R} \text{ for } j = 2, \dots, n, \tag{3.2.19}$$

whereas the boundary condition on the lift of $\partial\Omega$ readily gives $\frac{1}{i}\oint_{\partial\Omega} \hat{f}^2 dz = \oint_{\partial\Omega} |\hat{f}|^2 ds \geq 0$. Therefore

$$0 \leq \frac{1}{i}\oint_{\partial\Omega} \hat{f}^2 dz = 2\iint_{\Omega_r} 2m|\hat{f}|^2 dz - \sum_j 2\pi \hat{\mathscr{B}_j}^2 \leq 0,$$

so $\hat{f} \equiv 0$. □

Remark 3.2.16. As in the proof of the Proposition 3.2.15, it is easy to show that $h := \mathrm{Re} \int f^2 dz$ is globally well-defined in $\Omega \setminus \{a_1, \dots, a_n\}$. In terms of h, the boundary condition $f(z) \in \sqrt{\nu_{out}^{-1}}\mathbb{R}$ and the asymptotics around a_i is equivalent to

1. h is constant on $\partial\Omega$ and there is no $x_0 \in \partial\Omega$ such that $h > h(\partial\Omega)$ in a neighbourhood of x_0, and

2. h is bounded below near $a_2, \dots, a_n$ and $h^\dagger(w) := \mathrm{Re} \int^w \left(f(z) - Z^1_{-\frac{1}{2}}(z-a_1)\right)^2 dz$ is single valued and bounded near a_1.

We note that this boundary problem can easily be extended to the case where $\Omega = \mathbb{C}$, by requiring that h be finite and constant at infinity (we will in fact require that f decays exponentially fast, see also Lemma 3.3.2).

We now define quantities which reflect the geometry of Ω exploiting the boundary value problem above, which will turn out to be directly related to the Ising correlations through the connection which is precisely our main convergence result in Section 3.3.2. Determination of these quantities through isomonodromic deformation is the main subject of Section 3.4.

Definition 3.2.17. Given a solution $f_{[\Omega, a_1, \dots, a_n]}$ of the boundary value problem presented in Proposition 3.2.15 (the *continuous massive fermion*), define $\mathscr{A}_\Omega^{1,i} = \mathscr{A}_\Omega^{1,i}(a_1, \dots, a_n|m)$ as the real coefficients in the expansion

$$f_{[\Omega,a_1,\ldots,a_n]}(z) = Z^1_{-\frac{1}{2}}(z-a_1) + 2\mathscr{A}^1_\Omega Z^1_{\frac{1}{2}}(z-a_1) + 2\mathscr{A}^i_\Omega Z^i_{\frac{1}{2}}(z-a_1) + O\big((z-a)^{3/2}\big). \qquad (3.2.20)$$

In addition, in the case where $n=2$, define $\mathscr{B}_\Omega = \mathscr{B}_\Omega(a_1, a_2|m)$ as the coefficient

$$f_{[\Omega,a_1,a_2]}(z) = \mathscr{B}_\Omega Z^i_{-\frac{1}{2}}(z-a_1) + O\big((z-a)^{1/2}\big). \qquad (3.2.21)$$

For notational convenience, we do not assume $\mathscr{B}_\Omega > 0$ as in [CHI15]; instead, its sign depends on the sheet choice of $Z^i_{-\frac{1}{2}}(z-a_1)$, which we will explicitly fix whenever needed.

3.3 Discrete Analysis: Scaling Limit

In this section, we show the convergence of the discrete fermions introduced in Section 3.2 to their continuous counterparts. Then we show that the family of discrete fermions as $\delta \downarrow 0$ is precompact in Section 3.3.1, whose limit satisfies a unique characterisation as laid out in Proposition 3.2.15. These suffice to show convergence to the desired limit. Throughout this section, we assume a continuous mass $m<0$ is fixed.

3.3.1 Bulk Convergence

We finally undertake convergence of the discrete fermion $F_{[\Omega_\delta,a_1,\ldots,a_n]}$ to its continuous counterpart $f_{[\Omega,a_1,\ldots,a_n]}$ in scaling limit. First, we need to interpolate the discrete function defined on $[\Omega_\delta, a_1, \ldots, a_n]$ on the continuous domain $[\Omega, a_1, \ldots, a_n]$. While any reasonable interpolation (e.g. linear interpolation used in many papers dealing with massless case) should converge to the unique continuous limit, we will assume an interpolation scheme with a continuously differentiable $F_{[\Omega_\delta,a_1,\ldots,a_n]}$ as an easy way to show that the limit itself is continuous differentiable. While we do not explicitly carry it out, we could show the limit is smooth by using arbitrarily more regular interpolation scheme.

Proposition 3.3.1. *Suppose Ω is bounded, simply connected with smooth boundary or $\mathbb{C}$. For any compact subset $K \subset \Omega$, any infinite collection $\left(\frac{2}{\pi}\delta_k\right)^{-1/2} F_{\left[\Omega_{\delta_k},a_1,\ldots,a_n\right]} =: f_{\left[\Omega_{\delta_k},a_1,\ldots,a_n\right]}$ with $\delta_k \downarrow 0$ has a subsequence that (suitably interpolated as above) converges in $C^1(K)$-topology to a continuously differentiable limit.*

Proof. By Arzelà-Ascoli, it suffices to show that the discrete derivatives

$$\delta^{-1}\partial^\delta_x f_{[\Omega_\delta,a_1,\ldots,a_n]}(z) := \delta^{-1}\left[f_{[\Omega_\delta,a_1,\ldots,a_n]}(z+2\delta) - f_{[\Omega_\delta,a_1,\ldots,a_n]}(z)\right];$$
$$\delta^{-1}\partial^\delta_y f_{[\Omega_\delta,a_1,\ldots,a_n]}(z) := \delta^{-1}\left[f_{[\Omega_\delta,a_1,\ldots,a_n]}(z+2\delta i) - f_{[\Omega_\delta,a_1,\ldots,a_n]}(z)\right],$$

are equicontinuous on K. In view of Proposition 3.5.4, it suffices to show that the 'discrete L^2

norm' $\sum_{c\in\mathscr{C}(K_\delta)} \left|f_{[\Omega_\delta,a_1,\ldots,a_n]}(c)\right|^2\delta^2$ is bounded by a universal number.

Apply the discrete Green's formula as in (3.2.12) to $H_{[\Omega_\delta,a_1,\ldots,a_n]}$, then we get the following analogue to (3.2.15) (dividing both sides by δ),

$$\begin{aligned} cst &\geq \delta^{-1}\Delta^\delta H^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]}(a_1+\delta) && (3.3.1)\\ &\geq 2\sin\left(\frac{\pi}{4}-\Theta\right)\sum_{\substack{c\in\mathscr{C}[\Omega_\delta]\\ c\neq a_1+\delta+\frac{i^m\delta}{2}}}\delta A_{-\Theta}\left|f_{[\Omega_\delta,a_1,\ldots,a_n]}(c)\right|^2\\ &+\sum_{e\in\partial\mathscr{E}[\Omega_\delta]}2\delta\sin\left(\frac{\pi}{4}-2\Theta\right)\left|f_{[\Omega_\delta,a_1,\ldots,a_n]}(e)\right|^2,\\ &\geq 2\sin\left(\frac{\pi}{4}-\Theta\right)\sum_{\substack{c\in\mathscr{C}[\Omega_\delta]\\ c\neq a_1+\delta+\frac{i^m\delta}{2}}}\delta A_{-\Theta}\left|f_{[\Omega_\delta,a_1,\ldots,a_n]}(c)\right|^2 && (3.3.2)\end{aligned}$$

and since $A_{-\Theta}\sim -2\sqrt{2}m\delta$, we have the desired L^2 bound from the sum of $\delta A_{-\Theta}\left|f_{[\Omega_\delta,a_1,\ldots,a_n]}(c)\right|^2$. □

With a sequence K_m of increasing compact subsets such that $\bigcup_m K_m=\Omega\setminus\{a_1,\ldots,a_n\}$ and using diagonalisation, we can find a global subsequential limit $f_{[\Omega,a_1,\ldots,a_n]}$ with uniform convergence in compact subsets of Ω. We finish the proof of convergence by showing that a limit must satisfy the boundary value problem of Proposition 3.2.15, and thus is unique. We note that continuous differentiability, square integrability and the condition $\partial_{\bar z} f_{[\Omega,a_1,\ldots,a_n]}=m\overline{f_{[\Omega,a_1,\ldots,a_n]}}$ follows straightforwardly from Proposition 3.5.1 and Remark 3.5.2, so we are left to verify the boundary conditions, as laid out in Remark 3.2.16.

We first treat the following explicit case:

Lemma 3.3.2. *The one point spinor* $f_{[\mathbb{C}_\delta,0]}$ *converges to* $Z^1_{-\frac{1}{2}}(z=re^{i\theta})=\frac{e^{2mr}}{\sqrt{z}}$ *uniformly in compact subsets of* $[\mathbb{C}_\delta,0]$.

Proof. We show that a subsequential limit $f_{[\mathbb{C},0]}$ satisfies the unique characterisation of Remark 3.2.16, and thus has to be equal to $Z^1_{-\frac{1}{2}}$, an explicit solution. We first show that $f_{[\mathbb{C},0]}$ vanishes at infinity sufficiently fast to yield that $h_{[\mathbb{C},0]}:=\operatorname{Re}\int f^2_{[\mathbb{C},0]}dz$ is constant at infinity. But this follows from the identification of the discrete spinor with the massive harmonic measure (the hitting probability of an extinguished random walk) of the tip of a slit in Proposition 3.2.11 and the exponential estimates of Proposition 3.5.4.

It now remains to identify the singularity at 0 as $f_{[\mathbb{C},0]}\sim z^{-1/2}$. Note that the massless harmonic measure [Kes87, LaLi04] and thus the spinor [CHI15, Lemma 2.14] (for the identification $\vartheta(\delta)=\left(\frac{2}{\pi}\delta\right)^{1/2}$, see [GHP19, Lemma 5.14]) show that exact behaviour. Since the hitting probability of a massive random walk is dominated by the hitting probability of a simple random walk,

$f_{[\mathbb{C},0]} \cdot z^{1/2} \xrightarrow{z\to 0} \alpha$ for some $\alpha \in [0,1]$. But the probability that the massive random walk is extinguished can be made arbitrarily close to 0 as length scale becomes negligible compared to $\frac{1}{|m|}$, so the two harmonic measures become asymptotically equal as $z \to 0$ and we conclude $\alpha = 1$. □

By translation invariance of its definition, $f_{[\mathbb{C}_\delta,a_1]}(z)$ converges to $Z^1_{-\frac{1}{2}}(z-a_1)$ uniformly in compact subsets of $[\mathbb{C},a_1]$.

Proposition 3.3.3. *A subsequential limit $f_{[\Omega,a_1,\ldots,a_n]}$ of $f_{[\Omega_\delta,a_1,\ldots,a_n]}$ satisfies the conditions of Remark 3.2.16, and thus is unique.*

Proof. We first suppose $\Omega \neq \mathbb{C}$. Consider the renormalised discrete square integral $h_{[\Omega_\delta,a_1,\ldots,a_n]} := \left(\frac{2}{\pi}\delta\right)^{-1} H_{[\Omega_\delta,a_1,\ldots,a_n]}$. By Propositions 3.2.7 and 3.2.10, $h^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]} = \delta^{-1} H^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]}$ is discrete superharmonic on $\mathcal{V}[\Omega_\delta] \setminus \{a_1+\delta\}$ and takes the boundary value 0 on $\partial\mathcal{V}[\Omega_\delta]$. Moreover, by Proposition 3.3.1, $\Delta^\delta h^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]}(a_1+\delta) \leq cst =: c_0$. Thus, by superharmonicity, $h^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]}$ is lower bounded by $c_0\mathcal{G}_\delta$, where $\mathcal{G}_\delta$ is the domain Green's function $\mathcal{G}_\delta := \mathcal{G}_{\mathcal{V}[\Omega_\delta]}(\cdot, a_1+\delta)$ with Dirichlet boundary condition on $\partial\mathcal{V}[\Omega_\delta]$.

Since $\mathcal{G}_\delta$ is $O(\delta)$ on any vertex $a_{\text{int}} \in \mathcal{V}[\Omega_\delta]$ adjacent to $a \in \mathcal{V}[\Omega_\delta]$ (Lemma 3.5.3), we have that $O(\delta) = c_0\mathcal{G}_\delta(a_{\text{int}}) \leq h^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]}(a_{\text{int}}) \leq 0$ and $h^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]}(a_{\text{int}}) = -\left|f_{[\Omega_\delta,a_1,\ldots,a_n]}\left(\frac{a_{\text{int}}+a}{2}\right)\right|^2\delta = O(\delta)$ by (3.2.7). We see that therefore $f_{[\Omega_\delta,a_1,\ldots,a_n]}$ is uniformly bounded on boundary edges and corners (see Figure 3.1.1). Since $|f|^2$ is subharmonic by (3.5.5), $f_{[\Omega_\delta,a_1,\ldots,a_n]}$ is uniformly bounded on $\Omega_r := \Omega \setminus \bigcup_j B_r(a_j)$ for small fixed $r > 0$. Therefore by equicontinuity $h_{[\Omega_\delta,a_1,\ldots,a_n]} \to h_{[\Omega,a_1,\ldots,a_n]}$ uniformly on $\overline{\Omega_r}$, and $h_{[\Omega,a_1,\ldots,a_n]}$ is continuous up to the boundary. Note that since $\partial_{\bar z} f_{[\Omega,a_1,\ldots,a_n]} = m\overline{f_{[\Omega,a_1,\ldots,a_n]}}$, $h_{[\Omega,a_1,\ldots,a_n]} = \operatorname{Re}\int f^2_{[\Omega,a_1,\ldots,a_n]}dz$ satisfies

$$\Delta h_{[\Omega,a_1,\ldots,a_n]} = 4\partial_{\bar z}\partial_z h_{[\Omega,a_1,\ldots,a_n]} = 2\partial_{\bar z} f^2_{[\Omega,a_1,\ldots,a_n]} = 4m\left|f_{[\Omega,a_1,\ldots,a_n]}\right|^2. \tag{3.3.3}$$

We now verify the two remaining conditions of the boundary value problem.

1. $h_{[\Omega,a_1,\ldots,a_n]}$ in Ω_r is a continuous solution of a Poisson equation with bounded data $4m\left|f_{[\Omega,a_1,\ldots,a_n]}\right|^2$ and smooth boundary data; $h_{[\Omega,a_1,\ldots,a_n]}$ is continuously differentiable up to the boundary thanks to standard Green's function estimates (e.g. [GiTr15, Theorem 4.3]). From [ChSm12, Remark 6.3], which only uses the superharmonicity (in their convention, subharmonicity) of $h^\bullet_{[\Omega_\delta,a_1,\ldots,a_n]}$ and does not depend on a particular property of $h^\circ_{[\Omega_\delta,a_1,\ldots,a_n]}$, we see that there is no neighbourhood of $x_0 \in \partial\Omega$ where $h > 0$.

2. $h_{[\Omega_\delta,a_1,\ldots,a_n]}$ is bounded below by $c\mathcal{G}_\delta$, which is bounded from negative infinity away from a_1 in the scaling limit. So near $a_2,\ldots,a_n$, $h_{[\Omega,a_1,\ldots,a_n]}$ is bounded below. For the asymptotic near a_1, note that $f^\dagger_{[\Omega_\delta,a_1,\ldots,a_n]} := f_{[\Omega_\delta,a_1,\ldots,a_n]} - f_{[\mathbb{C}_\delta,a_1]}$ is s-holomorphic near a_1 with $f^\dagger_{[\Omega_\delta,a_1,\ldots,a_n]}(a_1+\frac{\delta}{2}) = 0$, so by Proposition 3.5.1 it is everywhere massive harmonic (unlike in the proof of Proposition 3.2.11, the zero prevents a singularity near

monodromy; see also [GHP19, Remark 2.6]). But both $f_{[\Omega_\delta,a_1,\ldots,a_n]}, f_{[\mathbb{C}_\delta,a_1]}$ are uniformly bounded, say, in a discrete circle $S_r := B_{r+5\delta}(a_1)\setminus B_r(a_1)$ for small $r>0$. As above, we conclude that $f^\dagger_{[\Omega_\delta,a_1,\ldots,a_n]}$ is uniformly bounded in $B_r(a_1)$. Given Lemma 3.3.2, this suffices to show boundedness and well-definedness for the continuous limits $f^\dagger, h^\dagger$.

If $\Omega=\mathbb{C}$, we first show that $f_{[\Omega_\delta,a_1,\ldots,a_n]}\to 0$ at infinity uniformly in δ and $h_{[\Omega_\delta,a_1,\ldots,a_n]}$ is constant and finite at infinity using Proposition 3.5.4 and the uniform L^2 bound (3.3.1). Since $f_{[\Omega_\delta,a_1,\ldots,a_n]}\to 0$ at infinity uniformly in δ, we may compare with the Green's function of a suitably large ball to obtain lower boundedness around $a_2,\ldots,a_n$. The above proof applies near a_1. □

3.3.2 Analysis near the Singularity

To show convergence of $F_{[\Omega_\delta,a_1,\ldots,a_n]}\left(a_1+\frac{3\delta}{2}\right)$, we model the expansion of $f_{[\Omega,a_1,\ldots,a_n]}$ around a_1 from Definition 3.2.17 in the discrete setting, and carefully analyse the magnitude of the difference. The candidate for discrete $Z^1_{-\frac{1}{2}}$ is clear: $F_{[\mathbb{C}_\delta,a_1]}$ is able to cancel out the singularity of $F_{[\Omega_\delta,a_1,\ldots,a_n]}$, and converges to $Z^1_{-\frac{1}{2}}$ by Lemma 3.3.2.

Then, we need to build a discrete analogue of $Z^1_{\frac{1}{2}}$ (or rather, some massive s-holomorphic function that has square root behaviour around 0). Following [CHI15, (3.12)], we define a discrete function $G_{[\mathbb{C}_\delta,a_1]}$ by discrete integrating the $F_{[\mathbb{C}_\delta,a_1]}$.

Proposition 3.3.4. *Construct the spinor $G_{[\mathbb{C}_\delta,a_1]}:\mathscr{C}^1[\Omega_\delta,a_1]\to\mathbb{R}$ by*

$$G_{[\mathbb{C}_\delta,a_1]}(z) := \delta\sum_{j=0}^{\infty}\Gamma^j F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta), \tag{3.3.4}$$

where $\Gamma(\delta|\Theta):=\tan^2\left(\frac{\pi}{4}+2\Theta\right)$, and $z-2j\delta$ is taken on the same sheet as z if $\pi(z)\notin a_1+\mathbb{R}_{>0}$ or if $\pi(z),\pi\left(z-2j\delta\right)\in a_1+\mathbb{R}_{>0}$, while $F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta)=0$ naturally as soon as $\pi\left(z-2j\delta\right)\in a_1+\mathbb{R}_{<0}$ (cf. Proposition 3.2.11).

It is massive harmonic with coefficient M_H^2 on $\mathbb{X}^+\cap\mathscr{C}^1[\mathbb{C}_\delta,0]$, and $\left(\frac{2}{\pi}\delta\right)^{-1/2}G_{[\mathbb{C}_\delta,a_1]}$ converges uniformly in compact subsets of $[\mathbb{C},a_1]$ to $g_{[\mathbb{C},a_1]}$ which has the asymptotic behaviour $g_{[\mathbb{C},a_1]}(z)\sim \mathrm{Re}\sqrt{z-a_1}$ near a_1.

Proof. We first show that the discrete integrand $\Gamma^j F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta)$ is summable. Without loss of generality, work on the sheet where $F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta)>0$. By Proposition 3.2.11, $F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta)$ is the probability of a massive random walk started at the real corner $z-2j\delta$ and extinguished at each step with probability $\left(1+\frac{2\sin^2 2\Theta}{\cos(4\Theta)}\right)^{-1}\frac{2\sin^2 2\Theta}{\cos(4\Theta)}$ surviving to hit $a_1+\frac{3\delta}{2}$ before $a_1+\mathbb{R}_{<0}$; denote by h_j the probability of the same event with $\Theta=0$, i.e. the massless harmonic measure. Project the two-dimensional walk into two independent one-dimensional walks respectively in the x,y-directions with step lengths δ. If p_{2j} is the probability of the one-dimensional massive random walk started at $x-2j\delta$ surviving to hit x, we have the upper bound

$$\Gamma^j F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta) \le \Gamma^j p_{2j} h_j.$$

However, p_{2j} is the massive harmonic measure of 0 as seen from $-2j$ in $\mathbb{Z}_{\le 0}$. From the boundary conditions $p_0 = 1$, $p_{-\infty} = 0$ and massive harmonicity in 1D $p_{j-2} + p_j - 2p_{j-1} = \frac{4\sin^2 2\Theta}{\cos 4\Theta} p_{j-1}$, it is straightforward to verify $p_j = \Gamma^{-j/2}$, and we have $\Gamma^j F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta) \le h_j$. By [CHI15, Lemma 3.4], $h_j = O(j^{-3/2})$ uniformly for z in a compact subset, and thus the sum is finite.

We know that $\left(\frac{2}{\pi}\delta\right)^{-1/2} F_{[\mathbb{C}_\delta,a_1]}(z')$ converges to $\mathrm{Re}\frac{e^{2mr'}}{\sqrt{z'}}$ on $\mathscr{C}^1[\Omega_\delta, a_1]$ uniformly in compact subsets away from a_1. Also note that in the scaling limit $x-2j\delta = x'$, Γ^j converges to $e^{-2m(x-x')}$. By above, if $z = x+iy$ is in a compact subset, the discrete integrand $\Gamma^{\frac{x-x'}{2\delta}} F_{[\mathbb{C}_\delta,a_1]}(z')$ decays to zero as $x' \to -\infty$ uniformly in z, so we conclude that $g_{[\mathbb{C}_\delta,a_1]} := \left(\frac{2}{\pi}\delta\right)^{-1/2} G_{[\mathbb{C}_\delta,a_1]}$ converges

$$g_{[\mathbb{C}_\delta,a_1]}\left(z = x+iy\right) \xrightarrow{\delta\to 0} \frac{1}{2}\int_{-\infty}^{x} e^{2m(x'-x)}\mathrm{Re}\frac{e^{2mr'}}{\sqrt{x'+iy'}}dx', \tag{3.3.5}$$

uniformly for z in a compact subset. As $z \to 0$, we can uniformly bound $e^{2m(x'-x)}e^{2mr'}$ close to 1, which gives the asymptotic of $g_{[\mathbb{C},a_1]}$ near z.

Massive harmonicity is clear unless $\pi(z) \in \mathbb{R}_{>0}$. If $\pi(z) \in a_1 + \mathbb{R}_{>0}$, $\left(\Delta^\delta - M_H^2\right) F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta) = 0$ if $\pi(z-2j\delta) \in a_1 + \frac{3\delta}{2} + \mathbb{R}_{\ge 0}$ while $F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta) = 0$ if $\pi(z-2j\delta) \in a_1 + \mathbb{R}_{<0}$, thus

$$\begin{aligned}\left(\Delta^\delta - M_H^2\right) G_{[\mathbb{C}_\delta,a_1]}(z) &= \delta \sum_{j=\lfloor(z-a_1)/2\delta\rfloor}^{\infty} \Gamma^j \left(\Delta^\delta - M_H^2\right) F_{[\mathbb{C}_\delta,a_1]}(z-2j\delta)\\ &= \delta\Gamma^{\lfloor(z-a_1)/2\delta\rfloor}\sum_{j=0}^{\infty}\Gamma^j\left(\Delta^\delta - M_H^2\right) F_{[\mathbb{C}_\delta,a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta),\end{aligned}$$

where the laplacian is taken on the planar slit domain $\mathbb{X}^+ \cap \mathscr{C}^1[\mathbb{C}_\delta, 0]$ with zero boundary values on the slit.

We need to show that the last sum vanishes.

$$\begin{aligned}&\sum_{j=0}^{N}\Gamma^j\left(\Delta^\delta - M_H^2\right) F_{[\mathbb{C}_\delta,a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta)\\ &= \sum_{j=0}^{N}\Gamma^j\sum_{s=\pm1}\left[F_{[\mathbb{C}_\delta,a_1]}(a_1 + \frac{3\delta}{2} - (2j+s)\delta + i\delta) - \tan\left(\frac{\pi}{4} - s2\Theta\right)F_{[\mathbb{C}_\delta,a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta)\right]\\ &+ \sum_{j=0}^{N}\Gamma^j\sum_{s=\pm1}\left[F_{[\mathbb{C}_\delta,a_1]}(a_1 + \frac{3\delta}{2} - (2j+s)\delta - i\delta) - \tan\left(\frac{\pi}{4} + s2\Theta\right)F_{[\mathbb{C}_\delta,a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta)\right],\end{aligned}$$

where two sums are done respectively above and below the slit. By massive discrete holomorphicity, we can convert the differences of real corner values into differences of imaginary corner values: we need to be careful, since the points in $\mathbb{X}^+$ directly above and below the cut are on opposite sheets. We can in fact think of $a_1 + \frac{\delta}{2}$ as lying on the slit, since the singularity (3.2.2) ascribes two values at $a_1 + \frac{\delta}{2}$, coming from above and below $a_1 + \mathbb{R}$. Above the slit, (3.5.1) implies

$$\begin{aligned}
&\sum_{s=\pm 1} \left[F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2j+s)\delta + i\delta) - \tan\left(\frac{\pi}{4} - s2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right] \\
&= -i \left[F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta + i\delta) - \tan\left(\frac{\pi}{4} + 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2j-1)\delta) \right] \\
&\quad + i \left[F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta + i\delta) - \tan\left(\frac{\pi}{4} - 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2j+1)\delta) \right] \\
&= i \tan\left(\frac{\pi}{4} + 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2j-1)\delta) - i \tan\left(\frac{\pi}{4} - 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2j+1)\delta).
\end{aligned}$$

Thanks to the factor of Γ^j, the sum telescopes (see Figure 3.5.1)

$$\begin{aligned}
&\sum_{j=0}^{N} \Gamma^j \sum_{s=\pm 1} \left[F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2j+s)\delta + i\delta) - \tan\left(\frac{\pi}{4} - s2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right] \\
&= i \tan\left(\frac{\pi}{4} + 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{5\delta}{2}) - i\Gamma^N \tan\left(\frac{\pi}{4} - 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2N+1)\delta).
\end{aligned}$$

The sum below the slit analogously gives

$$\begin{aligned}
&\sum_{j=0}^{N} \Gamma^j \sum_{s=\pm 1} \left[F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2j+s)\delta - i\delta) - \tan\left(\frac{\pi}{4} + s2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right] \\
&= -i \tan\left(\frac{\pi}{4} + 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{5\delta}{2}) + i\Gamma^N \tan\left(\frac{\pi}{4} - 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2N+1)\delta),
\end{aligned}$$

where $a_1 + \frac{3\delta}{2} - (2N+1)\delta$ is approached from below the sheet. Summing the two and taking $a_1 + \frac{3\delta}{2} - (2N+1)\delta$ from above the slit,

$$\sum_{j=0}^{N} \Gamma^j \left(\Delta^\delta - M_H^2\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - 2j\delta) = -2i\Gamma^N \tan\left(\frac{\pi}{4} - 2\Theta\right) F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2N+1)\delta).$$

Now, by Proposition 3.2.11, $-i \cdot F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2N+1)\delta)$ is the massive harmonic measure of the point $a_1 + \frac{\delta}{2}$ in the discrete plane $\mathbb{Y}^+ \cap \mathscr{C}^i[\mathbb{C}_\delta, 0]$ slit by $a_1 + \mathbb{R}_{>0}$. $-i \cdot \Gamma^N F_{[\mathbb{C}_\delta, a_1]}(a_1 + \frac{3\delta}{2} - (2N+1)\delta) = O(N^{-1/2})$ since we may bound it by the massless harmonic measure as above, and the sum decays to zero, as desired. □

Theorem 3.3.5. *As $\delta \to 0$, for $a_1, \dots, a_n$ at a definite distance from the boundary and each other, we have uniformly*

$$F_{[\Omega_\delta, a_1, \dots, a_n]}\left(a_1 + \frac{3\delta}{2}\right) = 1 + 2\delta \mathscr{A}^1_{\Omega_\delta}(a_1, \dots, a_n) + o(\delta), \tag{3.3.6}$$

$$\left| F_{[\Omega_\delta, a_1, a_2]}\left(a_2 + \frac{\delta}{2}\right)\right| = |\mathscr{B}_\Omega(a_1, a_2)| + o(1). \tag{3.3.7}$$

Proof. We adapt the strategy of [CHI15, Subsection 3.5]; the massive harmonic nature of our functions are hardly visible since at short length scale massive hitting probabilities approach simple random walk hitting probabilities. While we try to use the same notation for corresponding notions where appropriate, the functions we work with are all massive harmonic.

Note that, due to explicit constructions in Propositions 3.2.11 and 3.3.4, (3.3.6) can be written as

$$F_{[\Omega_\delta, a_1, \dots, a_n]}\left(a_1 + \frac{3\delta}{2}\right) = F_{[\mathbb{C}_\delta, a_1]}\left(a_1 + \frac{3\delta}{2}\right) + 2\mathscr{A}^1_{\Omega_\delta}(a_1, \dots, a_n)\, G_{[\mathbb{C}_\delta, a_1]}\left(a_1 + \frac{3\delta}{2}\right) + o(\delta).$$

Define the reflection $\mathscr{R}(\Omega)$ of Ω across the line $a_1 + \mathbb{R}$. Then there is a ball $B(a_1)$ around a_1 which belongs to $\Omega \cap \mathscr{R}(\Omega)$. Denote Λ to be the lift of the slit neighbourhood $B(a_1) \setminus [a_1 + \mathbb{R}_{<0}]$ such that both $F_{[\Omega_\delta, a_1, \dots, a_n]}, F_{[\mathscr{R}(\Omega_\delta), a_1, \dots, a_n]}$ have their fixed lift of the path origin $a_1 + \frac{\delta}{2}$ on Λ. By symmetry arguments about the path set $\Gamma(a_1 + \frac{\delta}{2}, z)$ similar to the proof of Proposition 3.2.11, we have $F_{[\Omega_\delta, a_1, \dots, a_n]}\left(a_1 + \frac{3\delta}{2}\right) = F_{[\mathscr{R}(\Omega_\delta), a_1, \dots, a_n]}\left(a_1 + \frac{3\delta}{2}\right)$, whereas the boundary values $F_{[\Omega_\delta, a_1, \dots, a_n]}(z), F_{[\mathscr{R}(\Omega_\delta), a_1, \dots, a_n]}(z)$ on the slit $\pi(z) \in a_1 + \mathbb{R}_{<0}$ cancel out to give $F_{[\Omega_\delta, a_1, \dots, a_n]}(z) + F_{[\mathscr{R}(\Omega_\delta), a_1, \dots, a_n]}(z) = 0$ for $z \in \mathscr{C}^1[\Omega_\delta, a_1] \cap \partial\Lambda$ (see also [CHI15, Subsection 3.5]). Then, restricted to $\mathscr{C}^1[\Omega_\delta, a_1] \cap \Lambda$, (recall $g_{[\mathbb{C}_\delta, a_1]} := \left(\frac{2}{\pi}\delta\right)^{-1/2} G_{[\mathbb{C}_\delta, a_1]}$)

$$K_\delta := \frac{1}{2}\left[f_{[\Omega_\delta, a_1, \dots, a_n]} + f_{[\mathscr{R}(\Omega_\delta), a_1, \dots, a_n]}\right] - f_{[\mathbb{C}_\delta, a_1]} - 2\mathscr{A}^1_{\Omega_\delta}(a_1, \dots, a_n)\, g_{[\mathbb{C}_\delta, a_1]},$$

is everywhere massive harmonic with zero boundary values on the slit $\mathscr{C}^1[\Omega_\delta, a_1] \cap \partial\Lambda$. We have to show

$$K_\delta(a_1 + \frac{3\delta}{2}) = f_{[\Omega_\delta, a_1, \dots, a_n]}(a_1 + \frac{3\delta}{2}) - 1 - 2\mathscr{A}^1_{\Omega_\delta}(a_1, \dots, a_n)\,\delta = o(\delta^{1/2}).$$

Noting the expansion of Definition 3.2.17, we see that, on Λ, away from a_1,

$$K_\delta \to \frac{1}{2}\left[f_{[\Omega, a_1, \dots, a_n]} + f_{[\mathscr{R}(\Omega), a_1, \dots, a_n]}\right] - Z^1_{-\frac{1}{2}} - 2\mathscr{A}^1_{\Omega_\delta}(a_1, \dots, a_n)\, \mathrm{Re}\sqrt{z - a_1} = o\left((z - a_1)^{1/2}\right),$$

so on the discrete circle $S_r = B_{r+5\delta}(a_1) \setminus B_r(a_1)$ for $r > 0$, $\max_{S_r} |K_\delta| = o(r^{1/2})$ as $r \to 0$. Sharp discrete Beurling estimate (see [CHI15, (3.4)] for the form used here) for harmonic functions may be used to dominate the value of the massive harmonic function S_δ at $a_1 + \frac{3\delta}{2}$, and we have

$$\left|K_\delta(a_1+\frac{3\delta}{2})\right| \leq cst\cdot\delta^{1/2}r^{-1/2}\max_{S_r}|K_\delta| = cst\cdot\delta^{1/2}o(1),$$

where $o(1)$ holds for $r\to 0$. We conclude the right hand side is $o(\delta^{1/2})$ as $\delta\to 0$.

For (3.3.7), without loss of generality assume $\mathscr{B}_\Omega(a_1,a_2)>0$. Define this time $\mathscr{R}(\Omega)$ as the reflection of Ω across $a_2+\mathbb{R}$. As above, write Λ for the lift of the slit disc $B(a_2)\setminus[a_2+\mathbb{R}_{<0}]$ such that we simultaneously define $F_{[\Omega_\delta,a_1,a_2]}(a_2+\frac{\delta}{2}) = F_{[\mathscr{R}(\Omega_\delta),a_1,a_2]}(a_2+\frac{\delta}{2}) =: i\mathscr{B}_\delta$ with $\mathscr{B}_\delta>0$. Again, by symmetry, $f_{[\Omega_\delta,a_1,a_2]}+f_{[\mathscr{R}(\Omega_\delta),a_1,a_2]}$ is zero on the boundary $\partial\Lambda\cap\mathscr{C}^i[\Omega_\delta,a_1,a_2]$, i.e. the lift of $a_2+\mathbb{R}_{<0}$. Define the massive harmonic measure of the point $a_1+\frac{\delta}{2}$ in the lattice $\Lambda\cap\mathscr{C}^i[\Omega_\delta,a_1,a_2]$ as W_δ. Then define

$$T_\delta := \frac{1}{2}\left[F_{[\Omega_\delta,a_1,\dots,a_n]}+F_{[\mathscr{R}(\Omega_\delta),a_1,\dots,a_n]}\right]-i\mathscr{B}_\delta W_\delta,$$

which is massive harmonic on $\Lambda\cap\mathscr{C}^i[\Omega_\delta,a_1,a_2]\setminus\left\{a_2+\frac{\delta}{2}\right\}$, takes the value 0 on $\partial\Lambda\cap\mathscr{C}^i[\Omega_\delta,a_1,a_2]$ and $\left\{a_2+\frac{\delta}{2}\right\}$. Since $\frac{1}{2}\left[f_{[\Omega_\delta,a_1,\dots,a_n]}+f_{[\mathscr{R}(\Omega_\delta),a_1,\dots,a_n]}\right]$ restricted to the imaginary corners converges to a continuous limit with asymptotic $i\cdot\mathrm{Re}\frac{\mathscr{B}_{\Omega_\delta}(a_1,a_2)}{\sqrt{z-a_2}}+o((z-a_2)^{-1/2})$ on the discrete circle $S_r=B_{r+5\delta}(a_2)\setminus B_r(a_2)$, it is easy to see that unless $\mathscr{B}_\delta\to\mathscr{B}_\Omega(a_1,a_2)$ and $T_\delta\left(a_2+\frac{\delta}{2}\right)=o(1)$, we can find a point in the bulk which is greater than any value on the boundary S_r, contradicting the maximum principle; see also [CHI15, (3.23)]. □

3.4 Continuum Analysis: Isomonodromy and Painlevé III

3.4.1 Continuous Analysis of the Coefficients

In this section, we carry out analysis of the continuum coefficients such as $\mathscr{A}_\Omega,\mathscr{B}_\Omega$ needed for the proof of the main theorem and the derivation of the Painlevé III transcendent in the next section. We will assume that a continuous mass parameter $m<0$ is fixed throughout this section.

Preliminaries: Massive Cauchy Formula.

In Section 3.2.3, we have seen that massive holomorphic functions admit a generalisation of Laurent-type expansions. Here we give explicit formulae related to the formal powers $Z_\nu^{1,i}$ and note that a Cauchy-type integral formula holds.

Recall $\partial_z=\frac{1}{2}e^{-i\theta}\left(\partial_r-ir^{-1}\partial_\theta\right),\partial_{\bar z}=\frac{1}{2}e^{i\theta}\left(\partial_r+ir^{-1}\partial_\theta\right)$. The following holds ([DLMF, Section

10]):

$$
\begin{aligned}
I_\nu'(r) &= I_{\nu\pm 1}(r) \pm \frac{\nu}{r} I_\nu(r),\\
-2\frac{\sin \nu\pi}{\pi r} &= I_\nu(r) I_{-\nu-1}(r) - I_{\nu+1}(r) I_{-\nu}(r).
\end{aligned} \tag{3.4.1}
$$

Define $W_\nu(re^{i\theta}) := e^{i\nu\theta} I_\nu(2|m|r)$. The formal powers $Z_\nu^{1,i}$ defined in can be written as

$$
\begin{aligned}
Z_\nu^1 &= \frac{\Gamma(\nu+1)}{|m|^\nu}\left(W_\nu + (\operatorname{sgn} m)\overline{W_{\nu+1}}\right), \\
Z_\nu^i &= \frac{\Gamma(\nu+1)}{|m|^\nu}\left(iW_\nu - i(\operatorname{sgn} m)\overline{W_{\nu+1}}\right).
\end{aligned} \tag{3.4.2}
$$

Proposition 3.4.1 ([Ber56, Section 6])**.** *Let f is an m-massive holomorphic function defined on a ramified disk $[B_R(a), a]$. The coefficients $A_\nu^{1,i}$ of $Z_\nu^{1,i}$ in the expansion (3.2.17) may be extracted by the line integrals:*

$$
\begin{aligned}
A_\nu^1 &= \frac{\pi}{4|m|\nu^2} Re \oint_C f(z) Z_{-1-\nu}^i(z-a)\,dz, \\
A_\nu^i &= -\frac{\pi}{4|m|(1+\nu)^2} Re \oint_C f(z) Z_{-1-\nu}^1(z-a)\,dz,
\end{aligned} \tag{3.4.3}
$$

where the line integral is taken on any smooth curve C going once around a.

Proof. As in the discrete case, if f, g are m-massive holomorphic functions, the real part $\operatorname{Re}\int f \cdot g\,dz$ is well defined: indeed, the increment around a closed curve ∂D inclosing a region D is $\int_{\partial D} f \cdot g\,dz = 2i\int \partial_{\bar z}(fg)\,d^2z = 2i\int m\operatorname{Re}(fg)\,d^2z \in i\mathbb{R}$ by the Green-Riemann formula. So the integral (3.4.3) has a well-defined value regardless of choice of C.

We will take $C = \partial B_r(a)$ for some small $r > 0$. In view of the definition (3.4.2), it is straightforward to verify first that any line integral of the form $\operatorname{Re}\int_C Z_\nu^1 Z_{\nu'}^1\,dz, \operatorname{Re}\int_C Z_\nu^i Z_{\nu'}^i\,dz$ vanish. Similarly one can verify the mixed integral

$$
\operatorname{Re}\int_C Z_\nu^1 Z_{\nu'}^i\,dz = \delta_{\nu+\nu',-1}\frac{-4|m|\nu^2}{\pi},
$$

with the help of (3.4.1) and the standard Gamma function equality [DLMF, Section 5.5]

$$
\Gamma(z)\Gamma(-1-z) = \frac{\Gamma(z)\Gamma(1-z)}{z(1+z)} = \frac{1}{z(1+z)}\frac{\pi}{\sin z\pi}.
$$

□

Finally, we note the derivatives of the formal powers.

$\partial_r W_\nu(re^{i\theta}) = e^{i\nu\theta}\cdot 2|m|\cdot\left(I_{\nu\pm1}(2|m|r) \pm \frac{\nu}{2|m|r} I_\nu(2|m|r)\right)$ and $\partial_\theta W_\nu(re^{i\theta}) = i\nu e^{i\nu\theta} I_\nu(2|m|r)$, and we see that

$$\partial_z W_\nu = |m| W_{\nu-1}, \partial_{\bar z} W_\nu = |m| W_{\nu+1},$$

and the corresponding identities for $Z_\nu^{1,i}$ follow. In fact, we will record, noting $\partial_x = \partial_z + \partial_{\bar z}, \partial_y = i(\partial_z - \partial_{\bar z})$,

$$\partial_x Z_\nu^1 = \nu Z_{\nu-1}^1 + (\nu+1)^{-1} m^2 Z_{\nu+1}^1, \partial_x Z_\nu^i = \nu Z_{\nu-1}^i + (\nu+1)^{-1} m^2 Z_{\nu+1}^i,$$
$$\partial_y Z_\nu^1 = \nu Z_{\nu-1}^i - (\nu+1)^{-1} m^2 Z_{\nu+1}^i, \partial_y Z_\nu^i = -\nu Z_{\nu-1}^1 + (\nu+1)^{-1} m^2 Z_{\nu+1}^1.$$

Preliminaries: Cauchy Transform and Harmonic Conjugate.

To work with generalised analytic functions, the *Cauchy transform* $\mathfrak{C} = \mathfrak{C}_\Omega : t(\cdot) \mapsto -\frac{1}{\pi}\int_\Omega \frac{t(w)}{w-z} d^2 w$ will be used as the inverse of the derivative $\partial_{\bar z}$. Define the Hölder seminorm $[t]_{C^\alpha(\Omega)}(w) := \sup_{z\in\Omega} \frac{|t(w)-t(z)|}{|w-z|^\alpha}$, $[t]_{C^\alpha(\Omega)} := \sup_{w\in\Omega}[t]_{C^\alpha(\Omega)}$ and the norm $|t|_{C^\alpha(\Omega)} := |t|_{C(\Omega)} + [t]_{C^\alpha(\Omega)}$. The following estimates are standard and may be shown by direct analysis of the kernel $\frac{1}{w-z}$.

Proposition 3.4.2. *Let $B = B_r(z)$ be a ball of radius $r > 0$.*

- *If $g \in L^p(B)$ for some $p > 2$, $\mathfrak{C}g \in C^\alpha(\mathbb{C})$ for $\alpha = \frac{p-2}{p}$, holomorphic outside of B, and vanishes at infinity, with*

$$\left|\mathfrak{C}g\right| \le cst \cdot r^\alpha \left|g\right|_{L^p(B)}; \left[\mathfrak{C}g\right]_{C^\alpha(\mathbb{C})} \le cst \cdot \left|g\right|_{L^p(B)}, \tag{3.4.4}$$

  where the constants only depend on p;

- *If $g \in C^\alpha(B)$, $\mathfrak{C}g$ is differentiable at z, with*

$$\left|\nabla \mathfrak{C}g(z)\right| \le cst\left(\left|g(z)\right| + r^\alpha [g]_{C^\alpha(B)}(z)\right); \tag{3.4.5}$$

  where the constant only depends on $\alpha \in (0,1]$.

Proof. See [Vek62, Sections 1.4-8]. Specifically, for (3.4.4) refer to [Vek62, Theorem 1.19]; (3.4.5) follows from [Vek62, (8.2), (8.7)]. □

Another standard analytic fact that we use is the Hölder regularity of harmonic conjugates.

Proposition 3.4.3 (Privalov's Theorem)**.** *Let D be a smooth bounded simply connected domain, and fix a conformal map $\varphi : D \to \mathbb{D}$ such that $M^{-1} < |\varphi'| < M$ for some $M > 0$.*

Let $t \in C^\alpha(\partial D)$ be a real-valued function on the boundary. Then there exists a holomorphic function $g \in C^\alpha(\bar D)$ such that $Img = t$ which is unique up to a real constant. Moreover, if $Reg = 0$ at any point in $\bar D$,

$$\left|g\right|_{C^\alpha(\bar D)} \le cst \cdot (1 + M^{2\alpha}) |t|_{C^\alpha(\partial D)},$$

where the constant only depends on α.

Proof. See, e.g., [GaMa08, Theorem 3.2], for the proof in the unit disc. Then it is straightforward to transfer the result to D using the conformal map φ' given that the Hölder seminorm transforms with a factor of $|\varphi'|^{\alpha}$. □

Analysis of the Continuous Observables.

We now study the continuous observables $f_{[\Omega,a_1,\dots,a_n]}(\cdot|m)$ with $m<0$ and its coefficients. Fix a conformal map $\varphi:\Omega\to\mathbb{D}$ and let $M>0$ be such that $M^{-1}<\varphi'<M$. Note that $\mathrm{diam}\Omega\le 2M$.

We first give the following lemma based on the Lemma 3.2.12 (similarity principle). The following proof in fact contains the idea of the proof of the principle, but strictly speaking we do rely on it, e.g. for the existence of the observable itself. Write $f_m(z):=f_{[\Omega,a_1,\dots,a_n]}(z|m)$.

Lemma 3.4.4. *There exists a unique function s in $\bar\Omega$ such that*

- *s is continuous in $\bar\Omega$;*
- *s is real on $\partial\Omega$ and $Re e^{-s(a)}=1$;*
- *$e^{-s}f_m$ is holomorphic in $[\Omega,a_1,\dots,a_n]$.*

Then, defining $c_j := Re\lim_{z\to a_j} e^{-s(z)}\sqrt{z-a_j}f_m$ (note $c_1=1$)

$$e^{-s(z)}f_m(z)=\sum_{j=1}^{n}c_b f_{[\Omega,a_1,\dots,a_n]}(z|0), \tag{3.4.6}$$

Moreover, for any $\alpha\in(0,1)$,

$$|s|_{C^\alpha}\le cst\cdot|m|(1+M^{1+2\alpha}),$$

where the constant only depends on α.

Proof. First assume existence of such s. Suppose there are two such functions s_1,s_2. Then $e^{-s_1(z)+s_2(z)}$ is bounded, real on $\partial\Omega$, and holomorphic in $\Omega\setminus\{a,b\}$; thus the value at a fixes it to be constantly 1. So $s_1-s_2\equiv 0$ given that $\mathrm{Im}(s_1-s_2)=0$ on $\partial\Omega$ and s is unique.

Then $\hat f(z):=e^{-s(z)}f_m(z)$ is holomorphic in $[\Omega,a_1,\dots,a_n]$ and $\sqrt{\nu_{out}}\hat f\in\mathbb{R}$ on the boundary $\partial\Omega$. Near a_j, $\mathrm{Re}\sqrt{z-a_j}\hat f(z)\sim c_j$. Thus we obtain (3.4.6) since the difference of both sides is zero by the uniqueness of the boundary value problem ([CHI15, Lemma 2.9], i.e. the massless version of Proposition 3.2.15).

Now we show the existence of s. Define $s_0:=\mathfrak{C}\left[\partial_{\bar z}f_m/f_m\right]$. Note that

$$\frac{\partial_{\bar z}f_m}{f_m}=\frac{\partial_{\bar z}f_{[\Omega,a,b]}(z|m)}{f_{[\Omega,a,b]}(z|m)}=m\left(\frac{\overline{f_{[\Omega,a,b]}(z|m)}}{f_{[\Omega,a,b]}(z|m)}\right) \tag{3.4.7}$$

is in $L^p(\Omega)$ for any p since f_m only vanishes at isolated points, so s_0 is bounded by Proposition 3.4.2.

We claim s_0 is differentiable almost everywhere in Ω. s_0 is differentiable at any point z_0 near which $\partial_{\bar{z}} \bar{f}_m / f_m$ satisfies a Hölder condition, since $\mathfrak{C}_\Omega \left[\partial_{\bar{z}} f_m / f_m\right]$ is the sum of $\mathfrak{C}_{\Omega \setminus B_r(z_0)} \left[\partial_{\bar{z}} f_m / f_m\right]$, which is holomorphic at z_0, and $\mathfrak{C}_{B_r(z_0)} \left[\partial_{\bar{z}} f_m / f_m\right]$, which is differentiable by Proposition 3.4.2. Since f_m is smooth on $[\Omega, a_1, \ldots, a_n]$, s_0 is differentiable away from a_j and at isolated points where f_m vanishes.

So $e^{-s_0} f_m$ is holomorphic almost everywhere, and by removable singularity it is holomorphic on $[\Omega, a_1, \ldots, a_n]$. Now let s_1 be a holomorphic function on Ω with boundary data $\mathrm{Im} s_1 = \mathrm{Im} s_0$ on $\partial\Omega$. We fix $\mathrm{Re} e^{s_1 - s_0} = 1$. Then $s := s_0 - s_1$ satisfies the desired properties.

For the Hölder estimate, recall that $\mathrm{diam}\Omega \le 2M$ and thus $\left|\partial_{\bar{z}} f_m / f_m\right|_{L^p(\Omega)} \le cst \cdot |m| M^{\frac{2}{p}}$. Then from Proposition 3.4.2 we have $|s_0|_{C^\alpha} \le cst \cdot (1 + M^\alpha) \cdot |m| M^{1-\alpha}$. Then the corresponding norm for s_1 is given by Proposition 3.4.3, and the sum gives the desired estimate. □

The above lemma allows us to give the following results in the case $n = 2$.

Lemma 3.4.5. *$|\mathscr{B}_\Omega(a_1, a_2|m)| \to 1$ as $\frac{|a_1 - a_2|}{dist(\{a_1, a_2\}, \partial\Omega)} \to 0$.*

Proof. By [CHI15, Remark 2.24], the result holds for $m = 0$.

Now consider the decomposition (3.4.6). Comparing the imaginary parts of the coefficient of $\frac{1}{\sqrt{z - a_2}}$ in both sides, we see that $\mathscr{B}_\Omega(a_1, a_2|m)\, \mathrm{Re}\left[e^{-s(a_2)}\right] = \mathscr{B}_\Omega(a_1, a_2|0)$. Since $\mathrm{Re} e^{-s(a_1)} = 1$ and e^{-s} is Hölder continuous by the previous lemma, we have $\left|\frac{\mathscr{B}_\Omega(a_1, a_2|m)}{\mathscr{B}_\Omega(a_1, a_2|0)}\right| \to 1$ as $|a_1 - a_2| \to 0$. □

Before we go on to give a more delicate estimate on the two-point observable, we note a few facts we use.

Remark 3.4.6. As mentioned above, the possible zeroes of f_m are problematic for the regularity of s. However, in the case where $n = 2$, f_m cannot vanish in Ω.

Indeed, $\hat{f} = e^{-s} f_m$ is a holomorphic function on $[\Omega, a_1, a_2]$ with the boundary condition $\sqrt{\nu_{out}} \hat{f} \in \mathbb{R}$. By the argument principle for solutions of a Hilbert boundary problem ([Wen92, Theorem 2.2]), one sees that $1 = \frac{1}{2} N_{\partial\Omega} + N_\Omega$, where 1 is the index of $\nu_{out} \cdot (z - a_1)(z - a_2)$ on $\partial\Omega$, $N_{\partial\Omega}, N_\Omega$ are number of zeroes of $\hat{f}^2 (z - a_1)(z - a_2)$ respectively on $\partial\Omega, \Omega$ counted with multiplicity. But since it is a square, any zero of $\hat{f}^2 (z - a_1)(z - a_2)$ is second order; the only possible scenario then is that $N_{\partial\Omega} = 2, N_\Omega = 0$. So $\hat{f}$ does not vanish in Ω, and since s is bounded, f_m does not.

Remark 3.4.7. Since $\hat{f}$ is a linear combination of the two observables with $m = 0$, we may estimate f_m using properties of them. The massless observables are conformally covariant: if

$\phi : \Omega \to \Omega'$ is a conformal map [CHI15, Lemmas 2.9, 2.21],

$$f_{[\Omega,a_1,\ldots,a_n]}(z|0) = f_{[\Omega',\phi(a_1),\ldots,\phi(a_n)]}(\phi(z)|0)\phi'(z)^{1/2};$$
$$f_{[\mathbb{H},a,b]}(z|0) = \frac{(2i\mathrm{Im}a)^{1/2}}{|b-\bar{a}|+|b-a|}\frac{\left[(\bar{b}-\bar{a})(\bar{b}-a)\right]^{1/2}(z-b)+[(b-a)(b-\bar{a})]^{1/2}(z-\bar{b})}{\sqrt{(z-a)(z-\bar{a})(z-b)(z-\bar{b})}}. \quad (3.4.8)$$

Now we list some properties of the half-plane observable which need simple verifications.

Lemma 3.4.8. *Suppose $a, b \in \mathbb{H}$ with $\mathrm{Im}a = \mathrm{Im}b = \epsilon$, $\mathrm{Re}a < \mathrm{Re}b$ (i.e. $|b-a| = \mathrm{Re}b - \mathrm{Re}a$)*

1. *Let $r \le \frac{1}{2}\min(\epsilon, d)$. Then*

$$\left|\sqrt{z-a}f_{[\mathbb{H},a,b]}(z)\right| \le cst \text{ for } z \in B_r(a),$$
$$\left|\sqrt{z-b}f_{[\mathbb{H},a,b]}(z)\right| \le cst \text{ for } z \in B_r(b),$$

with constants independent of the positions of a, b, r.

2. *Let $n_{a,b} \in \mathbb{R}$ be the zero of $f_{[\mathbb{H},a,b]}$.*

$$0 \le n_{a,b} - n_{b,a} \le |b-a| + 2\epsilon;$$
$$\left|n_{a,b} - a\right| \ge |b-a| + \epsilon.$$

3. *$\frac{f_{[\Omega,b,a]}(z|0)}{f_{[\Omega,a,b]}(z|0)}$ can be extended to a holomorphic function in Ω, and*

$$\left|\frac{f_{[\Omega,b,a]}(a|0)}{f_{[\Omega,a,b]}(a|0)}\right| \le 3; \quad \left|\left(\frac{f_{[\mathbb{H},b,a]}(z|0)}{f_{[\mathbb{H},a,b]}(z|0)}\right)'_{z=a}\right| \le \frac{cst}{|b-a|+\epsilon}.$$

Proof. These results all follow from the explicit formulae above.

1. Note that $|b-a| \le |b-\bar{a}| \le 2\epsilon + |b-a|$. We have

$$\left|\frac{\left[(\bar{b}-\bar{a})(\bar{b}-a)\right]^{1/2}}{|b-\bar{a}|+|b-a|}\right| \le \left|\frac{[|b-a|(|b-a|+2\epsilon)]^{1/2}}{2|b-a|+2\epsilon}\right| \le \frac{\sqrt{|b-a|}}{\sqrt{2|b-a|+2\epsilon}} < 1.$$

Then we estimate the two terms in (3.4.8) separately, noting that $\mathrm{Im}a = \epsilon \le |z-\bar{a}|$, and also $\left|\sqrt{z-b}\right| \le \left|\sqrt{z-\bar{b}}\right|$,

$$\left|\frac{(2i\mathrm{Im}a)^{1/2}(z-b)}{\sqrt{(z-a)(z-\bar{a})(z-b)(z-\bar{b})}}\right| \le \left|\frac{(2i\mathrm{Im}a)^{1/2}\sqrt{(z-b)}}{\sqrt{(z-a)(z-\bar{a})(z-\bar{b})}}\right| \le \frac{cst}{\left|\sqrt{z-a}\right|}. \quad (3.4.9)$$

Since $\left|z-\bar{b}\right| \le \epsilon + \frac{|b-a|}{2}$,

$$\left|\frac{\sqrt{|b-a|}}{\sqrt{2|b-a|+2\epsilon}}\frac{(2i\mathrm{Im}a)^{1/2}(z-\bar{b})}{\sqrt{(z-a)(z-\bar{a})(z-b)(z-\bar{b})}}\right| \le \left|\frac{cst\sqrt{|b-a|}}{\sqrt{(z-a)(z-b)}}\right|.$$

If $z \in B_r(a)$, we have the result from $|z-b| > \frac{1}{2}|a-b|$. If $z \in B_r(b)$, similarly note $|z-a| \geq \frac{1}{2}|a-b|$, and apply $\left|\sqrt{z-a}\right| > \left|\sqrt{z-b}\right|$ to (3.4.9).

2. From the formula on $f_{[\mathbb{H},a,b]}$, we may write

$$n_{a,b} = \frac{\mathrm{Re}\sqrt{(\bar{b}-\bar{a})(\bar{b}-a)}b}{\mathrm{Re}\sqrt{(\bar{b}-\bar{a})(\bar{b}-a)}} = \mathrm{Re}b - \frac{\mathrm{Im}\sqrt{\bar{b}-a}\mathrm{Im}b}{\mathrm{Re}\sqrt{\bar{b}-a}},$$
$$n_{b,a} = \mathrm{Re}a - \frac{\mathrm{Im}\sqrt{b-\bar{a}}\mathrm{Im}a}{\mathrm{Re}\sqrt{b-\bar{a}}} = \mathrm{Re}a + \frac{\mathrm{Im}\sqrt{\bar{b}-a}\mathrm{Im}a}{\mathrm{Re}\sqrt{\bar{b}-a}}.$$

So

$$0 \leq n_{a,b} - n_{b,a} \leq |b-a| + 2\epsilon,$$

and

$$\left|n_{a,b} - a\right| \geq (n_{a,b} - \mathrm{Re}a) + \mathrm{Im}a \geq |b-a| + \epsilon,$$

since $\sqrt{\bar{b}-a}$ belongs to the fourth quadrant and thus

$$0 < -\frac{\mathrm{Im}\sqrt{\bar{b}-a}}{\mathrm{Re}\sqrt{\bar{b}-a}} \leq 1.$$

3. Again examining the formula, we have, away from b, a (and then everywhere in $\mathbb{H}$ by removable singularity),

$$\frac{f_{[\mathbb{H},b,a]}(z|0)}{f_{[\mathbb{H},a,b]}(z|0)} = \frac{\mathrm{Re}\left[(\bar{a}-\bar{b})(\bar{a}-b)\right]^{1/2}(z-n_{b,a})}{\mathrm{Re}\left[(\bar{b}-\bar{a})(\bar{b}-a)\right]^{1/2}(z-n_{a,b})} =: \rho\frac{z-n_{b,a}}{z-n_{a,b}},$$

so

$$\left|\frac{f_{[\mathbb{H},b,a]}(a|0)}{f_{[\mathbb{H},a,b]}(a|0)}\right| = \left|1 + \frac{n_{a,b}-n_{b,a}}{a-n_{a,b}}\right| \leq 1 + \frac{|b-a|+2\epsilon}{|b-a|+\epsilon} \leq 3;$$
$$\left|\left(\frac{f_{[\mathbb{H},b,a]}(z|0)}{f_{[\mathbb{H},a,b]}(z|0)}\right)'_{z=a}\right| = \left|\frac{n_{a,b}-n_{b,a}}{\left(a-n_{a,b}\right)^2}\right| \leq \frac{|b-a|+2\epsilon}{(|b-a|+\epsilon)^2} \leq \frac{cst}{|b-a|+\epsilon}.$$

□

Recall that we fix a conformal map $\varphi : \Omega \to \mathbb{D}$ such that $M^{-1} \leq \varphi' \leq M$. Now fix the standard conformal map $\mathbb{D} \to \mathbb{H}$ such that $0 \in \mathbb{D}$ is mapped to $i \in \mathbb{H}$, and fix $\varphi_{\mathbb{H}} : \Omega \to \mathbb{H}$. Similar estimate $cst(R) \cdot M^{-1} \leq \varphi'_H \leq Cst(R) \cdot M$ holds in $\varphi^{-1}(B_R \cap \mathbb{H})$ for $R > 0$. Write $\mathscr{A}^1_\Omega(a_1, a_2|m) + i\mathscr{A}^i_\Omega(a_1, a_2|m) =: \mathscr{A}_\Omega(a_1, a_2|m)$.

Lemma 3.4.9. *Let $a_1, a_2 \in \varphi_{\mathbb{H}}^{-1}(B_R \cap \mathbb{H}) \subset \Omega$ be such that $Im\varphi_{\mathbb{H}}(a_1) = Im\varphi_{\mathbb{H}}(a_2) = \epsilon$. Then for* *any fixed* $0 < \gamma < 1$*, we have*

$$|\mathscr{A}_\Omega(a_1, a_2|m) - \mathscr{A}_\Omega(a_1, a_2|0)| \leq cst \cdot \left(\epsilon^{-\gamma} + |a_1 - a_2|^{-\gamma}\right),$$

where the constant only depends on M, R, m, γ*.*

Proof. We will refer to quantities only depending on M, R, m, γ as constants in this proof. To extract the desired difference $\mathscr{A}^\Delta := \mathscr{A}_\Omega(a_1, a_2|m) - \mathscr{A}_\Omega(a_1, a_2|0)$, we will study $e^{-2m|z-a_1|} f_m(z)/f_{[\Omega,a_1,a_2]}(z|0)$. Indeed, around a_1, we have

$$\begin{aligned}\frac{e^{-2m|z-a_1|} f_m(z)}{f_{[\Omega,a_1,a_2]}(z|0)} &= \frac{1}{e^{2m|z-a_1|}} \frac{\left(\frac{e^{2m|z-a_1|}}{\sqrt{z-a_1}} + 2\mathscr{A}_\Omega(a_1, a_2|m)\sqrt{z-a_1} + o(|z-a_1|^{1/2})\right)}{\left(\frac{1}{\sqrt{z-a_1}} + 2\mathscr{A}_\Omega(a_1, a_2|0)\sqrt{z-a_1} + o(|z-a_1|^{1/2})\right)} \\ &= 1 + 2\mathscr{A}^\Delta (z - a_1) + o(|z-a|),\end{aligned}$$

therefore $\mathscr{A}^\Delta$ is half the derivative of $\frac{e^{-2m|z-a_1|} f_m(z)}{f_{[\Omega,a_1,a_2]}(z|m)}$ at a_1.

By (3.4.6),

$$\frac{e^{-2m|z-a_1|} f_m(z|m)}{f_{[\Omega,a_1,a_2]}(z|0)} = e^{s(z)-2m|z-a_1|}\left(1 + c_2 \frac{f_{[\Omega,a_2,a_1]}(z|0)}{f_{[\Omega,a_1,a_2]}(z|0)}\right),$$

and we will estimate the derivatives of the two factors separately. Fix $\alpha = 1 - \gamma$.

For the second factor, note that $|c_2| = \left|\mathrm{Im} e^{-s(a_2)}\mathscr{B}_\Omega(a_1, a_2|m)\right| \leq \left|\mathrm{Im} e^{-s(a_2)}\right|$, but since s is uniform α-Hölder continuous and purely real on $\partial\Omega$, $\left|\mathrm{Im} e^{-s(a_2)}\right| \leq cst \cdot \mathrm{dist}(a_2, \partial\Omega)^\alpha \leq cst \cdot (M\epsilon)^\alpha$. Now by conformal covariance, $\frac{f_{[\Omega,a_2,a_1]}(z|0)}{f_{[\Omega,a_1,a_2]}(z|0)} = \frac{f_{[\Omega,\varphi_{\mathbb{H}}(a_2),\varphi_{\mathbb{H}}(a_1)]}(\varphi_{\mathbb{H}}(z)|0)}{f_{[\Omega,\varphi_{\mathbb{H}}(a_1),\varphi_{\mathbb{H}}(a_2)]}(\varphi_{\mathbb{H}}(z)|0)}$, so we may apply the third estimate in Lemma 3.4.8:

$$\begin{aligned} 1 + c_2 \frac{f_{[\Omega,a_2,a_1]}(a_1|0)}{f_{[\Omega,a_1,a_2]}(a_1|0)} &\leq 1 + cst \cdot (M\epsilon)^\alpha \leq cst, \\ \left(1 + c_2 \frac{f_{[\Omega,a_2,a_1]}(z|0)}{f_{[\Omega,a_1,a_2]}(z|0)}\right)'_{z=a_1} &\leq \frac{cst \cdot M^{\alpha+1}\epsilon^\alpha}{|a_2 - a_1| + \epsilon} \leq cst \cdot M^{\alpha+1}\epsilon^{\alpha-1}. \end{aligned} \tag{3.4.10}$$

Now for the first factor, we claim that $s(z) - 2m|z - a_1|$ is differentiable at a_1. Take $r = \frac{1}{2M}\min(|a_1 - a_2|, \epsilon)$. $f_m(z) = e^{-s(z)}\hat{f}(z)$, where $\hat{f}(z) = f_{[\Omega,a_1,a_2]} + c_2 f_{[\Omega,a_2,a_1]}$. By the first estimate of Lemma 3.4.8, $\sqrt{z-a_1}\hat{f}(z)$ is bounded on $B_r(a_1)$ (the factor of M in r is there so that $\varphi(B_r(a_1)) \subset \mathbb{H}$ satisfies the condition of the first estimate Lemma 3.4.8). Expand for $z \in B_{\frac{r}{4}}(a_1)$,

$$\left|\sqrt{z-a_1}\hat{f}(z) - e^{s(a_1)}\right| \leq \frac{cst}{r}|z - a_1|, \tag{3.4.11}$$

where we use the fact that $\sqrt{z-a_1}\hat{f}(z)\big|_{z=a_1} = e^{s(a_1)}$ and the derivative of $\sqrt{z-a_1}\hat{f}(z)$ is bounded by $\frac{cst}{r}$ uniformly in $B_{\frac{r}{4}}(a_1)$ by the Cauchy formula. Then again using the uniform

Hölder regularity of s,

$$\begin{aligned}\left|\sqrt{z-a_1}f_m(z)-1\right| &= \left|e^{-s(z)}\sqrt{z-a_1}\hat{f}(z)-e^{-s(z)+s(a_1)}+e^{-s(z)+s(a_1)}-1\right| \\ &= \left|e^{-s(z)}\right|\left|\sqrt{z-a_1}\hat{f}(z)-e^{s(a_1)}\right|+\left|e^{-s(z)+s(a_1)}-1\right| \\ &\le \frac{cst}{r}|z-a_1|+cst\cdot|z-a_1|^{\alpha}\le\frac{cst}{r^{\alpha}}|z-a_1|^{\alpha}.\end{aligned}$$

Note that by (3.4.11) there is a constant $\mathfrak{c}\in(0,\frac{1}{4}]$ such that $\left|\sqrt{z-a_1}f_m(z)\right|>cst>0$ on $B_{\mathfrak{c}r}(a_1)$. Thus we have in $B_{\mathfrak{c}r}(a_1)$

$$\begin{aligned}|\partial_{\bar z}(s(z)-2m|z-a_1|)| &= \left|\frac{m\bar{f}_m}{f_m}-\frac{m\sqrt{z-a_1}}{\overline{\sqrt{z-a_1}}}\right| \\ &= \left|m\frac{\bar{f}_m\overline{\sqrt{z-a_1}}-f_m\sqrt{z-a_1}}{f_m\overline{\sqrt{z-a_1}}}\right| \\ &\le \frac{cst\cdot|m|}{r^{\alpha}}|z-a_1|^{\alpha}.\end{aligned}$$

Then by (3.4.5), $\mathfrak{C}_{B_{\mathfrak{c}r}(a_1)}\partial_{\bar z}(s(z)-2m|z-a_1|)$ is differentiable at a_1, with the derivative bounded by $|s(a_1)|+cst\cdot|m|\le cst$.

The remainder $s(z)-2m|z-a_1|-\mathfrak{C}_{B_{\mathfrak{c}r}(a_1)}\partial_{\bar z}(s(z)-2m|z-a_1|)$ is holomorphic in $B_{\mathfrak{c}r}(a_1)$, with uniformly α-Hölder boundary values on $\partial B_{\mathfrak{c}r}(a_1)$. By Cauchy formula again, its derivative at a_1 is bounded by $\frac{cst}{r^{1-\alpha}}$. Therefore, $s(z)-2m|z-a_1|$ is differentiable and its derivative is bounded by $\frac{cst}{r^{1-\alpha}}$ for small r. Combining this estimate with (3.4.10) and writing $\gamma=1-\alpha$ gives the desired estimate. □

Finally, we show that the full plane observables are differentiable in the positions a_j of the spins, grounding the isomonodromic analysis in the next section.

Proposition 3.4.10. *The value of the full-plane observable $f_{[\mathbb{C},a_1,\dots,a_n]}(z)$ and thus its coefficients of the formal power series expansion around a_j are differentiable in the positions $a_1,\dots,a_n$.*

Proof. Differentiability of the coefficients follow directly from that of the observable value, since we can recover the coefficients using the Cauchy formula (3.4.3).

Without loss of generality, we show the x-derivative in a_1 exists. Now, set $h_0>4h>0$, $a_1^h=a_1+h$ and embed all $\left[\mathbb{C},a_1^h,\dots,a_n\right]$ in the double cover $\mathbb{C}^{2h}$ of $\mathbb{C}\setminus[B_{2h}(a_1)\cup\{a_2,\dots,a_n\}]$. Consider the difference $f^h:=f_{[\mathbb{C},a_1^h,\dots,a_n]}-f_{[\mathbb{C},a_1,\dots,a_n]}$ defined on $\mathbb{C}^{2h}$.

Clearly f^h is massive holomorphic, $\lim_{z\to a_j}\sqrt{z-a_j}f^h(z)=:i\mathscr{B}_j^h\in i\mathbb{R}$ for $j\ge2$, and f^h decays exponentially fast at infinity. So by applying the Green-Riemann theorem as in (3.2.18) to

$\left(f^h\right)^2$ on $\mathbb{C}^{h_0}$, we see that

$$i\oint_{\partial B_{h_0}(a_1)}\left(f^h\right)^2 dz = -2\iint_{\mathbb{C}\setminus B_{h_0}(a_1)} 2m|f^h|^2 dz + \sum_j 2\pi\left(\mathscr{B}_j^h\right)^2. \tag{3.4.12}$$

We claim that the left hand side, which is nonnegative and real since it is equal to the right hand side, is $O(h^2)$ on $\partial B_{h_0}(a_1)$ as $h \to 0$, which will be proven below. Then $\{\frac{1}{h}f^h\}_{h<h_0}$ is uniformly L^2-bounded in $\mathbb{C}^{h_0}$. By a continuous version of Proposition 3.5.4, there exists a subsequence h_k such that $\frac{1}{h_k}f^{h_k}$ converges uniformly in compact subsets; the limit satisfies the same boundary value problem as in Proposition 3.2.15 away from a_1. By diagonalising as $h_0 \to 0$, we may assume that there exists a limit f on $[\mathbb{C}, a_1, \ldots, a_n]$. It suffices to show that f is unique. But by the expansion (3.2.20), the form of a subsequential limit near a_1 is determined; indeed, the singular behaviour is

$$-\frac{1}{2}Z^1_{-\frac{3}{2}}(z-a_1) + 2\mathscr{A}^1_\Omega Z^1_{-\frac{1}{2}}(z-a_1) + 2\mathscr{A}^i_\Omega Z^i_{-\frac{1}{2}}(z-a_1) + (\text{regular part}),$$

so the difference of any two limits is zero by the uniqueness of the boundary value problem.

Once smoothness in the position of a_1 is proved, one may repeat the same arguments for other points, say, a_2. The difference is that the singularity at a_2 is $\frac{i\mathscr{B}_2}{\sqrt{z-a_2}}$, and we need to first show differentiability of $\mathscr{B}_2$ to make the argument work. But $f_{[\mathbb{C},a_2,a_1,\ldots,a_n]}$ and its coefficients is differentiable in a_2 by above, and $i\frac{(\mathscr{B}_2^h-\mathscr{B}_2)}{h}$ is equal to the coefficient of $\frac{1}{\sqrt{z-a_1}}$ in $\frac{1}{h}\left[f_{[\mathbb{C},a_2^h,a_1,\ldots,a_n]} - f_{[\mathbb{C},a_2,a_1,\ldots,a_n]}\right]$ by Green-Riemann's formula applied to $f_{[\mathbb{C},a_1,a_2^h,\ldots,a_n]}f_{[\mathbb{C},a_2^h,a_1,\ldots,a_n]}$, so its limit as $h \to 0$ exists.

Proof of Claim. Note that (3.4.12) shows that the integral is real and positive. Suppose $\frac{i}{h^2}\oint_{\partial B_{h_0}(a_1)}\left(f^h\right)^2 dz \to \infty$ as $h \to 0$, possibly along a subsequence (for notational convenience, we do not explicitly write the subsequence). Define $s^h_{h_0} := i\oint_{\partial B_{h_0}(a_1)}\left(f^h\right)^2 dz$, then $\frac{h^2}{s^h_{h_0}} \to 0$. Consider $Z^h := Z^1_{-\frac{1}{2}}(z-a_1^h) - Z^1_{-\frac{1}{2}}(z-a_1)$. Clearly, $\frac{1}{h}Z^h$ is bounded on $\partial B_{h_0}(a_1)$, so $\frac{i}{s^h_{h_0}}\oint_{\partial B_{h_0}(a_1)}\left(f^h - Z^h\right)^2 dz \to 1$. But $f^h - Z^h$ is massive holomorphic in $\mathbb{C}^{2h}$, so the Green-Riemann's formula gives

$$i\oint_{\partial B_{2h}(a_1)}\left(f^h - Z^h\right)^2 dz = i\oint_{\partial B_{h_0}(a_1)}\left(f^h - Z^h\right)^2 dz - 2\int 2m\left|f^h\right|^2 d^2z,$$

$$\frac{i}{s^h_{h_0}}\oint_{\partial B_{2h}(a_1)}\left(f^h - Z^h\right)^2 dz \geq \frac{i}{s^h_{h_0}}\oint_{\partial B_{h_0}(a_1)}\left(f^h - Z^h\right)^2 dz \xrightarrow{h\to 0} 1,$$

so it suffices to show that the left hand side tends to zero as $h \to 0$ to derive contradiction.

Fix h_0. Define $f_h^\dagger(z) := f_{[\mathbb{C},a_1^h,\ldots,a_n]}(z) - Z^1_{-\frac{1}{2}}(z-a_1^h)$ for $h < \frac{h_0}{4}$, which is bounded near a_1^h. Note that, since the L^2 norm of $f_{[\mathbb{C},a_1^h,\ldots,a_n]}(z)$ is bounded on $B_{\frac{3}{2}h_0}(a_1) \setminus B_{\frac{1}{2}h_0}(a_1)$ (uniformly in

$h < \frac{h_0}{4}$), $\left|f_{[\mathbb{C},a_1^h,\dots,a_n]}\right|$ is bounded (uniformly for $h < \frac{h_0}{4}$) on the circle $\partial B_{h_0}(a_1)$ by the continuous version of Proposition 3.5.4. Similarly, $Z^1_{-\frac{1}{2}}(z - a_1^h)$ is bounded on $\partial B_{h_0}(a_1)$ (uniformly for $h < \frac{h_0}{4}$). By Proposition 3.4.2 and the proof of Lemma 3.4.4, $\hat{f}_h := \exp\left[-\mathfrak{C}_{B_{h_0}(a_1)}\left[m\frac{\bar{f}_h^\dagger}{f_h^\dagger}\right]\right] f_h^\dagger$ is a holomorphic function in $\left[B_{h_0}(a_1), a_1^h\right]$ which remains bounded near a_1^h and bounded (uniformly for $h < \frac{h_0}{4}$) on $\partial B_{h_0}(a_1)$. Therefore, $\hat{f}_h^{\,2}$ is a holomorphic function on $B_{h_0}(a_1)$ which is zero at $a_1^h \in B_{2h}(a_1)$ and has a bounded derivative in $B_{2h}(a_1)$ (uniformly for $h < \frac{h_0}{4}$), applying Cauchy integral formula on $\partial B_{h_0}(a_1)$. Therefore, $\left|\hat{f}_h^2(z)\right| \leq cst \cdot h$ on $B_{2h}(a_1)$ for any $h < \frac{h_0}{4}$. Taking into account uniform boundedness of $\exp\left[\mathfrak{C}_{B_{h_0}(a_1)}\left[m\frac{\bar{f}_h^\dagger}{f_h^\dagger}\right]\right]$, $f_h^\dagger = \exp\left[\mathfrak{C}_{B_{h_0}(a_1)}\left[m\frac{\bar{f}_h^\dagger}{f_h^\dagger}\right]\right]\hat{f}_h$ is similarly bounded: $\left|f_h^\dagger(z)\right|^2 \leq cst \cdot h$ on $B_{2h}(a_1)$ for $h < \frac{h_0}{4}$. Therefore the integral

$$\frac{i}{s_{h_0}^h}\oint_{\partial B_{2h}(a_1)}\left(f^h - Z^h\right)^2 dz = \frac{i}{s_{h_0}^h}\oint_{\partial B_{2h}(a_1)}\left(f_h^\dagger - f_0^\dagger\right)^2 dz \leq \frac{cst \cdot h^2}{s_{h_0}^h},$$

tend to zero.

□

3.4.2 Derivation of Painlevé III

In this section, we take the convergence results established in Section 3.3 and derive established correlation results in the full plane, first shown in [WMTB76] and reformulated in terms of isomonodromic deformation in [SMJ77]. We will explicitly carry out the basic 2-point case following the presentation of [KaKo80, Sections III, IV], using the continuous limit of our discrete massive fermions which has been characterised in terms of a boundary value problem in Definition 3.2.17. We cannot directly cite their formulae, since instead of considering a complex space of functions which solve a two-dimensional Dirac equation, we cast them in terms of a real space of massive holomorphic functions because massive holomorphicity is an $\mathbb{R}$-linear notion. The resulting analysis is equivalent.

Isomonodromy.

We would first like to note how the functions behave under rotation around the origin. We will compose rotation of the coordinate system with multiplication by a phase factor and denote it by $R_\phi W_\nu(z) := W_\nu(e^{-i\phi}z)e^{-i\phi/2}$ and so on: first we see that $R_\phi W_\nu(z) = e^{-i(\nu+\frac{1}{2})\phi}W_\nu(z)$, and similarly

$$R_\phi Z_\nu^1 = \frac{\Gamma(\nu+1)}{|m|^\nu}\left(e^{-i(\nu+\frac{1}{2})\phi} W_\nu + (\operatorname{sgn} m)\, e^{i\left(\nu+\frac{1}{2}\right)\phi}\overline{W_{\nu+1}}\right) \tag{3.4.13}$$
$$= \cos\left[\left(\nu+\frac{1}{2}\right)\phi\right] Z_\nu^1 + \sin\left[\left(\nu+\frac{1}{2}\right)\phi\right] Z_\nu^i,$$
$$R_\phi Z_\nu^i = \cos\left[\left(\nu+\frac{1}{2}\right)\phi\right] Z_\nu^i - \sin\left[\left(\nu+\frac{1}{2}\right)\phi\right] Z_\nu^1.$$

Recall we fix $m<0$. Suppose $a>0$ is a positive real number, and consider the double cover $[\mathbb{C},-a,a]$. Consider the real vector space of m-massive holomorphic functions on the double cover which have singularity of order at most $3/2$ at each monodromy and decay at infinity. Around each monodromy, we can expand the singular part of a function in $Z^{1,i}_{-\frac{3}{2},-\frac{1}{2}}$, and from Proposition 3.2.15 we see in fact fixing the coefficients of $Z^{1,i}_{-\frac{3}{2}}, Z^1_{-\frac{1}{2}}$ at each monodromy fixes the function. 6 basis functions are given by the two fermions $f_1 := f_{[\mathbb{C},-a,a]}, f_2 := f_{[\mathbb{C},a,-a]}$ and their derivatives $\partial_x f_1, \partial_y f_1, \partial_x f_2, \partial_y f_2$. The idea is to express the variation of f_1 under movement of the monodromies $\pm a$ as a linear combination of these six functions, and to get a nontrivial equality by looking at the dependent coefficient of $Z^i_{-\frac{1}{2}}$. First, we augment the expansions (3.2.20), (3.2.21): $f_{[\mathbb{C},-a,a]}$ is equal to ($\mathscr{C}^{1,i}$ are real constants, unrelated to the discrete notation $\mathscr{C}^{1,i}[\Omega_\delta]$)

$$Z^1_{-\frac{1}{2}}(z+a) + 2\mathscr{A}^1 Z^1_{\frac{1}{2}}(z+a) + 2\mathscr{A}^i Z^i_{\frac{1}{2}}(z+a) + 2\mathscr{D}^1 Z^1_{\frac{3}{2}}(z+a)$$
$$+ 2\mathscr{D}^i Z^i_{\frac{3}{2}}(z+a) + O\left((z+a)^{5/2}\right) \text{ near } z=-a,$$
$$\mathscr{B}_{\mathbb{C}} Z^i_{-\frac{1}{2}}(z-a) + 2\mathscr{C}^1 Z^1_{\frac{1}{2}}(z-a) + 2\mathscr{C}^i Z^i_{\frac{1}{2}}(z-a) + 2\mathscr{E}^1 Z^1_{\frac{3}{2}}(z-a)$$
$$+ 2\mathscr{E}^i Z^i_{\frac{3}{2}}(z-a) + O\left((z-a)^{5/2}\right) \text{ near } z=a.$$

We need to make some reductions. Let us first note that $f_1(z)$ and $\bar{f}_1(\bar{z})$ both solves the boundary value problem of Proposition 3.2.15 on $[\mathbb{C},-a,a]$, and thus are equal to each other; since only Z_ν^i switches sign under the same transformation, we can conclude $\mathscr{A}^i = \mathscr{C}^1 = \mathscr{D}^i = \mathscr{E}^1 = 0$.

Similarly, $f_2(z)$ and $i f_1(e^{i\pi}z)$ are equal, assuming we carefully track the interaction between global rotation $z \mapsto e^{i\pi}z$ and the expansion bases $Z_\nu^\tau(\cdot \pm a)$; given a sign choice of $Z_\nu^\tau(z+a)$ (which is fixed by the coefficient $+1$ of $Z^1_{-\frac{1}{2}}(z+a)$) we will define $Z^\tau_{-\frac{1}{2}}(z-a) := iZ^\tau_{-\frac{1}{2}}(e^{i\pi}z+a)$, which then fixes signs for general ν by $Z_\nu^\tau(e^{i\pi}z \pm a) = \pm e^{i\nu\pi} Z_\nu^\tau(z \mp a)$. As a result we have $\mathscr{A}^{1,i}_{\mathbb{C}}(a,-a) = -\mathscr{A}^{1,i}_{\mathbb{C}}(-a,a) = -\mathscr{A}^{1,i}$, $\mathscr{B}_{\mathbb{C}}(a,-a) = -\mathscr{B}_{\mathbb{C}}(-a,a) = -\mathscr{B}$, $\mathscr{C}^{1,i}_{\mathbb{C}}(a,-a) = \mathscr{C}^{1,i}_{\mathbb{C}}(-a,a) = \mathscr{C}^{1,i}$, $\mathscr{D}^{1,i}_{\mathbb{C}}(a,-a) = \mathscr{D}^{1,i}_{\mathbb{C}}(-a,a) = \mathscr{D}^{1,i}$, $\mathscr{E}^{1,i}_{\mathbb{C}}(a,-a) = -\mathscr{E}^{1,i}_{\mathbb{C}}(-a,a) = -\mathscr{E}^{1,i}$.

In fact, given $Z_\nu^\tau(z+a)$, if we define $Z^\tau_{-\frac{1}{2}}(z \pm e^{i\phi}a) := e^{-\frac{i\phi}{2}} Z^\tau_{-\frac{1}{2}}(e^{-i\phi}z \pm a)$ for small $|\phi|$, $f_{[\mathbb{C},-ae^{i\phi},ae^{i\phi}]}(z) = R_\phi f_{[\mathbb{C},-a,a]}(z)$ again from (3.4.13):

$$
\begin{aligned}
&f_{[\mathbb{C},-ae^{i\phi},ae^{i\phi}]}(z) = R_\phi f_1(z) = Z^1_{-\frac{1}{2}}(z+ae^{i\phi})\\
&+2\cos\phi\mathscr{A}^1 Z^1_{\frac{1}{2}}(z+ae^{i\phi}) + 2\sin\phi\mathscr{A}^1 Z^i_{\frac{1}{2}}(z+ae^{i\phi}) + 2\cos 2\phi\mathscr{D}^1 Z^1_{\frac{3}{2}}(z+ae^{i\phi})\\
&+2\sin 2\phi\mathscr{D}^1 Z^i_{\frac{3}{2}}(z+ae^{i\phi}) + O\left((z+ae^{i\phi})^{5/2}\right) \text{ near } z=-a, \text{ and}\\
&f_{[\mathbb{C},-ae^{i\phi},ae^{i\phi}]}(z) = \mathscr{B}_\Omega Z^i_{-\frac{1}{2}}(z-ae^{i\phi}) + 2\cos\phi\mathscr{C}^i Z^i_{\frac{1}{2}}(z-ae^{i\phi}) - 2\sin\phi\mathscr{C}^i Z^1_{\frac{1}{2}}(z-ae^{i\phi})\\
&+2\cos 2\phi\mathscr{E}^i Z^i_{\frac{3}{2}}(z-ae^{i\phi}) - 2\sin 2\phi\mathscr{E}^i Z^1_{\frac{3}{2}}(z-ae^{i\phi}) + O\left(\left(z-ae^{i\phi}\right)^{5/2}\right) \text{ near } z=a.
\end{aligned}
$$

Expansion.

We are now ready to analyse the variation of $R_\phi f_1$ under both ∂_a and ∂_ϕ at $\phi = 0$.

Around $z=-a$,

$$
\begin{aligned}
f_1(z) &= Z^1_{-\frac{1}{2}}(z+a) + 2\mathscr{A}^1 Z^1_{\frac{1}{2}}(z+a) + 2\mathscr{D}^1 Z^1_{\frac{3}{2}}(z+a) + O\left((z+a)^{5/2}\right),\\
\partial_x f_1(z) &= -\frac{1}{2} Z^1_{-\frac{3}{2}}(z+a) + \mathscr{A}^1 Z^1_{-\frac{1}{2}}(z+a) + \left(3\mathscr{D}^1 + 2m^2\right) Z^1_{\frac{1}{2}}(z+a) + O\left((z+a)^{3/2}\right),\\
\partial_y f_1(z) &= -\frac{1}{2} Z^i_{-\frac{3}{2}}(z+a) + \mathscr{A}^1 Z^i_{-\frac{1}{2}}(z+a) + \left(3\mathscr{D}^1 - 2m^2\right) Z^i_{\frac{1}{2}}(z+a) + O\left((z+a)^{3/2}\right),
\end{aligned}
$$

while around a,

$$
\begin{aligned}
f_1(z) &= \mathscr{B} Z^i_{-\frac{1}{2}}(z-a) + 2\mathscr{C}^i Z^i_{\frac{1}{2}}(z-a) + 2\mathscr{E}^i Z^i_{\frac{3}{2}}(z-a) + O\left((z-a)^{5/2}\right),\\
\partial_x f_1(z) &= -\frac{\mathscr{B}}{2} Z^i_{-\frac{3}{2}}(z-a) + \mathscr{C}^i Z^i_{-\frac{1}{2}}(z-a) + \left(3\mathscr{E}^i + 2m^2\mathscr{B}\right) Z^i_{\frac{1}{2}}(z-a) + O\left((z-a)^{3/2}\right),\\
\partial_y f_1(z) &= \frac{\mathscr{B}}{2} Z^1_{-\frac{3}{2}}(z-a) - \mathscr{C}^i Z^1_{-\frac{1}{2}}(z-a) + \left(2m^2\mathscr{B} - 3\mathscr{E}^i\right) Z^1_{\frac{1}{2}}(z-a) + O\left((z-a)^{3/2}\right),
\end{aligned}
$$

and similar formulae hold for f_2 with $-a$ and a interchanged and the signs in front of $\mathscr{A},\mathscr{B},\mathscr{E}$ reversed. As for the varied functions, we have

$$
\begin{aligned}
\partial_a f_1(z) &= -\frac{1}{2} Z^1_{-\frac{3}{2}}(z+a) + \mathscr{A}^1 Z^1_{-\frac{1}{2}}(z+a) + \left(2\left(\partial_a \mathscr{A}^1\right) + 3\mathscr{D}^1 + 2m^2\right) Z^1_{\frac{1}{2}}(z+a)\\
&\quad + O\left((z+a)^{3/2}\right),\\
\partial_\phi R_\phi f_1(z) &= \frac{a}{2} Z^i_{-\frac{3}{2}}(z+a) - a\mathscr{A}^1 Z^i_{-\frac{1}{2}}(z+a) + \left(2\mathscr{A}^1 - 3a\mathscr{D}^1 + 2am^2\right) Z^i_{\frac{1}{2}}(z+a)\\
&\quad + O\left((z+a)^{3/2}\right),
\end{aligned}
$$

and

$$\partial_a f_1(z) = \frac{\mathscr{B}}{2} Z^i_{-\frac{3}{2}}(z-a) + \left(\partial_a \mathscr{B} - \mathscr{C}^i\right) Z^i_{-\frac{1}{2}}(z-a) + \left(2\left(\partial_a \mathscr{C}^i\right) - 3\mathscr{E}^i - 2m^2\mathscr{B}\right) Z^i_{\frac{1}{2}}(z-a) + O\left((z-a)^{3/2}\right),$$

$$\partial_\phi R_\phi f_1(z) = \frac{a\mathscr{B}}{2} Z^1_{-\frac{3}{2}}(z-a) - a\mathscr{C}^i Z^1_{-\frac{1}{2}}(z-a) + \left(2am^2\mathscr{B} - 3a\mathscr{E}^i - 2\mathscr{C}^i\right) Z^1_{\frac{1}{2}}(z-a) + O\left((z-a)^{3/2}\right).$$

The resulting expansions are

$$\partial_a f_1 = -\frac{2(\mathscr{A}^1\mathscr{B} + \mathscr{C}^i)\mathscr{B}}{1-\mathscr{B}^2} f_1 + \frac{1+\mathscr{B}^2}{1-\mathscr{B}^2}\partial_x f_1 - \frac{2\mathscr{B}}{1-\mathscr{B}^2}\partial_y f_2, \tag{3.4.14}$$

$$\partial_\phi R_\phi f_1 = -\frac{2a(\mathscr{A}^1\mathscr{B} + \mathscr{C}^i)}{1-\mathscr{B}^2} f_2 - \frac{2a\mathscr{B}}{1-\mathscr{B}^2}\partial_x f_2 - a\frac{1+\mathscr{B}^2}{1-\mathscr{B}^2}\partial_y f_1. \tag{3.4.15}$$

Derivation.

Comparing the coefficients of $Z^i_{-\frac{1}{2}}, Z^1_{\frac{1}{2}}, Z^i_{\frac{1}{2}}$, we get from (3.4.14)

$$\partial_a \mathscr{B} - \mathscr{C}^i = -\frac{2(\mathscr{A}^1\mathscr{B} + \mathscr{C}^i)\mathscr{B}}{1-\mathscr{B}^2}\mathscr{B} + \frac{1+\mathscr{B}^2}{1-\mathscr{B}^2}\mathscr{C}^i + \frac{2\mathscr{B}}{1-\mathscr{B}^2}\mathscr{A}^1,$$

$$2\left(\partial_a \mathscr{A}^1\right) + 3\mathscr{D}^1 + \frac{2m^2}{3} = -\frac{2(\mathscr{A}^1\mathscr{B} + \mathscr{C}^i)\mathscr{B}}{1-\mathscr{B}^2} 2\mathscr{A}^1 + \frac{1+\mathscr{B}^2}{1-\mathscr{B}^2}(3\mathscr{D}^1 + 2m^2) - \frac{2\mathscr{B}}{1-\mathscr{B}^2}\left(-2m^2\mathscr{B} + 3\mathscr{E}^i\right),$$

$$2\left(\partial_a \mathscr{C}^i\right) - 3\mathscr{E}^i - \frac{2m^2\mathscr{B}}{3} = -\frac{2(\mathscr{A}^1\mathscr{B} + \mathscr{C}^i)\mathscr{B}}{1-\mathscr{B}^2} 2\mathscr{C}^i + \frac{1+\mathscr{B}^2}{1-\mathscr{B}^2}\left(3\mathscr{E}^i + 2m^2\mathscr{B}\right) - \frac{2\mathscr{B}}{1-\mathscr{B}^2}(3\mathscr{D}^1 - 2m^2),$$

while for (3.4.15) we get

$$2\mathscr{A}^1 - 3a\mathscr{D}^1 + \frac{2am^2}{3} = -\frac{2a(\mathscr{A}^1\mathscr{B} + \mathscr{C}^i)}{1-\mathscr{B}^2} 2\mathscr{C}^i - \frac{2a\mathscr{B}}{1-\mathscr{B}^2}\left(-3\mathscr{E}^i - 2m^2\mathscr{B}\right) - a\frac{1+\mathscr{B}^2}{1-\mathscr{B}^2}(3\mathscr{D}^1 - 2m^2),$$

$$\frac{2am^2\mathscr{B}}{3} - 3a\mathscr{E}^i - 2\mathscr{C}^i = \frac{2a(\mathscr{A}^1\mathscr{B} + \mathscr{C}^i)}{1-\mathscr{B}^2} 2\mathscr{A}^1 - \frac{2a\mathscr{B}}{1-\mathscr{B}^2}(3\mathscr{D}^1 + 2m^2) - a\frac{1+\mathscr{B}^2}{1-\mathscr{B}^2}\left(2m^2\mathscr{B} - 3\mathscr{E}^i\right).$$

We now make the dependence in m explicit. Similarly to above, for any $k > 0$, $f_{[\mathbb{C},-ak^{-1},ak^{-1}]}(z|mk) = f_{[\mathbb{C},-a,a]}(kz|m)k^{1/2}$. Analysing the effect of this dilation, which leaves $r := am$ fixed, on the individual coefficients, we can write $\mathscr{A}^1(a,m) =: m\mathscr{A}_0(r)$, $\mathscr{B}(a,m) =: \mathscr{B}_0(r)$, $\mathscr{C}^i(a,m) =: m\mathscr{C}_0(r)$. Then we have $\partial_a\mathscr{A}^1 = m^2\mathscr{A}_0'$, $\partial_a\mathscr{B} = m\mathscr{B}_0'$, $\partial_a\mathscr{C}^1 = m^2\mathscr{C}_0'$. In terms of these functions, we have

$$\mathscr{B}_0' = -\frac{2(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)}{1-\mathscr{B}_0^2}\mathscr{B}_0^2 + \frac{2}{1-\mathscr{B}_0^2}\mathscr{C}_0 + \frac{2\mathscr{B}_0}{1-\mathscr{B}_0^2}\mathscr{A}_0 = 2\mathscr{A}_0\mathscr{B}_0 + 2\mathscr{C}_0, \tag{3.4.16}$$

$$\mathscr{A}_0' = -\frac{2\mathscr{B}_0(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)}{1-\mathscr{B}_0^2}\mathscr{A}_0 + \frac{4\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)} + \frac{3m^{-2}\mathscr{B}}{1-\mathscr{B}^2}(\mathscr{B}\mathscr{D}^1 - \mathscr{E}^i), \tag{3.4.17}$$

$$\mathscr{C}_0' = -\frac{2\mathscr{B}_0(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)}{1-\mathscr{B}_0^2}\mathscr{C}_0 + \frac{4\mathscr{B}_0}{\left(1-\mathscr{B}_0^2\right)} - \frac{3m^{-2}}{1-\mathscr{B}^2}(\mathscr{B}\mathscr{D}^1 - \mathscr{E}^i), \tag{3.4.18}$$

$$\mathscr{A}_0 = -\frac{2r(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)}{1-\mathscr{B}_0^2}\mathscr{C}_0 + \frac{4r\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)} - \frac{3am^{-1}\mathscr{B}}{1-\mathscr{B}^2}(\mathscr{B}\mathscr{D}^1 - \mathscr{E}^i), \tag{3.4.19}$$

$$\mathscr{C}_0 = -\frac{2r(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)}{1-\mathscr{B}_0^2}\mathscr{A}_0 + \frac{4r\mathscr{B}_0}{\left(1-\mathscr{B}_0^2\right)} + \frac{3am^{-1}}{1-\mathscr{B}^2}(\mathscr{B}\mathscr{D}^1 - \mathscr{E}^i). \tag{3.4.20}$$

Define $\mathscr{B}_0 =: \tanh h_0$. Then $\frac{\mathscr{B}_0'}{1-\mathscr{B}_0^2} = h_0'$ and $\frac{4\mathscr{B}_0^2}{(1-\mathscr{B}_0^2)^2} = \sinh^2 2h_0$. From (3.4.19),(3.4.20),

$$\mathscr{A}_0 + \mathscr{B}_0\frac{\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0}{1-\mathscr{B}_0^2} = \frac{\mathscr{A}_0 + \mathscr{B}_0\mathscr{C}_0}{1-\mathscr{B}_0^2} = -r\left[\frac{2(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)^2}{\left(1-\mathscr{B}_0^2\right)^2} - \frac{8\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)^2}\right], \tag{3.4.21}$$

and noting (3.4.16),

$$\mathscr{A}_0 = -\frac{1}{2}(\ln\cosh h_0)' - r\left[\frac{1}{2}\left(h_0'\right)^2 - 2\sinh^2 2h_0\right]. \tag{3.4.22}$$

To characterise h, first combine (3.4.17),(3.4.18) to get

$$\begin{aligned}\frac{\mathscr{A}_0' + \mathscr{C}_0'\mathscr{B}_0}{1-\mathscr{B}_0^2} &= -\frac{2(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)(\mathscr{A}_0\mathscr{B}_0 + \mathscr{B}_0^2\mathscr{C}_0)}{\left(1-\mathscr{B}_0^2\right)^2} + \frac{8\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)^2}, \\ &= -\left[\frac{2(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)^2}{\left(1-\mathscr{B}_0^2\right)^2} - \frac{8\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)^2}\right] + \frac{\mathscr{B}_0'\mathscr{C}_0}{1-\mathscr{B}_0^2},\end{aligned}$$

then differentiate (3.4.21) to get

$$\begin{aligned}&\frac{\mathscr{A}_0' + \mathscr{C}_0'\mathscr{B}_0 + \mathscr{B}_0'\mathscr{C}_0}{1-\mathscr{B}_0^2} + \frac{2\mathscr{B}_0\mathscr{B}_0'(\mathscr{A}_0 + \mathscr{B}_0\mathscr{C}_0)}{\left(1-\mathscr{B}_0^2\right)^2} \\ &= -\left[\frac{2(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)^2}{\left(1-\mathscr{B}_0^2\right)^2} - \frac{8\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)^2}\right] - r\left[\frac{2(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)^2}{\left(1-\mathscr{B}_0^2\right)^2} - \frac{8\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)^2}\right]'.\end{aligned}$$

Then combining the two we finally have

$$\frac{2\mathscr{B}_0'(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)}{\left(1-\mathscr{B}_0^2\right)^2} = -r\left[\frac{2(\mathscr{A}_0\mathscr{B}_0 + \mathscr{C}_0)^2}{\left(1-\mathscr{B}_0^2\right)^2} - \frac{8\mathscr{B}_0^2}{\left(1-\mathscr{B}_0^2\right)^2}\right]',$$

or

$$\left(h_0'\right)^2 = -r\left[\frac{1}{2}\left(h_0'\right)^2 - 2\sinh^2 2h_0\right]'.$$

Simplifying, we have $h_0'' + \frac{h_0'}{r} = 4\sinh 4h_0(r)$. This is equivalent to the Painlevé III equation $r\eta_0\eta_0'' = r\left(\eta_0'\right)^2 - \eta_0\eta_0' - 4r + 4r\eta_0^4$ by a change of variables $\eta_0 = e^{-2h_0}$ [KaKo80, (4.12)].

3.5 Appendix: Harmonicity Estimates

In this Appendix, we collect together the discrete analytic calculations and estimates used in the paper. Fix a discrete simply connected planar graph G_δ, which can be thought of as a subgraph of Ω_δ or $[\Omega_\delta, a_1, \ldots, a_n]$.

Proposition 3.5.1. *A massive s-holomorphic function* $F : \mathscr{E}\mathscr{C}[G_\delta] \to \mathbb{C}$ *is* massive discrete holomorphic, *that is to say*

$$\begin{aligned}\cos\left(\frac{\pi}{4}+2\Theta\right)F(r_+)-\cos\left(\frac{\pi}{4}-2\Theta\right)F(r_-)&=-i\left(\cos\left(\frac{\pi}{4}+2\Theta\right)F(i_+)-\cos\left(\frac{\pi}{4}-2\Theta\right)F(i_-)\right),\\ \cos\left(\frac{\pi}{4}-2\Theta\right)F(i'_+)-\cos\left(\frac{\pi}{4}+2\Theta\right)F(i'_-)&=-i\left(\cos\left(\frac{\pi}{4}-2\Theta\right)F(r'_+)-\cos\left(\frac{\pi}{4}+2\Theta\right)F(r'_-)\right),\end{aligned} \tag{3.5.1}$$

if there is a λ-corner c such that $r_\pm = c \pm \frac{\delta+\delta i}{2}$ (real corners) and $i_\pm = c \pm \frac{-\delta+\delta i}{2}$ (imaginary corners), or a $\bar{\lambda}$-corner c' such that $i'_\pm = c \pm \frac{\delta+\delta i}{2}$ and $r'_\pm = c \pm \frac{-\delta+\delta i}{2}$ (resp. imaginary and real corners).

It is massive harmonic, *i.e.*

$$\begin{aligned}\Delta^\delta F(c)&=2\left(\frac{\cos\left(\frac{\pi}{4}-2\Theta\right)}{\cos\left(\frac{\pi}{4}+2\Theta\right)}+\frac{\cos\left(\frac{\pi}{4}+2\Theta\right)}{\cos\left(\frac{\pi}{4}-2\Theta\right)}-2\right)F(c)\\ &=\left(\frac{8\sin^2 2\Theta}{\cos 4\Theta}\right)F(c)=:M_H^2F(c) \text{ for } c\in\mathscr{C}^{1,i}[G_\delta].\end{aligned} \tag{3.5.2}$$

In addition, its square satisfies

$$\partial_{\bar z}^\delta F^2(x)=\begin{cases}A_\Theta\sum_{n=0}^3\left|F\left(x+i^n\frac{\delta}{2}\right)\right|^2+B_\Theta\left|\partial_{\bar z}\bar F\right|^2(x) & x\in\mathscr{F}[\Omega_\delta]\setminus\{a_2,\ldots,a_n\}\\ -A_{-\Theta}\sum_{n=0}^3\left|F\left(x+i^n\frac{\delta}{2}\right)\right|^2-B_{-\Theta}\left|\partial_{\bar z}\bar F\right|^2(x) & x\in\mathscr{V}[\Omega_\delta]\setminus\{a_1+\delta\}\end{cases}. \tag{3.5.3}$$

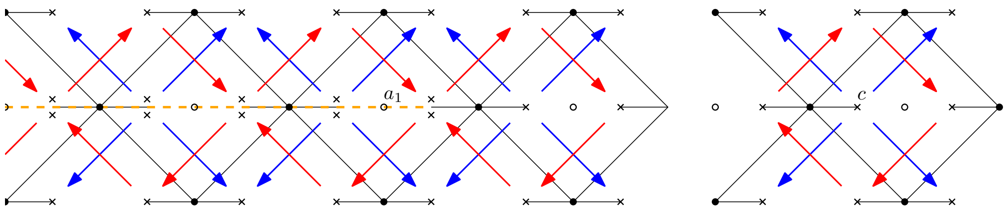

Figure 3.5.1 – Using holomorphicity to get harmonicity. Blue differences (laplacian) are turned into red, which telescope. Left: in the presence of a branch cut (orange). Right: simple planar case.

where $A_\Theta = \frac{2\left(\sqrt{2}\cos\left(\frac{\pi}{4}-2\Theta\right)-1\right)}{\sqrt{2}\cos^2\Theta\cos^2\left(\frac{\pi}{4}+2\Theta\right)}$, $B_\Theta = \frac{1}{2\sqrt{2}\cos^2\Theta}$.

Proof. For the first line in (3.5.1), note that by massive s-holomorphicity we have the edge values $F\left(\frac{r_+ + i_-}{2}\right) = e^{-i\Theta}\left[F(r_+) + F(i_-)\right]$ and $F\left(\frac{i_+ + r_-}{2}\right) = e^{i\Theta}\left[F(i_+) + F(r_-)\right]$. Since F is s-holomorphic at the λ-corner c, which is adjacent to both of them, writing
$e^{-i\Theta}\mathrm{Proj}_{e^{i\Theta}\lambda\mathbb{R}} F\left(\frac{r_+ + i_-}{2}\right) = e^{i\Theta}\mathrm{Proj}_{e^{-i\Theta}\lambda\mathbb{R}} F\left(\frac{i_+ + r_-}{2}\right)$, equivalent to
$\frac{1}{2}\left[e^{-2i\Theta}\left(F(r_+) + F(i_-)\right) + ie^{2i\Theta}\left(F(r_+) - F(i_-)\right)\right] = \frac{1}{2}\left[e^{2i\Theta}\left(F(i_+) + F(r_-)\right) + ie^{-2i\Theta}\left(-F(i_+) + F(r_-)\right)\right]$, and rearranging gives the result. For the second line, notice that iF is $(-\Theta)$-massive s-holomorphic if we move to the dual graph G_δ^* (i.e. $\mathcal{V}(G_\delta^*) := \mathcal{F}(G_\delta)$). Since this duality transformation converts $\bar{\lambda}$-corners into λ-corners, we can use the previous calculation.

For (3.5.2), suppose c is a real corner. Take four copies of the previous result (3.5.1) (see Figure 3.5.1) around for each of the four middle corners $c \pm \frac{\delta \pm i\delta}{2}$. Each of them involve c and one of the four neighbouring real corners $c \pm (\delta \pm i\delta)$; summing the four equations with scalar factors so that the coefficients of $F(c \pm (\delta \pm i\delta))$ in each equation is 1, the result is straightforward. The case where c is imaginary is immediate from duality as above.

For (3.5.3), take $x \in \mathcal{F}[G_\delta]$ and note that the value at each of the neighbouring edges $x + i^n \frac{\lambda\delta}{\sqrt{2}}$ can be reconstructed from two of the four corners $x + \frac{i^n\delta}{2}$. Explicitly, inverting s-holomorphicity projections give

$$\begin{aligned}\cos\left(\frac{\pi}{4} + 2\Theta\right)\lambda i^{n+1} F\left(x + i^n \frac{\lambda\delta}{\sqrt{2}}\right) &= e^{i\Theta}\bar{F}\left(x + \frac{i^n}{2}\delta\right) - e^{-i\Theta}\bar{F}\left(x + \frac{i^{n+1}}{2}\delta\right)\\ &= -i^n\left[e^{i\Theta}F\left(x + \frac{i^n}{2}\delta\right) - e^{-i\Theta}iF\left(x + \frac{i^{n+1}}{2}\delta\right)\right].\end{aligned}$$

noting that $\bar{F}\left(x + \frac{i^n}{2}\delta\right) = -i^n F\left(x + \frac{i^n}{2}\delta\right)$.

So multiplying the two lines

$$\begin{aligned}&\cos^2\left(\frac{\pi}{4} + 2\Theta\right) i^{2n+2}\lambda^2 F\left(x + i^n \frac{\lambda\delta}{\sqrt{2}}\right)^2 =\\ &-i^n \cdot \left[e^{2i\Theta}\left|F\left(x + \frac{i^n}{2}\delta\right)\right|^2 + ie^{-2i\Theta}\left|F\left(x + \frac{i^{n+1}}{2}\delta\right)\right|^2\right.\\ &\left.-F\left(x + \frac{i^n}{2}\delta\right)\bar{F}\left(x + \frac{i^{n+1}}{2}\delta\right) - i\bar{F}\left(x + \frac{i^n}{2}\delta\right)F\left(x + \frac{i^{n+1}}{2}\delta\right)\right]\\ &= -i^n \cdot \left[e^{2i\Theta}\left|F\left(x + \frac{i^n}{2}\delta\right)\right|^2 + ie^{-2i\Theta}\left|F\left(x + \frac{i^{n+1}}{2}\delta\right)\right|^2\right.\\ &\left.+2i^{n+1}F\left(x + \frac{i^n}{2}\delta\right)F\left(x + \frac{i^{n+1}}{2}\delta\right)\right].\end{aligned}$$

So

$$
\begin{aligned}
\cos^2\left(\frac{\pi}{4}+2\Theta\right)\partial^{\delta}_{\bar z}F(x)^2 = \cos^2\left(\frac{\pi}{4}+2\Theta\right)\sum_{n=0}^{3} i^n\lambda F\left(x+i^n\frac{\lambda\delta}{\sqrt{2}}\right)^2 \\
=\lambda^{-1}\left(e^{2i\Theta}+ie^{-2i\Theta}\right)\sum_{n=0}^{3}\left|F\left(x+\frac{i^n}{2}\delta\right)\right|^2 \\
+2\lambda^{-1}\sum_{n=0}^{3} i^{n+1}F\left(x+\frac{i^n}{2}\delta\right)F\left(x+\frac{i^{n+1}}{2}\delta\right) \\
=2\cos\left(\frac{\pi}{4}-2\Theta\right)\sum_{n=0}^{3}\left|F\left(x+\frac{i^n}{2}\delta\right)\right|^2+2\lambda^{-1}\sum_{n=0}^{3} i^{n+1}F\left(x+\frac{i^n}{2}\delta\right)F\left(x+\frac{i^{n+1}}{2}\delta\right).
\end{aligned}
$$

Now reuse the first relation

$$
\begin{aligned}
\cos\left(\frac{\pi}{4}+2\Theta\right)\lambda^{-1}i^{-n}F\left(x+i^n\frac{\lambda\delta}{\sqrt{2}}\right) &= -i^{-2n}\left[e^{i\Theta}\bar F\left(x+\frac{i^n}{2}\delta\right)-e^{-i\Theta}\bar F\left(x+\frac{i^{n+1}}{2}\delta\right)\right] \\
\cos\left(\frac{\pi}{4}+2\Theta\right)\lambda i^n\bar F\left(x+i^n\frac{\lambda\delta}{\sqrt{2}}\right) &= (-1)^{n+1}\left[e^{-i\Theta}F\left(x+\frac{i^n}{2}\delta\right)-e^{i\Theta}F\left(x+\frac{i^{n+1}}{2}\delta\right)\right] \\
\cos\left(\frac{\pi}{4}+2\Theta\right)\partial_{\bar z}\bar F(x) &= \left(e^{i\Theta}+e^{-i\Theta}\right)\sum_{n=0}^{3}(-1)^{n+1}F\left(x+\frac{i^n\delta}{2}\right) \\
&= 2\cos\Theta\sum_{n=0}^{3}(-1)^{n+1}F\left(x+\frac{i^n\delta}{2}\right).
\end{aligned}
$$

Taking squares

$$
\begin{aligned}
\cos^2\left(\frac{\pi}{4}+2\Theta\right)\left|\partial_{\bar z}\bar F(x)\right|^2 = 4\cos^2\Theta\left[\sum_{n=0}^{3}\left|F\left(x+\frac{i^n\delta}{2}\right)\right|^2\right. \\
\left.+\mathrm{Re}\sum_{n\neq n'}(-1)^{n+1}F\left(x+\frac{i^n\delta}{2}\right)(-1)^{n'+1}\bar F\left(x+\frac{i^{n'}\delta}{2}\right)\right] \\
=4\cos^2\Theta\left[\sum_{n=0}^{3}\left|F\left(x+\frac{i^n\delta}{2}\right)\right|^2\right. \\
\left.-\mathrm{Re}\sum_{n\neq n'}(-1)^{n+n'}i^{n'}F\left(x+\frac{i^n\delta}{2}\right)F\left(x+\frac{i^{n'}\delta}{2}\right)\right],
\end{aligned}
$$

but $i^{n'}F\left(x+\frac{i^n\delta}{2}\right)F\left(x+\frac{i^{n'}\delta}{2}\right)\in i\mathbb{R}$ if $|n-n'|=2$. The remaining 8 combinations of n, n' all give rise to purely real terms, and resumming gives

$$\operatorname{Re}\sum_{n\neq n'}(-1)^{n+n'}i^{n'}F\left(x+\frac{i^n\delta}{2}\right)F\left(x+\frac{i^{n'}\delta}{2}\right)=-\sum_{n=0}^{3}\left(i^n+i^{n+1}\right)F\left(x+\frac{i^n\delta}{2}\right)F\left(x+\frac{i^{n+1}\delta}{2}\right)$$

$$\cos^2\left(\frac{\pi}{4}+2\Theta\right)\left|\partial_{\bar z}\bar F(x)\right|^2=4\cos^2\Theta\left[\sum_{n=0}^{3}\left|F\left(x+\frac{i^n\delta}{2}\right)\right|^2\right.$$
$$\left.+\sum_{n=0}^{3}\sqrt{2}i^{n+1}\lambda^{-1}F\left(x+\frac{i^n\delta}{2}\right)F\left(x+\frac{i^{n+1}\delta}{2}\right)\right].$$

Comparing the two expressions

$$\cos^2\left(\frac{\pi}{4}+2\Theta\right)\left[2\sqrt{2}\cos^2\Theta\cdot\partial_{\bar z}^{\delta}F(x)^2-\left|\partial_{\bar z}\bar F(x)\right|^2\right]$$
$$=4\left(\sqrt{2}\cos\left(\frac{\pi}{4}-2\Theta\right)-1\right)\sum_{n=0}^{3}\left|F\left(x+\frac{i^n}{2}\delta\right)\right|^2,$$

we have the full result given duality. □

Remark 3.5.2. (3.5.1) is equivalent to massive s-holomorphicity in the sense that if we have such values of F on $\mathscr{C}^{1,i}[G_\delta]$ then it is easy to see from the proof that we have enough data to extend the values s-holomorphically first to $\mathscr{E}[G_\delta]$ and then the $\lambda,\bar\lambda$-corners. In other words, bound on $\mathscr{C}^{1,i}$ is equivalent to a global bound in an s-holomorphic function. On $\mathscr{E}[G_\delta]$, (3.5.1) becomes

$$\partial_{\bar z}^{\delta}F(x):=\sum_{n=0}^{3}i^ne^{i\pi/4}F\left(x+i^ne^{i\pi/4}\frac{\delta}{\sqrt{2}}\right)$$
$$=\sin\Theta\sec\left(\frac{\pi}{4}+2\Theta\right)\sum_{n=0}^{3}\overline{F\left(x+i^ne^{i\pi/4}\frac{\delta}{\sqrt{2}}\right)},$$

i.e. a discretised version of $\partial_{\bar z}f=m\bar f$ given $\Theta\sim\frac{m\delta}{2}$.

Lemma 3.5.3. *Suppose $\Omega'\subset\Omega$ are smooth simply connected domains. Any function H_0 on $\mathscr{V}\left[(\Omega\setminus\Omega')_\delta\right]$ which is harmonic and takes the boundary value 0 on $\partial\mathscr{V}[\Omega_\delta]$ and 1 on $\partial\mathscr{V}[\Omega'_\delta]$ satisfies $0\leq H_0(a_{int})\leq C(\Omega,\Omega')\delta$ on any $a_{int}\in\mathscr{V}\left[(\Omega\setminus\Omega')_\delta\right]$ adjacent to $\partial\mathscr{V}[\Omega_\delta]$ for a constant $C(\Omega,\Omega')$.*

Proof. We believe this lemma is standard. One possible proof would proceed by mapping $\Omega\setminus\Omega'$ to the annulus $B_1\setminus B_{r_0}$ for some $r_0>0$ using a Riemann map which smoothly extends to $\overline{B_1\setminus B_{r_0}}$. The radial function $\frac{1-r}{1-r_0}$ on $B_1\setminus B_{r_0}$ is superharmonic, so its composition with the Riemann map is (continuous) superharmonic on $\Omega\setminus\Omega'$; the restriction to $\mathscr{V}\left[(\Omega\setminus\Omega')_\delta\right]$ is discrete superharmonic for small enough δ since the discrete laplacian (suitably renormalised) and continuous laplacian are uniformly close on smooth functions, and we can use it to upper bound H_0. □

We frequently have local L^2-bounds for our function F; it turns out that thanks to massive harmonicity, this is sufficient for equicontinuity.

We estimate massive harmonic functions by using the *massive random walk*, a simple random walk which is extinguished at each step with probability $\left(1+\frac{2\sin^2 2\Theta}{\cos(4\Theta)}\right)^{-1}\frac{2\sin^2 2\Theta}{\cos(4\Theta)}$. Recall that the massive harmonic measure $\mathrm{hm}_A^a(z|\Theta)$ for a discrete domain A, $z\in A$, and $a\in\partial A\cup A$ is the probability of a massive random walk started at z hitting a before $\partial A\setminus\{a\}$. It is the unique Θ-massive harmonic function on A which takes the boundary value 1 at a and 0 on $\partial A\setminus\{a\}$. In the scaling limit $\delta\downarrow 0$ and $\Theta\sim\frac{m\delta}{2}$, the massive random walk is extinguished after an exponential step of mean $\frac{1}{2m^2\delta^2}$. Taking into account the square-root scaling for the random walk, this corresponds to a distance of order $\sqrt{2}\delta\cdot\frac{1}{\sqrt{2}|m|\delta}=\frac{1}{|m|}$. For more precise asymptotics than below, we refer to [BdTR17].

Proposition 3.5.4. *There are constants $C, C', c>0$ such that, for a real massive harmonic function $F:\mathscr{C}^1\left[(B_R)_\delta\right]\to\mathbb{C}$ (where $\delta<\frac{R}{4}$) with $\Delta^\delta F=M_H^2(\Theta)F$,*

$$\left|F\left(-\frac{\delta}{2}\right)\right|\le Ce^{cmR}\sqrt{\frac{L}{R}}, \tag{3.5.4}$$
$$\delta^{-1}\left|F\left(\frac{\delta}{2}+i\delta\right)-F\left(-\frac{\delta}{2}\right)\right|\le C'e^{cmR}\sqrt{\frac{L}{R^3}},$$

where $L=\sum_{c\in\mathscr{C}^1[(B_R)_\delta]}|F(c)|^2\delta^2$.

Proof. For the first bound, note that $F^2\ge 0$ is subharmonic:

$$\Delta^\delta F^2(c)=\left(2M_H^2+\frac{M_H^4}{4}\right)F^2(c)+\frac{1}{2}\sum_{c\sim x,y}\left[F(x)-F(y)\right]^2. \tag{3.5.5}$$

So we can use the mean value property for harmonic functions: for $0<r<R$, write the discrete circle $S_r=\mathscr{C}^1\left[(B_R)_\delta\right]\cap(B_{r+4\delta}\setminus B_r)$

$$\left|F\left(-\frac{\delta}{2}\right)\right|^2\le\frac{cst}{r}\sum_{c\in S_r}|F(c)|^2\delta,$$

multiplying by δ and summing over the $O(\delta)$ discrete circles S_r such that their union equals $B_R\setminus B_{R/2}$

$$\left|F\left(-\frac{\delta}{2}\right)\right|^2\le\frac{cst}{R}\sum_{c\in B_R\setminus B_{R/2}}\left|F\left(-\frac{\delta}{2}\right)\right|^2\delta^2\le\frac{cst\cdot L}{R}. \tag{3.5.6}$$

For the desired bounds, note that by first applying (3.5.6) to smaller balls of radii $R/2$ we can opt for a bound of the form

$$\left|F\left(-\frac{\delta}{2}\right)\right| \leq cst \cdot e^{cmR} \max_{B_{R/2}} |F|, \tag{3.5.7}$$
$$\delta^{-1}\left|F\left(\frac{\delta}{2}+i\delta\right)-F\left(-\frac{\delta}{2}\right)\right| \leq cst \cdot e^{cmR} \frac{\max_{B_{R/2}} |F|}{R/2}.$$

Consider the first estimate. By the maximum and minimum principles, we may bound

$$-\max_{B_{R/2}} |F| \operatorname{hm}_{B_{R/2}}^{S_{R/2}} \leq F \leq \max_{B_{R/2}} |F| \operatorname{hm}_{B_{R/2}}^{S_{R/2}}.$$

The massive harmonic measure $\operatorname{hm}_{B_{R/2}}^{S_{R/2}}(c)$ is the hitting probability of $S_{R/2}$ of the massive random walk started at c. For the bound at $-\frac{\delta}{2}$, simply note that the probability of a massive random walk reaching a box at distance d decays exponentially fast in $|m|\,d$. (see e.g. the projection argument in the proof of Proposition 3.3.4).

For the second, by decomposing $F(c) = \sum_{c' \in S_{R/2}} \operatorname{hm}_{B_{R/2}}^{\{c'\}}(c) F(c')$ for $c \in B_{R/2}$, with $\operatorname{hm}_{B_{R/2}}^{\{c'\}} = \operatorname{hm}_{B_{R/2}}^{\{c'\}}(\cdot|m)$ being the massive harmonic function on $B_{R/2}$ whose boundary value is 0 on $S_{R/2} \setminus \{c'\}$ and 1 at $c' \in S_{R/2}$, it suffices to show $\left|\operatorname{hm}_{B_{R/2}}^{\{c'\}}(\frac{\delta}{2}+i\delta) - \operatorname{hm}_{B_{R/2}}^{\{c'\}}(-\frac{\delta}{2})\right| \leq cst \cdot \frac{\delta^2 e^{cmR}}{R^2}$. We know that the hitting probability for the simple random walk (i.e. the harmonic measure of the point c', with $m=0$) satisfies the desired estimate (e.g. [ChSm11, Proposition 2.7]): as $\delta \to 0$, the difference of the probabilities P_1, P_2 of simple random walk started at neighbouring points near 0 reaching c' before other points in $S_{R/2}$ is bounded above by $cst \cdot P_1 \cdot \frac{\delta}{R} \leq cst \cdot \frac{\delta^2}{R^2}$. For the massive random walk, these instances (say, coupled with the same exponential clock) need to survive to contribute to the difference; therefore the difference decays by an additional exponential factor. □

Remark 3.5.5. The second bound in (3.5.4) is valid for differences in other directions as well, since massive harmonicity and the bound are rotationally invariant. By considering smaller balls within B_R, we in fact deduce uniform bounds for F and its discrete derivative in, say, $B_{R/2}$. Then, defining $D_\lambda^\delta F(c) := F(c+\delta+i\delta) - F(c), D_{\bar\lambda}^\delta F(c) := F(c+\delta-i\delta) - F(c)$, which are massive harmonic functions uniformly bounded in $B_{R/2}$, and using the bound (3.5.7) on them, we actually have bound on discrete derivatives of second order in, say, $B_{R/4}$. Recursively, we see that derivatives of any order can be locally bounded.

4 Conclusion

We conclude by reviewing contributions made in this thesis and outlining potential future directions of development.

4.1 Lattice Local Fields and Spin Patterns

Correlations of the energy density $\epsilon_{z^\delta} := \frac{\sqrt{2}}{2} - \sigma_{z^\delta}\sigma_{z^\delta+\delta}$ and the spin σ_{z^δ} converge to those predicted by Conformal Field Theory. These lattice fields correspond to two so-called nonconstant *primary* local fields of the Ising CFT [DMS97], obeying precise predictions verified by [HoSm13, CHI15].

However, in the discrete model, there are clearly more observable local quantities, i.e. lattice local fields: for example, the product of three spins in a row. How do these *lattice local fields* fit into the CFT correspondence? The second chapter of this thesis provides a convergence result that may serve as a partial answer to this question. Concretely, it proves that the quantities in a bounded simply connected domain Ω

$$\mathbb{E}^+_{\beta_c,\Omega_\delta}\left[\sigma_{\delta z_1}\cdots\sigma_{\delta z_{2n}}\right], \text{ and } \frac{\mathbb{E}^+_{\beta_c,\Omega_\delta}\left[\sigma_0\sigma_{\delta z_1}\cdots\sigma_{\delta z_{2n}}\right]}{\mathbb{E}^+_{\beta_c,\Omega_\delta}\left[\sigma_0\right]},$$

where $0, z_1, \ldots, z_{2n} \in \Omega_{\delta=1}$, converge in the scaling limit at first order to explicit quantities independent of Ω, while the second order $O(\delta)$ correction depends explicitly on the conformal geometry of Ω. Such even- and odd-correlations form a basis for any local function of the spin configuration σ. Convergence for the correlations in fact implies a similar convergence for the probability of any specific *spin pattern* $\rho \in \{\pm 1\}^B$ for some finite set of sites B. Here the scaling of $1/8$ amounts to breaking the $\sigma \mapsto -\sigma$ symmetry, so with explicit C_P, C'_P we have

$$\mathbb{P}^{+}_{\beta_c,\Omega_\delta}\left[\sigma|_{\delta B} = \pm\rho\right] = \mathbb{P}^{+}_{\beta_c,\mathbb{Z}^2}\left[\sigma|_{B} = \pm\rho\right] + C_P(A,\rho,\Omega)\delta + O(\delta),$$
$$\mathbb{P}^{+}_{\beta_c,\Omega_\delta}\left[\sigma|_{\delta B} = \rho\right] = \mathbb{P}^{+}_{\beta_c,\mathbb{Z}^2}\left[\sigma|_{B} = \rho\right] + C'_P(A,\rho,\Omega)\delta^{1/8} + O(\delta^{1/8}).$$

The coefficients C_P, C'_P are conformally covariant with dimensions $1, \frac{1}{8}$, and they could in fact be considered as the one-point functions of the corresponding (truncated then renormalised) fields. The results of this chapter suggest that the product of an even number of local spins scales like the energy and the product of an odd number of local spins scales like the spin, at least at the one-point correlation level. Given the representation of the Virasoro algebra at the lattice level [HKV17], it seems natural to expect that all the CFT local fields can be realised in terms of lattice local fields by applying the discrete Virasoro generators to the spin-products treated by our result. This suggests the following conjecture (page 20): any nonzero lattice local field, suitably renormalised, converge to a continuous field in the Ising CFT, and all continuous fields are obtained this way.

In addition, the results also highlight the richness of the discrete theory. The explicit quantities in the full plane $\mathbb{Z}^2$ and the correction terms are not found in the continuous CFT, but are obtained through analysis of the discrete fermion. A simple consequence of our analysis is a scheme to explicitly calculate any finite dimensional distribution in the Ising measure in $\mathbb{Z}^2$.

A crucial ingredient in enabling such calculation is the explicit identification of the *discrete harmonic measure of the tip of the slit plane*, which gives the hitting measure of the point 0 by a simple random walk on the lattice $\delta(1+i)\mathbb{Z}^2 \setminus \mathbb{R}_{\leq 0}$. Beyond its random walk interpretation, it has yielded an explicit formula for a normalisation factor implicitly used in the proof of the spin correlation result of [CHI15]. Furthermore, this result is important in the construction of lattice representations in [HKV17].

The harmonic measure in turn can be used to construct discrete versions of the inverse complex square root and square root functions. Although the discrete square root involves a definition in terms of an infinite sum, we have succeeded in developing an algorithm based on s-holomorphicity and rotational symmetry to compute it exactly using recursion. What makes the subsequent use of the discrete square root in this chapter original is that it identifies *two* discrete counterparts of the function $i\sqrt{z}$ (see also [Dub15]); they must be balanced in a very nontrivial manner to yield the precise convergence result obtained.

4.2 Massive Limit in Bounded Domains

Given the full-plane results on massive scaling limit and the emergence of CFT in the critical scaling limit in general domains, a natural question of a massive scaling limit in general domains Ω arises. Does spin correlation converge, and if it does how is it affected by the geometry of Ω? In the third chapter of this thesis, we answer this question in the case of a *subcritical* massive scaling limit: scale $\beta(\delta) = \beta_c - \frac{m}{2}\delta$ with a constant $m < 0$, then

$$< z_1, \ldots, z_n >_{m,\Omega} := \lim_{\delta \to 0} \delta^{-n/8} \mathbb{E}^{+}_{\beta(\delta),\Omega_\delta} [\sigma_{z_1^\delta}, \ldots, \sigma_{z_n^\delta}],$$

exists in the case where Ω is a smooth bounded simply connected domain. This scaling function depends on the shape of Ω in a much more complex manner than in the critical case, where it depends on only a finite number of values of the conformal map ϕ and its derivative. This is indeed expected, since the loss of scale invariance suggests that even a mere dilatation of the domain would have a highly nontrivial effect on the fields.

As noted in the introduction, the methods that have yielded the convergence results in the full plane are not readily generalisable to a general domain Ω. Our proof instead uses s-holomorphic analysis of the fermionic observable. While the fact that the discrete fermionic observables may also be defined in the massive setup and that they satisfy a perturbed notion of s-holomorpicity has been known and even used to yield significant results, integrability of the square or its properties have not been investigated before.

Hence, the analysis required a generalisation of the full s-holomorphic analysis of the critical case into the massive setup. The approximate harmonicity of the integral of the square does not generalise, and precompactness arguments based on harmonicity need to be replaced by arguments based on an L^2-bound; identification by the appropriate boundary value problem in the continuum becomes highly nontrivial, lacking the conformal invariance of the Laplace equation. Furthermore, any continuous limit f satisfies a perturbed notion $\partial_{\bar{z}} f = m\bar{f}$ of massive holomorphicity named *generalised analyticity* [Ber56, Vek62], whose theory is in many ways parallel to but strictly weaker than the theory of holomorphic functions. A crucial observation used in the proof relies on exploiting the boundary condition $\nu_{out}^{-1/2}$ in a new way both to control the discrete function and uniquely identify the continuous limit.

The fermion being a central object in both the discrete and continuous Ising CFT, casting the massive theory in the fermionic language hints at a massive perturbation of CFT. Indeed, in the terminology of the previous chapter, all the lattice local fields endure in the form of local functions of the spin; it is natural to conjecture that they converge to continuous objects, forming perturbed versions of CFT fields. Cardy and Mussardo [CaMu90] in turn gave arguments on the existence of a perturbation of the space of CFT fields in the massive continuous theory. However, it is as of yet unclear whether the existing results in terms of isomonodromic analysis may be connected to CFT methods. As a step towards a more substantial investigation, the chapter contains an independent justification of the full-plane convergence and the emergence of the Painlévé transcendent in the two-point case. While the continuum analysis is equivalent, it derives the result solely using massive s-holomorphic fermions and is readily accessible to readers familiar with the s-holomorphic analysis of the model.

4.3 Further Developments

In this thesis, we have tried to further examine and contribute to the understanding of the correspondence between the Ising model and its continuous Conformal Field Theory beyond the theory of critical scaling limits of the spin and the energy density. There are many directions one may pursue in the theme based on the results of this thesis. The question most relevant to this thesis would be a mixture of the questions considered in the two chapters: what is the one-point behaviour of lattice local fields in the massive limit? Then we may go further, to develop a general theory of correlations between lattice local fields centred at distinct points $z_1,\ldots,z_n$, both in the critical and the massive Ising model. One case that is both simple and foundational is that of energy density correlation; a massive generalisation of Hongler's work [Hon10] should be readily attacked given the ideas presented in this thesis.

Another interesting line of inquiry concerns the limits of the $+/-$ interfaces in the massive regime, given results [MaSm10, ChWa19, NoWe09] on the massive loop erased random walk (mLERW) and massive percolation. The primary obstacle here is the lack of explicitness. In the critical case, limits of the interfaces may be identified in terms of holomorphic fermions which serve as *martingale observables* of the growing curves, and convergence in part consists of showing that the discrete martingale observable converges to the explicit continuous formula which is conformally covariant. The interfaces in the massive model are much less tractable in explicit terms without conformal symmetry. Preliminary analysis based on ideas and techniques established in this thesis suggests that even a continuous heuristic on the limiting interface is a nontrivial question, since generalised analytic functions are not explicit in general. Discrete a priori information on the regularity of the interface might be helpful; some are already available in terms of crossing probability bounds [DGP14]. In contrast, the discrete analysis (say, the precompactness of the discrete observables) needs to be modified and developed in specifics but the general strategies of this thesis should still have relevance.

Considering the manifestation of universality highlights many intriguing directions of generalisation. In this regard, the fact that lattice local fields beyond the spin and the energy density are inherently tied to the shape of the underlying lattice is interesting. The general framework of s-holomorphicity is applicable in general isoradial lattices, but there are some arguments specific to the square lattice which need to be generalised: construction of the discrete square

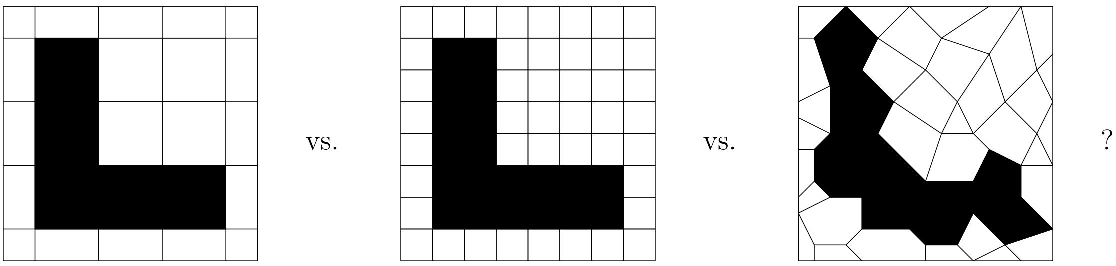

Figure 4.3.1 – Comparable lattice local fields on distinct lattices.

root is one. Assuming a convergence result, a naïve yet reasonable theme of inquiry would be comparisons between lattice local fields which have comparable 'shape' overall but defined on distinct lattices (see Figure 4.3.1). Moreover, ideas of s-holomorphic analysis have been generalised to even broader classes, to *circle patterns* [Lis17] and *s-embeddings* [Che18]. In particular, s-embedding sheds a light on the connection between critical Ising weights and the underlying abstract topological graph; how the massive setup would fit into this picture is a very interesting question analytically and geometrically.

Perhaps the most natural question given the results of the third chapter is the analysis of the supercritical case, where $m > 0$ and $\beta(\delta) < \beta_c$. Many of the inequalities that worked for our advantage in the subcritical case switch direction, and we do not even expect to have a well-defined spin-fermion correlation in the full plane. Preliminarily, we note that the integral of the square may still be defined, and that it satisfies a maximum/minimum principle.

In fact, the integral of the square is quite a robust construction. It may still be defined in the case where we have varying temperature β on different edges. The fact that the mass m is a constant was very useful in our analysis, since massive holomorphic functions become discrete *massive harmonic* functions; they may be analysed in terms of massive simple random walk, i.e. a random walk which is extinguished at each step with a given probability. However, there is a good reason to allow the length scale ξ to vary spatially. In particular, a simple change-of-variables argument shows that treating log-harmonic mass m is tantamount to treating the constant mass case in all simply connected domains. In such a setup, massive holomorphicity becomes an approximate discrete relation, and the equation $\Delta f = m^2 f$ is replaced by a general discrete elliptic equation.

Bibliography

[ADS15] M. Aizenman, H. Duminil-Copin, V. Sidoravicius, *Random Currents and Continuity of Ising Model's Spontaneous Magnetization*. Comm. Math. Phys. 334(2):719-742, 2015.

[BBC16] L. Baratchart, A. Borichev, S. Chaabi, *Pseudo-holomorphic functions at the critical exponent*. J. Eur. Math. Soc. 18(9):1919–1960, 2016.

[Bax89] R. Baxter, *Exactly solved models in statistical mechanics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989.

[BeDC12] V. Beffara, H. Duminil-Copin, *Smirnov's Fermionic Observable away from Criticality*, Ann. Probab., 40(6):2667–2689, 2012.

[BPZ84] A. A. Belavin, A. M. Polyakov, A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*. Nucl. Phys. B 333-380, 1984.

[BeHo16] S. Benoist, C. Hongler, *The scaling limit of critical Ising interfaces is CLE(3)*, arXiv:1604.06975, 2016.

[BeHo18] S. Benoist, C.Hongler, in preparation.

[Ber56] L. Bers, *An outline of the theory of pseudoanalytic functions*. Bull. Amer. Math. Soc. 62(4):291-331, 1956.

[Bo07] C. Boutillier, *Pattern densities in non-frozen dimer models*. Comm. Math. Phys. 271(1):55–91, 2007.

[BodT09] C. Boutillier, B. de Tilière, *The critical Z-invariant Ising model via dimers: locality property*, Comm. Math. Phys. 301(2):473-516, 2011.

[BodT08] C. Boutillier, B. de Tilière, *The critical Z-invariant Ising model via dimers: the periodic case*, Prob. Theory and Related Fields. 147(3):379-413, 2010.

[BdTR17] C. Boutillier, B. de Tilière, K. Raschel, *The Z-invariant massive Laplacian on isoradial graphs*, Invent. Math. 208(1):109-189, 2017.

[BuGu93] T. Burkhardt, I. Guim, *Conformal theory of the two-dimensional Ising model with homogeneous boundary conditions and with disordered boundary fields*, Phys. Rev. B (1), 47:14306-14311, 1993.

[CGN15] F. Camia, C. Garban, C. Newman, *Planar Ising magnetization field I. Uniqueness of the critical scaling limit.* Ann. Probab., 42(2):528-571, 2015.

[CGN16] F. Camia, C. Garban, C. Newman, *Planar Ising magnetization field II. Properties of the critical and near-critical scaling limits.* Ann. Inst. H. Poincaré Probab. Statist., 52(1):146-161, 2016.

[Car84] J. Cardy, *Conformal invariance and Surface Critical Behavior*, Nucl. Phys. B 240:514-532, 1984.

[CaMu90] J. Cardy and G. Mussardo, *Form factors of descendents operator in perturbed Conformal Field Theories.* Nucl. Phys. B 340:387–402, 1990.

[Che16] D. Chelkak, *Robust discrete complex analysis: A toolbox.* Ann. Probab. 44(1):628-683, 2016.

[Che18] D. Chelkak, *Planar Ising model at criticality: state-of-the-art and perspectives*, Proceedings of ICM2018, 2018.

[CCK17] D. Chelkak, D. Cimasoni, A. Kassel, *Revisiting the combinatorics of the 2D Ising model*, arXiv:1507.08242, 2017.

[CDHKS14] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, S. Smirnov, *Convergence of Ising Interfaces to Schramm's SLE Curves*, C. R. Math. Acad. Sci. Paris, 352(2):157-161, 2014.

[CGS17] D. Chelkak, A. Glazman, S. Smirnov, *Discrete stress-energy tensor in the loop O(n) model.* arXiv:1604.06339, 2017.

[ChHo16] D. Chelkak, C. Hongler, in preparation.

[CHI15] D. Chelkak, C. Hongler, K. Izyurov, *Conformal invariance of spin correlations in the planar Ising model*, Ann. Math., 181(3):1087–1138, 2015.

[CHI18] D. Chelkak, C. Hongler, K. Izyurov, in preparation.

[ChIz13] D. Chelkak, K. Izyurov, *Holomorphic Spinor Observables in the Critical Ising Model.* Comm. Math. Phys. 322(2):302-303, 2013.

[ChSm11] D. Chelkak, S. Smirnov, *Discrete complex analysis on isoradial graphs.* Adv. Math., 228:1590–1630, 2011.

[ChSm12] D. Chelkak, S. Smirnov, *Universality in the 2D Ising model and conformal invariance of fermionic observables.* Invent. Math., 189(3):515-580, 2012.

[ChWa19] D. Chelkak, Y. Wan, *On the convergence of massive loop-erased random walks to massive SLE(2) curves.* arXiv:1903.08045, 2019.

[CFL28] R. Courant, K. Friedrichs, H. Lewy, *Uber die partiellen Differenzengleichungen der mathematischen Physik*. Math. Ann., 100:32-74, 1928.

[dT18] B. de Tilière, *The Z-Dirac and massive Laplacian operators in the Z-invariant Ising model*, preprint.

[DIK13] P. Deift, A. Its, I. Krasovsky, *Toeplitz Matrices and Toeplitz Determinants under the Impetus of the Ising Model: Some History and Some Recent Results*. Comm. Pure Appl. Math., 66(9):1360-1438, 2013.

[DMS97] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Graduate texts in contemporary physics. Springer-Verlag New York, 1997.

[DHN11] H. Duminil-Copin, C. Hongler, P. Nolin, *Connection probabilities and RSW-type bounds for the FK Ising model*, Comm. Pure Appl. Math., 64(9):1165–1198, 2011.

[DGHMT16] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, V. Tassion, *Discontinuity of the phase transition for the planar random-cluster and Potts models with q>4*. arXiv:1611.09877, 2016.

[DGP14] H. Duminil-Copin, C. Garban and G. Pete, *The near-critical planar FK-Ising model*. Comm. in Math. Phys. 326(1):1–35, 2014.

[DLMF] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, eds., *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.20 , 2018.

[Dub11] J. Dubédat, *Exact Bosonization of the Ising Model*. arXiv:1112.4399, 2011.

[Dub15] J. Dubédat, *Dimers and families of Cauchy-Riemann operators I*. J. Amer. Math. Soc., 28:1063-1167, 2015.

[Fis66] M. Fisher, *On the Dimer Solution of Planar Ising Models*. J. Math. Phys., 7:1776-1781, 1966.

[FIKN06] A. Fokas, A. Its, A. Kapaev, V. Novokshenov, *Painleve Transcendents: The Riemann-Hilbert Approach*. American Mathematical Society, 2006.

[FrVe17] S. Friedli, Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge University Press, Cambridge, 2017.

[GaMa08] J. Garnett, D. Marshall, *Harmonic Measure*. Cambridge University Press, Cambridge, 2008.

[GHP19] R. Gheissari, C. Hongler and S.C. Park, *Ising Model: Local Spin Correlations and Conformal Invariance*, Commun. Math. Phys. 367(3):771-833, 2019.

[GiTr15] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Springer, 2015.

[Gri06] G. Grimmett, *The Random-Cluster Model*. Volume 333 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2006.

[Hen99] M. Henkel, *Conformal Invariance and Critical Phenomena*. Springer-Verlag, Berlin, 1999.

[Hon10] C. Hongler, *Conformal Invariance of Ising model Correlations,* Ph.D. Thesis, 2010.

[HoSm13] C. Hongler, S. Smirnov, *The energy density in the critical planar Ising model*, Acta Math., 211(2):191-225, 2013.

[HoKy10] C. Hongler, K. Kytölä, *Ising Interfaces and Free Boundary Conditions*, J. Amer. Math. Soc., 26:1107– 1189, 2013.

[HKV17] C. Hongler, K. Kytölä, F. Viklund, *Conformal Field Theory at the Lattice Level: Discrete Complex Analysis and Virasoro Structure*, arXiv:1307.4104, 2017.

[HKZ15] C. Hongler, K. Kytölä, A. Zahabi, *Discrete holomorphicity and Ising model operator formalism*, Contemporary Mathematics, 644(X):79-115, 2015.

[Isi25] E. Ising, *Beitrag zur Theorie des Ferromagnetismus*. Zeitschrift für Physik, 31:253-258, 1925.

[Izy17] K. Izyurov, *Critical Ising interfaces in multiply-connected domains*. Probab. Theory Relat. Fields, 167(1-2):379-415, 2017.

[KaCe71] L. Kadanoff, H. Ceva, *Determination of an operator algebra for the two-dimensional Ising model*. Phys. Rev. B (3), 3:3918-3939, 1971.

[KaKo80] L. Kadanoff, M. Kohmoto, *SMJ's analysis of Ising model correlation functions*, Ann. Phys. 126:371-398, 1980.

[Kau49] B. Kaufman, *Crystal statistics. II. Partition function evaluated by spinor analysis*. Phys. Rev., II. Ser., 76:1232-1243, 1949.

[KaOn49] B. Kaufman, L. Onsager, *Crystal statistics. III. Short-range order in a binary Ising lattice*. Phys. Rev., II. Ser., 76:1244-1252, 1949.

[KaOn50] B. Kaufman, L. Onsager, *Crystal statistics. IV. Long-range order in a binary crystal*. Unpublished typescript, 1950.

[Ken00] R. Kenyon, *Conformal invariance of domino tiling*. Ann. Probab., 28:759-795, 2000.

[Kes87] H. Kesten, *Hitting probabilities of random walks on* $\mathbb{Z}^d$, Stochastic Processes Appl. 25:165-184, 1987.

[KrWa41] H. A. Kramers and G. H. Wannier. *Statistics of the two-dimensional ferromagnet. I.* Phys. Rev. (2), 60:252-262, 1941.

[LaLi04] G. F. Lawler, V. Limic, *The Beurling Estimate for a Class of Random Walks*. Electron. J. Probab. 9:846-861, 2004.

[Lel55] J. Lelong-Ferrand. *Représentation conforme et transformations à intégrale de Dirichlet bornée*. Gauthier-Villars, Paris, 1955.

[Len20] W. Lenz, *Beitrag zum Verständnis der magnetischen Eigenschaften in festen Körpern*. Phys. Zeitschr., 21:613-615, 1920.

[Lis17] M. Lis, *Circle patterns and critical Ising models*. arXiv:1712.08736, 2017.

[LuSl12] E. Lubetzky, A. Sly, *Critical Ising on the square lattice mixes in polynomial time*. Comm. Math. Phys. 313:815-836, 2012.

[MaSm10] N. Makarov, S. Smirnov, *Off-critical lattice models and massive SLEs*. Exner, Pavel(ed.) XVIth International Congress on Mathematical Physics, Prague, 3-8 August 2009. 362-371, Singapore: World Sci. Publ., 2010.

[Mer01] C. Mercat, *Discrete Riemann surfaces and the Ising model*. Comm. Math. Phys. 218:177-216, 2001.

[McWu73] B. M. McCoy, and T. T. Wu, *The two-dimensional Ising model*. Harvard University Press, Cambridge, Massachusetts, 1973.

[Mus10] G. Mussardo, *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics*. Oxford University Press, Oxford, 2010.

[NoWe09] P. Nolin, W. Werner, *Asymmetry of near-critical percolation interfaces*. J. Amer. Math. Soc. 22:797819, 2009.

[Ons44] L. Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition*, Phys Rev (2), 65:117-149, 1944.

[Pal07] J. Palmer, *Planar Ising correlations*. Birkhäuser, 2007.

[PaTr83] J. Palmer and C. A. Tracy, *Two-Dimensional Ising Correlations: The SMJ Analysis*. Adv. in Appl. Math. 4:46–102, 1983.

[Pei36] R. Peierls, *On Ising's model of ferromagnetism*. Math. Proc. Camb. Phil. Soc. 32:477–481, 1936.

[Sch00] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*. Israel J. Math., 118:221–288, 2000.

[Seg88] G. Segal, *The definition of conformal field theory*. Differential geometrical methods in theoretical physics, 250:165–171, 1988

[SMJ77] M. Sato, T. Miwa, M. Jimbo, *Studies on holonomic quantum fields, I-IV*. Proc. Japan Acad. Ser. A Math. Sci., 53(1):6–10, 53(1):147-152, 53(1):153-158, 53(1):183-185, 1977.

[SML64] T. Schultz, D. Mattis and E. Lieb, *Two-dimensional Ising model as a soluble problem of many fermions*. Rev. Mod. Phys., 36:856–871, 1964.

[Smi06] S. Smirnov, *Towards conformal invariance of 2D lattice models*. Sanz-Solé, Marta (ed.) et al., Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume II: Invited lectures, 1421-1451. Zürich: European Mathematical Society (EMS), 2006.

[Smi10] S. Smirnov, *Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model*. Ann. Math., 172(2):1435-1467, 2007.

[Vek62] I. N. Vekua, *Generalized Analytic Functions*, Pergamon Press, 1962.

[Wen92] G. Wen, *Conformal Mappings and Boundary Value Problems*. American Mathematical Society, Providence, RI, 1992.

[WMTB76] T. Wu, B. McCoy, C. Tracy, E. Barouch, *Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region*. Phys. Rev. B, 13:316-374, 1976.

[Yan52] C. N. Yang, *The spontaneous magnetization of a two-dimensional Ising model*. Phys. Rev. (2), 85:808-816, 1952.

[Zam87] A.B. Zamolodchikov, *Higher-order integrals of motion in two-dimensional models of the field theory with a broken conformal symmetry*, JETP Lett., 46(4):160-164, 1987

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