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The fine Tate–Shafarevich group

BY CHRISTIAN WUTHRICH

Séction de mathématiques, CSAG, École Polytechnique Fédérale,
1015 *Lausanne, Switzerland.*

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Abstract

Within the Tate–Shafarevich group of an elliptic curve E defined over a number field K, there is a canonical subgroup defined by imposing stronger conditions at the places above a given prime p. This group appears naturally in the Iwasawa theory for E. We propose a study of what one can say about the relation to the full Tate–Shafarevich group. Some numerical examples are included, as well as a few conjectures.

1. *Introduction*

Let E be an elliptic curve defined over a number field K. We fix a prime number p. Everything will depend on p, but we will omit it from our notations. For an abelian group A, the p-primary part of A will be denoted by $A(p)$. The p-primary part of the Tate–Shafarevich group $Ш(E/K)(p)$ is defined to be the kernel of the localisation map

$$0 \longrightarrow Ш(E/K)(p) \longrightarrow H^1(K, E)(p) \longrightarrow \oplus_v H^1(K_v, E)(p).$$

We will define in Section 2 a certain canonical subgroup of $Ш(E/K)(p)$ by imposing stronger conditions at the places above p. It will be called the fine Tate–Shafarevich group, denoted by another Russian letter $Ж(E/K)$. It is actually defined as the union of a sequence of subgroups $Ж^k(E/K)$ in $Ш(E/K)[p^k]$ and it represents the "Tate–Shafarevich part" of the fine Selmer group $\mathcal{R}(E/K)$ as defined in [**CS05**]. See Section 2 for the details.

The motivation for studying this group comes from the Iwasawa theory of E. Its behaviour is closely linked to Euler systems for the Tate-module $T_pE = \varprojlim E[p^k]$, like the one by Kato [**Kat04**]. Suppose E is defined over $\mathbb{Q}$. If the L-series of E does not vanish at 1, then Kato's Euler system $\mathbf{c}$ is such that $\mathbf{c}_{\mathbb{Q}} \in H^1(\mathbb{Q}, T_pE)$ is not torsion. By [**Rub00**, theorem 2·2·3], the fine Selmer group $\mathcal{R}(E/\mathbb{Q})$, which is then equal to $Ж(E/\mathbb{Q})$, is

finite. Moreover the divisibility of $\mathbf{c}_{\mathbb{Q}}$ by p may be used to give an upper bound on the size of $Ш(E/\mathbb{Q})$.

For more details on the Iwasawa theory of the fine Selmer group, we refer the reader to [**Wut04**], [**Wut05**] and [**PR95**].

This paper focuses on the analysis of the fine Tate–Shafarevich group over the number field K. The first part concerns its relation to the full Tate–Shafarevich group. For $K = \mathbb{Q}$, we prove in Theorem 3·4 that $Ш(E/\mathbb{Q})$ is non-trivial exactly if $Ш(E/\mathbb{Q})$ is non-trivial, and that they are equal if the rank of the curve is positive (see Theorem 3·5). Numerical examples are given in Section 4 for curves of rank 0 and $p = 2$. We find three examples of curves E defined over $\mathbb{Q}$ with $Ш(E/\mathbb{Q})(2) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ which show that all possible orders (namely 1, 2 or 4) can occur for $Ш^1(E/\mathbb{Q})$. The fine Tate–Shafarevich groups $Ш(E/\mathbb{Q})$ have order 2, for the first two curves, and 4 for the last example.

We present a version of the Cassels–Tate pairing due to Flach [**Fla90**] for our subgroup $Ш(E/K)$ in Section 6. Then the fine version of the Mordell–Weil group $E(K)$, also obtained by imposing conditions at the places above p, is considered. We end the article with some conjectures on the fine Tate–Shafarevich group $Ш(E/K)$. In contrast to the situation over $\mathbb{Q}$, it is not true for larger fields that the fine subgroup of the Tate–Shafarevich group is often relatively big. In fact, for $\mathbb{Z}_p$-extensions it is conjectured that the growth of the fine Tate–Shafarevich group is significantly smaller.

2. *Definitions*

For any $k \geqslant 1$, we have a short exact sequence coming from the p^k-descent

$$0 \longrightarrow E(K)/p^k E(K) \xrightarrow{\kappa} S^k(E/K) \longrightarrow Ш(E/K)[p^k] \longrightarrow 0 \qquad (2\cdot1)$$

where $S^k(E/K)$ is the Selmer group defined as the kernel of the map

$$H^1(K, E[p^k]) \longrightarrow \bigoplus_{\upsilon} H^1(K_\upsilon, E)[p^k]$$

with the sum running over all places υ in K. This Selmer group contains a subgroup $R^k(E/K)$, which we will call the *fine Selmer group*, obtained by imposing stronger conditions at the places above p, i.e. the sequence

$$0 \longrightarrow R^k(E/K) \longrightarrow S^k(E/K) \longrightarrow \bigoplus_{\upsilon|p} H^1(K_\upsilon, E[p^k])$$

is exact. The subgroup $R^k(E/K)$ is sometimes called the "strict" or "restricted" Selmer group, but in order to avoid possible confusions with different definitions, we prefer the terminology in [**CS05**]. Similarly we can define a fine subgroup of the Mordell–Weil group. Namely the following kernel

$$0 \longrightarrow M^k(E/K) \longrightarrow E(K)/p^k E(K) \longrightarrow \bigoplus_{\upsilon|p} E(K_\upsilon)/p^k E(K_\upsilon), \qquad (2\cdot2)$$

which can also be written as the intersection of $R^k(E/K)$ with the image of the Kummer map κ in (2·1) inside $S^k(E/K)$.

Now, we can proceed to define the group $Ш^k(E/K)$ as the quotient of $R^k(E/K)$ by $M^k(E/K)$. We call it the *fine Tate–Shafarevich group*. So we have an exact sequence

$$0 \longrightarrow M^k(E/K) \xrightarrow{\kappa} R^k(E/K) \longrightarrow Ш^k(E/K) \longrightarrow 0 \qquad (2\cdot3)$$

similar to the Kummer sequence (2·1). In order to prove that it is really a subgroup of the Tate–Shafarevich group, we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(K)/p^kE(K) & \longrightarrow & S^k(E/K) & \longrightarrow & Ш(E/K)[p^k] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow 0 & & \\ 0 & \longrightarrow & \oplus E(K_\upsilon)/p^kE(K_\upsilon) & \longrightarrow & \oplus H^1(K_\upsilon, E[p^k]) & \longrightarrow & \oplus H^1(K_\upsilon, E)[p^k] & \longrightarrow & 0 \end{array}$$

(the $\oplus$ stands for the sum over all places υ above p). We can extract an exact sequence

$$0 \longrightarrow M^k(E/K) \longrightarrow R^k(E/K) \longrightarrow Ш(E/K)[p^k] \xrightarrow{\delta^k} C^k \tag{2·4}$$

where C^k is the cokernel of the left vertical map in the above diagram. As a consequence, one can view $Ш^k(E/K)$ as a subgroup of $Ш(E/K)[p^k]$ with quotient inside C^k.

Limits

As usual, it is interesting to consider the limits as k tends to infinity. We write $\mathcal{S}(E/K)$ for the inductive limit $\varinjlim S^k(E/K)$ obtained from the inclusion $E[p^k] \hookrightarrow E[p^{k+1}]$. We denote the projective limit $\varprojlim S^k(E/K)$ induced from the map $[p]\colon E[p^{k+1}] \twoheadrightarrow E[p^k]$ by $\mathfrak{S}(E/K)$. The same limits can be used for the subgroups $R^k(E/K)$ and $M^k(E/K)$.

$$\begin{aligned} \mathcal{S}(E/K) &= \varinjlim S^k(E/K) & \mathfrak{S}(E/K) &= \varprojlim S^k(E/K) \\ \mathcal{R}(E/K) &= \varinjlim R^k(E/K) & \mathfrak{R}(E/K) &= \varprojlim R^k(E/K) \\ \mathcal{M}(E/K) &= \varinjlim M^k(E/K) & \mathfrak{M}(E/K) &= \varprojlim M^k(E/K). \end{aligned}$$

For the fine Tate–Shafarevich group, we will denote the limit $\varinjlim Ш^k(E/K)$ simply by $Ш(E/K)$. It is a subgroup of the p-primary part $Ш(E/K)(p)$ of the Tate–Shafarevich group. We have two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}(E/K) & \longrightarrow & \mathcal{R}(E/K) & \longrightarrow & Ш(E/K) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathfrak{M}(E/K) & \longrightarrow & \mathfrak{R}(E/K) & \longrightarrow & \varprojlim Ш^k(E/K) & \longrightarrow & 0 \end{array}$$

since all terms in (2·3) are finite. Both $\varprojlim Ш^k(E/K)$ and $T_pШ(E/K) = \varprojlim Ш[p^k]$ are subgroups of $T_pШ(E/K)$, where, for an abelian group A, we write T_pA for the projective limit $\varprojlim A[p^k]$. Of course, it is widely believed that these three groups are trivial. One can show that.

LEMMA 2·1. *The group* $\varprojlim Ш^k(E/K)$ *has finite index in* $T_pШ(E/K)$.

A proof can be found in [**Wut04**, lemma II·1].

LEMMA 2·2. *The maps*

$$Ш^1(E/K) \longrightarrow Ш^2(E/K) \longrightarrow \cdots$$

are injective.

Proof. This is obvious from the following diagram. Here C^k is as in (2·4).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ш^k(E/K) & \longrightarrow & Ш(E/K)[p^k] & \xrightarrow{\delta^k} & C^k \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Ш^{k+1}(E/K) & \longrightarrow & Ш(E/K)[p^{k+1}] & \xrightarrow{\delta^{k+1}} & C^{k+1} \end{array} \tag{2·5}$$

where the map on the right-hand side is induced by the maps

$$[p]\colon E(K_\upsilon)/p^kE(K_\upsilon) \longrightarrow E(K_\upsilon)/p^{k+1}E(K_\upsilon).$$

The following lemma tells us in some cases when the above sequence stabilises.

LEMMA 2·3. *Let E be an elliptic curve over $\mathbb{Q}$ of rank 0 with $E(\mathbb{Q}_p)[p] = 0$. If p^k kills all elements in $Ш(E/\mathbb{Q})(p)$, then*

$$\text{Ж}^k(E/\mathbb{Q}) = \text{Ж}(E/\mathbb{Q}).$$

Proof. Under the hypothesis on k, the map in the middle of (2·5) becomes an isomorphism. Because $E(\mathbb{Q})/p^kE(\mathbb{Q}) = 0$, the group C^k is equal to $E(\mathbb{Q}_p)/p^kE(\mathbb{Q}_p)$. Hence the map induced by $[p]$ on $C^k \to C^{k+1}$ is injective if $E(\mathbb{Q}_p)[p] = 0$.

Even with weaker assumptions, it is usually easy to verify for which k the sequence of $\text{Ж}^k(E/K)$ stabilises.

Remark. Instead of imposing stronger conditions only at places above p, we could more generally consider a finite set Σ of places in K, containing the places above p, and then demand that elements in $S^k(E/K)$ have trivial localisation in the group $H^1(K_\upsilon, E[p^k])$ for all places in Σ. We would obtain even smaller subgroups $R^k_\Sigma(E/K)$ and $\text{Ж}^k_\Sigma(E/K)$. Note that $\mathcal{R}_\Sigma(E/K) = \varinjlim R^k_\Sigma(E/K)$ does not differ from $\mathcal{R}(E/K)$ because the groups $E(K_\upsilon) \otimes {}^{\mathbb{Q}_p}\!/_{\mathbb{Z}_p}$ are trivial for all places $\upsilon \nmid p$.

3. *Comparison*

LEMMA 3·1. *Let K_υ be a finite extension of $\mathbb{Q}_p$ of degree n_υ. Then*

$$\#\left(E(K_\upsilon)/p^kE(K_\upsilon)\right) = p^{n_\upsilon \cdot k} \cdot \#\left(E(K_\upsilon)[p^k]\right).$$

Proof. Note that, if A is a subgroup of an abelian group B of finite index, then

$$\frac{\#A/p^kA}{\#A[p^k]} = \frac{\#B/p^kB}{\#B[p^k]},$$

provided all terms are well-defined. Denote by $E^\circ(K_\upsilon)$ the K_υ-rational points on the connected component of the Néron-model of E and by $\widehat{E}$ the formal group associated to E (see [**Sil96**]). Since $E(K_\upsilon)$ has finite index in $E^\circ(K_\upsilon)$ and in $\widehat{E}(\mathfrak{m}_\upsilon^a)$, for any power of the maximal ideal $\mathfrak{m}_\upsilon$ in the ring of integers in K_υ, we obtain that

$$\begin{aligned}\frac{\#E(K_\upsilon)/p^kE(K_\upsilon)}{\#E(K_\upsilon)[p^k]} &= \frac{\#E^\circ(K_\upsilon)/p^kE(K_\upsilon)}{\#E^\circ(K_\upsilon)[p^k]} = \frac{\#\widehat{E}(\mathfrak{m}_\upsilon)/p^k\widehat{E}(\mathfrak{m}_\upsilon)}{\#\widehat{E}(\mathfrak{m}_\upsilon)[p^k]} \\ &= \frac{\#\widehat{E}(\mathfrak{m}_\upsilon^a)/p^k\widehat{E}(\mathfrak{m}_\upsilon^a)}{\#\widehat{E}(\mathfrak{m}_\upsilon^a)[p^k]}.\end{aligned}$$

Now, if a is large enough, the formal logarithm gives an isomorphism from $\widehat{E}(\mathfrak{m}_\upsilon^a)$ to $\mathfrak{m}_\upsilon^a$ and we have $\widehat{E}(\mathfrak{m}_\upsilon^a)[p^k] = 0$ and $\#\widehat{E}(\mathfrak{m}_\upsilon^a)/p^k\widehat{E}(\mathfrak{m}_\upsilon^a) = p^{kn_\upsilon}$.

PROPOSITION 3·2. *The index of $\text{Ж}^k(E/K)$ inside $Ш(E/K)[p^k]$ is bounded by*

$$[Ш(E/K)[p^k] : \text{Ж}^k(E/K)] \leqslant p^{[K:\mathbb{Q}]\cdot k} \cdot \prod_{\upsilon|p} \#E(K_\upsilon)[p^k].$$

Proof. We saw that the quotient of the two groups in question is contained in the cokernel C^k of the map from $E(K)/p^kE(K)$ to $\oplus E(K_v)/p^kE(K_v)$. The previous lemma proves that the target of this map has size bounded by the right-hand side of the formula in the proposition.

COROLLARY 3·3. *If E is defined over $\mathbb{Q}$ and $E(\mathbb{Q}_p)[p] = 0$, then*

$$\left[Ш(E/\mathbb{Q})[p] \,:\, \text{Ж}^1(E/\mathbb{Q})\right] \leqslant p.$$

THEOREM 3·4. *Let E be an elliptic curve defined over $\mathbb{Q}$. Suppose $Ш(E/\mathbb{Q})(p)$ is finite, then $\text{Ж}(E/\mathbb{Q})$ is non-trivial if and only if $Ш(E/\mathbb{Q})(p)$ is non-trivial.*

Proof. The injection of $\text{Ж}(E/\mathbb{Q})$ into $Ш(E/\mathbb{Q})(p)$ has cokernel in $C = \varinjlim C^k$ which is a quotient of $E(\mathbb{Q}_p) \otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$. But this last group is isomorphic to ${}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$. If $Ш(E/\mathbb{Q})(p)$ is finite and non-trivial, then it is of the form $A \oplus A$ for some abelian p-group A by the Cassels–Tate pairing. So $Ш(E/\mathbb{Q})(p)$ can not be embedded into ${}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$.

THEOREM 3·5. *Let E be an elliptic curve defined over $\mathbb{Q}$ of positive rank, then we have $\text{Ж}(E/\mathbb{Q}) = Ш(E/\mathbb{Q})(p)$.*

Proof. The group C in the proof of the previous theorem is the cokernel of the localisation map $E(\mathbb{Q}) \otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p} \twoheadrightarrow E(\mathbb{Q}_p) \otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$. The target of this map equals $\widehat{E}(p^2\mathbb{Z}_p) \otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$, which is isomorphic to ${}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$ via the p-adic elliptic logarithm map. If the rank of E is positive, the localisation map is non-trivial: a sufficiently large multiple of any point of infinite order in $E(\mathbb{Q})$ will belong to $\widehat{E}(p^2\mathbb{Z}_p)$ and has non-zero logarithm. Since $E(\mathbb{Q}_p) \otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$ is divisible the map has to be surjective and so $C = 0$.

If the rank is zero, $\text{Ж}(E/\mathbb{Q})$ may be strictly smaller than $Ш(E/\mathbb{Q})(p)$ as we will see in the numerical examples in the next section.

We see here that the fine Tate–Shafarevich group for an elliptic curve over $\mathbb{Q}$ tends to be rather large in comparison with the whole Tate–Shafarevich group. This is likely to be specific to $\mathbb{Q}$ as we will announce a conjecture for certain larger fields in Section 8.

4. *Numerical examples*

In this section we give four numerical examples of fine Tate–Shafarevich groups for $p = 2$. The four curves E are all defined over $\mathbb{Q}$. They all have rank 0, as can be verified using Kolyvagin's result or a 3-descent; in fact for each curve we have $E(\mathbb{Q}) = {}^{\mathbb{Z}}/_{2\mathbb{Z}} \oplus {}^{\mathbb{Z}}/_{2\mathbb{Z}}$. Then we perform a complete 2-descent on E as explained in [**Sil96**, proposition X·1·4].

The complete 2-descent proves for the first three examples that $Ш(E/\mathbb{Q})[2] = {}^{\mathbb{Z}}/_{2\mathbb{Z}} \oplus {}^{\mathbb{Z}}/_{2\mathbb{Z}}$. Hence the fine Tate–Shafarevich group $\text{Ж}^1(E/\mathbb{Q})$ can have 1, 2 or 4 elements. The examples are chosen as to show that all three cases can occur. Furthermore we can prove that the 2-primary part $Ш(E/\mathbb{Q})(2)$ has order 4 using a 4-descent or the Cassels-Tate pairing implemented in `magma`. So the fine Tate–Shafarevich group $\text{Ж}(E/\mathbb{Q})$ can have either 2 or 4 elements by Corollary 3·3.

The details of the computations are only included for the first example.

An example with a trivial $\text{Ж}^1(E/\mathbb{Q})$

Let $E/\mathbb{Q}$ be the elliptic curve O3 of conductor 930 in Cremona's table [**Cre97**]

$$E: \quad y^2 + xy = x^3 - 19'220\,x - 1'027'200.$$

The Mordell–Weil group $E(\mathbb{Q})$ is generated by two points $T_1 = (-80, 40)$ and $T_2 = (-321/4, 321/8)$ of order 2. The analytic order of $Ш(E/\mathbb{Q})$ is 4.

Let us change the equation to the non-minimal equation

$$y^2 = (x + 320) \cdot (x + 321) \cdot (x - 640)$$

in which T_1 has coordinates $(-320, 0)$ and T_2 is equal to $(-321, 0)$. Let Σ be the set $\{2, 3, 5, 31\}$ which includes all primes of bad reduction for E. The group $\mathbb{Q}(\Sigma, 2)$ is defined to be the group of non-zero integers only divisible by elements in Σ modulo squares; which in our case is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$. The above choice of a basis of $E(\mathbb{Q})[2]$ allows us to identify the Selmer group $S^1(E/\mathbb{Q})$ with a subgroup of the square of $\mathbb{Q}(\Sigma, 2)$. In fact, the image of T_1 is represented by $(-15, 1)$ while T_2 is sent to $(-1, 1)$ in $\mathbb{Q}(\Sigma, 2)^2$. One sees immediately that T_1 belongs to the fine Mordell-Weil group $M^1(E/\mathbb{Q})$ since both -15 and 1 are squares in $\mathbb{Q}_2^\times$, in other words there is a 4-torsion point

$$S_1 = (2^3 + 2^7 + 2^8 + \mathbf{O}(2^{10}), 2^3 + 2^8 + 2^9 + \mathbf{O}(2^{10}))$$

defined over $\mathbb{Q}_2$ such that $2 \cdot S_1 = T_1$.

Now the first Selmer group $S^1(E/\mathbb{Q})$ is generated by the image of T_1 and T_2 and the elements $(5, 1)$ and $(2, 1)$ in $\mathbb{Q}(\Sigma, 2)^2$. We can conclude that $R^1(E/\mathbb{Q})$ is equal to $(-15, 1) \cdot \mathbb{Z}/2\mathbb{Z} = M^1(E/\mathbb{Q})$, and hence that $Ш^1(E/\mathbb{Q})$ is trivial.

We now use the diagram (2·5) to compute the whole of $Ш(E/\mathbb{Q})$. We need to compute C^k first. Denote by $\Phi(E)(\mathbb{Q}_2)$ the group of connected components of the Néron-model over $\mathbb{Q}_2$. For our example $\Phi(E)(\mathbb{Q}_2)$ is a cyclic group of order 4; the reduction type is I_4. The torsion group of $E(\mathbb{Q}_2)$ is $\mathbb{Z}/4\mathbb{Z}\, S_1 \oplus \mathbb{Z}/2\mathbb{Z}\, T_2$. Since S_1 maps to the generator of $\Phi(E)(\mathbb{Q}_2)$, we know that $E(\mathbb{Q}_2) = E^\circ(\mathbb{Q}_2) \oplus \mathbb{Z}/4\mathbb{Z}\, S_1$ with E° being the connected component of the identity of the Néron model. The torsion point T_2 lies in the first layer of the formal group in the minimal model, and so we have $E(\mathbb{Q}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}/4\mathbb{Z}\, S_1 \oplus \mathbb{Z}/2\mathbb{Z}\, T_2$. We conclude that C^k is isomorphic to $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\, S_1$. The map induced by [2] is the injection $\mathbb{Z}/2^k\mathbb{Z} \hookrightarrow \mathbb{Z}/2^{k+1}\mathbb{Z}$ on the first factor and the trivial map on the second:

$$\begin{array}{ccc} C^k & \longrightarrow & C^{k+1} \\ \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} S_1 & \xrightarrow[(1,0)]{} & \mathbb{Z}/2^{k+1}\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} S_1. \end{array}$$

The element $(5, 1)$ in $Ш(E/\mathbb{Q})[2]$ is mapped to the element $\delta^1(5, 1)$ in C^1 represented by

$$Q_1 = (2^{-2} + 1 + 2^6 + 2^7 + 2^9 + \mathbf{O}(2^{10}), 2^{-3} + 2 + 2^2 + 2^3 + 2^6 + 2^7 + 2^9 + \mathbf{O}(2^{10}))$$

which belongs to the first factor of $C^1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\, S_1$. The image of the other generator $(2, 1)$ can be represented by S_1. Hence the first element of $Ш(E/\mathbb{Q})[2]$ always maps to a non-trivial element in C^k and does therefore not belong to $Ш^k(E/\mathbb{Q})$ for any k, while the latter element has trivial image in C^2 and thus belongs to $Ш^2(E/\mathbb{Q})$.

We can conclude the following:

$$Ш^1(E/\mathbb{Q}) = 0 \quad \text{and} \quad Ш^2(E/\mathbb{Q}) = Ш(E/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}.$$

A $Ш^1(E/\mathbb{Q})$ with 2 elements

Let E be the curve E5 of conductor 210

$$E: \quad y^2 + xy = x^3 - 120'050\, x - 16'020'000.$$

The Mordell–Weil group $E(\mathbb{Q})$ is generated by two points $T_1 = (400, -200)$ and $T_2 = (-801/4, 801/8)$ of order 2. A computation like for the first example shows that $M^1(E/\mathbb{Q}) = 0$ and that $R^1(E/\mathbb{Q}) = \text{Ш}^1(E/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$. Using that $\text{Ш}(E/\mathbb{Q})(2)$ has 4 elements one shows that $\text{Ш}^k(E/\mathbb{Q}) = \text{Ш}(E/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$ for all $k \geqslant 1$.

An example where $\text{Ш}^1(E/\mathbb{Q})$ has 4 elements

Let E be the elliptic curve

$$E: \quad y^2 + xy = x^3 - x^2 - 3'861\,x - 91'368.$$

It is labelled 1287E2 in the tables of Cremona. For this curve we have that $M^1(E/\mathbb{Q}) = 0$ and $R^1(E/\mathbb{Q}) = \text{Ш}^1(E/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = \text{Ш}(E/\mathbb{Q})[2]$. Here we get $\text{Ш}(E/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

A bigger example

Finally, we include an example[1] of a larger Tate–Shafarevich group. The curve E

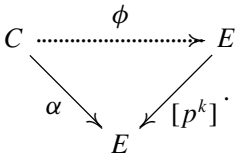
$$E: \quad y^2 + x\,y = x^3 - x^2 - 213'230'796\,x + 1'183'712'688'107$$

of conductor $3^2 \cdot 13^2 \cdot 17 \cdot 157^2$ is a twist of the curve $17A2$. The Mordell–Weil group has rank 0 and it has 4 rational 2-torsion points. The Birch and Swinnerton–Dyer conjecture predicts the order of $\text{Ш}(E/\mathbb{Q})$ to be 16.

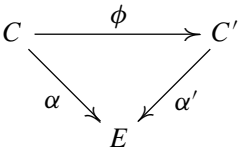
This time, we find that $S^1(E/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^6$ and so $\text{Ш}(E/\mathbb{Q})[2] = (\mathbb{Z}/2\mathbb{Z})^4$. It can be verified as before that the 2-primary part of $\text{Ш}(E/\mathbb{Q})$ consists of these 16 elements. The computation of the fine Selmer group shows that $R^1(E/\mathbb{Q}) = \text{Ш}^1(E/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^3$. The fine Tate–Shafarevich is equal to $\text{Ш}(E/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^3$.

5. *Geometric interpretation*

As usual, an element in the Selmer group and in the Tate–Shafarevich group may be interpreted geometrically in terms of torsors of E. Let F be an extension of K. An p^k-covering over F is a morphism of curves $\alpha: C \to E$ defined over F such that there is an isomorphism ϕ over $\bar{F}$ having the property that the following diagram commutes.



Note that C is a torsor of E. Two p^k-coverings α and α' are isomorphic if there is a ϕ defined over F such that



commutes. In particular, a p^k-covering α is isomorphic to the trivial p^k-covering given by $[p^k]: E \to E$ if and only if there is a point defined over F in the fibre of α above $O \in E(F)$.

[1] I am grateful to Christophe Delaunay for helping me to find this example.

It is well known that the elements of $H^1(F, E[p^k])$ can be represented in a unique way as isomorphism classes of p^k-coverings.

PROPOSITION 5·1. *The fine Selmer group $R^k(E/K)$ is in bijection with the pointed set of isomorphism classes of p^k-covering $\alpha: C \to E$ over K such that.*

(i) *$C(K_\upsilon)$ is non empty for all places υ in K, and,*
(ii) *for all places υ above p, there is a K_υ-rational point P_υ in the fibre $\alpha^{-1}(O)$ above $O \in E(K)$.*

The point P_υ is only defined up to addition by a K_υ-rational p^k-torsion point. For the fine Tate–Shafarevich group, we have the following description.

PROPOSITION 5·2. *The fine Tate–Shafarevich group $Ш^k(E/K)$ can be viewed as the pointed set of isomorphism classes of principal homogeneous spaces C under E defined over K such that*:

(i) *$C(K_\upsilon)$ is non empty for all places*;
(ii) *there is a K-rational effective divisor D of degree p^k on C; and*
(iii) *for all places υ above p, the divisor D contains a K_υ-rational point P_υ in its support.*

6. *Duality*

Cassels constructed a pairing on the Tate–Shafarevich group, whose kernel consists of the infinitely divisible elements. It follows that a finite Tate–Shafarevich group has square order. The construction of Cassels was extended by Flach in [**Fla90**]. In order to state the duality for the fine Selmer group, we need to define another generalised version of the Selmer group, namely the group $Q(E/K)$ defined by the exact sequence

$$0 \longrightarrow Q(E/K) \longrightarrow H^1(K, E(p)) \longrightarrow \prod_{\upsilon \nmid p} H^1(K_\upsilon, E)(p) \times \prod_{\upsilon | p} H^1(K_\upsilon, E(p))/\,\mathrm{div}.$$

Here we denote by $A/\,\mathrm{div}$ the quotient of A by its maximal divisible subgroup whenever A is a discrete abelian group. The enlarged Selmer group $Q(E/K)$ contains the usual Selmer group $Ш(E/K)$. Let $Ч(E/K)(p)$ be the quotient of $Q(E/K)$ by the image of $E(K) \otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$.

THEOREM 6·1. *There is a alternating pairing on $R(E/K) \times Q(E/K)$ with values in ${}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$ whose kernel on each side consists of the divisible elements. In particular this identifies $R(E/K)/\,\mathrm{div}$ with the dual of $Ч(E/K)/\,\mathrm{div}$.*

This is just a reformulation of the main theoren in [**Fla90**] when using the local conditions $W_\upsilon = 0$ for all places υ.

From the Poitou–Tate sequence [**NSW00**, theorem 8·6·13] we may deduce two other facts. Let Σ be any finite set of places containing the ones above p, and those where E has bad reduction. The group $R(E/K)$ is dual to the kernel of

$$H^2(G_\Sigma(K), T_pE) \longrightarrow \bigoplus_{\upsilon \in \Sigma} H^2(K_\upsilon, T_pE)$$

where $G_\Sigma(K)$ stands for the Galois group of the maximal extension of K which is unramified outside Σ. The target of the map is the dual of the finite group $\oplus_{\upsilon \in \Sigma} E(K_\upsilon)(p)$. Furthermore

there is an exact sequence

$$0 \longrightarrow \mathcal{R}(E/K) \longrightarrow H^1(G_\Sigma(K), E(p)) \longrightarrow \bigoplus_{v\in\Sigma} H^1(K_v, E(p))$$
$$\longrightarrow H^1(G_\Sigma(K), T_pE)^\wedge \longrightarrow H^2(G_\Sigma(K), E(p)) \longrightarrow 0.$$

Here $M^\wedge$ denotes the Pontryagin dual of M.

7. *The fine Mordell–Weil group*

We now turn our interest to the other side of the fine Selmer group, namely the fine Mordell-Weil group. In the limit, we have a compact version $\mathfrak{M}(E/K)$ and a discrete version $\mathcal{M}(E/K)$. Note that, even though the group $E(K)\otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}$ is divisible, the group $\mathcal{M}(E/K)$ is not necessarily divisible as we see in the following example. Let E be the elliptic curve 37A1 given by the equation

$$E: \quad y^2 + y = x^3 - x \tag{7·1}$$

defined over $\mathbb{Q}$, whose Mordell–Weil group $E(\mathbb{Q})$ is generated by the point of infinite order $P = (0, 0)$. We shall see that the fine Mordell–Weil group $\mathcal{M}(E/\mathbb{Q})$ for the prime $p = 179$ is a finite non-trivial group. The curve has good, ordinary, non-anomalous reduction at p. The point $81 \cdot P$ is the first multiple of P that lies in the kernel of reduction modulo p, but it even lies in the second layer of the formal group, i.e. the denominator of the x-coordinate is divisible by p^4. In other words, the point P is divisible by p in $E(\mathbb{Q}_p)$. Hence $\mathcal{M}(E/\mathbb{Q}) = {}^{\mathbb{Z}}/_{p\mathbb{Z}}$.

More generally, the size of the group $\mathcal{M}(E/K)/\operatorname{div}$ measures the divisibility of the generators in $E(\mathbb{Q}_p)$. For curves of rank 1 over $\mathbb{Q}$ it can be expressed in terms of the p-adic logarithm of the generator. See [**CM94**, lemma 9]; the first table in this paper can be used to find further examples like the one above.

If A is an abelian group, we denote by $A^\star$ the p-adic completion $\varprojlim A/p^kA$ of A.

THEOREM 7·1. *We have the following exact sequence*

$$0 \longrightarrow \mathfrak{M}(E/K) \longrightarrow T_p\mathcal{M}(E/K) \longrightarrow T \longrightarrow \operatorname{Tors}_{\mathbb{Z}_p}(D) \longrightarrow \mathcal{M}(E/K)/\operatorname{div} \longrightarrow 0$$

where D is the cokernel of the map $E(K)^\star \longrightarrow \oplus_{v|p} E(K_v)^\star$ and T is the cokernel of the corresponding localisation map on the p-primary part, namely $E(K)(p) \rightarrowtail \oplus_{v|p} E(K_v)(p)$.

Proof. We split the exact sequence

$$0 \longrightarrow \mathcal{M}(E/K) \longrightarrow E(K)\otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p} \longrightarrow \oplus\, E(K_v)\otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p} \longrightarrow C \longrightarrow 0$$

by defining B to be the image of the middle map. Now, the middle terms are divisible and so is B. This provides us with two exact sequences

$$0 \longrightarrow T_p\mathcal{M}(E/K) \longrightarrow T_p\big(E(K)\otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}\big) \longrightarrow T_pB \longrightarrow \mathcal{M}(E/K)/\operatorname{div} \longrightarrow 0 \tag{7·2}$$

$$0 \longrightarrow T_pB \longrightarrow \oplus\, T_p(E(K_v)\otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}) \longrightarrow T_pC \longrightarrow 0. \tag{7·3}$$

We apply the snake lemma to the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(K)(p) & \longrightarrow & E(K)^\star & \longrightarrow & T_p(E(K)\otimes {}^{\mathbb{Q}_p}\!/_{\mathbb{Z}_p}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \oplus E(K_\upsilon)(p) & \longrightarrow & \oplus E(K_\upsilon)^\star & \longrightarrow & \oplus T_p(E(K_\upsilon)\otimes {}^{\mathbb{Q}_p}\!/_{\mathbb{Z}_p}) & \longrightarrow & 0 \end{array}$$

Defining A to be the cokernel of the vertical map on the right-hand side, we find the sequence

$$0 \longrightarrow \mathfrak{M}(E/K) \longrightarrow T_p\mathcal{M}(E/K) \longrightarrow T \longrightarrow D \longrightarrow A \longrightarrow 0$$

containing the first four terms of the sequence in the theorem. Furthermore, by the two sequences involving (7·2) and (7·3), we see that the cokernel A is in the following sequence

$$0 \longrightarrow \mathcal{M}(E/K)/\operatorname{div} \longrightarrow A \longrightarrow T_pC \longrightarrow 0.$$

The fact that T_pC is $\mathbb{Z}_p$-free proves that $\operatorname{Tors}_{\mathbb{Z}_p}(A) = \mathcal{M}(E/K)/\operatorname{div}$.

Note that if E is defined over $\mathbb{Q}$ and the rank is positive then D is torsion.

COROLLARY 7·2. *If the curve E is defined over $\mathbb{Q}$ and $E(\mathbb{Q}_p)[p]$ is trivial, then the canonical map from $\mathfrak{M}(E/\mathbb{Q})$ to $T_p\mathcal{M}(E/K)$ is an isomorphism.*

Note that this does not hold in general, as we can see in the following example. Let E be the curve

$$E: \quad y^2 + x\,y = x^3 + 1 \tag{7·4}$$

of conductor 433 and let $p = 3$. The curve E has no non-trivial torsion point defined over $\mathbb{Q}$, but two linearly independent points of infinite order, namely $P_1 = (-1, 0)$ and $P_2 = (0, 1)$. Over $\mathbb{Q}_3$ there is a 3-torsion point on E, namely

$$S = \big(2 + 2\cdot 3^3 + 3^6 + 2\cdot 3^7 + 3^9 + \mathbf{O}(3^{10}),\, 2\cdot 3^2 + 3^3 + 3^5 + 2\cdot 3^8 + \mathbf{O}(3^{10})\big).$$

The reduction of E at $p = 3$ is good anomalous with 6 points on the reduced curve $\widetilde{E}(\mathbb{F}_3)$. The reduction of the point P_2 generates $\widetilde{E}(\mathbb{F}_3)$. On the other hand the point $P_1 + 2\cdot P_2 = (13/9, 38/27)$ is in the first layer of the formal group $\widehat{E}(3\mathbb{Z}_3)$. It follows that $E(\mathbb{Q})^\star \twoheadrightarrow E(\mathbb{Q}_3)^\star$ is surjective. On the other hand the group T is not trivial since there is this point S of order 3 that it not defined over $\mathbb{Q}$. From the sequence in the above theorem, we conclude that the index of $\mathfrak{M}(E/\mathbb{Q})$ in $T_p\mathcal{M}(E/\mathbb{Q})$ is equal to p.

As a consequence of the previous theorem, we find some more information on the fine Mordell–Weil group

COROLLARY 7·3. *The compact fine Mordell–Weil group $\mathfrak{M}(E/K)$ is a free $\mathbb{Z}_p$-module of finite rank. If the rank r of the curve is zero then $\mathfrak{M}(E/K) = \mathcal{M}(E/K) = 0$, otherwise, if r is positive, then*

$$r - [K:\mathbb{Q}] \leqslant \operatorname{corank}_{\mathbb{Z}_p} \mathcal{M}(E/K) = \operatorname{rank}_{\mathbb{Z}_p} \mathfrak{M}(E/K) \leqslant r - 1.$$

If $K = \mathbb{Q}$, we have equality.

Proof. The only thing that is left to prove is the left-hand inequality. Let P be a point of infinite order on $E(K)$. A certain multiple of P will lie in all the kernels $\widehat{E}(\mathfrak{m}_\upsilon)$ of reduction modulo all places above p. Since there is an injection from $\widehat{E}(\mathfrak{m}_\upsilon)$ into $E(K_\upsilon)^\star$ for all $\upsilon \mid p$, this multiple will not lie in $\mathfrak{M}(E/K)$. (Compare to the proof of Theorem 3·5.)

In particular, for curves of rank 0, the sequence in Theorem 7·1 becomes simply an isomorphism between T and $\mathrm{Tors}_{\mathbb{Z}_p}(D)$. If E is defined over $\mathbb{Q}$ and has rank 1 then we get a sequence

$$0 \longrightarrow T \longrightarrow D \longrightarrow \mathcal{M}(E/\mathbb{Q}) \longrightarrow 0$$

of finite groups.

8. *Conjectures*

The purpose of this last section is to announce some conjectures on the finiteness of the fine Tate–Shafarevich group. Of course, there is a first basic

CONJECTURE 8·1. *The fine Tate–Shafarevich group $\text{Ш}(E/K)$ of an elliptic curve over a number field K is finite. For a given E/K it is trivial for all but a finite number of primes p.*

It is a weaker form of what should be believed by any gentleman namely that $\text{Ш}(E/K)$ is already finite. Note that over $\mathbb{Q}$ for a curve of positive rank the two statements are equivalent by Theorem 3·5.

The original interest in this group comes from Iwasawa theory for elliptic curves. Let E be an elliptic curve, for simplicity, defined over $\mathbb{Q}$. Using the methods of Perrin-Riou [**PR95**] one is able to compute the growth of the size of the fine Tate–Shafarevich group in certain towers of number fields. See [**Wut05**] and [**Wut04**] for more details. Let $\mathbb{Q}(\mu[p^{n+1}])$ be the cyclotomic field of p^{n+1}-th roots of unity and let ${}_n\mathbb{Q}$ be the unique subfield of $\mathbb{Q}(\mu[p^{n+1}])$ with Galois group $\mathrm{Gal}({}_n\mathbb{Q}:\mathbb{Q})$ isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. The cyclotomic $\mathbb{Z}_p$-extension ${}_\infty\mathbb{Q}$ of $\mathbb{Q}$ is defined to be the union of all ${}_n\mathbb{Q}$ as n varies over the positive integers.

Denote by e_n the integer such that $\#\text{Ш}(E/{}_n\mathbb{Q}) = p^{e_n}$. By a famous theorem of Kato [**Kat04**], we know that the integer e_n grows like $\mu \cdot p^n + \lambda \cdot n + \mathbf{O}(1)$ for some integers $\mu \geqslant 0$ and $\lambda \geqslant 0$. There is a first conjecture, a special case of conjecture A in [**CS05**], due to Coates and Sujatha.

CONJECTURE 8·2. *The growth of the order of $\text{Ш}(E/{}_n\mathbb{Q})$ is of the form $e_n = \lambda \cdot n + \mathbf{O}(1)$, i.e. $\mu = 0$.*

The striking thing about this conjecture is that a similar statement for $\text{Ш}(E/{}_n\mathbb{Q})$ is known to be false ever since the beginnings of Iwasawa theory for elliptic curves, see [**Maz72**]. For our case of a curve over $\mathbb{Q}$, the conjecture is verified if E admits an isogeny of degree p defined over $\mathbb{Q}$. This follows from [**CS05**, corollary 3·5].

As an example, we might consider the curve of conductor 11 given by

$$E: \quad y^2 + y = x^3 - x^2 - 7'820\,x - 263'580$$

for which it is known that the order of $\text{Ш}(E/{}_n\mathbb{Q})(5)$ for $p = 5$ grows like $5^{2\cdot 5^n}$, i.e. $\mu = 2$ and $\lambda = 0$. Nevertheless, the order of $\text{Ш}(E/{}_n\mathbb{Q})$ stays bounded, see [**Wut05**].

The numerical evidence computed in [**Wut04**] suggest that it might be possible that the following question has a positive answer.

QUESTION 8·3. *Is the order of $\text{Ш}(E/{}_n\mathbb{Q})$ bounded independently of n, i.e. do we have that $\mu = \lambda = 0$?*

In other words, we could ask if $\text{Ш}(E/{}_\infty\mathbb{Q})$ is finite. For the supersingular case the eventuality that the answer might be "yes" was mentioned to me by Kurihara and Pollack.

Nevertheless, there is one hint that it could be wrong in the extension to the situation of ordinary reduction. In the case of a global field of positive characteristic different from p, say K is a function field of a curve over a finite field, the fine Tate–Shafarevich group of an elliptic curve E over K naturally coincides with the full Tate–Shafarevich group, since there are no places above p. Moreover it is equal to the Brauer–Grothendieck group $\mathrm{Br}(\mathcal{E})$ of the elliptic surface $\mathcal{E}$ whose generic fibre is E, see [**Tat95**]. The analogue of $Ш(E/_\infty\mathbb{Q})$ is the the Brauer group $\mathrm{Br}(\bar{\mathcal{E}})$ of the surface $\mathcal{E}$ but over the algebraic closure of the finite base field. There are certainly many examples of infinite $\mathrm{Br}(\bar{\mathcal{E}})$.

On the other hand, we know that $\mathrm{Br}(\bar{\mathcal{E}})$ is co-finitely generated over $\mathbb{Z}_p$ and so Conjecture 8·2 is known for function fields.

But at least, we can be sure that a weaker form of the question should have a positive answer, namely.

CONJECTURE 8·4. *For all but a finite number of primes p, the order of $Ш(E/_n\mathbb{Q})$ is bounded independently of n.*

Compare with the conjectures in [**Wut05**] on the density of such primes.

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