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Point sets containing their triangle centers

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Abstract Let S be a set of at least five points in the plane, not all on a line. Suppose that for any three points $a, b, c \in S$ the nine-point center of triangle abc also belongs to S. We show that S must be dense in the plane. We also consider several problems about partitioning the plane into two sets containing their triangle centers.

Keywords Dense point set · Nine-point center · Triangle centers · Iterated process

Mathematics Subject Classification (2000) 52A35

1 Introduction

According to Ambrus and Bezdek (2007), the investigation of geometric iterative processes was initiated by L.F. Tóth. The main idea is as follows: start with a certain subset of a Euclidean space and then add to it at each step new points according to a prescribed geometric rule. The problem is to describe the structure of the limit set, or just to prove that it has a specific property, usually that it is dense. Many problems stemming from this general idea have been considered (e.g. Bezdek and Pach 1985; Cooper and Walters 2010; Grüne and Kamali 2008; Ismailescu and Radoičić 2004).

Iorio et al. (2005) proposed the following version of iterative processes: at each step add the circumcenters (incenters, orthocenters, centroids, respectively) of all non-degenerate triangles determined by three points from the existing set. They proved

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that in the case of the circumcenters the iterative process results in a dense subset of the plane, while for the incenters and centroids they found that the limit set must be dense in its convex hull. The corresponding problem for the orthocenters turned out to be more difficult and it was solved only later by Ambrus and Bezdek (Ambrus and Bezdek 2006; Ambrus 2006). They showed that any subset of the plane containing its orthocenters must be either a subset of a rectangular hyperbola or everywhere dense in the plane. Concerning higher dimensions, analogous theorems for the incenter and circumcenter versions of the problem were shown in Ambrus and Bezdek (2006, 2007).

In the first part of this paper we consider a similar problem based on the nine-point center iterations. The *nine-point center* of a triangle abc is the center of the circle that passes through the midpoints of the sides of abc (Johnson 1960). We prove the following result.

Theorem 1 (Nine-point center) *Let S be a set of at least five points in the plane, not all on a line, such that for every three non-collinear points in S, the nine-point center of the triangle determined by these three points is also in S. Then S is everywhere dense in the plane.*

The method we use in the proof shows an interesting combination of calculus and plane geometry. Theorem 1 gives an answer analogous to that of the circumcenter problem. We explain this by the fact that the circumcenter and the nine-point center are triangle centers that can lie outside of the triangle, unlike the centroid and the incenter which are always strictly inside the triangle.

While working on this paper we also tried to deduce a similar result which would strengthen the previous results about the circumcenters and incenters, but we did not get to a final conclusion. Instead we can offer the following problem.

Problem 1 Let S be a set of at least three points in the plane, not all on a line, such that for every three non-collinear points in S, either the circumcenter or the incenter of the triangle determined by these three points is also in S. Is S necessarily dense in its convex hull?

In the second part of the paper we introduce a new variant of the problems, by asking whether it is possible to partition the plane in a nontrivial way into two sets satisfying the above properties (i.e., containing their triangle centers). First we prove that there is no nontrivial partition in the case of the orthocenters. We learned about this problem from M. Treml.

Theorem 2 (Orthocenter) *Suppose that the plane is colored in two colors so that for any three non-collinear points a, b, c of the same color the orthocenter of triangle abc has the same color. Then the whole plane is colored in one color.*

It seems to be more difficult to decide whether the same conclusion holds for an arbitrary (finite or countable) number of colors.

Problem 2 Suppose that the plane is colored in finitely (countably) many colors so that for any three non-collinear points a, b, c of the same color the orthocenter of triangle abc has the same color. Does it necessarily follow that the whole plane is colored in one color?

We show an analogous theorem for the circumcenters in an arbitrary dimension.

Theorem 3 (Circumcenter) *Suppose that $\mathbb{R}^n$ ($n \geq 2$) is colored in two colors so that for any set $A = \{a_1, \ldots, a_{n+1}\}$ of $n+1$ affinely independent points of the same color the circumcenter of the simplex spanned by A has the same color. Then the whole space $\mathbb{R}^n$ is colored in one color.*

It can seem plausible that similar statements are also true for other triangle/simplex centers. However, it is easy to see that for the centroids and incenters a simple partition into two halfspaces (with some freedom on the boundary hyperplane) constitutes a counterexample. The question is whether this is basically the only counterexample. We show that in the case of the centroids the answer is no, if we assume the axiom of choice.

Theorem 4 (Centroid) (*AC assumed*) *There exists a partition of $\mathbb{R}^n$ into two everywhere dense sets such that for any $n+1$ affinely independent points from one set the centroid of the simplex spanned by them belongs to the same set.*

Finally, we are left with the following two problems.

Problem 3 Is it possible to partition the plane into two parts in a nontrivial way so that for any three non-collinear points a, b, c from one part the incenter of triangle abc also belongs to the same part? An analogous question can be also asked for $\mathbb{R}^n$.

Problem 4 Suppose that the plane is colored in two colors so that for any three non-collinear points a, b, c of the same color the nine-point center of triangle abc has the same color. Is the whole plane necessarily colored in one color?

2 Sets containing their nine-point centers

In this section we prove Theorem 1.

Calculus lemma. First, we show a calculus lemma, which is used in the proof. The lemma can be viewed as a version of the Banach fixed point theorem.

Let $I \subset \mathbb{R}$ be a closed interval. Recall that a function $f : I \to I$ is called a *contraction with constant k* (where $k < 1$) if

$$|f(x) - f(y)| \leq k|x - y|, \quad \text{for all } x, y \in I.$$

Lemma 5 *Let $I \subset \mathbb{R}$ be a closed interval and $f : I \to I$ a contraction with constant k. Consider a sequence $(x_n)_{n\geq 1}$, such that $x_1 \in I$ and*

$$x_{n+1} = f(x_n) + \epsilon_n \quad (n \geq 1),$$

where $(\epsilon_n)_{n\geq 1}$ is a given sequence (it is assumed that $x_n \in I$ for all n). If $\lim_{n\to\infty} \epsilon_n = 0$, then x_n converges to c, a unique fixed point of f in I.

Proof By the Banach fixed point theorem we know that function f has a unique fixed point $c \in I$. By using induction (and triangle inequality) it is easy to deduce the inequality:

$$|x_n - c| \leq \sum_{i=1}^{n-1} |\epsilon_i| \cdot k^{n-1-i} + k^{n-1}|x_1 - c|, \quad \text{for } n \geq 2. \tag{1}$$

Let $\epsilon > 0$ be arbitrary. Since ϵ_n converges, there is an M such that $|\epsilon_n| \leq M$ for all n. We can choose an n_0 such that for all $n \geq n_0$ we have $|\epsilon_n| \leq \epsilon$, (again, since $\epsilon_n \to 0$) and, at the same time, $k^n M \leq \epsilon$ and $k^{n-1}|x_1 - c| \leq \epsilon$. Going back to (1), for $n \geq 2n_0$ we have

$$\begin{aligned} |x_n - c| &\leq \sum_{i=1}^{n-1} |\epsilon_i| \cdot k^{n-1-i} + k^{n-1}|x_1 - c| \\ &\leq \epsilon(1 + k + \cdots + k^{n-1-n_0}) + k^{n-n_0} M(1 + k + \cdots + k^{n_0-2}) + k^{n-1}|x_1 - c| \\ &\leq \frac{2\epsilon}{1-k} + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we conclude that $\lim_{n\to\infty} x_n = c$. □

Technical lemmas. We state several geometric lemmas, omitting their proofs, which are straightforward and a bit technical (e.g., one can use the method of coordinates).

Lemma 6 *If a, b, c are points lying on a line l and x is a point not lying on line l, then the nine-point center of at least one of the triangles abx, acx, bcx does not lie on l.*

Lemma 7 *Let abc be a triangle with $\angle bac = \frac{\pi}{6}$. If the nine-point center of $\triangle abc$ lies on line ab, then abc is an isosceles triangle with base ac.*

Lemma 8 *Let abc be a triangle with $\angle bac = \frac{\pi}{6}$ and $\angle cba > \frac{\pi}{2}$. If n is the nine-point center of $\triangle abc$, then $\angle ban \notin \{\frac{\pi}{6}, \frac{\pi}{3}\}$.*

Lemma 9 *Let abc be a triangle with $\angle bac = \frac{\pi}{3}$. If n is the nine-point center of $\triangle abc$, then $\angle ban = \angle can = \frac{\pi}{6}$.*

Main lemmas. We introduce two definitions which will be useful throughout the proof. An isosceles triangle with an angle of $2\pi/3$ is called a *nice* triangle. We say that a point c is *excellent* for ab if abc is a non-degenerate triangle with non-obtuse angles at vertices a and b.

In order to prove Theorem 1 without loss of generality we can assume that the set S is closed, since from the fact that S satisfies the conditions of the theorem it follows that $\overline{S}$ satisfies them as well. So, throughout this section *we assume that S is a closed set* satisfying the conditions of the theorem and the goal is to prove that $S = \mathbb{R}^2$.

The following lemma is the heart of the proof of Theorem 1.

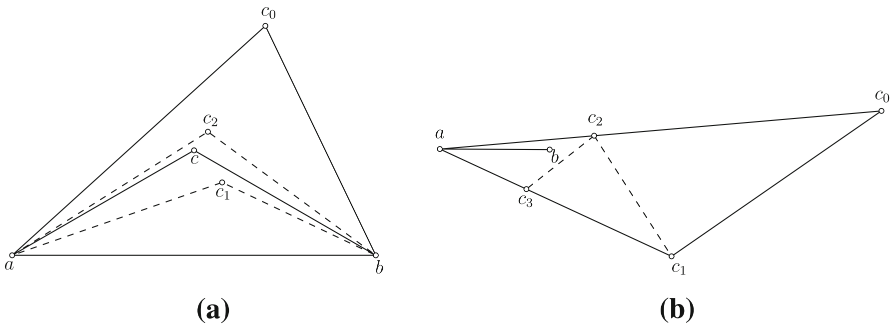

Fig. 1 **a** Lemma 10, case 1, the sequence of iterated nine-point centers converging to c; **b** Lemma 10, case 2.1, sequence of iterated nice triangle vertices eventually becomes excellent

Lemma 10 *For any two points $a, b \in S$, there is some $c \in S$ such that $\triangle abc$ is nice with base ab.*

Proof WLOG we can assume that $a = (-1, 0)$ and $b = (1, 0)$. We distinguish two cases.

1. *There is a point $c_0 \in S$ that is excellent for ab.* Define recursively the sequence of points $(c_k)_{k\geq 0}$ as follows: c_{k+1} is the nine-point center of $\triangle c_k ab$ for $k \geq 1$ (see Fig. 1a). Let $c_0 = (p, q)$. Note that $p \in [-1, 1]$, since c_0 is excellent for ab. It is easy to get the following recursive formula for $c_k = (x_k, y_k)$:

$$c_k = \left(\frac{x_{k-1}}{2}, \frac{1 - x_{k-1}^2}{4y_{k-1}} + \frac{y_{k-1}}{4} \right).$$

Hence,

$$x_k = \frac{p}{2^k} \quad \text{and} \quad y_{k+1} = \frac{y_k}{4} + \frac{2^{2k} - p^2}{2^{2k+2} y_k} \quad \text{for } k \geq 0.$$

Obviously, all y_k have the same sign as $y_0 = q$. WLOG suppose that they are positive. We have that

$$y_{k+1} \geq 2\sqrt{\frac{y_k}{4} \cdot \frac{2^{2k} - p^2}{2^{2k+2} y_k}} = \frac{1}{2}\sqrt{1 - \frac{p^2}{2^{2k}}} \geq \frac{\sqrt{15}}{8},$$

for $k \geq 2$. Let $I = [\frac{\sqrt{15}}{8}, \infty)$. Function f given by $f(x) = \frac{1}{4}(x + \frac{1}{x})$ maps I to I and is a contraction with constant $49/60$, since

$$|f'(x)| = \left| \frac{1}{4} - \frac{1}{4x^2} \right| \leq \frac{49}{60}.$$

On the other hand,

$$\lim_{k\to\infty} \frac{p^2}{2^{2k+2}y_k} = 0,$$

so we can apply Lemma 5 to conclude that $\lim_{k\to\infty} y_k = 1/\sqrt{3}$ (a unique fixed point of f in I). Also, $\lim_{k\to\infty} x_k = 0$, and putting this together we get that $\lim_{k\to\infty} c_k = (0, 1/\sqrt{3})$, i.e., the sequence (c_k) converges to the point c such that $\triangle abc$ is nice. Since S is closed, $c \in S$ and we are done in this case.

2. *No point in S is excellent for ab*. It turns out that this case cannot happen. For the sake of obtaining a contradiction we assume the contrary. Let S' (resp. S'') be the set of points $p \in S$ such that $\pi/2 < \angle pba < \pi$ (resp. $\pi/2 < \angle pab < \pi$). By assumption all points from S lie either in $S' \cup S''$ or on line ab. WLOG we may assume that S' is non-empty. Again we split the analysis into several subcases.
   2.1 *There is a point* $c_0 \in S$ *such that* $0 < \angle bac_0 < \pi/6$. Clearly, $c_0 \in S'$, i.e., $\angle c_0ba > \pi/2$. Hence, point b is excellent for ac_0 and from the previous case we know that $c_1 \in S$, where c_1 is a point such that $\triangle ac_0c_1$ is nice with base ac_0 and b and c_1 are on the same side of ac_0 (see Fig. 1b). Since c_1 is not excellent for ab, we conclude that b is excellent for c_1, so c_2 lies in S, where c_2 is a point such that $\triangle ac_1c_2$ is nice with base ac_1 and b and c_2 are on the same side of ac_1. By proceeding in this manner we get an infinite sequence of points $c_0, c_1, c_2, \ldots$ lying on segments ac_0 and ac_1. Obviously, $\lim_{k\to\infty} ac_k = 0$, and all points c_k for large enough k are excellent for ab, which is a contradiction.
   2.2 *There is a point* $c_0 \in S$ *such that* $\pi/6 < \angle bac_0 < \pi/2$ *and* $\angle bac_0 \neq \pi/3$. Similarly as in the previous case we can get a point $c_1 \in S$ such that $0 < \angle bac_1 < \pi/6$, so this case reduces to the previous one.
   2.3 *There is a point* $c_0 \in S$ *such that* $\angle bac_0 = \pi/6$. Let c_1 be the nine-point center of $\triangle abc_0$. If $\angle bac_1 \neq 0$, then $c_1 \in S'$ and $\angle bac_1 \notin \{\pi/6, \pi/3\}$ (by Lemma 8), so we are done by cases 2.1 and 2.2. Let us suppose that $\angle bac_1 = 0$, i.e., c_1 lies on line ab. By Lemma 7 this is possible only if $\triangle abc_0$ is nice (with base ac_0) and $c_1 = b$. Therefore, there are two possible positions for point c_0. WLOG we can assume that c_0 lies in the upper half-plane. Denote by $\overline{c_0}$ the point symmetric to c_0 with respect to line ab. Since there are at least five points in S, there is at least one point $x \in S$ different from $a, b, c_0, \overline{c_0}$.
      2.3.1 $x \in S''$. We are done, unless $\angle xba \in \{\frac{\pi}{6}, \frac{\pi}{3}\}$.
      If $\angle xba = \frac{\pi}{6}$, then we are done, unless $\triangle abx$ is nice with base bx. So in this case there are two possible positions for x, but in any case point a is excellent for c_0x (see Fig. 2a, b). Hence, the point y that forms with c_0 and x a nice triangle (with base c_0x) also belongs to S. One can check that y is excellent for ab. Contradiction.
      If $\angle xba = \frac{\pi}{3}$, then by Lemma 9 the nine-point center of $\triangle abx$ is a point that brings us back to the previous case.

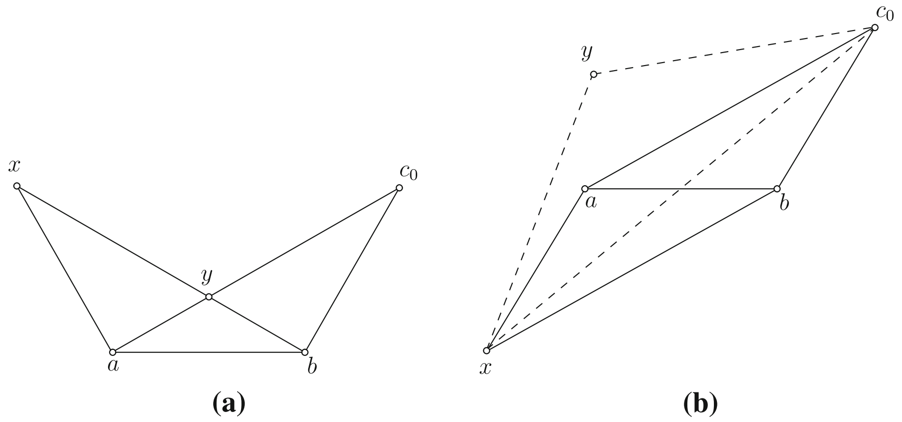

Fig. 2 Lemma 10, case 2.3, for any of two different positions of x point y is excellent for ab, in the second case $\angle bay = \pi/2$

2.3.2 *x lies on line ab*. By Lemma 6 the nine-point center of at least one of the triangles xac_0 and xbc_0 does not lie on line ab. Moreover, it is easy to see that at least one of them does not coincide with c_0 or $\overline{c_0}$ and also does not lie on line ab. This returns us to one of the already considered cases.

2.4 *There is a point $c_0 \in S$ such that $\angle bac_0 = \pi/3$*. By Lemma 9 the nine-point center of $\triangle abc_0$ must be a point satisfying the condition of the previous case.

□

Having Lemma 10, it is easy to prove the following lemma, from which Theorem 1 will follow immediately.

Lemma 11 *For any two points $a, b \in S$, there is a point $c \in S$, such that c lies on segment ab and $ca/cb \in \{\frac{1}{2}, 2\}$.*

Proof By Lemma 10 there is $n \in S$ such that $\triangle abn$ is nice with base ab. Consider the points x and x' such that $\triangle bnx$ and $\triangle bnx'$ are nice (with base bn) and x' lies on ab, while x is on the other side of bn (see Fig. 3). By Lemma 10 applied to bn we conclude that either x or x' must be in S. If $x' \in S$, then we are done, so let us assume that $x \in S$. Let u be a point on segment bn such that $bu : nu = 2$. Since u is the nine-point

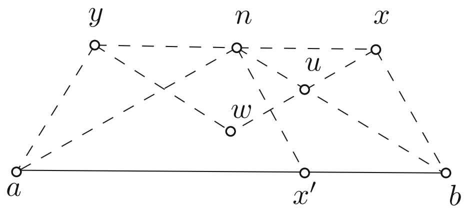

Fig. 3 Lemma 11, iterated nice triangles

center of abx, it must be in S. Let y be a point symmetric to x w.r.t. n. Analogously, we can assume that $y \in S$. Now u is excellent for xy, and therefore $w \in S$, where w is symmetric to x w.r.t. u. Finally, w is excellent for bn and therefore $x' \in S$. □

Proof of Theorem 1 Recall that we are proving Theorem 1 under the additional condition that S is a closed set. Once we are done with this proof, it is enough to apply the theorem to set $\overline{S}$ to conclude that it holds in full generality.

By Lemma 11 we have that $S = conv(S)$, which implies that S contains a disc D. Let a be an arbitrary point in the plane. Consider a circle C with center a which intersects the interior of D. Let x, y, z be three arbitrary points in $C \cap int(D)$. If we choose x, y, z to be close enough to each other, then the unique triangle $x'y'z'$, which has x, y, z as the midpoints of its sides, must be contained in $int(D)$. Therefore, $x', y', z' \in S$ and a is the nine-point center of triangle $x'y'z'$. Thus, $a \in S$ and, since a was arbitrary, we conclude that $S = \mathbb{R}^2$. □

Remark It is easy to see from the proof that the only exceptional configuration of three points is a triangle with angles $\pi/6, \pi/6, 2\pi/3$ and the only exceptional configuration of four points is the vertex set of an equilateral triangle with its center.

3 Partitioning the Euclidean space into sets containing their simplex centers

Here we prove Theorems 2, 3 and 4.

Proof of Theorem 2 Suppose the contrary. Then there exists a line l which contains points of different colors. We can find four points m, n, p, q in that order on line l such that m and n have the same color, while p and q have different colors.

Choose line l to be the x-axis and let $m = (0, 0)$, $n = (a, 0)$, $p = (b, 0)$, $q = (c, 0)$. Without loss of generality we assume that $a < b < c$. Consider the points

$$r = \left(\frac{bc}{b+c-a}, \frac{bc}{b+c-a} - a\right) \quad \text{and} \quad s = \left(\frac{bc}{b+c-a}, -\frac{bc}{b+c-a}\right).$$

It is easy to check that points m, n, r, s form an orthocentric system as well as points p, q, r, s (Fig. 4a). Now, no matter how the points r and s are colored, we have an orthocentric system with three points of one color and the fourth point of different color, which is a contradiction. □

Proof of Theorem 3 Let us denote the colors by 1 and -1. The proof is based on the following two simple observations.

(1) Any $(n-2)$-sphere contains n affinely independent points of the same color.
(2) If S is an $(n-1)$-sphere with center of color i, then for any hyperplane H there is at most one hyperplane H' parallel to H such that $H \cap S$ contains n affinely independent points of color $-i$.

In order to prove (1) it is enough to notice that we can choose $2n-1$ points on the $(n-2)$-sphere so that any n of them are affinely independent. Then by the pigeonhole principle we can find n points of the same color. As for the claim (2), suppose that there

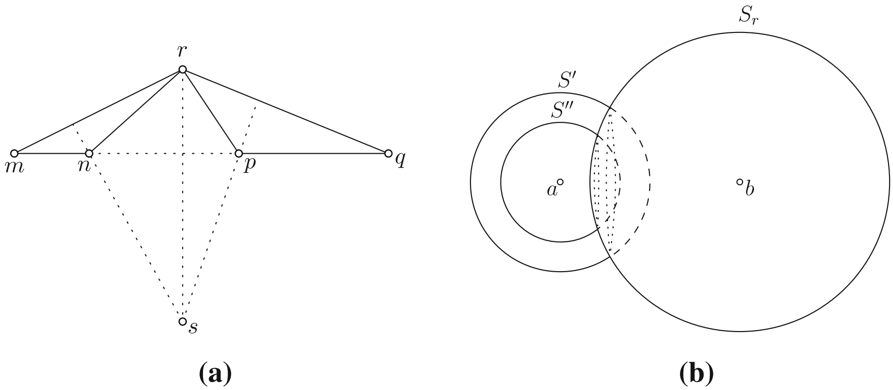

Fig. 4 **a** Proof of Theorem 2; **b** proof of Theorem 3

are two such hyperplanes H' and H''. Then we can find affinely independent points $a_1, \dots, a_n$ of color $-i$ in $H' \cap S$ that span H' and by taking a point $a_{n+1} \in H'' \cap S$ of the same color we get a simplex $a_1 \dots a_{n+1}$ whose all vertices have color $-i$ and whose circumcenter, center of S, has color i, which is a contradiction.

Now we prove the theorem by contradiction. Suppose that there are two points of different colors: a of color 1 and b of color -1. For simplicity assume that $|ab| = 1$. Consider spheres S' and S'' centered at a with radii $\frac{1}{2}$ and $\frac{1}{3}$. Let H be a hyperplane orthogonal to ab. For any $r \in (0, 1)$ let S_r be a $(n-1)$-sphere with center b and radius r (Fig. 4b). It is easy to see that $S_r \cap S'$ and $S_r \cap S''$ are $(n-2)$-spheres lying in hyperplanes parallel to H. By (2) we can choose $r \in (0, 1)$ so that $S_r \cap S'$ as well as $S_r \cap S''$ does not contain n affinely independent points of color -1. By (1) we conclude that $S_r \cap S'$ and $S_r \cap S''$ contain n affinely independent points of color 1. But now (2) applied to the sphere S_r gives a contradiction. □

The following proof of Theorem 4 is largely inspired by the discussion that we had on MathOverflow (2010) forum (the idea of the proof is due to Thomas Kragh).

Proof of Theorem 4 First, we prove that there exists a basis $B_n = \{v_\alpha\}$ for $\mathbb{R}^n$, considered as a vector space over the rationals $\mathbb{Q}$, which is everywhere dense in $\mathbb{R}^n$. Let $I_1, I_2, \dots$ be the list of all open balls with rational centers and radii. We construct inductively a set $\{v_1, v_2, \dots\}$ of linearly independent vectors such that $v_k \in I_k$ for all k. Let $v_1 \in I_1$ be arbitrary. Suppose we have already chosen $v_1 \in I_1, \dots, v_k \in I_k$ that are linearly independent. Now there are only countably many vectors in I_{k+1} that are equal to a linear combination of vectors $v_1, \dots, v_k$ with rational coefficients. Hence, we can pick $v_{k+1} \in I_{k+1}$ so that $v_1, \dots, v_k, v_{k+1}$ are linearly independent. The inductive step is done and, therefore, we have a countable set of linearly independent vectors, which is everywhere dense in $\mathbb{R}^n$. By the well-known theorem we can extend this set to a basis B_n of $\mathbb{R}^n$.

Now we can finish the proof of the theorem. We partition $\mathbb{R}^n$ into two parts A and B by using the basis B_n in the following way. For an arbitrary $v \in \mathbb{R}^n$, let $v = r_1 v_{\alpha_1} + \dots + r_k v_{\alpha_k}$ be its unique representation in the basis B_n. If $r_1 + \dots + r_k \geq 0$, we put v in the set A, otherwise, we put it in B. It is easy to check that this partition satisfies the centroid-condition and also both A and B are clearly everywhere dense, since the basis is everywhere dense. □

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