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Cases of equality and strict inequality in the extended Hardy–Littlewood inequalities*

Hichem Hajaiej
IACS–FSB, Section de Mathématiques, EPFL, CH-1015 Lausanne, Switzerland (hichem.hajaiej@epfl.ch)

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Extended Hardy–Littlewood inequalities are

$$\int_{\mathbb{R}^N} H(|x|, u_1(x), \dots, u_m(x))\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} H(|x|, u_1^*(x), \dots, u_m^*(x))\,\mathrm{d}x, \qquad (*)$$

where $\{u_i\}_{1\leqslant i\leqslant m}$ are non-negative functions and $\{u_i^*\}_{1\leqslant i\leqslant m}$ denote their Schwarz symmetrization.

In this paper, we determine appropriate conditions under which equality in $(*)$ occurs if and only if $\{u_i\}_{1\leqslant i\leqslant m}$ are Schwarz symmetric.

1. Introduction

Cases of equality in symmetrization inequalities have various applications in mathematical physics [5,6,20,21], and economy [7,8]; some connections with Brenie's map and Monge's transportation problem were pointed out in [1,2,7,8,12,13]. However, results related to this topic are not numerous.

Lieb showed in an innovative paper the uniqueness of minimizing solution of Choquard's nonlinear equation establishing cases of equality in the Riesz rearrangement inequality with a fixed, strictly decreasing kernel [20]. For the more general case, A. Burchard determined cases of equality in a famous paper [5], proving and generalizing a conjecture by Lieb under optimal hypotheses. Techniques developed by Burchard to solve this complicated problem are very subtle but cannot apply to the extended Hardy–Littlewood inequalities. The study of cases of equality in $(*)$ is as important and exciting as in the Riesz rearrangement inequality. Lieb and Loss, two pioneers of this domain, developed tricky methods to determine cases of equality in $(*)$ when $H(t,s) = ts$ or $H(t,s) = -J(t-s)$. For this more complicated case, their method is based on the decomposition of J into J_+ and J_-, where

$$J_+(t) = \begin{cases} 0 & \text{for } t \leqslant 0, \\ J(t) & \text{for } t > 0, \end{cases} \qquad J_-(t) = \begin{cases} J(t) & \text{for } t \leqslant 0, \\ 0 & \text{for } t > 0. \end{cases}$$

Thus, it suffices to study cases of equality for J_+ and J_- separately. Assuming that $J' > 0$ and consequently that $J_+' > 0$ for $t > 0$ and $-J_-' > 0$ for $t < 0$, they simplify shrewdly the establishment of cases of equality in $(*)$ for this special

*Dedicated to the senior of symmetrization inequalities, E. H. Lieb, on the occasion of his 70th birthday.

case. This work was completed in [21, theorem 3.5, p. 75]. Let us point out that the method of Lieb and Loss hangs closely on the strict positivity of J_+' and $-J_-'$. Consequently, it cannot apply to the general case $(*)$ without severe restrictions on H. Note also that necessary integrability assumptions were not mentioned in their results [9, theorem 3.4, p. 74, theorem 3.5, p. 75]. Inspired by their work, we extend their results to the more general case $(*)$ following a self-contained approach. Our method also enables us to generalize their result in the case discussed above.

Our principal ingredient is to reduce the study of cases of equality in $(*)$ to that of integrands which are the product of derivatives of H and step functions.

Let us explain heuristically (details are given in § 4) how to do so in the case where $H = \psi(t, s)$ is such that

$$\psi(u(x), v(x)) = \int_0^\infty \partial_2\psi(u(x), s)1_{\{s\leqslant v(x)\}}\,\mathrm{d}s, \tag{S1}$$

where u and v are two functions vanishing at infinity. Then

$$\int_{\mathbb{R}^N} \psi(u(x), v(x))\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi(u^*(x), v^*(x))\,\mathrm{d}x \quad (< \infty)$$

if and only if

$$\int_{\mathbb{R}^N}\biggl(\int_0^\infty \partial_2\psi(u(x), s)1_{\{s\leqslant v(x)\}}\,\mathrm{d}s\biggr)\mathrm{d}x = \int_{\mathbb{R}^N}\biggl(\int_0^\infty \partial_2\psi(u^*(x), s)1_{\{s\leqslant v^*(x)\}}\,\mathrm{d}s\biggr)\mathrm{d}x.$$

Under appropriate conditions on the integrands, we can exchange the ds and dx integrations, we then obtain

$$\int_0^\infty\int_{\mathbb{R}^N} \partial_2\psi(u(x), s)1_{\{v(x)\geqslant s\}}\,\mathrm{d}x\,\mathrm{d}s = \int_0^\infty\int_{\mathbb{R}^N} \partial_2\psi(u^*(x), s)1_{\{v^*(x)\geqslant s\}}\,\mathrm{d}x\,\mathrm{d}s. \tag{S2}$$

Under well-known conditions on $\partial_2\psi$, the Hardy–Littlewood inequality implies that

$$\int_{\mathbb{R}^N} \partial_2\psi(u(x), s)1_{\{v(x)\geqslant s\}}\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \partial_2\psi(u^*(x), s)1_{\{v^*(x)\geqslant s\}}\,\mathrm{d}x. \tag{S3}$$

Assuming (S3), equation (S2) is true if and only if, for almost every $s \geqslant 0$,

$$\int_{\mathbb{R}^N} \partial_2\psi(u(x), s)1_{\{v(x)\geqslant s\}}\,\mathrm{d}x = \int_{\mathbb{R}^N} \partial_2\psi(u^*(x), s)1_{\{v^*(x)\geqslant s\}}\,\mathrm{d}x. \tag{S4}$$

Evidently, it is much easier to determine under which conditions (S4) holds if and only if $u = u^*$ a.e. and $v = v^*$ a.e. $(*)$ $v = v^*$ a.e. than to do it for $(*)$.

In § 3, we develop a self-contained approach enabling us to solve $(*)$ for an appropriate class of functions (lemma 3.12). Consequently, we determine cases of equality in $(*)$ for a large class of integrands $H = \psi(t, s)$.

Applications of our results in the nonlinear Schrödinger equation, in the calculus of variations and in economics, discussed in § 6, are a good illustration of their importance. Particularly interesting connections between our theorems and Monge's transportation problem are studied in § 7.

Throughout the following, $u = u^*$ a.e. is a strictly decreasing function.

In § 4, we establish hypotheses under which

$$\int_{\mathbb{R}^N} \psi(u(x), v(x))\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi(u(x), v^*(x))\,\mathrm{d}x \tag{1.1}$$

if and only if $v = v^*$ a.e.

Our main result is theorem 4.1, where we prove that the above assertion is true if ψ is absolutely continuous with respect to the second variable with $\partial_2\psi$ strictly increasing with respect to the first variable and $\partial_2\psi$, u, v and v^* satisfying two integrability assumptions.

In the special case $\psi(t, s) = ts$, the required conditions are verified if

$$\int u(x)v^*(x)\,\mathrm{d}x < \infty.$$

In this case,

$$\int_{\mathbb{R}^N} u(x)v(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} u(x)v^*(x)\,\mathrm{d}x \quad \text{if and only if } v = v^* \text{ a.e.}$$

Let us point out that the strict monotonicity assumptions in theorem 4.1 (and in all results stated with these conditions) cannot be weakened. Indeed, simple examples show that, if u is not strictly increasing, (1.1) holds even though v differs from v^* everywhere. Moreover, if we drop the integrability assumptions, the conclusion of theorem 4.1 may fail (see remark 4.4).

As a first consequence of theorem 4.1, we determine a large class of functions J for which

$$\int_{\mathbb{R}^N} J(u(x), v(x))\,\mathrm{d}x = \int_{\mathbb{R}^N} J(u(x), v^*(x))\,\mathrm{d}x \tag{1.2}$$

if and only if $v = v^*$ a.e. (theorem 5.6).

A particularly interesting corollary of theorem 5.6 is theorem 3.5 of [21], namely, if J is a non-negative strictly convex function such that $J(0) = 0$, $J(u)$ and $J(u - v) \in L^1(\mathbb{R}^N)$, where v is a function vanishing at infinity, then (1.2) holds if and only if $v = v^*$ a.e. (theorem 5.7). Let us point out that integrability assumptions in theorem 5.7 are essential. These conditions were not stated in theorem 3.5 of [21]; it seems to us that they were assumed tacitly.

Note also that the very particular case $J(x) = |x|^p$, $p > 1$, was crucial for producing optimizing sequences for functionals having appropriate symmetry properties in the elegant approach of Carlen and Loss [6].

In § 6, we focus our attention on

$$\int_{\mathbb{R}^N} G(|x|, v(x))\,\mathrm{d}x = \int_{\mathbb{R}^N} G(|x|, v^*(x))\,\mathrm{d}x. \tag{1.3}$$

By setting $u(x) = 1/|x|$ and $G(|x|, s) = \psi(1/|x|, s)$ in (1.1), we deduce easily, from theorem 4.1, hypotheses on G under which (1.3) is true if and only if $v = v^*$ a.e.

Now let us give a foretaste of some applications of results stated in § 6. The study of the strict inequality

$$\int_{\mathbb{R}^N} G(|x|, v(x))\,\mathrm{d}x < \int_{\mathbb{R}^N} G(|x|, v^*(x))\,\mathrm{d}x \tag{1.4}$$

is crucial in order to characterize the minimizers of $(P_c) : \inf_{v\in S_c} J(v)$, where J is the functional defined on $H^1(\mathbb{R}^N)$ by

$$J(v) = \int_{\mathbb{R}^N} (\tfrac{1}{2}|\nabla v|^2 - G(|x|, v(x)))\,\mathrm{d}x \tag{1.5}$$

and $S_c = \{v \in H^1(\mathbb{R}^N) : |v|_2^2 = c^2\}$ for $c > 0$. In [18], we established hypotheses under which (P_c) is attained by the Schwarz symmetric function. In this paper, we show that all minimizers for (P_c) have this property. Our approach hinges on the following strict inequality:

$$\text{for any } v \in H^1(\mathbb{R}^N),\ J(v^*) < J(v) \text{ unless } v = v^* \text{ a.e.} \tag{1.6}$$

On the other hand, we know [4] that, for any $v \in H^1(\mathbb{R}^N)$,

$$|\nabla v^*|_2 \leqslant |\nabla v|_2, \tag{1.7}$$

and in theorem 3.5 of [17], we established, together with Stuart, conditions under which

$$\int_{\mathbb{R}^N} G(|x|, v(x))\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} G(|x|, v^*(x))\,\mathrm{d}x.$$

Suppose that the hypotheses of this result are satisfied. Then (1.6) holds if and only if

$$\text{for any } v \in H^1(\mathbb{R}^N),\ |\nabla v^*|_2 < |\nabla v|_2 \text{ unless } v = v^* \text{ a.e,} \tag{1.8}$$

or, for any $v \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} G(|x|, v(x))\,\mathrm{d}x < \int_{\mathbb{R}^N} G(|x|, v^*(x))\,\mathrm{d}x \text{ unless } v = v^* \text{ a.e.} \tag{H}$$

Clearly, (1.8) is not true (for more details, see [4]). Since the first alternative is false, we focus our study on (H), obtaining theorems 6.1 and 6.3, which are well suited for dealing with (P_c) and more generally for other problems arising in mathematical physics (see § 6 for more details).

The techniques we will use to determine cases of equality in $(*)$ for integrands H, depending on more than two variables, are different. They will be discussed in depth in further publications. The paper concludes with a challenging open question concerning connections between symmetrization inequalities and transportation problems.

2. Notation

All statements about measurability refer to the Lebesgue measure, mes, on $\mathbb{R}^N$ or $[0, \infty)$.

For $r \geqslant 0$, $B(0, r) = \{x \in \mathbb{R}^N : |x| < r\}$ and $B'(0, r)$ is either $B(0, r)$ or $\overline{B(0, r)}$.

There is a constant $V_N > 0$ such that $\operatorname{mes} B(0, r) = V_N r^N$ for all $r > 0$. For a measurable subset A of $\mathbb{R}^N$ with $\operatorname{mes} A < \infty$, $A^* = B(0, r)$, where $V_N r^N = \operatorname{mes} A$. Note that A^* is open even though A may not be.

If A and B are two measurable subsets of $\mathbb{R}^N$ such that $A = B \cup N$, where $\operatorname{mes} N = 0$, then we write $A \equiv B$.

Let M_N denote the set of all extended real-valued functions which are measurable on $\mathbb{R}^N$. For $u \in M_N$ and $t \in \mathbb{R}$, $d_u(t) = \operatorname{mes}\{x \in \mathbb{R}^N : u(x) > t\}$ is the distribution function of u and

$$F_N = \{u \in M_N : 0 \leqslant u \leqslant \infty \text{ a.e on } \mathbb{R}^N \text{ and } d_u(t) < \infty \text{ for all } t > 0\}$$

is the set of Schwarz symmetrizable functions.

Following the terminology of [15], we say that an element $u \in F_N$ is Schwarz symmetric if there exists a non-increasing function $h : (0, \infty) \to [0, \infty)$ such that $u(x) = h(|x|)$ for almost every $x \in \mathbb{R}^N$. Its Schwarz symmetrization is denoted by u^*.

In an integral where no domain of integration is indicated, the integration extends over all $\mathbb{R}^N$.

3. Preliminaries

In this section, we will make frequent use of the following basic result.

LEMMA 3.1. *Let Y and Z be two integrable functions defined on $[0, \infty)$ such that*

$$Y \leqslant Z \quad \text{and} \quad \int_0^\infty Y(s)\,\mathrm{d}s = \int_0^\infty Z(s)\,\mathrm{d}s.$$

Then $Y \equiv Z$ for almost every $s \geqslant 0$.

DEFINITION 3.2. A radially symmetric function $u : \mathbb{R}^N \to \mathbb{R}$ is strictly decreasing when $u(x) > u(y)$ if $|x| < |y|$.

REMARK 3.3. If $u \in F_N$ is a strictly decreasing function, then $u = u^*$ a.e., $u > 0$ and $\lim_{|x|\to\infty} u(x) = 0$.

LEMMA 3.4. *Let A be a measurable subset of $\mathbb{R}^N$ having finite measure with $A^* = B(0, r_A)$. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative, non-increasing function such that there exists s_0 satisfying $\{h(|x|) \geqslant s_0\} = B'(0, r_A)$ and $\int h(|x|)1_{A^*}(x)\,\mathrm{d}x < \infty$. Then*

$$\int h(|x|)1_A(x)\,\mathrm{d}x = \int h(|x|)1_{A^*}(x)\,\mathrm{d}x \text{ if and only if } A \equiv A^*.$$

Proof.

$$\int h(|x|)1_A(x)\,\mathrm{d}x = \int \left(\int_0^{h(|x|)} \mathrm{d}s\right) 1_A(x)\,\mathrm{d}x.$$

Using Tonelli's theorem, we obtain

$$\int h(|x|)1_A(x)\,\mathrm{d}x = \int_0^\infty \left(\int 1_{\{h(|x|)\geqslant s\}} 1_A(x)\,\mathrm{d}x\right)\mathrm{d}s. \tag{3.1}$$

In a same way, we have that

$$\int h(|x|)1_{A^*}(x)\,\mathrm{d}x = \int_0^\infty \left(\int 1_{\{h(|x|)\geqslant s\}} 1_{A^*}(x)\,\mathrm{d}x\right)\mathrm{d}s. \tag{3.2}$$

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On the other hand, for any $s \geqslant 0$,

$$\int 1_{\{h(|x|)\geqslant s\}} 1_A(x)\,\mathrm{d}x \leqslant \int 1_{\{h(|x|)\geqslant s\}} 1_{A^*}(x)\,\mathrm{d}x. \tag{3.3}$$

Indeed, if $\{h(|x|) \geqslant s\} = \mathbb{R}^N$, (3.3) is trivial. Otherwise, $\{h(|x|) \geqslant s\}$ has a finite measure and (3.3) is an immediate consequence of the Hardy–Littlewood inequality (see proposition 2.3 of [16], for example), since $\{h(|x|) \geqslant s\}^* = \{h(|x|) \geqslant s\}$. Using (3.3), it follows from lemma 3.1 that (3.1) is equal to (3.3) if and only if, for almost every $s \geqslant 0$,

$$\int 1_{\{h(|x|)\geqslant s\}} 1_A(x)\,\mathrm{d}x = \int 1_{\{h(|x|)\geqslant s\}} 1_{A^*}(x)\,\mathrm{d}x. \tag{3.4}$$

For any measurable set M of $\mathbb{R}^N$ with finite measure and $s \geqslant 0$, set

$$F_M(s) = \int 1_{\{h(|x|)\geqslant s\}} 1_M(x)\,\mathrm{d}x.$$

By the monotonicity of h, there exists $r(s) \geqslant 0$ such that $\{h(|x|) \geqslant s\} = B'(0, r(s))$ for all $s \geqslant 0$, where

$$r(s) = \frac{1}{V_N^{1/N}} \operatorname{mes}^{1/N}\{h(|x|) \geqslant s\}.$$

It follows immediately that $r(s)$ is a left continuous function. The dominated convergence theorem enables us to conclude that

$$F_M \text{ is a left continuous function on } [0, \infty). \tag{3.5}$$

Equation (3.4), together with (3.5), implies that $F_A(s) = F_{A^*}(s)$ for all $s \geqslant 0$. In particular, for $s = s_0 : F_A(s_0) = F_{A^*}(s_0)$, that it is to say, $\operatorname{mes} A \cap A^* = \operatorname{mes} A^* = \operatorname{mes} A$, which ends the proof. □

Remark 3.5. If h satisfies the hypotheses of lemma 3.4, then $h(|x|) < s_0$ if $|x| > r_A$.

Corollary 3.6. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly decreasing function. Then, for any measurable subset A of $\mathbb{R}^N$ having finite measure such that $\int h(|x|)1_{A^*}(x)\,\mathrm{d}x < \infty$,*

$$\int h(|x|)1_A(x)\,\mathrm{d}x = \int h(|x|)1_{A^*}(x)\,\mathrm{d}x$$

if and only if $A \equiv A^*$.

Proof. If h is non-negative and strictly decreasing, then, for any $r > 0$, there exists $s_r > 0$ such that $\{h(|x|) \geqslant s_r\} = B'(0, r)$. We then can apply lemma 3.4 to any measurable set A having finite measure. □

Remark 3.7. The strict monotonicity of h is essential in order to obtain the result for all measurable subsets of $\mathbb{R}^N$ having finite measure. Indeed, if h does not satisfy this condition, we can easily construct a measurable set $A \not\equiv A^*$ such that $\int h(|x|)1_A(x)\,\mathrm{d}x = \int h(|x|)1_{A^*}(x)\,\mathrm{d}x$.

COROLLARY 3.8. *Let h and A be as in the previous corollary. Then*

$$\int h(|x|)1_A(x)\,\mathrm{d}x < \int h(|x|)1_{A^*}(x)\,\mathrm{d}x \text{ unless } A \equiv A^*.$$

Proof. By lemma 3.7 of [17], we have that $\int h(|x|)1_A(x)\,\mathrm{d}x \leqslant \int h(|x|)1_{A^*}(x)\,\mathrm{d}x$. This, together with corollary 3.6, enables us to conclude the proof. □

LEMMA 3.9. *Let $k : \mathbb{R}^N \to \mathbb{R}$ be a function such that the following conditions hold.*

(i) $k(x) = k(y)$ *if* $|x| = |y|$.

(ii) $k(x) \leqslant k(y)$ *if* $|x| > |y|$.

(iii) $l = \lim_{|x|\to\infty} k(x) > -\infty$. *Then* $\int k(x)1_A(x)\,\mathrm{d}x \leqslant \int k(x)1_{A^*}(x)\,\mathrm{d}x$ *for any measurable subset A of $\mathbb{R}^N$ with finite measure.*

Proof. Consider $h(|x|) = k(x) - l$. Clearly, h is well defined by (i), non-negative and non-increasing by (ii). By lemma 3.1 of [17],

$$\int (k(x) - l)1_A(x)\,\mathrm{d}x \leqslant \int (k(x) - l)1_{A^*}(x)\,\mathrm{d}x, \tag{3.6}$$

$\int k(x)1_A(x)\,\mathrm{d}x = \int (k(x) - l)1_A(x)\,\mathrm{d}x + l \operatorname{mes} A$, by lemma 3.1 of [17], since $\operatorname{mes} A < \infty$. By replacing A by A^*, we obtain

$$\int k(x)1_{A^*}(x)\,\mathrm{d}x = \int (k(x) - l)1_{A^*}(x)\,\mathrm{d}x + l \operatorname{mes} A^*.$$

This ends the proof, since $\operatorname{mes} A = \operatorname{mes} A^*$. □

LEMMA 3.10. *Let $k : \mathbb{R}^N \to \mathbb{R}$ be a strictly decreasing function verifying (iii) of lemma 3.9 and A be a measurable subset of $\mathbb{R}^N$ with finite measure such that $\int k(x)1_{A^*}(x)\,\mathrm{d}x < \infty$. Then the following conditions hold:*

(1) $\displaystyle\int k(x)1_A(x)\,\mathrm{d}x = \int k(x)1_{A^*}(x)\,\mathrm{d}x$ *if and only if* $A \equiv A^*$;

(2) $\displaystyle\int k(x)1_A(x)\,\mathrm{d}x < \int k(x)1_{A^*}(x)\,\mathrm{d}x$ *unless* $A \equiv A^*$.

LEMMA 3.11. *Let f and g be two non-negative functions defined on $\mathbb{R}^N$ such that $\{f(x) \geqslant s\} \equiv \{g(x) \geqslant s\}$ for almost every $s \geqslant 0$. Then $f = g$ a.e.*

Proof. $f(x) = \int_0^\infty 1_{\{s\leqslant f(x)\}}\,\mathrm{d}s = \int_0^\infty 1_{\{f(x)\geqslant s\}}\,\mathrm{d}s$. We can suppose without loss of generality that, for almost every $s \geqslant 0$,

$$\{f(x) \geqslant s\} = \{g(x) \geqslant s\} \cup E^s,$$

where $\operatorname{mes} E^s = 0$. It then follows that $f(x) = g(x) + \int_0^\infty 1_{E^s}(x)\,\mathrm{d}s$.

$\int 1_{E^s}(x)\,\mathrm{d}x = \operatorname{mes} E^s = 0$, then $\int_0^\infty \int 1_{E^s}(x)\,\mathrm{d}x\,\mathrm{d}s = 0$, by the Tonelli theorem, and we can exchange the $\mathrm{d}x$ and $\mathrm{d}s$ integrations; we then obtain

$$\int\int_0^\infty 1_{E^s}(x)\,\mathrm{d}s\,\mathrm{d}x = 0,$$

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which is equivalent to $\int_0^\infty 1_{E^s}(x)\,\mathrm{d}s = 0$ for almost every $x \in \mathbb{R}^N$, proving that $f = g$ a.e. $\square$

LEMMA 3.12. *Let $k : \mathbb{R}^N \to \mathbb{R}$ be a strictly decreasing function such that*

$$\lim_{|x|\to\infty} k(x) > -\infty,$$

and let f be an element of F_N. Suppose that, for almost every $s \geqslant 0$,

$$\int k(x) 1_{\{f^*(x) \geqslant s\}}\,\mathrm{d}x < \infty.$$

Then

$$\int k(x) 1_{\{f(x) \geqslant s\}}\,\mathrm{d}x = \int k(x) 1_{\{f^*(x) \geqslant s\}}\,\mathrm{d}x \quad \text{for almost every } s \geqslant 0,$$

if and only if $f = f^*$ a.e.

Proof. This is an immediate consequence of lemmas 3.10 and 3.11, since $\{f(x) \geqslant s\}^* = \{f^*(x) \geqslant s\}$ for all $s \geqslant 0$. $\square$

REMARK 3.13. Apart from having their own role in the next section, the results proved in this section have many other important applications and developments, which will be discussed in further publications.

4. Our main results

Throughout this section we prove some results for which we cannot find a reference. It is quite surprising that such theorems do not exist in the literature, as they have particularly interesting far-reaching consequences.

In this section u is a strictly decreasing element of F_N.

THEOREM 4.1. *Let $\psi : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ be a function verifying:*

(1) *there exists $F : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ such that $\psi(t, s) = \int_0^s F(t, w)\,\mathrm{d}w$ for all $t > 0$, $s \geqslant 0$;*

(2) *for almost every $s \geqslant 0$, $F(\cdot, s)$ is strictly increasing on $(0, \infty)$;*

(3) $\displaystyle\iint_0^\infty |F(u(x), s)| 1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x$ *and* $\displaystyle\iint_0^\infty |F(u(x), s)| 1_{\{s \leqslant v^*(x)\}}\,\mathrm{d}s\,\mathrm{d}x$ *are finite;*

(4) $\displaystyle\lim_{|x|\to\infty} F(u(x), s) = L_s > -\infty$ *for almost every $s \geqslant 0$.*

Let $v \in F_N$. Then $-\infty < \int \psi(u(x), v(x))\,\mathrm{d}x \leqslant \int \psi(u(x), v^*(x))\,\mathrm{d}x < \infty$ and

$$\int \psi(u(x), v(x))\,\mathrm{d}x = \int \psi(u(x), v^*(x))\,\mathrm{d}x$$

if and only if $v = v^*$ a.e.

Proof. First, observe that (3) enables us to use Fubini's theorem ensuring that $\psi(u(x), v(x))$ and $\psi(u(x), v^*(x))$ are two integrable functions on $\mathbb{R}^N$. Now suppose that $\int \psi(u(x), v(x))\,\mathrm{d}x = \int \psi(u(x), v^*(x))\,\mathrm{d}x$, and

$$\int \psi(u(x), v(x))\,\mathrm{d}x = \iint_0^\infty F(u(x), s)1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x. \tag{4.1}$$

By Fubini's theorem,

$$\int \psi(u(x), v(x))\,\mathrm{d}x = \int_0^\infty \int F(u(x), s)1_{\{v(x) \geqslant s\}}\,\mathrm{d}x\,\mathrm{d}s. \tag{4.2}$$

Similarly, we have that

$$\int \psi(u(x), v^*(x))\,\mathrm{d}x = \int_0^\infty \int F(u(x), s)1_{\{v^*(x) \geqslant s\}}\,\mathrm{d}x\,\mathrm{d}s. \tag{4.3}$$

For almost every $s \geqslant 0$, $x \mapsto F(u(x), s)$ satisfies all hypotheses of lemma 3.9. It follows that

$$\int F(u(x), s)1_{\{v(x) \geqslant s\}}\,\mathrm{d}x \leqslant \int F(u(x), s)1_{\{v^*(x) \geqslant s\}}\,\mathrm{d}x. \tag{4.4}$$

Combining (4.2)–(4.4), we obtain, using lemma 3.4 that, for almost every $s \geqslant 0$,

$$\int F(u(x), s)1_{\{v(x) \geqslant s\}}\,\mathrm{d}x = \int F(u(x), s)1_{\{v^*(x) \geqslant s\}}\,\mathrm{d}x.$$

The conclusion is an immediate application of lemma 3.12. □

The first part of the result is an immediate consequence of (4.2)–(4.4).

Remark 4.2. Assume that, for any $t > 0$, $a > 0$, we have the following conditions:

(F1) $\psi(t, \cdot)$ is continuous on $[0, a]$;

(F2) $\partial_2\psi(t, \cdot)$ exists for any $s \in [0, a] \setminus A$, where A is countable;

(F3) $\partial_2\psi(t, \cdot) \in L^1([0, a])$;

(F4) $\psi(t, 0) = 0$.

Then (1) of theorem 4.1 is satisfied by $F(t, w) = \partial_2\psi(t, w)$. For an illuminating discussion concerning the optimality of (F1) $\to$ (F4) to obtain (1) (see [11, p. 110]).

Clearly, if $\partial_2\Psi(t, \cdot)$ exists for any $s \in [0, a]$ and $t > 0$, ψ is non-negative and $\psi(t, 0) = 0$, then (1) of theorem 4.1 is satisfied by $F(t, w) = \partial_2\psi(t, w)$. This is an immediate consequence of the general fundamental theorem of calculus, where the positivity of ψ ensures that its Henstock–Kurzweil integral coincides with Lebesgue integral (for more details, see [11, p. 82]).

If (F1) $\to$ (F3), and (2)–(4) of theorem 4.1 hold, then the conclusion of theorem 4.1 is valid if

$$\int \psi(u(x), 0)\,\mathrm{d}x < \infty.$$

Indeed, in this case, for any $t > 0$, $s \geqslant 0$,

$$\psi(t,s) - \psi(t,0) = \int_0^s F(t,w)\,\mathrm{d}w.$$

Thus,

$$\begin{aligned}\int \psi(u(x),v(x)) &= \int \biggl(\int_0^\infty F(u(x),s)1_{\{s\leqslant v(x)\}}\,\mathrm{d}s\biggr) + \psi(u(x),0)\,\mathrm{d}x \\ &= \iint_0^\infty F(u(x),s)1_{\{s\leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x + \int \psi(u(x),0)\,\mathrm{d}x,\end{aligned}$$

showing that

$$\int \psi(u(x),v(x))\,\mathrm{d}x = \int \psi(u(x),v^*(x))\,\mathrm{d}x$$

if and only if

$$\iint_0^\infty F(u(x),s)1_{\{s\leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x = \iint_0^\infty F(u(x),s)1_{\{s\leqslant v^*(x)\}}\,\mathrm{d}s\,\mathrm{d}x.$$

We then continue in exactly the same way as in the proof of theorem 4.1.

If we suppose that F can be extended to $\mathbb{R}_+ \times \mathbb{R}_+$ and that (2) of theorem 4.1 holds for any $t \geqslant 0$, then (4) is satisfied, since $\lim_{|x|\to\infty} F(u(x),s) \geqslant F(0,s) > -\infty$.

If $\psi \in C^2(\mathbb{R}_+^* \times \mathbb{R}_+)$, then (2) is equivalent to $\partial_1\partial_2\psi > 0$.

Evidently, (2) is only needed for $t \in \operatorname{Im} u$ and for almost every $s \in \operatorname{Im} v \cup \operatorname{Im} v^*$.

Corollary 4.3. *Let $n \geqslant 1$, u as mentioned in the beginning of § 4 and $v \in F_N$ such that $\int u(x)v^*(x)\,\mathrm{d}x < \infty$. Then*

(i) $\displaystyle\int u(x)v^n(x)\,\mathrm{d}x = \int u(x)(v^*(x))^n\,\mathrm{d}x$ *if and only if $v = v^*$ a.e.*

(ii) $\displaystyle\int u(x)v^n(x)\,\mathrm{d}x < \int u(x)(v^*(x))^n\,\mathrm{d}x$ *unless $v = v^*$ a.e.*

Proof.

(i) Set $F(t,s) = nts^{n-1}$ and apply the previous theorem.

(ii) This follows by combining (i) and the Hardy–Littlewood inequality.

□

Remark 4.4. The integrability assumption in (i) (and consequently (ii)) is essential. If it is not satisfied, the conclusion in (i) (and consequently in theorem 4.1) may fail.

Take $N = 1$, $u(x) = 1/|x|^2$. Let $\varepsilon > 0$ and consider

$$A_\varepsilon = \{x \in \mathbb{R} : |x| < \tfrac{1}{2}\varepsilon\} \cup \{x \in \mathbb{R} : \tfrac{3}{2}\varepsilon < |x| < 2\varepsilon\}.$$

Then

$$\int u(x)1_{A_\varepsilon}(x)\,\mathrm{d}x = \int u(x)1_{A_\varepsilon^*}(x)\,\mathrm{d}x,$$

but A_ε differs from A_ε^* for any $\varepsilon > 0$.

Note also that this integrability condition was not mentioned in theorem 3.4 of [21, p. 74].

Now, we state a crucial result for the next section.

THEOREM 4.5. *Let $\varphi : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ be a function such that $\varphi(t,s) = \psi(t,s) + \alpha(t)$. The conclusion of the previous theorem remains valid if (ψ, u, v, v^*) satisfy hypotheses of theorem 4.1 and $\alpha(u)$ is integrable.*

5. Case of convex integrands

We begin this section proving the symmetrization inequality

$$\int J(f^*(x) - v^*(x))\,\mathrm{d}x \leqslant \int J(f(x) - v(x))\,\mathrm{d}x. \tag{5.1}$$

This inequality, called non-expansivity of the rearrangement, was first proved separately by Chiti [9] and Crandall and Tartar [10] in the case where J is a non-negative convex function such that $J(t) = J(-t)$ and $J(0) = 0$. A slight generalization of this result is due to Lieb and Loss [21], who were able to drop the assumption that J is symmetric. The next result recovers all theorems mentioned above.

THEOREM 5.1. *Let $J : \mathbb{R} \to \mathbb{R}$ be a convex function such that $J(0) = 0$. Then, for f and v elements of F_N verifying that*

$$\int J_-(f(x))\,\mathrm{d}x \quad \textit{and} \quad \int J_-(-v(x))\,\mathrm{d}x$$

are finite, (5.1) holds.

Proof. We define $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by $F(t,s) = -J(t-s)$. It suffices to prove that F has the property (CZR) (see [17]). Then theorem 5.2 of [17] ensures the conclusion. Let us recall that, if J is convex, then, for any $A < B < C < D$,

$$\frac{J(B) - J(A)}{B - A} \leqslant \frac{J(D) - J(C)}{D - C}.$$

$F(t,s) = -J(t-s)$ has the property (CZR) if and only if

$$J(b-d) - J(b-c) - J(a-d) - J(a-c) \leqslant 0, \tag{5.2}$$

for all $0 \leqslant a \leqslant b$ and $0 \leqslant c \leqslant d$. If $a = b$ or $c = d$, then (5.2) is evident, otherwise there are two subcases:

(i) if $a - d < a - c < b - d < b - c$, then

$$\frac{J(a-c) - J(a-d)}{(a-c) - (a-d)} \leqslant \frac{J(b-c) - J(b-d)}{(b-c) - (b-d)}$$

and (5.2) holds;

(ii) if $a - d < b - d < a - c < b - c$, then

$$\frac{J(b-d) - J(a-d)}{(b-d) - (a-d)} \leqslant \frac{J(b-c) - J(a-c)}{(b-c) - (a-c)}$$

and we conclude in the same way as before. □

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REMARK 5.2. Let us mention that, in our result, we do not require the non-negativity of J. If J verifies this hypothesis, then the integrability assumptions needed in theorem 5.1 are trivially satisfied and we deduce the result of Lieb and Loss.

COROLLARY 5.3 (Rosenbloom–Crowe–Zweibel, Lieb–Loss). *Let $J : \mathbb{R} \to \mathbb{R}_+$ be a convex function such that $J(0) = 0$. Then*

$$\int J(f^*(x) - v^*(x))\,\mathrm{d}x \leqslant \int J(f(x) - v(x))\,\mathrm{d}x$$

for all $f, v \in F_N$.

COROLLARY 5.4. *For any f, $v \in F_N \cap L^p(\mathbb{R}^N)$, $p \geqslant 1$,*

$$|f^* - v^*|_p \leqslant |f - v|_p.$$

Before treating cases of equality and strict inequality in (5.1), let us recall some crucial properties of convex functions.

(P_0) $J : \mathbb{R} \to \mathbb{R}$ is a convex function.

(P_1) J is absolutely continuous on every compact subinterval of $\mathbb{R}$ and J' (it is defined almost everywhere in $\mathbb{R}$) is increasing in the domain where it is defined.

(P_2) J is differentiable almost everywhere, $J' \in L^1([a, b])$ for all $-\infty < a < b < \infty$ and $J(b) - J(a) = \int_a^b J'(s)\,\mathrm{d}s$, where J' is a non-decreasing function where it is defined.

(P_0), (P_1) and (P_2) are equivalent. For more details, see [11, p. 109].

REMARK 5.5. Note also that J'_r and J'_l, the right and the left derivatives of a convex function J, are defined everywhere. Thus, without loss of generality, we can suppose that J also has this property (otherwise, we replace J by J'_r or J'_l).

In all the following results, u is a strictly decreasing element of F_N.

THEOREM 5.6. *Let $J : \mathbb{R} \to \mathbb{R}$ be a strictly convex function such that $J(0) = 0$ and $v \in F_N$. Suppose that*

(i) *$J(u)$ is integrable;*

(ii) $\displaystyle\iint_0^\infty |J'(u(x)-s)| 1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x$ *and* $\displaystyle\iint_0^\infty |J'(u(x)-s)| 1_{\{s \leqslant v^*(x)\}}\,\mathrm{d}s\,\mathrm{d}x$ *are finite*

Then equality in (5.1) holds if and only if $v = v^*$ a.e.

Proof. By (P_2), for all $t, s \geqslant 0$, $J(t - s) = \int_0^{t-s} J'(r)\,\mathrm{d}r$. By the change of variables $w = t - r$, we have that

$$\begin{aligned} J(t - s) &= -\int_t^s J'(t - w)\,\mathrm{d}w = \int_0^t J'(t - w)\,\mathrm{d}w - \int_0^s J'(t - w)\,\mathrm{d}w \\ &= \int_0^t J'(p)\,\mathrm{d}p - \int_0^s J'(t - w)\,\mathrm{d}w = J(t) - \int_0^s J'(t - w)\,\mathrm{d}w. \end{aligned}$$

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Now define $\varphi : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\varphi(t,s) = -J(t-s) = \int_0^s J'(t-w)\,\mathrm{d}w - J(t).$$

Then φ satisfies all the hypotheses of theorem 4.5. Indeed, it is easy to check that $\psi : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$, defined by

$$\psi(t,s) = \int_0^s J'(t-w)\,\mathrm{d}w,$$

verifies the required conditions in theorem 4.1. □

An interesting consequence of this result is theorem 3.5 of [21].

Theorem 5.7. *Let $J : \mathbb{R} \to \mathbb{R}$ be a non-negative, strictly convex function such that $J(0) = 0$. Then, for any $v \in F_N$ such that $J(u)$ and $J(u-v) \in L^1(\mathbb{R}^N)$, equality in (5.1) occurs if and only if $v = v^*$ a.e.*

Proof. Under the assumptions of theorem 5.7, $J'(s) \geqslant 0$ for all $s \geqslant 0$ and $J'(s) \leqslant 0$ for all $s \leqslant 0$. We now prove that (J,u,v) satisfy all the hypotheses of theorem 5.6:

$$\begin{aligned}\iint_0^\infty &|J'(u(x)-s)| 1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_{\{v(x)\leqslant u(x)\}} \int_0^\infty |J'(u(x)-s)| 1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x \\ &\quad + \int_{\{v(x) > u(x)\}} \int_0^\infty |J'(u(x)-s)| 1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

If $v(x) \leqslant u(x)$,

$$(J'(u(x)-s)1_{\{s\leqslant v(x)\}})_+ = J'(u(x)-s)1_{\{s\leqslant v(x)\}}$$

and

$$(J'(u(x)-s)1_{\{s\leqslant v(x)\}})_- = 0.$$

Thus

$$\begin{aligned}\int_{\{v(x)\leqslant u(x)\}} \int_0^\infty &|J'(u(x)-s)| 1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_{\{v(x)\leqslant u(x)\}} (J(u(x)) - J(u(x)-v(x)))\,\mathrm{d}x.\end{aligned}$$

If $v(x) > u(x)$,

$$\begin{aligned}(J'(u(x)-s)1_{\{s\leqslant v(x)\}})_+ &= J'(u(x)-s)1_{\{s\leqslant u(x)\}}1_{\{s\leqslant v(x)\}}, \\ (J'(u(x)-s)1_{\{s\leqslant v(x)\}})_- &= -J'(u(x)-s)1_{\{s\leqslant v(x)\}}1_{\{s\geqslant u(x)\}}\end{aligned}$$

and

$$\int_0^\infty |J'(u(x)-s)| 1_{\{s\leqslant v(x)\}}\,\mathrm{d}s = J(u(x)) + J(u(x)-v(x)).$$

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Thus

$$\iint_0^\infty |J'(u(x)-s)| 1_{\{s\leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x \\ = \int J(u(x))\,\mathrm{d}x + \int_{\{v(x)>u(x)\}} J(u(x)-v(x))\,\mathrm{d}x \\ - \int_{\{v(x)\leqslant u(x)\}} J(u(x)-v(x))\,\mathrm{d}x.$$

This equality remains true when we replace v by v^*. We then conclude by remarking that, by theorem 5.1, we certainly have

$$\int J(u-v^*) \leqslant \int J(u-v).$$

□

Remark 5.8. In [21], Lieb and Loss stated theorem 5.7 without integrability assumptions. It seems to the author that these conditions were used tacitly in the proof of theorem 3.5 of [21]. In the following example, we prove that these conditions cannot be ruled out. Take $J(x)=x^2$, $u(x)=1/|x|^{1/4}$ and let v be any characteristic function such that $v \neq v^*$ everywhere. Clearly,

$$\int J(u(x)-v(x))\,\mathrm{d}x = \int J(u(x)-v^*(x))\,\mathrm{d}x = \infty.$$

Carlen and Loss assumed that the conclusion of theorem 5.7 is true if one supposes that $\int J(u(x)-v(x))\,\mathrm{d}m < \infty$, where $\mathrm{d}m = \mathrm{e}^{-\pi|x|^2}\,\mathrm{d}x$. The examples given above prove that this is not true. Nevertheless, they only used this result in the case where $J(x)=|x|^p$, $p>1$, and $u,v \in L^p(\mathbb{R}^N)$. Clearly, integrability assumptions are satisfied here.

Theorem 5.9. *Let $J:\mathbb{R}\to\mathbb{R}_+$ be a strictly convex function such that $J(0)=0$. Then, for any $v\in F_N$ such that $J(u)$ and $J(u-v)$ are integrable,*

$$\int J(u(x)-v(x))\,\mathrm{d}x < \int J(u(x)-v^*(x))\,\mathrm{d}x \quad \textit{unless } v=v^* \textit{ a.e.}$$

Proof. We merely combine theorem 5.1 and theorem 5.7. □

Corollary 5.10. *Let $v \in F_N \cap L^p(\mathbb{R}^N)$. Then $|u-v^*|_p < |u-v|_p$, where $p>1$, unless $v=v^*$ a.e.*

6. Applications of our results

In this section, we study cases of equality and strict inequality in $(*)$, where $H\equiv G$ is a Carathéodory function defined on $(0,\infty)\times\mathbb{R}_+\to\mathbb{R}$. All results stated in the following are consequences of theorem 4.1, so we will not give detailed proofs. Some applications of such results in the calculus of variations and nonlinear Schrödinger equations are then given.

THEOREM 6.1. *Let $G : (0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ be a function such that there exists $g : (0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ verifying $G(r,t) = \int_0^t g(r,s)\,\mathrm{d}s$ for all $r \in (0,\infty)$ and $t \in \mathbb{R}_+$. Suppose that*

(i) *for almost every $s \geqslant 0$, $g(\cdot,s)$ is strictly decreasing on $(0,\infty)$;*

(ii) $\displaystyle\iint_0^\infty |g(|x|,s)| 1_{\{s \leqslant v(x)\}}\,\mathrm{d}s\,\mathrm{d}x$ *and* $\displaystyle\iint_0^\infty |g(|x|,s)| 1_{\{s \leqslant v^*(x)\}}\,\mathrm{d}s\,\mathrm{d}x$ *are finite;*

(iii) $\displaystyle\lim_{|x|\to\infty} g(|x|,s) = l_s > -\infty$ *for almost every $s \geqslant 0$.*

Then, for any $v \in F_N$,

$$-\infty < \int G(|x|, v(x))\,\mathrm{d}x \leqslant \int G(|x|, v^*(x))\,\mathrm{d}x < \infty$$

and

$$\int G(|x|, v(x))\,\mathrm{d}x = \int G(|x|, v^*(x))\,\mathrm{d}x$$

if and only if $v = v^*$.

Proof. Set $F(r,s) = g(1/r,s)$ for all $r > 0$ and $s \geqslant 0$. We can check easily that F verifies all their hypotheses of theorem 4.1 with $v(x) = 1/|x|$. Hence,

$$\int G(|x|, v(x))\,\mathrm{d}x = \int G(|x|, v^*(x))\,\mathrm{d}x$$

if and only if

$$\int \psi\left(\frac{1}{|x|}, v(x)\right)\mathrm{d}x = \int \psi\left(\frac{1}{|x|}, v^*(x)\right)\mathrm{d}x,$$

where $\psi(r,t) = \int_0^t F(r,s)\,\mathrm{d}s = \int_0^t g(1/r,s)\,\mathrm{d}s$ for all $r > 0$ and $s \geqslant 0$. Then the conclusion of our assertion follows from theorem 4.1. □

REMARK 6.2. If $G \in C^2((0,\infty) \times \mathbb{R}_+)$, then (i) is equivalent to $\partial_1\partial_2 G < 0$. This hypothesis is called strict monotonicity of order 2 in economy (see [8], for example).

Note that the hypothesis

(CV) $|g(r,s)| \leqslant K(s + s^l)$, where $K > 0$, $l > 1$, for all $r > 0$, $s \geqslant 0$ and $v \in L^2(\mathbb{R}^N) \cap L^{l+1}(\mathbb{R}^N)$,

implies (ii) of theorem 6.1. Note also that (CV), together with (i), implies (iii).

We now can state the following interesting result in the context of the calculus of variations.

THEOREM 6.3 (CV). *Let $G : (0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ be a function such that there exists $g : (0,\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $G(r,t) = \int_0^t g(r,s)\,\mathrm{d}s$. Suppose that (i) of theorem 6.1 and (CV) of remark 6.2 hold. Then, for any $v \in L^2(\mathbb{R}^{\mathrm{N}}) \cap L^{l+1}(\mathbb{R}^{\mathrm{N}}) \cap F_N$,*

$$-\infty < \int G(|x|, v(x))\,\mathrm{d}x \leqslant \int G(|x|, v^*(x))\,\mathrm{d}x < \infty$$

and

$$\int G(|x|, v(x))\,\mathrm{d}x = \int G(|x|, v^*(x))\,\mathrm{d}x$$

if and only if $v = v^*$ *a.e.*

6.1. Some applications

6.1.1. *Nonlinear Schrödinger equation*

In [18], Stuart and the author present a broad extension in proving that the constrained problem

$$(P_c): \quad \inf\left\{J(u) : u \in H^1(\mathbb{R}^N) \text{ and } \int u^2(x)\,\mathrm{d}x = c^2\right\},$$

where $c > 0$ is prescribed and

$$J(u) = \int_{\mathbb{R}^N} (\tfrac{1}{2}|\nabla u|^2 - G(|x|, u(x)))\,\mathrm{d}x$$

always has a Schwarz symmetric minimizer. Under some regularity assumptions about G, a minimizer for (P_c) satisfies the Euler–Lagrange equation

$$(E_c): \quad \Delta u(x) + \partial_2 G(|x|, u(x)) + \lambda u(x) = 0 \text{ on } \mathbb{R}^N,$$

where λ is a Lagrange multiplier.

Our result concerning (P_c) in [18] gives natural conditions ensuring that (E_c) has a Schwarz symmetric ground state.

Now, a natural question is, Under which conditions are all ground states of (E_c) Schwarz symmetric? Theorem 6.3 answers us. Namely, suppose for example that G satisfies hypotheses of theorem 3.1 of [18] and theorem 6.3, then, we can easily prove that all ground states of (E_c) are Schwarz symmetric.

This result has interesting consequences in the study of the nonlinear Schrödinger equation

$$\left.\begin{aligned} \mathrm{i}\partial_t \Phi + \Delta\Phi + G(|x|, |\Phi|) = 0, \qquad & t \geqslant 0,\ x \in \mathbb{R}^N, \\ \Phi(0, x) = \Phi_0(x). \qquad & \end{aligned}\right\} \tag{NLS}$$

Note that (E_c) arises in (NLS). A standard principle in physics, [14, 23], asserts that the most stable state of (NLS) is the Schwarz symmetric ground state showing that our results concerning (H) are important in dealing with the orbital stability of (NLS).

6.1.2. *Economics*

Let us also mention an economic application and a generalized Spence–Mirrlees condition of our results. In economics, the Spence–Mirrless condition is $(*)$ with a strict inequality, $H(r, u_1, \dots, u_m) = h(u_1, u_2)$ and $N = 1$. It is a powerful tool in the theory of incentives and it is also crucial to prove the reallocation principle [7].

Note also that cases of equality and strict inequality are related to duality problems introduced by Gangbo [12, 13] and the characterization of the minimizers of Monge's problem [8]. More details are given in the sequel.

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7. Concluding remarks

Techniques developed in this paper enable us to treat cases of equality in $(*)$ when $m > 2$ and $u_1, \dots, u_{m-1}$ are already symmetric and decreasing. Results are obtained in the same way as in theorem 4.1, so we will not mention them. It is more interesting to obtain cases of equality in $(*)$ when only one of the $\{u_i\}_{1 \leqslant i \leqslant m}$ is Schwarz symmetric (note that, if none of the u_i are Schwarz symmetric, equality can hold in $(*)$ even though the other functions are different from their Schwarz symmetrization). An in-depth discussion will be given in a further publication.

Let us mention that the absolute continuity of G (or ψ) is crucial in our approach. However, this condition can be relaxed when v is a step function. In this case, all results established in [15–17] remain valid for cases of equality and strict inequality, adding a strict monotonicity assumption and an integrability condition.

Here is one example of what we mean.

THEOREM 7.1. *Let $G : (0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ be a Carathéodory function having the property that there is a subset Γ of $(0,\infty)$ having one-dimensional measure zero such that*

(i) *$G_-(|x|, 0)$ is integrable on $\mathbb{R}^N$;*

(ii) *G satisfies (CZR2) (see [17]) with strict inequality;*

(iii) *there exists a continuous function $g : [0,\infty) \to \mathbb{R}$ such that*

$$g(0) = 0 \text{ and } \lim_{r\to\infty} \{G(r,a) - G(r,b)\} \leqslant g(a) - g(b) \text{ for all } a, b \in [0,\infty),\ b \geqslant a.$$

Let $v \in E_N$ be such that $\int g_-(v(x))\,\mathrm{d}x$ and $\int G(|x|, v^*(x))\,\mathrm{d}x < \infty$. Then

$$\int G(|x|, v(x))\,\mathrm{d}x < \int G(|x|, v^*(x))\,\mathrm{d}x \quad \text{unless } v = v^* \text{ a.e.}$$

7.1. Connection with Monge's optimal transportation problem

For some monotonic rearrangements, cases of equality and strict inequality in the 'extended Hardy–Littlewood inequalities' can be determined even though H is not absolutely continuous. A more precise description follows.

In [8], Carlier studied Monge's optimal transportation problem with $(n+1)$ marginals in $\mathbb{R}$: given $(n+1)$ Borel probability measures $(\mu_0, \dots, \mu_n)$, he considered

$$(\mathrm{M}) : \inf_{s \in \Gamma} \int_{\mathbb{R}} H(t, s_1(t), \dots, s_n(t))\,\mathrm{d}\mu_0(t),$$

where $\Gamma = \{s = (s_1, \dots, s_n),$ each $s_i : \mathbb{R} \to \mathbb{R}$ is Borel and $\mu_i = \mu_0 s_i^{-1}$ for $i = 1, \dots, n\}$.

Assuming that

(C1) $H \in C(T, \mathbb{R})$, where $T = T_0 \times \dots \times T_n$, T_0 is compact, $T_i = \operatorname{supp} \mu_i$, is a compact subset of $\mathbb{R}$ for each $i = 1, \dots, n$;

(C2) $\mu_0(\{t\}) = 0$ for any $t \in T_0$;

(C3) H is strictly monotone of order 2 on T (see definition 1 of [8]).

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Solving the dual problem of (M), he proved that

(1) (M) admits a unique (up to μ_0-a.e. equivalence) solution $(s_1, \dots, s_n)$;

(2) each component s_i of s is non-decreasing.

This powerful result is closely related to the 'extended Hardy–Littlewood inequalities'. Indeed, take $\mu_i = \mu_0 x_i^{-1}$, where $x = (x_1, \dots, x_n)$ are n bounded Borel real-valued functions on T_0, and define

$$s_i(t_0) = \inf\{t_i \in T_i : u_i(t_i) > t_0\} \quad \mu_0\text{-a.e } t_0,$$

where

$$u_i(t_i) = \inf\{t_0 \in T_0 : \mu_0([t_0, \infty)) \leqslant \mu_i([t_i, \infty))\}.$$

Then, as shown in [8], the optimality and uniqueness of s, which is a non-decreasing rearrangement of x, tell us that

$$\int_{T_0} H(t, x_1(t), \dots, x_n(t))\, \mathrm{d}\mu_0(t) \geqslant \int_{T_0} H(t, s_1(t), \dots, s_n(t))\, \mathrm{d}\mu_0(t),$$

and this inequality is strict unless each x_i is non-decreasing.

Now, considering

$$(\mathrm{S}) : \sup_{s \in \Gamma} \int_{T_0} H(t, s_1(t), \dots, s_n(t))\, \mathrm{d}\mu_0(t),$$

we can easily prove, under minor modifications of the subtle Carlier method, that under (C1)–(C3), we have

$$\int_{T_0} H(t, x_1(t), \dots, x_n(t))\, \mathrm{d}\mu_0(t) \leqslant \int_{T_0} H(t, r_1(t), \dots, r_n(t))\, \mathrm{d}\mu_0(t)$$

and this inequality is strict unless each x_i is non-increasing (here r_i is a non-increasing rearrangement of x_i, which is the Schwarz one if we take $|t|$ instead of t, $T_0 = [-a, a]$, where $a > 0$, and x_i a non-negative Borel bounded function).

These observations bring us to the following question. If $T_0 = \mathbb{R}^N$, $T_i = \mathbb{R}$ for each $i = 1, \dots, n$, is there an appropriate class of functions H for which the right-hand side of $(*)$ is the unique solution of a well-posed Monge transportation problem (S)?

It seems to us that approaches developed by Carlier [7,8], Gangbo and Mc Cann [12], Gangbo and Swiech [13] and Brenier [1,2] (to name only a few) cannot help us to answer the above question.

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