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The number of positive solutions of a non-linear problem with discontinuous non-linearity

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Synopsis

For the non-linear problem

$$\begin{cases} -u''(x)=\lambda f(u(x)) & \text{for} \quad 0<x<1 \\ u(0)=u(1)=0 \end{cases}$$

where f is a discontinuous function at 1, we show that the number of non-trivial positive solutions, for a given real number $\lambda \geqq 0$, is related to the graph of a continuous function g. Then, by studying the function g it is possible in some special cases to give, for any $\lambda \geqq 0$, the minimal or exact number of non-trivial positive solutions.

1. Introduction

We consider the non-linear two point boundary-value problem

$$\begin{cases} -u''(x)=\lambda f(u(x)) & \text{for} \quad 0<x<1 \\ u(0)=u(1)=0 \end{cases} \tag{1.1}$$

where $f:[0,+\infty)\to[0,+\infty)$ is given. We assume that there exist two continuously differentiable functions $h:[0,1]\to[0,+\infty)$ and $k:[1,+\infty)\to(0,+\infty)$ such that

$$h(0)=0, \tag{H1}$$

$$h(p)>0 \quad \text{for all} \quad p\in(0,1], \tag{H2}$$

$$h(1)\neq k(1), \tag{H3}$$

$$f(p)=\begin{cases} h(p) & \text{if} \quad p\in[0,1) \\ k(p) & \text{if} \quad p\in(1,+\infty). \end{cases} \tag{H4}$$

The value of f at 1 need not be related to h and k, but $f(1)$ should be positive.

DEFINITION. A solution of problem (1.1) is a pair $(u,\lambda)\in C^1([0,1])\times[0,+\infty)$

such that

$$u(x) \geqq 0 \quad \text{for all} \quad x \in [0, 1],$$

$$u(0) = u(1) = 0,$$

$$u' \text{ is absolutely continuous on } [0, 1]$$

and

$$-u''(x) = \lambda f(u(x)) \quad \text{for almost all} \quad x \in [0, 1].$$

We denote by S the subset of $C^1([0, 1]) \times [0, +\infty)$ consisting of all the solutions of problem (1.1). Since $f(0) = 0$, the set of trivial solutions $\{(0, \lambda) \in C^1([0, 1) \times [0, +\infty) \mid \lambda \geqq 0\}$ belongs to S. Let $S^+ = \{(u, \lambda) \in S \mid \|u\| \neq 0\}$ where $\|u\| = \max_{x \in [0,1]} |u(x)|$.

In order to prove the existence of solutions in S^+, we consider for all $n \in \mathbb{N}$ the following problem:

$$\begin{cases} -u''(x) = \lambda f_n(u(x)) & \text{for} \quad 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{$1.1_{(n)}$}$$

where $f_n : [0, +\infty) \to (0, +\infty)$ is defined by $f_n(t) = f(t) + (1/n)$ for all t in $[0, +\infty)$. Let S_n be the subset of $C^1([0, 1]) \times (0, +\infty)$ consisting of all the solutions of problem $(1.1)_{(n)}$. Then (see [**1**, **2**, **3**]), we know that for any $\rho > 0$, there exists a unique (u_n, λ_n) in S_n such that $\|u_n\| = \rho$. Moreover (u_n, λ_n) has the following properties:

(1) $\lambda_n = g_n^2(\|u_n\|)$, where $g_n : (0, +\infty) \to (0, +\infty)$ is a continuous function defined by $g_n(p) = \sqrt{2} \int_0^p \{F_n(p) - F_n(\omega)\}^{-\frac{1}{2}} d\omega$, $F_n(\omega) = \int_0^\omega f_n(s)\, ds$;
(2) $u_n'(x) > 0$ for all $x \in [0, \frac{1}{2})$ and $u'(x) < 0$ for all $x \in (\frac{1}{2}, 1]$;
(3) $u_n(\frac{1}{2}) = \rho$ and $u_n'(\frac{1}{2}) = 0$;
(4) if $\rho = 1$, then $\{x \in [0, 1] \mid u_n(x) = 1\} = \{\frac{1}{2}\}$;
if $\rho > 1$, then there exists $x_0 \in (0, \frac{1}{2})$ such that $\{x \in [0, 1] \mid u_n(x) = 1\} = \{x_0, 1 - x_0\}$;
(5) $u_n(x) = u_n(1 - x)$ for all $x \in [0, 1]$.

Now that these definitions have been given, let us state our main results.

In Section 3, we show that for any $\rho > 0$ there exists a unique solution (u, λ) in S^+ such that $\|u\| = \rho$. We also state that S^+ is a continuum in $C^1([0, 1]) \times [0, +\infty)$ and furthermore S^+ is a continuous curve in $C^1([0, 1]) \times [0, +\infty)$ which can be parameterized by $\|u\|$. The last result of Section 2 is that if $h(1) < k(1)$ and $\lim_{p \to +\infty} p^{-1} f(p) = 0$, then there are always values of λ for which there exist at least two distinct solutions of problem (1.1) in S^+.

In Section 3, we study the case $h'(0) = \lim_{p \to 0^+} p^{-1} f(p) = \alpha > 0$. We show that $\mathscr{C} = S^+ \cup \{(0, \pi^2/\alpha)\}$ is a continuum in $C^1([0, 1]) \times [0, +\infty)$ and furthermore $\mathscr{C}$ is a continuous curve in $C^1([0,1])\times[0,+\infty)$ which can be parameterized by $\|u\|$. We also state that if $\lim_{p\to+\infty} p^{-1}f(p)=0$, then for any $\lambda\geqq\pi^2/\alpha$ there is at least one solution of problem (1.1) in $\mathscr{C}$. For the two following cases:

$$f(p)\geqq pf'(p) \quad \text{for all} \quad p\in[0,1)\cup(1,+\infty) \quad \text{and} \quad h(1)>k(1), \tag{A}$$

$$f(p)\leqq pf'(p) \quad \text{for all} \quad p\in[0,1)\cup(1,+\infty) \quad \text{and} \quad h(1)<k(1), \tag{B}$$

we give, for any $\lambda\geqq 0$, the exact number of solutions of problem (1.1) in $\mathscr{C}$. At the end of Section 3, we give a theorem in which, for some values of λ, there exist at least three distinct solutions of problem (1.1) in $\mathscr{C}$.

In Section 4, we study the case $h'(0)=\lim_{p\to 0^+} p^{-1}f(p)=0$. We show the existence of a number $\hat{\lambda}\geqq 0$ such that, for any $\lambda>\hat{\lambda}$, there is at least one solution of problem (1.1) in S^+. Moreover, if $\lim_{p\to+\infty} p^{-1}f(p)=0$, there exists a positive number λ_1 such that, for any $\lambda>\lambda_1$ there are at least two distinct solutions of problem (1.1) in S^+, for $\lambda=\lambda_1$ there is at least one solution in S^+ and for $\lambda\in[0,\lambda_1)$ there is no solution in S^+. As in Section 3, for the case (B): $f(p)\leqq pf'(p)$ for all $p\in[0,1)\cup(1,+\infty)$ and $h(1)<k(1)$, we give, for any $\lambda\geqq 0$, the exact number of solutions of problem (1.1) in S^+.

The problem (1.1) has already been studied in the following three articles. In [**4**], Laetsch studies the problem (1.1) with the assumption that f is a continuous function on $[0,+\infty)$. In [**3**], Stuart studies it with the same assumptions as in the present paper, but supposes that $h(0)>0$. In [**5**], Nistri treats the case $h(p)=0$ for all $p\in[0,1]$ and $k(p)>0$ on $[1,+\infty)$.

2. General properties of S^+

We start this section by giving a theorem concerning the structure of any solution (u,λ) in S^+.

THEOREM 2.1. *Let $(u,\lambda)\in S^+$, then:*

(1) *$\lambda>0$ and $u(x)>0$ for all $x\in(0,1)$;*
(2) *$u'(x)>0$ for all $x\in[0,\tfrac12)$ and $u'(x)<0$ for all $x\in(\tfrac12,1]$;*
(3) *$u(\tfrac12)=\|u\|$ and $u'(\tfrac12)=0$;*
(4) *if $\|u\|=1$, then $\{x\in[0,1] \mid u(x)=1\}=\{\tfrac12\}$;*

if $\|u\|>1$, there exists x_0 in $(0,\tfrac12)$ such that $\{x\in[0,1] \mid u(x)=1\}=\{x_0,1-x_0\}$;

(5) *$u(x)=u(1-x)$ for all $x\in[0,1]$.*

The proof of this theorem is given in [**1**].

Let $F(\omega)=\int_0^\omega f(s)\,ds$ for $\omega>0$. Since $f(s)>0$ for all $s>0$, F is a strictly increasing function on $(0,+\infty)$.

Let $g(p)=\sqrt{2}\int_0^p\{F(p)-F(\omega)\}^{-\frac12}\,d\omega$ for $p>0$. We note that $g(p)<+\infty$ for $p>0$ and g is a continuous function on $(0,+\infty)$. Moreover $\lim_{n\to+\infty} g_n(p)=g(p)$ for all $p>0$.

LEMMA 2.2 *Suppose that* $(u,\lambda)\in S^+$. *Then* $\lambda^{\frac{1}{2}}=g(\|u\|)$.

Proof. Let us assume that $\|u\|>1$. By Theorem 2.1, there exists a number $x_0\in(0,\frac{1}{2})$ such that $u(x_0)=1$ and $-u''(x)=\lambda f(u(x))$ for all $x\in[0,x_0)\cup(x_0,\frac{1}{2}]$. We obtain

$$-\tfrac{1}{2}(u'(x))^2=\lambda F(u(x))+c_1 \quad \text{for all} \quad x\in[0,x_0)$$

$$-\tfrac{1}{2}(u'(x))^2=\lambda F(u(x))+c_2 \quad \text{for all} \quad x\in(x_0,\tfrac{1}{2}].$$

Since $u\in C^1([0,1])$ and F is continuous on $(0,+\infty)$, it follows that $c_1=c_2=-\lambda F(\|u\|)$.

Thus $u'(x)=\sqrt{2\lambda}\{F(\|u\|)-F(u(x))\}^{\frac{1}{2}}$ for all $x\in[0,\frac{1}{2}]$, and consequently $\lambda^{\frac{1}{2}}=g(\|u\|)$.

If $\|u\|\leqq 1$, a similar argument shows that $\lambda^{\frac{1}{2}}=g(\|u\|)$. This completes the proof.

This relationship was introduced in this connection by Laetsch [**4**].

COROLLARY 2.3. (1) *Suppose that* (u_1,λ_1) *and* (u_2,λ_2) *are two solutions of problem* (1.1) *in* S^+ *such that* $\|u_1\|=\|u_2\|$. *Then* $\lambda_1=\lambda_2$ *and* $u_1(x)=u_2(x)$ *on* $[0,1]$.

(2) *Let* $\lambda>0$. *Suppose that* (u_1,λ) *and* (u_2,λ) *are two distinct solutions of problem* (1.1) *in* S^+. *Then either* $u_1(x)<u_2(x)$ *on* $(0,1)$ *or* $u_1(x)>u_2(x)$ *on* $(0,1)$.

Now that all these preliminary results have been obtained, we can state out main existence theorem.

THEOREM 2.4. *For any* $\rho>0$, *there exists exactly one solution of problem* (1.1) *such that* $\|u\|=\rho$.

Proof. Since the unicity is given by Corollary 2.3, we only need to prove the existence of a solution. Let $\rho>0$, then for any $n\in\mathbb{N}$, there exists (u_n,λ_n) in S_n such that $\|u_n\|=\rho$ and $\lambda_n=g_n^2(\rho)$. Since $\lim_{n\to+\infty}g_n^2(\rho)=g^2(\rho)$ we have that $\{\lambda_n\}_{n\geqq 1}$ converges and that $\lambda\equiv\lim_{n\to+\infty}\lambda_n=g^2(\rho)$. Let $\bar{\lambda}=\sup\{\lambda_n|n\in\mathbb{N}\}$ and $l(\rho)=\sup\{f(t)|t\in[0,\rho]\}$, then $\bar{\lambda}<+\infty$ and $l(\rho)>0$.

We thus obtain $|u_n'(x)-u_n'(y)|=\lambda_n|\int_x^y f_n(u_n(s))\,ds|\leqq\bar{\lambda}(l(\rho)+1)|x-y|$ for all x,y in $[0,1]$ and all $n\in\mathbb{N}$. If we put $y=\frac{1}{2}$, we have $\|u_n'\|\leqq\bar{\lambda}(l(\rho)+1)$ for all $n\in\mathbb{N}$. Therefore, by the Ascoli–Arzelà theorem, there exists a subsequence $\{u_{n_j}\}_{j\geqq 1}$ of $\{u_n\}_{n\geqq 1}$ which converges to u in $C^1([0,1])$. It follows that $u(\frac{1}{2})=\|u\|=\rho$, $u(0)=u(1)=0$ and $u'(x)\geqq 0$ for all $x\in[0,\frac{1}{2}]$. Since $u(x)=u(1-x)$ for all $x\in[0,1]$, it remains to show that $u'(x)=-\lambda\int_{\frac{1}{2}}^x f(u(s))\,ds$ for all $x\in[0,\frac{1}{2}]$.

(a) Suppose that $\rho\leqq 1$, then

$$u'(x)=\lim_{j\to+\infty}u_{n_j}'(x)=\lim_{j\to+\infty}-\lambda_{n_j}\int_{\frac{1}{2}}^{x}\{h(u_{n_j}(s))+(1/n_j)\}\,ds$$

$$=-\lambda\int_{\frac{1}{2}}^{x}h(u(s))\,ds$$

for all $x \in [0, \frac{1}{2}]$. Therefore

$$\{x \in [0,1] \mid u(x)=1\} \subset \{\tfrac{1}{2}\},$$

and we have

$$\int_{\frac{1}{2}}^{x} h(u(s))\,ds = \int_{\frac{1}{2}}^{x} f(u(s))\,ds \quad \text{on } [0, \tfrac{1}{2}].$$

(b) Suppose that $\rho > 1$. Then, there exists a number x_0 in $(0, \frac{1}{2})$ such that

$$\{x \in [0,1] \mid u(x)=1\} = \{x_0, 1-x_0\}.$$

(1) For any $x \in (0, x_0)$, there exists $j_x \in \mathbb{N}$ such that $u_{n_j}(s) < 1$ for all $j \geqq j_x$ and all $s \in [0, x]$. Thus,

$$u'(x) = \lim_{j \to +\infty} \left[-\lambda_{n_j} \int_0^x \left\{ h(u_{n_j}(s)) + \frac{1}{n_j} \right\} ds + u'_{n_j}(0) \right]$$

$$= -\lambda \int_0^x f(u(s))\,ds + u'(0) \quad \text{for all} \quad x \in [0, x_0).$$

Since u' is a continuous function on $[0,1]$, we have $u'(x) = -\lambda \int_0^x f(u(s))\,ds + u'(0)$ for all $x \in [0, x_0]$.

(2) A similar argument shows that $u'(x) = -\lambda \int_{\frac{1}{2}}^{x} f(u(s))\,ds$ on $[x_0, \frac{1}{2}]$.

And it follows from (1) and (2), that $u'(x) = -\lambda \int_{\frac{1}{2}}^{x} f(u(s))\,ds$ for all $x \in [0, \frac{1}{2}]$. This completes the proof of this theorem.

Theorem 2.4 allows us to consider the function $\sigma : (0, +\infty) \to C^1([0,1])$ which is defined by: $(\sigma(\rho), g^2(\rho)) \in S^+$ and $\|\sigma(\rho)\| = \rho$. On S^+ we consider the topology induced from $C^1([0,1]) \times [0, +\infty)$.

THEOREM 2.5. *The one-to-one map* $\psi : (0, +\infty) \to S^+$ *defined by* $\psi(\rho) = (\sigma(\rho), g^2(\rho))$ *is continuous.*

An immediate consequence of this last theorem is that S^+ is a continuum in $C^1([0,1]) \times [0, +\infty)$ and furthermore S^+ is a continuous curve in $C^1([0,1]) \times [0, +\infty)$ which can be parameterized by $\|u\|$.

It follows, from Lemma 2.2 and Theorem 2.4, that the number of solutions of problem (1.1) in S^+ is given, for any $\lambda > 0$, by the graph of g^2. With the purpose of obtaining better information about this graph, we give two different representations of g'.

First representation of g'

We note that, for $p > 0$,

$$g(p) = \sqrt{2}\, p^{\frac{1}{2}} \int_0^1 R(p,t)^{-\frac{1}{2}}\,dt$$

where

$$R(p,t) = \int_t^1 f(pz)\,dz \quad \text{for all } (p,t) \in (0, +\infty) \times [0,1].$$

Let

$$\Delta_{p_0} = \begin{cases} \left[\dfrac{p_0}{2}, \bar{p}_0\right] & \text{if} \quad p_0 \in (0, 1) \\ [\bar{p}_0, 2p_0] & \text{if} \quad p_0 \in (1, +\infty). \end{cases}$$

Where $\bar{p} = (1 + p_0)/2$ for all $p_0 \in (0, +\infty)$.

LEMMA 2.6. *For all* $p_0 \in (0, 1) \cup (1, +\infty)$, *there exists a positive constant* $d(p_0)$ *such that*

$$R(p, t)^{-\frac{3}{2}} \left| \int_0^1 f'(pz) z \, dz \right| \leqq d(p_0)(1-t)^{-\frac{1}{2}} \quad \textit{for all} \quad (p, t) \in \Delta_{p_0} \times [0, 1).$$

LEMMA 2.7. *Let* $T(p) = \int_0^1 R(p, t)^{-\frac{1}{2}} dt$. *Then* T *is continuously differentiable on* $(0, 1) \cup (1, +\infty)$. *Furthermore*

$$T'(p) = -\tfrac{1}{2} \int_0^1 R(p, t)^{-\frac{3}{2}} \int_t^1 f'(pz) z \, dz \, dt \quad \textit{if} \quad p \in (0, 1)$$

and

$$T'(p) = -\tfrac{1}{2} \int_0^1 R(p, t)^{-\frac{3}{2}} \int_t^1 f'(pz) z \, dz \, dt - \tfrac{1}{2} p^{-2} \{k(1) - h(1)\} \int_0^{1/p} R(p, t)^{-\frac{3}{2}} dt$$

if $p \in (1, +\infty)$.

Proof. (1) If $p \in (0, 1)$, it follows by using Lemma 2.6, that

$$T'(p) = \frac{d}{dp} \int_0^1 R(p, t)^{-\frac{1}{2}} dt = \int_0^1 \frac{\partial R(p, t)^{-\frac{1}{2}}}{\partial p} dt = -\tfrac{1}{2} \int_0^1 R(p, t)^{-\frac{3}{2}} \int_t^1 f'(pz) z \, dz \, dt.$$

T' is continuous on $(0, 1)$.

(2) If $p > 1$, then

$$T(p) = \int_0^1 R(p, t)^{-\frac{1}{2}} dt = \int_0^{1/p} R(p, t)^{-\frac{1}{2}} dt + \int_{1/p}^1 R(p, t)^{-\frac{1}{2}} dt.$$

And it follows, by using Lemma 2.6, that

$$T'(p) = -\tfrac{1}{2} \int_0^1 R(p, t)^{-\frac{3}{2}} \int_t^1 f'(pz) z \, dz \, dt - \tfrac{1}{2} p^{-2} \{k(1) - h(1)\} \int_0^{1/p} R(p, t)^{-\frac{3}{2}} dt.$$

T' is continuous on $(1, +\infty)$.

Now, we are able to give the first representation of g'.

COROLLARY 2.8. *g is continuously differentiable on* $(0,1)\cup(1,+\infty)$. *Furthermore*

$$g'(p)=\sqrt{2}\left\{\tfrac{1}{2}p^{-\frac{1}{2}}\int_0^1 R(p,t)^{-\frac{1}{2}}\,dt-\tfrac{1}{2}p^{\frac{1}{2}}\int_0^1 R(p,t)^{-\frac{3}{2}}\int_t^1 f'(pz)z\,dz\,dt\right\}\quad \text{if}\quad p\in(0,1)$$

and

$$g'(p)=\sqrt{2}\left\{\tfrac{1}{2}p^{-\frac{1}{2}}\int_0^1 R(p,t)^{-\frac{1}{2}}\,dt-\tfrac{1}{2}p^{\frac{1}{2}}\int_0^1 R(p,t)^{-\frac{3}{2}}\int_t^1 f'(pz)z\,dz\,dt\right.$$
$$\left.-\tfrac{1}{2}p^{-\frac{3}{2}}\{k(1)-h(1)\}\int_0^{1/p} R(p,t)^{-\frac{3}{2}}\,dt\right\}\quad \text{if}\quad p\in(1,+\infty).$$

Second representation of g'

LEMMA 2.9.

$$g'(p)=\sqrt{2}\,f(p)\left\{-\tfrac{1}{2}\int_0^{\frac{1}{2}}\{F(p)-F(\omega)\}^{-\frac{3}{2}}\,d\omega+\left\{\frac{1}{k(1)}-\frac{1}{h(1)}\right\}\{F(p)-F(1)\}^{-\frac{1}{2}}\right.$$
$$\left.+\frac{1}{h(\frac{1}{2})}\{F(p)-F(\tfrac{1}{2})\}^{-\frac{1}{2}}-\int_{\frac{1}{2}}^{p}\{F(p)-F(\omega)\}^{-\frac{1}{2}}\frac{f'(\omega)}{f^2(\omega)}\,d\omega\right\}\quad \textit{for all}\quad p>1.$$

Proof. Since

$$g(p)=\sqrt{2}\left\{\int_0^1\{F(p)-F(\omega)\}^{-\frac{1}{2}}\,d\omega+\frac{2}{k(1)}\{F(p)-F(1)\}^{\frac{1}{2}}\right.$$
$$\left.-2\int_1^p\{F(p)-F(\omega)\}^{\frac{1}{2}}\frac{f'(\omega)}{f^2(\omega)}\,d\omega\right\}\quad \text{for all}\quad p>1,$$

by differentiation we obtain the assertion.

COROLLARY 2.10.

$$\lim_{p\to 1^+} g'(p)=\begin{cases}+\infty & \text{if}\quad h(1)>k(1)\\ -\infty & \text{if}\quad h(1)<k(1)\end{cases}.$$

Having shown, in Theorem 2.4, that there is exactly one solution of problem (1.1) in S^+ for each value of $\|u\|$, let us now ask for which values of $\lambda>0$ there is a solution. The next lemma helps us to answer this question.

LEMMA 2.11. *Suppose that* $\lim_{p\to+\infty} p^{-1}f(p)=0$. *Then* $\lim_{p\to+\infty} g(p)=+\infty$.

Proof. Let $n\in\mathbb{N}-\{1\}$. Then there exists a positive number p_n such that

$$\{F(p)-F(\omega)\}\leqq\frac{1}{2n}(p^2-\omega^2)\quad \text{for all}\quad p\geqq\omega\geqq p_n.$$

Let $\bar{p}_n = np_n$, then if $p \geqq \bar{p}_n$ we obtain

$$g(p) \geqq 2\sqrt{n} \int_{p/n}^{p} \frac{d\omega}{\sqrt{(p^2-\omega^2)}} = 2\sqrt{n}\left\{\frac{\pi}{2} - \operatorname{Arc\,sin}\frac{1}{n}\right\} \geqq 2\sqrt{n}.$$

Therefore $\lim_{p \to +\infty} g(p) = +\infty$.

THEOREM 2.12. *Suppose that* $\lim_{p\to+\infty} p^{-1}f(p) = 0$ *and that* $h(1) < k(1)$. *Then there exist numbers* λ_1 *and* λ_2 *with* $0 < \lambda_1 < \lambda_2$ *such that, for each* $\lambda \in (\lambda_1, \lambda_2)$, *there are at least two distinct solutions of problem* (1.1) *in* S^+.

Proof. Since g^2 is continuous on $[1, +\infty)$, $\lim_{p\to+\infty} g(p) = +\infty$ (Lemma 2.11) and $\lim_{p\to 1^+} g'(p) = -\infty$ (Corollary 2.10), there exists a number $\rho_1 > 1$ such that $0 < \lambda_1 = \min\{g^2(p) | p \geqq 1\} = g^2(\rho_1) < g^2(1) = \lambda_2$. Therefore, for any $\lambda \in (\lambda_1, \lambda_2)$, there exist two numbers ρ and $\bar{\rho}$ with $1 < \rho < \rho_1 < \bar{\rho}$ such that $g^2(\rho) = g^2(\bar{\rho}) = \lambda$. It follows that $(\sigma(\rho), \lambda)$ and $(\sigma(\bar{\rho}), \lambda)$ are two distinct solutions of problem (1.1) in S^+. This completes the proof.

3. Study of S^+ when $h'(0) > 0$

Let $\mathscr{C}$ denote the subset of $C^1([0,1]) \times [0, +\infty)$ defined by $\mathscr{C} = S^+ \cup \{(0, \pi^2/\alpha)\}$ where $\alpha = \lim_{p\to 0^+} p^{-1}f(p) = h'(0)$. On $\mathscr{C}$ we consider the topology induced from $C^1([0,1]) \times [0, +\infty)$.

Let

$$\tilde{g}(p) = \begin{cases} g(p) & \text{if } p > 0 \\ \dfrac{\pi}{\sqrt{\alpha}} & \text{if } p = 0 \end{cases} \quad \text{and} \quad \tilde{\psi}(p) = \begin{cases} \psi(p) & \text{if } p > 0 \\ \left(0, \dfrac{\pi^2}{\alpha}\right) & \text{if } p = 0. \end{cases}$$

Then it is easy to see that $\tilde{g}: [0, +\infty) \to (0, +\infty)$ is a continuous function and hence $\tilde{\psi}: [0, +\infty) \to \mathscr{C}$ is a continuous function. It immediately follows that $\mathscr{C}$ is a continuum in $C^1([0,1]) \times [0, +\infty)$, and furthermore $\mathscr{C}$ is a continuous curve in $C^1([0,1] \times [0, +\infty)$ which can be parametrized by $\|u\|$.

PROPOSITION 3.1. *Suppose that* $\lim_{p\to+\infty} p^{-1}f(p) = 0$. *Then, for any* $\lambda \geqq \pi^2/\alpha$, *there is at least one solution of problem* (1.1) *in* $\mathscr{C}$.

Proof. We see from Lemma 2.11 that $\lim_{p\to+\infty} \tilde{g}(p) = +\infty$. The result follows immediately from this and the continuity of $\tilde{g}$ on $[0, +\infty)$.

PROPOSITION 3.2. *Suppose that* $\liminf_{p\to+\infty} p^{-1}f(p) = \beta > 0$. *Then, there exists a positive constant* γ *such that* $f(p) \geqq \gamma p$ *for all* $p \geqq 0$ *and* $\{\lambda \geqq 0 |$ *there exists* $(u, \lambda) \in \mathscr{C}\} \subset (0, \pi^2/\gamma]$.

Proof. Since $\liminf_{p\to+\infty} p^{-1}f(p) = \beta > 0$, there exist two positive numbers $\bar{\beta}$ and p_0 such that $f(p) \geqq \bar{\beta}p$ for all $p \geqq p_0$. Since $\lim_{p\to 0^+} p^{-1}f(p) = \alpha > 0$, the number

$m=\inf\{p^{-1}f(p)|p\in(0,p_0]\}$ is positive. Let $\gamma=\min\{\bar{\beta},m\}$, then $f(p)\geqq\gamma p$ for all $p\geqq 0$.

If $(u,\lambda)\in\mathscr{C}$ with $\|u\|\neq 0$, then $\{F(\|u\|)-F(\omega)\}\geqq\gamma/2(\|u\|^2-\omega^2)$ for all $\omega\in[0,\|u\|]$, and we obtain

$$\sqrt{\lambda}=\sqrt{2}\int_0^{\|u\|}\{F(\|u\|)-F(\omega)\}^{-\frac{1}{2}}\,d\omega\leqq\frac{\pi}{\sqrt{\gamma}}$$

If $(u,\lambda)\in\mathscr{C}$ with $\|u\|=0$, then $\lambda=(\pi^2/\alpha)\leqq(\pi^2/\gamma)$.

This completes the proof of this proposition.

Now, we study the structure of $\mathscr{C}$ under the assumption (A): $f(p)\geqq pf'(p)$ for all $p\in[0,1)\cup(1,+\infty)$ and $h(1)>k(1)$. Under this assumption, $p^{-1}f(p)$ is a non-increasing function on $[0,1)\cup(1,+\infty)$. Let $m=\lim_{p\to+\infty}p^{-1}f(p)$, then it is easy to show that

$$\xi=\lim_{p\to+\infty}\tilde{g}(p)=\begin{cases}\dfrac{\pi}{\sqrt{m}} & \text{if}\quad m\neq 0\\ +\infty & \text{if}\quad m=0.\end{cases}$$

THEOREM 3.3. *Let $a\in[0,1]$. Suppose that:*

(i) *f satisfies assumption* (A),
(ii) $f(p)=\alpha p$ *for all* $p\in(0,a)$,
(iii) $f(p)\neq\alpha p$ *for all* $p\in(a,1)$,

then

(1) $\{(u,\lambda)\in\mathscr{C}|\lambda\in[0,\pi^2/\alpha)\cup[\xi^2,+\infty)\}=\varnothing$,
(2) $\{(u,\lambda)\in\mathscr{C}|\lambda=\pi^2/\alpha\}=\{(\gamma\sin\pi,\pi^2/\alpha)|\gamma\in[0,a]\}$,
(3) *for any* $\lambda\in(\pi^2/\alpha,\xi^2)$, *there exists exactly one solution of problem* (1.1) *in* $\mathscr{C}$.

Proof. By assumption (i), we have that $\int_t^1 f'(pz)z\,dz\leqq p^{-1}R(p,t)$ for all $(p,t)\in(0,+\infty)\times[0,1]$. Suppose that $a\neq 1$, then for all $p\in(a,1)$ there exist two numbers t_p and $\bar{t}_p$ in $(0,1)$ with $t_p<\bar{t}_p$ such that $\int_t^1 f'(pz)z\,dz<p^{-1}R(p,t)$ for all $t\in(t_p,\bar{t}_p)$. And we obtain, by the first representation of g', that $\tilde{g}'(p)>0$ for all $p\in(a,1)\cup(1,+\infty)$. Thus, this theorem becomes an immediate consequence of Lemma 2.2 and Theorem 2.4.

In the same way, we can study the structure of $\mathscr{C}$ under the following assumption (B): $f(p)\leqq pf'(p)$ for all $p\in[0,1)\cup(1,+\infty)$ and $h(1)<k(1)$. In this case, $p^{-1}f(p)$ is a non-decreasing function on $[0,1)\cup(1,+\infty)$. Let $r=\lim_{p\to+\infty}p^{-1}f(p)$, then it is easy to show that

$$\xi=\lim_{p\to+\infty}\tilde{g}(p)=\begin{cases}\dfrac{\pi}{\sqrt{r}} & \text{if}\quad r\in[0,+\infty)\\ 0 & \text{if}\quad r=+\infty.\end{cases}$$

THEOREM 3.4. *Let $a \in [0, 1]$. Suppose that:*

(i) *f satisfies assumption* (B),
(ii) *$f(p) = \alpha p$ for all $p \in (0, a)$,*
(iii) *$f(p) \neq \alpha p$ for all $p \in (a, 1)$,*

then

(1) $\{(u, \lambda) \in \mathscr{C} \mid \lambda \in [0, \xi^2] \cup (\pi^2/\alpha, +\infty)\} = \varnothing$,
(2) $\{(u, \lambda) \in \mathscr{C} \mid \lambda = \pi^2/\alpha\} = \{(\gamma \sin \pi, \pi^2/\alpha) \mid \gamma \in [0, a]\}$,
(3) *for any $\lambda \in (\xi^2, \pi^2/\alpha)$, there exists exactly one solution of problem* (1.1) *in $\mathscr{C}$.*

In our next theorem, we give an example of a function f for which there are, for some $\lambda > 0$, at least three distinct solutions of problem (1.1) in $\mathscr{C}$.

THEOREM 3.5. *Let $a \in [0, 1)$. Suppose that*

(i) $f(p) \geqq pf'(p)$ *for all* $p \in [0, 1)$,
(ii) $f(p) = \alpha p$ *for all* $p \in [0, a]$,
(iii) $f(p) \neq \alpha p$ *for all* $p \in (a, 1)$,
(iv) $h(1) < k(1)$,
(v) $\lim_{p \to +\infty} p^{-1} f(p) = 0$.

Then, there exist numbers λ_1 and λ_2 with $0 < \lambda_1 < \lambda_2$ such that for any $\lambda \in (\lambda_1, \lambda_2)$ there exist at least three distinct solutions of problem (1.1) *in $\mathscr{C}$.*

Proof. Since $\tilde{g}^2$ is continuous on $[1, +\infty)$, $\lim_{p \to +\infty} \tilde{g}(p) = +\infty$ (Lemma 2.11) and $\lim_{p \to 1^+} \tilde{g}'(p) = -\infty$ (Corollary 2.10), we have that $\bar{\lambda} = \min\{\tilde{g}^2(p) \mid p \geqq 1\}$ exists and that $0 < \bar{\lambda} < \tilde{g}^2(1) = \lambda_2$. We obtain from assumptions (i) and (iii) that $\lambda_2 > \pi^2/\alpha$. Let $\lambda_1 = \max\{\bar{\lambda}, (\pi^2/\alpha)\}$, then $0 < \lambda_1 < \lambda_2$ and there exists a number $\bar{\rho}_1 > 1$ such that $\tilde{g}^2(\bar{\rho}_1) = \lambda_1$. Let $\lambda \in (\lambda_1, \lambda_2)$. Since $\tilde{g}^2$ is a continuous function on $[0, +\infty)$ and $\lim_{p \to +\infty} \tilde{g}^2(p) = +\infty$ (Lemma 2.11), there exist numbers ρ_1, ρ_2 and ρ_3 with $0 < \rho_1 < 1 < \rho_2 < \bar{\rho}_1 < \rho_3$ such that $\lambda = \tilde{g}^2(\rho_1) = \tilde{g}^2(\rho_2) = \tilde{g}^2(\rho_3)$. Therefore $(\sigma(\rho_1), \lambda)$, $(\sigma(\rho_2), \lambda)$ and $(\sigma(\rho_3), \lambda)$ are three distinct solutions of problem (1.1) in $\mathscr{C}$.

4. Study of S^+ when $h'(0) = 0$

Since $h'(0) = 0$, it follows immediately that $\lim_{p \to 0^+} g(p) = +\infty$.

PROPOSITION 4.1. *There exists a number $\hat{\lambda} \geqq 0$ such that for any $\lambda > \hat{\lambda}$ there is at least one solution of problem* (1.1) *in S^+.*

PROPOSITION 4.2. *Suppose that $\lim_{p \to +\infty} p^{-1} f(p) = 0$. Then, there exists a positive number λ_1 such that for any $\lambda > \lambda_1$ the problem* (1.1) *has at least two distinct solutions in S^+, for $\lambda = \lambda_1$ at least one and for $\lambda \in [0, \lambda_1)$ none.*

Proof. Since $\lim_{p \to 0^+} g^2(p) = +\infty$, $\lim_{p \to +\infty} g^2(p) = +\infty$ (Lemma 2.11) and g^2 is a continuous function on $(0, +\infty)$, $\lambda_1 = \min\{g^2(p) \mid p > 0\}$ exists and for any

$\lambda > \lambda_1 > 0$ there are numbers ρ, ρ_1 and $\bar{\rho}$ with $0 < \rho < \rho_1 < \bar{\rho}$ such that $g^2(\rho_1) = \lambda_1$ and $g^2(\rho) = g^2(\bar{\rho}) = \lambda$. Thus, $(\sigma(\rho), \lambda)$ and $(\sigma(\bar{\rho}), \lambda)$ are two distinct solutions of problem (1.1) in S^+. Furthermore, $(\sigma(\rho_1), \lambda_1)$ is also a solution of problem (1.1) in S^+. This completes the proof of this proposition.

Now, we study the structure of S^+ under assumption (B): $f(p) \leqq pf'(p)$ for all $p \in [0, 1) \cup (1, +\infty)$ and $h(1) < k(1)$. In this case, $p^{-1}f(p)$ is a non-decreasing function on $(0, 1) \cup (1, +\infty)$. Let $m = \lim_{p \to +\infty} p^{-1} f(p)$, then

$$\xi = \lim_{p \to +\infty} \tilde{g}(p) = \begin{cases} \dfrac{\pi}{\sqrt{m}} & \text{if} \quad m \in (0, +\infty) \\ 0 & \text{if} \quad m = +\infty \end{cases}.$$

THEOREM 4.3. *Suppose that f satisfies assumption* (B). *Then*

(1) $\{(u, \lambda) \in S^+ \mid \lambda \in [0, \xi^2]\} = \varnothing$,

(2) *for any* $\lambda > \xi^2$, *there exists exactly one solution of problem* (1.1) *in* S^+.

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