

# Finitely presented wreath products and double coset decompositions

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Received: 30 August 2005 / Accepted: 13 March 2006 / Published online: 22 June 2006  
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**Abstract** We characterize which permutational wreath products  $G \ltimes W^{(X)}$  are finitely presented. This occurs if and only if  $G$  and  $W$  are finitely presented,  $G$  acts on  $X$  with finitely generated stabilizers, and with finitely many orbits on the cartesian square  $X^2$ . On the one hand, this extends a result of G. Baumslag about infinite presentation of standard wreath products; on the other hand, this provides nontrivial examples of finitely presented groups. For instance, we obtain two quasi-isometric finitely presented groups, one of which is torsion-free and the other has an infinite torsion subgroup. Motivated by the characterization above, we discuss the following question: which finitely generated groups can have a finitely generated subgroup with finitely many double cosets? The discussion involves properties related to the structure of maximal subgroups, and to the profinite topology.

**Keywords** Wreath products · Two-transitive actions · Double coset decompositions · Graph products · Subgroup separability · Engulfing Property · Maximal subgroups

**Mathematics Subject Classifications (2000)** Primary 20E22 · Secondary 20B22 · 20F05 · 20E28

## 1 Introduction

Let  $G$  be a group, and  $X$  a  $G$ -set. Let  $W$  be another group. Then  $G$  acts on the direct sum  $W^{(X)}$  by permutations of factors. The (*permutational*) *wreath product*  $W \wr_X G$  is defined to be the semidirect product  $W^{(X)} \rtimes G$ . When the action of  $G$  on  $X$  is simply transitive, it is called the *standard* wreath product (this special case is sometimes called the wreath product) and denoted by  $W \wr G$ .

By a result of Baumslag [2], a standard wreath product  $W \wr G$  with  $W \neq \{1\}$  and  $G$  infinite is never finitely presented. In contrast, permutational wreath products provide nontrivial examples:

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**Theorem 1.1** *If  $W \neq \{1\}$ , the wreath product  $W \wr_X G$  is finitely presented if and only if the following conditions are satisfied*

- (i)  *$W$  and  $G$  are finitely presented,*
- (ii)  *$G$  acts on  $X$  with finitely generated stabilizers, and*
- (iii) *the product action of  $G$  on the Cartesian square  $X^2$  has finitely many orbits.*

Note that this result extends Baumslag's result: indeed, if  $G$  acts simply transitively on  $X$ , then (iii) implies that  $X$  is finite.

We indicate (see Examples 3.4, 3.5, 3.6) groups  $G$  with an infinite  $G$ -set  $X$  satisfying the hypotheses of Theorem 1.1, which thus provides new examples of finitely presented groups. For instance, it allows to prove the existence of two quasi-isometric finitely presented groups, one of which is torsion-free and the other has an infinite torsion subgroup (see Proposition 2.12).

A general question, motivated by Theorem 1.1, is: what are pairs  $(G, X)$  satisfying the hypotheses of Theorem 1.1? Trivial examples are pairs  $(G, X)$  where  $G$  is finitely presented and  $X$  a finite  $G$ -set, thus we focus on nontrivial cases, namely those for which  $X$  is infinite.

Section 3 is devoted to discuss obstructions, for a given group  $G$ , to the existence of an infinite  $G$ -set  $X$  satisfying (ii) and (iii) of Theorem 1.1, respectively satisfying (iii). It is, in the major part, written as a survey, including many examples. For instance, if  $G$  is a finitely generated linear solvable group, there exists no infinite  $G$ -set satisfying (iii) of Theorem 1.1; while if  $G$  is a free group, there exists an infinite  $G$ -set satisfying (iii) of Theorem 1.1, but none can satisfy both (ii) and (iii).

## 2 Finitely presented wreath products

### 2.1 Proof of Theorem 1.1

For completeness, we first recall the following easy result.

**Proposition 2.1** *If  $X \neq \emptyset$  and  $W \neq \{1\}$ , the wreath product  $W \wr_X G$  is finitely generated if and only if  $G$  and  $W$  are finitely generated, and  $G$  has a finite number of orbits on  $X$ .*

*Proof* If the conditions are satisfied, and if  $n$  denotes the number of  $G$ -orbits in  $X$ , then  $W \wr_X G$  can be written as a quotient of the free product  $W^{*n} * G$ , where  $W^{*n}$  denotes the free product of  $n$  copies of  $W$ .

Conversely, suppose that  $W \wr_X G$  is finitely generated. Being a quotient of  $W \wr_X G$ ,  $G$  is also finitely generated. Since  $X$  is nonempty,  $W$  embeds in  $W \wr_X G$ , hence is countable. If  $W$  is not finitely generated, it can be written as the union of a strictly increasing sequence of subgroups  $W_n$ . Therefore  $W \wr_X G$  is the union of the strictly increasing sequence of subgroups  $W_n \wr_X G$ , and hence is not finitely generated. Finally, if  $X$  has infinitely many  $G$ -orbits, it can be written as the union of a strictly increasing sequence  $(X_n)$  of  $G$ -invariant subsets. So  $G$  is the union of the strictly increasing sequence of subgroups  $W \wr_{X_n} G$  (using that  $W \neq \{1\}$ ).  $\square$

Let us now look at a presentation for the wreath product  $W \wr_X G$ . For the sake of simplicity, we first suppose that  $G$  acts transitively on  $X$ , so that we can write  $X = G/H$ . It is easy to check that a presentation for  $W \wr_{G/H} G$  is given by

$$\langle G, W, \mid [H, W], [W, gWg^{-1}] \forall g \in G - H \rangle. \quad (2.1)$$

\* This concise notation must be understood as:  $W \wr_{G/H} G$  is the quotient of the free product  $G * W$  by the given relations.

Using the relation  $[H, W] = \{1\}$ , it is immediate that, in the family of relations  $[W, gWg^{-1}]$  with  $g \in G - H$ , it suffices to take into account those  $g$  in  $G/H - \{H\}$ . In fact, we can do better: we can consider only those  $g$  in  $H \backslash G/H - \{H\}$ : this is obtained by conjugating the relation  $[W, gWg^{-1}]$  by an element of  $H$ . With these remarks, we can prove:

**Theorem 2.2** *Let  $G, W$  be finitely presented groups. Let  $G$  act on a set  $X$ , with finitely generated stabilizers. Suppose that the product action of  $G$  on  $X^2$  has a finite number of orbits. Then  $W \wr_X G$  is finitely presented.*

*Proof* We begin by the case when  $G$  is transitive on  $X$ , so that we can write  $X = G/H$ . Since  $W$  and  $H$  are finitely generated,  $[H, W] = \{1\}$  in the presentation (2.1) reduces to a finite number of relators. The hypothesis that the product action of  $G$  on  $X^2$  has a finite number of orbits reads as:  $H \backslash G/H$  is finite. Then the result follows from the remarks above: the family of relations  $[W, gWg^{-1}]$  of the presentation (2.1) reduces to the finite family  $[W, g_i W g_i^{-1}]$ , where  $(g_i)$  is a finite system of representants of the double classes modulo  $H$  in  $G$ , except the class  $\{H\}$ .

We now indicate how to deal with the case when  $G$  is not necessarily transitive on  $X$ , which makes no essential difference. Choosing a point in each orbit, we can write  $X = \coprod_{i \in I} G/H_i$ , where  $I = G \backslash X$ . For all  $i \in I$ , consider a copy  $W_i$  of  $W$ . Then it is easy to check that a presentation for  $W \wr_X G$  is given by the quotient of the free product of  $G$  and all  $W_i$  by the relations:

$$\begin{aligned} [H_i, W_i] \quad (i \in I), \quad [W_i, gW_i g^{-1}] \quad (i \in I, g \in G - H_i), \\ [W_i, gW_j g^{-1}] \quad (i, j \in I, i \neq j, g \in G). \end{aligned}$$

If we forget for a few seconds the two latter families of relations, we get the generalized free product with amalgamation  $G *_{(H_i)} (H_i \times W_i)$ . Given that  $I$  is finite, that  $G$  and  $W$  are finitely presented, and that all  $H_i$  are finitely generated, this free product with amalgamation is clearly finitely presented.

Choose  $R \subset G$  such that, for every  $i, j \in I$ , every double coset  $H_i g H_j$  is equal to  $H_i g' H_j$  for some  $g' \in R$ . Then the last two families of relations follow from their subfamilies when  $g$  ranges over  $R$ . On the other hand, the  $G$ -action on  $X^2$  having a finite number of orbits is equivalent to saying that all double quotients  $H_i \backslash G/H_j$  are finite, so that  $R$  can be chosen finite. Thus, since  $W$  is finitely generated, these reduce to finitely many relations.  $\square$

We are now going to show that the converse of Theorem 2.2 is true. We need some elementary preliminaries on graph products.

Let  $\Gamma$  be a graph, that is, a set  $\Gamma^0 = X$ , whose elements are called *vertices*, along with a subset  $\Gamma^1$  of subsets of cardinality two of  $\Gamma^0$ , called *edges*. For each  $i \in \Gamma^0$ , let  $W_i$  be a group. Following [13], the graph product  $P = (W_i)_{i \in X}^{(\Gamma)}$  of all  $W_i$  is by definition the quotient of the free product of all  $W_i$  by the relations  $[W_i, W_j] = \{1\}$  if  $\{i, j\} \in \Gamma^1$ . Denote by  $\sigma_i$  the obvious morphism  $W_i \rightarrow P$ . Observe that if  $\Gamma$  is the totally disconnected graph, then  $P$  is the free product of all  $W_i$ , and if  $\Gamma$  is the complete graph, then  $\Gamma$  is the direct sum (sometimes called the restricted direct product) of all  $W_i$ . When all  $W_i$  are equal to a single group  $W$ ; we denote the graph product by  $W^{(\Gamma)}$ .

### Lemma 2.3

- (1) For all  $i$ ,  $\sigma_i: W_i \rightarrow P$  is injective.
- (2) If  $\{i, j\} \notin \Gamma^1$ , the natural morphism  $\sigma_i * \sigma_j \rightarrow P$  is injective.
- (3) If  $\{i, j\} \in \Gamma^1$ , the natural morphism  $\sigma_i \times \sigma_j \rightarrow P$  is injective.

- (4) If  $\{i, k\} \notin \Gamma^1$  and  $\{j, k\} \notin \Gamma^1$ , the natural morphism  $W_i * W_j * W_k \rightarrow P$ , or  $(W_i \times W_j) * W_k \rightarrow P$  (according as whether  $\{i, j\}$  belongs to  $\Gamma^1$ ) is injective.

*Proof* It suffices to observe that all these morphisms are split, as we see by taking the quotient of  $P$  by the normal subgroup generated by all  $W_\ell$  for  $\ell \neq i$  (resp. for  $\ell \neq i, j$ ) (resp. for  $\ell \neq i, j, k$ ).  $\square$

Lemma 2.3 has the following consequence. Let  $\Gamma'$  be another graph structure on the same set of vertices:  $\Gamma'^0 = \Gamma^0 = X$ . Suppose in addition that  $\Gamma'^1 \supseteq \Gamma^1$ . There is a natural morphism  $p$  from  $P = (W_i)_{i \in X}^{(\Gamma)}$  to  $P' = (W_i)_{i \in X}^{(\Gamma')}$ , which is obviously surjective. Lemma 2.3(2) yields:

**Lemma 2.4** *Suppose that  $W_i \neq \{1\}$  for all  $i \in X$ . Then the morphism  $p$  is bijective if and only if  $\Gamma'^1 = \Gamma^1$ .*

*Proof* Suppose that  $p$  is bijective. Let  $\{i, j\}$  be an edge in  $\Gamma^1$ . Then  $[W_i, W_j] = \{1\}$  in  $P$ . By injectivity, we get that  $[W_i, W_j] = \{1\}$  in  $P'$ . Since  $W_i \neq \{1\}$  and  $W_j \neq \{1\}$ , we obtain that  $W_i$  and  $W_j$  cannot generate their free product in  $P'$ , so that, by Lemma 2.3(2),  $\{i, j\} \in \Gamma'^1$ . The converse is clear.  $\square$

Now denote by  $Q$  the kernel of the natural morphism  $P = (W_i)_{i \in X}^{(\Gamma)} \rightarrow \bigoplus_{i \in X} W_i$ . We want to show that  $Q$  often contains a free nonabelian group. Assume, from now on, that  $W_i \neq \{1\}$  for all  $i$ . It already follows from Lemma 2.4 that if  $\Gamma$  is not the complete graph, then  $Q \neq \{1\}$ . Now denote by  $\Gamma_{\text{op}}$  the complement graph; namely,  $\Gamma_{\text{op}}^0 = \Gamma^0 = X$ , and, for all  $i \neq j \in X$ ,  $\{i, j\} \in \Gamma_{\text{op}}^1$  if and only if  $\{i, j\} \notin \Gamma^1$ . Note that a decomposition of  $\Gamma$  (resp.  $\Gamma_{\text{op}}$ ) into connected components corresponds to a decomposition of  $P$  into a free product (resp. a direct sum).

**Lemma 2.5** *Suppose that  $W_i \neq \{1\}$  for all  $i$ . The following are equivalent.*

- (i)  $Q$  does not contain any nonabelian free subgroup.
- (ii) All connected components of  $\Gamma_{\text{op}}$  have at most 2 elements, and whenever  $\{i, j\}$  is a 2-element connected component of  $\Gamma_{\text{op}}$ , then  $W_i$  and  $W_j$  are isomorphic to  $C_2$ , the cyclic group on two elements.

*Proof* Suppose that (i) holds. Let  $J$  be the union of 1-element connected components of  $\Gamma_{\text{op}}$ , and  $K \subset X - J$  a subset intersecting each 2-element connected component of  $\Gamma_{\text{op}}$  in exactly one element. Then  $Q$  can be identified to the kernel of the natural morphism  $D_\infty^{(K)} \rightarrow (C_2 \times C_2)^{(K)}$ , where  $D_\infty \simeq C_2 * C_2$  denotes the infinite dihedral group, and thus  $Q$  is abelian (isomorphic to  $\mathbf{Z}^{(K)}$ ) and cannot contain free subgroups.

Conversely, suppose that (ii) is satisfied.

(a) Suppose that there exists a connected component of  $\Gamma_{\text{op}}$  with at least 2 elements, and with at least one element  $i$  such that  $W_i$  is not cyclic on two elements. Pick  $j$  such that  $\{i, j\} \in \Gamma_{\text{op}}^1$ . The following fact is immediate.

**Fact 2.6** Let  $G$  be a group with at least three elements. Then it has a subgroup isomorphic to either  $\mathbf{Z}$ ,  $C_p$  (the cyclic group of prime odd order  $p$ ),  $C_4$ , or  $C_2 \times C_2$ .

Pick any nontrivial cyclic subgroup  $Z_j$  in  $W_j$ , and any subgroup  $Z_i$  of  $W_i$  as in Fact 2.6. By Lemma 2.3, there is a natural embedding of  $Z_i * Z_j$  into  $P$ , which is mapped to the abelian group  $Z_i \times Z_j$  in  $\bigoplus_{i \in X} W_i$ . Since  $Z_i * Z_j$  contains a nonabelian free subgroup,

so does its derived subgroup which is contained in  $Q$ , so that  $Q$  contains a nonabelian free subgroup.

(b) Otherwise, suppose that there exists a connected component of  $\Gamma_{\text{op}}$  with at least 3 elements. Take  $i, j, k \in X$ , distinct, such that  $\{i, k\}$  and  $\{j, k\}$  belong to  $\Gamma_{\text{op}}^1$ . We can suppose that  $W_i, W_j, W_k$  are cyclic on two elements, otherwise we can argue as in (a). By Lemma 2.3, we get an embedding of  $(C_2 \times C_2) * C_2$  or  $C_2 * C_2 * C_2$  into  $P$ , mapping to the abelian subgroup  $C_2 \times C_2 \times C_2$  in  $\bigoplus_{i \in X} W_i$ . As in a), since both  $(C_2 \times C_2) * C_2$  and  $C_2 * C_2 * C_2$  contain nonabelian free subgroups, we obtain that  $Q$  contains a nonabelian free subgroup.  $\square$

When  $\Gamma$  is the totally disconnected graph, Lemma 2.5 reduces as:

**Lemma 2.7** *Let  $(W_i)_{i \in X}$  be a family of nontrivial groups, and let  $Q$  be the kernel of the natural morphism from the free product of all  $W_i$  to the direct sum of all  $W_i$ . Suppose that  $X$  has at least 2 elements, and, if all  $W_i$  are cyclic on 2 elements, that  $X$  has at least 3 elements. Then  $Q$  contains a non-abelian free subgroup.*  $\square$

**Lemma 2.8** *Let  $X$  be a set, and  $\Gamma_n$  an increasing family of graph structures on  $X$ : that is,  $\Gamma_n^0 = X$ , and  $\Gamma_n^1 \subseteq \Gamma_{n+1}^1$  for all  $n$ . Suppose that  $X$  can be written as a finite disjoint union  $X = \coprod_{i=1}^k X_i$  such that, for all  $n$ , the complement graph  $(\Gamma_n)_{\text{op}}$  can be written as a disjoint union of subgraphs  $\Lambda_{n,i}$ , with  $\Lambda_{n,i}^0 = X_i$  and  $\Lambda_{n,i}$  has constant finite degree. Then the sequence  $(\Gamma_n)$  is eventually constant.*

*Proof* Let  $d_{n,i}$  denote the degree of  $\Lambda_{n,i}$ . The sequence  $(\sum_{i=1}^k d_{n,i})_{n \in \mathbb{N}}$  decreases, hence is eventually constant. Thus eventually, all sequences  $(d_{n,i})_{n \in \mathbb{N}}$  are constant. Observe that if  $d_{n,i} = d_{n+1,i}$ , then  $\Lambda_{n,i} = \Lambda_{n+1,i}$ . Accordingly, the sequence  $(\Gamma_n)$  is eventually constant.  $\square$

Now suppose that all  $W_i$  are equal to a single group  $W \neq \{1\}$ , and suppose that a group  $G$  acts on  $\Gamma$ , i.e., acts on  $\Gamma^0 = X$  preserving  $\Gamma^1$ . Then the semidirect product  $W^{(\Gamma)} \rtimes G$  is well-defined.

We have to describe, given a  $G$ -set, what are the graph structures preserved by  $G$ . Let  $X$  be a set. Define an *edge set* on  $X$  to be a subset of  $X \times X$  which is symmetric and does not intersect the diagonal; an edge set obviously defines a structure of graph on  $X$ . Suppose now that  $X$  is a  $G$ -set. Decompose  $X$  into its  $G$ -orbits:  $X = \coprod X_i$  ( $i \in I$ ), and choose some base-point  $x_i$  in each  $X_i$  so that we can write  $X_i = G/H_i$ .

**Lemma 2.9** *If  $E$  is a  $G$ -invariant edge set on  $X$ , and if  $(i, j) \in I^2$ , define  $B_{ij} = B_{ij}(E) = \{g \in G, (x_i, gx_j) \in E\}$ . Then the subsets  $B_{ij} \subset G$  satisfy: for all  $i, j \in I$ ,  $B_{ij}^{-1} = B_{ji}$ ,  $H_i B_{ij} = B_{ij}$ ,  $H_i \cap B_{ii} = \emptyset$ .*

*Conversely, for every family  $(V_{ij})_{i,j \in I}$  of subsets of  $G$  satisfying these three conditions, there exists a unique  $G$ -invariant edge set  $E$  such that  $V_{ij} = B_{ij}(E)$  for all  $i, j \in I$ , given by  $(gx_i, g'x_j) \in E$  if and only if  $g^{-1}g' \in V_{ij}$ .*

*Proof* All verifications are straightforward.  $\square$

We can now prove the converse of Theorem 2.2. It is essentially contained in the following slightly stronger result:

**Proposition 2.10** *Let  $G, W$  be groups, and  $X$  a  $G$ -set with finitely many orbits. Suppose that  $W \neq \{1\}$ ,  $X \neq \emptyset$ , and that one of the following conditions is satisfied.*

- (1) The group  $G$  has infinitely many orbits on  $X^2$ .
- (2) For some  $x \in X$ , the stabilizer  $G_x$  is not finitely generated.

Then, for every finitely presented group mapping onto  $W \wr_X G$ , the kernel contains a nonabelian free subgroup. In particular,  $W \wr_X G$  is not finitely presented.

*Proof* We keep the notation introduced above:  $X = \coprod G/H_i$ .

Suppose that (1) is satisfied. Then, for some  $k, \ell$ ,  $H_k \backslash G/H_\ell$  is infinite. Define, for  $i, j \in I$ ,  $n \in \mathbb{N}$ , subsets  $V_{ij}^n$  of  $G$  as follows.

If  $k \neq \ell$ , take a strictly increasing sequence  $(U_n)$  of finite subsets of  $H_k \backslash G/H_\ell$  whose union is all of  $H_k \backslash G/H_\ell$ . Define  $V_{k\ell}^n = U_n$ ,  $V_{\ell k}^n = U_n^{-1}$ .

If  $k = \ell$ , take a strictly increasing sequence  $(U_n)$  of finite subsets of  $H_k \backslash G/H_k - \{H_k\}$  which are symmetric under inversion, so that the union of all  $U_n$  is all of  $H_k \backslash G/H_k - \{H_k\}$ . Define  $V_{kk}^n = U_n$ .

In both cases, for all  $i, j$  such that  $\{i, j\} \neq \{k, \ell\}$ , define  $V_{ij}^n$  to be all of  $H_i \backslash G/H_j$  if  $i \neq j$ , and  $H_i \backslash G/H_i - \{H_i\}$  if  $i = j$ .

Let  $E_n$  be the  $G$ -invariant edge set on  $X$  corresponding, by Lemma 2.9, to the family  $(V_{ij}^n)_{i,j \in I}$ , and denote by  $X_n$  the corresponding graph. Observe that  $(E_n)$  is a strictly increasing sequence of  $G$ -invariant edge sets whose union is the full edge set  $E_\infty = X^2 - \text{diag}(X)$ . Hence, the sequence of surjective morphisms between finitely generated groups  $W^{(X_n)} \rtimes G \rightarrow W^{(X_{n+1})} \rtimes G$  converges to  $W^{(X_\infty)} \rtimes G = W^{(X)} \rtimes G$ . This already proves that  $W^{(X)} \rtimes G$  is not finitely presented: more precisely, if a finitely presented group maps onto  $W^{(X)} \rtimes G$ , then the map factors through  $W^{(X_n)} \rtimes G$  for some  $n$ .

Now, if the kernel of  $W^{(X_n)} \rightarrow W^{(X)}$  does not contain a nonabelian free subgroup, then, by Lemma 2.5, the complement graph of  $X_n$  has all its vertices of degree at most 1. Since this degree is constant on every  $G$ -orbit of  $X$ , the hypotheses of Lemma 2.8 are satisfied, and thus the sequence of graphs  $(X_n)$  stabilizes, a contradiction. Therefore, for all  $n$ , the kernel of  $W^{(X_n)} \rtimes G \rightarrow W^{(X)} \rtimes G$  does not contain any nonabelian free subgroup. Since, for every finitely presented group mapping onto  $W \wr_X G$ , the map must factor through  $W^{(X_n)} \rtimes G$  for some  $n$ , we obtain the desired conclusion.

Suppose that (2) is satisfied: fix  $i$  such that  $H_i$  is not finitely generated. Write  $H_i$  as a strictly increasing union of subgroups  $H_{i,n}$ . Define  $X_n$  as the disjoint union  $\coprod_{j \neq i} G/H_j \sqcup G/H_{i,n}$ , and endow it with the edge set defined as:  $x \sim y$  unless  $x = y$  or  $x, y \in G/H_{i,n}$  and  $x \in yH_i$ . Let  $Q$  be the kernel of the natural map  $W^{(X_n)} \rightarrow W^{(X)}$ . It coincides with the kernel of the natural map from the graph product  $W^{(G/H_{i,n})}$  to  $W^{(G/H_i)}$ , and hence contains the kernel of the natural map from the free product  $W^{*H_i/H_{i,n}}$  to  $W$ . Noting that  $H_{i,n}$  has infinite index in  $H_i$ , by Lemma 2.7,  $Q$  contains a nonabelian free subgroup. Accordingly, the kernel of  $W^{(X_n)} \rtimes G \rightarrow W^{(X)} \rtimes G$  also contains a nonabelian free subgroup for all  $n$ , and since  $W^{(X_n)} \rtimes G$  is a sequence of finitely generated groups converging to  $W \wr_X G$ , we can conclude as we did for (1): if a finitely presented group maps onto  $W^{(X)} \rtimes G$ , then the map factors through  $W^{(X_n)} \rtimes G$  for some  $n$ .  $\square$

**Theorem 2.11** *Let  $G, W$  be groups. Let  $G$  act on a nonempty set  $X$ . Suppose that  $W \wr_X G$  is finitely presented. Then  $G$  and  $W$  are finitely presented, and, if  $W \neq \{1\}$ , then the action of  $G$  on  $X$  has finitely generated stabilizers, and the product action of  $G$  on  $X^2$  has a finite number of orbits.*

*Proof* By Proposition 2.1,  $G$  and  $W$  are finitely generated, and  $G$  has finitely many orbits on  $X$ .

Now observe that  $G$  is finitely presented, since it is obtained from  $W \wr_X G = W^{(X)} \rtimes G$  by killing a finite generating subset of  $W^I \subset W^{(X)}$ , where  $I \subset X$  is a finite set which contains one point in each orbit.

Suppose now that  $W$  is not finitely presented. Then there is a sequence of noninjective surjective morphisms  $W_n \rightarrow W_{n+1}$  between finitely generated groups, whose limit is  $W$ . Then, the sequence of noninjective surjective morphisms between finitely generated groups:  $W_n \wr_X G \rightarrow W_{n+1} \wr_X G$  converges to  $W \wr_X G$ , contradicting that  $W \wr_X G$  is finitely presented. Now Proposition 2.10 allows us to conclude.  $\square$

## 2.2 Applications

Our main application consists in proving that the property of being torsion-free is not weakly geometric among finitely presented groups. The examples of [10] are standard wreath products, so are infinitely presented.

Let  $F$  be the Thompson group of the dyadic interval (see Example 3.4), and  $F_{1/2}$  the stabilizer of  $1/2$ . The homogeneous space  $F/F_{1/2}$  can be identified with the set  $I$  of all dyadic numbers contained in the interval  $(0, 1)$ , and the action of  $F$  is transitive on ordered pairs  $(a, b)$ ,  $a < b$ , that is,  $F_{1/2}$  has exactly three cosets in  $F$ .

**Proposition 2.12** *The finitely presented groups  $\mathbf{Z} \wr_I F$  and  $D_\infty \wr_I F$  are bi-Lipschitz-equivalent. The first is torsion-free, while the second contains an infinite subgroup of exponent 2.*

*Proof* The finite presentation follows from Theorem 2.2. The second assertion reduces, by Proposition 4.2, to the fact that  $\mathbf{Z}$  and  $D_\infty$  are bi-Lipschitz-equivalent. The last assertion is clear.  $\square$

Let  $S$  be any nonabelian simple, finitely presented group (possibly finite). Let  $G$  be a finitely presented group, with an infinite index, finitely generated subgroup  $H$ , such that  $H \backslash G/H$  is finite, and such that the action of  $G$  on  $G/H$  is faithful.

Set  $\Gamma = S \wr_{G/H} G$ . This group has the following properties:

**Proposition 2.13** (1)  $\Gamma$  is finitely presented.  
(2) Any nontrivial normal subgroup of  $\Gamma$  contains  $N = S^{(G/H)}$ .

*Proof* (1) follows from Theorem 2.2.

(2) Since the action of  $G$  on  $N$  is purely outer (that is, the morphism  $G \rightarrow \text{Out}(N)$  is injective), every nontrivial normal subgroup of  $G$  intersects nontrivially  $N$ . On the other hand, any normal subgroup  $N'$  intersecting nontrivially  $N$  contains it: let us recall the standard argument. For  $x \in G/H$  and  $s \in S$ , denote by  $\delta_x(s)$  the function  $X \rightarrow S$  sending  $x$  to  $s$  and every  $y \neq x$  to 1. If  $(s_x)_{x \in G/H}$  be a nontrivial element in  $N' \cap N$ , then, taking the commutator with a suitable  $\delta_x(s)$ , we obtain that  $N'$  contains  $\delta_x(t)$  for some  $x \in G/H$  and some  $1 \neq t \in S$ . Such an element clearly generates  $N$  as a normal subgroup.  $\square$

Note that the normal subgroup lattice structure of  $\Gamma$  is obtained from that of  $G$  by adding a point “at the bottom”.

An example of a direct application of Proposition 2.10 is the following well-known result, initially proved in [33].

**Corollary 2.14** *The free  $d$ -solvable group  $R_{d,n}$  on  $n$  generators ( $d, n \geq 2$ ) is not finitely presented.*



*Proof* It suffices to observe that if  $A$  is a finitely presented group which maps onto  $R_{d,n}$ , then  $A$  contains a free subgroup of rank two. Indeed,  $R_{d,n}$  maps onto  $\mathbf{Z} \wr \mathbf{Z}$ , while every finitely presented group mapping onto  $\mathbf{Z} \wr \mathbf{Z}$  must contain a free subgroup by Proposition 2.10.  $\square$

### 3 Subgroups of finite biindex and related properties

#### 3.1 Definitions and examples

Theorems 2.2 and 2.11 raise the following question: which finitely presented groups  $G$  have an infinite index finitely generated subgroup  $H$  such that  $G$  acts on  $(G/H)^2$  with a finite number of orbits? It is also natural to ask the same question without assuming  $H$  finitely generated. These questions seem to have never been systematically investigated, but related properties give useful information for our purposes; for instance subgroup separability, which has been extensively studied for other motivations, such as the generalized word problem. Hence, the purpose of the following definitions is to present various obstructions for a group  $G$  to have an almost 2-transitive action on an infinite set.

**Definition 3.1** Define a *pair of groups* as a pair  $(G, H)$ , where  $G$  is a group and  $H$  a subgroup.

We say that a pair is *finitely presented* if  $G$  is finitely presented and  $H$  is finitely generated.

We say  $H$  has *finite biindex* in  $G$  if  $H \backslash G/H$  is finite. We also say that the pair  $(G, H)$  is *almost 2-transitive*; this is equivalent to say that  $G$  has finitely many orbits on  $(G/H)^2$ .

We say  $H$  is *almost maximal* in  $G$  if there are only finitely many subgroups of  $G$  containing  $H$ . We also say that the pair  $(G, H)$  is *almost primitive*.

We say that a subgroup  $H$  of  $G$  has *finite proindex* if the profinite closure of  $H$  in  $G$  (that is, the intersection of all finite index subgroups of  $G$  containing  $H$ ) has finite index in  $G$ .

**Lemma 3.2** For pairs  $(G, H)$ , we have the implications:  $(H \text{ has finite index}) \Rightarrow (H \text{ has finite biindex}) \Rightarrow (H \text{ is almost maximal}) \Rightarrow (H \text{ has finite proindex})$ .

*Proof* The first one is trivial. For the second one, suppose that  $H$  has finite biindex  $m$ . Every subgroup containing  $H$  is an union of double cosets of  $H$ ; accordingly the number of possible subgroups is bounded by  $2^m$ . For the third implication, observe that if a group has profinite closure of infinite index, this profinite closure must be the intersection of infinitely many finite index subgroups.  $\square$

**Remark 3.3** None of these implications is an equivalence, even when  $G$  is finitely presented.

- For examples of infinite index subgroups of finite biindex, see Examples 3.4, 3.5, and 3.6 below.
- If  $G \neq \{1\}$  has no proper subgroup of finite index (for instance,  $G$  is infinite and simple), then  $\{1\}$  has finite proindex in  $G$ , but is not almost maximal.
- Recall that, for a group  $G$  and a subgroup  $H$ , the pair  $(G, H)$  is called a *Hecke pair* if, for all  $g \in G$ ,  $gHg^{-1}$  and  $H$  are commensurable, i.e., they have a common finite index subgroup; equivalently this means that the orbits of  $H$  in  $G/H$  are finite. On the other hand,  $H$  having finite biindex means that there are finitely many such orbits. Thus if  $(G, H)$  is a Hecke pair and  $H$  has infinite index, then  $H$  has infinite biindex. Now it is known that, for any prime  $p$ ,  $(\mathrm{SL}_2(\mathbf{Z}[1/p]), \mathrm{SL}_2(\mathbf{Z}))$  is a Hecke pair, and that  $\mathrm{SL}_2(\mathbf{Z})$  is a maximal subgroup of infinite index in  $\mathrm{SL}_2(\mathbf{Z}[1/p])$ , hence also has infinite biindex.



I only know a restricted sample of faithful almost 2-transitive finitely presented pairs.

**Example 3.4** Let  $G$  be the Thompson group  $F$ . This is the group of piecewise linear increasing homeomorphisms of  $[0, 1]$  with singularities in  $\mathbf{Z}[1/2] \cap [0, 1]$  and slopes powers of 2. This group is finitely presented and torsion-free, does not contain any nonabelian free subgroup, and has simple derived subgroup (see [8]). The group  $F$  acts on  $[0, 1] \cap \mathbf{Z}[1/2]$ , fixing 0 and 1, and acting transitively on pairs  $(a, b) \in \mathbf{Z}[1/2]$  satisfying  $0 < a < b < 1$ . The stabilizer  $F_{1/2}$  of  $1/2$  is easily seen to be isomorphic to  $F \times F$ . So the pair  $(F, F_{1/2})$  is almost 2-transitive and finitely presented.

**Example 3.5** Let  $G$  be the Thompson group  $T$  (see [8]) of the circle, which is finitely presented and simple. This is the group of piecewise linear oriented homeomorphisms of the circle  $\mathbf{R}/\mathbf{Z}$  with singularities in  $\mathbf{Z}[1/2]/\mathbf{Z}$  and slopes powers of 2. The stabilizer  $H$  of  $0 = 1 \in \mathbf{R}/\mathbf{Z}$  is isomorphic to the Thompson group  $F$  of Example 3.4. Then  $T$  acts two-transitively on  $T/F = \mathbf{Z}[1/2]/\mathbf{Z}$ .

**Example 3.6** (*Houghton groups*) Fix an integer  $n \geq 1$ . Let  $\mathbf{N}$  denote the non-negative integers, and set  $\Omega_n = \mathbf{N} \times \{1, \dots, n\}$ . We think at  $\Omega_n$  as the disjoint union of  $n$  copies  $\mathbf{N}_1, \dots, \mathbf{N}_n$  of  $\mathbf{N}$ . Let  $G_n$  be the group of all permutations  $\sigma$  of  $\Omega_n$  such that, for all  $i$ ,  $\sigma(\mathbf{N}_i) \Delta \mathbf{N}_i$  is finite, and  $\sigma$  is eventually a translation on  $\mathbf{N}_i$ .

When  $n = 1$ ,  $G_1$  is the group of permutations with finite support of  $\mathbf{N}_1$ , while  $G_n$  is finitely generated if  $n \geq 2$  and finitely presented if  $n \geq 3$  (see [7]; Brown attributes the finite presentation when  $n = 3$  to R. Burns and D. Solitar; the finite generation is due to Houghton). For an explicit presentation when  $n = 3$ , see [19].

Note that, for  $n \geq 2$ , the derived (resp. second derived) subgroup of  $G$  coincides with the group of permutations (resp. even permutations) with finite support of  $\Omega_n$ . In particular, the action of  $G_n$  on  $\Omega_n$  is  $k$ -transitive for all  $k$ .

On the other hand, as an extension of  $\mathbf{Z}^{n-1}$  by a locally finite group,  $G_n$  is elementary amenable (but not virtually solvable). The stabilizer  $H_n$  of a point is isomorphic to  $G_n$ ; in particular, it is finitely generated for  $n \geq 2$ .

**Example 3.7** In [32], a 3-manifold group  $\Gamma$  together with an infinite index surface subgroup  $\Lambda$  are exhibited; it is proved in [23] that  $\Lambda \backslash \Gamma / \Lambda$  is finite. (I do not know if the  $\Gamma$ -action on  $\Gamma / \Lambda$  is faithful.)

**Example 3.8** A refinement by D. Wise [35] of a construction of Rips shows that, for every finitely presented group  $Q$ , there exists a finitely presented, residually finite, torsion-free,  $C'(1/6)$  small cancellation group  $G$  and a surjective map  $p: G \rightarrow Q$ , such that  $\text{Ker}(p)$  is a finitely generated subgroup of  $G$ .

Accordingly, if  $K$  is a finitely generated subgroup of finite biindex and infinite index in  $Q$ , then  $p^{-1}(K)$  is a finitely generated subgroup of finite biindex and infinite index in  $G$ .

Thus, starting from any of the above examples, we obtain examples of almost 2-transitive finitely presented pairs  $(G, H)$  with  $G/H$  infinite and  $G$  residually finite, torsion-free, word hyperbolic, satisfying the  $C'(1/6)$  small cancellation property.

### 3.2 Related definitions

We first introduce some obstructions to the existence of a infinite index subgroup of finite biindex.

**Definition 3.9** We say that a group  $G$  has Property (PF) [respectively (MF), resp. (BF)] if every finite proindex (resp. almost maximal, resp. finite biindex) subgroup  $H$  has finite index.

We also recall that a group is (ERF) if every subgroup is closed for the profinite topology (ERF stands for “Extended Residually Finite”).

As a consequence of Lemma 3.2, we have the following implications.

$$ERF \implies PF \implies MF \implies BF$$

Note that these properties are inherited by quotients. Note also that Properties ERF and PF are invariant by commensurability, and that Property ERF is also inherited by subgroups. We show below (Proposition 3.14) that, for finitely generated groups, Properties (PF) and (MF) are equivalent.

**Example 3.10** (1) The Thompson group of Example 3.4 is 2-generated and does not have Property (BF). In particular, nonabelian free groups do not have Property (BF).

(2) In [22], it is proved that a finitely generated group which is linear over a commutative ring, and not virtually solvable, has a maximal subgroup of infinite index, thus does not have Property (MF).

(3) By a result of Olshanskii [25], any non-elementary word hyperbolic group has an infinite quotient with no proper subgroup of finite index. In particular, it has a maximal subgroup of infinite index, hence does not satisfy Property (MF).

(4) Hall [18] has exhibited finitely generated 3-solvable groups with infinite index maximal subgroups, hence without Property (MF).

(5) If  $G$  is a virtually solvable group which is not virtually polycyclic, then it is proved in [1] that  $G$  has a subgroup  $H$  conjugate to a proper subgroup of itself. In particular,  $G$  is not ERF.

(6) A virtually polycyclic group is ERF (Malcev [21]). It is not known if there are other examples of finitely generated ERF groups.

(7) We prove (Proposition 3.20) that if a finitely generated group  $\Gamma$  is an extension with virtually polycyclic quotient and nilpotent kernel, then  $\Gamma$  has Property (MF). In particular, this holds when  $\Gamma$  is a linear virtually solvable group.

(8) The first Grigorchuk group  $\Gamma$  has Property (PF) (Pervova [28], Grigorchuk and Wilson [16]). It is not ERF: indeed, it has a subgroup isomorphic to a direct sum  $\bigoplus_{n \geq 1} C_{2^n}$ , thus mapping onto the quasi-cyclic group  $C_{2^\infty}$  which is not residually finite. Accordingly,  $\Gamma$  has a subgroup which is not ERF, hence  $\Gamma$  is neither ERF. On the other hand, it is an open question to find a group of subexponential growth which does not have Property (PF); equivalently to find a group of subexponential growth with a maximal subgroup of infinite index.

We now introduce similar obstructions to the existence of a *finitely generated* infinite index subgroup of finite biindex.

**Definition 3.11** We say that a group  $G$  has Property (LPF) [respectively (LMF), resp. (LBF)] if every finite proindex (resp. almost maximal, resp. finite biindex) *finitely generated* subgroup  $H$  has finite index.

We also recall that a group is LERF if every finitely generated subgroup is closed for the profinite topology. (LERF is also called “subgroup separable”). In these four abbreviations, the additional letter L stands for “locally”.

Again as a consequence of Lemma 3.2, we have the following implications.

$$\begin{array}{ccccccc} ERF & \implies & PF & \implies & MF & \implies & BF \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ LERF & \implies & LPF & \implies & LMF & \implies & LBF \end{array}$$

Note that the properties in the second row are no longer inherited by quotients: indeed, free groups are LERF (see the example below) but do not have Property BF (see Example 3.10). Note also that Properties LERF and LPF are invariant by commensurability, and that Property LERF is also inherited by subgroups.

In the literature, a group is defined to have the *engulfing* Property if every proper finitely generated subgroup is contained in a proper finite index subgroup. Clearly, a group has Property (LPF) if and only if all its finite index subgroups have the engulfing Property.

**Example 3.12** (1) A free nonabelian group is LERF [17].

- (2) The first Grigorchuk group  $\Gamma$  is LERF (Pervova [28], Grigorchuk and Wilson [16]). On the other hand, non-residually finite groups of subexponential growth appear in [11].
- (3) If  $G$  is the Baumslag-Solitar group  $BS(1, p)$  ( $|p| \geq 2$ ), then  $G$  is not LERF, since its subgroup  $\mathbb{Z}[1/p]$  is not LERF (its quotient by a cyclic subgroup is divisible). If  $G$  is a standard wreath product  $A \wr \mathbb{Z}$ , with  $A$  finitely generated abelian, then  $G$  is LERF but not ERF (Proposition 3.19).
- (4) In [23], an example of a free-by-cyclic 3-manifold group which fails to satisfy Property (LPF) is given.
- (5) If  $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$  is a lattice, then  $\Gamma$  has Property (LMF). More precisely, for every finitely generated subgroup of infinite index  $\Lambda \subset \Gamma$ , there exists a strictly decreasing sequence of subgroups  $\Lambda \subset \Lambda_n \subset \Gamma$ . All this follows from the proof of [15, Theorem 1.3]. On the other hand, it is not known if  $\Gamma$  is always LERF, or even has (LPF).
- (6) Example 3.8, and the bare existence of finitely generated groups without Property (LBF) (Examples 3.4, 3.5, 3.6) imply the existence of torsion-free word hyperbolic groups without Property (LBF).

### 3.3 Nearly maximal subgroups

Recall [29] that a subgroup  $H$  of a group  $G$  is *nearly maximal* if  $H$  is maximal among infinite index subgroups of  $G$ . A standard verification shows that every infinite index subgroup of a finitely generated group  $G$  is contained in a nearly maximal subgroup.

**Observation 3.13** If  $H$  is a nearly maximal subgroup of a group  $G$ , then either  $H$  is closed or has finite proindex in  $G$ .

This obvious result has the following consequence. Suppose that a finitely generated group does not have Property (PF). Let  $H$  be an infinite index subgroup with finite proindex. Then  $H$  is contained in a nearly maximal subgroup  $M$ . Clearly,  $M$  has also finite proindex. So the only subgroups containing  $M$  are those which contain  $\bar{M}$ , and there are finitely many, so that  $M$  is almost maximal. This proves:

**Proposition 3.14** *Let  $G$  be a finitely generated group. The following are equivalent.*

- (i)  $G$  has Property (PF).
- (ii) Every nearly maximal subgroup of  $G$  is profinitely closed.
- (iii)  $G$  has Property (MF).

**Remark 3.15** (1) On the other hand, it is not clear whether Property (LMF) implies Property (LPF). I actually conjecture that it is not true. A possible counterexample could be a free product  $G * G$ , where  $G$  is any nontrivial finitely generated group without any nontrivial finite quotient, but I do not know how to prove Property (LMF) for such a group. Note also that although  $SL_n(\mathbb{Z})$  ( $n \geq 3$ ) is known not to have Property (LPF) [34], whether it has Property (LMF) is open (by [22] it does not have Property (MF)).

(2) Note that the (infinitely generated) quasi-cyclic group  $C_{p^\infty} = \mathbf{Z}[1/p]/\mathbf{Z}$  is (PF) but not (MF).

A consequence of Proposition 3.14 is that, for finitely generated groups, Property (MF) is a commensurability invariant. I do not know if this is true for Property (LMF); however, we have:

**Proposition 3.16** *Property (LMF) is inherited by subgroups of finite index.*

**Lemma 3.17** *Let  $G$  be a group without Property (LMF). Then  $G$  has a finitely generated nearly maximal subgroup which is almost maximal.*

*Proof* Let  $H_0$  be a finitely generated, almost maximal subgroup of infinite index. If  $H_0$  is not nearly maximal, it is properly contained in a subgroup  $K_1$  of infinite index; define  $H_1$  as the subgroup generated by  $H_0$  and one element in  $K_0 - H_0$ . Go on defining an increasing sequence of finitely generated subgroups of infinite index. This process stops, since  $H$  is almost maximal. So, for some  $n$ ,  $H_n$  is nearly maximal, and, since it contains  $H_0$ , it is almost maximal.  $\square$

*Proof of Proposition 3.16* Note that an infinitely generated group necessarily has Property (LMF). Let  $G$  be a finitely generated group, and  $H$  a subgroup of finite index. Suppose that  $H$  does not have Property (LMF). By Lemma 3.17, let  $M$  be a finitely generated, nearly maximal, almost maximal subgroup of  $H$ . Then  $M$  is contained in a nearly maximal subgroup  $M'$  of  $G$ . Since  $M'$  has infinite index in  $G$  and  $H$  has finite index in  $G$ , the subgroup  $M' \cap H$  has infinite index in  $H$ , so that  $M' \cap H = M$ . In particular,  $M$  has finite index in  $M'$ , so that  $M'$  is also finitely generated.

It is clear that  $M'$  is not profinitely closed in  $G$ : otherwise, so would be  $M = M' \cap H$ , and  $M$  would also be closed in  $H$ .  $\square$

### 3.4 Finitely generated solvable groups

**Lemma 3.18** *Let  $G$  be a finitely generated group which has a surjective morphism  $p$  onto an abelian group  $A$ , with abelian kernel  $K$ . Let  $H$  be a subgroup of  $G$  such that  $p(H) = A$ . Then  $H$  is closed for the profinite topology.*

*Proof* The assumption implies that  $H \cap K$  is normal in  $G$ . Maybe replacing  $G$  by  $G/(H \cap K)$ , we can suppose that  $H \cap K = \{1\}$ , so that  $G = H \rtimes K$ . To see that  $H$  is closed for the profinite topology, it clearly suffices to show that its profinite closure has trivial intersection with  $K$ . Thus, let  $k \in K - \{1\}$  belong to the profinite closure of  $H$ .

Since  $G$  is finitely generated and metabelian, it is residually finite [18]. So, there exists a finite index subgroup  $L$  of  $K$ , normal in  $G$ , such that  $k \notin L$ . Then  $H \rtimes L$  contains  $H$ , has finite index in  $G$ , and does not contain  $k$ . This is a contradiction.  $\square$

**Proposition 3.19** *Let  $G$  be a standard wreath product  $A \wr \mathbf{Z}$ , with  $A \neq \{1\}$  finitely generated abelian. Then  $G$  is LERF, but not ERF.*

*Proof* (1) It is not ERF because the subgroup  $A^{(\mathbf{N})}$  of  $A^{(\mathbf{Z})} \rtimes \mathbf{Z}$  is not closed for the profinite topology, since it is conjugate to a proper subgroup of itself.

(2) Let  $H$  be a finitely generated subgroup of  $G$ . Let us show that  $H$  is closed for the profinite topology.

First case:  $H$  is not contained in  $A^{(\mathbb{Z})}$ . Then the projection of  $H$  in  $\mathbb{Z}$  is a subgroup  $n\mathbb{Z}$  of  $\mathbb{Z}$  ( $n \geq 1$ ). It clearly suffices to show that  $H$  is closed in  $A^{(\mathbb{Z})} \rtimes n\mathbb{Z}$ , and this is a consequence of Lemma 3.18.

Second case:  $H$  is contained in  $A^{(\mathbb{Z})}$ . Clearly,  $A^{(\mathbb{Z})}$  is closed in the profinite topology. Therefore we have to consider  $h \in A^{(\mathbb{Z})} - H$  and show that  $h$  is not contained in the profinite closure of  $H$ . Take a finite subset  $F$  of  $\mathbb{Z}$  containing all supports of  $h$  and generators of  $H$ . Let  $n$  be greater than the diameter of  $F$ . Replacing  $G$  by its finite index subgroup  $A^{(\mathbb{Z})} \rtimes \mathbb{Z}$ , we can suppose that  $H$  and  $h$  are contained in  $A_0$ , where  $A^{(\mathbb{Z})} = \bigoplus_{i \in \mathbb{Z}} A_i$ . Then  $h$  is a nontrivial element in the abelianization of the quotient of  $G$  by the normal subgroup generated by  $H$ . In particular,  $h$  does not belong to the profinite closure of  $H$ .  $\square$

Example 3.12(4) indicates that it is not obvious how to generalize Proposition 3.19. It would be interesting to characterize LERF groups among finitely generated solvable groups; even in the case of metabelian groups this is open.

Here is now a result about Property (MF) for a class of finitely generated solvable groups.

**Proposition 3.20** *Let  $G$  be group, which is nilpotent-by-(virtually polycyclic), i.e., lies in a extension with nilpotent kernel and virtually polycyclic quotient. Then  $G$  has Property (MF).*

Note that every finitely generated, virtually solvable group which is linear over a field is nilpotent-by-(virtually abelian), hence belongs to this class. In particular, this encompasses a result of Margulis and Soifer (the easier implication in [22, Main Theorem]). The main ingredient to prove Proposition 3.20 is the following deep result:

**Theorem 3.21** (Roseblade [31]) *Let  $H$  be a virtually polycyclic group, and let  $M$  be a simple  $\mathbb{Z}H$ -module. Then  $M$  is finite.*

*Proof of Proposition 3.20* Let  $G$  be a finitely generated group,  $N$  a nilpotent, normal subgroup, such that  $G/N$  is virtually polycyclic.

Suppose by contradiction that  $G$  does not have Property (MF). Passing to a subgroup of finite index if necessary, we can suppose that  $G$  has a maximal subgroup  $M$  of infinite index. We can suppose that  $M$  contains no nontrivial normal subgroup of  $G$ . The centre  $Z(N)$  of  $N$  is normal in  $G$ . Since  $M$  does not contain any nontrivial normal subgroup,  $M$  does not contain  $Z(N)$ . By maximality,  $MZ(N) = G$ . Thus, since  $M \cap Z(N)$  is normalized by both  $M$  and  $Z(N)$ , it is a normal subgroup of  $G$  contained in  $M$ , hence is trivial. Accordingly,  $G$  is the semidirect product of  $M$  by  $Z(N)$ . Since  $M$  is a maximal subgroup,  $Z(N)$  is a simple  $M$ -module, and actually a simple  $M/(M \cap N)$ -module since  $N$  acts trivially on its centre. Since  $M/(M \cap N)$  is a subgroup of  $G/N$ , it is virtually polycyclic, so that by Theorem 3.21,  $Z(N)$  is finite. Hence  $M$  has finite index in  $G$ , contradiction.  $\square$

**Remark 3.22** P. Hall has constructed [18] a 3-solvable group  $G$  with a maximal, finitely generated subgroup of infinite index. In particular,  $G$  does not have Property (LMF), so that “nilpotent-by-polycyclic” cannot be replaced by “3-solvable” in Proposition 3.20.

I do not know if there exists a finitely generated solvable group without Property (BF). However, using standard arguments, we have the following result.

**Proposition 3.23** *The following are equivalent.*

- (1) *There exists a finitely generated  $n$ -solvable group without Property (BF).*
- (2) *There exists a finitely generated  $n$ -solvable group without Property (LBF).*

- (3) *There exists a finitely generated  $(n - 1)$ -solvable group  $\Gamma$ , and an infinite  $\Gamma$ -module  $V$ , such that the action of  $\Gamma$  on  $V$  has finitely many orbits.*

*Proof* (2)  $\Rightarrow$  (1) is trivial.

(3)  $\Rightarrow$  (2). Observe that  $\Gamma$  is a finitely generated subgroup of finite biindex in  $\Gamma \ltimes V$ .

Suppose (1). Let  $G$  be a finitely generated solvable group, and  $M$  a subgroup of finite biindex and infinite index. Replacing  $M$  by a larger subgroup if necessary, we can suppose it nearly maximal, and replacing  $G$  by the profinite closure of  $M$  if necessary, we can suppose  $M$  maximal. Moreover, taking the quotient by a normal subgroup if necessary, we can suppose the only normal subgroup of  $G$  contained in  $M$  is  $\{1\}$ , i.e.,  $G$  acts faithfully on  $G/M$ .

Let  $A$  be the last nontrivial term of the derived series of  $G$ . Then  $A$  is a normal subgroup and  $A \neq \{1\}$ , so that  $A$  is not contained in  $M$ . Accordingly,  $MA = G$ . Observe that  $M \cap A$  is normalized both by  $M$  (since  $A$  is normal) and by  $A$  (since  $A$  is abelian). It follows that  $M \cap A$  is normal in  $G$ ; therefore  $M \cap A = \{1\}$ , and  $G \simeq M \ltimes A$ . Since  $M$  has finite biindex in  $G$ ,  $M$  acts with finitely many orbits on  $A$ .  $\square$

For  $n \geq 3$ , we leave as open whether the equivalent statements of Proposition 3.23 are true. For  $n \leq 2$ , they are false as a consequence of Proposition 3.20. We record this in the following:

- Question 3.24** (1) Does there exist a finitely generated, solvable group without Property (LBF)?  
 (2) Does there exist a finitely presented solvable group without Property (LBF)?  
 (3) Does there exist a finitely presented solvable group without Property (MF)?

The existence of a finitely generated solvable group without Property (LBF) would permit to construct solvable finitely presented wreath products, and would imply, arguing as in Proposition 2.12, that the class of virtually solvable groups is not invariant under quasi-isometries within the class of finitely presented groups.

### 3.5 Amalgams and obstructions to Property ((L)BF)

The following theorem is due to M. Hall in the case of free groups, P. Scott in the case of surface groups, and to Brunner, Burns, and Solitar [4] for the general case.

**Theorem 3.25** *Let  $G$  be the amalgam of two free groups over a cyclic subgroup. Then  $G$  is LERF.*

In contrast, Burger and Mozes [5] have constructed amalgams of two free groups over a finite index subgroup which are finitely presented simple groups. I do not know if these groups have Property (LMF). These examples indicate that amalgams may have very different behaviours, so that it seems that no general statement can be made. The following result is a particular case of Theorem 2 in [20].

**Theorem 3.26** (Karrass, Solitar (1973)) *Let  $G$  be a finitely generated group which splits as a nontrivial amalgam over a finite subgroup. Then  $G$  has Property (LBF).*

**Example 3.27** Let  $G$  be an infinite group, all of whose subgroups are either finite or of finite index. Then  $G$  clearly has Property (BF). If, moreover,  $G$  has no proper subgroup of finite index, then  $G$  has a maximal subgroup which is finite; in particular  $G$  does not have Property (LMF). There exist nontrivial examples of such groups: infinite two-generator

groups all of whose nontrivial proper subgroups are isomorphic to  $\mathbf{Z}/p\mathbf{Z}$ ,  $p$  a big prime, have been constructed by Olshanskii (see [24]). All known examples of such groups are infinitely presented.

### 3.6 Fibre products

Let  $G_1, G_2, Q$  be groups and  $p_i : G_i \rightarrow Q$  a surjection for  $i = 1, 2$ . We are interested in the pair  $(G, H)$ , where  $G = G_1 \times G_2$  and  $H$  is the fibre product  $G_1 \times_Q G_2 = \{(x, y) \in G_1 \times G_2, p_1(g_1) = p_2(g_2)\}$ .

**Proposition 3.28** (1) *There is a natural order-preserving bijection between the set of subgroups of  $G_1 \times G_2$  containing  $H = G_1 \times_Q G_2$  and the set of normal subgroups of  $Q$ . It induces a bijection between finite index normal subgroups of  $Q$  and finite index subgroups of  $G_1 \times G_2$  containing  $G_1 \times_Q G_2$ .*

(2) *Suppose that  $G_1$  and  $G_2$  are finitely generated. If  $Q$  is finitely presented, then  $G_1 \times_Q G_2$  is finitely generated. Conversely, if  $G_1$  and  $G_2$  are in addition finitely presented and if  $G_1 \times_Q G_2$  is finitely generated, then  $Q$  is finitely presented.*

*Proof* (1) If  $K$  is a subgroup of  $G_1 \times G_2$  containing  $H = G_1 \times_Q G_2$ , then set  $u(K) = p_1(K \cap (G_1 \times \{1\}))$ . This is a normal subgroup of  $Q$ , because  $K \cap (G_1 \times \{1\})$  is normal in  $G_1$  (identified with  $G_1 \times \{1\}$ ). Indeed, let  $x$  belong to  $K \cap (G_1 \times \{1\})$ . This means that  $(x, 1) \in K$ . Fix  $y \in G_1$  and let us check that  $xyx^{-1} \in K \cap (G_1 \times \{1\})$ , i.e.,  $(yxy^{-1}, 1) \in K$ . Choose  $a \in G_2$  such that  $p_2(a) = p_1(y)$ . Then  $(y, a) \in H \subseteq K$ , so that  $(yxy^{-1}, 1) = (y, a)(x, 1)(y, a)^{-1}$  also belongs to  $K$ .

If  $N$  is a normal subgroup of  $Q$ , set  $v(Q) = G_1 \times_{Q/N} G_2$ . This is a subgroup of  $G_1 \times G_2$  containing  $H$ . We claim that  $u$  and  $v$  are inverse bijections (clearly, they preserve the order).

- $K \subseteq v(u(K))$ : Let  $(x, y)$  belong to  $K$ . Write  $p_2(y) = p_1(a)$  for some  $a \in G_1$ , so that  $(a, y) \in H \subseteq K$ . Then  $(xa^{-1}, 1) = (x, y)(a, y)^{-1} \in K$ . Thus  $p_1(xa^{-1}) = p_1(x)p_2(y)^{-1} \in u(K)$ . This means that  $(x, y) \in G_1 \times_{Q/u(K)} G_2 = v(u(K))$ .
- $v(u(K)) \subseteq K$ : Let  $(x, y)$  belong to  $v(u(K))$ . This means that  $p_1(x)p_2(y)^{-1} \in u(K)$ , i.e.,  $p_1(x)p_2(y)^{-1} = p_1(a)$  for some  $a \in G_1$  such that  $(a, 1) \in K$ . Therefore  $(xa^{-1}, y) \in H \subseteq K$ , so that  $(x, y) = (xa^{-1}, y)(a, 1) \in K$ .
- $N \subseteq u(v(N))$ : Let  $\alpha$  belong to  $N$ . Write  $\alpha = p_1(x)$  for some  $x \in X$ . Then  $(x, 1) \in G_1 \times_{Q/N} G_1 = v(N)$ , so that  $\alpha \in u(v(N))$ .
- $u(v(N)) \subseteq N$ : Let  $\alpha$  belong to  $u(v(N))$ . This means that  $\alpha = p_1(x)$ , for some  $x \in G_1$  such that  $(x, 1) \in G_1 \times_{Q/N} G_2$ , so that  $p_1(x) = \alpha \in N$ .

(2) Suppose that  $G_1$  and  $G_2$  are finitely generated, and  $Q$  is finitely presented. For  $i = 1, 2$ , write  $N_i = \text{Ker}(p_i)$ . Since  $Q$  is finitely presented and  $G_i$  finitely generated,  $N_i$  is generated as a normal subgroup in  $G_i$  by a finite subset  $R_i$ . Besides, take a finite subset  $S$  of  $H = G_1 \times_Q G_2$  such that  $p_i(S)$  generates  $Q$  for  $i = 1, 2$ . Then  $(R_1 \times \{1\}) \cup (\{1\} \times R_2) \cup S$  is a finite generating subset for a subgroup  $M$  of  $H$ . We claim that  $M = H$ . Let  $(x, y)$  belong to  $H$ . The hypothesis on  $S$  implies that there exists  $z \in G_2$  and  $a \in N_1$  such that  $(ax, z) \in M$ . Since  $(ax, z) \in M \subseteq H$ ,  $p_1(x) = p_2(z)$ , so that  $z^{-1}y \in N_2$ . Hence  $(x, y) = (a, 1)^{-1}(ax, z)(1, z^{-1}y)$ . We claim that  $(a, 1) \in M$ . Indeed,  $a \in N_1$ , and, using that  $p_1(S)$  generates  $G_1$  and  $R_1$  generates  $N_1$  as a normal subgroup, we obtain that  $(a, 1) \in M$ . Similarly,  $(1, z^{-1}y) \in M$ , and therefore  $(x, y) \in M$ .

Conversely, suppose that  $G_1$  and  $G_2$  are finitely presented and suppose that  $H$  is finitely generated. There exists a finitely presented group  $Q'$ , a surjective map  $q : Q' \rightarrow Q$ , surjective



maps  $q_i : G_i \rightarrow Q'$ ,  $i = 1, 2$ , such that  $p_i = q \circ q_i$  for  $i = 1, 2$ . If  $Q$  is not finitely presented, then the kernel of  $q$  can be written as a union of an increasing sequence of subgroups  $M_n$ , normal in  $Q'$ . By (1), the normal subgroups intermediate between  $\text{Ker}(q_1)$  and  $\text{Ker}(p_1)$  correspond bijectively with the subgroups intermediate between  $G_1 \times_{Q'} G_2$  and  $G_1 \times_Q G_2$ . Accordingly, the latter is not finitely generated.  $\square$

**Corollary 3.29** *Let  $G_1, G_2$  be finitely generated groups. The subgroup  $G_1 \times_Q G_2$  has finite proindex (resp. is almost maximal) in  $G_1 \times G_2$  if and only if  $Q$  has a minimal finite index subgroup (resp.  $Q$  has a finite number of normal subgroups).*

**Remark 3.30** The question of finite presentability of a fibre product  $G_1 \times_Q G_2$  is not trivial at all. It is easy to show that  $G_1$  and  $G_2$ , and  $Q$  must necessarily be finitely presented, but the converse is not true. For instance, take the Baumslag-Solitar group  $BS(1, p) = \mathbf{Z} \rtimes_p \mathbf{Z}[1/p]$  with  $p \geq 2$ , which has presentation  $\langle x, y \mid {}^x y = y^p \rangle$ . There are two morphisms  $p_+, p_-$  of this group onto  $\mathbf{Z}$ . This gives, up to isomorphism, two possible fiber products over  $\mathbf{Z}$ , which we denote by  $BS(1, p) \times_{\mathbf{Z}++} BS(1, p)$  and  $BS(1, p) \times_{\mathbf{Z}+-} BS(1, p)$ . Then the former is finitely presented, while the second is not. The first has presentation  $\langle x, y, z \mid [y, z] = 1, {}^x y = y^p, {}^x z = z^p \rangle$ , while the second has a finitely generated central extension by  $\mathbf{Z}[1/p]$ , given by the semidirect product of the diagonal subgroup of  $\text{SL}_2(\mathbf{Z}[1/p])$  by the Heisenberg group  $H_3(\mathbf{Z}[1/p])$ , hence is not finitely presented. For more about the finite presentation of fibre products, see [3, 6, 14].

**Proposition 3.31** *The subgroup  $H = G_1 \times_Q G_2$  has finite biindex in  $G_1 \times G_2$  if and only if  $Q$  has a finite number of conjugacy classes.*

*Proof* It suffices to check that every double coset of  $H$  contains an element of  $G_1 \times \{1\}$ , and that two elements  $(x, 1)$  and  $(y, 1)$  of  $G_1 \times \{1\}$  are in the same double coset if and only if the images of  $x$  and  $y$  in  $Q$  are conjugate.  $\square$

**Remark 3.32** Examples of infinite, finitely generated groups with finitely many conjugacy classes have been constructed by S. Ivanov (see [24, Theorem 41.2]), and examples with exactly one nontrivial conjugacy class have recently been announced by Osin [26]. But it is an open question to find infinite finitely presented groups with finitely many conjugacy classes.

### 3.7 Hereditary properties

**Lemma 3.33** *If  $N$  is normal in  $G$ , then the following statements are equivalent: (i)  $N$  is almost maximal, (ii)  $N$  has finite index.*

*Proof* It suffices to show (i)  $\Rightarrow$  (ii), which is equivalent to the statement: every infinite group has infinitely many subgroups. If  $G/N$  is torsion, then it is the union of its finite subgroups, so they are infinitely many. Otherwise,  $G/N$  contains an infinite cyclic subgroup, which contains infinitely many subgroups.  $\square$

**Lemma 3.34** *Suppose  $H_1, H_2$  are subgroups of  $G$ ,  $H_1 \subset H_2$ . Suppose that  $H_1$  has finite biindex (resp. is almost maximal) in  $G$ . Then  $H_2$  has finite biindex (resp. is almost maximal) in  $G$  and  $H_1$  has finite biindex (resp. is almost maximal) in  $H_2$ .*

*Proof* The statement for with “almost maximal” is trivial.

Suppose that  $H_1$  has finite biindex in  $G$ . Trivially, so has  $H_2$ . Write  $G = \bigcup_{i \in I} H_1 g_i H_1$ , with  $I$  finite, and set  $J = \{i \in I, g_i \in H_2\}$ . Then  $H_2 = \bigcup_{i \in J} H_1 g_i H_1$ , so that  $H_1$  has finite biindex in  $G$ .  $\square$

**Lemma 3.35** *Suppose  $H_1, H_2$  are subgroups of  $G$ ,  $H_1$  is contained as a subgroup of finite index in  $H_2$ , and  $H_2$  has finite biindex in  $G$ . Then  $H_1$  has finite biindex in  $G$ .*

*Proof* Write  $G = \bigcup_i H_2 g_i H_2$ ,  $H_2 = \bigcup_j h_j H_1 = \bigcup_k H_1 h'_k$ , all these unions being finite. Then  $G = \bigcup_{i,j,k} H_1 h'_k g_i h_j H_1$ .  $\square$

**Remark 3.36** The converse of Lemma 3.34 is false in both cases.

In the case of finite biindex, consider  $G = \text{SL}(2, K)$ , where  $K$  is an algebraically closed field,  $T$  is the subgroup of upper triangular matrices in  $G$ , and  $D$  denotes the diagonal matrices in  $G$ . Then  $G$  is two-transitive on  $G/T \simeq P^1(K)$ , and  $T$  is two-transitive on  $T/D \simeq K$ , the affine line. But, by a dimension argument, the action of  $D$  on  $G/D$  cannot have a finite number of orbits. On the other hand, I do not know any counterexample with  $G$  finitely generated.

For a counterexample with almost maximal subgroups, which also shows that the analogue of Lemma 3.35 is false with almost maximal subgroups, take an infinite group  $G$  with a finite maximal subgroup  $H$ . Such groups are constructed in [24]. So  $H$  is almost maximal in  $G$  and  $\{1\}$  has finite index in  $H$ . But, by Lemma 3.33  $\{1\}$  is not almost maximal in  $G$ .

**Remark 3.37** Here is a trivial consequence of Lemma 3.34. Let  $G_1$  be a group,  $G_2$  is a finite index subgroup of  $G_1$ , and  $H$  a subgroup of  $G_2$ . Then, if  $H$  has finite biindex (resp. is almost maximal) in  $G_2$ , it has also finite biindex (resp. is almost maximal) in  $G_1$ . The point is that I do not know, in both cases, if the converse is true.

**Lemma 3.38** *Suppose that, for  $i = 1, 2$ ,  $H_i$  has finite biindex (resp. is almost maximal) in  $G_i$ . Then  $H_1 \times H_2$  has finite biindex (resp. is almost maximal) in  $G_1 \times G_2$ .*

*Proof* This is obvious with finite biindex. Suppose that, for  $i = 1, 2$ ,  $H_i$  is almost maximal in  $G_i$ . If there are infinitely many subgroups containing  $H_1 \times H_2$ , infinitely many have the same intersection  $K_i$  and projection  $P_i$  on  $G_i$  for  $i = 1, 2$ . Note that  $K_i$  is normal in  $P_i$ . Since, as a consequence of Lemma 3.34,  $K_i$  is almost maximal in  $P_i$ , this implies, by Lemma 3.33, that  $P_i/K_i$  is finite for  $i = 1, 2$ . Since only finitely many subgroups can exist between  $K_1 \times K_2$  and  $P_1 \times P_2$ , we have a contradiction.  $\square$

**Lemma 3.39** *If  $H$  has finite biindex (resp. is almost maximal) in  $G$  and  $N$  is a normal subgroup of  $G$ , then  $H/(H \cap N)$  has finite biindex (resp. is almost maximal) in  $G/N$ .*

*Proof* For the case of finite biindex, pass the expression  $G = \bigcup_{i=1}^n H g_i H$  to the quotient. The statement for almost maximal subgroups is trivial.  $\square$

**Proposition 3.40** *Properties (BF) and (LBF) are inherited from finite index subgroups.*

*Proof* Let  $G$  be a group and  $N$  a finite index subgroup. Suppose that  $N$  is (L)BF. Let  $H$  be a (finitely generated) almost maximal subgroup in  $G$ . By Lemma 3.35,  $H \cap N$  has finite biindex in  $G$  (and is also finitely generated), so has finite biindex in  $N$  by Lemma 3.34. Since  $N$  has Property (L)BF,  $H \cap N$  has finite index in  $N$ , so that  $H$  has finite index in  $G$ .  $\square$

I do not know if Properties (BF) and (LBF) are inherited by finite index subgroups. This motivates the following question.

**Question 3.41** Let  $G$  be a group,  $N$  a subgroup of finite index, and  $H$  a subgroup of  $N$ . If  $H$  has finite biindex in  $N$ , must it have finite biindex in  $G$ ?

### 3.8 Faithful almost 2-transitive pairs

We could define weaker analogs of Properties (PF) through (LBF), say  $(fPF)$ , etc., by only considering subgroups  $H$  such that the action of  $G$  on  $G/H$  is faithful.

Not much is known about these properties for infinite groups. Dixon [9] has shown that, in a suitable sense, a generic subgroup on  $n \geq 2$  generators of the symmetric group  $\text{Sym}(\mathbf{N})$  is free and 2-transitive, showing that free groups also have *faithful* 2-transitive actions on infinite sets and therefore do not have Property (fBF). The non-existence of a faithful primitive action, which is, for infinite groups, a priori slightly stronger than Property (fMF), is widely investigated in [12].

Examples 3.4, 3.5, and 3.6 provide essentially the only examples of finitely presented groups which I know not to have Property (fLBF). In the finitely generated case, we also have the groups  $G \times G$  when the infinite group  $G$  has finitely many conjugacy classes, and has trivial center (this latter assumption is always satisfied modulo a finite normal subgroup). Note that, in all these examples,  $G$  has very few normal subgroups: for  $F$  and the Houghton groups,  $G$  has simple derived subgroup;  $T$  is itself simple, and the groups with finitely many conjugacy classes have finitely many normal subgroups. We therefore ask:

**Question 3.42** Does there exist a residually finite group that acts almost 2-transitively and faithfully on an infinite set, with finitely generated stabilizers?

The answer is yes when “almost 2-transitively” is replaced by “primitively”, as shows the example, pointed out in [12], of the action of  $\text{PSL}_2(\mathbf{Z}[1/p])$  on  $\text{PSL}_2(\mathbf{Z}[1/p])/\text{PSL}_2(\mathbf{Z})$ .

## Appendix A. Length of words in wreath products

We consider the wreath product  $A = W \wr_{G/H} G$ , where  $W$  and  $G$  are finitely generated. We write  $W$  additively although it is not necessarily abelian. We write the elements of  $A$ :  $(f, c)$ , where  $f \in W^{(G/H)}$  and  $c \in G$ ; we denote by  $x_0$  the base-point of  $G/H$ . If  $w \in W$  and  $c \in G$ , by the abusive notation  $(w, c)$ , we mean  $(\delta_{x_0}(w), c)$ , where  $\delta_{x_0}$  is the natural inclusion of  $W$  into the  $x_0$ -component of  $W^{(G/H)}$ . The product in  $A$  is given by  $(f_1, c_1)(f_2, c_2) = (f_1 + c_1 f_2, c_1 c_2)$ . Fix a symmetric finite generating subset  $S$  of  $G$ . We call a path of length  $n$  in  $G$  a sequence  $(g_0, \dots, g_n)$  such that  $g_0 = 1$  and  $g_{i+1}g_i^{-1} \in S$  for all  $i = 0, \dots, n-1$ . For any finite subset  $F$  of  $X$  and  $c \in G$ , let  $K(F, c)$  be the minimal length of a path  $(g_0, \dots, g_n)$  in  $G$  such that  $g_n = c$  and  $F \subset \{g_0 x_0, \dots, g_n x_0\}$ . On the other hand, fix a finite symmetric generating subset  $T$  of  $W$ , and denote by  $|\cdot|$  the corresponding word length. For  $f \in W^{(X)}$ , set  $|f| = \sum_{x \in X} |f(x)|$ . Fix, as generating subset of  $A$ , the union of  $(t, 1)$  for  $t \in T$  and  $(0, s)$  for  $s \in S$ . Denote again by  $|\cdot|$  the word length in  $A$ . The following lemma is obtained in [27] in the case of standard wreath products.

**Lemma A.1** For  $f \in W^{(G/H)}$  and  $c \in G$ , we have  $|(f, c)| = K(\text{supp}(f), c) + |f|$ .

*Proof* Set  $n = K(\text{supp}(f), c)$ . Let  $1 = g_0, g_1, \dots, g_n$  be a path of length  $n$  such that  $g_n = c$  and, whenever  $x \in \text{supp}(f)$ ,  $x = g_i x_0$  for some  $i$ . For all  $i$ , set  $g_{i+1} = s_i g_i$  with  $s_i \in S$ , and  $x_i = g_i x_0$ . Then

$$\begin{aligned} (f, c) &= (f(x_0), s_1)(f(x_1), s_2) \cdots (f(x_{n-1}), s_n)(f(x_n), 1) \\ &= (f(x_0), 1)(0, s_1)(f(x_1), 1)(0, s_2) \cdots (f(x_{n-1}), 1)(0, s_n). \end{aligned}$$

Thus,  $(f, c)$  can be expressed as a product of  $K(\text{supp}(f), c) + |f|$  generators. Accordingly, for all  $(f, c)$ ,  $|(f, c)| \leq K(\text{supp}(f), c) + |f|$ .

Conversely suppose that  $(f, c)$  can be expressed as a product of a minimal number  $n$  of generators. Putting generators of  $W_{x_0}$  together, we get

$$\begin{aligned}(f, c) &= (w_1, 1)(0, s_1)(w_2, 1)(0, s_2) \cdots (w_m, 1)(0, s_m)(w_{m+1}, 1) \\ &= (w_1, s_1) \cdots (w_m, s_m)(w_{m+1}, 1)\end{aligned}$$

with each  $w_i \in W$ ,  $s_i \in S$ . Set  $g_i = \prod_{j=1}^i s_j$ .

Then  $(g_0, g_1, \dots, g_m)$  is a path joining 1 to  $g_m = c$ . Besides,  $f = w_1 + g_1 w_2 + \cdots + g_m w_{m+1}$ , so that  $\text{supp}(f)$  is contained in  $\{g_0 x_0, g_1 x_0, \dots, g_{m-1} x_0, g_m x_0\}$ . Accordingly,  $K(\text{supp}(f), c) \leq m$ , and  $n = \sum_i |w_i| + m \geq |f| + K(\text{supp}(f), c)$ .  $\square$

This immediately implies the following result, first observed by A. Erschler-Dyubina [10] in the case of standard wreath products.

**Proposition A.2** *Let  $G$  be a group,  $H$  a subgroup, and  $W_1, W_2$  two bi-Lipschitz-equivalent groups. Then  $W_1 \wr_{G/H} G$  and  $W_2 \wr_{G/H} G$  are bi-Lipschitz-equivalent.*

**Acknowledgements** I am grateful to Laurent Bartholdi and Luc Guyot for valuable discussions and encouragement. I thank Victor Guba and Mark Sapir for discussions about Thompson's groups. I also thank Pierre de la Harpe and Romain Tessera for useful corrections.

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