

The stability of the resonance set for a problem with jumping non-linearity

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Synopsis

For $Q \in L^2(0, 1)$ we investigate the set $\Gamma \in \mathbb{R}^2$ of pairs (α, β) for which the problem

$$\int_0^1 u'(x)v'(x) dx - \int_0^1 (u(x)v(x))' Q(x) dx = \int_0^1 (\alpha u(x)^+ - \beta u(x)^-) v(x) dx$$

$\forall v \in H_0^1(0, 1)$ has a nontrivial solution $u \in H_0^1(0, 1)$ which has exactly one zero in $(0, 1)$ and is positive near $x=0$. We show that Γ is stable in a certain sense under small perturbations of Q . The dependence of Γ upon Q is illustrated by an example.

1. Introduction

In recent years many authors have studied nonlinear problems of the type

$$Tu = f(u) + h, \quad (1)$$

where $T: D(T) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is a self-adjoint operator with compact resolvent in $L^2(\Omega)$, Ω an open domain of \mathbb{R}^N and $f: \mathbb{R} \rightarrow \mathbb{R}$ a continuous function such that the following limits exist:

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{s} = \alpha \quad \text{and} \quad \lim_{s \rightarrow -\infty} \frac{f(s)}{s} = \beta.$$

The set of all $h \in L^2(\Omega)$ such that (1) has solution depends upon the values of α and β .

Among the numerous results obtained for this question we recall that of T. Gallouët and O. Kavian [1, 2] and B. Ruf [3]. Let $\sigma = \{\lambda_i: \lambda_i < \lambda_{i+1} \forall i \in \mathbb{N}\}$ be the spectrum of T . Suppose that λ_k is a simple eigenvalue with $k \geq 2$ and let ψ_k be an eigenfunction corresponding to λ_k such that $\int_{\Omega} \psi_k^2 = 1$. Then for $(\alpha, \beta) \in (\lambda_{k-1}, \lambda_{k+1}) \times (\lambda_{k-1}, \lambda_{k+1}) = S_k$ there exists a unique pair $(u, C) \in D(T) \times \mathbb{R}$ such that

$$\left. \begin{aligned} Tu &= \alpha u^+ - \beta u^- + C \psi_k \\ \int_{\Omega} u \psi_k &= 1, \end{aligned} \right\} \quad (2)$$

where $s^+ = \max\{s, 0\}$ and $s^- = \max\{-s, 0\}$. The function $C: S_k \rightarrow \mathbb{R}$ so defined is continuous and strictly decreasing with respect to each of the variables α and β . In particular $\Gamma = \{(\alpha, \beta) \in S_k: C(\alpha, \beta) = 0\}$ is a continuous curve passing through (λ_k, λ_k) and it is the graph of a strictly decreasing function $\eta(\alpha)$. The curve Γ splits S_k into two connected components. For $(\alpha, \beta) \in S_k$ the following holds. If

$C(\alpha, \beta)C(\beta, \alpha) > 0$, then for all $h \in L^2(\Omega)$ (1) has a solution. If $C(\alpha, \beta)C(\beta, \alpha) < 0$, there exists $h \in L^2(\Omega)$ such that (1) has no solution.

In [4] C. A. Stuart has shown that if $\Omega \subset \mathbb{R}^N$ is bounded with smooth boundary, $T = -\Delta$ and $D(T) = W^{2,p}(\Omega) \cap \dot{W}^{1,p}(\Omega)$ for some $p > N/2$, then C is a C^1 -function on S_k . He gives formulae for $\partial C / \partial \alpha$, $\partial C / \partial \beta$ and $\eta'(\alpha)$.

Note that the normalisation in (2) enables us to distinguish the two symmetric curves $C(\alpha, \beta) = 0$ and $C(\beta, \alpha) = 0$. In fact if u is a solution of (2) with $C = 0$, then $v = -u$ solves $Tv = \beta v^+ - \alpha v^-$ and $\int_{\Omega} v \psi_k = -1$.

The result of [1, 2] and [3] motivates a more precise study of Γ . In this paper we consider the following problem: given $Q \in L^2(0, 1)$, describe the set of pairs (α, β) such that

$$\int_0^1 u'(x)v'(x) dx - \int_0^1 (u(x)v(x))' Q(x) dx = \int_0^1 (\alpha u(x)^+ - \beta u(x)^-)v(x) dx \quad (3)$$

for all $v \in H_0^1(0, 1)$ has a nontrivial solution $u \in H_0^1(0, 1)$ which has exactly one zero in $(0, 1)$.

If $Q \in C^1([0, 1])$, then (3) reduces to

$$\begin{cases} -u''(x) + q(x)u(x) = \alpha u(x)^+ - \beta u(x)^- \\ u(0) = u(1) = 0, \end{cases} \quad (4)$$

where $q(x) = Q'(x)$ for all $x \in [0, 1]$. As we shall see, the solution of (3) can be normalised as in (2) so that we are concerned with a problem of type (2) with $C = 0$ and $k = 2$. The unique zero $z \in (0, 1)$ of the solution of (3) splits the equation (3) into two linear problems

$$\int_0^z u'(x)v'(x) dx - \int_0^z (u(x)v(x))' Q(x) dx = \alpha \int_0^z u(x)v(x) dx \quad \forall v \in H_0^1(0, z) \quad (5)$$

$$\int_z^1 u'(x)v'(x) dx - \int_z^1 (u(x)v(x))' Q(x) dx = \beta \int_z^1 u(x)v(x) dx \quad \forall v \in H_0^1(z, 1). \quad (6)$$

These are studied in the next section. The first eigenvalues of (5) and (6), the only ones we are interested in, are shown to be stable under small perturbations of z and Q . In Section 3 Γ is constructed using z as parameter. It is proved that Γ is stable in a certain sense under small perturbations of Q . In Section 4 an example illustrates the effect of Q on Γ . In particular Γ is of class C^2 in this example and it shows that the sign of $\eta''(\alpha)$ may change. In virtue of the perturbation result of Section 3 Γ can be approximately reproduced by the classical boundary value problem (4).

2. The linear equation

Let $H_0^1 = H_0^1(0, 1)$ denote the usual space of real valued functions which have a generalised derivative in $L^2 = L^2(0, 1)$ and which vanish at $x = 0$ and $x = 1$. Then H_0^1 is a Hilbert space with respect to the scalar product $(u, v) = \int_0^1 u'(x)v'(x) dx$ and the norm of an element $u \in H_0^1$ is $\|u\|_{1,2} = \sqrt{(u, u)}$. Recall that $H_0^1 \subset C_0([0, 1])$ is a continuous embedding and $\|u\|_{\infty} = \max_{x \in [0,1]} |u(x)| \leq \|u\|_{1,2} \quad \forall u \in H_0^1$. As

usual the formula $\langle Lu, v \rangle = (u, v)$ defines an isomorphism L from H_0^1 into its dual H^{-1} and $\|L\| = 1$. Since there is an embedding $J: H_0^1 \rightarrow H^{-1}$, defined by $\langle Ju, v \rangle = \int_0^1 u(x)v(x) dx \forall u, v \in H_0^1$, we can write an element u of H_0^1 and the element Ju of H^{-1} with the same symbol.

For $Q \in L^2$ define the generalised multiplication $M(Q): H_0^1 \rightarrow H^{-1}$ by $\langle M(Q)u, v \rangle = - \int_0^1 (u(x)v(x))' Q(x) dx \forall u, v \in H_0^1$. Then $M(Q)$ is symmetric and bounded. In fact boundedness follows from

$$|\langle M(Q)u, v \rangle| = 2 \left| \int_0^1 u(x)u'(x)Q(x) dx \right| \leq 2 \|u\|_\infty \|u\|_{1,2} \|Q\|_2 \leq 2 \|u\|_{1,2}^2 \|Q\|_2.$$

For $Q \in L^2$ set $T(Q) = L + M(Q)$.

LEMMA 2.1 (Gårding's inequality). *For $Q \in L^2$ given, there exists $\lambda_0 \in \mathbb{R}$ such that*

$$\langle T(Q)u, u \rangle - \lambda \langle u, u \rangle \geq \frac{1}{2} \|u\|_{1,2}^2 \forall \lambda \leq \lambda_0, \forall u \in H_0^1.$$

Proof. Let $Q \in L^2$ be given. For $u \in H_0^1$ recall that $\|u\|_\infty^2 \leq \|u\|_2 \|u\|_{1,2}$ and then $|\langle M(Q)u, u \rangle| \leq 2 \|u\|_2^{\frac{1}{2}} \|u\|_{1,2}^{\frac{3}{2}} \|Q\|_2$. An application of Young's inequality implies that $\|u\|_2^{\frac{1}{2}} \|u\|_{1,2}^{\frac{3}{2}} \leq \frac{1}{4} \alpha^3 \|u\|_2^2 + \frac{3}{4} \alpha^{-1} \|u\|_{1,2}^2 \forall \alpha > 0$ and consequently

$$\begin{aligned} \langle T(Q)u, u \rangle - \lambda \langle u, u \rangle &\geq \langle Lu, u \rangle - |\langle M(Q)u, u \rangle| - \lambda \langle u, u \rangle \\ &\geq (1 - \frac{3}{2} \alpha^{-1} \|Q\|_2) \|u\|_{1,2}^2 - (\frac{1}{2} \alpha^3 \|Q\|_2 + \lambda) \langle u, u \rangle. \end{aligned}$$

Setting $\alpha = 3 \|Q\|_2$, $\lambda_0 = -\frac{27}{2} \|Q\|_2^4$, the lemma is proved. \square

To state some properties of $T(Q)$ we prove the following.

PROPOSITION 2.2. (a) *The spectrum of $T(Q)$ consists of an increasing sequence of eigenvalues which tends to infinity.*

(b) *The eigenvalues of $T(Q)$ are simple.*

Proof. (a) Let $\lambda_0 \in \mathbb{R}$ be as in Lemma 2.1, fix $\lambda < \lambda_0$ and define the symmetric bilinear form $a: H_0^1 \times H_0^1 \rightarrow \mathbb{R}$ by $a(u, v) = \langle T(Q)u, v \rangle - \lambda \langle u, v \rangle$. Since $\frac{1}{2} \|u\|_{1,2}^2 \leq a(u, u) \leq \|T(Q) - \lambda I\| \|u\|_{1,2}^2$, $a(\cdot, \cdot)$ is a scalar product on H_0^1 and defines an equivalent norm of $\|\cdot\|_{1,2}$. In order to verify that $T(Q) - \lambda I: H_0^1 \rightarrow H^{-1}$ is bijective let $\phi \in H^{-1}$ be given. By the Representation Theorem of Riesz-Fréchet there exists a unique $u \in H_0^1$ such that $\langle \phi, v \rangle = a(u, v) = \langle T(Q)u, v \rangle - \lambda \langle u, v \rangle$ for all $v \in H_0^1$. Hence the resolvent $(T(Q) - \lambda I)^{-1}: H^{-1} \rightarrow H^{-1}$ exists and is compact by the compact embedding $H_0^1 \subset H^{-1}$. Thus the spectrum of $T(Q)$ consists of an increasing sequence of eigenvalues which tends to infinity.

(b) Let α be an eigenvalue of $T(Q)$ and let $u \in H_0^1$ be a corresponding eigenfunction. One has

$$\int_0^1 u'(x)v'(x) dx - \int_0^1 (u(x)v(x))' Q(x) dx = \alpha \int_0^1 u(x)v(x) dx \forall v \in H_0^1. \quad (7)$$

Integration by parts implies that

$$\int_0^1 v'(x) \left[u'(x) - u(x)Q(x) + \int_0^x (u'(t)Q(t) + \alpha u(t)) dt \right] dx = 0 \forall v \in H_0^1.$$

Hence there exists a constant c such that

$$u'(x) = u(x)Q(x) - \int_0^x (u'(t)Q(t) + \alpha u(t)) dt + c \quad \text{almost everywhere on } (0, 1). \quad (8)$$

Let u_1, u_2 be two solutions of (7) and c_1, c_2 their respective constants. We set $w = c_2 u_1 - c_1 u_2$ and show that $w \equiv 0$ on $[0, 1]$. In fact w satisfies

$$w'(x) = w(x)Q(x) - \int_0^x (w'(t)Q(t) + \alpha w(t)) dt \quad \text{almost everywhere on } (0, 1).$$

Multiply this equation by $w'(x)$ and integrate it over $(0, \varepsilon)$, $0 < \varepsilon < 1$, then

$$\int_0^\varepsilon w'^2(x) dx = \int_0^\varepsilon w(x)w'(x)Q(x) dx - \int_0^\varepsilon w'(x) \int_0^x (w'(t)Q(t) + \alpha w(t)) dt dx.$$

Note that $|w(x)| \leq \int_0^x |w'(t)| dt \leq \sqrt{\varepsilon} \|w'\|_{L^2(0, \varepsilon)}$ almost everywhere on $(0, \varepsilon)$ and estimate the terms on the right hand side:

$$\begin{aligned} \left| \int_0^\varepsilon w(x)w'(x)Q(x) dx \right| &\leq \|w\|_{L^\infty(0, \varepsilon)} \|w'\|_{L^2(0, \varepsilon)} \|Q\|_{L^2(0, \varepsilon)} \\ &\leq \sqrt{\varepsilon} \|Q\|_{L^2(0, \varepsilon)} \|w'\|_{L^2(0, \varepsilon)}^2 \\ \left| \int_0^\varepsilon w'(x) \int_0^x w'(t)Q(t) dt dx \right| &\leq \|w'\|_{L^2(0, \varepsilon)} \|Q\|_{L^2(0, \varepsilon)} \int_0^\varepsilon |w'(x)| dx \\ &\leq \sqrt{\varepsilon} \|Q\|_{L^2(0, \varepsilon)} \|w'\|_{L^2(0, \varepsilon)}^2 \\ \left| \int_0^\varepsilon w'(x) \int_0^x \alpha w(t) dt dx \right| &\leq \varepsilon |\alpha| \|w\|_{L^\infty(0, \varepsilon)} \int_0^\varepsilon |w'(x)| dx \leq \varepsilon^2 |\alpha| \|w'\|_{L^2(0, \varepsilon)}^2. \end{aligned}$$

Finally one obtains $\|w'\|_{L^2(0, \varepsilon)}^2 \leq \sqrt{\varepsilon} (2 \|Q\|_2 + |\alpha|) \|w'\|_{L^2(0, \varepsilon)}^2$ and for $\varepsilon_0 = (4 \|Q\|_2 + 2 |\alpha|)^{-2}$ we find that $w' = 0$ almost everywhere on $(0, \varepsilon_0)$. Hence $w \equiv 0$ on $(0, \varepsilon_0)$. Repeating the same argument for the intervals $(\varepsilon_0, 2\varepsilon_0)$, $(2\varepsilon_0, 3\varepsilon_0), \dots$ we deduce that $w \equiv 0$ on $[0, 1]$. Thus u_1 and u_2 are linearly dependent and α is a simple eigenvalue. \square

The next lemma concerns the stability of the first eigenvalue of $T(Q)$, denoted by $\alpha(Q)$, under small perturbations of $Q \in L^2$.

LEMMA 2.3. *Given $P \in L^2$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, if $Q \in L^2$ satisfies $\|P - Q\|_2 < \delta$, then $|\alpha(P) - \alpha(Q)| < \varepsilon$.*

Proof. By Lemma 2.1 there exists $\lambda \in \mathbb{R}$ such that $\|u\|_{1,2}^2 \leq 2 \langle T(P)u, u \rangle - 2\lambda \langle u, u \rangle$ for all $u \in H_0^1$, then by the minimal characterisation of the first eigenvalue

$$\begin{aligned} \alpha(Q) &= \min_{\|u\|_2=1} \langle T(Q)u, u \rangle = \min_{\|u\|_2=1} (\langle T(P)u, u \rangle + \langle M(Q - P)u, u \rangle) \\ &\leq \min_{\|u\|_2=1} (\langle T(P)u, u \rangle + \|M(Q - P)\| \|u\|_{1,2}^2) \\ &\leq \min_{\|u\|_2=1} (\langle T(P)u, u \rangle + 4 \|Q - P\|_2 (\langle T(P)u, u \rangle - \lambda)) \\ &= \alpha(P) + 4 \|Q - P\|_2 (\alpha(P) - \lambda). \end{aligned}$$

Similarly $\alpha(Q) \geq \alpha(P) - 4 \|Q - P\|_2 (\alpha(P) - \lambda)$ and setting $\delta = \varepsilon/4(\alpha(P) - \lambda)^{-1}$ the lemma is proved. \square

For $Q \in L^2$ and $l \in (0, 1)$ consider the problem

$$\int_0^l u'(x)v'(x) dx - \int_0^l (u(x)v(x))' Q(x) dx = \alpha \int_0^l u(x)v(x) dx \quad \forall v \in H_0^1(0, l). \quad (9)$$

Set $y = x/l$, $\hat{u}(y) = u(x)$, $\hat{v}(y) = v(x)$ and define $Q_l(y) = Q(x)$, then (9) is equivalent to

$$\int_0^1 \hat{u}'(y)\hat{v}'(y) dy - \int_0^1 (\hat{u}(y)\hat{v}(y))' l Q_l(y) dy = l^2 \alpha \int_0^1 \hat{u}(y)\hat{v}(y) dy \quad \forall \hat{v} \in H_0^1. \quad (10)$$

LEMMA 2.4. Let $P \in L^2$, $m \in (0, 1)$ and $\varepsilon > 0$ be given. There exist $\delta_1, \delta_2 > 0$ such that, if $Q \in L^2$ and $l \in (0, 1)$ satisfy $\|P - Q\|_2 < \delta_1$, respectively $|m - l| < \delta_2$, then $\|mP_m - lQ_l\|_2 < \varepsilon$.

Proof. Let $P \in L^2$, $m \in (0, 1)$ and $\varepsilon > 0$ be given. Since $C([0, 1])$ is densely embedded in L^2 , for all $\varepsilon_1 > 0$ there exists $p \in C([0, 1])$ such that $\|P - p\|_2 < \varepsilon_1$. For $Q \in L^2$ and $l \in (0, 1)$, by the triangle inequality,

$$\begin{aligned} \|mP_m - lQ_l\|_2 &\leq m \|P_m - p_m\|_2 + m \|p_m - p_l\|_2 \\ &\quad + m \|p_l - P_l\|_2 + m \|P_l - Q_l\|_2 + |m - l| \|Q_l\|_2. \end{aligned}$$

Note that $m \|P_m - p_m\|_2 \leq \sqrt{m} \|P - p\|_2 < \varepsilon_1$ and $\|p_m - p_l\|_2 < \varepsilon_1$ if $|m - l|$ is sufficiently small. Treating the other terms similarly, for $\varepsilon_1 > 0$ sufficiently small, there exist $\delta_1, \delta_2 > 0$ such that $\|mP_m - lQ_l\|_2 < l\varepsilon$ whenever $\|P - Q\|_2 < \delta_1$ and $|m - l| < \delta_2$. \square

THEOREM 2.5. Let $\alpha(l, Q)$ denote the first eigenvalue of (9); then $A: (0, 1) \times L^2 \rightarrow \mathbb{R}$ defined by $A(l, Q) = \alpha(l, Q)$ is continuous.

Theorem 2.5 is a direct consequence of formula (10), Lemma 2.3 and Lemma 2.4.

LEMMA 2.6. Suppose that $c \in \mathbb{R}$ is given, then the first eigenfunction $u = u(l, Q) \in H_0^1(0, l)$ of (9) can be normalised such that

$$u'(x) = u(x)Q(x) - \int_l^x (u'(t)Q(t) + \alpha(l, Q)u(t)) dt + c \quad \text{almost everywhere on } (0, l). \quad (11)$$

Furthermore $U: (0, 1) \times L^2 \rightarrow \mathbb{R}$ defined by $U(l, Q) = \|u(l, Q)\|_2$ is continuous.

Proof. Recall that a solution $u \in H_0^1(0, l)$ of (9) satisfies (11) for some $c \in \mathbb{R}$. If $u \neq 0$, then, proceeding as in the proof of Proposition 2.2(b), we conclude that $c \neq 0$. The linearity of the problem then implies that every constant $c \in \mathbb{R}$ can be achieved.

Setting $y = x/l$, $\hat{u}(y) = u(x)$ and $Q_l(y) = Q(x)$, (11) reduces to

$$\begin{aligned} \hat{u}'(y) &= \hat{u}(y)lQ_l(y) - \int_1^y (\hat{u}'(s)lQ_l(s) \\ &\quad + l^2\alpha(l, Q)\hat{u}(s)) ds + lc \quad \text{almost everywhere on } (0, 1). \end{aligned} \quad (12)$$

Define the bounded linear functional $L_{l,Q}: H_0^1 \rightarrow \mathbb{R}$ by

$$L_{l,Q}(v) = \int_0^1 \left[v(y)lQ_l(y) - \int_1^y (v'(s)lQ_l(s) + l^2\alpha(l, Q)v(s)) ds \right] dy.$$

Since the eigenspace of (9) is one-dimensional, the norm of the solution $\hat{u}(l, Q)$ is uniquely determined by $L_{l,Q}(\hat{u}(l, Q)) + lc = 0$. Fix $m \in (0, 1)$ and $P \in L^2$, then by standard perturbation theory there exists $\hat{v}(l, Q) \in \text{span}\{\hat{u}(l, Q)\}$ such that $\|\hat{v}(l, Q) - \hat{u}(m, P)\|_{1,2} \rightarrow 0$ as $|l - m| \rightarrow 0$, $\|Q - P\|_2 \rightarrow 0$. One verifies that $\hat{u}(l, Q) = -cl\hat{v}(l, Q)/L_{l,Q}(\hat{v}(l, Q))$. By the continuity of $L_{l,Q}$ with respect to l and Q we have $\|\hat{u}(l, Q)\|_{1,2} \rightarrow \|\hat{u}(m, P)\|_{1,2}$ as $|l - m| \rightarrow 0$, $\|Q - P\|_2 \rightarrow 0$. Passing to the original coordinates the lemma is proved. \square

3. The nonlinear equation

Let $\alpha(z, Q)$ and $\beta(z, Q)$ denote the first eigenvalues of (5) and (6) and let the corresponding eigenfunctions $u_1 \in H_0^1(0, z)$ and $u_2 \in H_0^1(z, 1)$ satisfy the normalisations

$$u_1'(x) = u_1(x)Q(x) - \int_z^x (u_1'(t)Q(t) + \alpha(z, Q)u_1(t)) dt + c \quad \text{almost everywhere on } (0, z)$$

$$u_2'(x) = u_2(x)Q(x) - \int_z^x (u_2'(t)Q(t) + \beta(z, Q)u_2(t)) dt + c \quad \text{almost everywhere on } (z, 1)$$

where $c < 0$ is a given number. Define

$$u(x) = \begin{cases} u_1(x) & \text{for } x \in [0, z] \\ u_2(x) & \text{for } x \in (z, 1]. \end{cases}$$

As in the proof of Proposition 2.2(b), there exists $K > 0$ such that $\|u_1'\|_{L^2(z-\varepsilon, z)}^2 \leq \sqrt{\varepsilon} K \|u_1'\|_{L^2(z-\varepsilon, z)}^2 - cu_1(z-\varepsilon)$. Thus $u_1(z-\varepsilon) > 0$ if $0 < \varepsilon < K^{-2}$ and it follows that $u_1 > 0$ on $(0, z)$. In the same way one verifies that $u_2 < 0$ on $(z, 1)$. Since for all $v \in H_0^1$,

$$\begin{aligned} \int_0^1 v'(x) \left[u'(x) - u(x)Q(x) + \int_z^x (u'(t)Q(t) + \alpha(z, Q)u(t)^+ - \beta(z, Q)u(t)^-) dt \right] dx \\ = \int_0^1 v'(x)c dx = 0, \end{aligned}$$

then u is a solution of (3) and therefore $\alpha(z, Q)$ and $\beta(z, Q)$ are parameters for (3) to have nontrivial solutions. By Theorem 2.5 the set $\Gamma(Q)$ defined by $\Gamma(Q) = \{(\alpha(z, Q), \beta(z, Q)) \in \mathbb{R}^2: z \in (0, 1)\}$ is a continuous curve. Let $\lambda_k(Q)$, $k = 1, 2, \dots$ denote the eigenvalues of $T(Q)$ and let $\psi_k(Q)$ be eigenfunctions of $T(Q)$ corresponding to $\lambda_k(Q)$ such that $\|\psi_k(Q)\|_2 = 1$ and $\psi_2(Q)(x) > 0$ for x near 0.

PROPOSITION 3.1. If $(\alpha, \beta) \in \Gamma(Q)$, then (3) has an unique solution $u \in H_0^1$ such that

$$\langle u, \psi_2(Q) \rangle = 1. \quad (13)$$

Proof. Let z_0 be the unique zero of $\psi_2(Q)$, $u \in H_0^1$ a solution of (3) with $u(x) > 0$ for x near 0 and let $z \in (0, 1)$ denote the unique zero of u . Suppose first that $z_0 < z < 1$, then by the inclusions $H_0^1(0, z_0) \subset H_0^1(0, z)$, $H_0^1(z, 1) \subset H_0^1(z_0, 1)$ and by the minimal characterisation of the first eigenvalue, $\alpha(z, Q) < \lambda_2(Q) < \beta(z, Q)$. One has

$$\begin{aligned} \lambda_2(Q) \langle u, \psi_2(Q) \rangle &= \langle u, T(Q)\psi_2(Q) \rangle \\ &= \langle T(Q)u, \psi_2(Q) \rangle = \langle \alpha(z, Q)u^+ - \beta(z, Q)u^-, \psi_2(Q) \rangle \\ &= \alpha(z, Q) \langle u, \psi_2(Q) \rangle + (\alpha(z, Q) - \beta(z, Q)) \langle u^-, \psi_2(Q) \rangle. \end{aligned}$$

Hence

$$\langle u, \psi_2(Q) \rangle = \frac{\alpha(z, Q) - \beta(z, Q)}{\lambda_2(Q) - \alpha(z, Q)} \langle u^-, \psi_2(Q) \rangle > 0.$$

The case $0 < z < z_0$ is handled in the same manner and the case $z = z_0$ is trivial. Thus $\langle u, \psi_2(Q) \rangle > 0$ whenever $u(x) > 0$ for x near 0 and by the positive homogeneity of (3), (13) can be satisfied. Uniqueness follows from Proposition 2.2(b). \square

THEOREM 3.2. Let $P \in L^2$, $\varepsilon > 0$ and $a, b \in (0, 1)$, $a < b$, be given. There exists $\delta > 0$ such that, if $Q \in L^2$ satisfies $\|P - Q\|_2 < \delta$, then for all $z \in [a, b]$, $((\alpha(z, P) - \alpha(z, Q))^2 + (\beta(z, P) - \beta(z, Q))^2)^{\frac{1}{2}} < \varepsilon$.

Proof. Set

$$\begin{aligned} \delta_\alpha(z) &= \sup \left\{ \gamma \in \mathbb{R} : |\alpha(z, P) - \alpha(z, Q)| < \frac{\sqrt{2}\varepsilon}{2} \forall Q \in L^2 \text{ such that } \|P - Q\|_2 < \gamma \right\}, \\ \delta_\beta(z) &= \sup \left\{ \gamma \in \mathbb{R} : |\beta(z, P) - \beta(z, Q)| < \frac{\sqrt{2}\varepsilon}{2} \forall Q \in L^2 \text{ such that } \|P - Q\|_2 < \gamma \right\} \end{aligned}$$

and let us show that $\delta_1 = \inf \{\delta_\alpha(z) : z \in [a, b]\} > 0$. Then likewise $\delta_2 = \inf \{\delta_\beta(z) : z \in [a, b]\} > 0$ and by setting $\delta = \min \{\delta_1, \delta_2\}$ the theorem is proved. By way of contradiction, suppose that $\delta_\alpha(z_i) \rightarrow 0$ for a sequence $(z_i) \subset [a, b]$. Then $z_i \rightarrow \hat{z} \in [a, b]$ for a subsequence. By Theorem 2.5 there exists $\nu > 0$ such that $|\alpha(\hat{z}, P) - \alpha(z, Q)| < \sqrt{2}\varepsilon/4$ for all $Q \in L^2$ such that $\|P - Q\|_2 < \nu$ and for all $z \in (\hat{z} - \nu, \hat{z} + \nu)$. Consequently for those z and Q

$$|\alpha(z, Q) - \alpha(z, P)| \leq |\alpha(z, Q) - \alpha(\hat{z}, P)| + |\alpha(\hat{z}, P) - \alpha(z, P)| < \frac{\sqrt{2}\varepsilon}{2}.$$

Hence $\delta_\alpha(z) \geq \nu > 0$ for all $z \in (\hat{z} - \nu, \hat{z} + \nu)$. This is in contradiction to the assumption. \square

Remark 1. $\Gamma(Q)$ is unbounded on both sides. More precisely $\lim_{z \rightarrow 0^+} \alpha(z, Q) = \infty$ and $\lim_{z \rightarrow 1^-} \beta(z, Q) = \infty$. In fact

$$\lim_{z \rightarrow 0^+} \|zQ_z\|_2^2 = \lim_{z \rightarrow 0^+} \int_0^1 z^2 Q(zx)^2 dx = \lim_{z \rightarrow 0^+} z \int_0^z Q(y)^2 dy = 0,$$

then by (10) and Theorem 2.5 $\lim_{z \rightarrow 0^+} z^2 \alpha(z, Q) = \pi^2$. Thus $\lim_{z \rightarrow 0^+} \alpha(z, Q) = \infty$ and similarly $\lim_{z \rightarrow 1^-} \beta(z, Q) = \infty$.

Remark 2. Recall that $\alpha(\cdot, Q)$ is strictly decreasing, $\beta(\cdot, Q)$ is strictly increasing and $\lim_{z \rightarrow 1^-} \alpha(z, Q) = \lambda_1(Q)$. Hence there exists a continuous and strictly decreasing function $\eta: (\lambda_1(Q), \infty) \rightarrow \mathbb{R}$ such that $\Gamma(Q) = \{(\alpha, \eta(\alpha)): \alpha \in (\lambda_1(Q), \infty)\}$.

THEOREM 3.3. *There exist an open interval $J \subset (\lambda_1(Q), \lambda_3(Q))$ and a C^1 -function $\eta: J \rightarrow \mathbb{R}$ such that $\Gamma(Q) \cap (\lambda_1(Q), \lambda_3(Q)) \times (\lambda_1(Q), \lambda_3(Q)) = \{(\alpha, \eta(\alpha)): \alpha \in J\}$. If $u(\alpha, \beta) \in H_0^1$ denotes the solution of (3) and (13), then*

$$\eta'(\alpha) = -\frac{\|u(\alpha, \eta(\alpha))^+\|_2^2}{\|u(\alpha, \eta(\alpha))\|_2^2} \quad \text{for all } \alpha \in J.$$

Proof. By the result of C. A. Stuart [4] the theorem is true for $Q \equiv 0$ and it generalises easily to $Q \in C^1([0, 1])$. If $Q \in L^2$, then there exist $Q^{(n)} \in C^1([0, 1])$, $n = 1, 2, \dots$ such that $\|Q - Q^{(n)}\|_2 \rightarrow 0$ as $n \rightarrow \infty$. By Remark 2 there exist open intervals $J \subset (\lambda_1(Q), \lambda_3(Q))$, $J_n \subset (\lambda_1(Q^{(n)}), \lambda_3(Q^{(n)}))$ and continuous functions $\eta: J \rightarrow \mathbb{R}$, $\eta_n: J_n \rightarrow \mathbb{R}$ such that $\Gamma(Q) \cap (\lambda_1(Q), \lambda_3(Q))^2 = \{(\alpha, \eta(\alpha)): \alpha \in J\}$ and $\Gamma(Q^{(n)}) \cap (\lambda_1(Q^{(n)}), \lambda_3(Q^{(n)}))^2 = \{(\alpha, \eta_n(\alpha)): \alpha \in J_n\}$, $n = 1, 2, \dots$. By [4] the η_n are of class C^1 . Let $u(\alpha, \eta(\alpha))$, $u_n(\alpha, \eta_n(\alpha)) \in H_0^1$ denote the solutions of (3) and (13) for Q , respectively for $Q^{(n)}$; then

$$\eta'_n(\alpha) = -\frac{\|u_n(\alpha, \eta_n(\alpha))^+\|_2^2}{\|u_n(\alpha, \eta_n(\alpha))\|_2^2} \quad n = 1, 2, \dots \quad (14)$$

Note that $-\infty < \eta'_n(\alpha) < 0$ for all $\alpha \in J_n$, $n = 1, 2, \dots$. Define the change of variables $\hat{\alpha} = 1/\sqrt{2}(\alpha - \beta)$, $\hat{\beta} = 1/\sqrt{2}(\alpha + \beta)$; then for a C^1 -function $\beta(\alpha)$, satisfying $-\infty < \beta'(\alpha) < 0$, the derivative transforms as follows:

$$\hat{\beta}'(\hat{\alpha}) = \frac{1 + \beta'(\alpha)}{1 - \beta'(\alpha)} \quad \text{and} \quad |\hat{\beta}'(\hat{\alpha})| < 1.$$

Set $\hat{u}(\hat{\alpha}) = u(\alpha, \eta(\alpha))$, $\hat{u}_n(\hat{\alpha}) = u_n(\alpha, \eta_n(\alpha))$, $n = 1, 2, \dots$, then (14) is equivalent to

$$\hat{\eta}'_n(\hat{\alpha}) = \frac{\|\hat{u}_n(\hat{\alpha})^-\|_2^2 - \|\hat{u}_n(\hat{\alpha})^+\|_2^2}{\|\hat{u}_n(\hat{\alpha})\|_2^2}.$$

Note that (14) does not depend upon the particular normalisation of u_n . Fix $c < 0$ and normalise u_n^+ by (11) for all n . Since u_n^- matches u_n^+ , the u_n^- are normalised in the same way with the same constant c . By the Lebesgue Dominated Convergence Theorem, Lemma 2.6 and Theorem 3.2

$$\begin{aligned} \hat{\eta}(\hat{\alpha}) - \hat{\eta}(\hat{\alpha}_0) &= \lim_{n \rightarrow \infty} (\hat{\eta}_n(\hat{\alpha}) - \hat{\eta}_n(\hat{\alpha}_0)) = \lim_{n \rightarrow \infty} \int_{\hat{\alpha}_0}^{\hat{\alpha}} \hat{\eta}'_n(s) ds = \int_{\hat{\alpha}_0}^{\hat{\alpha}} \left(\lim_{n \rightarrow \infty} \hat{\eta}'_n(s) \right) ds \\ &= \int_{\hat{\alpha}_0}^{\hat{\alpha}} \frac{\|\hat{u}(s)^-\|_2^2 - \|\hat{u}(s)^+\|_2^2}{\|\hat{u}(s)\|_2^2} ds, \end{aligned}$$

for all $\hat{\alpha}$, $\hat{\alpha}_0$ such that $\alpha(\hat{\alpha}, \hat{\eta}(\hat{\alpha}))$, $\alpha(\hat{\alpha}_0, \hat{\eta}(\hat{\alpha}_0)) \in J$. Again by Lemma 2.6 the integrand of the last formula is continuous and hence $\hat{\eta}$ is of class C^1 . Differentiating $\hat{\eta}$ and passing to the original coordinates proves the theorem. \square

We summarise the results of this section in the following corollary using the notation of Theorem 3.3.

COROLLARY 3.4. *Let $Q, Q^{(n)} \in L^2$ be such that $\|Q - Q^{(n)}\|_2 \rightarrow 0$ as $n \rightarrow \infty$ and let $I \subset (\lambda_1(Q), \lambda_3(Q))$ be a compact interval. There exists an integer n_0 such that $(\lambda_1(Q^{(n)}), \lambda_3(Q^{(n)})) \supset I$ for all $n > n_0$ and $\lim_{n \rightarrow \infty} \|\eta - \eta_n\|_{C^1(I)} = 0$.*

4. An example

Given $a \in (0, 1)$ and $p \in \mathbb{R}$, set

$$Q(x) = \begin{cases} 0 & \text{for } x \in [0, a] \\ p & \text{for } x \in (a, 1]. \end{cases}$$

We have

$$\langle M(Q)u, v \rangle = - \int_0^1 (u(x)v(x))' Q(x) dx = pu(a)v(a) \quad \text{for all } u, v \in H_0^1,$$

and we are concerned with the multiplication by the Dirac delta distribution up to the factor p . For $z \in [a, 1)$, (5) and (6) take the form

$$\int_0^z u'(x)v'(x) dx + pu(a)v(a) = \alpha \int_0^z u(x)v(x) dx \quad \text{for all } v \in H_0^1(0, z) \quad (15)$$

$$\int_z^1 u'(x)v'(x) dx = \beta \int_z^1 u(x)v(x) dx \quad \text{for all } v \in H_0^1(z, 1) \quad (16)$$

and for $z \in (0, a]$ the situation is similar. We describe the set $\Gamma(Q) = \{(\alpha(z, Q), \beta(z, Q)) \in \mathbb{R}^2 : z \in (0, 1)\}$ near the value $z = a$. We fix p for the time being, omit the variable Q and set $\alpha(z) = \alpha(z, Q)$ and $\beta(z) = \beta(z, Q)$. It is easily seen that $\alpha(a) = \pi^2/a^2$, thus by the continuity of $\alpha(\cdot)$, $\alpha(z) > 0$ for z near a . Hence the first eigenfunction of (15) must take the form

$$f(x) = \begin{cases} \sin \sqrt{\alpha} x & \text{for } x \in [0, a] \\ \frac{\sin \sqrt{\alpha} a}{\sin \sqrt{\alpha}(z-a)} \sin \sqrt{\alpha}(z-x) & \text{for } x \in (a, z]. \end{cases}$$

To determine the dependence of α on z and p , note that by (8)

$$f'(a^+) - f'(a^-) - pf(a) = 0, \quad (17)$$

or

$$-\sqrt{\alpha} \sin \sqrt{\alpha} a \operatorname{ctg} \sqrt{\alpha}(z-a) - \sqrt{\alpha} \cos \sqrt{\alpha} a - p \sin \sqrt{\alpha} a = 0.$$

For (α, z) near $(\pi^2/a^2, a)$ this relation is equivalent to finding zeros of

$$F(\alpha, z) = (p \operatorname{tg} \sqrt{\alpha} a + \sqrt{\alpha}) \operatorname{tg} \sqrt{\alpha}(z-a) + \sqrt{\alpha} \operatorname{tg} \sqrt{\alpha} a.$$

Note that $F(\pi^2/a^2, a) = 0$. Let $F_\alpha, F_z, F_{\alpha z}, \dots$ denote the partial derivatives of F . One verifies that

$$F_\alpha\left(\frac{\pi^2}{a^2}, a\right) = \frac{a}{2} \quad \text{and} \quad F_z\left(\frac{\pi^2}{a^2}, a\right) = \frac{\pi^2}{a^2}.$$

By the Implicit Function Theorem there exists a C^1 -function $\phi(z)$ such that $\phi(a) = \pi^2/a^2$ and $F(\phi(z), z) = 0$ for z near a . In particular $\alpha(z) = \phi(z)$ for $z \geq a$ and $\alpha'(a^+) = \phi'(a) = -2\pi^2/a^3$. Concerning (16), one has $\beta(z) = \pi^2/(1-z)^2$ for $z \in [a, 1)$. By Theorem 3.3 there exists an open interval $J \subset \mathbb{R}$ and a C^1 -function $\eta: J \rightarrow \mathbb{R}$ such that $\Gamma(Q) \cap (\lambda_1(Q), \lambda_3(Q))^2 = \{(\alpha, \eta(\alpha)): \alpha \in J\}$; that is to say $\eta(\alpha(z)) = \beta(z)$ for $z \in [a, 1)$. In particular

$$\eta\left(\frac{\pi^2}{a^2}\right) = \beta(a) = \frac{\pi^2}{(1-a)^2} \quad \text{and} \quad \eta'\left(\frac{\pi^2}{a^2}\right) = \beta'(a^+)(\alpha'(a^+))^{-1} = -\left(\frac{a}{1-a}\right)^3.$$

Let us turn to the second derivatives. From $F_\alpha\phi' + F_z = 0$ we deduce that

$$F_{\alpha\alpha}\phi'^2 + 2F_{z\alpha}\phi' + F_{zz} + F_\alpha\phi'' = 0,$$

or

$$\frac{\phi''}{\phi'^3} = F_z^{-1}[F_{\alpha\alpha} + 2F_{z\alpha}(\phi')^{-1} + F_{zz}(\phi')^{-2}].$$

By straightforward calculation

$$F_{\alpha\alpha}\left(\frac{\pi^2}{a^2}, \alpha\right) = \frac{a^3}{4\pi^2}, \quad F_{z\alpha}\left(\frac{\pi^2}{a^2}, a\right) = \frac{ap}{2} + 1, \quad F_{zz}\left(\frac{\pi^2}{a^2}, a\right) = 0.$$

Consequently

$$\begin{aligned} \eta''\left(\left(\frac{\pi^2}{a^2}\right)^{-}\right) &= \beta''(a^+)(\phi'(a))^{-2} - \beta'(a^+)\phi''(a)(\phi'(a))^{-3} \\ &= \frac{6\pi^2}{(1-a)^4} \frac{a^6}{4\pi^4} + \frac{2\pi^2}{(1-a)^3} \frac{a^5}{\pi^4} \left(\frac{3}{4} + \frac{ap}{2}\right) \\ &= \frac{3a^5}{2\pi^2(1-a)^4} + \frac{a^6p}{\pi^2(1-a)^3}. \end{aligned}$$

To calculate the limit at the right, recall the identity $\eta''(\alpha) = -(\eta'(\alpha))^3 \times (\eta^{-1})''(\eta(\alpha))$ where η^{-1} denotes the inverse of η . By a symmetry argument it is easily seen that $(\eta^{-1})''((\pi^2/(1-a)^2)^{-})$ is obtained from $\eta''((\pi^2/a^2)^{-})$ replacing a by $1-a$. Hence

$$\begin{aligned} \eta''\left(\left(\frac{\pi^2}{a^2}\right)^{+}\right) &= -\eta'\left(\frac{\pi^2}{a^2}\right)^3 (\eta^{-1})''\left(\left(\frac{\pi^2}{(1-a)^2}\right)^{-}\right) \\ &= \left(\frac{a}{1-a}\right)^9 \left[\frac{3(1-a)^5}{2\pi^2a^4} + \frac{(1-a)^6p}{\pi^2a^3}\right] = \eta''\left(\left(\frac{\pi^2}{a^2}\right)^{-}\right). \end{aligned}$$

This implies that η is of class C^2 near $\alpha = \pi^2/a^2$ and

$$\eta''\left(\frac{\pi^2}{a^2}\right) = \frac{3a^5}{2\pi^2(1-a)^4} + \frac{a^6p}{\pi^2(1-a)^3}.$$

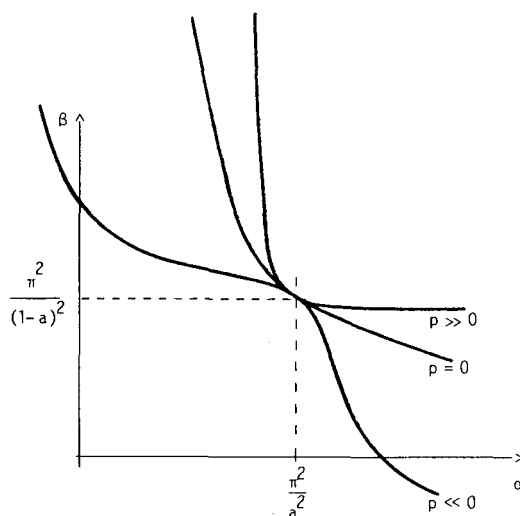


Figure 1

The second term of this formula is the contribution of the Dirac distribution.

Note that $\eta''(\pi^2/a^2) > 0$ if and only if $p > -\frac{3}{2a(1-a)}$, which for $\Gamma(Q)$ yields the diagram in Figure 1.

Finally we investigate $\Gamma(Q)$ under the limits $p \rightarrow \pm\infty$. Fix $z_0 \in (a, 1)$ and define $\alpha_0(p) = \alpha(z_0, Q)$, $\beta_0(p) = \beta(z_0, Q)$. Note that β_0 is constant, α_0 is continuous and strictly increasing. It is easy to see that $\alpha_0(p) < 0$ for sufficiently negative p . In this case the solution of (15) takes the form

$$f(x) = \begin{cases} \text{sh } \sqrt{-\alpha} x & \text{for } x \in [0, a] \\ \frac{\text{sh } \sqrt{-\alpha} a}{\text{sh } \sqrt{-\alpha}(z_0 - a)} \text{sh } \sqrt{-\alpha}(z_0 - x) & \text{for } x \in (a, z_0]. \end{cases}$$

By (17),

$$-\sqrt{-\alpha} \text{sh } \sqrt{-\alpha} a \text{th } \sqrt{-\alpha}(z_0 - a) - \sqrt{-\alpha} \text{ch } \sqrt{-\alpha} a - p \text{sh } \sqrt{-\alpha} a = 0$$

and it follows that $\lim_{p \rightarrow -\infty} \alpha_0(p) = -\infty$. On the other hand $\alpha_0(p) > 0$ for large p , then again by (17)

$$-\sqrt{\alpha} \text{ctg } \sqrt{\alpha}(z_0 - a) - \sqrt{\alpha} \text{ctg } \sqrt{\alpha} a - p = 0.$$

Thus by the monotony of α_0 , $\lim_{p \rightarrow \infty} \alpha_0(p) = \min \left\{ \frac{\pi^2}{(z_0 - a)^2}, \frac{\pi^2}{a^2} \right\}$.

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