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Stationary states of two-dimensional magnetohydrodynamic turbulence: non-dissipative limit

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A class of exact stationary statistical states for the inviscid magnetohydrodynamic equations in two dimensions and in various geometries is found and the corresponding fluctuation spectra are calculated. Some solutions agree with previous computations in the canonical ensemble while other solutions are found. In particular, the Navier–Stokes limit is recovered and maximum cross helicity solutions exist in two dimensions. The difficulty of proving existence and uniqueness of statistical solutions for non-dissipative two-dimensional turbulence is quoted in terms of rugged constants and associated Gibbs measure.

1. Introduction

There is still considerable interest in the statistical theory of turbulence of two-dimensional flow in electrostatic guiding-centre plasma, Navier–Stokes fluids and, lately, in two-dimensional turbulence in magnetohydrodynamics, both in the non-dissipative limit and in the case of finite viscosity and finite conductivity.

The main motivation for such two-dimensional studies stems from the fact that they are amenable to high resolution numerical methods. Moreover, the two-dimensional case still represents an interface between plasma and fluid turbulence (Dupree 1974; Edwards 1973).

In a series of recent studies, turbulence in magnetohydrodynamics (Fyfe & Montgomery 1976, 1978; Fyfe *et al.* 1977*a*, *b*) for a two-dimensional model, and electron fluid turbulence for a one-dimensional model have been investigated by methods borrowed from fluid turbulence theory. Numerical results have been obtained in the solutions of the magnetohydrodynamic equations for some values of initial parameters. In the dissipative regime, energy and vector potential cascade respectively to high wavenumbers and to lower wavenumbers. Kolmogoroff-type spectral power laws, analogous to those for a pure Navier–Stokes fluid, have been obtained; this has been suggested and verified by numerical solutions of the magnetohydrodynamic equations. Macroscopic

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structures involving long-wavelength, self-generated magnetic fields have recently been obtained even in a non-dissipative forced model (Montgomery & Vahala 1978). In this work, we continue to be concerned with the nondissipative limit but in different geometries. Instead of using Gibbs ensemble for the statistics of the random Fourier coefficients of the non-dissipative 'turbulent' fields, we merely investigate time-asymptotic statistical homogeneous solutions of the associated Hopf equation for the generating function of the model. In so doing, we generalize a method employed earlier by Cook & Taylor for the pure Navier–Stokes case. Only recently, the Hopf equation was successfully employed in deriving kinetic equations in the pure Navier–Stokes case (Mond & Knorr 1978).

The paper is organized as follows: in § 2 we define the model, and the Hopf equation for the generating function in different geometries is developed. The condition of stationarity gives us a set of four algebraic equations for the fluctuation spectra. Some solutions are found in § 3; here we pay special attention to the Navier–Stokes limit not recovered in early approaches. New solutions for the spectra are found. The method is also applied to a one-dimensional model of electron fluid turbulence; an exact statistical equilibrium is also found in this case. To compare with statistical mechanics, we then informally discuss in § 4 the existence and uniqueness of flows obeying the magnetohydrodynamic equations. These are constructed by means of the Gibbs canonical measure and known rugged constants. In the simple case of toroidal magnetic fields and in a special case of pure toroidal magnetic fields, such methods do not allow one to infer the presence of a classical dynamical system if the usual Gibbs measure is employed. Conclusions are drawn in § 5.

2. Models and stationary states

The Lundquist equations of two-dimensional incompressible non-dissipative magnetohydrodynamics, in appropriate units and in Elsasser's symmetrized form with the fields $\mathbf{u} = \mathbf{v}+\mathbf{B}$ and $\mathbf{u}^* = \mathbf{v}-\mathbf{B}$ are given by (Kraichnan 1965):

$$\left.\begin{aligned} \partial\mathbf{u}/\partial t+(\mathbf{u}^*.\nabla)\,\mathbf{u} &= -\boldsymbol{\nabla}p_{\mathrm{tot}}, \\ \partial\mathbf{u}^*/\partial t+(\mathbf{u}.\nabla)\,\mathbf{u}^* &= -\boldsymbol{\nabla}p_{\mathrm{tot}}. \end{aligned}\right\} \tag{1}$$

In (1) it is assumed that the velocity field $\mathbf{v} = \mathbf{v}(x,y) = (v_1(x,y), v_2(x,y))$ has only x and y components and for the magnetic field $\mathbf{B}$, we will consider both the case of pure poloidal field $\mathbf{B}(x,y) = (B_1(x,y), B_2(x,y), 0)$ or pure toroidal field

$$\mathbf{B}(x,y) = (0,0,B_3(x,y).$$

All quantities are assumed to vary only in the (x,y) directions $(\partial/\partial z \equiv 0)$, with the usual periodic boundary conditions in the (x,y) plane, If $\mathbf{u} = \mathbf{u}^*$, then (1) describes the pure Navier–Stokes case. Let $\boldsymbol{\omega} = \nabla\times\mathbf{v} = (0,0,\omega_3(x,y))$ be the vorticity, $\mathbf{j} = \nabla\times\mathbf{B} = (0,0,j_3(x,y))$ in the pure poloidal case and let

$$\tilde{\boldsymbol{\omega}} = \nabla\times(\nabla\times\mathbf{B}) = (0,0,\tilde{\omega}_3(x,y))$$

in the pure toroidal case; the statistical properties of the flow described by the above equations may then be investigated starting from the dynamical equations for the scalars $\omega_k(t)$ and $j_k(t)$ in the poloidal case and $\omega_k(t)$ and $\tilde{\omega}_k(t)$ in the toroidal case, the Fourier transformed vorticity density, electrical current density and Fourier decomposition of $\tilde{\boldsymbol{\omega}} = \nabla \times (\nabla \times \mathbf{B})$, respectively. Writing for any field $\mathbf{f} = \sum_{\mathbf{k}} \mathbf{f}_k(t)\, e^{i\mathbf{k}.\mathbf{x}}$, we have

$$\left.\begin{aligned} \frac{\partial \omega_k}{\partial t} &= \sum_{\mathbf{r},\mathbf{p}} M_{1rp}^{-k}(\omega_r \omega_p - \delta j_r j_p), \\ \frac{\partial j_k}{\partial t} &= \sum_{\mathbf{r},\mathbf{p}} M_{2rp}^{-k}(j_r \omega_p - \omega_r j_p). \end{aligned}\right\} \tag{2}$$

For the poloidal case $\delta = 1$, while the toroidal case is recovered by putting $\delta = 0$ and replacing j_k by $\tilde{\omega}_k$. In (2),

$$M_{1rp}^{-k} = \tfrac{1}{2}\hat{e}_z.(\mathbf{r} \times \mathbf{p})\,(p^{-2} - r^{-2})\,\delta_{-k,r,p},$$

$$M_{2rp}^{-k} = \tfrac{1}{2}\hat{e}_z.(\mathbf{r} \times \mathbf{p})\left(\frac{k^2}{r^2 p^2}\right)\delta_{-k,r,p},$$

where $\delta_{k,r,p} = 1$ if $\mathbf{k} + \mathbf{r} + \mathbf{p} = 0$, otherwise zero. The flow invariants may be discussed in terms of the two basic symmetry properties of the coefficients M_{1rp}^{-k}, given by

$$\frac{M_{1rp}^{\;k}}{k^2} + \frac{M_{1pk}^{\;r}}{r^2} + \frac{M_{1kr}^{\;p}}{p^2} = 0, \tag{3}$$

$$M_{1rp}^{\;k} + M_{1pk}^{\;r} + M_{1kr}^{\;p} = 0. \tag{4}$$

Notice that $M_{2rp}^{\;k}$ is related to $M_{1rp}^{\;k} = (r^2 - p^2).k^{-2} M_{2rp}^{\;k}$ and the above two properties of $M_{1rp}^{\;k}$ will be employed. Nevertheless, for any triplet $\mathbf{k}$, $\mathbf{r}$, $\mathbf{p}$, such that $\mathbf{k} + \mathbf{r} + \mathbf{p} = 0$, a useful property for $M_{2rp}^{\;k}$, which follows from (3) and (4), may be formulated and reads

$$\frac{M_{2kr}^{\;p}}{p^4} - \frac{M_{2pk}^{\;r}}{r^4} = 0, \tag{5}$$

where k, r, p may be permuted.

In the following treatment one need not construct any rugged invariant or constant of motion (Fyfe & Montgomery 1976). Rather, one merely makes use of and exploits the basic symmetry properties expressed by (3), (4) and (5). In any case, it may be easily shown that (3) and (4) are both needed, simultaneously, and are sufficient to ensure conservation of the invariant of the motion. For the poloidal case three rugged constants have been found, namely the total energy, cross helicity and vector square potential. In the toroidal case, one can easily verify the existence of four rugged invariants.

Now, instead of assuming Gibbs equilibrium statistics for the random Fourier coefficients of the fields and making use of rugged invariants in order to compute the time-asymptotic fluctuation spectra, we consider more generally the characteristic functional defined as

$$G(\{\alpha_k\}, \{\beta_k\}) = \langle \exp\,[i(\sum_k \alpha_k \omega_k + \sum_k \beta_k j_k)]\rangle,$$

where the brackets denote an average with respect to some unspecified measure. Here (i) G defines the statistics of the random Fourier coefficients of the 'turbulent' fields $\boldsymbol{\omega}$, $\mathbf{j}$, $\tilde{\omega}$, (ii) G contains all information about the ensemble and (iii) the derivatives of G generate all moments of the ω_k, j_k or $\tilde{\omega}_k$. In particular,

$$\langle \omega_k \omega_{-k} \rangle = -\left[\frac{\partial^2 G}{\partial \alpha_k \, \partial \alpha_{-k}}\right]_{\{\alpha_k\}=\{\beta_k\}=0},$$

$$\langle j_k j_{-k} \rangle = -\left[\frac{\partial^2 G}{\partial \beta_k \, \partial \beta_{-k}}\right]_{\{\alpha_k\}=\{\beta_k\}=0},$$

$$\langle \omega_k j_{-k} \rangle = -\left[\frac{\partial^2 G}{\partial \alpha_k \, \partial \beta_{-k}}\right]_{\{\alpha_k\}=\{\beta_k\}=0}. \tag{6}$$

The linear equation satisfied by G, the Hopf equation for this problem (Hopf 1952), then reads

$$i\frac{\partial G}{\partial t} = \sum_{\mathbf{r},\mathbf{p},\mathbf{k}} \alpha_{-k} M_{1rp}^{\ k}\left(\frac{\partial^2 G}{\partial \alpha_r \, \partial \alpha_p} - \left(\frac{\partial^2 G}{\partial \beta_r \, \partial \beta_p}\right) + \sum_{\mathbf{r},\mathbf{p},\mathbf{k}} \beta_{-k} M_{2rp}^{\ k}\left(\frac{\partial^2 G}{\partial \beta_r \, \partial \alpha_p} - \frac{\partial^2 G}{\partial \alpha_r \, \partial \beta_p}\right). \tag{7}$$

A Liouville equation may still be obtained by the Fourier transform of G. Many stationary solutions of (7), homogeneous as well as inhomogeneous, may exist. The appearance of time-asymptotic inhomogeneous statistical solutions will be investigated in another work (Merlini *et al.* 1979). Here we investigate homogeneous solutions and consider those in which G is of the form

$$G = G(-\tfrac{1}{2}\sum_k (\alpha_k \alpha_{-k} Q_k + \beta_k \beta_{-k} P_k + \tfrac{1}{2}\alpha_k \beta_{-k} R_k + \tfrac{1}{2}\alpha_{-k}\beta_k R_{-k})), \tag{8}$$

where $R_k = R_{-k}$, for which the covariance $\langle \omega_k \omega_{-k}\rangle \propto Q_k$, $\langle j_k j_{-k}\rangle \propto P_k$ (or $\langle \tilde{\omega}_k \tilde{\omega}_{-k}\rangle \propto P_k$) and $\langle \omega_k j_{-k}\rangle \propto R_k$ (or $\langle \tilde{\omega}_k \tilde{\omega}_{-k}\rangle \propto R_k$).

It is then easily shown that G of expression (8) is a stationary solution of (7) provided that Q_k, P_k, R_k satisfy (for all $\mathbf{r}$, $\mathbf{p}$, $\mathbf{k}$ such that $\mathbf{k}+\mathbf{r}+\mathbf{p} = 0$) the following equations:

$$M_{2rp}^{\ k}\left(P_r \frac{R_p}{2} - \frac{R_r}{2} P_p\right) + M_{2pk}^{\ r}\left(P_p \frac{R_k}{2} - \frac{R_p}{2} P_k\right) + M_{2kr}^{\ p}\left(P_k \frac{R_r}{2} - \frac{R_k}{2} P_r\right) = 0, \tag{9}$$

$$M_{1rp}^{\ k}\left(\frac{R_r R_p}{4} - P_r P_p\right) + M_{2pk}^{\ r}\left(P_p Q_k - \frac{R_p R_k}{4}\right) + M_{2kr}^{\ p}\left(\frac{R_k R_r}{4} - Q_k P_r\right) = 0, \tag{10}$$

$$M_{1rp}^{\ k}\left(\frac{R_r}{2} Q_p - P_r \frac{R_p}{2}\right) + M_{1kr}^{\ p}\left(Q_k \frac{R_r}{2} - \frac{R_k}{2} P_r\right) + M_{2pk}^{\ r}\left(\frac{R_p}{2} Q_k - Q_p \frac{R_k}{2}\right) = 0, \tag{11}$$

$$M_{1rp}^{\ k}\left(Q_r Q_p - \frac{R_r R_p}{4}\right) + M_{1pk}^{\ r}\left(Q_p Q_k - \frac{R_p R_k}{4}\right) + M_{1kr}^{\ p}\left(Q_k Q_r - \frac{R_k R_r}{4}\right) = 0. \tag{12}$$

Notice that the left-hand sides of (9)–(12) are, respectively, the coefficients of terms involving a product of 0, 1, 2 and 3 coefficients α. It should also be mentioned that, in the case of the pure toroidal magnetic field ($\delta = 0$ and $j_k \equiv \tilde{\omega}_k$), there are no PP terms in (10), no P terms in (11) and no RR terms in (12).

Any solution of the set of equations (9)–(12) for the spectra Q_k, P_k, R_k, belongs to the general set of stationary solutions of (7) in which G is of the form expressed by (8), i.e. a function whose argument is a linear combination of the bilinear expectation value for the fluctuation spectra of ω_k, j_k or $\tilde{\omega}_k$. We now find explicitly a family of solutions of (9)–(12).

3. Explicit solutions

3.1. *The Navier–Stokes limit*

Let $R_k = P_k = 0$ for all k; then the set of equations (9)–(12) reduces to

$$\frac{M_{1rp}{}^{k}}{Q_k} + \frac{M_{1pk}{}^{r}}{Q_r} + \frac{M_{1kr}{}^{p}}{Q_p} = 0. \tag{13}$$

Equation (13) subsumes conservation of energy and enstrophy for the fluid. The two-parameter solution gives for the stationary spectrum $Q_k = k^2/(\alpha+\beta k^2)$, i.e. the Navier–Stokes limit (Kraichnan 1967; Cook & Taylor 1971). Thus, within the framework of our treatment, such a limit is still recovered contrary to results of earlier studies (Fyfe & Montgomery 1976).

3.2. *Pure toroidal magnetic field* ($\delta = 0, j_k \rightarrow \tilde{\omega}_k$)

We first seek solutions with zero cross correlation between $\boldsymbol{\omega}$ and $\tilde{\boldsymbol{\omega}}$; from (9)–(12), one finds that $Q_k = k^2/(\alpha+\beta k^2)$ and for P_k, $M_{2pk}{}^{r}/P_r - M_{2kr}{}^{p}/P_p = 0$ for all r and p such that $\mathbf{k}+\mathbf{r}+\mathbf{p} = 0$. Using the identity (5), one obtains $P_k = k^4/\gamma$, with γ independent of α and β, whence

$$Q_k = \frac{k^2}{\alpha+\beta k^2}, \quad P_k = \frac{k^4}{\gamma}, \quad R_k \equiv 0.$$

Such a family still describes the Navier–Stokes limit for the fluid with a flat equipartition of the magnetic field fluctuation energy for 'passive' field $\mathbf{B}$, since $\langle B_k B_{-k}\rangle = \langle \tilde{\omega}_k \tilde{\omega}_{-k}\rangle/k^4 = 1/\gamma$. This solution is analogous to the zero cross-helicity limit in the pure poloidal case (§ 3.3) below, but the role of magnetic field and fluid are interchanged. A more general solution such that $R_k = 2f_k Q_k$ with f_k arbitrary, and Q_k as given above is easily found. The equation for f_k reads

$$\frac{M_{1rp}{}^{k}}{Q_k} + \frac{M_{1kr}{}^{p}}{Q_p} + \frac{M_{2pk}{}^{r}}{Q_r}\frac{(f_p - f_k)}{f_r} = 0. \tag{14}$$

Using the definition of $M_{2pk}{}^{r}$, $f_k = ck^2$, P_k is then determined to be $P_k = (d+c^2Q_k)\,k^4$ where c, d are arbitrary constants. Thus,

$$\begin{aligned} Q_k &= \langle \omega_k \omega_{-k}\rangle = k^2/(\alpha+\beta k^2), \\ R_k &= 2ck^2 Q_k, \\ P_k &= (d + c^2 Q_k)\,k^4. \end{aligned}$$

Let $d = 0$; in this case we see that the magnetic field fluctuation-energy $\langle B_k B_{-k}\rangle = c^2 Q_k$ is proportional to the enstrophy spectrum of the fluid while the vector square potential is proportional to the fluid energy. This case is somewhat analogous to the modified guiding centre plasma (Calinon & Merlini 1978).

3.3. *Pure poloidal magnetic field*

We discuss three different cases:

(i) Solutions such that $R_k = \alpha P_k$. Here, (9) is identically satisfied; one must solve

$$M_{1rp}^{\ \ k} P_r P_p \left(\frac{\alpha^2}{4} - 1\right) + M_{2pk}^{\ \ r}\left(P_p Q_k - \frac{\alpha^2}{4} P_p P_k\right) + M_{2kr}^{\ \ p}\left(P_r P_k \frac{\alpha^2}{4} - Q_k P_r\right) = 0, \quad (15)$$

$$M_{1rp}^{\ \ k} \frac{\alpha}{2}(P_r Q_p - P_r P_p) + M_{1kr}^{\ \ p} \frac{\alpha}{2}(Q_k P_r - P_k P_r) + M_{2kp}^{\ \ r} \frac{\alpha}{2}(P_p Q_k - Q_p P_k) = 0, \quad (16)$$

$$M_{1rp}^{\ \ k}\left(Q_r Q_p - \frac{\alpha^2}{4} P_r P_p\right) + M_{1pk}^{\ \ r}\left(Q_p Q_k - P_r P_k \frac{\alpha^2}{4}\right) + M_{1kr}^{\ \ p}\left(Q_k Q_r - \frac{\alpha^2}{4} P_k P_r\right) = 0. \quad (17)$$

First let $\alpha = 0$; equation (17) then gives $Q_k = k^2/(ak^2+b)$. Substitution of Q_k into (15) provides the equation

$$M_{1rp}^{\ \ k} \frac{(ak^2+b)}{k^2} + \frac{M_{2kr}^{\ \ p}}{P_p} - \frac{M_{2pk}^{\ \ r}}{P_r} = 0.$$

Identities (3)–(5) suggest the ansatz $P_k = k^4/(\gamma + f_k)$, with f_k some function of k.

A solution is easily found with $\alpha = 0$ and $f_k = bk^2$. Thus,

$$R_k = 0, \quad Q_k = \frac{k^2}{b}, \quad P_k = \frac{k^4}{bk^2+\gamma} \qquad (18)$$

is an exact statistical equilibrium with a flat equipartition of the fluid kinetic energy. In this case, we note that $Q_k = P_k(1+\gamma/k^2 b)$. A more general solution in which Q_k and P_k have the same relation, but with $\alpha \neq 0$ is easily found; after some algebra, we obtain

$$\left.\begin{aligned} R_k &= \alpha_1 P_k, \\ Q_k &= P_k(1+\alpha_2/k^2), \\ P_k &= \frac{\alpha_3 k^4}{(1-\frac{1}{4}\alpha^2)k^2 + \alpha^2}. \end{aligned}\right\} \qquad (19)$$

Equations (19) comprise a three-temperature family of solutions previously obtained by a different method (Fyfe & Montgomery 1976). Nevertheless, other solutions appear within the framework of our treatment, different from the one above. Thus

(ii) $\alpha \neq 0$ but $P_p/Q_p = P_k/Q_k = c$ in (16); then $c = 1$. Equation (17) gives $Q_k = P_k = k^2/(ak^2+b)$ and (15) is satisfied choosing $\alpha^2 = 4$.

So a new solution is obtained and is given by

$$\left.\begin{aligned} Q_k &= P_k = k^2/(ak^2+b), \\ R_k &= \pm 2Q_k. \end{aligned}\right\} \qquad (20)$$

Note that in this case, there is equipartition between fluid and magnetic field energies, a very different situation from case (i) above. A typical *configuration* which contributes to such equilibria is given by $\mathbf{u} \simeq \mathbf{B}$. A numerical computation of such maximum cross-helicity equilibria is needed in order to investigate its stability.

(iii) We turn now to the pure poloidal case by investigating yet another family of solutions adapted to the Navier–Stokes limit with non-zero vector square potential but with possibly zero cross correlation between magnetic and velocity fields. This question is connected with a kind of symmetry breaking, owing to the presence of the vector square potential invariant. We demand that the dynamical equations for ω_k should still be given by Navier–Stokes equations, i.e. we impose $\mathbf{B}$ fields with the constraint $\langle \nabla \times (\nabla \times \mathbf{B} \times \mathbf{B}) \rangle \equiv 0$. Then equations (9)–(12) reduce to those of the toroidal case but with $\tilde{\omega}_k$ still given by j_k. The solutions are the same and one obtains

$$\left.\begin{aligned} Q_k &= k^2/(\alpha + \beta k^2), \\ R_k &= 2ck^2 Q_k, \\ P_k &= (d + c^2 Q_k)\, k^4. \end{aligned}\right\} \tag{21}$$

In conclusion, we obtain a solution such that $R_k = 0$ with

$$P_k/k_4 = \langle j_k j_{-k} \rangle / k^4 = d,$$

i.e. a constant equipartition of the vector square potential and the limiting procedure of vanishing $\mathbf{B}$ fields ($c = d = 0$) is still possible in our treatment. Moreover, the more general solution (21) indicates that cross helicity and vector square potential spectra may be proportional to that of the enstrophy of the fluid. So, it is in the limit of non-zero cross helicity that the vector square potential goes over into the enstrophy of the fluid.

(iv) One-dimensional electron fluid turbulence.

Fyfe & Montgomery (1978) recently formulated a one-dimensional model of electrostatic turbulence in a compressible electron fluid plasma, with immobile ionic background; a locally adiabatic equation of state with ratio of specific heat $\gamma = 2$ was assumed. Their theoretical treatment was done in some approximations, i.e. in the non-interacting limit. We now compute the asymptotic fluctuation spectra within the above framework. We start from the fluid equations given by (Fyfe & Montgomery 1978)

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \tag{22}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -E - \frac{\partial n}{\partial x} + \nu \frac{\partial^2 v}{\partial x^2}. \tag{23}$$

We are still concerned here with the non-dissipative limit $\nu \to 0$. Eliminating the electric field with the help of Poisson's equation and introducing the quantity $\omega = \partial v/\partial x$ and $\rho = n - 1$, i.e. the net charge density, the Fourier transformed equations are given by

$$\dot{\omega}_k + \tfrac{1}{2} \sum_{q+q'=k} \omega_q \omega_{q'} \frac{k^2}{qq'} = \rho_k (1 + k^2), \tag{24}$$

$$\dot{\rho}_k + \tfrac{1}{2} \sum_{q+q'=k} \left(\rho_q \omega_{q'} \frac{k}{q'} + \rho_{q'} \omega_q \frac{k}{q} \right) = -\omega_k. \tag{25}$$

For $k = 0$ we have $\dot{\omega}_0 = \rho_0, \dot{\rho}_0 = -\omega_0$. We impose charge neutrality, which gives $\rho_0 = \dot{\rho}_0 = 0$ and thus $\dot{\omega}_0 = \omega_0 = 0$, such that (24) and (25) hold for each k. (In our treatment the $k = 0$ component does not enter into the problem.) Notice that elimination of density and velocity field is possible in this model, so that one obtains the following equation satisfied by the electric field above:

$$\ddot{E}(1-E_x)^2 + 2\dot{E}\dot{E}_x(1-E_x) + \dot{E}^2 E_{xx} = (E_{xx}-E)(1-E_x)^3. \tag{26}$$

Equation (26) points to the strong nonlinearity of the problem. In any case, we are able to find an exact statistical equilibrium. Let $G = \langle \exp[i \sum_k (\alpha_k \rho_k + \beta_k \omega_k)] \rangle$ be the generating function; the Hopf equation for this model becomes

$$-i\frac{\partial G}{\partial t} = i\Big(\sum_k \Big(\alpha_k \frac{\partial}{\partial \beta_k} - \beta_k(1+k^2)\frac{\partial}{\partial \alpha_k}\Big) + \sum_k \alpha_k \Big(\frac{1}{2}\sum_{q+q'=k} \frac{k}{q'}\frac{\partial}{\partial \alpha_q}\frac{\partial}{\partial \beta_{q'}} + \frac{k}{q'}\frac{\partial}{\partial \alpha_{q'}}\frac{\partial}{\partial \beta_q}\Big)$$
$$+ \sum_k \beta_k \Big(\frac{1}{2}\sum_{q+q'=k}\frac{k^2}{qq'}\frac{\partial}{\partial \beta_q}\frac{\partial}{\partial \beta_{q'}}\Big)\Big) G. \tag{27}$$

To compare with numerical computations (Fyfe & Montgomery 1978) we look at the simple case of stationary solutions for G such that

$$G = G(\sum_k \alpha_k \alpha_{-k} P_k + \sum_k \beta_k \beta_{-k} Q_k) \tag{28}$$

in which the covariance $P_k \sim \langle \rho_k \rho_{-k} \rangle$, $Q_k \sim \langle \omega_k \omega_{-k} \rangle$ and $\langle \rho_k \omega_{-k} \rangle = 0$ for all k. The equations for the spectra are

$$Q_k = (1+k^2) P_k, \tag{29}$$

$$Q_q Q_{q'} \frac{k^2}{qq'} + Q_k Q_{q'} \frac{q^2}{kq'} + Q_k Q_q \frac{q'^2}{kq} = 0, \tag{30}$$

for all triplets $k, q_1 q'$ with $k+q+q' = 0$, i.e.

$$\frac{1}{qq'k}\Big(\frac{k}{Q_k/k^2} + \frac{q}{Q_q/q^2} + \frac{q'}{Qq'/q'^2}\Big) = 0. \tag{31}$$

There is only one solution given by $Q_k = c$ and $P_k = ck^2/(1+k^2)$. The presence of only one parameter c in this exact statistical equilibrium, is connected with the existence of only one non-zero constant of motion, the energy, guaranteed by $k+q+q' = 0$, the fluctuation energy being distributed equally between the kinetic and electrostatic parts since

$$\langle v_k v_{-k} \rangle = Q_k/k^2 = c \quad \text{and} \quad \langle \rho_k \rho_{-k} \rangle + \langle E_k E_{-k} \rangle = \langle \rho_k \rho_{-k} \rangle (1+k^{-2}) = c,$$

with the electric field spectrum given by $\langle E_k E_{-k} \rangle = c/(1+k^2)$. The numerical verification of the above spectra has been recently given (Fyfe & Montgomery 1978) and the electric field fluctuation spectrum above is essentially the same as that for a Vlasov plasma (Knorr 1977).

The time-asymptotic statistics corresponding to the above solution is easily recognized if one drops the $\rho\omega$ cross terms in (24) and (25). Then since energy is conserved by nonlinear interactions between modes, it is easily shown that the expression $\sum_k (v_{k-}v_k + \rho_k\rho_{-k} + E_k E_{-k})$ is the total energy. Looking at (26) for the electric field alone, this corresponds to the equation

$$\ddot{E} + \dot{E}\dot{E}_x = -E + E_{xx}, \tag{32}$$

where most of the nonlinear terms (of order 3 and 4) in the original equation (26) have been dropped, with the treatment corresponding to the non-interacting limit in which the conserved energy is biquadratic in the fields. In fact from (24) and (25), we have

$$\frac{d\epsilon}{dt} = 0 \quad \text{with} \quad \epsilon = \frac{1}{2}\int dx(v^2 + E^2 + \rho^2),$$

the energy of the system. It is expected that a more refined stochastic treatment of (26) without dropping the nonlinear terms $v^2\rho$ typical of this problem should give rise to time-asymptotic statistics not too different from those calculated up to now.

4. Magnetohydrodynamic equations and classical dynamical system

In this section we relate our theory to that of statistical mechanics associated with the Liouville equation of systems of an infinite number of interacting particles and discuss qualitative properties of the magnetohydrodynamic equations if viewed as a classical dynamical system.

In discussing statistical solutions of the magnetohydrodynamic equations, two parallel approaches are possible: the first approach consists in truncating the basic equations between some maximum and minimum wave vector, studying numerically their solutions with a sufficient set of initial configurations, and investigating some important physical properties which are uniform in L, i.e. properties of the flow described formally by the basic equations, which do not depend on the number L of modes considered. This would justify the numerical approximations. This direction has been followed in recent works (Fyfe *et al.* 1976, 1977*a*, *b*) for the non-dissipative as well as dissipative case with external forcing. Different initial conditions were chosen.

In the second approach, less quantitative, investigations do not rely upon any truncation in k-space. Here one deals with a system with an infinite number of interacting modes analogous to systems with an infinite number of interacting particles in statistical mechanics. It is, therefore, tempting to discuss and exploit this analogy and translate some considerations for Liouville equations of an infinite system of particles to the magnetohydrodynamic equations. The magnetohydrodynamic equations given by (1) are rather formal. To see if they define a classical dynamical system, one should be able to find some probability measure μ on the space of initial conditions Ω (conveniently chosen), which is invariant under the dynamics described formally by the equations themselves (S_t will denote the time evolution). Having found such a measure the question is whether the equations have a meaning valid for all μ and define a flow in such a way that the triplet (Ω, S_t, μ) becomes a classical dynamical system (Gallavotti 1976*a*, *b*).

We now go directly into the situation of magnetohydrodynamics and in the following we choose as field variables the scalar potential ϕ such that

$$\nabla \times \mathbf{u} = (\Delta\phi)(x)$$

and the vector potential a, such that $(\Delta \mathbf{a})(x) = \boldsymbol{\nabla} \times \mathbf{B}$. The choice of Gibbs measure in magnetohydrodynamics is guided by a principle of maximal simplicity and in analogy with the Gibbs measure for a system of interacting particles. It is at this moment that rugged constants came into play; for the two-dimensional magnetohydrodynamics these are the energy E, the cross helicity H and the vector square potential A. They are

$$\left.\begin{aligned} E &= \frac{1}{2}\int_Q |\nabla\phi|^2\, d^2x + \frac{1}{2}\int_Q |\nabla \times \mathbf{a}|^2\, d^2x, \\ H &= \frac{1}{2}\int_Q (\nabla \times \mathbf{a}).(\boldsymbol{\nabla} \times \boldsymbol{\phi})\, d^2x, \\ A &= \frac{1}{2}\int_Q \mathbf{a}^2\, d^2x. \end{aligned}\right\} \tag{33}$$

In some geometry (see, for example, the toroidal case) the enstrophy of the fluid is still conserved and reads

$$S = \frac{1}{2}\int_Q (\Delta\phi)^2\, d^2x.$$

Q denotes a two-dimensional torus $Q = [0, 1] \times [0, 1]$ and for each field we assume the decomposition $\mathbf{f}(x) = \sum_{\mathbf{k}} \mathbf{f}_k e^{i\mathbf{k}\cdot\mathbf{x}}$, $\mathbf{k} = 2\pi\mathbf{n}$, $\mathbf{n} = (n_1, n_2)$, $n_i \in \mathbb{Z}$. We then obtain the basic dynamical equations in the form

$$\left.\begin{aligned} \frac{\partial a_k}{\partial t} &= \sum_{\substack{\mathbf{r},\mathbf{p} \\ \mathbf{k}+\mathbf{r}+\mathbf{p}=0}} \tfrac{1}{2}\hat{e}_z.(\mathbf{r}\times\mathbf{p})\,(a_r\phi_p - \phi_r a_p) = B^a_K(\phi_1 a), \\ \frac{\partial \phi_k}{\partial t} &= \sum_{\substack{\mathbf{r},\mathbf{p} \\ \mathbf{k}+\mathbf{r}+\mathbf{p}=0}} \tfrac{1}{2}\hat{e}_z.(\mathbf{r}\times\mathbf{p})\,(\phi_r\phi_p - a_r a_p)\frac{(r^2-p^2)}{k^2} = B^\phi_K(\phi_1 a). \end{aligned}\right\} \tag{34}$$

The existence of the vector square potential invariant guaranteed by the symmetry property expressed by our equation (5), does not have the same direct physical interpretation as E and H and involves only one (the magnetic) of the field variables. It appears less exotically in investigating basic properties of a discrete modified guiding centre plasma (Calinon & Merlini 1978); moreover, as in the poloidal case, with viscosity and resistivity (Fyfe *et al.* 1977*a*, *b*) it may be used to discuss cascades in the toroidal case.

In the algebraically simple computation of the spectra worked out above, we have not used explicitly any particular measure. We must keep in mind, however, that we were limited to functions G depending bilinearly on ϕ, a or j and ω, and necessarily homogeneous, i.e. $\langle\phi\rangle = \langle a\rangle = 0$. We will consider now the canonical measure. The basic equations for ϕ and a appear as a generalization of the case $a = 0$ and they have essentially the same structure. We omit here the general poloidal case, which may be discussed along the same lines and limit ourselves to the toroidal case (§ 3.2). Since, in this case, the scalar potential is decoupled from the vector potential, a canonical measure which comes into play is constructed with the two invariants T, the kinetic energy and S_f, the vorticity, where

$$S_f(\phi) = \frac{1}{2}\int_Q d^2x f(\Delta\phi),$$

$S_f(\phi)$ coinciding with the enstrophy if $f(\Delta\phi) = (\Delta\phi)^2$. For such a case, in k-space, we then have the product measure

$$\mu_{\beta,\gamma}(\phi) = \prod_{\mathbf{k}\neq(0,0)} [\exp(-(\beta k^2+\gamma k^4)\,|\phi_k|^2)] \Big/ \left(\frac{\pi}{\beta k^2+\gamma k^4}\right)^{-\frac{1}{2}} d^2\phi_k \tag{35}$$

on the space $\mathscr{S}'$ of sequence $\{\phi_k\}$ of rapid decay; a method at our disposal in order to gain some information in our case is that of Lanford (see Gallavotti 1976*a, b*) for a system of an infinite number of particles); the principal requirement in discussing the basic property on existence and uniqueness of flow is to check the inequality

$$\int (d\phi)\,.\,\mu_{\beta\gamma}\,|B_k^\phi(\phi_1 0)| < \infty.$$

This says that the magnetohydrodynamic equations have a meaning at $t = 0$, valid everywhere with respect to $\mu_{\beta\gamma}$. Now the best inequality at our disposal (Gallavotti 1976*a*), which still holds in the toroidal case and has its analogue in the case of a system of interacting particles (Marchioro, Pellegrinotti & Presutti 1975), indicates that for almost all sequences $\{\phi_k\}$, one can suppose that

$$|\phi_k| \leqslant \frac{c(\phi)}{(\beta k^2+\gamma k^4)^{\frac{1}{2}}} \ln|\mathbf{k}|. \tag{36}$$

The above bound on the decreasing of ϕ_k as a function of k is, for mathematical reasons, the best one. Moreover, the set

$$M = \left\{\phi\,|\,\exists c(\phi) \quad \text{with} \quad |\phi_k| < \frac{c(\phi)}{(\beta k^2+\gamma k^4)^{\frac{1}{2}}}\ln|\mathbf{k}|\right\}$$

has probability one with respect to $\mu_{\beta\gamma}$, i.e. $\mu_{\beta,\gamma}(M) = 1$, as may be easily verified by explicit calculation.

Then, following the method, and using the above bound, there is a value λ such that

$$|B_k^\phi(\phi)| < \lambda\, c(\phi)^2 \sum_{\mathbf{p}+\mathbf{r}+\mathbf{k}=0} \ln|\mathbf{p}|\,.\,\ln|\mathbf{r}|\,\frac{(\mathbf{p}\times\mathbf{r})}{\mathbf{k}^2}\,.\,\frac{(p^2-r^2)}{\mathbf{p}^2.\mathbf{r}^2}$$

and the right-hand side diverges; that the existence of solution for the magnetohydrodynamic equations in $\mathscr{S}'$, hold fast everywhere with respect to $\mu_{\beta,\gamma}$ (which is not good enough) cannot be checked. So, even in the toroidal case, in the non-dissipative limit, the problem of proving the existence, everywhere with respect to the Gaussian measure $\mu_{\beta,\gamma}$, of flow on $\mathscr{S}'$ generated by the solutions of the magnetohydrodynamic equations is not solved. It is expected that deviation from the Gaussian measure, even if small, or use of other methods is necessary, which may alter the conclusion above; we believe that this fact should reinforce recent conclusions (Montgomery 1977), that the zero viscosity limit of magnetohydrodynamic or Navier–Stokes equations is not trivial.

5. Conclusion

The time-asymptotic statistics of the two-dimensional magnetohydrodynamic equations in various geometries as well as that of a one-dimensional electron-fluid plasma, have been investigated by use of the associated Hopf equation. From our study, (i) some *homogeneous* stationary solutions have been obtained; (ii) some limiting case such as the Navier–Stokes limit and maximum cross helicity are solutions of the Hopf equation; (iii) in the one-dimensional case of fluid 'turbulence', an exact homogeneous solution has been found which agrees with the results of numerical computations. The procedure described indicates the possibility of extending the above methods to investigate the appearance and the transition to *inhomogeneous* time-asymptotic or equilibrium solutions of the model considered, by techniques borrowed from classical field theory. If, for some values of the basic parameters, the state of the system can change from homogeneous, then statistical inhomogeneous pressure, density or field profiles should appear in analogy with the appearance of inhomogeneous solutions of the BBGKY equilibrium hierarchy for the modified guiding centre plasma.

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