

## PARABOLIC HIGGS BUNDLES AND $\Gamma$ -HIGGS BUNDLES

INDRANIL BISWAS, SOURADEEP MAJUMDER  and MICHAEL LENNOX WONG

(Received 31 August 2012; accepted 5 June 2013; first published online 19 August 2013)

Communicated by D. Chan

### Abstract

We investigate parabolic Higgs bundles and  $\Gamma$ -Higgs bundles on a smooth complex projective variety.

2010 *Mathematics subject classification*: primary 14D23; secondary 14H60.

*Keywords and phrases*: parabolic Higgs bundle, root stack,  $\Gamma$  Higgs bundle.

### 1. Introduction

Let  $X$  be a compact connected Riemann surface and  $E$  a holomorphic vector bundle on  $X$ . The infinitesimal deformations of  $E$  are parametrized by  $H^1(X, \text{End } E)$ , where  $\text{End } E = E \otimes E^*$  is the sheaf of endomorphisms of the vector bundle  $E$ . By Serre duality, we have  $H^1(X, \text{End } E)^* = H^0(X, (\text{End } E) \otimes \Omega_X^1)$ , where  $\Omega_X^1$  is the holomorphic cotangent bundle of  $X$ . A Higgs field on  $E$  is defined to be a holomorphic section of  $(\text{End } E) \otimes \Omega_X^1$ ; they were introduced by Hitchin [Hi87a, Hi87b]. A Higgs bundle is a holomorphic vector bundle equipped with a Higgs field. Hitchin proved that stable Higgs bundles of rank  $r$  and degree zero on  $X$  are in bijective correspondence with the irreducible flat connections on  $X$  of rank  $r$  [Hi87a]. He also proved that the moduli space of Higgs bundles on  $X$  of rank  $r$  is a holomorphic symplectic manifold, and the space of holomorphic functions on this holomorphic symplectic manifold gives it the structure of an algebraically completely integrable system [Hi87b]. Simpson arrived at Higgs bundles via his investigations of variations of Hodge structures [Si88]. He extended the results of Hitchin to Higgs bundles over higher dimensional complex projective manifolds.

A parabolic structure on a holomorphic vector bundle  $E$  on  $X$  is roughly a system of weighted filtrations of the fibers of  $E$  over some finitely many given points. A parabolic vector bundle is a holomorphic vector bundle equipped with a parabolic

---

MLW would like to express his gratitude to the Fonds québécois de la recherche sur la nature et les technologies for the support of a Bourse de recherche postdoctorale (B3).

© 2013 Australian Mathematical Publishing Association Inc. 1446-7887/2013 \$16.00

structure; parabolic vector bundles were introduced by Mehta and Seshadri [MS80]. Parabolic vector bundles with Higgs structure were introduced by Yokogawa [Yo95].

Our aim here is to investigate the parabolic vector bundles equipped with a Higgs structure. More precisely, we study the relationship between the parabolic Higgs bundles and the Higgs vector bundles on a root stack.

Root stacks are important examples of smooth Deligne–Mumford stacks; see [Cad07, Bo07] for root stacks.

Let  $Y$  be a smooth complex projective variety on which a finite group  $\Gamma$  acts as a group of automorphisms satisfying the condition that the quotient  $X = \Gamma \backslash Y$  is also a smooth variety. There is a bijective correspondence between the *parabolic Higgs bundles* on  $X$  and the  $\Gamma$ -*Higgs bundles* on  $Y$ . We prove that the parabolic Higgs bundles on  $X$  are identified with the Higgs bundles on the associated root stack.

The organization of the paper is as follows. In Section 3 we review the notions of parabolic bundle and  $\Gamma$ -vector bundle and define the parabolic Higgs bundle and  $\Gamma$ -Higgs bundles. As was done for parabolic vector bundles in [Bi97] we describe an equivalence between the category of  $\Gamma$ -Higgs bundles on  $Y$  and parabolic Higgs bundle on  $X$ .

Section 4.1 describes the construction of a root stack as done in [Cad07]. In Section 4.2, we investigate vector bundles on root stacks. Section 4.3 generalizes Theorem 3.5 to the case of the root stack over the base space.

## 2. Preliminaries

Let  $Y$  be a smooth complex projective variety of dimension  $m$  endowed with the action  $\lambda : \Gamma \times Y \rightarrow Y$  of a finite group  $\Gamma$  such that:

- (1)  $X := \Gamma \backslash Y$  is also a smooth variety; and
- (2) the projection map  $\pi : Y \rightarrow X$  is a Galois covering with Galois group  $\Gamma$ .

The closed subset of  $Y$  consisting of points with nontrivial isotropy subgroups for the action of  $\Gamma$  is a divisor  $\widetilde{D} \subset Y$  [Bi97, Lemma 2.8]. Let  $D \subset X$  be its (reduced) image under  $\pi$  and  $D = D_1 + \cdots + D_h$  its decomposition into irreducible components. We will always be working with the assumption that the  $D_\mu$ ,  $1 \leq \mu \leq h$ , are smooth, and  $D$  is a normal crossing divisor; this means that all the intersections of the irreducible components of  $D$  are transversal. For an effective divisor  $D'$  on  $Y$ , by  $D'_{\text{red}}$  we will denote the corresponding reduced divisor. So  $D'_{\text{red}}$  is obtained from  $D'$  by setting all the multiplicities to be one. We set

$$\widetilde{D}_\mu := (\pi^* D_\mu)_{\text{red}}, \quad 1 \leq \mu \leq h, \quad \widetilde{D} := \sum_{\mu=1}^h \widetilde{D}_\mu.$$

There exist  $k_\mu, r \in \mathbb{N}$  for  $1 \leq \mu \leq h$  such that  $\pi^* D_\mu = k_\mu r \widetilde{D}_\mu$ ; with this,

$$\pi^* D = r \sum_{\mu=1}^h k_\mu \widetilde{D}_\mu.$$

It should be clarified that there are many choices of  $k_\mu$  and  $r$ . We will write  $\bar{D} := \sum_{\mu=1}^h k_\mu \bar{D}_\mu$ , so that  $\pi^* D = r \bar{D}$ .

**LEMMA 2.1.** *With the assumption that  $D$  is a normal crossing divisor with smooth components, given a point  $y \in Y$  with  $\pi(y) \in \bigcap_1^l D_\mu$ , one can choose coordinates  $w_1, \dots, w_m$  in an analytic neighborhood of  $y$  and  $z_1, \dots, z_m$  in an analytic neighborhood of  $\pi(y)$  such that  $D_\mu$  is defined by  $z_\mu$  for  $1 \leq \mu \leq l$ ,  $\bar{D}_\mu$  is defined by  $w_\mu$  for  $1 \leq \mu \leq l$  and  $\pi$  is given in the local coordinates by*

$$z_1 = w_1^{k_{1r}}, \dots, z_l = w_l^{k_{lr}}, \quad z_{l+1} = w_{l+1}, \dots, z_m = w_m.$$

**PROOF.** This is proved in [Na, page 11, Theorem 1.1.14].  $\square$

Recall [Del70, Section II.3] that the sheaf  $\Omega_X^1(\log D)$  of logarithmic differentials with poles at  $D$  is the locally free sheaf on  $X$ , a basis for which in the neighborhood of a point in  $\bigcap_1^l D_\mu$  with coordinates chosen as in the lemma is given by

$$\frac{dz_\mu}{z_\mu}, 1 \leq \mu \leq l, \quad dz_\mu, l+1 \leq \mu \leq m.$$

Therefore, the dual  $\Omega_X^1(\log D)^*$  is the subsheaf of the holomorphic tangent bundle  $TX$  given by the sheaf of vector fields that preserve  $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ .

We have the following analogue of Hurwitz's theorem.

**LEMMA 2.2.** *With  $\pi : Y \rightarrow X$  as above, one has*

$$\Omega_Y^1(\log \bar{D}) \cong \pi^* \Omega_X^1(\log D).$$

**PROOF.** This follows immediately from [EV92, page 33, Lemma 3.21].  $\square$

Observe that we have inclusions of sheaves

$$\begin{aligned} \Omega_X^1 &\subseteq \Omega_X^1(\log D) \subseteq \Omega_X^1(D) := \Omega_X^1 \otimes \mathcal{O}_X(D), \\ \Omega_Y^1 &\subseteq \Omega_Y^1(\log \bar{D}) \subseteq \Omega_Y^1(\bar{D}) := \Omega_Y^1 \otimes \mathcal{O}_Y(\bar{D}). \end{aligned}$$

Fixing an irreducible component  $D_\mu$  of  $D$ , there is a residue map (see [Del70, Section II.3.7])

$$\text{Res}_{D_\mu} : \Omega_X^1(\log D_\mu) \rightarrow \mathcal{O}_{D_\mu}.$$

In local coordinates  $z_1, \dots, z_m$  on  $X$  where  $D_\mu$  is defined by  $z_\mu = 0$ , if  $\omega$  is a section of  $\Omega_X^1(\log D_\mu)$  with local expression

$$\omega = f_1 dz_1 + \dots + f_\mu \frac{dz_\mu}{z_\mu} + \dots + f_m dz_m,$$

where the  $f_i$ ,  $1 \leq i \leq m$ , are holomorphic functions, then the residue has the local expression

$$\text{Res}_{D_\mu} \omega = f_\mu|_{z_\mu=0}.$$

### 3. Parabolic Higgs bundles and $\Gamma$ -Higgs bundles

**3.1. Parabolic Higgs bundles.** Let  $E$  be a torsion-free coherent sheaf on  $X$ . We recall that a *parabolic structure* on  $E$  with respect to the divisor  $D$  is the data of a filtration

$$E = E_{\alpha_1} \supset E_{\alpha_2} \supset \cdots \supset E_{\alpha_l} \supset E_{\alpha_{l+1}} = E(-D),$$

where  $0 \leq \alpha_1 < \cdots < \alpha_l < 1$  are real numbers called *weights* (see [MY92, Definition 1.2]). The  $\alpha_j$  will be chosen without redundancy in the sense that if  $\epsilon > 0$ , then  $E_{\alpha_j + \epsilon} \neq E_{\alpha_j}$ . We will often shorten  $E_{\alpha_j}$  to  $E_j$ . The sheaf  $E$  together with a parabolic structure is called a *parabolic sheaf* and is often denoted by  $E_*$ . If  $E$  is a locally free sheaf, then we will call  $E_*$  a *parabolic vector bundle*. See [MY92] for more on parabolic sheaves.

We will always assume that the parabolic weights are rational numbers whose denominators all divide  $r \in \mathbb{N}$ , that is,  $\alpha_j \in (1/r)\mathbb{Z}$ , for  $1 \leq j \leq l$ ; this way, we may write  $\alpha_j = m_j/r$  for some integers  $0 \leq m_j \leq r-1$ . It should be clarified that there are many choices for  $r$ . Further, we will make the same assumptions as in [Bi97, Assumptions 3.2].

A *parabolic Higgs field*, respectively *strongly parabolic Higgs field*, will be defined as a section  $\phi \in H^0(X, (\text{End } E) \otimes \Omega_X^1(\log D))$  satisfying

$$\phi \wedge \phi = 0$$

and

$$(\text{Res}_{D_\mu} \phi)(E_j|_{D_\mu}) \subseteq E_j|_{D_\mu}, \text{ respectively } (\text{Res}_{D_\mu} \phi)(E_j|_{D_\mu}) \subseteq E_{j+1}|_{D_\mu}, \quad (3.1)$$

for  $1 \leq j \leq l, 1 \leq \mu \leq h$ . By a *parabolic Higgs bundle* we will mean a pair  $(E_*, \phi)$  consisting of a parabolic vector bundle  $E_*$  and a strongly parabolic Higgs field  $\phi$ .

**REMARK 3.1.** Observe that this definition of a parabolic Higgs field differs from that given in [Yo95, Definition 2.2], where one takes  $\phi \in H^0(X, (\text{End } E) \otimes \Omega_X^1(D))$ .

**3.2.  $\Gamma$ -Higgs bundles.** Let  $W$  be a vector bundle on  $Y$  admitting an action  $\Lambda : \Gamma \times W \rightarrow W$  compatible with the action  $\lambda$  on  $Y$ . If we think of  $W$  as a space with projection  $r : W \rightarrow Y$ , then this means that

$$\begin{array}{ccc} \Gamma \times W & \xrightarrow{\Lambda} & W \\ \mathbb{1}_\Gamma \times r \downarrow & & \downarrow r \\ \Gamma \times Y & \xrightarrow{\lambda} & Y \end{array}$$

commutes. Alternatively, if we think of  $W$  as a locally free sheaf then this means that there is an isomorphism

$$L : \lambda^* W \xrightarrow{\sim} p_Y^* W$$

of sheaves on  $\Gamma \times Y$  satisfying a suitable cocycle condition. When such an action exists, we will call  $W$  a  $\Gamma$ -vector bundle. In this realization, if  $W'$  is another  $\Gamma$ -vector

bundle with  $L' : \lambda^* W' \rightarrow p_Y^* W'$  giving the action on  $W'$ , then compatible actions on  $W \oplus W'$  and  $W \otimes W'$  are readily defined since direct sums and tensor products commute with pullbacks.

For each  $\gamma \in \Gamma$ , the restrictions  $L_{\{\gamma\} \times Y} : \lambda_\gamma^* W \xrightarrow{\sim} W$  yield isomorphisms  $L_\gamma : W \rightarrow \lambda_{\gamma*} W$  (by adjunction) satisfying

$$L_e = \mathbb{1}_W \quad \text{and} \quad \lambda_{\gamma*} L_\delta \circ L_\gamma = L_{\gamma\delta}$$

for all  $\gamma, \delta \in \Gamma$ . In our case, since  $\Gamma$  is discrete, knowledge of the  $L_\gamma$  is enough to reconstruct  $L$ .

**EXAMPLE 3.2.** There are three examples of  $\Gamma$ -bundles that will be of particular interest to us.

- (a) The action  $\lambda$  on  $Y$  induces a natural action on the sheaf of differentials  $\Omega_Y^1$  which will be compatible with  $\lambda$ .
- (b) Since  $X = \Gamma \backslash Y$ , we have  $\pi \circ \lambda = \pi \circ p_Y$  as maps  $\Gamma \times Y \rightarrow X$ . Thus, if  $E$  is any vector bundle on  $X$ , there is a canonical isomorphism  $\lambda^* \pi^* E \xrightarrow{\sim} p_Y^* \pi^* E$ . Hence the pullback  $\pi^* E$  carries a  $\Gamma$ -action for which the action on the fibers is induced by the action on  $Y$ .
- (c) By the previous example,  $\mathcal{O}_Y(\pi^* D) = \pi^* \mathcal{O}_X(D)$  carries a compatible  $\Gamma$ -action. Since  $\widetilde{D} \subseteq \pi^* D$  is a  $\Gamma$ -invariant subset we have an induced action on the line bundle  $\mathcal{O}_Y(\widetilde{D})$  making it into a  $\Gamma$ -line bundle.

Let  $W, W'$  be as above. A homomorphism  $\Phi : W \rightarrow W'$  is said to *commute with the  $\Gamma$ -actions* or is a  $\Gamma$ -homomorphism if the diagram

$$\begin{array}{ccc} \lambda^* W & \xrightarrow{\lambda^* \Phi} & \lambda^* W' \\ L \downarrow & & \downarrow L' \\ p_Y^* W & \xrightarrow{p_Y^* \Phi} & p_Y^* W' \end{array}$$

commutes.

If  $\Phi \in H^0(Y, (\text{End } W) \otimes \Omega_Y^1)$  is a Higgs field on  $W$ , that is,  $\Phi \wedge \Phi = 0$ , then we will call it a  $\Gamma$ -Higgs field if as a map  $W \rightarrow W \otimes \Omega_Y^1$  it commutes with the  $\Gamma$ -actions, where  $W \otimes \Omega_Y^1$  has the tensor product action. Thus, for every  $\gamma \in \Gamma$ , there is a commutative diagram.

$$\begin{array}{ccc} W & \xrightarrow{\Phi} & W \otimes \Omega_Y^1 \\ L_\gamma \downarrow & & \downarrow \bar{L}_\gamma \\ \lambda_{\gamma*} W & \xrightarrow{\lambda_{\gamma*} \Phi} & \lambda_{\gamma*} (W \otimes \Omega_Y^1) \end{array} \quad (3.2)$$

If  $\Phi$  is a  $\Gamma$ -Higgs field, the pair  $(W, \Phi)$  will be referred to as a  $\Gamma$ -Higgs bundle.

**3.3. From  $\Gamma$ -Higgs bundles to parabolic Higgs bundles.** We now begin with a  $\Gamma$ -Higgs bundle  $(W, \Phi)$  and from it construct a parabolic Higgs bundle  $(E_*, \phi)$ . The underlying vector bundle  $E$  is defined as  $E := \pi_* W^\Gamma$ , the sheaf of  $\Gamma$ -invariant sections of  $\pi_* W$ , and as in [B97, Section 2c], the parabolic structure on  $E$  is defined by

$$E_j := \pi_* W \left( \sum_{\mu=1}^h \lfloor -k_\mu r \alpha_j \rfloor \widetilde{D}_\mu \right)^\Gamma.$$

Suppose  $\Phi \in H^0(Y, (\text{End } W) \otimes \Omega_Y^1)$  is a  $\Gamma$ -Higgs field on  $W$ . We will think of  $\Phi$  as a homomorphism  $\Phi : W \longrightarrow W \otimes \Omega_Y^1$ . Since  $\Omega_Y^1 \subseteq \Omega_Y^1(\log \widetilde{D})$ ,

$$\pi_*(W \otimes \Omega_Y^1) \subseteq \pi_*(W \otimes \Omega_Y^1(\log \widetilde{D})) = \pi_*(W \otimes \pi^* \Omega_X^1(\log D)) = \pi_* W \otimes \Omega_X^1(\log D),$$

where the first equality is due to Lemma 2.2 and the last step by the projection formula. Therefore,  $\phi := \pi_* \Phi$  may be considered as a map  $\pi_* W \longrightarrow \pi_* W \otimes \Omega_X^1(\log D)$ , and we have a candidate for a parabolic Higgs field.

Let  $U \subseteq X$  be open and let  $s$  be an invariant section of  $\pi_* W$  over  $U$ , so that we may think of  $s$  as a section  $\widehat{s}$  of  $W$  over  $\pi^{-1}(U)$  with  $L_\gamma \widehat{s} = \widehat{s}$  for all  $\gamma \in \Gamma$ . Then by definition  $\phi s := \widehat{\Phi \widehat{s}}$ , and for  $\gamma \in \Gamma$ , using (3.2),

$$\widetilde{L}_\gamma(\phi s) = \widetilde{L}_\gamma(\widehat{\Phi \widehat{s}}) = \widehat{\Phi(L_\gamma \widehat{s})} = \widehat{\Phi \widehat{s}} = \phi s,$$

so  $\phi s$  is a  $\Gamma$ -invariant section, and hence  $\phi : E \longrightarrow E \otimes \Omega_X^1(\log D)$ .

**PROPOSITION 3.3.** *To a  $\Gamma$ -Higgs bundle  $(W, \Phi)$  there is a naturally associated parabolic Higgs bundle  $(E_*, \phi)$ .*

**PROOF.** We have constructed  $(E_*, \phi)$ . We must prove that  $\phi$  is strongly parabolic. This is a condition on the residues of  $\phi$  along the components of the divisor  $D$ , so we may concentrate on those points of  $D_\mu$  that do not belong to any other component of  $D$ . Therefore, we may assume that we are in the neighborhood of a point  $y$  of  $\widetilde{D}_1$  that lies on no other  $\widetilde{D}_\mu$ . In this neighborhood, for  $1 \leq j \leq l$ ,

$$E_j = \pi_* W(-m_j k_1 \widetilde{D}_1)^\Gamma.$$

We now choose coordinates on  $Y$  and  $X$  as in Lemma 2.1, so that the divisor  $\widetilde{D}_1$  is defined by  $w_1$  and the divisor  $D_1$  is defined by  $z_1$ ; we will write  $p := k_1 r$  so that  $\pi$  is given in these coordinates by

$$z_1 = w_1^p, \quad z_2 = w_2, \dots, z_m = w_m.$$

In these coordinates, near  $y$ , we may write

$$\Phi = A_1 dw_1 + \dots + A_m dw_m$$

for some holomorphic sections  $A_i$  of  $\text{End } W$ . We may then consider  $A_1 dw_1 = (1/p)A_1 w_1 dz_1/z_1$  as a locally defined map  $W \rightarrow W \otimes \mathcal{O}_Y(-\widetilde{D}_1) \otimes \pi^* \Omega_X^1(\log D_1)$ , or more generally, as a map

$$W(-m_j k_1 \widetilde{D}_1) \rightarrow W(-(m_j k_1 + 1) \widetilde{D}_1) \otimes \pi^* \Omega_X^1(\log D_1)$$

for  $1 \leq j \leq l$ . It is easily verified that  $W(-(m_j k_1 + 1) \widetilde{D}_1) \subseteq W(-rk_1(\alpha_j + \epsilon) \widetilde{D}_1)$ , where  $0 \leq \epsilon \leq 1/rk_1$ . So taking invariants we see that  $\pi_* A_1 dw_1 = (1/p)A_1 w_1 dz_1/z_1$  gives a locally defined map

$$E_j \rightarrow E_{\alpha_j + (1/rk_1)} \otimes \Omega_X^1(\log D_1) = E_{j+1} \otimes \Omega_X^1(\log D_1).$$

Since, by definition,

$$\text{Res}_{D_1} \phi = \frac{1}{p} w_1 A_1|_{z_1=0},$$

and noting that  $(\text{Res}_{D_1} \phi)(E_j|_{D_1}) \subseteq E_{j+1}|_{D_1}$ , it follows that the strong parabolicity condition (3.1) is satisfied.

That  $\phi \wedge \phi = 0$  is easily seen, since if  $s$  is any section of  $E$ , then

$$(\phi \wedge \phi)s = (\Phi \wedge \Phi)\widehat{s} = 0$$

since  $\Phi \wedge \Phi = 0$ . □

**3.4. From parabolic Higgs bundles to  $\Gamma$ -Higgs bundles.** Recall that we are working under assumptions as in [Bi97, Assumptions 3.2], hence we can use the construction from [Bi97, Section 3b] in the following proposition.

**PROPOSITION 3.4.** *Given a parabolic Higgs bundle  $(E_*, \phi)$  on  $X$ , we can associate a  $\Gamma$ -Higgs bundle  $(W, \Phi)$  on  $Y$ .*

**PROOF.** We will begin by constructing a parabolic vector bundle on  $X$  of rank  $m$ . The holomorphic vector bundle underlying the parabolic vector bundle is  $\Omega_X^1$ . To define the parabolic structure, take any irreducible component  $D_i$  of  $D$ . Let  $\iota: D_i \hookrightarrow X$  be the inclusion map. We have a short exact sequence of vector bundles on  $D_i$

$$0 \rightarrow N_{D_i}^* \rightarrow \iota^* \Omega_X^1 \rightarrow \Omega_{D_i}^1 \rightarrow 0,$$

where  $N_{D_i}$  is the normal bundle of  $D_i$ . Note that the Poincaré adjunction formula says that  $N_{D_i} = \iota^* \mathcal{O}_X(D_i)$ . The quasiparabolic filtration over  $D_i$  is the above filtration

$$N_{D_i}^* \subset \iota^* \Omega_X^1.$$

The parabolic weights are 0 and  $(rk_i - 1)/rk_i$ . More precisely,  $N_{D_i}^*$  has parabolic weight  $(rk_i - 1)/rk_i$  and the parabolic weight of the quotient  $\Omega_{D_i}^1$  is zero. Note that the nonzero parabolic weight  $(rk_i - 1)/rk_i$  has multiplicity one. This parabolic vector bundle will be denoted by  $\widetilde{\Omega}_X^1$ .

The action of  $\Gamma$  on  $Y$  induces an action of  $\Gamma$  on the vector bundle  $\Omega_Y^1$  making it a  $\Gamma$ -bundle. From the construction of  $\widetilde{\Omega}_X^1$  it can be deduced that the parabolic vector

bundle corresponding to the  $\Gamma$ -bundle  $\Omega_Y^1$  is  $\widetilde{\Omega}_X^1$ . To prove this, first note that if  $U_0 \subset X$  is a Zariski open subset such that the complement  $U_0^c$  is of codimension at least two, and  $V, W$  are two algebraic vector bundles on  $X$  that are isomorphic over  $U_0$ , then  $V$  and  $W$  are isomorphic over  $X$ ; using Hartog's theorem, any isomorphism  $V|_{U_0} \rightarrow W|_{U_0}$  extends to a homomorphism  $V \rightarrow W$ , and similarly, we have a homomorphism  $W \rightarrow V$ , and these two homomorphisms are inverses of each other because they are so over  $U_0$ . Next note that

$$(\pi_* \Omega_Y^1)^\Gamma = \Omega_X^1$$

because  $(\pi_* \Omega_Y^1)^\Gamma = \Omega_X^1$  over the complement of the singular locus of  $D$ . Therefore,  $\Omega_X^1$  is the vector bundle underlying the parabolic bundle corresponding to the  $\Gamma$ -bundle  $\Omega_Y^1$ . It is now straightforward to check that the parabolic weights are of the above type.

Let  $W$  be the  $\Gamma$ -bundle on  $Y$  corresponding to the parabolic vector bundle  $E_*$  on  $X$  (using [Bi97, Section 3b]). Let  $\phi$  be a strongly parabolic Higgs field on  $E_*$ . It is straightforward to check that  $\phi$  defines a homomorphism of parabolic vector bundles

$$\phi' : E_* \rightarrow E_* \otimes \widetilde{\Omega}_X^1,$$

where  $E_* \otimes \widetilde{\Omega}_X^1$  is the parabolic tensor product of  $E_*$  and  $\widetilde{\Omega}_X^1$ .

Since the correspondence between parabolic bundles and  $\Gamma$ -vector bundles is compatible with the operation of tensor product, we conclude that the parabolic tensor product  $E_* \otimes \widetilde{\Omega}_X^1$  corresponds to the  $\Gamma$ -bundle  $W \otimes \Omega_Y^1$ . Therefore, the above homomorphism  $\phi'$  pulls back to a  $\Gamma$ -equivariant homomorphism  $\Phi$  from  $W$  to  $W \otimes \Omega_Y^1$ .  $\square$

**THEOREM 3.5.** *We have an equivalence of categories between  $\Gamma$ -Higgs bundles on  $Y$  and parabolic Higgs bundles on  $X$  which satisfy the assumptions as in [Bi97, Assumptions 3.2].*

**PROOF.** Proof is clear from Propositions 3.3 and 3.4 and [Bi97, Sections 2c, 3b].  $\square$

**REMARK 3.6.** In Borne's formalism, the parabolic bundle  $E_*$  may be considered as a functor  $((1/r)\mathbb{Z})^{\text{op}} \rightarrow \mathfrak{Vect}(X)$ , with

$$\frac{j}{r} \mapsto E_j(D),$$

and composing with  $\pi^* : \mathfrak{Vect}(X) \rightarrow \mathfrak{Vect}(Y)$ , we get a functor  $((1/r)\mathbb{Z})^{\text{op}} \rightarrow \mathfrak{Vect}(Y)$ . We also have a covariant functor  $(1/r)\mathbb{Z} \rightarrow \mathfrak{Vect}(Y)$  given by

$$\frac{j}{r} \mapsto \mathcal{O}_Y(m_{j-1}\overline{D}).$$

Therefore, we obtain a functor  $((1/r)\mathbb{Z})^{\text{op}} \times \frac{1}{r}\mathbb{Z} \rightarrow \mathfrak{Vect}(Y)$

$$\frac{j}{r} \mapsto \pi^* E_j(D) \otimes \mathcal{O}_Y(m_{j-1}\overline{D}).$$



An end for this functor [Ma98, Section IX.5] consists of a vector bundle  $V \in \text{Ob}\mathfrak{Vect}(Y)$  and diagrams for  $i \leq j$

$$\begin{array}{ccc}
 & \pi^* E_i(D) \otimes \mathcal{O}_Y(m_{i-1} \bar{D}) & \\
 V \swarrow & & \searrow \\
 & \pi^* E_i(D) \otimes \mathcal{O}_Y(m_{j-1} \bar{D}) & \\
 V \searrow & & \swarrow \\
 & \pi^* E_j(D) \otimes \mathcal{O}_Y(m_{j-1} \bar{D}) &
 \end{array}$$

such that the diagram is terminal among all such diagrams. It is not difficult to check that  $W$  is a universal end for the functor defined above, that is, it is an end, and given an end  $V$  as in the diagram, there is a unique morphism  $V \rightarrow W$  which yields the appropriate commuting diagrams.

#### 4. Root stacks

The notion of a root stack is something of a generalization of the notion of an orbifold with cyclic isotropy groups over a divisor. Of course, our main interest in this construction is in the case when  $X$  is a smooth complex projective variety, but giving the definition for an arbitrary  $\mathbb{C}$ -scheme imposes no further conceptual or technical difficulties, so we will give the definition and describe some of the basic properties in this generality. We largely follow the presentations of [Bo07] and [Cad07] here (as well as [The], [Vis08] for generalities), so we direct the reader requiring further illumination on issues raised below to these references.

**4.1. Definition and construction.** We fix a  $\mathbb{C}$ -scheme  $X$ , an invertible sheaf  $L$  on  $X$  and  $s \in H^0(X, L)$ , so that if  $s$  is nonzero, it defines an effective divisor  $D$  on  $X$ . We will also fix  $r \in \mathbb{N}$ . Let  $\mathfrak{X} = \mathfrak{X}_{(L,r,s)}$  denote the category whose objects are quadruples

$$(f : U \rightarrow X, N, \phi, t), \quad (4.1)$$

where  $U$  is a  $\mathbb{C}$ -scheme,  $f$  is a morphism of  $\mathbb{C}$ -schemes,  $N$  is an invertible sheaf on  $U$ ,  $t \in H^0(U, N)$  and  $\phi : N^{\otimes r} \xrightarrow{\sim} f^* L$  is an isomorphism of invertible sheaves with  $\phi(t^{\otimes r}) = f^* s$ . A morphism

$$(f : U \rightarrow X, N, \phi, t) \rightarrow (g : V \rightarrow X, M, \psi, u)$$

consists of a pair  $(k, \sigma)$ , where  $k : U \rightarrow V$  is a  $\mathbb{C}$ -morphism making

$$\begin{array}{ccc}
 U & \xrightarrow{k} & V \\
 f \searrow & & \swarrow g \\
 & X &
 \end{array}$$

commute and  $\sigma : N \xrightarrow{\sim} k^*M$  is an isomorphism such that  $\sigma(t) = k^*(u)$ . Moreover, the following diagram must commute:

$$\begin{array}{ccc} N^{\otimes r} & \xrightarrow{\phi} & f^*L \\ \sigma^{\otimes r} \downarrow & & \downarrow \text{can} \\ k^*M^{\otimes r} & \xrightarrow{k^*\psi} & k^*g^*L \end{array}$$

If

$$(g : V \longrightarrow X, M, \psi, u) \xrightarrow{(l, \tau)} (h : W \longrightarrow X, J, \rho, v)$$

is another morphism, then the composition is defined as

$$(l, \tau) \circ (k, \sigma) := (l \circ k, k^*\tau \circ \sigma), \quad (4.2)$$

using the canonical isomorphism  $(l \circ k)^*J \cong k^*l^*J$ .

We will often use the symbols  $\mathfrak{f}, g$  to denote objects of  $\mathfrak{X}$ . If it is understood that  $\mathfrak{f} \in \mathfrak{X}_U$ , then by  $\mathfrak{f}$  we will denote the quadruple  $\mathfrak{f} = (f : U \longrightarrow X, N_{\mathfrak{f}}, \phi_{\mathfrak{f}}, t_{\mathfrak{f}})$ .

The category  $\mathfrak{X}$  comes with a functor  $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$  which simply takes  $\mathfrak{f}$  to  $U$  and  $(k, \sigma)$  to  $h$ .

**PROPOSITION 4.1** [Cad07, Theorem 2.3.3]. *The morphism of categories  $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$  makes  $\mathfrak{X}$  a Deligne–Mumford stack.*

**REMARK 4.2.** The previous statement implies that  $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$  is a category fibered in groupoids. Let  $\mathfrak{f} \in \text{Ob } \mathfrak{X}_U$  be an object of  $\mathfrak{X}$  lying over  $U$  as given in (4.1) and let  $g : V \longrightarrow U$  be a morphism of schemes. A choice of pullback  $g^*\mathfrak{f} \in \text{Ob } \mathfrak{X}_V$  can easily be described by the tuple

$$(g \circ f : V \longrightarrow X, g^*N_{\mathfrak{f}}, g^*\phi_{\mathfrak{f}}, g^*t_{\mathfrak{f}})$$

and the Cartesian arrow  $g^*\mathfrak{f} \longrightarrow \mathfrak{f}$  is given by  $(g, \mathbb{1}_{g^*N_{\mathfrak{f}}})$ .

**EXAMPLE 4.3** [Cad07, Example 2.4.1]. Suppose  $X = \text{Spec } A$  is an affine scheme,  $L = \mathcal{O}_X$  is the trivial bundle and  $s \in H^0(X, \mathcal{O}_X) = A$  is a function. Consider  $U = \text{Spec } B$ , where  $B = A[t]/(t^r - s)$ . Then  $U$  admits an action of the group of  $r$ th roots of unity (more precisely, of the group scheme of the  $r$ th roots of unity)  $\mu_r$  of order  $r$ , where the induced action of  $\zeta \in \mu_r$ , a generator, is given by

$$\zeta \cdot a = a, a \in A, \quad \zeta \cdot t = \zeta^{-1}t.$$

In this case, the root stack  $\mathfrak{X}_{(\mathcal{O}_X, s, r)}$  coincides with the quotient stack  $[U/\mu_r]$ . Thus, as a quotient by a finite group (scheme), the map  $U \longrightarrow \mathfrak{X}$  is an étale cover.

**REMARK 4.4.** If  $X$  is any  $\mathbb{C}$ -scheme, and  $L, s$  are as before, we may take an open affine cover  $\{X_i = \operatorname{Spec} A_i\}$  such that  $L|_{X_i} \cong \mathcal{O}_{X_i}$  and  $s|_{X_i}$  corresponds to  $s_i \in A_i$ . Then by the example above,

$$\coprod_i U_i \longrightarrow \mathfrak{X}$$

is an étale cover, where  $U_i = \operatorname{Spec} A[t_i]/(t_i^r - s_i)$ .

There is also a functor  $\pi : \mathfrak{X} \longrightarrow \mathfrak{Sch}/X$ , whose action on objects and morphisms is given by

$$\mathfrak{f} \longmapsto f : U \longrightarrow X, \quad (k, \sigma) \longmapsto k;$$

this yields a 1-morphism over  $\mathfrak{Sch}/\mathbb{C}$ , which we will often simply write as  $\pi : \mathfrak{X} \longrightarrow X$ .

**4.2. Vector bundles and differentials on a root stack.** Recall (for example [G01, Definition 2.50], [LM00, Lemme 12.2.1], [Vis89, Definition 7.18]) that a quasi-coherent sheaf  $\mathcal{F}$  on  $\mathfrak{X}$  consists of the data of a quasi-coherent sheaf  $\mathcal{F}_{\mathfrak{f}}$  for each étale morphism  $\mathfrak{f} : U \longrightarrow \mathfrak{X}$  along with isomorphisms  $\alpha_k = \alpha_k^{\mathcal{F}} : \mathcal{F}_{\mathfrak{f}} \longrightarrow k^* \mathcal{F}_{\mathfrak{g}}$  for any commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{k} & V \\ & \searrow \mathfrak{f} & \swarrow \mathfrak{g} \\ & \mathfrak{X} & \end{array} \quad (4.3)$$

such that for a composition  $U \xrightarrow{k} V \xrightarrow{h} W \longrightarrow \mathfrak{X}$  one has

$$\alpha_{h \circ k} = k^* \alpha_h \circ \alpha_k. \quad (4.4)$$

A (global) section  $s \in H^0(\mathfrak{X}, \mathcal{F})$  of  $\mathcal{F}$  over  $\mathfrak{X}$  is the data of a global section  $s_{\mathfrak{f}} \in H^0(U, \mathcal{F}_{\mathfrak{f}})$  for each étale morphism  $\mathfrak{f} : U \longrightarrow \mathfrak{X}$  such that for a diagram (4.3) as above, one has

$$\alpha_k(s_{\mathfrak{f}}) = k^* s_{\mathfrak{g}}.$$

A quasi-coherent sheaf  $\mathcal{F}$  on  $\mathfrak{X}$  is a subsheaf of a quasi-coherent sheaf  $\mathcal{G}$  if  $\mathcal{F}_{\mathfrak{f}} \subseteq \mathcal{G}_{\mathfrak{f}}$  for all étale  $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ .

**LEMMA 4.5.** *In the situation of Example 4.3, where  $X = \operatorname{Spec} A$  is affine,  $U := \operatorname{Spec} A[t]/(t^r - s)$  and  $\mathfrak{X} = [U/\mu]$ , then for a quasi-coherent sheaf  $\mathcal{F}$  on  $\mathfrak{X}$ ,  $\mathcal{F}_U$  admits a  $\mu_r$ -action compatible with that on  $U$ .*

**PROOF.** We have  $U \times_{\mathfrak{X}} U \cong U \times \mu$  and under this isomorphism, the two projection maps from  $U \times_{\mathfrak{X}} U$  correspond to the maps  $p_U, \lambda : U \times \mu \longrightarrow U$ , where  $p_U$  is the projection onto  $U$  and  $\lambda$  is the action on  $U$ . Then the required action is defined by the composition

$$p_U^* \mathcal{F}_U \xrightarrow{\alpha_{p_U}^{-1}} \mathcal{F}_{U \times_{\mathfrak{X}} U} \xrightarrow{\alpha_{\lambda}} \lambda^* \mathcal{F}_U.$$

This concludes the proof.  $\square$

**4.2.1. The sheaf of differentials on  $\mathfrak{X}$ .** The sheaf of differentials  $\Omega_{\mathfrak{X}}^1 = \Omega_{\mathfrak{X}/\mathbb{C}}^1$  can be defined as follows. If  $\mathfrak{f} : U \rightarrow \mathfrak{X}$  is an étale map, then we simply set

$$\Omega_{\mathfrak{X}, \mathfrak{f}}^1 := \Omega_{U/\mathbb{C}}^1.$$

If we are given a diagram (4.3), then from the composition  $U \xrightarrow{k} V \rightarrow \text{Spec } \mathbb{C}$ , one obtains a sequence

$$0 \rightarrow k^* \Omega_{V/\mathbb{C}} \rightarrow \Omega_{U/\mathbb{C}} \rightarrow \Omega_{U/V} \rightarrow 0,$$

which is left exact [The, More on Morphisms, Ch. 33, Lemma 9.9] and whose last term is zero since  $k$  is necessarily étale. This defines isomorphisms  $\alpha_k$ . The requirement (4.4) will be met because of the universal properties these morphisms possess.

**4.2.2. The tautological invertible sheaf on  $\mathfrak{X}$ .** The root stack  $\mathfrak{X}$  possesses a tautological invertible sheaf  $\mathcal{N}$ . For an étale morphism  $\mathfrak{f} : U \rightarrow \mathfrak{X}$ , we simply take

$$\mathcal{N}_{\mathfrak{f}} := N_{\mathfrak{f}}.$$

Given a diagram (4.3), one has an isomorphism  $(\mathbb{1}_U, \sigma) : \mathfrak{f} \rightarrow k^*g$  and one may take

$$\alpha_k^{\mathcal{N}} := \sigma : N_{\mathfrak{f}} \rightarrow k^*N_g.$$

The expression in the second component of (4.2) implies that (4.4) is satisfied. This defines the invertible sheaf  $\mathcal{N}$  on  $\mathfrak{X}$ . Furthermore, by definition, we also get a tautological section  $t$  of  $\mathcal{N}$  over  $\mathfrak{X}$  by simply taking

$$t_{\mathfrak{f}} := t_{\mathfrak{f}}.$$

**4.3. Higgs fields on root stacks.** Let  $X$  be as in Section 2, so that it is a smooth complex projective variety;  $D$  will be a normal crossing divisor with smooth components. Let  $s \in H^0(X, \mathcal{O}_X(D))$  be a section with  $(s) = D$ . We also fix  $r \in \mathbb{N}$ . In all that follows  $\mathfrak{X} = \mathfrak{X}_{(\mathcal{O}_X(D), r, s)}$  will be the associated root stack as constructed in Section 4.1.

**REMARK 4.6.** Consider the fuller situation of Section 2, where  $\pi : Y \rightarrow X$  be a Galois cover of smooth complex projective varieties and there is a divisor  $\overline{D}$  on  $Y$  such that  $\pi^*D = r\overline{D}$ . Then there is an isomorphism  $\phi : \mathcal{O}_Y(\overline{D})^{\otimes r} \xrightarrow{\sim} \pi^*\mathcal{O}_X(D)$  and a section  $t \in H^0(Y, \mathcal{O}_Y(\overline{D}))$  such that  $\phi(t^{\otimes r}) = \pi^*s$ . Therefore, the quadruple  $(\pi : Y \rightarrow X, \mathcal{O}_Y(\overline{D}), \phi, t)$  defines a morphism

$$\widehat{\pi} : Y \rightarrow \mathfrak{X}.$$

**4.3.1. Higgs fields.** Let  $\mathcal{V}$  be a vector bundle on  $\mathfrak{X}$ . A *Higgs field*  $\Phi$  on  $\mathcal{V}$  is a homomorphism  $\Phi : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_{\mathfrak{X}}^1$ . This means that for each étale morphism  $\mathfrak{f} : U \rightarrow \mathfrak{X}$  we have a homomorphism  $\Phi_{\mathfrak{f}} : \mathcal{V}_{\mathfrak{f}} \rightarrow \mathcal{V}_{\mathfrak{f}} \otimes \Omega_U^1$  such that given a diagram (4.3), we

obtain a commutative square.

$$\begin{array}{ccc}
 \mathcal{V}_{\mathfrak{f}} & \xrightarrow{\Phi_{\mathfrak{f}}} & \mathcal{V}_{\mathfrak{f}} \otimes \Omega_U^1 \\
 \alpha_k^{\mathcal{V}} \downarrow & & \downarrow \alpha_k^{\mathcal{V}} \otimes \alpha_k^{\Omega^1} \\
 k^* \mathcal{V}_{\mathfrak{g}} & \xrightarrow{k^* \Phi_{\mathfrak{g}}} & k^* \mathcal{V}_{\mathfrak{f}} \otimes k^* \Omega_V^1
 \end{array} \quad (4.5)$$

**THEOREM 4.7.** *There is an equivalence of categories of Higgs bundles on  $\mathfrak{X}$  and parabolic Higgs bundles on  $X$ .*

**PROOF.** We remark that a parabolic structure is given locally, so we may assume that  $X = \text{Spec} A$  is affine and that the parabolic divisor  $D$  is defined by  $s \in A$ . Then as in Example 4.3, we may take  $U = \text{Spec} B$  where  $B = A[t]/(t^r - s)$ , so that  $\mathfrak{X} = [U/\mu]$ . In this case, the map  $\mathfrak{f}: U \rightarrow \mathfrak{X}$  is étale; we will write  $f: U \rightarrow X$  for the underlying map induced from  $A \rightarrow B$ . Given a vector bundle  $\mathcal{V}$ , by Lemma 4.5, the bundle  $\mathcal{V}_{\mathfrak{f}}$  on  $U$  carries a compatible  $\mu$ -action. The fact that  $\Phi_{\mathfrak{f}}$  commutes with this action comes from the existence of the diagram (4.5) for the two projection morphisms  $U \times_{\mathfrak{X}} U \rightarrow U$ . Thus, we are reduced to the case of  $\Gamma$ -bundles when  $\Gamma = \mu$ , which comes from Propositions 3.3 and 3.4.  $\square$

### Acknowledgement

We are very grateful to the referee for helpful comments to improve the exposition.

### References

- [Bi97] I. Biswas, ‘Parabolic bundles as orbifold bundles’, *Duke Math. J.* **88**(2) (1997), 305–325.
- [Bo07] N. Borne, ‘Fibrés paraboliques et champ des racines’, *Int. Math. Res. Not.* **2007**(16) (2007), Article ID rnm049, 38 pages.
- [Cad07] C. Cadman, ‘Using stacks to impose tangency conditions on curves’, *Amer. J. Math.* **129**(2) (2007), 405–427.
- [Del70] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Mathematics, 163 (Springer, Berlin, 1970).
- [EV92] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, DMV Seminar, 20 (Birkhäuser, Basel, 1992).
- [G01] T. L. Gómez, ‘Algebraic stacks’, *Proc. Indian Acad. Sci. Math. Sci.* **111**(1) (2001), 1–31.
- [Hi87a] N. J. Hitchin, ‘The self-duality equations on a Riemann surface’, *Proc. Lond. Math. Soc.* **55** (1987), 59–126.
- [Hi87b] N. J. Hitchin, ‘Stable bundles and integrable systems’, *Duke Math. J.* **54**(1) (1987), 91–114.
- [LM00] G. Laumon and L. Moret-Bailly, *Champs Algébriques* (Springer, Berlin, 2000).
- [Ma98] S. Mac Lane, *Categories for the Working Mathematician*, 2nd edn. Graduate Texts in Mathematics, 5 (Springer, New York, 1998).
- [MY92] M. Maruyama and K. Yokogawa, ‘Moduli of parabolic stable sheaves’, *Math. Ann.* **293** (1992), 77–99.
- [MS80] V. Mehta and C. Seshadri, ‘Moduli of vector bundles on curves with parabolic structures’, *Math. Ann.* **248** (1980), 205–239.
- [Na] M. Namba, *Branched Coverings and Algebraic Functions*, Pitman Research Notes in Mathematics, 161 (Longman Scientific & Technical, Harlow, 1987).

- [Si88] C. T. Simpson, ‘Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization’, *J. Amer. Math. Soc.* **1** (1988), 867–918.
- [The] The stacks project authors, *Stacks Project*, [http://math.columbia.edu/algebraic\\_geometry/stacks-git](http://math.columbia.edu/algebraic_geometry/stacks-git).
- [Vis89] A. Vistoli, ‘Intersection theory on algebraic stacks and on their moduli spaces’, *Invent. Math.* **97** (1989), 613–670.
- [Vis08] A. Vistoli, Notes on Grothendieck topologies, fibred categories and descent theory, <http://homepage.sns.it/vistoli/descent.pdf>, 2008.
- [Yo95] K. Yokogawa, ‘Infinitesimal deformations of parabolic Higgs sheaves’, *Internat. J. Math.* **6** (1995), 125–148.

INDRANIL BISWAS, School of Mathematics,  
Tata Institute of Fundamental Research, Homi Bhabha Road,  
Mumbai 400005, India  
e-mail: [indranil@math.tifr.res.in](mailto:indranil@math.tifr.res.in)

SOURADEEP MAJUMDER, School of Mathematics,  
Tata Institute of Fundamental Research, Homi Bhabha Road,  
Mumbai 400005, India  
e-mail: [souradip@math.tifr.res.in](mailto:souradip@math.tifr.res.in)

MICHAEL LENNOX WONG, Chair of Geometry,  
Mathematics Section, Ecole Polytechnique Fédérale de Lausanne,  
Station 8, 1015 Lausanne, Switzerland  
e-mail: [michael.wong@epfl.ch](mailto:michael.wong@epfl.ch)