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# On fields of $u$-invariant 4

By

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**Abstract.** This note is motivated by the problem of determining the $u$-invariant of a field $F$ of characteristic different from two when it is known that $u(F(\sqrt{-1})) = 4$. A criterion is given to decide whether $u(F) \leq 4$ in this situation.

**1. Introduction.** In the algebraic theory of quadratic forms various open problems and several spectacular breakthroughs of the last two decades are related to the $u$-invariant (cf. [10, Chapter 8]). Let us consider a field extension $F/R$ of transcendence degree $n \geqq 1$ where $R$ is real closed (e.g., $R = \mathbb{R}$). In [9], Pfister formulated the conjecture that in this situation $u(F) \leqq 2^n$. For the case where $F$ is nonreal this is a special case of Lang's conjecture in [7] according to which $F$ would be a $\mathcal{C}_n$-field. The two conjectures are motivated by the known fact that the quadratic extension $F(\sqrt{-1})$ is a $\mathcal{C}_n$-field, in particular $u(F(\sqrt{-1})) \leqq 2^n$. This fact can be used to obtain the following bounds on $u(F)$ using the results in [5]: if $F$ is nonreal or if $n \leqq 2$, then $u(F) \leqq 3 \cdot 2^{n-1}$; if $F$ is real and $n > 2$, then $u(F) < 4^n - 2^n$. So far Pfister's conjecture that $u(F) \leqq 2^n$ is settled only for $n = 1$. For $n = 2$ one knows that $u(F) \in \{0, 1, 2, 4, 6\}$ (cf. [5]). This case is investigated in [9], but the problem is still open, even in the particular case of the rational function field in two variables $\mathbb{R}(X, Y)$.

This indicates that we still do not know that much about the quadratic form theory over the function field of a surface over a real closed field. Therefore it is natural to seek for inspiration from other types of fields which have similar properties with respect to quadratic forms. There are at least two interesting types of such fields: first, fraction fields of completions of the local ring of functions in a closed point of a surface over a real closed field, e.g., the field of formal power series in two variables $\mathbb{R}((X, Y))$; second, function fields of curves over the power series field $\mathbb{R}((t))$ or, more generally, over a field with a henselian discrete valuation with real closed residue field. Fields of the first type were studied in [3] where it was shown that they satisfy a certain Hasse principle for quadratic forms of dimension at least three. Fields of the second type were investigated recently in [12] and [1].

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The main result of this article is a criterion to decide whether $u(F) \leqq 4$ for a field $F$ of characteristic different from two such that $u(F(\sqrt{-1})) = 4$ (3.1). The criterion is a strengthening of a criterion formulated by Pfister in [9, Proposition 6]. It turns out to be trivially satisfied in the case where $F$ has Pythagoras number $p(F) \leqq 2$ (3.3). Using this one easily sees that certain fields of the two types mentioned above have $u$-invariant equal to 4 (3.4). Note that the condition that $p(F) \leqq 2$ means that the norm form of the extension $F(\sqrt{-1})/F$ represents all totally positive elements of $F$. In view of this observation we obtain a slight generalization of (3.3): if $K/F$ is a quadratic extension such that the norms of $K/F$ are exactly the sums of squares in $F$, then $u(F) \leqq u(K)$ holds (3.5).

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**2. Terminology.** Throughout this article, let $F$ be a field of characteristic different from 2. Let $F^\times$ denote the multiplicative group of $F$ and $\sum F^{\times 2}$ the subgroup of non-zero sums of squares in $F$. Elements of $F$ which are sums of squares are said to be *totally positive*. The terms 'form' and 'quadratic form' shall always refer to a regular quadratic form. The main references for the theory of quadratic forms over fields are [11] and [6]. Notations and facts used here and not given explicitly are standard and can be found in these books.

The *u-invariant* of the field $F$ was defined by Elman and Lam in [5] as

$$u(F) \;=\; \sup\{\dim(\varphi) \mid \varphi \text{ anisotropic torsion form over } F\}.$$

The *Pythagoras number* of the field $F$ is defined by

$$p(F) = \sup\left\{ l \in \mathbb{N} \mid \exists\, x \in \sum F^{\times 2} \text{ s.t. } x \text{ not a sum of less than } l \text{ squares} \right\}.$$

In both cases, the supremum is taken in the set $\mathbb{N} \cup \{\infty\}$.

For any quadratic form $\varphi$ over $F$ we denote by $D_F(\varphi)$ the set of non-zero elements of $F$ which are represented by $\varphi$. Recall that $\varphi$ is *universal* if $D_F(\varphi) = F^\times$, i.e., if $\varphi$ represents all non-zero elements of $F$. In particular, any isotropic form is universal. For $m \in \mathbb{N}$ we also write $D_F(m)$ for the set of non-zero elements in $F$ which can be written as sums of $m$ squares in $F$.

A quadratic form $\varphi$ over $F$ is said to be *weakly isotropic* (resp. *torsion*) if for some $n \geqq 1$ the multiple $n \times \varphi = \varphi \perp \ldots \perp \varphi$ ($n$ times) is isotropic (resp. hyperbolic) over $F$.

If $\varphi$ is a quadratic form and $d$ is its determinant, then

$$d_\pm(\varphi) \;=\; \begin{cases} d & \text{if } \dim(\varphi) \equiv 0, 1 \mod 4 \\ -d & \text{if } \dim(\varphi) \equiv 2, 3 \mod 4 \end{cases}$$

is called the *discriminant* (also 'signed determinant') of $\varphi$. An *Albert form* is a quadratic form of dimension 6 and of trivial discriminant. Given a form $\varphi$ which is similar to a subform of a Pfister form $\pi$ with $\frac{1}{2}\dim(\pi) < \dim(\varphi)$ is called a *Pfister neighbor* (of $\pi$).

For $n \in \mathbb{N}$, we denote by $I^nF$ the $n$th power of the fundamental ideal in the Witt ring $WF$ of $F$. By $X(F)$ we denote the space of orderings of $F$. By the Artin-Schreier Theorem, $X(F) \neq \emptyset$ if and only if $-1 \notin \sum F^{\times 2}$ and in this case the field $F$ is said to be *real*, otherwise *nonreal*. The real closure of $F$ with respect to an ordering $P \in X(F)$ is denoted by $F_P$.

**3. A criterion.** We know that the condition that $u(F(\sqrt{-1})) = 4$ implies that $I^3F$ is torsion-free and $u(F) \in \{0, 1, 2, 4, 6\}$. Moreover, if in this situation $u(F) = 6$, then there exists an Albert form over $F$ which is torsion and anisotropic.

**Theorem 3.1.** *Assume that $u(F(\sqrt{-1})) = 4$. A necessary and sufficient condition to have that $u(F) \leqq 4$ is that, for any $a \in D_F(3)$ and $d \in D_F(2)$, the form $\langle 1, 1, -a, -ad\rangle$ is universal over $F$.*

Proof. Assume first that $a \in D_F(3)$ and $d \in D_F(2)$ do exist such that the form $\psi = \langle 1, 1, -a, -ad\rangle$ is not universal over $F$. Let $t \in F^\times$ be an element which is not represented by $\psi$. Since $I^3F$ is torsion-free, any 3-fold Pfister form over $F$ is either hyperbolic or definite at some ordering $P$ of $F$. Since $\psi \perp \langle -t\rangle$ is totally indefinite and anisotropic, it follows that this 5-dimensional form is not a Pfister neighbor. Therefore $\psi \perp \langle -t\rangle$ does not represent its discriminant $-dt$. Then $\varphi = \psi \perp \langle -t, dt\rangle = \langle 1, 1, -a, -ad, -t, dt\rangle$ is anisotropic. Since $\varphi$ is a torsion form, we obtain that $u(F) \geqq \dim(\varphi) = 6$.

To prove the converse implication assume now that $u(F) = 6$. Then there exists an Albert form $\varphi$ over $F$ which is torsion and anisotropic. Since $u(F(\sqrt{-1})) = 4$, $\varphi$ becomes isotropic over $F(\sqrt{-1})$. Hence $\varphi$ contains a subform $\langle z, z\rangle$ with $z \in F^\times$. Since $I^3F$ is torsion-free, we have $z\varphi \cong \varphi$ and $2 \times \varphi$ is hyperbolic. Therefore we have $\varphi \cong \langle 1, 1\rangle \perp \psi$ for some 4-dimensional form $\psi$. Then $2 \times \psi$ is isotropic and a well known argument yields that $\psi$ contains a binary form $\beta$ such that $2 \times \beta$ is hyperbolic (cf. [5, Proposition 2.2]). We denote by $d$ the discriminant of $\beta$ and observe that $d$ is a sum of two squares in $F$. Comparing discriminants, we see that we can write $\varphi \cong \langle 1, 1, -a, -ad\rangle \perp \beta$ for some $a \in F^\times$. Now, since both $\varphi$ and $\beta$ are torsion, so is $\langle 1, 1, -a, -ad\rangle$. In particular, $a$ is a sum of squares in $F$. As $I^3F$ is torsion-free we have $p(F) \leqq 4$, thus $a$ is a sum of four squares in $F$. We may write $a = x + y$ where each $x$ and $y$ are sums of two squares and $x \neq 0$. Then $xa = x^2 + xy$ is a sum of three squares. Multiplying $\varphi$ with $x$ we obtain $\varphi \cong x\varphi \cong \langle 1, 1, -ax, -axd\rangle \perp x\beta$. As we can replace $x\beta$ by $\beta$ and $ax$ by $a$, we may assume that $a$ is a sum of three squares. Now, since $\varphi$ is anisotropic, $\langle 1, 1, -a, -ad\rangle$ is not universal over $F$. □

Remark 3.2. In [9, Proposition 6] it is shown, under some hypothesis which in particular implies that $u(F(\sqrt{-1})) \leqq 4$, that one has $u(F) \leqq 4$ if and only if every form $\langle 1, -t_1\rangle \perp a\langle 1, -t_2\rangle$ with $t_1, t_2 \in \sum F^{\times 2}$ and $a \in F^\times$ is universal over $F$. The above theorem shows that it actually suffices to consider all such forms where $a = 1$.

**Corollary 3.3.** *If $u(F(\sqrt{-1})) = 4$ and $p(F) \leqq 2$, then $u(F) \leqq 4$.*

Proof. This follows from the theorem, since by the additional hypothesis that $p(F) \leqq 2$, for any $a \in \sum F$, the form $\langle 1, 1, -a\rangle$ over $F$ is isotropic and thus universal. □

Examples 3.4. (1) In [2] it was proven that $p(\mathbb{R}((X, Y))) = 2$ and that $u(\mathbb{C}((X, Y))) = 4$. Using the corollary this gives an elementary argument that $u(\mathbb{R}((X, Y))) = 4$. In [3, Theorem 4.4] it was shown more generally that $u(F) \leqq 4$ holds when $F$ is the fraction field of an excellent 2-dimensional local domain with real closed residue field.

(2) It is not difficult to see that the field $\mathbb{R}((X))(Y)$ has Pythagoras number 2. This follows from Milnor's exact sequence (cf. [6, Chapter 9, Section 3]), together with the fact that $\mathbb{R}((X))$ is hereditarily pythagorean. More generally, let $R$ be a real closed field and $C$ an elliptic curve over $R((t))$ which has good reduction to $\mathbb{R}$ with respect to $t$. In [12] it is shown that the Pythagoras number of the function field $F = \mathbb{R}((t))(C)$ is two. Furthermore $u(F(\sqrt{-1})) = 4$, since $F(\sqrt{-1})$ is a $\mathcal{C}_2$-field. From (3.3) we then obtain that $u(F) \leqq 4$. It is then not difficult to show that $u(F) = 4$ in this case.

The above corollary can be generalized a little.

**Proposition 3.5.** *Let $K/F$ be a quadratic extension where $K$ is nonreal and not quadratically closed and such that every totally positive element of $F$ is a norm of $K/F$. Then $u(F) \leqq u(K)$.*

Proof. We fix $t \in F^\times$ such that $K = F(\sqrt{-t})$. As $K$ is nonreal, $t$ is a sum of squares in $F$. By hypothesis, the norm form of $K/F$ represents all sums of squares over $F$. In other terms, we have $D_F(\langle 1, t\rangle) = \sum F^{\times 2}$. This implies that any form $\tau$ over $F$ is weakly isotropic if and only if $\langle 1, t\rangle \otimes \tau$ is isotropic.

Assume now that $u(F) > u(K)$. Then there exists an anisotropic torsion form $\varphi$ over $F$ such that $\dim(\varphi) > u(K)$. Since $\varphi$ must become isotropic over $K$, it follows that $\varphi$ contains the norm form $\langle 1, t\rangle$ up to a scalar factor. After scaling $\varphi$, we may assume that $\varphi = \langle 1, t\rangle \perp \vartheta$. By the hypotheses we have that $u(K) \geqq 2$, thus $\dim(\varphi) \geqq 3$. If $F$ were nonreal, then $\langle 1, t\rangle$ would be universal, and this would be in contradiction to $\varphi$ being anisotropic.

Hence we may assume for the rest that $F$ is real. Using the '$\beta$-decomposition' argument in [5], we may write $\vartheta \cong \gamma \perp \beta_1 \perp \ldots \perp \beta_r$ where $\gamma$ is a form such that $\langle 1, t\rangle \otimes \gamma$ is anisotropic while each $\beta_i$ $(1 \leqq i \leqq r)$ is a 2-dimensional form such that $\langle 1, t\rangle \otimes \beta_i$ is hyperbolic. Then $\gamma$ is not weakly isotropic, while any $\beta_i$ $(1 \leqq i \leqq r)$ is a torsion form. Since $\varphi$ is a torsion form over $F$, it follows now that $\langle 1, t\rangle \perp \gamma$ is torsion. Since $\gamma$ is not weakly isotropic, we get that $\dim(\gamma) = 2$. Since $\langle 1, t\rangle$ represents only totally positive elements of $F$ and since $\langle 1, t\rangle \perp \gamma$ is torsion, $\gamma$ represents only totally negative elements of $F$. But as $\langle 1, t\rangle$ represents all totally positive elements of $F$, we conclude that $\langle 1, t\rangle \perp \gamma$ is isotropic, which is in contradiction to $\varphi$ being anisotropic. □

Remarks 3.6. (1) If $F$ has Pythagoras number $p(F) \leqq 2$, then $K = F(\sqrt{-1})$ is a quadratic extension satisfying the hypotheses of the proposition. Therefore, (3.3) can also be derived from (3.5).

(2) At least in the case where the base field $F$ is nonreal, the statement of (3.5) is not new. In this case we have $K = F(\sqrt{t})$ where $t$ belongs to the Kaplansky radical of $F$ (cf. [4]); in particular, any anisotropic quadratic form over $F$ of dimension at least 3 remains anisotropic over $K$, which is a stronger observation than that $u(F) \leqq u(K)$.

(3) Assume that $F$ is a real field with finite square class group $F^{\times}/F^{\times 2}$. If the Witt ring $WF$ is of elementary type, then a quadratic extension $K/F$ such as in the statement of the proposition always exists. This may be useful in view of the Elementary Type Conjecture (cf. [8]), especially in connection with the question about the possible values for $u(F)$ when $F^{\times}/F^{\times 2}$ is finite.

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