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Pfister involutions

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Abstract. The question of the existence of an analogue, in the framework of central simple algebras with involution, of the notion of Pfister form is raised. In particular, algebras with orthogonal involution which split as a tensor product of quaternion algebras with involution are studied. It is proven that, up to degree 16, over any extension over which the algebra splits, the involution is adjoint to a Pfister form. Moreover, cohomological invariants of those algebras with involution are discussed.

Keywords. Algebras with involution; Pfister forms; cohomological invariants.

0. Introduction

An involution on a central simple algebra is nothing but a twisted form of a symmetric or alternating bilinear form up to a scalar factor ([KMRT98], ch. 1). Hence the theory of central simple algebras with involution naturally appears as an extension of the theory of quadratic forms, which is an important source of inspiration for this subject.

We do not have, for algebras with involution, such a nice algebraic theory as for quadratic forms, since orthogonal sums are not always defined, and are not unique when defined [Dej95]. Nevertheless, in view of the fundamental role played by Pfister forms in the theory of quadratic forms, and also of the nice properties they share, it seems natural to try and find out whether an analogous notion exists in the setting of algebras with involution.

The main purpose of this paper is to raise this question, which was originally posed by David Tao [Tao]; this is done in §2. In particular, this leads to the consideration of algebras with orthogonal involution which split as a tensor product of r quaternion algebras with involution. One central question is then the following: consider such a product of quaternions with involution, and assume the algebra is split. Is the corresponding involution adjoint to a Pfister form? The answer is positive up to $r = 5$. A survey of this question is given in §2.4. In §4, we give a direct proof of this fact for $r = 4$. Before that, we study in §3, the existence of cohomological invariants for some of the algebras with involution which can naturally be considered as generalisations of Pfister quadratic forms.

1. Notations

Throughout this paper, the base field F is supposed to be of characteristic different from 2, and K denotes a field extension of F. We refer the reader to [Sch85], [Lam73] and [KMRT98] for more details on what follows in this section.

1.1 *Cohomology*

Let K_s be a separable closure of the field K, and let us denote by Γ_K the absolute Galois group $\Gamma_K = \mathrm{Gal}(K_s/K)$. The Galois cohomology groups of Γ_K with coefficients in $\mathbb{Z}/2$ will be denoted by $H^i(K) = H^i(\Gamma_K, \mathbb{Z}/2)$. For any $a \in K^\star$, we denote by (a) the image in $H^1(K)$ of the class of a in $K^\star/K^{\star 2}$ under the canonical isomorphism $K^\star/K^{\star 2} \simeq H^1(K)$, and by $(a_1, \dots, a_i) \in H^i(K)$ the cup-product $(a_1) \cup (a_2) \cup \cdots \cup (a_i)$. In particular, the element $(a_1, a_2) \in H^2(K)$ corresponds, under the canonical isomorphism $H^2(K) \simeq \mathrm{Br}_2(K)$, where $\mathrm{Br}_2(K)$ denotes the 2-torsion part of the Brauer group of K, to the Brauer class of the quaternion algebra $(a_1, a_2)_K$.

Consider now a smooth integral variety X over F, and denote by $F(X)$ its function field. An element $\alpha \in H^i(F(X))$ is said to be *unramified* if for each codimension one point x in X, with local ring $\mathcal{O}_x$ and residue field κ_x, the element α belongs to the image of the natural map $H^i_{et}(\mathcal{O}_x, \mathbb{Z}/2) \to H^i(F(X))$, or equivalently its image under the residue map $\partial_x : H^i(F(X)) \to H^{i-1}(\kappa_x)$ is zero (see [CT95], Theorem 4.1.1). We denote by $H^i_{nr}(F(X)/F)$ the subgroup of $H^i(F(X))$ of unramified elements.

1.2 *Quadratic forms*

The quadratic forms considered in this paper are non-degenerate. If q is a quadratic form over K, we let $K(q)$ be the function field of the corresponding projective quadric. The field $K(q)$ is the generic field over which an anisotropic form q acquires a non-trivial zero.

Consider a diagonalisation $\langle a_1, \dots, a_n\rangle$ of a quadratic form q. We denote by $d(q)$ the signed discriminant of q, that is $d(q) = (-1)^{\frac{n(n-1)}{2}} a_1 \dots a_n \in K^\star/K^{\star 2}$, and by $C(q)$ its Clifford algebra (see [Sch85] or [Lam73] for a definition and structure theorems). We recall that $C(q)$ is a $\mathbb{Z}/2$-graded algebra, and we denote by $C_0(q)$ its even part.

For any $a_1, \dots, a_r \in K^\star$ we denote by $\langle\langle a_1, \dots, a_r\rangle\rangle$ the r-fold Pfister form $\otimes_{i=1}^r \langle 1, -a_i\rangle$. We let $P_r(K)$ be the set of r-fold Pfister forms over K, and $GP_r(K)$ be the set of quadratic forms over K which are similar to an r-fold Pfister form. Pfister forms are also characterized, up to similarities, by the following property:

Theorem 1.1 (([Kne76], Theorem 5.8) and [Wad72]). *Let q be a quadratic form over F. The following assertions are equivalent*:

(i) *The dimension of q is a power of 2, and $q_{F(q)}$ is hyperbolic*;
(ii) *The quadratic form q is similar to a Pfister form.*

From the above theorem, one easily deduces:

COROLLARY 1.2

Let q be a quadratic form over F. The following assertions are equivalent:

(i) *The dimension of q is a power of 2 and for any field extension K/F, if q_K is isotropic, then it is hyperbolic*;
(ii) *q is similar to a Pfister form.*

We denote by e_r the map $P_r(K) \to H^r(K)$ defined by Arason [Ara75] as $e_r(\langle\langle a_1, \dots, a_r\rangle\rangle) = (a_1, \dots, a_r)$.

Let $W(K)$ be the Witt ring of the field K, and denote by $I(K)$ the fundamental ideal of $W(K)$, which consists of classes of even-dimensional quadratic forms. Its rth power $I^r(K)$ is additively generated by r-fold Pfister forms. For $r = 1, 2$ and 3, the invariant e_r extends to a surjective homomorphism $I^r(K) \to H^r(K)$ with kernel $I^{r+1}(K)$ (see [Mer81] for $r = 2$ and [MS90] for $r = 3$). It follows from this that the class of an even-dimensional quadratic form q belongs to $I^2(K)$ (resp. $I^3(K)$) if and only if $e_1(q) = 0$ (resp. $e_1(q) = 0$, $e_2(q) = 0$). If we assume moreover that q is of dimension 4 (resp. 8), this is equivalent to saying that q is similar to a Pfister form.

Moreover, the maps e_1 and e_2 are actually defined (as maps) over the whole Witt ring $W(K)$, and can be explicitly described in terms of classical invariants of quadratic forms. Indeed, e_1 associates to the class of a quadratic form q its signed discriminant $d(q) \in K^\star/K^{\star 2}$. Moreover, the image under e_2 of the class of the same form q is the Brauer class of its Clifford algebra $C(q)$ if the dimension of q is even, and the Brauer class of $C_0(q)$ if the dimension of q is odd.

1.3 *Algebras with involution*

An involution τ on a central simple algebra B over K is an anti-automorphism of order 2 of the ring B. We only consider here involutions of the first kind, that is K-linear ones. For any field extension L/K, we denote by B_L the L-algebra $B \otimes_K L$, by τ_L the involution $\tau \otimes \mathrm{Id}$ of B_L and by $(B, \tau)_L$ the pair (B_L, τ_L).

Consider now a splitting field L of B, that is an extension L/K such that B_L is the endomorphism algebra of some L-vector space V. The involution τ_L is the adjoint involution ad_b with respect to some bilinear form $b : V \times V \to L$, which is either symmetric or skew-symmetric. The type of the form b does not depend on the choice of the splitting field L; the involution τ is said to be of orthogonal type if b is symmetric, and of symplectic type if it is skew-symmetric.

Let Q be a quaternion algebra over K. It admits a unique involution of symplectic type, which we call the canonical involution of Q, and which is defined by $\gamma_Q(x) = \mathrm{Trd}_Q(x) - x$, where Trd_Q is the reduced trace on Q. We denote by Q^0 the subspace of pure quaternions, that is those $q \in Q$ satisfying $\mathrm{Trd}_Q(q) = 0$, or equivalently, $\gamma_Q(q) = -q$. For any pure quaternion $q \in Q^0$, we have $q^2 \in F$. For any orthogonal involution σ on Q, there exists a pure quaternion $q \in Q$ such that $\sigma = \mathrm{Int}(q) \circ \gamma_Q$, where $\mathrm{Int}(q)$ is the inner automorphism associated to q, defined by $\mathrm{Int}(q)(x) = qxq^{-1}$.

If the degree of B is even, and if τ is of orthogonal type, we denote by $d(\tau) \in K^\star/K^{\star 2}$ the discriminant of τ, and by $C(B, \tau)$ its Clifford algebra ([KMRT98], §7, 8). In the split orthogonal case $(B, \tau) = (\mathrm{End}_F(V), \mathrm{ad}_q)$, they correspond respectively to the discriminant of q and its even Clifford algebra $C_0(q)$. Note that by the structure theorem ([KMRT98], (8.10)), if the discriminant of τ is trivial, then the Clifford algebra $C(B, \tau)$ is a direct product of two K-central simple algebras, $C(B, \tau) = C_+ \times C_-$.

A right ideal I of a central simple algebra with involution (B, τ) is called isotropic if $\sigma(I)I = \{0\}$. The algebra with involution (B, τ) is called isotropic if it contains a non trivial isotropic right ideal, and hyperbolic if it contains a non trivial isotropic right ideal of maximal dimension (that is of reduced dimension $\frac{1}{2}\deg(B)$) ([KMRT98], § 6).

In [Tao94], David Tao associates to an algebra with orthogonal involution (B, τ) a variety which, in the split orthogonal case $(B, \tau) = (\mathrm{End}_K(V), \mathrm{ad}_q)$, is the projective quadric associated to q. This variety is called the involution variety of (B, τ); its function field is the generic field over which B splits and τ becomes isotropic.

2. Three classes of algebras with involution

From now on, we consider a central simple algebra A over F, endowed with an involution σ of orthogonal type. We denote by F_A the function field of the Severi–Brauer variety of A, which is known to be a generic splitting field for A. After scalar extension to F_A, the involution σ becomes the adjoint involution with respect to some quadratic form over F_A, which we denote by q_σ. Note that this form is uniquely defined up to a scalar factor in $F_A^\star$.

In view of the definition and properties of Pfister forms, it seems natural, for our purpose, to consider the three classes of algebras with involution introduced in this section.

2.1 *Pfister involutions*

DEFINITION 2.1

The algebra with orthogonal involution (A, σ) is called a Pfister algebra with involution if σ_{F_A} is adjoint to a Pfister form.

Remark 2.2.

(i) If (A, σ) is a Pfister algebra with involution, the degree of A is a power of 2.
(ii) Since the form q_σ is uniquely defined up to a scalar factor, (A, σ) is a Pfister algebra with involution if and only if $q_\sigma \in GP(F_A)$. Moreover, two similar Pfister forms are actually isometric (as follows from ([Sch85], ch. 4, 1.5). Hence, in this particular case, there is a canonical choice for the quadratic form q_σ; we may assume it is a Pfister form, in which case it is uniquely defined up to isomorphism.
(iii) Since any 2-dimensional quadratic form is similar to a Pfister form, any degree 2 algebra with orthogonal involution is a Pfister algebra with involution.

The field F_A is a generic splitting field for A. Hence, we may deduce from the definition and from Corollary 1.2 the following proposition:

PROPOSITION 2.3

The following assertions are equivalent:

(i) (A, σ) *is a Pfister algebra with involution*;

(ii) *For any field extension K/F which splits A, the involution σ_K is adjoint to a Pfister form*;
(iii) *The degree of A is a power of 2 and for any field extension K/F which splits A, if σ_K is isotropic, then it is hyperbolic*;
(iv) *The degree of A is a power of 2 and after extending scalars to the function field of its involution variety, (A, σ) becomes hyperbolic.*

2.2 *Involutions of type $I \Rightarrow H$*

As recalled in §1, 'isotropy implies hyperbolicity' is a characterization of Pfister forms. Hence, we may also consider algebras with involution satisfying the same property:

DEFINITION 2.4

The algebra with orthogonal involution (A, σ) is said to be of type $I \Rightarrow H$ if the degree of A is a power of 2 and for any field extension K/F, if $(A, \sigma)_K$ is isotropic, then it is hyperbolic.

Remark 2.5.

(i) Again the condition is empty in degree 2. Any degree 2 algebra with orthogonal involution is of type $I \Rightarrow H$.
(ii) From the previous proposition, one deduces that any involution of type $I \Rightarrow H$ is a Pfister involution. Moreover, if A is split, then the two definitions are equivalent.

2.3 *Product of quaternions with involution*

Up to similarities, Pfister forms are those quadratic forms which diagonalise as a tensor product of two dimensional forms. Hence, we now consider algebras with involution which split as a tensor product of degree 2 algebras with involution.

DEFINITION 2.6

The algebra with orthogonal involution (A, σ) is called a product of quaternions with involution if there exists an integer r and quaternion algebras with involution (Q_i, σ_i) for $i = 1, \ldots, r$ such that $(A, \sigma) \simeq \otimes_{i=1}^{r} (Q_i, \sigma_i)$.

Remark 2.7.

(i) If (A, σ) is a product of quaternions with involution, then the degree of A is a power of 2.
(ii) Since σ is of orthogonal type, the number of indices i for which σ_i is of symplectic type is necessarily even.
(iii) In [KPS91], it is proven that a tensor product of two quaternion algebras with orthogonal involutions admits a decomposition as a tensor product of quaternion algebras with symplectic (hence canonical) involutions. Hence, any product of quaternions with involution admits a decomposition as above in which all the σ_i if r is even, and all but one if r is odd, are the canonical involutions of Q_i.

As above, the condition is empty in degree 2, any degree 2 algebra with orthogonal involution is a product of quaternions with involution. In degrees 4 and 8, we have the following characterizations:

Theorem 2.8 [KPS91]. *Let (A, σ) be a degree 4 algebra with orthogonal involution. It is a product of quaternions with involution if and only if the discriminant of σ is 1.*

Theorem 2.9 ([KMRT98], (42.11)). *Let (A, σ) be a degree 8 algebra with orthogonal involution. It is a product of quaternions with involution if and only if the discriminant of σ is trivial and one component of the Clifford algebra of (A, σ) splits.*

2.4 *Shapiro's conjecture*

It seems a natural question to try and find out whether the three classes of algebras with involution introduced above are equivalent. This is obviously the case in degree 2. The following proposition will be proven in §3.3:

PROPOSITION 2.10

Let (A, σ) be an algebra of degree at most 8 with orthogonal involution. The following are equivalent:

(i) *(A, σ) is a Pfister algebra with involution*;
(ii) *(A, σ) is a product of quaternions with involution*;
(iii) *(A, σ) is of type $I \Rightarrow H$.*

Nevertheless, the general question of the equivalence of these three classes of algebras with involution is largely open in higher degree. The most significant result is due to Shapiro. In his book 'Composition of quadratic forms' he makes the following conjecture:

Conjecture 2.11 ([Sha00], (9.17)). Let (A, σ) be a product of r quaternions with involution. If A is split, then (A, σ) admits a decomposition as a tensor product of r quaternion algebras with involution in which each quaternion algebra is split.

Moreover, he proves the following theorem:

Theorem 2.12 ([Sha00], Claim in p. 166 and Ch. 9). *Conjecture* 2.11 *is true if $r \leq 5$.*

It is easy to see that Shapiro's conjecture is true for some r if and only if any product of r quaternions with involution is a Pfister algebra with involution. Hence the previous theorem implies.

COROLLARY 2.13

Any product of $r \leq 5$ quaternions with involution is a Pfister algebra with involution.

Shapiro does not give a direct proof of this conjecture. He is actually interested in another conjecture, which he calls the Pfister factor conjecture, and which gives a characterization of r-fold Pfister forms in terms of the existence of vector-spaces of maximal dimension in the group of similarities of these forms (see [Sha00], (2.17)) for a precise statement).

He proves the Pfister factor conjecture for $r \leq 5$, using tools from the algebraic theory of quadratic forms, and also proves it is equivalent to Conjecture 2.11.

In fact, for $r \leq 3$, we have a little bit more: as already mentioned in Proposition 2.10, (A, σ) is a product of quaternions with involution if and only if it is a Pfister algebra with involution. A proof of this fact, using cohomological invariants, which was already noticed by David Tao, will be given in §3, where we study the general question of cohomological invariants of Pfister involutions.

In §4, we give a direct proof of 2.11 in the $r = 4$ case, based on the study of some trace forms of product of quaternions with involution. Since this paper was submitted, Serhir and Tignol [ST] found another direct proof of this conjecture for $r \leq 5$, using the discriminant of symplectic involutions defined by Berhuy, Monsurro and Tignol [BMT].

3. Cohomological invariants

From the point of view of quadratic form theory, cohomological invariants seem a natural tool for studying these questions. In § 3.1, we define an invariant of a Pfister involution, with values in the unramified cohomology group of the function field of the generic splitting field of the underlying algebra. We then study the question of the existence of an analogous invariant with values in the cohomology group of the base field.

3.1 *Invariant e_i for Pfister algebras with involution*

Throughout this section, (A, σ) is a Pfister algebra with involution over F. The degree of A is 2^i, and we assume q_σ is an i-fold Pfister form over F_A (see Remark 2.2). Let us consider the Arason invariant $e_i(q_\sigma) \in H^i(F_A)$. We have the following:

Theorem 3.1. *The invariant $e_i(q_\sigma)$ belongs to the unramified cohomology group $H^i_{nr}(F_A/F)$.*

Proof. Given a codimension one point x of the Severi–Brauer variety X_A of A, its residue field κ_x splits A. Hence, the involution σ_{κ_x} is the adjoint involution with respect to a quadratic form q_x which is a Pfister form uniquely determined by σ_{κ_x} (see Remark 2.2 (ii)).

Let us now consider the completions $\widehat{\mathcal{O}_x}$ and $\widehat{F(X_A)}$ of $\mathcal{O}_x$ and $F(X_A)$ at the discrete valuation associated to x. Since $\widehat{\mathcal{O}_x}$ is complete, the field $\widehat{F(X_A)}$ is isomorphic to $\kappa_x((t))$, and for the same reason as above, the involution $\sigma_{\widehat{F(X_A)}}$ is adjoint to a unique Pfister form $q_{\widehat{F(X_A)}}$, which is the form q_x extended to $\kappa_x((t))$.

From this, we get that $e_i(q_{\widehat{F(X_A)}})$ is the image of $e_i(q_x)$ under the natural map $H^i(\kappa_x) \to H^i(\kappa_x((t)))$. By ([CT95], §3.3), since the corresponding ring is complete, this implies that the image of $e_i(q_{\widehat{F(X_A)}})$ under the residue map $\partial_x : H^i(\widehat{F(X_A)}) \to H^{i-1}(\kappa_x)$ is trivial. Finally, again by ([CT95], §3.3), $\partial_x(e_i(q_\sigma)) = \partial_x(e_i(q_{\widehat{F(X_A)}}))$, and this proves the theorem. $\square$

Of course, it would be nicer to have an invariant with values in the cohomology group of the base field. To be more precise, let us denote by $E_i(A)$ the kernel of the restriction map $H^i(F) \to H^i(F_A)$ and by Φ the injection:

$$\Phi : H^i(F)/E_i(A) \to H^i_{nr}(F_A).$$

We may ask the following question: Does $e_i(q_\sigma)$ belong to the image of Φ? In §3.2, we prove that this is the case for $i = 0, 1$ and 2, and we give an interpretation of the corresponding invariant in $H^i(F)/E_i(A)$ in terms of classical invariants of orthogonal involutions. In §3.4, we prove this is not the case anymore for $i = 3$.

3.2 *Invariants e_0, e_1 and e_2*

Let us consider now any algebra with orthogonal involution (A, σ). As recalled in §1, the first three Arason invariants e_0, e_1 and e_2 for quadratic forms play a particular role. Indeed, they are actually defined as maps over the whole Witt ring $W(F)$, and they can be described in terms of classical invariants of quadratic forms. In view of this, we may give the following definition:

DEFINITION 3.2

Let (A, σ) be an algebra with orthogonal involution over F. We let

$$e_0(A, \sigma) = \overline{\deg(A)} \in \mathbb{Z}/2\mathbb{Z} \simeq H^0(F).$$

If the degree of A is even (that is $e_0(A, \sigma) = 0$), we let

$$e_1(A, \sigma) = d(\sigma) \in F^\star/F^{\star 2} = H^1(F),$$

where $d(\sigma)$ denotes the discriminant of σ.

Remark 3.3. Note that, as opposed to what happens for quadratic forms, the invariant e_1 is only defined when e_0 is trivial. This is a consequence of the fact that the discriminant of a quadratic form is an invariant up to similarity, and hence an invariant of the corresponding adjoint involution, only if the form has even dimension.

Assume now that $e_0(A, \sigma) = e_1(A, \sigma) = 0$, which means (A, σ) has even degree and trivial discriminant. From the structure theorem recalled in §1, the Clifford algebra $C(A, \sigma)$ is isomorphic to a direct product of two central simple algebras over F, $C(A, \sigma) = C_+ \times C_-$, which give rise to two Brauer classes $[C_+]$ and $[C_-]$ in $\mathrm{Br}_2(F)$. The definition of e_2 then relies on the following proposition:

PROPOSITION 3.4 ([KMRT98], (9.12))

In $\mathrm{Br}_2(F)$, *we have* $[C_+] + [C_-] \in \{0, [A]\}$.

Indeed this implies that the two classes actually coincide in the quotient of $\mathrm{Br}_2(F)$ by the subgroup $\{0, [A]\}$, which is exactly $E_2(A)$. Hence, we give the following definition:

DEFINITION 3.5

Let (A, σ) be an algebra with orthogonal involution over F of even degree and trivial discriminant. We let

$$e_2(A, \sigma) = [C_+] = [C_-] \in \mathrm{Br}_2(F)/E_2(A).$$

Next, we prove the following:

PROPOSITION 3.6

Let (A,σ) be a split algebra with orthogonal involution, $(A,\sigma)=(\mathrm{End}_F(V),\mathrm{ad}_q)$. When they are defined, the invariants $e_0(A,\sigma)$, $e_1(A,\sigma)$ and $e_2(A,\sigma)$ coincide respectively with $e_0(q)$, $e_1(q)$ and $e_2(q)$.

Proof. This is clear for e_0 and e_1. For e_2, first note that if A is split, then $e_2(A,\sigma)$ actually belongs to $\mathrm{Br}_2(F)$. Moreover, $e_2(A,\sigma)$ is only defined when e_0 and e_1 are trivial, in which case the form q is of even dimension and trivial discriminant. From the structure theorem for Clifford algebra (see for instance ([Lam73], 5, §2) or ([Sch85], 9(2.10)) we get that in this situation, we may represent $C(q)$ as $M_2(B)$, for some central simple algebra B over F, and the even part $C_0(q)$ corresponds to diagonal matrices, $C_0(q)\simeq B\times B$, so that $e_2(q)=[C(q)]=[B]=e_2(A,\sigma)$. □

From this proposition, we easily deduce:

COROLLARY 3.7

Let (A,σ) be an algebra with orthogonal involution such that $e_i(A,\sigma)$ is defined for some $i\leq 2$. The invariant $e_i(A,\sigma)$ maps to $e_i(q_\sigma)$ under the morphism $\Phi:\ H^i(F)/E_i(A)\to H^i(F_A)$.

Hence, those invariants e_i may be used to characterize degree 4 and 8 Pfister involutions. Indeed, consider an algebra with orthogonal involution (A,σ), of degree 2^i for some $i\in\{2,3\}$. By definition, it is a Pfister algebra with involution if and only if the form q_σ belongs to $GP_i(F_A)$. As recalled in §1, this is also equivalent to saying that $e_1(q_\sigma)=0$ if $i=2$, and $e_1(q_\sigma)=e_2(q_\sigma)=0$ if $i=3$. From this we get the following:

PROPOSITION 3.8

The degree 4 algebra with orthogonal involution (A,σ) is a Pfister algebra with involution if and only if $e_1(A,\sigma)=0$. The degree 8 algebra with orthogonal involution (A,σ) is a Pfister algebra with involution if and only if $e_1(A,\sigma)=e_2(A,\sigma)=0$.

Using this, we are now able to prove Proposition 2.10.

3.3 *Proof of Proposition 2.10*

Comparing Proposition 3.8 with Theorems 2.8 and 2.9 we get the equivalence between (i) and (ii), using ([KMRT98], (9.14)) in the degree 8 case. Moreover, as already noticed in Remark 2.5, any involution of type $I\Rightarrow H$ is a Pfister involution. Hence, it only remains to prove that a product of r quaternions with involution with $r\leq 3$ is of type $I\Rightarrow H$. Let us consider such a product of quaternions with involution (A,σ) and assume it is isotropic. Then, A cannot be a division algebra and has index at most 2^{r-1}.

If A is split, σ is the adjoint involution with respect to an isotropic Pfister form. Hence it is hyperbolic, and this concludes the proof in that case.

Assume now that the index of A is 2^{r-1}, and let D be a division algebra Brauer-equivalent to A. We may represent (A,σ) as $(\mathrm{End}_D(M),\mathrm{ad}_h)$, where (M,h) is a rank 2 hermitian module over D. Again, since σ is isotropic, h is isotropic, hence hyperbolic because of its rank, and this concludes the proof in that case. If $r=2$, we are done, and it only remains to consider the case when $r=3$ and A has index 2. Let Q be a quaternion division algebra

Brauer-equivalent to A, denote by γ its canonical involution, and let (M, h) be a skew-hermitian module over (Q, γ) such that $(A, \sigma) = (\mathrm{End}_Q(M), \mathrm{ad}_h)$. Denote by C the conic associated to Q, and by L its function field, which is known to be a generic splitting field for Q, and hence for A. Since A_L is split, σ_L, and hence h_L are hyperbolic. By ([PSS01], Proposition 3.3) (see also [Dej01]), this implies that h itself, and hence σ is hyperbolic, and the proof is complete.

3.4 *About the e_3 invariant*

As opposed to what happens for e_0, e_1 and e_2, there does not exist any invariant in $H^3(F)/E_3(A)$ which is a descent of $e_3(q_\sigma)$ for degree 8 Pfister algebras with involution, as the following theorem shows:

Theorem 3.9. *There exists a degree 8 Pfister algebra with involution for which the invariant $e_3(q_\sigma)$ does not belong to the image of the morphism $\Phi : H^3(F)/E_3(A) \to H^3_{nr}(F(X_A))$.*

Proof. In his paper 'Simple algebras and quadratic forms', Merkurjev ([Mer92], proof of Theorem 4) constructs a division algebra A, which is a product of three quaternion algebras, $A = Q_1 \otimes Q_2 \otimes Q_3$, and with centre a field F of cohomological dimension at most 2. In particular, we have $H^3(F) = 0$. Consider any orthogonal decomposable involution $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ on A. By Proposition 2.10, (A, σ) is a degree 8 Pfister algebra with involution. Moreover, by a result of Karpenko ([Kar00], Theorem 5.3), since A is a division algebra, the involution σ remains anisotropic over $F(X_A)$. Hence, q_σ is an anisotropic 3-fold Pfister form, and $e_3(q_\sigma)$ is non-trivial. Since $H^3(F) = 0$, this is enough to prove that $e_3(q_\sigma)$ does not belong to the image of Φ. $\square$

Remark 3.10. Using Merkurjev's construction of division product of quaternions with involution mentioned in the proof of Theorem 3.9, one may construct explicit elements in the unramified cohomology $H^i_{nr}(F_A/F)$ for any $i \geq 3$ for which Shapiro's conjecture is known, which do not come from $H^i(F)$.

4. Product of four quaternions with involution

In this section, we give a direct proof of Shapiro's conjecture for $r = 4$, i.e. we prove that any product of four quaternions with involution is a Pfister algebra with involution.

By Proposition 2.3 and Corollary 1.2, it suffices to prove the following proposition:

PROPOSITION 4.1

Let (A, σ) be a product of four quaternions with involution. If A is split and σ is isotropic, then it is adjoint to a hyperbolic quadratic form.

Let $(A, \sigma) = \otimes_{i=1}^4 (Q_i, \gamma_i)$, and assume A is split and σ is isotropic. By Remark 2.7 (ii), we may assume that each γ_i is the canonical involution on Q_i. Let us denote by $(D, \gamma) = (Q_1, \gamma_1) \otimes (Q_2, \gamma_2)$. We start with a lemma which gives a description of (A, σ):

Lemma 4.2. *There exists an invertible element $u \in D^\star$ satisfying $\gamma(u) = u$, $\mathrm{Trd}_D(u) = 0$ and $\mathrm{Nrd}_D(u) \in F^{\star 2}$ such that*

$$(Q_3, \gamma_3) \otimes (Q_4, \gamma_4) \simeq (D, \mathrm{Int}(u^{-1}) \circ \gamma)$$

and

$$(A,\sigma)\simeq(\mathrm{End}_F(D),\mathrm{ad}_{q_u}),$$

where q_u is the quadratic form defined on D by $q_u(x)=\mathrm{Trd}_D(xu\gamma(x))$.

Proof. Since A is split, $Q_3\otimes Q_4$ is isomorphic to D, and $\gamma_3\otimes\gamma_4$ corresponds under this isomorphism to an orthogonal involution γ' on D. There exists an invertible γ-symmetric element $u\in D$ such that $\gamma'=\mathrm{Int}(u^{-1})\circ\gamma$. Moreover, since $\gamma_3\otimes\gamma_4$ is decomposable, by Theorem 2.8, its discriminant is trivial. Hence, so is the discriminant of γ', and by ([KMRT98], (7.3)(1)), we get that $\mathrm{Nrd}_D(u)\in F^{\star 2}$.

Using this, we now get that (A,σ) is isomorphic to

$$(D\otimes D,\gamma\otimes\mathrm{Int}(u^{-1})\circ\gamma).$$

By ([KMRT98], (11.1)), under the canonical isomorphism $D\otimes D\simeq\mathrm{End}_F(D)$, the involution $\gamma\otimes\mathrm{Int}(u^{-1})\circ\gamma$ is adjoint to the quadratic form $q_u : D\to F$ defined by $q_u(x)=\mathrm{Trd}_D(xu\gamma(x))$, and it only remains to prove that we may assume $\mathrm{Trd}_D(u)=0$.

Since the isotropic involution σ is adjoint to q_u, the quadratic form also is isotropic. Moreover, by a general position argument, there exists an invertible element $y\in D$ such that $q_u(y)=\mathrm{Trd}_D(yu\gamma(y))=0$. Then, the map $D\to D$, $x\mapsto xy^{-1}$ is an isometry between q_u and the quadratic form $q_{yu\gamma(y)} : x\mapsto\mathrm{Trd}_D(xyu\gamma(y)\gamma(x))$. One may easily check that this new element $yu\gamma(y)$ satisfies all properties of the lemma, including $\mathrm{Trd}_D(yu\gamma(y))=0$, and this ends the proof. □

To get Proposition 4.1, we now have to prove that the quadratic form q_u is hyperbolic. This follows easily from the following lemma:

Lemma 4.3. *The quadratic space (D,q_u) contains a totally isotropic subspace of dimension* 5.

Indeed, by the computations of classical invariants for tensor product of algebras with involution given in ([KMRT98], (7.3)(4) and p. 150), we have $e_0(A,\sigma)=e_1(A,\sigma)=e_2(A,\sigma)=0$. Hence, by Proposition 3.6, the quadratic form q_u has trivial e_0, e_1 and e_2 invariants, and as recalled in §1, this implies that it lies in $I^3(K)$.

Since q_u is 16 dimensional, the previous lemma implies its anisotropic dimension is at most 6. By Arason–Pfister's theorem, this implies that q_u is hyperbolic, and thus concludes the proof of Proposition 4.1.

Proof of Lemma 4.3. For any $z\in D^\star$, we denote by q_z the quadratic form $D=Q_1\otimes Q_2\to F$, $x\mapsto\mathrm{Trd}_D(xz\gamma(x))$. We first prove the following fact:

Claim 4.4. Let $z\in D^\star$ satisfy $\mathrm{Trd}_D(z)=0$. We then have

(i) Q_1 is totally isotropic for q_z;
(ii) for all $x\in Q_1$, $\mathrm{Trd}_D(xz)=0$.

Indeed, for any $x\in Q_1$, we have $x\gamma(x)=x\gamma_1(x)=\mathrm{Nrd}_D(x)$. Hence, $q_z(x)=\mathrm{Trd}_D(xzx^{-1}\mathrm{Nrd}_D(x))=\mathrm{Nrd}_D(x)\mathrm{Trd}_D(z)=0$. Moreover, considering the corresponding bilinear form, we also get that for any $x,y\in Q_1$, $\mathrm{Trd}_D(xz\gamma(y))=0$, and this gives the second part of the claim by taking $y=1$.

Let us now denote by ϕ the isomorphism $(Q_3, \gamma_3) \otimes (Q_4, \gamma_4) \simeq (D, \mathrm{Int}(u^{-1}) \circ \gamma)$ of Lemma 4.2. For any pure quaternion $q \in Q_4^0$, we let W_q be the image under ϕ of the 3-dimensional subspace $\{x \otimes q,\ x \in Q_3^0\}$ of $Q_3 \otimes Q_4$. We then have

Claim 4.5. The subspace $\gamma(W_q)$ of D is totally isotropic for q_u.

Indeed, from the corresponding properties for $\{x \otimes q,\ x \in Q_3^0\}$, any element $y \in W_q$ satisfies $y^2 \in F$ and $\mathrm{Int}(u^{-1}) \circ \gamma(y) = y$. Hence, we have $\gamma(y)u = uy$ and $q_u(\gamma(y)) = \mathrm{Trd}_D(\gamma(y)uy) = \mathrm{Trd}_D(uy^2) = y^2\mathrm{Trd}_D(u) = 0$.

Let us now denote by T the kernel of the linear form on D defined by $z \mapsto \mathrm{Trd}_D(u\gamma(z))$. Clearly, $T \cap \gamma(W_q)$ has dimension at least 2. Fix a 2 dimensional subspace $V_q \subset T \cap \gamma(W_q)$. We then have

Claim 4.6. The subspace $Q_1 + V_q$ of D is totally isotropic for q_u.

Indeed, Q_1 is totally isotropic by Claim 4.4, and since $V_q \subset \gamma(W_q)$, it also is by Claim 4.5. Moreover, for any $z \in V_q \subset T$, we have $\mathrm{Trd}_D(u\gamma(z)) = 0$. Hence by Claim 4.4(ii), $\mathrm{Trd}_D(xu\gamma(z)) = 0$ for any $x \in Q_1$. Hence Q_1 and V_q are orthogonal, and we get the claim.

To finish with, it only remains to prove that there exists some $q \in Q_4^0$ such that $Q_1 + V_q$ has dimension greater than 5, i.e. V_q is not contained in Q_1. But if V_q is contained in Q_1, then it is contained in Q_1^0 which has dimension 3. One may then choose another pure quaternion $q' \in Q_4^0$ which is linearly independent from q. This way, we get another 2 dimensional subspace $V_{q'}$ which is in direct sum with V_q, and which cannot also be contained in Q_1^0. □

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