$1/f$ Noise Generated with the Branching Process Model

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***Abstract*--It was demonstrated that the branching process model is useful to generate a series having a $1/f$ spectrum in a wide range of frequency more than seven decades.**

Keywords-- $1/f$ noise, branching process, computer simulation

I . INTRODUCTION

In the stochastic process such as the Markov process, an event in the process is strongly influenced by the event happened just before. The time series composed of these events shows usually a $1/f^2$-like spectrum. In the present work, a medium, in which many particles exist and each single particle may be divided into several particles and be absorbed, is considered, where the particle number at a moment is decided stochastically by the number just before. If these particles are observed by a detector, the detecting events may show another kind of stochastic behavior. This is the basic idea of this work and is illustrated in Fig. 1.

The particle numbers at time t_1 and t_2 in Fig. 1 are 7 and 10, respectively. A detection may correlated with another detection through the branching paths as given by *a*, *b* and *c* in the figure. The detection *d* has no correlation with *a*, *b* and *c*, because it is on another branching chain different from that of *a*, *b* and *c*. The length of the path between the detections has statistical correlation with the physical time interval. The time interval, for example, between *a* and *b* is approximately equivalent to that between *b* and *c*, but the correlation between *b* and *c* may be far weaker than that between *a* and *b*, because the path between *b* and *c* is longer than that between *a* and *b*.

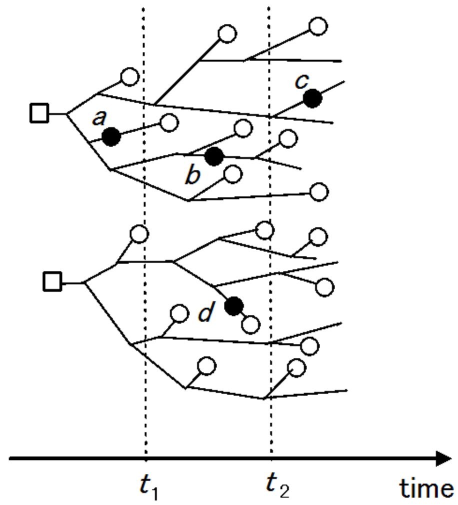

Fig. 1. Chain of the branching process. The square, circle and black spot represent a particle immigrated randomly in the medium, absorption and detection of a particle, respectively. The path of a particle is shown by a full line.

II . DETECTION PROBABILITY

We suppose a medium in which a particle may be subjected to absorption, branching multiplicative reaction, and detection with the probabilities λ_a, λ_m and λ_d, respectively. The statistics of the particles and detections in the medium can be obtained in close forms. Their detailed mathematical expressions are given in my previous works [1] and [2] by using the branching process model. In the present work, we focused only on the case that exactly two particles are produced by a branching reaction, a detection of a particle has no influence on the particle number and the absorption rate is equivalent to the branching rate, i.e., $\lambda_a = \lambda_m$. The case that a particle is absorbed by detection and the case of $\lambda_a \neq \lambda_m$ are discussed in [1], [2], [3] and [4].

We consider the probability $P_k(m,n,t)$ that *m* detection counts have been recorded during the time interval (0, *t*) and *n* particles are found in the medium at time $t > 0$ after *k* particles exist at $t = 0$. It is rather complicated to obtain this probability for every single value of *m*>0, but the probability $P_k(0,n,t)$ can be described closely as

$$P_k(0,n,t) = K_k^{(0,n)} = \sum_{l=0}^{n} p(0,l,t) \cdot K_{k-1}^{(0,n-l)}, \quad (1)$$

where

$$K_0^{(0,j)} = \delta_{j,0} \quad (2)$$

and

$$p(0,l,t) = \begin{cases} \eta_0 \xi_0 V & (l=0) \\ \frac{(\eta_0-\xi_0)^2 e^{-\theta_0 t}}{(\eta_0-\xi_0 e^{-\theta_0 t})^2} & (l=1) \\ V \cdot p(0,l-1,t) & (l \geq 2) \end{cases}. \quad (3)$$

Here

$$V = \frac{1-e^{-\theta_0 t}}{\eta_0-\xi_0 e^{-\theta_0 t}}, \quad (4)$$

$$\begin{aligned} \theta_0 &= \lambda_a \sqrt{\varepsilon(\varepsilon+4)} \\ \eta_0 &= \frac{(1+\varepsilon+\sqrt{\varepsilon(\varepsilon+4)})}{2} \\ \xi_0 &= \frac{(1+\varepsilon-\sqrt{\varepsilon(\varepsilon+4)})}{2} \end{aligned} \quad (5)$$

and

$$\varepsilon = \frac{\lambda_d}{\lambda_a}. \quad (6)$$

The probability that we get some detections during the time interval (0, *t*) and *n* particles are found in the medium at $t > 0$ after we had *k* particles at $t = 0$ is given by

$$\sum_{m=1}^{\infty} P_k(m,n,t) = P_k(n,t) - P_k(0,n,t), \quad (7)$$

where $P_k(n,t)$ is the probability that *n* particles are found in the medium at time $t > 0$ after we had *k* particles at $t = 0$. When the probability (7) is much smaller than $P_k(0,n,t)$, the probability recording more than two counts can be negligible and the following relation holds approximately,

$$P_k(1,n,t) \cong P_k(n,t) - P_k(0,n,t). \quad (8)$$

The probability $P_k(n,t)$ is given in [1], [2] and [3] as

$$P_k(n,t) = K_k^n = \sum_{l=0}^{n} p(l,t) \cdot K_{k-1}^{n-l}, \quad (9)$$

where

$$K_0^i = \delta_{i,0}, \quad (10)$$

$$p(l,t) = \begin{cases} W & (l=0) \\ \left(\frac{W}{\lambda_a t}\right)^2 & (l=1) \\ W \cdot p(l-1,t) & (l \geq 2) \end{cases} \quad (11)$$

and

$$W = \frac{\lambda_a t}{\lambda_a t+1}. \quad (12)$$

Ⅲ. COMPUTER SIMULATIONS

Whether a particle detection has occurred or not in a very short time interval was decided successively by using the Monte Carlo method with the probabilities described in (1), (8) and (9), and the series formed by the time intervals between two successive detections (count series) was generated in the case of $\varepsilon = 1$. The obtained series was analyzed using the fast Fourier transformation technique (FFT) and the power spectral density (PSD) was calculated. One of the results is shown in Fig. 2, where the frequency is related to the detection counts not to time. The PSD behaves like a $1/f$ distribution for six decades or more of frequency when the series size is shorter than 2097152. The PSD behavior starts to deviate from the $1/f$ line in a low-frequency range when the series size is over 4194304. The condition $\varepsilon = 1$ means that each particle is detected once, on average, before absorbed in the medium. When $\varepsilon < 1$, the PSD deviates from $1/f$ line at a shorter series size [3]. On the other hand, when $\varepsilon > 1$, the PSD is very similar to the case of $\varepsilon = 1$ [3].

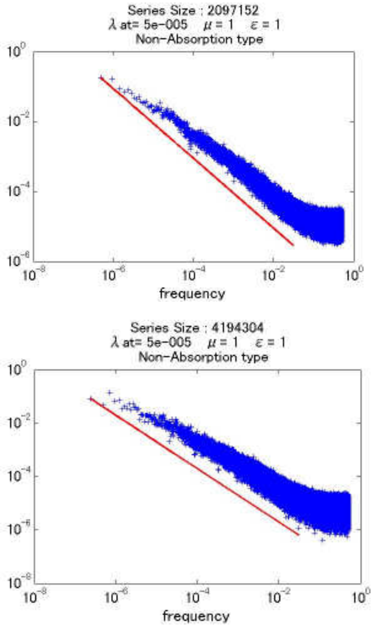

Fig. 2. The PSD of the count series for the case of $\varepsilon = 1$. The series sizes are 2097152 and 4193404 for the upper and lower PSD, respectively. The straight lines give the $1/f$ behavior.

IV. RESULTS AND DISCUSSIONS

There are at least two limitations on performing the computer simulations. The existing particle number should be avoided to be zero, because no branching will arise from no particle situation. A very large number of particles takes unreasonably long time to process on a computer, and so it should be set a limit on the number of particles. In the present simulations the upper limit of the number of particles (maximum particle number) was set to 1000. The effect of the limitation of the particle number on the PDS was examined, the results of which is shown in Fig. 3, where the count series were generated under the same condition with the case in Fig. 2 but the upper limits were set to 200, 400 and 700. As can be seen in Fig. 3, the frequency range with the $1/f$ behavior of the spectrum is so sensitive to the limit, and is increasing steadily with the maximum particle number. It indicates clearly that the frequency range with the $1/f$ behavior can be extended more if the upper limit is set to larger than 1000. It is not sure that this trend in Fig. 3 continues on and on endlessly, but if the limit is set up to 2000 or more, the PSD may behave like $1/f$ for about eight decades of frequency.

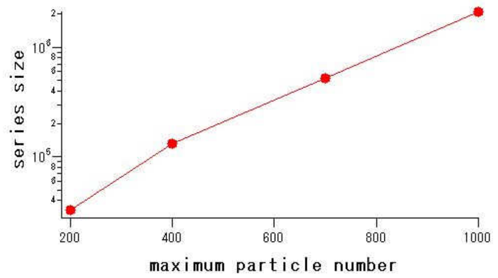

Fig. 3. Relation of the longest series size with the $1/f$ behavior and the maximum particle number.

In order to see the above expectation, two count series with the upper limits set to 2000 and 3000 were generated in the case of $\varepsilon = 1$. Their obtained results are shown in Figs. 4 and 5, respectively. In Fig. 4, the PSD of the series size of 33554432 behaves like a $1/f$ distribution, but it turns off from the $1/f$ line in a low-frequency range at a longer series size. In Fig. 5, when the series size is 67108864, the PSD has a $1/f$ distribution over seven decades of frequency, but at a longer size of the series it is not clear whether the PSD deviates from the $1/f$ line or not. It was difficult to generate a fully long series in this case because of limited computing time, and so the statistical precision of the FFT results in Fig. 5 is insufficient. It can be said, however, that the frequency range having the $1/f$ behavior spreads out wider compared with the case that the upper limit is

2000.

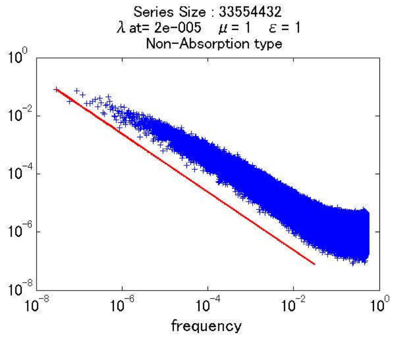

Fig. 4. The PSD of the count series when the maximum particle number is set to 2000. The series size is 33554432, and the straight line gives the $1/f$ behavior.

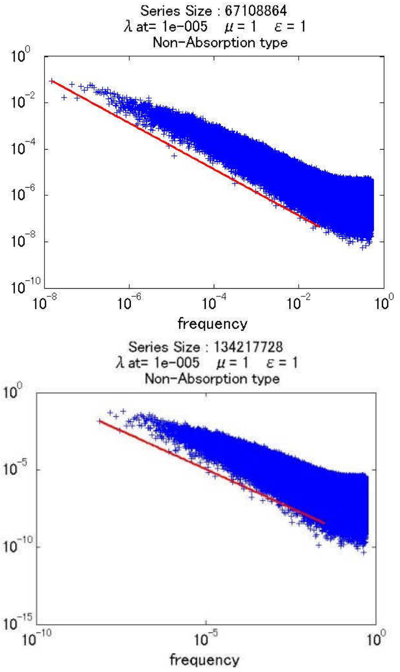

Fig. 5. The PSD of the count series when the maximum particle number is set to 3000. The series size are 67108864 (upper) and 134217728 (lower). The straight lines give the $1/f$ behavior.

V. CONCLUSION

The branching process model has been applied to discuss the $1/f$ problem, and the $1/f$ behavior has been demonstrated in the spectrum of the obtained series in a wide range, as wide as seven decades, of frequency. This frequency range is expected to expand up to eight decades or more by simulating on a higher-speed computer. It is expected, from the present results, that the $1/f$ behavior may be realized by observing a part of the familiar $1/f^2$ phenomena.

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