Non-local, non-convex functionals converging to Sobolev norms

Haïm Brezis^{a,b,c}, Hoai-Minh Nguyen^{d,*}^a Department of Mathematics, Rutgers University, Hill Center, Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA^b Departments of Mathematics and Computer Science, Technion, Israel Institute of Technology, 32.000 Haifa, Israel^c Laboratoire Jacques-Louis Lions, Sorbonne Universités, UPMC Université Paris-6, 4 place Jussieu, 75005 Paris, France^d Department of Mathematics, EPFL SB CAMA, Station 8 CH-1015 Lausanne, Switzerland

ARTICLE INFO

Article history:

Received 3 September 2019

Accepted 5 September 2019

Communicated by Vicentiu Radulescu

Keywords:

Non-local

Non-convex

Pointwise convergence

 Γ -convergence

Sobolev norms

ABSTRACT

We study the pointwise convergence and the Γ -convergence of a family of non-local, non-convex functionals Λ_δ in $L^p(\Omega)$ for $p > 1$. We show that the limits are multiples of $\int_\Omega |\nabla u|^p$. This is a continuation of our previous work where the case $p = 1$ was considered.

© 2019 Elsevier Ltd. All rights reserved.

1. Introduction and statement of the main results

Assume that $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is defined at every point of $[0, +\infty)$, φ is continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$ where it admits a limit from the left and from the right, and $\varphi(0) = 0$. Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) denote a domain which is either bounded and smooth, or $\Omega = \mathbb{R}^d$. Given a measurable function u on Ω , and a parameter $\delta > 0$, we define the following non-local functionals, for $p > 1$,

$$\Lambda(u, \Omega) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{p+d}} dx dy \quad \text{and} \quad \Lambda_\delta(u, \Omega) := \delta^p \Lambda(u/\delta, \Omega). \quad (1.1)$$

To simplify the notation, we will often delete Ω and write $\Lambda_\delta(u)$ instead of $\Lambda_\delta(u, \Omega)$.

As in [3], we consider the following four assumptions on φ :

$$\varphi(t) \leq at^{p+1} \text{ in } [0, 1] \text{ for some positive constant } a, \quad (1.2)$$

$$\varphi(t) \leq b \text{ in } \mathbb{R}_+ \text{ for some positive constant } b, \quad (1.3)$$

$$\varphi \text{ is non-decreasing}, \quad (1.4)$$

* Corresponding author at: Department of Mathematics, EPFL SB CAMA, Station 8 CH-1015 Lausanne, Switzerland.

E-mail addresses: brezis@math.rutgers.edu (H. Brezis), hoai-minh.nguyen@epfl.ch (H.-M. Nguyen).

and

$$\gamma_{d,p} \int_0^\infty \varphi(t) t^{-(p+1)} dt = 1, \text{ where } \gamma_{d,p} := \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p d\sigma \text{ for some } e \in \mathbb{S}^{d-1}. \quad (1.5)$$

In this paper, we study the pointwise and the Γ -convergence of Λ_δ as $\delta \rightarrow 0$ for $p > 1$. This is a continuation of our previous work [3] where the case $p = 1$ was investigated in great details. Concerning the pointwise convergence of Λ_δ , our main result is

Theorem 1. *Let $d \geq 1$ and $p > 1$. Assume (1.2), (1.3), and (1.5) (the monotonicity assumption (1.4) is not required here). We have*

(i) *There exists a positive constant $C_{p,\Omega}$ such that*

$$\Lambda_\delta(u, \Omega) \leq C_{p,\Omega} \int_\Omega |\nabla u|^p dx \quad \forall u \in W^{1,p}(\Omega), \forall \delta > 0; \quad (1.6)$$

moreover,

$$\lim_{\delta \rightarrow 0} \Lambda_\delta(u, \Omega) = \int_\Omega |\nabla u|^p dx \quad \forall u \in W^{1,p}(\Omega). \quad (1.7)$$

(ii) *Assume in addition that φ satisfies (1.4). Let $u \in L^p(\Omega)$ be such that*

$$\liminf_{\delta \rightarrow 0} \Lambda_\delta(u, \Omega) < +\infty, \quad (1.8)$$

then $u \in W^{1,p}(\Omega)$.

Remark 1. Theorem 1 provides a characterization of the Sobolev space $W^{1,p}(\Omega)$ for $p > 1$:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega); \liminf_{\delta \rightarrow 0} \Lambda_\delta(u) < +\infty \right\}.$$

This fact is originally due to Bourgain and Nguyen [1,4] when $\varphi = \hat{\varphi}_1 := c\mathbb{1}_{(1,+\infty)}$ for an appropriate constant c .

There are some similarities but also striking differences between the cases $p > 1$ and $p = 1$.

(a) First note a similarity. Let $p = 1$ and φ satisfy (1.2)–(1.4), and assume that $u \in L^1(\Omega)$ verifies

$$\liminf_{\delta \rightarrow 0} \Lambda_\delta(u, \Omega) < +\infty,$$

then $u \in BV(\Omega)$ (see [1,3]).

(b) Next is a major difference. Let $p = 1$. There exists $u \in W^{1,1}(\Omega)$ such that, for all φ satisfying (1.2)–(1.4), one has

$$\lim_{\delta \rightarrow 0} \Lambda_\delta(u, \Omega) = +\infty$$

[3, Pathology 1]. In particular, (1.6) and (1.7) do not hold for $p = 1$. An example in the same spirit was originally constructed by Ponce and is presented in [4]. Other pathologies occurring in the case $p = 1$ can be found in [3, Section 2.2].

As we will see later, the proof of (1.6) involves the theory of maximal functions. The use of this theory was suggested independently by Nguyen [4] and Ponce and van Schaftingen (unpublished communication to the authors). The proof of (1.6) uses the same strategy as in [4].

We point out that assertion (ii) fails without the monotonicity condition (1.4) on φ . Here is an example e.g. with $\Omega = \mathbb{R}$. Let $\varphi = c\mathbb{1}_{(1,2)}$ for an appropriate, positive constant c . Let $u = \mathbb{1}_{(0,1)}$. One can easily check that $\Lambda_\delta(u) = 0$ for $\delta \in (0, 1/2)$ and it is clear that $u \notin W^{1,p}(\mathbb{R})$ for $p > 1$.

Concerning the Γ -convergence of A_δ , our main result is

Theorem 2. *Let $d \geq 1$ and $p > 1$. Assume (1.2)–(1.5). Then*

$$A_\delta(\cdot, \Omega) \text{ } \Gamma\text{-converges in } L^p(\Omega) \text{ to } A_0(\cdot, \Omega) := \kappa \int_{\Omega} |\nabla \cdot|^p dx,$$

as $\delta \rightarrow 0$, for some constant κ which depends only on p and φ , and verifies

$$0 < \kappa \leq 1. \quad (1.9)$$

Theorem 2 was known earlier when $\varphi = \hat{\varphi}_1$ [5,6].

The paper is organized as follows. Theorem 1 is proved in Section 2 and the proof of Theorem 2 is given in Section 3. Throughout the paper, we denote

$$\varphi_\delta(t) := \delta^p \varphi(t/\delta) \text{ for } p > 1, \delta > 0, t \geq 0.$$

2. Proof of Theorem 1

In view of the fact that $\liminf_{t \rightarrow +\infty} \varphi(t) > 0$, assertion (1.8) is a direct consequence of [1, Theorem 1]; note that [1, Theorem 1] is stated for $\Omega = \mathbb{R}^d$ but the proof can be easily adapted to the case where Ω is bounded. It could also be deduced from Theorem 2.

We now establish assertions (1.6) and (1.7). The proof consists of two steps.

Step 1: Proof of (1.6) and (1.7) when $\Omega = \mathbb{R}^d$ and $u \in W^{1,p}(\mathbb{R}^d)$. Replacing y by $x + z$ and using polar coordinates in the z variable, we find

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{p+d}} dy = \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi_\delta(|u(x + h\sigma) - u(x)|)}{h^{p+1}} d\sigma. \quad (2.1)$$

We have

$$\begin{aligned} & \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi_\delta(|u(x + h\sigma) - u(x)|)}{h^{p+1}} d\sigma \\ &= \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\delta^p \varphi(|u(x + h\sigma) - u(x)|/\delta)}{h^{p+1}} d\sigma. \end{aligned} \quad (2.2)$$

Rescaling the variable h gives

$$\begin{aligned} & \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\delta^p \varphi(|u(x + h\sigma) - u(x)|/\delta)}{h^{p+1}} d\sigma \\ &= \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi(|u(x + \delta h\sigma) - u(x)|/\delta)}{h^{p+1}} d\sigma. \end{aligned} \quad (2.3)$$

Combining (2.1), (2.2), and (2.3) yields

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+p}} dy = \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi(|u(x + \delta h\sigma) - u(x)|/\delta)}{h^{p+1}} d\sigma. \quad (2.4)$$

Note that

$$\lim_{\delta \rightarrow 0} \frac{|u(x + \delta h\sigma) - u(x)|}{\delta} = |\langle \nabla u(x), \sigma \rangle| h \text{ for a.e. } (x, h, \sigma) \in \mathbb{R}^d \times [0, +\infty) \times \mathbb{S}^{d-1}. \quad (2.5)$$

Here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. Since φ is continuous at 0 and on $(0, +\infty)$ except at a finite number of points, it follows that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{1}{h^{p+1}} \varphi(|u(x + \delta h\sigma) - u(x)|/\delta) &= \frac{1}{h^{p+1}} \varphi(|\langle \nabla u(x), \sigma \rangle| h) \\ &\text{for a.e. } (x, h, \sigma) \in \mathbb{R}^d \times (0, +\infty) \times \mathbb{S}^{d-1}. \end{aligned} \quad (2.6)$$

Rescaling once more the variable h gives

$$\int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^{p+1}} \varphi(|\langle \nabla u(x), \sigma \rangle| h) d\sigma = |\nabla u(x)|^p \int_0^\infty \varphi(t) t^{-(p+1)} dt \int_{\mathbb{S}^{d-1}} |\langle \sigma, e \rangle|^p d\sigma; \quad (2.7)$$

here we have also used the obvious fact that, for every $V \in \mathbb{R}^d$, and for any fixed $e \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} |\langle V, \sigma \rangle|^p d\sigma = |V|^p \int_{\mathbb{S}^{d-1}} |\langle e, \sigma \rangle|^p d\sigma.$$

Thus, by the normalization condition (1.5), we obtain

$$\int_{\mathbb{R}^d} dx \int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^{p+1}} \varphi(|\langle \nabla u(x), \sigma \rangle| h) d\sigma = \int_{\mathbb{R}^d} |\nabla u|^p dx. \quad (2.8)$$

Set

$$\tilde{\varphi}(t) = \begin{cases} at^{p+1} & \text{for } t \in [0, 1), \\ b & \text{for } t \in [1, +\infty). \end{cases}$$

Then

$$\tilde{\varphi} \text{ is non-decreasing and } \varphi \leq \tilde{\varphi}. \quad (2.9)$$

Note that, for a.e. $(x, h, \sigma) \in \mathbb{R}^d \times (0, +\infty) \times \mathbb{S}^{d-1}$,

$$\frac{|u(x + \delta h \sigma) - u(x)|}{\delta} \leq \frac{1}{\delta} \int_0^{h\delta} |\langle \nabla u(x + s\sigma), \sigma \rangle| ds \leq hM(\nabla u, \sigma)(x), \quad (2.10)$$

where

$$M(\nabla u, \sigma)(x) := \sup_{t>0} \frac{1}{t} \int_0^t |\langle \nabla u(x + s\sigma), \sigma \rangle| ds.$$

Combining (2.4) and (2.10), we derive from (2.9) that

$$\begin{aligned} \Lambda_\delta(u) &\leq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_0^\infty \frac{\tilde{\varphi}(h|M(\nabla u, \sigma)(x)|)}{h^{p+1}} dh dx d\sigma \\ &= \int_0^{+\infty} \tilde{\varphi}(t) t^{-(p+1)} dt \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} |M(\nabla u, \sigma)(x)|^p dx d\sigma. \end{aligned} \quad (2.11)$$

We claim that, for $\sigma \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{R}^d} |M(\nabla u, \sigma)(x)|^p dx \leq C_p \int_{\mathbb{R}^d} |\nabla u(x)|^p dx. \quad (2.12)$$

For notational ease, we will only consider the case $\sigma = e_1$. By the theory of maximal functions (see e.g. [7]), one has, for $g \in L^p(\mathbb{R})$,

$$\int_{\mathbb{R}} \left| \sup_{t>0} \int_{\xi-t}^{\xi+t} |g(s)| ds \right|^p d\xi \leq C_p \int_{\mathbb{R}} |g(\xi)|^p d\xi.$$

Using this inequality with $g(x_1) = \partial_{x_1} u(x_1, x')$ for $x' \in \mathbb{R}^{d-1}$, we obtain

$$\int_{\mathbb{R}} |M(\nabla u, e_1)(x_1, x')|^p dx_1 \leq C_p \int_{\mathbb{R}} |\partial_{x_1} u(x_1, x')|^p dx_1.$$

Integrating with respect to x' yields

$$\int_{\mathbb{R}^d} |M(\nabla u, e_1)(x)|^p dx \leq C_p \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\partial_{x_1} u(x_1, x')|^p dx_1 dx' \leq C_p \int_{\mathbb{R}^d} |\nabla u(x)|^p dx,$$

and (2.12) follows.

Using (2.12), we deduce from (2.11) that

$$A_\delta(u) \leq C_{p,d} \int_{\mathbb{R}^d} |\nabla u|^p dx,$$

which is (1.6). From (2.6), (2.7), (2.8), and (2.10) we derive, using the dominated convergence theorem, that

$$\lim_{\delta \rightarrow 0} A_\delta(u) = \int_{\mathbb{R}^d} |\nabla u|^p dx.$$

This completes Step 1.

Step 2: Proof of (1.6) and (1.7) when Ω is bounded and $u \in W^{1,p}(\Omega)$. We first claim that

$$\lim_{\delta \rightarrow 0} A_\delta(u) = \int_{\Omega} |\nabla u|^p \text{ for } u \in W^{1,p}(\Omega). \quad (2.13)$$

Indeed, consider an extension of u in $\mathbb{R}^d$ which belongs to $W^{1,p}(\mathbb{R}^d)$, and is still denoted by u . By the same method as in the case $\Omega = \mathbb{R}^d$, we have

$$\lim_{\delta \rightarrow 0} \int_{\Omega} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{p+d}} dy = \int_{\Omega} |\nabla u|^p dx \quad (2.14)$$

and, for $D \Subset \Omega$ and $\varepsilon > 0$,

$$\lim_{\delta \rightarrow 0} \int_D dx \int_{B(x,\varepsilon)} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{p+d}} dy = \int_D |\nabla u|^p dx. \quad (2.15)$$

Combining (2.14) and (2.15) yields (2.13).

We next show that

$$A_\delta(u) \leq C_{p,\Omega} \int_{\Omega} |\nabla u|^p dx \text{ for } u \in W^{1,p}(\Omega). \quad (2.16)$$

Without loss of generality, we may assume that $\int_{\Omega} u = 0$. Consider an extension U of u in $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |\nabla U|^p dx \leq C_{p,\Omega} \int_{\Omega} |\nabla u|^p dx.$$

Such an extension exists since Ω is smooth and $\int_{\Omega} u = 0$, see, e.g., [2, Chapter 9]. Using the fact

$$A_\delta(u, \Omega) \leq A_\delta(U, \mathbb{R}^d) \leq C_{p,d} \int_{\mathbb{R}^d} |\nabla U|^p dx,$$

we get (2.16). The proof is complete. $\square$

3. Proof of Theorem 2

We first recall the meaning of Γ -convergence. One says that $A_\delta(\cdot, \Omega) \xrightarrow{\Gamma} A_0(\cdot, \Omega)$ in $L^p(\Omega)$ as $\delta \rightarrow 0$ if

(G1) For each $g \in L^p(\Omega)$ and for every family $(g_\delta) \subset L^p(\Omega)$ such that (g_δ) converges to g in $L^p(\Omega)$ as $\delta \rightarrow 0$, one has

$$\liminf_{\delta \rightarrow 0} A_\delta(g_\delta, \Omega) \geq A_0(g, \Omega).$$

(G2) For each $g \in L^p(\Omega)$, there exists a family $(g_\delta) \subset L^p(\Omega)$ such that (g_δ) converges to g in $L^p(\Omega)$ as $\delta \rightarrow 0$, and

$$\limsup_{\delta \rightarrow 0} A_\delta(g_\delta, \Omega) \leq A_0(g, \Omega).$$

Denote Q the unit open cube, i.e., $Q = (0, 1)^d$ and set

$$U(x) = d^{-1/2} \sum_{j=1}^d x_j \text{ in } Q,$$

so that $|\nabla U| = 1$ in Q .

In the following two subsections, we establish properties (G1) and (G2) where κ is the constant defined by

$$\kappa = \inf \liminf_{\delta \rightarrow 0} \Lambda_\delta(v_\delta, Q). \quad (3.1)$$

Here the infimum is taken over all families of functions $(v_\delta) \subset L^p(Q)$ such that $v_\delta \rightarrow U$ in $L^p(Q)$ as $\delta \rightarrow 0$.

3.1. Proof of Property (G1)

We begin with

Lemma 1. *Let $d \geq 1$, $p > 1$, S be an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, and let g be an affine function. Then*

$$\inf \liminf_{\delta \rightarrow 0} \Lambda_\delta(g_\delta, S) = \kappa |\nabla g|^p |S|, \quad (3.2)$$

where the infimum is taken over all families $(g_\delta) \subset L^p(S)$ such that $g_\delta \rightarrow g$ in $L^p(S)$ as $\delta \rightarrow 0$.

Proof. The proof of Lemma 1 is based on the definition of κ in (3.1) and a covering argument. It is identical to the one of the first part of [3, Lemma 6]. The details are omitted. $\square$

The proof of Property (G1) for $p > 1$ relies on the following lemma with roots in [6].

Lemma 2. *Let $d \geq 1$, $p > 1$, and $\varepsilon > 0$. There exist two positive constants $\hat{\delta}_1, \hat{\delta}_2$ such that for every open cube $\tilde{Q}$ which is an image of Q by a dilation, for every $a \in \mathbb{R}^d$, every $b \in \mathbb{R}$, and every $h \in L^p(\tilde{Q})$ satisfying*

$$\int_{\tilde{Q}} |h(x) - (\langle a, x \rangle + b)|^p dx \leq \hat{\delta}_1 |a|^p |\tilde{Q}|^{p/d}, \quad (3.3)$$

one has

$$\Lambda_\delta(h, \tilde{Q}) \geq (\kappa - \varepsilon) |a|^p |\tilde{Q}| \text{ for } \delta \in (0, \hat{\delta}_2 |a| |\tilde{Q}|^{1/d}). \quad (3.4)$$

Hereafter, as usual, we denote $f_A f = \frac{1}{|A|} \int_A f$.

Proof. By a change of variables, without loss of generality, it suffices to prove Lemma 2 in the case $\tilde{Q} = Q$, $|a| = 1$, and $b = 0$. We prove this by contradiction. Suppose that this is not true. There exist $\varepsilon_0 > 0$, a sequence of measurable functions $(h_n) \subset L^p(Q)$, a sequence $(a_n) \subset \mathbb{R}^d$, and a sequence (δ_n) converging to 0 such that $|a_n| = 1$,

$$\int_Q |h_n(x) - \langle a_n, x \rangle|^p \leq \frac{1}{n}, \quad \text{and} \quad \Lambda_{\delta_n}(h_n, Q) < \kappa - \varepsilon_0.$$

Without loss of generality, we may assume that (a_n) converges to a for some $a \in \mathbb{R}^d$ with $|a| = 1$. It follows that (h_n) converges to $\langle a, \cdot \rangle$ in $L^p(Q)$. Applying Lemma 1 with $S = Q$ and $g = \langle a, \cdot \rangle$, we obtain a contradiction. The conclusion follows. $\square$

The second key ingredient in the proof of Property (G1) is the following useful property of functions in $W^{1,p}(\mathbb{R}^d)$.

Lemma 3. Let $d \geq 1$, $p > 1$, and $u \in W^{1,p}(\mathbb{R}^d)$. Given $\varepsilon_1 > 0$, there exist a subset $B = B(\varepsilon_1)$ of Lebesgue points of u and ∇u , and an integer $\ell = \ell(\varepsilon_1) \geq 1$ such that

$$\int_{\mathbb{R}^d \setminus B} |\nabla u|^p dx \leq \varepsilon_1 \int_{\mathbb{R}^d} |\nabla u|^p dx, \quad (3.5)$$

and, for every open cube Q' with $|Q'|^{1/d} \leq 1/\ell$ and $Q' \cap B \neq \emptyset$, and for every $x \in Q' \cap B$,

$$\frac{1}{|Q'|^p} \int_{Q'} |u(y) - u(x) - \langle \nabla u(x), y - x \rangle|^p dy \leq \varepsilon_1 \quad (3.6)$$

and

$$|\nabla u(x)|^p \geq (1 - \varepsilon_1) \int_{Q'} |\nabla u(y)|^p dy. \quad (3.7)$$

Proof. We first recall the following property of $W^{1,p}(\mathbb{R}^d)$ functions (see e.g., [8, Theorem 3.4.2]): for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \rightarrow 0} \frac{1}{r^p} \int_{Q(x,r)} |u(y) - u(x) - \langle \nabla u(x), y - x \rangle|^p dy = 0, \quad (3.8)$$

where $Q(x, r) := x + (-r, r)^d$ for $x \in \mathbb{R}^d$ and $r > 0$.

Given $n \in \mathbb{N}$, define, for a.e. $x \in \mathbb{R}^d$,

$$\rho_n(x) = \sup \left\{ \frac{1}{r^p} \int_{Q(x,r)} |u(y) - u(x) - \langle \nabla u(x), y - x \rangle|^p dy; r \in (0, 1/n) \right\} \quad (3.9)$$

and

$$\tau_n(x) = \sup \left\{ \int_{Q(x,r)} |\nabla u(y) - \nabla u(x)|^p dy; r \in (0, 1/n) \right\}. \quad (3.10)$$

Note that, by (3.8), $\rho_n(x) \rightarrow 0$ for a.e. $x \in \mathbb{R}^d$ as $n \rightarrow +\infty$. We also have, $\tau_n(x) \rightarrow 0$ for a.e. $x \in \mathbb{R}^d$ as $n \rightarrow +\infty$ (and in fact at every Lebesgue point of ∇u). For $m \geq 1$, set

$$D_m = \left\{ x \in (-m, m)^d; x \text{ is a Lebesgue point of } u \text{ and } \nabla u, \text{ and } |\nabla u(x)| \geq 1/m \right\}.$$

Since

$$\lim_{m \rightarrow +\infty} \int_{\mathbb{R}^d \setminus D_m} |\nabla u|^p dx = 0,$$

there exists $m \geq 1$ such that

$$\int_{\mathbb{R}^d \setminus D_m} |\nabla u|^p dx \leq \frac{\varepsilon_1}{2} \int_{\mathbb{R}^d} |\nabla u|^p dx. \quad (3.11)$$

Fix such an m . By Egorov's theorem, there exists a subset $B \subset D_m$ such that (ρ_n) and (τ_n) converge to 0 uniformly on B , and

$$\int_{D_m \setminus B} |\nabla u|^p dx \leq \frac{\varepsilon_1}{2} \int_{\mathbb{R}^d} |\nabla u|^p dx. \quad (3.12)$$

Combining (3.11) and (3.12) yields (3.5).

By the triangle inequality, we have, for every non-empty, open cube Q' and a.e. $x \in \mathbb{R}^d$ (in particular for $x \in Q' \cap B$),

$$\left(\int_{Q'} |\nabla u(y)|^p dy \right)^{1/p} \leq \left(\int_{Q'} |\nabla u(y) - \nabla u(x)|^p dy \right)^{1/p} + |\nabla u(x)| \leq \frac{|\nabla u(x)|}{(1 - \varepsilon_1)^{1/p}}, \quad (3.13)$$

provided

$$\left(\int_{Q'} |\nabla u(y) - \nabla u(x)|^p dy \right)^{1/p} \leq \left(\frac{1}{(1 - \varepsilon_1)^{1/p}} - 1 \right) 1/m \quad \text{and} \quad |\nabla u(x)| \geq 1/m.$$

Since (ρ_n) and (τ_n) converge to 0 uniformly on B and $|\nabla u(x)| \geq 1/m$ for $x \in B$, it follows from (3.13) that there exists an $\ell \geq 1$ such that (3.6) and (3.7) hold when $|Q'|^{1/d} \leq 1/\ell$ and $Q' \cap B \neq \emptyset$, and $x \in Q' \cap B$. The proof is complete. $\square$

We are ready to give the

Proof of Property (G1). We only consider the case $\Omega = \mathbb{R}^d$. The other case can be handled as in [3] and is left to the reader. We follow the same strategy as in [6].

In order to establish Property (G1), it suffices to prove that

$$\liminf_{k \rightarrow +\infty} \Lambda_{\delta_k}(g_k, \mathbb{R}^d) \geq \kappa \int_{\mathbb{R}^d} |\nabla g|^p dx \quad (3.14)$$

for every $g \in L^p(\mathbb{R}^d)$, $(\delta_k) \subset \mathbb{R}_+$ and $(g_k) \subset L^p(\mathbb{R}^d)$ such that $\delta_k \rightarrow 0$ and $g_k \rightarrow g$ in $L^p(\mathbb{R}^d)$.

Without loss of generality, we may assume that $\liminf_{k \rightarrow +\infty} \Lambda_{\delta_k}(g_k, \mathbb{R}^d) < +\infty$. It follows from [6] that $g \in W^{1,p}(\mathbb{R}^d)$. Fix $\varepsilon > 0$ (arbitrary) and let $\hat{\delta}_1$ be the positive constant in Lemma 2. Set, for $m \geq 1$,

$$A_m = \left\{ x \in \mathbb{R}^d; x \text{ is a Lebesgue point of } g \text{ and } \nabla g, \text{ and } |\nabla g(x)| \leq 1/m \right\}.$$

Since

$$\lim_{m \rightarrow +\infty} \int_{A_m} |\nabla g|^p dx = 0,$$

there exists $m \geq 1$ such that

$$\int_{A_m} |\nabla g|^p dx \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla g|^p dx. \quad (3.15)$$

Fix such an integer m . By Lemma 3 applied to $u = g$ and $\varepsilon_1 = \min\{\varepsilon/2, \hat{\delta}_1/(2m)^p\}$, there exist a subset B of Lebesgue points of g and ∇g , and a positive integer ℓ such that

$$\int_{\mathbb{R}^d \setminus B} |\nabla g|^p dx \leq \varepsilon_1 \int_{\mathbb{R}^d} |\nabla g|^p dx \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla g|^p dx, \quad (3.16)$$

and for every open cube Q' with $|Q'|^{1/d} \leq 1/\ell$ and $Q' \cap B \neq \emptyset$, and, for every $x \in Q' \cap B$,

$$\frac{1}{|Q'|^{p/d}} \int_{Q'} |g(y) - g(x) - \langle \nabla g(x), y - x \rangle|^p dy \leq \varepsilon_1 \leq \hat{\delta}_1/(2m)^p \quad (3.17)$$

and

$$|\nabla g(x)|^p |Q'| \geq (1 - \varepsilon_1) \int_{Q'} |\nabla g|^p dy \geq (1 - \varepsilon) \int_{Q'} |\nabla g|^p dy. \quad (3.18)$$

Fix such a set B and such an integer ℓ . Set

$$B_m := B \setminus A_m.$$

Since $\mathbb{R}^d \setminus (B \setminus A_m) \subset (\mathbb{R}^d \setminus B) \cup A_m$, it follows that

$$\int_{\mathbb{R}^d \setminus B_m} |\nabla g|^p dx = \int_{\mathbb{R}^d \setminus (B \setminus A_m)} |\nabla g|^p dx \leq \int_{\mathbb{R}^d \setminus B} |\nabla g|^p dx + \int_{A_m} |\nabla g|^p dx.$$

We deduce from (3.15) and (3.16) that

$$\int_{\mathbb{R}^d \setminus B_m} |\nabla g|^p dx \leq \varepsilon \int_{\mathbb{R}^d} |\nabla g|^p dx. \quad (3.19)$$

Set $P_\ell = \frac{1}{\ell}\mathbb{Z}^d$. Let Ω_ℓ be the collection of all open cubes with side length $1/\ell$ whose vertices belong to P_ℓ and denote

$$\mathbf{J}_\ell = \left\{ Q' \in \Omega_\ell; Q' \cap B_m \neq \emptyset \right\}.$$

Take $Q' \in \mathbf{J}_\ell$ and $x \in Q' \cap B_m$. Since $g_k \rightarrow g$ in $L^p(Q')$, from (3.17), we obtain, for large k ,

$$\frac{1}{|Q'|^{p/d}} \int_{Q'} |g_k(y) - g(x) - \langle \nabla g(x), y - x \rangle|^p dy < \hat{\delta}_1/m^p \leq \hat{\delta}_1 |\nabla g(x)|^p,$$

since $|\nabla g(x)| \geq 1/m$ for $x \in B_m \subset \mathbb{R}^d \setminus A_m$. Next, we apply Lemma 2 with $\tilde{Q} = Q'$, $h = g_k$, $a = \nabla g(x)$, $b = g(x)$, and large k ; we have

$$A_\delta(g_k, Q') \geq (\kappa - \varepsilon) |\nabla g(x)|^p |Q'| \text{ for } \delta \in (0, \hat{\delta}_2 |\nabla g(x)|^p |Q'|^{1/d}),$$

which implies, by (3.18),

$$\liminf_{k \rightarrow +\infty} A_{\delta_k}(g_k, Q') \geq (\kappa - \varepsilon)(1 - \varepsilon) \int_{Q'} |\nabla g|^p dy. \quad (3.20)$$

Since

$$\liminf_{k \rightarrow +\infty} A_{\delta_k}(g_k, \mathbb{R}^d) \geq \sum_{Q' \in \mathbf{J}_\ell} \liminf_{k \rightarrow +\infty} A_\delta(g_k, Q'),$$

it follows from (3.20) that

$$\begin{aligned} \liminf_{k \rightarrow +\infty} A_{\delta_k}(g_k, \mathbb{R}^d) &\geq (\kappa - \varepsilon)(1 - \varepsilon) \sum_{Q' \in \mathbf{J}_\ell} \int_{Q'} |\nabla g|^p dx \\ &\geq (\kappa - \varepsilon)(1 - \varepsilon) \int_{B_m} |\nabla g|^p dx \stackrel{(3.19)}{\geq} (\kappa - \varepsilon)(1 - \varepsilon)^2 \int_{\mathbb{R}^d} |\nabla g|^p dx; \end{aligned}$$

in the second inequality, we have used the fact B_m is contained in $\bigcup_{Q' \in \mathbf{J}_\ell} Q'$ up to a null set. Since $\varepsilon > 0$ is arbitrary, one has

$$\liminf_{k \rightarrow +\infty} A_{\delta_k}(g_k, \mathbb{R}^d) \geq \kappa \int_{\mathbb{R}^d} |\nabla g|^p dx.$$

The proof is complete. $\square$

3.2. Proof of Property (G2)

The proof of Property (G2) for $p > 1$ is the same as the one for $p = 1$ given in [3]. The details are omitted.

Acknowledgments

This work was completed during a visit of H.-M. Nguyen at Rutgers University. He thanks H. Brezis for the invitation and the Department of Mathematics for its hospitality.

References

- [1] J. Bourgain, H.-M. Nguyen, A new characterization of Sobolev spaces, C. R. Acad. Sci. Paris 343 (2006) 75–80.
- [2] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.
- [3] H. Brezis, H.-M. Nguyen, Non-local functionals related to the total variation and applications in image processing, Ann. PDE 4 (2018) 77.
- [4] H.-M. Nguyen, Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006) 689–720.
- [5] H.-M. Nguyen, Γ -Convergence and Sobolev norms, C. R. Acad. Sci. Paris 345 (2007) 679–684.
- [6] H.-M. Nguyen, Γ -Convergence, Sobolev norms, and BV functions, Duke Math. J. 157 (2011) 495–533.
- [7] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, in: Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, 1993.
- [8] W. Ziemer, Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation, in: Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989.