Unipotent Overgroups in Simple Algebraic Groups

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Abstract. In this article we survey recent results on the structure of centers of centralizers of unipotent elements u in simple linear algebraic groups G. We bring forth the case of bad characteristic treated by the first author as well as a new case-free proof in characteristic 0 of the second author giving a lower bound for $\dim Z(C_G(u))$ in case u is an even element. We also point out properties of the group $Z(C_G(u)^\circ)$.

1 Introduction

Throughout G will denote a simple linear algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$, u will denote a unipotent element of G and e a nilpotent element in the Lie algebra $\mathrm{Lie}(G)$. We will say p is *bad for* G if $p = 2$ and G is not of type A_n or if $p = 3$ and G is of exceptional type or if $p = 5$ and G is of type E_8; otherwise we say p is *good for* G. Characteristic 0 is considered to be good for all G.

It was shown in [Tes95] and [PST00] that if $p > 0$ and $u \in G$ has order p, with the exception of precisely one conjugacy class of elements, u lies in an A_1-type subgroup of G. In particular, u lies in a closed connected 1-dimensional subgroup of G. The exception is the $\tilde{A}_1^{(3)}$ class in G of type G_2 when $\mathrm{char}(k) = 3$. Even in the one exceptional case, u lies in a 1-dimensional closed connected subgroup of G. Moreover, if p is a good prime for G, it follows from [Sei00, Theorem 1.2] that there exists a 1-dimensional subgroup U containing u which has particularly nice properties, for example:

$$C_G(u) = C_G(U) = C_G(\mathrm{Lie}(U)) \tag{1.1}$$

It is natural to ask: does there exist a canonically defined overgroup of u satisfying the above equalities, either in bad characteristic, or when u no longer has order p? The conditions $C_G(U) = C_G(u)$ means that the subgroup U lies in $Z(C_G(u)) = C_G(C_G(u))$. The structure of abelian algebraic groups, and in particular abelian connected unipotent groups, shows that one should aim for a t-dimensional group if $o(u) = p^t$.

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The first work in this direction was done by Richard Proud in [Pro01], who showed:

Theorem 1.1. *[Pro01, Main Theorem] Let G be a simple algebraic group defined over an algebraically closed field k and assume* $\mathrm{char}(k) = p > 0$ *is good for G. Let $u \in G$ be of order p^t, $t > 1$. Then there exists a closed connected abelian t-dimensional unipotent subgroup $W \leq G$ with $u \in W$.*

This existence result does not point to any particularly canonical properties of the overgroup, whereas the group $C_G(C_G(u))$ is a canonically defined abelian overgroup. But is it unipotent? And what about the connected component, which is also a canonically defined group, associated to u - does it even contain u? These questions were addressed in [Pro] and [Sei04]. They showed (independently) the following result, where for an abelian group H we denote by H_u the subgroup of unipotent elements.

Proposition 1.2. *Let G and u be as above. Then $Z(C_G(u)) = Z(G) \times Z(C_G(u))_u$. Moreover, if p is good for G, then $Z(C_G(u))_u = \big(Z(C_G(u))\big)^\circ$. In particular, if p is good, the group $\big(Z(C_G(u))\big)^\circ$ is a (canonically defined) connected abelian unipotent overgroup of u.*

Continuing with the case of good characteristic, we establish here the following corollary of [MT09], showing that we have Equality (1.1) above when we replace U by $Z(C_G(u))^\circ$.

Proposition 1.3. *Let G be a simple algebraic group defined over an algebraically closed field of characteristic p. Let $u \in G$ be unipotent and set $Z = Z(C_G(u))$. If p is good for G then*

$$C_G(u) = C_G(Z^\circ) = C_G(\mathrm{Lie}(Z^\circ)).$$

In what follows we discuss some recent work on describing $Z(C_G(u))$. The analysis is different depending on whether the characteristic is a bad prime for G or not. We consider as well the case where the field is of characteristic 0, as we obtain new results even in this setting.

2 Good characteristic

In this section we consider the case where $\mathrm{char}(k)$ is good for G. For some statements we will need the notion of *very good prime*: p is *very good* for G if it is good and in addition does not divide $n+1$ when G is of type A_n.

A certain number of very powerful tools are available under our current assumption on $\mathrm{char}(k)$:

- Springer maps: Given G and u, we fix a Springer map, a G-equivariant homeomorphism $\varphi : \mathcal{U} \to \mathcal{N}$, between the variety of unipotent elements in G and the variety of nilpotent elements in $\mathrm{Lie}(G)$ (for a discussion of this see

[McN03, Proposition 29]). Such a bijection exists as long as $\mathrm{char}(k)$ is good for G (and in fact is an isomorphism of varieties as long as the characteristic is very good for G). So we have $C_G(u) = C_G(\varphi(u))$, which allows us to reduce questions about centralizers of unipotent elements to the study of centralizers of nilpotent elements in $\mathrm{Lie}(G)$.

- Smoothness of centralizers. We can use the result of Slodowy [Slo80, p.38]: if $\mathrm{char}(k)$ is a very good prime for G, then $\mathrm{Lie}(C_G(x)) = C_{\mathrm{Lie}G}(x)$ where x is either a unipotent element of G or a nilpotent element in $\mathrm{Lie}(G)$.
- The Bala-Carter-Pommerening classification of unipotent classes in G and nilpotent orbits in $\mathrm{Lie}(G)$ (see [BC76a], [BC76b], [Pom77] and [Pom80]).
- For each nilpotent element $e \in \mathrm{Lie}(G)$, there exists an associated cocharacter. The definition is given below and the existence is given by [Pom77, Satz (3.1)] or [Pre03, Theorem A].

We can now prove Proposition 1.3.

Proof (Proof of Proposition 1.3.). The first equality follows from Proposition 1.2. The containment

$$C_G(Z^\circ) \subset C_G(\mathrm{Lie}(Z^\circ))$$

is clear. For the reverse inclusion, we fix a Springer map φ as above and set $e = \varphi(u)$; so $C_G(e) = C_G(u)$. If G is of type A_n then it follows from the explicit description of Z° given in [LT11, §4.1] that $e \in \mathrm{Lie}(Z)$. If G is not of type A_n, then using [MT09, Theorem A], we again have $e \in \mathrm{Lie}(Z)$. In all cases $e \in \mathrm{Lie}(Z^\circ)$, so $C_G(\mathrm{Lie}(Z^\circ)) \subset C_G(e)$.

In order to state some results from [LT11], we will require the following two definitions.

Definition 2.1. *Let H be a connected reductive algebraic group. We say that a nilpotent element $e \in \mathrm{Lie}(H)$ is a* distinguished *nilpotent element in* $\mathrm{Lie}(H)$ *if $C_H(e)^\circ$ contains no noncentral semisimple elements or, equivalently, each torus of $C_H(e)$ lies in $Z(H)$.*

Note that taking S to be a maximal torus of $C_G(e)$, we have that e is distinguished in the Lie algebra of the reductive subgroup $C_G(S)$ (a Levi subgroup of G).

Definition 2.2. *Let $e \in \mathrm{Lie}(G)$ be nilpotent. A morphism $\tau : k^* \to G$ is said to be an* associated cocharacter for e *if*

i. $\tau(c)e = c^2 e$ *for all $c \in k^*$, and*
ii. $\mathrm{im}(\tau) \subseteq [L, L]$*, for some Levi subgroup of G, such that e is distinguished in* $\mathrm{Lie}(L)$*.*

One can show that any two cocharacters associated to e are conjugate by an element of $C_G(e)^\circ$(see [Jan04, Lemma 5.3]). Moreover, an associated cocharacter determines a unique *weighted Dynkin diagram* where each node is labelled with an integer $i_\alpha \in \{0,1,2\}$. (This is analogous to the usual Kostant-Dynkin theory in characteristic 0.) We will call this the *weighted Dynkin diagram of* e.

Assume for the moment that the characteristic is very good for G. Since $\mathrm{im}(\tau)$ normalizes $C_G(e)$ it acts on the subspace $\mathrm{Lie}(C_G(e))$ with a certain set of (integral) weights. The assumption on $\mathrm{char}(k)$ implies that $\mathrm{Lie}(C_G(e)) = C_{\mathrm{Lie}G}(e)$. We denote by $C_{\mathrm{Lie}G}(e)_+$ the subalgebra spanned by the set of $\mathrm{im}(\tau)$-weight spaces associated to strictly positive weights. We also have $C_G(e) = CR$, a semidirect product of $R = R_u(C_G(e))$ and a reductive (not necessarily connected) group C. In fact, $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$ (see [Jan04, Proposition 5.10]) and $\mathrm{Lie}(R) = C_{\mathrm{Lie}G}(e)_+$. The following description of $\mathrm{Lie}(Z(C_G(e)))$ allows one to determine this object computationally.

Theorem 2.3. *[LT11, Theorem 3.9] Let* e, τ, *and* C *be as above and assume* $\mathrm{char}(k)$ *is very good for* G. *Then* $\mathrm{Lie}(Z(C_G(e))) = \big(Z(C_{\mathrm{Lie}G}(e)_+)\big)^C$, *that is the fixed points of* C *acting on* $Z(C_{\mathrm{Lie}G}(e)_+)$.

In order to apply Theorem 2.3 one has to: find a basis for $Z(C_{\mathrm{Lie}G}(e)_+)$ (if G is classical, this can be deduced from the basis for $Z(C_{\mathrm{Lie}G}(e))$ given by Yakimova in [Yak09, §2]), determine the fixed point space of the connected reductive group C° acting there, and find representatives for the component group C/C° and let them act as well. In the exceptional groups, lengthy case-by-case considerations are required for most results.

In order to state one of the main results we will need the following additional definition.

Definition 2.4. *We write* Δ_e *for the weighted Dynkin diagram of* e *and* $n_2(\Delta_e)$ *for the number of weights equal to* 2 *in* Δ_e. *We say that* e *is* even *if all weights of* τ *are even, so the weighted Dynkin diagram has all labels either 0 or 2. (For example, distinguished nilpotent elements are even.)*

Theorem 2.5. *[LT11, Theorem 2] Let* G *be as above and assume* $\mathrm{char}(k)$ *is good for* G. *Let* $e \in \mathrm{Lie}(G)$ *be an even nilpotent element. Then*

$$\dim Z(C_G(e)) = n_2(\Delta_e) = \dim Z(C_G(\mathrm{im}(\tau))).$$

In [LT11], the dimension of $Z(C_G(e))$ was determined for e in any nilpotent orbit. Indeed, [LT11, Theorem 4] gives a formula for $\dim Z(C_G(e))$ as a function of the weighted Dynkin diagram Δ_e. The formula for noneven elements is more technical and we refer the reader to the original article for the precise statement.

In the same article, the authors establish the following connection between the degrees of the invariant polynomials of the Weyl group for G and the weights of an associated cocharacter τ for e on $C_{\mathrm{Lie}G}(e)_+$. Again, the proof for the exceptional groups follows from lengthy case-by-case analysis.

Theorem 2.6. *[LT11, Theorem 1] Let* $e \in \mathrm{Lie}(G)$ *be a distinguished nilpotent element, with associated cocharacter* τ. *Let* $d_1, \cdots, d_\ell$ *be the degrees of the invariant polynomials of the Weyl group of* G, *ordered such that* d_ℓ *is* ℓ *if* G *is of* *type D_ℓ, and otherwise d_ℓ is* $\max\{d_i\}$*, and $d_i < d_j$ if $i < j < \ell$. Then the weights of* $\operatorname{im}(\tau)$ *on* $\operatorname{Lie}(Z(C_G(e)))$ *are the $n_2(\Delta_e)$ integers $2d_i - 2$ for $i \in S_\Delta$, where*

$$S_\Delta = \begin{cases} \{1, \dots, n_2(\Delta_e) - 1, \ell\} & \text{if } G \text{ is of type } D_\ell \text{ and } \Delta = \cdots {}^2_2; \\ \{1, \dots, n_2(\Delta_e) - 1, n_2(\Delta_e)\} & \text{otherwise.} \end{cases}$$

Here $\cdots {}^2_2$ stands for any weighted Dynkin diagram of type D_ℓ for which the last two nodes (in Bourbaki notation) have label 2.

Open Problems 1 *1. Find a proof in characteristic* 0 *(free of case-by-case analysis) that for an even nilpotent element e we have* $\dim Z(C_G(e)) = n_2(\Delta_e)$*. In what follows we give a proof for the inequality* $\dim Z(C_G(e)) \geq n_2(\Delta_e)$*.*

2. Give a case-free proof of the more general formula for $\dim Z(C_G(e))$ *in [LT11, Theorem 4].*

3. Theorem 2.6 was deduced from the case-by-case considerations in [LT11]. If $k = \mathbb{C}$ and e is regular then this result was established in [Kos59] (see §4 for the definition of regular*). Give a case-free proof of Theorem 2.6, at least over fields of characteristic 0.*

2.1 A proof of an inequality in characteristic 0

Here we assume that e is a non-zero *even* element and that k is the field of complex numbers $\mathbb{C}$. With this assumption, there exist h and f in $\operatorname{Lie}(G)$ such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$ and in particular e, h, f span an $\mathfrak{sl}_2$-subalgebra of $\operatorname{Lie}(G)$ which we denote by $\mathfrak{a}$ (see [Jac58]). Moreover, $C_{\operatorname{Lie}G}(e) = C_{\operatorname{Lie}G}(\mathfrak{a}) \oplus \mathfrak{r}$ where $\mathfrak{r}$ is nilpotent and $C_{\operatorname{Lie}G}(\mathfrak{a})$ is a reductive subalgebra. Moreover h spans the Lie algebra of $\operatorname{im}(\tau)$ where τ is an associated cocharacter for e. So $C_G(\operatorname{im}(\tau)) = C_G(h)$.

Lemma 2.7. *Let $e \in \operatorname{Lie}(G)$ be a nonzero even nilpotent element. Then* $\dim C_G(e) = \dim C_{\operatorname{Lie}G}(e)$ *is equal to the dimension of the* 0 *weight space for* $\operatorname{ad}(h)$*, and hence, for all $t \in \mathbb{C}^*$ we have*

$$\begin{aligned} \dim C_{\operatorname{Lie}G}(e) &= \dim C_{\operatorname{Lie}G}(h) = \dim C_G(h) = \dim C_G(th) = \dim C_{\operatorname{Lie}G}(th) \\ &= \dim C_{\operatorname{Lie}G}(th + e). \end{aligned}$$

Proof. Since we are assuming that e is an even element, in a decomposition of $\operatorname{Lie}(G)$ into a direct sum of irreducible $\mathfrak{a}$-submodules, each irreducible summand has an even highest weight and the zero weight occurs with multiplicity 1 in each irreducible summand. Moreover, $C_{\operatorname{Lie}G}(e)$ is precisely the set of fixed points for $\operatorname{ad}(e)$ acting on $\operatorname{Lie}(G)$, and there is a 1-dimensional subspace of such vectors in each irreducible summand. This establishes the first equality. The second, third and fourth equalities are clear. The last equality follows from the fact that th and $th + e$ are conjugate under the action of the closed connected (P)SL_2-subgroup $A \subseteq G$ with $\operatorname{Lie}(A) = \mathfrak{a}$.

In characteristic 0 it is not difficult to prove the Lie algebra version of Proposition 1.2. In fact, we have $Z(C_{\mathrm{Lie}G}(e)) \subseteq \mathfrak{r}$. Using this and some basic facts about centralizers and centers, we can show:

Lemma 2.8. $\mathrm{Lie}(Z(C_G(e))) = \big(Z(C_{\mathrm{Lie}G}(e))\big)^C$ *where* $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$.

For the proof of the following inequality we use an argument based upon ideas in [Kos59, Theorem 5.7].

Proposition 2.9. *If* $e \in \mathrm{Lie}G$ *is a non-zero even element then* $\dim Z(C_G(e)) \geq n_2(\Delta_e) = \dim Z(C_G(h))$.

Proof. Set $f = \dim Z(C_{\mathrm{Lie}G}(th+e))$ and $d = \dim C_{\mathrm{Lie}G}(h) = \dim C_{\mathrm{Lie}G}(e) = \dim C_{\mathrm{Lie}G}(th+e)$, for all $t \in \mathbb{C}$ by Lemma 2.7. Choose a sequence $\{t_n\} \subset \mathbb{C}$ with $t_n h + e$ converging to e. We consider the Grassmannian of d-dimensional subspaces of $\mathrm{Lie}(G)$, which is a compact space. Hence, eventually after refining $\{t_n\}$, we may assume that the sequence of d-dimensional subalgebras $C_{\mathrm{Lie}G}(t_n h + e)$ converges to a d-dimensional subspace $\mathfrak{u}$ in the Grassmannian. Let $w_1, \ldots, w_d$ be a basis of $\mathfrak{u}$, and for each $1 \leq i \leq d$, choose $\{w_i^n \in C_{\mathrm{Lie}G}(t_n h + e)\}_{n \in \mathbb{N}}$ such that $\{w_i^n\}$ converges to w_i. Since $[w_i^n, t_n h + e] = 0$ for all n, we have $[w_i, e] = 0$ for all i. Hence $\mathfrak{u} \subseteq C_{\mathrm{Lie}G}(e)$. But by our assumption that e is an even element and Lemma 2.7, we have that $d = \dim C_{\mathrm{Lie}G}(e)$ and so $\mathfrak{u} = C_{\mathrm{Lie}G}(e)$.

After again extracting a subsequence, we may assume that the sequence of f-dimensional subspaces $Z(C_{\mathrm{Lie}G}(t_n h + e))$ converges to an f-dimensional subspace $\mathfrak{z}$ of $\mathrm{Lie}(G)$. As before we have that $\mathfrak{z} \subseteq C_{\mathrm{Lie}G}(e)$ and we claim that $\mathfrak{z} \subseteq Z(C_{\mathrm{Lie}G}(e))^C$. Take a basis $\{z_1, \ldots, z_f\}$ of $\mathfrak{z}$ and for each j a sequence $\{z_j^n \in Z(C_{\mathrm{Lie}G}(t_n h + e))\}_{n \in \mathbb{N}}$ such that $\{z_j^n\}$ converges to z_j. In particular, $[z_j^n, w_i^n] = 0$ for all n, since $w_i^n \in C_{\mathrm{Lie}G}(t_{n_k} h + e)$. So $[z_j, w_i] = 0$ for all i, j which shows that $z_j \in Z(C_{\mathrm{Lie}G}(e))$. Moreover

$$Z(C_{\mathrm{Lie}G}(t_n h + e)) = \mathrm{Lie}(Z(C_G(t_n h + e))),$$

since $C_G(t_n h+e)$ is connected, and this latter is equal to $\mathrm{Lie}(C_G(t_n h+e))^{C_G(t_n h+e)}$. But since

$$C = C_G(e) \cap C_G(h) \subseteq C_G(t_n h + e) \text{ for all } t_n,$$

we also have that C fixes all elements in $Z(C_{\mathrm{Lie}G}(t_n h + e))$ and so C fixes the z_j^n and hence fixes z_j for all j. This shows that the f-dimensional subspace $\mathfrak{z}$ indeed lies in $Z(C_{\mathrm{Lie}G}(e))^C$. By Lemma 2.8, it follows that $\mathfrak{z} \subseteq \mathrm{Lie}(Z(C_G(e))$ and we conclude that $\dim Z(C_G(e)) \geq n_2(\Delta_e)$.

3 Bad characteristic

In this section we focus on the case where G is of exceptional type and the characteristic of k is bad for G. The most recent reference on unipotent and nilpotent classes, with an extensive treatment of bad characteristic, is [LS12]. In this setting almost all main tools used in the analysis for good characteristic fail. The difficulties are:

- No Springer isomorphism. Liebeck and Seitz show that if the characteristic is not 2 then there is a bijective correspondence between nilpotent orbits and unipotent classes which comes close to a Springer map (see [LS12, Theorem 1]). They also show that in the case of characteristic 2 there is an injective map from unipotent to nilpotent orbits satisfying some useful properties.
- Centralizers are not smooth, that is, for $u \in G$ unipotent, we do not always have $\mathrm{Lie}(C_G(u)) = C_{\mathrm{Lie}G}(u)$. So now studying $C_G(u)$ and $Z(C_G(u))$ cannot be 'linearized'.
- For u unipotent, we do not necessarily have u in $C_G(u)^\circ$ and so $Z(C_G(u))^\circ$ will not work as a canonically defined connected abelian overgroup of u. Springer showed in [Spr66b] that for $u \in G$ regular and $\mathrm{char}(k)$ a bad prime for G, then $u \notin C_G(u)^\circ$. All classes u for which $u \notin C_G(u)^\circ$ were determined in [LS12, Corollary 4].
- Associated cocharacters still exist but several of the useful properties of their good-characteristic counterparts are lost. For more details see [LS12, Theorem 9.1].

In order to complete the analysis of $Z(C_G(u))$ by extending it to bad characteristic, the following description of $Z(C_G(u))^\circ$ can be used to algorithmically determine this group.

Theorem 3.1. *[Sim13, Theorem A] Let B be a Borel subgroup of a simple algebraic group G defined over an algebraically closed field and set $U = R_u(B)$. Let $u \in G$ be a unipotent element and suppose that B contains a Borel subgroup of $C_G(u)$. Then*

$$\big(Z(C_G(u))\big)^\circ = C_{Z(C_U(u)^\circ)^\circ}(T_u, \tilde{A})^\circ$$

where T_u is a maximal torus of $C_B(u)$ and $\tilde{A}$ is a set of coset representatives for $C_G(u)^\circ$ in $C_G(u)$.

A consequence of this is

Corollary 3.2. *[Sim13, Corollary 2.8] With notation as in Theorem 3.1 and under the assumption that the characteristic of the field is 0 we have* $\mathrm{Lie}Z(C_G(u)) = Z(C_{\mathrm{Lie}G}(u))^{\tilde{A}}$.

Using a Springer map one has a similar statement for nilpotent elements which gives a proof of Lemma 2.8. This corollary shows that the component group of the centralizer plays an important role in determining the double centralizer and can be viewed as a justification for the difficulty of showing $\dim Z(C_G(e)) \leq n_2(\Delta_e)$ other than through a case-by-case analysis.

In order to apply Theorem 3.1, one first needs to find a Borel subgroup which contains a Borel subgroup of the centralizer. Once this is determined one can computationally obtain $Z(C_U(u)^\circ)^\circ$. This was carried out case by case in [Sim13] for each class of unipotent elements in the exceptional algebraic groups defined over fields of bad characteristic:

Theorem 3.3. *[Sim13, Theorem D] Suppose that G is of exceptional type and that the characteristic of k is bad for G. Let $u \in G$ be a unipotent element. Then* $\dim Z(C_G(u))$ *is explicitly determined; the tables are given in [Sim13, §9].*

Moreover, the analysis determines if u lies in $Z(C_G(u))^\circ$, and when u lies in $Z(C_G(u)^\circ)^\circ$. In particular, we find that u does not necessarily lie in $Z(C_G(u))^\circ$, even if u does lie in $C_G(u)^\circ$. Note that in all cases, we have

$$Z(C_G(u))^\circ \subset Z(C_G(u)^\circ)^\circ \subset Z(C_G(u)^\circ) \subset C_G(u)^\circ \subset C_G(u).$$

Clearly, when $u \notin C_G(u)^\circ$, we have $u \notin Z(C_G(u))^\circ$. But in fact, there exist u with $u \in C_G(u)^\circ$, but $u \notin Z(C_G(u))^\circ$. There are examples which show that each of the above inclusions may be proper:

Corollary 3.4. *Let G be an exceptional algebraic group and suppose that the characteristic is 2. Except for the classes A_2 and A_2A_1 in E_6, an element u lies in $Z(C_G(u))^\circ$ if and only if it has order 2.*

Open Problems 2 *The problem of determining* $\dim Z(C_G(u))$*, for u a unipotent element in an orthogonal or symplectic group over a field of characteristic* 2*, is still open.*

4 Related questions

The results mentioned so far can be used to address related questions. The questions considered here revolve around several characterizations of regular unipotent elements.

Definition 4.1. *We call an element x of G* regular *if* $\dim C_G(x)$ *is minimal.*

If u is unipotent, then there are various characterizations of the condition that $\dim C_G(u)$ is minimal. More precisely, if the characteristic is good for G then we have

$$\begin{aligned} u \text{ regular unipotent} &\Leftrightarrow \dim C_G(u) = \operatorname{rank}(G) && (4.2)\\ &\Leftrightarrow G.u \text{ dense in the variety of unipotent elements of } G && (4.3)\\ &\Leftrightarrow C_G(u) \text{ abelian} && (4.4)\\ &\Leftrightarrow C_G(u)^\circ \text{ abelian} && (4.5)\\ &\Leftrightarrow \dim Z(C_G(u)) = \operatorname{rank}(G) && (4.6)\\ &\Leftrightarrow \dim Z(C_G(u)^\circ) = \operatorname{rank}(G) && (4.7)\end{aligned}$$

whereas in bad characteristic only the first three equivalences remain true.

For the first two equivalences we refer the reader to [SS70, Chapter III]. In [Spr66a], Springer proved that for a regular unipotent element u, $C_G(u)^\circ$ is abelian. Then Lou showed in [Lou68] that for regular unipotent elements the full centralizer $C_G(u)$ is abelian. In [Kur83] Kurtzke showed that in good characteristic a unipotent element $u \in G$ is regular if and only if $C_G(u)^\circ$ is abelian. Lawther extended these results to cover bad characteristics, so the equivalence in (4.4) is true:

Theorem 4.2. *[Law12, Theorem 2] Let u be a unipotent element in G. Then u is regular if and only if $C_G(u)$ is abelian.*

The result of Kurtzke does not generalize however. In (4.5) we only have '$\Rightarrow$' in general. Our analysis gives a second proof for exceptional groups in bad characteristic of the following result.

Theorem 4.3. *[Law12, Theorem 1] For $u \in G$ unipotent, with $C_G(u)^\circ$ abelian, then either u is regular, or $u \in G = G_2$, $p = 3$ and u lies in the class of subregular elements.*

The implications '$\Rightarrow$' in (4.6) and (4.7) follow from (4.4), (4.5) and (4.2). That the reversed implications do not hold in general can be deduced from Theorem 3.3.

Corollary 4.4. *Let G be of exceptional type and suppose that $\mathrm{char}(k)$ is bad for G. If $u \in G$ is unipotent then*

i. $\dim Z(C_G(u)^\circ) \leq \mathrm{rank}(G)$ *unless $p = 2$, G is of type F_4 and u is in the $F_4(a_3)$ class; in this case* $\dim Z(C_G(u)^\circ) = 6$.
ii. $\dim Z(C_G(u)) = \mathrm{rank}(G)$ *if and only if u is in the regular class or (G, p, class of u) is one of the triples $(G_2, 3, \tilde{A}_1^{(3)})$, $(F_4, 2, F_4(a_1))$ and $(F_4, 2, C_3(a_1)^{(2)})$; in these cases* $\dim Z(C_G(u)) = \dim Z(C_G(u)^\circ)$.

In fact, it is not known whether the inequality $\dim Z(C_G(u)) \leq \mathrm{rank}(G)$ holds in general. From [LT11, Theorem 4] and Theorem 3.3 we deduce the following

Corollary 4.5. *Assume $p \neq 2$ if G is of type B_n, C_n or D_n. Then* $\dim Z(C_G(u)) \leq \mathrm{rank}(G)$.

Open Problems 3 *1. Give a case-free proof of the inequality* $\dim Z(C_G(u)) \leq \mathrm{rank}(G)$*, in characteristic 0, or even in good positive characteristic.*
2. Determine whether $\dim Z(C_G(u)) \leq \mathrm{rank}(G)$ *in the cases which are excluded in Corollary 4.5.*

We conclude by showing that if the characteristic is good for G then the implications '$\Leftarrow$' in (4.6) and (4.7) hold. The implication '$\Leftarrow$' in (6) follows from [LT11, Theorem 4]. We therefore turn our attention to $Z(C_G(u)^\circ)$.

For exceptional groups, if the characteristic is good, the dimension of $Z(C_G(u)^\circ)$ can be deduced from the tables in [LT11], where a basis for $\mathcal{Z}^\natural = Z(C_{\mathrm{Lie}G}(e)_+)^{C^\circ}$ is given. We need the following lemma.

Lemma 4.6. *Assume $\mathrm{char}(k)$ is a very good prime for G. Then*

$$\mathrm{Lie}(Z(C_G(e)^\circ)) = Z(C_{\mathrm{Lie}G}(e)_+)^{C^\circ}.$$

Proof. By [Sim13, Lemma 2.6], $Z(C_G(u)^\circ) = Z(C_G(u)^\circ)_u \times (Z(G) \cap C_G(u)^\circ)$, for all unipotent elements $u \in G$; so via a Springer map, the analogous statement is true for all nilpotent elements $e \in \mathrm{Lie}G$. In particular, setting $C_G(e) = CR$, where $R = R_u(C_G(e))$ as in §2, we have $Z(C_G(e)^\circ)^\circ \subseteq R$. So

$$\begin{aligned}\mathrm{Lie}Z(C_G(e)^\circ) &= \mathrm{Lie}(C_G(e)^\circ)^{C^\circ R} \cap \mathrm{Lie}R \\ &= (\mathrm{Lie}C \oplus \mathrm{Lie}R)^{C^\circ R} \cap \mathrm{Lie}R \\ &= (\mathrm{Lie}R)^{RC^\circ} \\ &= (\mathrm{Lie}Z(R))^{C^\circ}.\end{aligned}$$

Now we argue as in the proofs of Propositions [LT11, 3.7,3.8,3.9], to see that $Z(\mathrm{Lie}R) = \mathrm{Lie}(Z(R))$. So finally we have $\mathrm{Lie}(Z(C_G(e)^\circ)) = Z(\mathrm{Lie}R)^{C^\circ} = Z(C_{\mathrm{Lie}G}(e)_+)^{C^\circ} = \mathcal{Z}^\natural$.

In particular, we now deduce from [LT11, Section 11] that for the exceptional groups defined over fields of characteristic 0 or good characteristic p, we have $\dim Z(C_G(u)^\circ) = \mathrm{rank}(G)$ if and only if u is regular.

We finish by establishing this result for the classical groups.

Lemma 4.7. *Let G be a classical group defined over a field of characteristic* 0 *or of good characteristic p. Let $u \in G$ be unipotent. Then* $\dim Z(C_G(u)^\circ) = \mathrm{rank}(G)$ *if and only if u is regular.*

Proof. Using a Springer map, we will consider $e \in \mathrm{Lie}G$ nilpotent. If e is regular nilpotent, then by Theorem 4.2, $C_G(e)$ is abelian and so we have the result.

Now take $e \in \mathrm{Lie}G$ a non regular nilpotent element.

If G is of type A_n, then $C_G(e)$ is connected and so $Z(C_G(e)^\circ) = Z(C_G(e))$ and Theorem [LT11, Theorem 4] shows that the latter is of dimension strictly less than $\mathrm{rank}(G)$.

For G of type B_n, C_n and D_n, set $Y = \langle e, e^3, \dots \rangle$, the subspace of $\mathrm{Lie}G$ spanned by the odd powers of e. By [Yak09, Theorem 2.3], $Z(C_{\mathrm{Lie}G}(e)) = Y$, unless G is of type B_n or D_n and e has at least 3 Jordan blocks on the natural module for G, 2 of which are blocks of odd sizes strictly bigger than 1. In the exceptional cases, $Z(C_{\mathrm{Lie}G}(e)) = Y \oplus \langle x \rangle$ for some $0 \neq x \in \mathrm{Lie}G$. Using the known Jordan block structure of nilpotent elements in $\mathrm{Lie}G$ acting on the natural module for G (for example as given in [Car85, Chapter 13]), we deduce that $\dim Y < \mathrm{rank}(G)$ for all nonregular nilpotent elements e, and $\dim Y + 1 < \mathrm{rank}(G)$, unless G has type B_n and e has a Jordan block of size $2n-1$ on the natural kG-module. In the latter case, e has exactly three blocks of sizes $2n-1$, 1 and 1, and so [Yak09, Theorem 2.3] implies that $Z(C_{\mathrm{Lie}G}(e)) = Y$. Since $\mathrm{Lie}(Z(C_G(e)^\circ)) \subset Z(\mathrm{Lie}(C_G(e)^\circ)) = Z(\mathrm{Lie}C_G(e)) = Z(C_{\mathrm{Lie}G}(e))$, we have that $\dim Z(C_G(e)^\circ) < \mathrm{rank}(G)$.

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