

A NEW FAMILY OF IRREDUCIBLE SUBGROUPS OF THE ORTHOGONAL ALGEBRAIC GROUPS

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ABSTRACT. Let $n \geq 3$, and let Y be a simply connected, simple algebraic group of type D_{n+1} over an algebraically closed field K . Also let X be the subgroup of type B_n of Y , embedded in the usual way. In this paper, we correct an error in a proof of a theorem of Seitz (Mem. Amer. Math. Soc. **67** (1987), no. 365), resulting in the discovery of a new family of triples (X, Y, V) , where V denotes a finite-dimensional, irreducible, rational KY -module, on which X acts irreducibly. We go on to investigate the impact of the existence of the new examples on the classification of the maximal closed connected subgroups of the classical algebraic groups.

1. INTRODUCTION

Let K be an algebraically closed field of characteristic $p \geq 0$. In the 1950s, Dynkin determined the maximal closed connected subgroups of the simple classical type linear algebraic groups defined over K , assuming $\text{char}(K) = 0$ (see [11, 12]); in 1987, Seitz [23] established an analogous classification in the case where $\text{char}(K) > 0$. The main step in both of these classifications is the determination of all triples (X, Y, V) , where Y is a simple linear algebraic group defined over K , X is a proper closed connected subgroup of Y , and V is a non-trivial irreducible, finite-dimensional (p -restricted if $\text{char}(K) = p > 0$) KY -module on which X acts irreducibly. The determination of these so-called “irreducible triples” is covered in the work of Dynkin [11, 12] (in case $\text{char}(K) = 0$), Seitz [23] (in case $\text{char}(K) > 0$ and Y is a classical group), and Testerman [26] (in case $\text{char}(K) > 0$ and Y is of exceptional type). The existence of an irreducible triple of the form (X, Y, V) as above, arising from a rational representation $\rho : Y \rightarrow \text{GL}(V)$, indicates that $\rho(X)$ is *not* maximal in the smallest classical group $\text{Isom}(V)$ containing both $\rho(X)$ and $\rho(Y)$, while the large majority of tensor-indecomposable irreducible representations of a simple algebraic group give rise to maximal subgroups of the smallest classical group containing the image.

Recently, the second author’s PhD student Nathan Scheinmann discovered an irreducible triple which does not appear in [23, Theorem 1, Table 1]. Namely, take K to be of characteristic 3, and $X = B_3$ embedded in the usual way in $Y = D_4$ as the stabilizer of a non-singular 1-space on the 8-dimensional natural module for

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Y . Consider the irreducible KY -module with highest weight $\lambda_1 + \lambda_2 + \lambda_3$ (here λ_i , $1 \leq i \leq 4$, is a set of fundamental weights for D_4 , λ_1 is the highest weight of the natural KD_4 -module, and we label Dynkin diagrams as in [3]). The restriction of the highest weight to a maximal torus of B_3 shows the existence of a B_3 -composition factor of highest weight $\omega_1 + \omega_2 + \omega_3$ (ω_i , $i = 1, 2, 3$, a set of fundamental weights for B_3) and consulting [19], one sees that these modules are both of dimension 384, and hence X acts irreducibly on V .

The absence of this example from [23, Table 1] is the result of an error in the proof of [23, 8.7]. Here, we correct the error in this proof and, in so doing, establish the existence of a whole new family of modules V for the group $Y = D_{n+1}$ on which $X = B_n$ acts irreducibly. For a fixed n and a fixed p , there are finitely many modules V , but for each n there exist infinitely many primes p for which there is a new example. The precise description of the family is given in Theorem 1.2 below. In addition to this infinite family, our investigations revealed one further example of an irreducibly acting subgroup which does not appear in [23, Table 1], namely the group $Y = C_4$, defined over a field of characteristic 2, acting on V , the irreducible module with highest weight λ_3 , has a subgroup $X = B_3$, contained in a maximal rank subgroup of type $D_4 \subset C_4$, both acting irreducibly on V . The triple (D_4, C_4, V) appears in [23, Table 1], as well as the triple $(B_3, D_4, V|_{D_4})$. However, the triple (B_3, C_4, V) is omitted.

The goal of this paper is two-fold: first we concentrate on the embedding $X = B_n \subset Y = D_{n+1}$ and determine all p -restricted irreducible representations of Y whose restriction to X is irreducible, thereby correcting [23, 8.7]; see Theorem 1.2 below. The second goal of the paper is to show that the existence of the new examples has no *further* influence on the main results of [23] and [26]. Indeed, the proofs of the main theorems in these two articles depend on an inductive hypothesis, concerning the list of examples for smaller rank groups. The new family of examples for the pair (B_n, D_{n+1}) , as well as the one “new” example for the pair (B_3, C_4) , alters the inductive hypothesis and therefore requires one to take these new examples into consideration when working through all other possible embeddings. This is precisely what has been carried out in the proofs of Proposition 1.5, Proposition 1.7, and Theorem 1.8.

Remark 1.1. Combining the results [23, Theorem 1], [26, Main Theorem], Theorem 1.2, Proposition 1.5, Proposition 1.7, and Theorem 1.8, we conclude that the only new examples to be added to [23, Table 1] are those for pairs (X, Y) as follows:

- (B_n, D_{n+1}) , $n \geq 3$, and irreducible modules described in Theorem 1.2, and
- (B_3, C_4) , when $p = 2$ and B_3 is embedded in a short root maximal rank D_4 subgroup of C_4 , each acting irreducibly on the irreducible KC_4 -module with highest weight λ_3 . (Here λ_i , $1 \leq i \leq 4$, is a set of fundamental weights for C_4 and we label Dynkin diagrams as in [3]).

Theorem 1.2 covers the embedding $B_n \subset D_{n+1}$ and the case of $B_3 \subset C_4$ is discussed at the end of Section 4. This assertion of the completeness of the rectified list is dependent upon our Hypothesis 1.4 below, where we state explicitly which results from [23] are assumed for the proofs of our main results.

Statement of results. Let $Y = \text{Spin}_{2n+2}(K)$ ($n \geq 2$) be a simply connected, simple algebraic group of type D_{n+1} over K . Also let X be the subgroup of type B_n , embedded in Y in the usual way, as the stabilizer of a non-singular 1-dimensional

subspace of the natural module for Y . Fix T_Y a maximal torus of Y and $B_Y \subset Y$ a Borel subgroup containing T_Y . Denote by $\{\lambda_1, \dots, \lambda_{n+1}\}$ the corresponding set of fundamental weights for T_Y , ordered as in [3], where the natural $(2n+2)$ -dimensional KY -module has highest weight λ_1 . Let σ be a graph automorphism of Y stabilizing T_Y , and with $X = Y^\sigma$, the group of σ -fixed points. Our first main result is the following; the proof is given in Section 3.

Theorem 1.2. *Let $Y = \text{Spin}_{2n+2}(K)$ be a simply connected, simple algebraic group of type D_{n+1} over K , $n \geq 2$, and let X be the subgroup of type B_n , as above. Consider a non-trivial, irreducible KY -module V having p -restricted highest weight $\lambda = \sum_{r=1}^{n+1} a_r \lambda_r$. Then X acts irreducibly on V if and only if λ or $\sigma(\lambda)$ is equal to $\sum_{r=1}^n a_r \lambda_r$, with $a_n \neq 0$, such that for all $1 \leq i < j \leq n$ such that $a_i a_j \neq 0$ and $a_r = 0$ for all $i < r < j$, we have $p \mid (a_i + a_j + j - i)$.*

The set of weights which is listed in [23, Table 1] for the pair (B_n, D_{n+1}) is

$$\{c\lambda_n, a\lambda_k + b\lambda_n : abc \neq 0, p \mid (a + b + n - k)\}.$$

So we see that the new examples are a generalization of those found by Seitz, where one congruence condition is replaced by a set of congruence conditions. (Note that there are new examples only if $n \geq 3$.) It is perhaps informative to point out precisely what error occurs in the proof of [23, 8.7], where the embedding $B_n \subset D_{n+1}$ is considered. In the proof, Seitz defines a certain vector in the irreducible KY -module of highest weight λ and shows that this vector is annihilated by all simple root vectors in the Lie algebra of X , which then implies that X does not act irreducibly. However if λ satisfies the congruence conditions, the vector is in fact the zero vector in V and so does not give rise to a second composition factor as claimed.

The omission of the triple (B_3, C_4, λ_3) from [23, Table 1] is of a different nature, and occurs in the proof of [23, 15.13]. In the first part of the proof, there is a reduction to the case where X acts irreducibly on the natural KY -module with highest weight μ , and the precise embedding we have here is for $\mu = \delta_3$ (following Seitz's notation, δ_3 is the third fundamental dominant weight), so $Y = C_4$ or D_4 . Then Seitz argues that $\lambda = \lambda_k$ for some k . Since he is assuming that V is not the natural KY -module, he invokes the inductive hypothesis and deduces that $\lambda = \lambda_3$. Now Seitz concludes that V is a spin module for Y . Evidently, he considers only the embedding of X in D_4 , omitting to include the case of X in C_4 .

Returning to the family of examples described in Theorem 1.2, it is natural to ask how one might discover the given set of congruence conditions, and here we must give credit to the work of Ford in [13], where he studied irreducible triples of the form (H, G, V) , G a simple classical type algebraic group over K , H a disconnected closed subgroup of G with H° simple, and V an irreducible KG -module on which H acts irreducibly. He discovered a family of irreducible triples for the embedding $D_n.2 \subset B_n$, where the highest weight of the irreducible KB_n -module satisfies similar congruence conditions. His methods were later applied by Cavallin in [9] when studying irreducible KB_n -modules having precisely two D_n -composition factors.

The second goal of this article is to show that the existence of the new examples for the pair $(X, Y) = (B_n, D_{n+1})$, described by Theorem 1.2, and the further omitted example (B_3, C_4) has no *further* influence on the main theorems in [23, 26]. To explain the issue which must be addressed and our approach to the

problem, we must describe to some extent the strategy of the proof of [23, Theorem 1]. First note that the assumption that X acts irreducibly on some KY -module implies that X is semisimple. One of the main techniques used to determine the triples (X, Y, V) as above involves arguing inductively, working with a suitable embedding $P_X = Q_X L_X \subset P_Y = Q_Y L_Y$ of parabolic subgroups, where $Q_X = R_u(P_X) \subset R_u(P_Y) = Q_Y$. Indeed, [23, 2.1] implies that if X acts irreducibly on V , then the derived subgroup L'_X acts irreducibly on the commutator quotient $V/[V, Q_Y]$, an irreducible KL'_Y -module. Moreover, the highest weight of $V/[V, Q_Y]$ as a KL'_Y -module is the restriction of the highest weight of V to an appropriate maximal torus of L'_Y . (This is a variation of a result of [24].) Thus, Seitz and Testerman proceed by induction on the rank of X ; Seitz treats the case X of type A_1 by ad hoc methods, exploiting the fact that all weights of an irreducible KA_1 -module are of multiplicity one. Now Theorem 1.2 above introduces a new family of examples of irreducible triples. As a consequence, one needs to reinvestigate all embeddings $X \subset Y$ where the pair $B_m \subset D_{m+1}$, $m \geq 3$, may arise when considering the projection of a Levi factor L'_X of X into a simple component of a Levi factor L'_Y of Y , under the additional hypothesis that X acts irreducibly on a KY -module whose highest weight has restriction to the D_{m+1} -component of L'_Y among the new examples described by Theorem 1.2. This is precisely what we consider in Proposition 1.5 below. A similar analysis must be carried out for the one new example (B_3, C_4) as a potential embedding of Levi factors. This easier case is covered by Proposition 1.7. In order to state the results, we introduce the following terminology.

Definition 1.3. We will say a p -restricted dominant weight λ for $Y = D_{m+1}$, $m \geq 3$, satisfies the congruence conditions if

- (i) $\lambda = \sum_{i=1}^m a_i \lambda_i$,
- (ii) $a_m \neq 0$,
- (iii) there exists $i < j \leq m - 1$ with $a_i a_j \neq 0$, and
- (iv) for all $1 \leq i < j \leq m$ such that $a_i a_j \neq 0$ and $a_r = 0$ for all $i < r < j$, we have $p \mid (a_i + a_j + j - i)$.

Note that the above congruence conditions are precisely those satisfied by the highest weights in Theorem 1.2 but *not* appearing in [23, Table 1]. (See the remark following the statement of Theorem 1.2.)

For the proofs of Proposition 1.5, Proposition 1.7, and Theorem 1.8, we require the following inductive hypotheses.

Hypothesis 1.4. Assume $\text{char}(K) = p > 0$. Let G be a simple algebraic group defined over K and H a semisimple, proper, closed subgroup of G , where the pair (H, G) is one of the following:

- (i) (H, B_n) , $n \geq 3$,
- (ii) (B_ℓ, A_n) , $\ell \geq 2$,
- (iii) (C_3, D_n) , $n \geq 4$,
- (iv) (C_3, A_5) .

Let V be a p -restricted irreducible KG -module, with corresponding representation $\rho: G \rightarrow GL(V)$. Then H acts irreducibly on V if and only if the triple $(\rho(H), \rho(G), V)$ appears in [23, Table 1], where the highest weight of V is given up to graph automorphisms of G .

The classical case. Let Y be of classical type. The next two results ensure that, under the assumption of Hypothesis 1.4 for all embeddings $H \subset G$ with $\text{rank}(G) < \text{rank}(Y)$, the only new examples of irreducible triples (X, Y, V) are those described in Remark 1.1.

Proposition 1.5. *Let Y be a simply connected, simple algebraic group of type D_{n+1} , $n \geq 4$, and X a semisimple, proper, closed subgroup of Y acting irreducibly on a p -restricted irreducible KY -module V of highest weight λ , V not the natural module for Y . Assume Hypothesis 1.4 for all embeddings $H \subset G$ with $\text{rank}(G) < \text{rank}(Y)$. Moreover, if X is simple, assume the following conditions are satisfied:*

- (i) P_X is a proper parabolic subgroup of X , with Levi factor L_X of type B_{m-1} , $m \geq 4$.
- (ii) P_Y is a parabolic subgroup of Y with $P_X \subset P_Y$ and $R_u(P_X) \subset R_u(P_Y)$.
- (iii) For a Levi factor L_Y of P_Y , writing $L'_Y = L_1 L_2 \cdots L_t$, a commuting product of simple groups, L_t is of type D_m , and L'_X projects non-trivially into L_t .
- (iv) For the irreducible KL'_Y -module $V/[V, Q_Y]$, write $V/[V, Q_Y] = M_1 \otimes \cdots \otimes M_t$, where M_i is an irreducible KL_i -module.
- (v) The highest weight of the KL_t -module M_t satisfies the congruence conditions.

Then one of the following holds:

- (a) $X \subset B_k \times B_{n-k}$ for some $0 < k < n$, and $\lambda = \lambda_n$ or λ_{n+1} , or
- (b) $X = B_n$ and the embedding of X in Y is the usual embedding of B_n in D_{n+1} , that is, B_n is the stabilizer of a non-singular 1-space on the natural $(2n+2)$ -dimensional KY -module.

Moreover, in case (a) above, the subgroup $B_k \times B_{n-k}$ acts irreducibly on the KY -modules of highest weights λ_n and λ_{n+1} .

Remark 1.6. Now to go on to determine the irreducible triples satisfying (a), we rely on Hypothesis 1.4(i), and for those satisfying (b), we apply Theorem 1.2.

Proposition 1.7. *Assume $p = \text{char}(K) = 2$. Let Y be a simply connected, simple algebraic group of type C_n , $n \geq 5$, and X a semisimple, proper, closed subgroup of Y with proper Levi factor of type B_3 . Let V be a p -restricted irreducible KY -module of highest weight λ , V not the natural module for Y . Assume Hypothesis 1.4 for all embeddings $H \subset G$ with $\text{rank}(G) < \text{rank}(Y)$. Moreover, assume the following conditions are satisfied:*

- (i) P_X is a proper parabolic subgroup of X , with Levi factor L_X of type B_3 .
- (ii) P_Y is a parabolic subgroup of Y with $P_X \subset P_Y$ and $R_u(P_X) \subset R_u(P_Y)$.
- (iii) For a Levi factor L_Y of P_Y , writing $L'_Y = L_1 L_2 \cdots L_t$, a commuting product of simple groups, L_t is of type C_4 , and L'_X projects non-trivially into L_t .
- (iv) For the irreducible KL'_Y -module $V/[V, Q_Y]$, write $V/[V, Q_Y] = M_1 \otimes \cdots \otimes M_t$, where M_i is an irreducible KL_i -module.
- (v) The highest weight of the KL_t -module M_t is the third fundamental dominant weight for L_t .

Then X acts reducibly on V .

The exceptional case. We now turn to the consideration of the case where Y is a simply connected, simple algebraic group of exceptional type over K and X is a proper closed, connected subgroup of Y acting irreducibly on some p -restricted irreducible KY -module. As usual, X is then semisimple, and once again, we must consider the possibility of a parabolic embedding $P_X \subset P_Y$, with Levi factor L_X of P_X , of type B_m , Levi factor L_Y of P_Y , having a simple factor of type D_{m+1} , with the action on the commutator quotient arising from a weight which satisfies the congruence conditions. (Note that the pair (B_3, C_4) will never occur as an embedding of Levi factors when Y is exceptional.) In particular, Y is of type E_n for $n = 6, 7$ or 8 .

Theorem 1.8. *Let Y be a simply connected, simple algebraic group of type E_n , $6 \leq n \leq 8$, defined over K and let X be a semisimple, proper, closed, connected subgroup of Y , having a proper parabolic subgroup with Levi factor of type B_m for some $m \geq 3$. Assume Hypothesis 1.4 for all embeddings $H \subset G$ with $\text{rank}(G) < \text{rank}(Y)$. Let V be a non-trivial irreducible KY -module with p -restricted highest weight λ . Then X acts irreducibly on V if and only if $Y = E_6$, $X = F_4$, and one of the following holds:*

- (i) $\lambda = (p-3)\lambda_1$ or $\lambda = (p-3)\lambda_6$, with $p > 3$.
- (ii) $\lambda = \lambda_1 + (p-2)\lambda_3$ or $\lambda = (p-2)\lambda_5 + \lambda_6$, with $p > 2$.

Note that the existence of the examples arising in Theorem 1.8 had already been established by Testerman [26, Main Theorem]. The proof of the “only if” direction requires us to treat, eventually ruling out, several new potential configurations that arise from Theorem 1.2 in the inductive process, as explained in Section 5.

About the proofs. We conclude this section with a brief discussion of the methods and further remarks on our inductive assumption (Hypothesis 1.4). In order to prove Theorem 1.2, we first show that it is enough to work with the Lie algebras of Y and X . Indeed, as λ is p -restricted, the irreducible KY -module V is generated by a maximal vector v^+ for B_Y as a module for the universal enveloping algebra \mathfrak{U}_Y of $\text{Lie}(Y)$. Therefore in order to show that $V|_X$ is irreducible, it suffices to show that $\mathfrak{U}_Y v^+ = \mathfrak{U}_X v^+$, where \mathfrak{U}_X is the universal enveloping algebra of $\text{Lie}(X)$. We rely on the fact that any irreducible module for Y is self-dual as a KX -module (see 3.1 below), and apply the techniques developed by Ford in [13], further investigated by Cavallin in [9], to establish this generation result.

For the proof of Proposition 1.5, we carry out an analysis used by Seitz in [23, Section 8], but applied specifically to the group $Y = D_{m+1}$. He first shows that a proper closed connected subgroup X acts irreducibly on a non-trivial irreducible KY -module only if either X acts irreducibly and tensor indecomposably on the natural module for Y , or the triple (X, Y, V) is known. This part of our proof is not at all original, but we include it for completeness. At this point, however, our proof proceeds along different lines; we compare the commutator series for two different parabolic embeddings and obtain conditions on the highest weight which are compatible with the given congruence conditions only if the pair (X, Y) is (B_m, D_{m+1}) , which is handled by Theorem 1.2. The proof of Proposition 1.7 is much simpler given that we are dealing with a fixed-rank embedding.

For the proof of Theorem 1.8, we proceed differently than in [26]; we use the classification of the maximal closed positive-dimensional subgroups of the exceptional type algebraic groups, given in [18], which was not available when [26] was

written. Hence, we first consider the case where X is maximal, find only the two examples of the theorem and conclude using the main result of [26] for the group $Y = F_4$.

In addition to Hypothesis 1.4, we rely upon two further results in [23], namely [23, Theorem 4.1] and [23, 6.1]. The first result classifies the irreducible triples (X, Y, V) when $\text{rank}(X) = \text{rank}(Y)$, the second covers the case where $\text{rank}(X) = 1$. The proofs of these results are completely independent of the results in [23, Section 8]. Finally, we will use the results of [23, Section 2] concerning parabolic embeddings and commutator series in irreducible modules for semisimple groups.

2. PRELIMINARIES

In this section, we introduce the notation that shall be used in the remainder of the paper, and recall some basic properties of rational modules for simple linear algebraic groups. We rely on the standard reference [16] for a treatment of this general theory.

2.1. Notation. Let K be an algebraically closed field of characteristic $p \geq 0$, and let G be a simply connected, simple linear algebraic group over K . (All algebraic groups considered here will be linear algebraic groups, even if we omit to say so explicitly.) Also fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U denotes the unipotent radical of B . Let $\text{rank}(G) = \ell$ and let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be the corresponding base of the root system $\Phi = \Phi^+ \sqcup \Phi^-$ of G , where Φ^+ and Φ^- denote the sets of positive and negative roots, respectively. Throughout we use the ordering of simple roots as in [3]. Let \mathscr{W} be the Weyl group of G , $\mathscr{W} = N_G(T)/T$, and for $\alpha \in \Phi$, denote by s_α the corresponding reflection. In addition, let

$$X(T) = \text{Hom}(T, K^*)$$

denote the character group of T and write $(-, -)$ for a fixed \mathscr{W} -invariant inner product on the space $X(T)_\mathbb{R} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Also let $\lambda_1, \dots, \lambda_\ell$ be the fundamental dominant weights for T corresponding to our choice of base Π , that is, $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $1 \leq i \leq j \leq \ell$, where

$$\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$$

for $\lambda, \alpha \in X(T)$, $\alpha \neq 0$. Set $X^+(T) = \{\lambda \in X(T) : \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Pi\}$ and call a character $\lambda \in X^+(T)$ a *dominant T -weight* (or simply *dominant weight*, if the choice of torus is clear in the context). Finally, we say that $\mu \in X(T)$ is *under* $\lambda \in X(T)$ (and we write $\mu \preccurlyeq \lambda$) if $\lambda - \mu = \sum_{r=1}^{\ell} c_r \alpha_r$ for some $c_1, \dots, c_\ell \in \mathbb{Z}_{\geq 0}$. We also write $\mu \prec \lambda$ to indicate that $\mu \preccurlyeq \lambda$ and $\mu \neq \lambda$.

2.2. Rational modules. In this section, we recall some elementary facts on weights and multiplicities, as well as basic properties of Weyl and irreducible modules for G . Let V be a finite-dimensional, rational KG -module. Then

$$V = \bigoplus_{\mu \in X(T)} V_\mu,$$

where, for $\mu \in X(T)$, $V_\mu = \{v \in V : tv = \mu(t)v \text{ for all } t \in T\}$. A weight $\mu \in X(T)$ is called a *weight of V* if $V_\mu \neq 0$, in which case V_μ is said to be its corresponding *weight space*. Also, we denote by $m_V(\mu)$ the *multiplicity of μ in V* , and let $\Lambda(V) = \{\mu \in X(T) : V_\mu \neq 0\}$ denote the set of weights of V and write

$\Lambda^+(V) = \Lambda(V) \cap X^+(T)$ for the set of dominant weights of V . It is well known that each weight of V is \mathscr{W} -conjugate to a unique dominant weight in $\Lambda^+(V)$. Also, if $\lambda \in X^+(T)$, then $w\lambda \preccurlyeq \lambda$ for every $w \in \mathscr{W}$, and all weights in a \mathscr{W} -orbit have the same multiplicity.

A non-zero vector $v^+ \in V$ is called a *maximal vector* of weight $\lambda \in \Lambda(V)$ for the pair (B, T) if $v^+ \in V_\lambda$ and $Bv^+ \subseteq \langle v^+ \rangle_K$. Now for $\lambda \in X^+(T)$ a dominant weight, we write $V_G(\lambda)$ for the Weyl module having highest weight λ , and denote by $L_G(\lambda)$ the unique irreducible quotient of $V_G(\lambda)$. In other words,

$$L_G(\lambda) = V_G(\lambda) / \text{rad}(\lambda),$$

where $\text{rad}(\lambda)$ is the unique maximal submodule of $V_G(\lambda)$, called the *radical* of $V_G(\lambda)$. We write $\Lambda(\lambda)$ for $\Lambda(V_G(\lambda))$ and $\Lambda^+(\lambda)$ for $\Lambda^+(V_G(\lambda))$. Also, we denote by $H^0(\lambda)$ the induced KG -module having highest weight λ . Recall that $H^0(\lambda)$ has a unique simple submodule, isomorphic to $L_G(\lambda)$, and that

$$\Lambda(H^0(\lambda)) = \Lambda(-w_0\lambda),$$

where w_0 denotes the longest element in \mathscr{W} . For $\mu \in X^+(T)$, we write $[V, L_G(\mu)]$ to denote the number of times the irreducible KG -module $L_G(\mu)$ appears as a composition factor of V . We also use the notation

$$U_\alpha = \{x_\alpha(c) : c \in K\}$$

to denote the T -root subgroup of G corresponding to the root $\alpha \in \Phi$ (that is, $x_\alpha : K \rightarrow G$ is a morphism of algebraic groups inducing an isomorphism onto $\text{Im}(x_\alpha)$, such that $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ for $t \in T$ and $c \in K$). Finally, we fix a Chevalley basis $\mathscr{B} = \{f_\alpha, h_r, e_\alpha : \alpha \in \Phi^+, 1 \leq r \leq \ell\}$ for the Lie algebra $\text{Lie}(G)$ of G , compatible with our choice of $T \subset B$, where $e_\alpha \in \text{Lie}(G)_\alpha$, $f_\alpha \in \text{Lie}(G)_{-\alpha}$ are root vectors for $\alpha \in \Phi^+$ and $h_r = [e_{\alpha_r}, f_{\alpha_r}]$ for $1 \leq r \leq \ell$. The proof of the following result can be deduced from applying the Poincaré-Birkhoff-Witt Theorem [4] to [10, A. 6.4].

Lemma 2.1. *Let $\lambda \in X^+(T)$ be a p -restricted dominant weight for T , and let $V = L_G(\lambda)$. Also let $v^+ \in V$ be a maximal vector of weight λ for B , and let $\mu \in \Lambda(V)$. Then for any fixed ordering \leq on Φ^+ , we have*

$$V_\mu = \left\langle f_{\gamma_1} \cdots f_{\gamma_k} v^+ : k \in \mathbb{Z}_{\geq 0}, \gamma_1, \dots, \gamma_k \in \Phi^+, \gamma_1 \leq \dots \leq \gamma_k, \sum_{r=1}^k \gamma_r = \lambda - \mu \right\rangle_K.$$

We conclude this section by illustrating how Lemma 2.1 can provide information on weight multiplicities in certain irreducible KG -modules in the case where G is of type A_ℓ ($\ell \geq 2$) over K . Consider the dominant T -weights $\lambda = a\lambda_1 + b\lambda_\ell$ ($a, b \geq 1$) and $\mu = (a-1)\lambda_1 + (b-1)\lambda_\ell$. Writing $V = L_G(\lambda)$, an application of Lemma 2.1 then shows that V_μ is spanned by

$$(1) \quad \{f_{\alpha_1+\dots+\alpha_r} f_{\alpha_{r+1}+\dots+\alpha_\ell} v^+, 1 \leq r \leq \ell-1\} \cup \{f_{\alpha_1+\dots+\alpha_\ell} v^+\},$$

where v^+ is a maximal vector in V for B . (We used the fact that $f_{\alpha_i+\dots+\alpha_j} v^+ = 0$ for $1 < i \leq j < \ell$ together with the commutator formula.) Finally, we set

$$V_{1,\ell} = \langle f_{\alpha_1+\dots+\alpha_r} f_{\alpha_{r+1}+\dots+\alpha_\ell} v^+ : 1 \leq r \leq \ell-1 \rangle_K.$$

Proposition 2.2. *Assume G is of type A_ℓ over K for some $\ell \in \mathbb{Z}_{\geq 2}$ and consider the dominant T -weight $\lambda = a\lambda_1 + b\lambda_\ell \in X^+(T)$, $1 \leq a, b < p$. Set $V = L_G(\lambda)$. Also*

let $\mu = (a-1)\lambda_1 + (b-1)\lambda_\ell$. Then the following assertions are equivalent:

- (i) The weight μ affords the highest weight of a composition factor of $V_G(\lambda)$.
- (ii) The inequality $m_{V_G(\lambda)}(\mu) > m_V(\mu)$ is satisfied.
- (iii) The generators in (1) are linearly dependent.
- (iv) The element $f_{\alpha_1+\dots+\alpha_\ell}v^+$ belongs to $V_{1,\ell}$.
- (v) The divisibility condition $p \mid a+b+\ell-1$ is satisfied.

Proof. Clearly (i) implies (ii). Conversely, the only dominant weights $\nu \in \Lambda^+(V)$ such that $\mu \prec \nu \prec \lambda$ are $\lambda - \alpha_1$ (if $a > 1$) and $\lambda - \alpha_\ell$ (if $b > 1$). These weights have multiplicity 1 in $V_G(\lambda)$ and hence none of them can afford the highest weight of a composition factor of $V_G(\lambda)$ by [22]. Consequently (ii) implies (i). Now an application of Freudenthal's formula yields $m_{V_G(\lambda)}(\mu) = \ell$, thus showing that $\ell = m_{V_G(\lambda)}(\mu) > m_V(\mu)$ if and only if the ℓ generators in (1) are linearly dependent. Therefore (ii) and (iii) are equivalent as well. Finally, let $(\eta_r)_{1 \leq r \leq \ell} \in K^\ell$ and set

$$w^+ = \sum_{r=1}^{\ell-1} \eta_r f_{\alpha_1+\dots+\alpha_r} f_{\alpha_{r+1}+\dots+\alpha_\ell} v^+ + \eta_\ell f_{\alpha_1+\dots+\alpha_\ell} v^+.$$

A straightforward calculation shows that $Bw^+ = \langle w^+ \rangle_K$ if and only if $p \mid a+b+\ell-1$ and $\eta_\ell \neq 0$, in which case $w^+ = 0$ (since V is irreducible and $w^+ \notin V_\lambda$). This shows that (iii), (iv), and (v) are equivalent, thus completing the proof. \square

3. PROOF OF THEOREM 1.2

Let K be an algebraically closed field having characteristic $p \geq 0$ and let $Y = \text{Spin}_{2n+2}(K)$ be a simply connected, simple algebraic group of type D_{n+1} over K , with $n \geq 2$. Let $X \subset Y$ be the subgroup of type B_n , embedded in the usual way, as the stabilizer of a non-singular 1-dimensional subspace of the natural $(2n+2)$ -dimensional module for Y . Fix T_Y a maximal torus of Y and T_X a maximal torus of X such that $T_X \subset T_Y$ and let $T_Y \subset B_Y$, $T_X \subset B_X$ denote Borel subgroups of Y , X , respectively, with $B_X \subset B_Y$. Let $\Pi(Y) = \{\alpha_1, \dots, \alpha_{n+1}\}$ be the corresponding base for the root system of Y , and denote by $\{\lambda_1, \dots, \lambda_{n+1}\}$ the corresponding set of fundamental weights for T_Y , where the natural KY -module has highest weight λ_1 . Let σ be a graph automorphism of Y stabilizing T_Y , and with $X = Y^\sigma$, the group of σ -fixed points. Finally, let $\Pi(X) = \{\beta_1, \dots, \beta_n\}$ be the base for the corresponding root system of X , associated with the choice of Borel subgroup B_X , and denote by $\{\omega_1, \dots, \omega_n\}$ the associated set of fundamental dominant weights for T_X .

3.1. Preliminary considerations. For σ as above and for a KY -module V , let ${}^\sigma V$ denote the vector space V equipped with the Y -action $gv = \sigma(g)v$ for $g \in Y$, $v \in V$. Clearly ${}^\sigma V$ is irreducible if and only if V is.

Lemma 3.1. *Let V be an irreducible, finite-dimensional, rational KY -module. Then $V|_X$ is self-dual.*

Proof. Let $\lambda \in X^+(T_Y)$ be the highest weight of V . Then V^* has highest weight $-\omega_0\lambda$. If $n+1$ is even, then $-\omega_0 = 1$ by [25, Exercise 78] and so V is self-dual as a KY -module, from which the desired result follows. If on the other hand $n+1$ is odd, then ${}^\sigma V \cong V^*$ by [25, Lemma 78], yielding $(V|_X)^* = (V^*)|_X \cong ({}^\sigma V)|_X = V|_X$, since $\sigma(x) = x$ for all $x \in X$. The result follows. \square

Let $\mathcal{B}_Y = \{f_\alpha, h_r, e_\alpha : \alpha \in \Phi^+(Y), 1 \leq r \leq n+1\}$ be a Chevalley basis for the Lie algebra $\text{Lie}(Y)$ of Y , compatible with our choice of $T_Y \subset B_Y$, as in Section 2.2. As in [23, Section 8], we may assume that the B_n -type subalgebra $\text{Lie}(X)$ of $\text{Lie}(Y)$ is generated by the root vectors

$$(2) \quad \begin{aligned} e_{\beta_r} &= e_{\alpha_r} \text{ for } 1 \leq r \leq n-1, \\ e_{\beta_n} &= e_{\alpha_n} + e_{\alpha_{n+1}}, \\ f_{\beta_r} &= f_{\alpha_r} \text{ for } 1 \leq r \leq n-1, \\ f_{\beta_n} &= f_{\alpha_n} + f_{\alpha_{n+1}}. \end{aligned}$$

In particular, we get that $\alpha_j|_{T_X} = \beta_j$ for $1 \leq j \leq n-1$, while $\alpha_{n+1}|_{T_X} = \alpha_n|_{T_X} = \beta_n$, so that $\lambda_j|_{T_X} = \omega_j$ for $1 \leq j \leq n-1$, $\lambda_n|_{T_X} = \lambda_{n+1}|_{T_X} = \omega_n$ by [14, Section 13.2, p. 69]. Also for $1 \leq i \leq j \leq n+1$, write $f_{i,j} = f_{\alpha_i + \dots + \alpha_j}$, where we set $f_{i,i} = f_{\alpha_i}$ for $1 \leq i \leq n+1$ and $f_{n,n+1} = 0$ by convention. In a similar fashion, for $1 \leq k \leq n$, we set

$$\hat{f}_{k,n+1} = f_{\alpha_k + \dots + \alpha_{n-1} + \alpha_{n+1}},$$

where again we adopt the convention $\hat{f}_{n,n+1} = f_{\alpha_{n+1}}$. Finally, for $1 \leq i < j \leq n-1$, we set

$$F_{i,j} = f_{\alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}},$$

where $F_{i,i+1} = f_{\alpha_i + 2\alpha_{i+1} + \dots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}}$, $F_{i,n-1} = f_{\alpha_i + \dots + \alpha_{n-2} + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}}$ for every $1 \leq i \leq n-2$. We will require the following relations in $\text{Lie}(X)$.

Lemma 3.2. *Adopting the notation introduced above, we have*

- (i) $f_{\beta_r + \dots + \beta_s} = \pm f_{r,s}$ for $1 \leq r \leq s \leq n-1$,
- (ii) $f_{\beta_r + \dots + \beta_n} = \pm(f_{r,n} \pm \hat{f}_{r,n+1})$ for $1 \leq r \leq n$,
- (iii) $f_{r,n+1} \in \text{Lie}(X)$ for $1 \leq r \leq n-1$,
- (iv) $F_{r,s+1} \in \text{Lie}(X)$ for $1 \leq r \leq s \leq n-2$.

Proof. We start by showing (i), arguing by induction on $0 \leq s-r \leq n-2$. If $r = s$, then the assertion immediately follows from (2), so we assume $1 \leq r < s \leq n-1$ in the remainder of the proof. We then successively get

$$\begin{aligned} f_{\beta_r + \dots + \beta_s} &= \pm(f_{\beta_{r+1} + \dots + \beta_s} f_{\beta_r} - f_{\beta_r} f_{\beta_{r+1} + \dots + \beta_s}) \\ &= \pm N_1(f_{r+1,s} f_{\alpha_r} - f_{\alpha_r} f_{r+1,s}) \\ &= \pm N_1(N_2 f_{r,s} + f_{\alpha_r} f_{r+1,s} - f_{\alpha_r} f_{r+1,s}) \\ &= \pm N_1 N_2 f_{r,s} \end{aligned}$$

for some $N_1, N_2 \in \{\pm 1\}$, where the second equality follows from (2) and our induction assumption. Therefore (i) holds as desired. For the second assertion, we again argue by induction on $0 \leq n-r \leq n-1$. In the case where $r = n$, then the result holds by (2), hence we assume $1 \leq r \leq n-1$ in the remainder of the proof. We then successively get

$$\begin{aligned} f_{\beta_r + \dots + \beta_n} &= \pm(f_{\beta_{r+1} + \dots + \beta_n} f_{\beta_r} - f_{\beta_r} f_{\beta_{r+1} + \dots + \beta_n}) \\ &= \pm N_1(f_{r+1,n} f_{\alpha_r} + N_2 \hat{f}_{r+1,n+1} f_{\alpha_r} - f_{\alpha_r} f_{r+1,n} - N_2 f_{\alpha_r} \hat{f}_{r+1,n+1}) \\ &= \pm N_1(N_3 f_{r,n} + N_2 \hat{f}_{r+1,n+1} f_{\alpha_r} - N_2 f_{\alpha_r} \hat{f}_{r+1,n+1}) \\ &= \pm N_1(N_3 f_{r,n} + N_2 N_4 \hat{f}_{r,n+1}) \\ &= \pm N_1 N_3(f_{r,n} + N_2 N_3 N_4 \hat{f}_{r,n+1}) \end{aligned}$$

for some $N_1, N_2, N_3, N_4 \in \{\pm 1\}$, where again the second equality follows from (2) and our induction assumption. Therefore (ii) holds as well. Next we show the third assertion, letting $1 \leq r \leq n-1$ be fixed, and setting $\mu = \beta_r + \cdots + \beta_{n-1} + 2\beta_n$. The aforementioned root restrictions yield $\text{Lie}(X)_\mu \subset \text{Lie}(Y)_{\alpha_r + \cdots + \alpha_{n+1}}$, and since the latter T_Y -weight space is 1-dimensional, we get that $\text{Lie}(X)_\mu$ is at most 1-dimensional as well. Now as $f_{\beta_r + \cdots + \beta_n + 2\beta_{n+1}} \in \text{Lie}(X)_\mu$ and $\text{Lie}(Y)_{\alpha_r + \cdots + \alpha_{n+1}} = \langle f_{r,n+1} \rangle_K$, we get that $f_{\beta_r + \cdots + \beta_n + 2\beta_{n+1}}$ is a non-zero multiple of $f_{r,n+1}$, from which (iii) follows. Finally, the assertion (iv) can be dealt with in a similar fashion. \square

In the remainder of this section, we let $V = L_Y(\lambda)$ be a non-trivial, irreducible KY -module having p -restricted highest weight $\lambda = \sum_{r=1}^{n+1} a_r \lambda_r \in X^+(T_Y)$, and fix a maximal vector v^+ in V for B_Y . Setting $\lambda|_{T_X} = \omega$, one observes that v^+ is a maximal vector of weight ω in V for B_X , since $B_X \subset B_Y$. The following result provides a necessary and sufficient condition for V to be irreducible in the case where $a_n \neq 0 = a_{n+1}$.

Lemma 3.3. *Let $V = L_Y(\lambda)$ be as above, and assume $a_n \neq 0 = a_{n+1}$. Then X acts irreducibly on V if and only if $V = \text{Lie}(X)v^+$.*

Proof. First assume X acts irreducibly on V , so that $V|_X \cong L_X(\omega)$, and observe that since $a_n a_{n+1} = 0$, the T_X -weight ω is p -restricted. Therefore the Lie algebra $\text{Lie}(X)$ of X acts irreducibly on V by [10, Theorem 1], from which the desired assertion follows. Conversely, assume $V = \text{Lie}(X)v^+$. Then $V = \langle Xv^+ \rangle$ and hence $V|_X$ has a quotient isomorphic to $L_X(\omega)$ by [16, II, Lemma 2.13 (b)]. Consequently $V^*|_X$ contains a KX -submodule isomorphic to $L_X(\omega)$. Now $V^*|_X \cong V|_X$ by Lemma 3.1, showing the existence of a submodule U of $V|_X$ such that $U \cong L_X(\omega)$. Since $(V|_X)_\omega = \langle v^+ \rangle_K$, we get that $v^+ \in U$ and so $V = \langle Xv^+ \rangle \subset U \cong L_X(\omega)$ as desired. \square

In view of Lemma 3.3, a necessary condition for X to act irreducibly on $V = L_Y(\lambda)$, with λ such that $a_n \neq 0 = a_{n+1}$, is for $f_\gamma v^+$ to belong to $\text{Lie}(X)v^+$ for every $\gamma \in \Phi^+(Y)$. We conclude this section by showing that $V|_X$ is irreducible if and only if $f_{r,n}v^+ \in \text{Lie}(X)v^+$ for every $1 \leq r \leq n$ (see Proposition 3.5 below). We first need the following preliminary lemma.

Lemma 3.4. *Let $V = L_Y(\lambda)$ be as above, with $a_n \neq 0 = a_{n+1}$, and assume $f_\gamma v^+ \in \text{Lie}(X)v^+$ for every $\gamma \in \Phi^+(Y)$. Then $f_\gamma f_{\eta_1} \cdots f_{\eta_s} v^+ \in \text{Lie}(X)v^+$ for every $\gamma \in \Phi^+(Y)$ and $\eta_1, \dots, \eta_s \in \Phi^+(X)$.*

Proof. We proceed by induction on $s \geq 1$. First take $\eta \in \Phi^+(X)$ and consider $f_\gamma f_\eta v^+$, with $\gamma \in \Phi^+(Y)$. Since $f_\gamma f_\eta v^+ = [f_\gamma, f_\eta]v^+ + f_\eta f_\gamma v^+$ and as $f_\gamma v^+ \in \text{Lie}(X)v^+$ by assumption, it suffices to show that $[f_\gamma, f_\eta]v^+ \in \text{Lie}(X)v^+$. If $f_\gamma \in \text{Lie}(X)$, then clearly $[f_\gamma, f_\eta]v^+$ lies in $\text{Lie}(X)v^+$, so assume $f_\gamma \notin \text{Lie}(X)$. By Lemma (3.2), we then have that

$$\gamma \in \{\alpha_n, \alpha_{n+1}\} \cup \{\alpha_i + \cdots + \alpha_n, \alpha_i + \cdots + \alpha_{n-1} + \alpha_{n+1}\}_{1 \leq i \leq n-1}.$$

Take first $\gamma = \sum_{r=i}^n \alpha_r$ for some $1 \leq i \leq n$, so $f_\gamma = f_{i,n}$, and consider $[f_\gamma, f_\eta]v^+$. If the latter equals zero, then we immediately get the desired result. So we may assume the contrary and thus we have f_η is one of the following:

- (i) $f_{j,i-1}$, $1 \leq j \leq i-1$.
- (ii) $f_{\alpha_n} + f_{\alpha_{n+1}}$ (if $i \neq n$).

- (iii) $f_{k,n} \pm \hat{f}_{k,n+1}$, $i + 1 \leq k \leq n$.
- (iv) $f_{j,n} \pm \hat{f}_{j,n+1}$, $1 \leq j \leq i - 1$.

We now calculate $[f_\gamma, f_\eta]v^+$ in each case.

- (i) $[f_{i,n}, f_{j,i-1}]v^+ = \pm f_{j,n}v^+$, which lies in $\text{Lie}(X)v^+$ by assumption.
- (ii) $[f_{i,n}, f_{\alpha_n} + f_{\alpha_{n+1}}]v^+ = \pm f_{i,n+1}v^+$, which lies in $\text{Lie}(X)v^+$ since $f_{i,n+1} \in \text{Lie}(X)$ by Lemma (3.2)(iii).
- (iii) $[f_{i,n}, f_{k,n} \pm \hat{f}_{k,n+1}]v^+ = \pm F_{i,k}v^+$, which again lies in $\text{Lie}(X)v^+$ as $F_{i,k} \in \text{Lie}(X)$ by Lemma (3.2)(iv).
- (iv) $[f_{i,n}, f_{j,n} \pm \hat{f}_{j,n+1}]v^+ = \pm F_{j,i}v^+$, which as in the previous case lies in $\text{Lie}(X)v^+$.

Consequently $f_\gamma f_\eta v^+ \in \text{Lie}(X)v^+$ for $\gamma = \sum_{r=i}^n \alpha_r$ as desired. Arguing in a similar fashion, one shows that the same holds for $\gamma = \sum_{r=i}^{n-1} \alpha_r + \alpha_{n+1}$ and $\gamma = \alpha_{n+1}$ as well, where $1 \leq i \leq n - 1$. Therefore the lemma holds in the situation where $s = 1$, and hence we assume $s > 1$ in the remainder of the proof. For $\eta_1, \dots, \eta_s \in \Phi^+(X)$, we have

$$f_\gamma f_{\eta_1} \cdots f_{\eta_s} v^+ = [f_\gamma, f_{\eta_1}]f_{\eta_2} \cdots f_{\eta_s} v^+ + f_{\eta_1} f_\gamma f_{\eta_2} \cdots f_{\eta_s} v^+.$$

Now $f_\gamma f_{\eta_2} \cdots f_{\eta_s} v^+ \in \text{Lie}(X)v^+$ by induction and hence so does $f_{\eta_1} f_\gamma f_{\eta_2} \cdots f_{\eta_s} v^+$. On the other hand, we either have $[f_\gamma, f_{\eta_1}] = 0$ or $[f_\gamma, f_{\eta_1}] = \xi f_\delta$ for some $\xi \in K$ and $\delta \in \Phi^+(Y)$ by (i), (ii), (iii), (iv) above, in which case $[f_\gamma, f_{\eta_1}]f_{\eta_2} \cdots f_{\eta_s} v^+ = \xi f_\delta f_{\eta_2} \cdots f_{\eta_s} v^+ \in \text{Lie}(X)v^+$ by induction, thus completing the proof. \square

We now establish the following necessary and sufficient condition for $V|_{\text{Lie}(X)}$ to be generated by v^+ as a $\text{Lie}(X)$ -module (and hence for $V|_X$ to be irreducible by Lemma 3.3), where V has highest weight λ with $a_n \neq 0 = a_{n+1}$.

Proposition 3.5. *Let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight $\lambda = \sum_{r=1}^n a_r \lambda_r$, with $a_n \neq 0$. Then $V = \text{Lie}(X)v^+$ if and only if $f_{r,n}v^+ \in \text{Lie}(X)v^+$ for every $1 \leq r \leq n$.*

Proof. If $V = \text{Lie}(X)v^+$, then clearly $f_\gamma v^+ \in \text{Lie}(X)v^+$ for every $\gamma \in \Phi^+(Y)$. Hence in particular $f_{r,n}v^+ \in \text{Lie}(X)v^+$ for every $1 \leq r \leq n$ as desired. Conversely, suppose that $f_{r,n}v^+ \in \text{Lie}(X)v^+$ for every $1 \leq r \leq n$ and notice that since $\lambda \in X^+(T_Y)$ is p -restricted, we have

$$V = \langle f_{\gamma_1} \cdots f_{\gamma_s} v^+ : s \in \mathbb{Z}_{\geq 0}, \gamma_1, \dots, \gamma_s \in \Phi^+(Y) \rangle_K$$

by [10, Theorem 1]. Hence in order to show that $V = \text{Lie}(X)v^+$, it suffices to show that $f_{\gamma_1} \cdots f_{\gamma_s} v^+ \in \text{Lie}(X)v^+$ for every $s \in \mathbb{Z}_{>0}$ and $\gamma_1, \dots, \gamma_s \in \Phi^+(Y)$. Assume for a contradiction that this is not the case and let $m \in \mathbb{Z}_{\geq 0}$ be minimal such that there exists $\gamma_1, \dots, \gamma_m \in \Phi^+(Y)$ with $f_{\gamma_1} \cdots f_{\gamma_m} v^+ \notin \text{Lie}(X)v^+$. Lemma (3.2) implies $f_\gamma v^+ \in \text{Lie}(X)v^+$ for $\gamma \in \Phi^+(Y)$ and so $m \geq 2$. Now by minimality, we have $f_{\gamma_2} \cdots f_{\gamma_m} v^+ \in \text{Lie}(X)v^+$ and hence an application of Lemma 3.4 completes the proof. \square

3.2. Conclusion. Let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted non-zero highest weight $\lambda = \sum_{r=1}^{n+1} a_r \lambda_r \in X^+(T_Y)$, and set $\omega = \lambda|_{T_X}$. In this section, we will complete the proof of Theorem 1.2. We first show that for certain weights λ , it is straightforward to see that $V|_X$ is reducible. Although the proof of the following proposition can be found in [23, Section 8], we include it here for completeness.

Proposition 3.6. *Let $\lambda \in X^+(T_Y)$ and let V be as above. Then the following assertions hold:*

- (i) *If $a_n a_{n+1} \neq 0$, or if $a_n = a_{n+1} = 0$, then $V|_X$ is reducible.*
- (ii) *If $\lambda \in \{a\lambda_n, a\lambda_{n+1} : a \in \mathbb{Z}_{>0}\}$, then X acts irreducibly on V .*
- (iii) *If $V|_X$ is irreducible, so $a_n a_{n+1} = 0$ by (i), and λ is not as in (ii), then taking $1 \leq k < n$ maximal such that $a_k \neq 0$, we have $p \mid (a_k + a_n + a_{n+1} + n - k)$.*

Proof. For (i), first consider the case where $a_n a_{n+1} \neq 0$. Here the T_X -weight $\omega' = \omega - \beta_n$ has multiplicity at most 1 in $L_X(\omega)$, while each of $\lambda - \alpha_n$ and $\lambda - \alpha_{n+1}$ is a T_Y -weight of V restricting to ω' . Therefore the latter occurs in a second KX -composition factor of V , showing that $V|_X$ is reducible as desired. Next assume $a_n = a_{n+1} = 0$, and let $1 \leq k \leq n - 1$ be maximal such that $a_k \neq 0$. Then the T_X -weight $\omega'' = \omega - (\beta_k + \cdots + \beta_n)$ has multiplicity at most 1 in $L_X(\omega)$ (since the corresponding weight space is generated by $f_{\beta_k + \cdots + \beta_n} v^+$), while each of $\lambda - (\alpha_k + \cdots + \alpha_n)$ and $\lambda - (\alpha_k + \cdots + \alpha_{n-1} + \alpha_{n+1})$ restricts to ω'' . Consequently $V|_X$ is reducible in this case as well, and (i) holds as desired.

Now turn to (ii) and let $\lambda = a\lambda_n$ for some $a \in \mathbb{Z}_{>0}$, and assume for a contradiction that $V|_X$ is reducible. By [23, 8.5], there exist $1 \leq i \leq n$ and a maximal vector $w^+ \in V$ for B_X such that

$$0 \neq e_{\alpha_i + \cdots + \alpha_n} w^+ \in \langle v^+ \rangle_K.$$

In particular $w^+ \in V_{\lambda - (\alpha_i + \cdots + \alpha_n)}$, and one checks that the only T_Y -weight of V restricting to $\omega - (\beta_i + \cdots + \beta_n)$ is $\lambda - (\alpha_i + \cdots + \alpha_n)$. So $w^+ \in V_{\lambda - (\alpha_i + \cdots + \alpha_n)} = \langle f_{\alpha_i + \cdots + \alpha_n} v^+ \rangle_K$, thus yielding $e_{\beta_i} w^+ = e_{\alpha_i} w^+ \neq 0$, a contradiction. Therefore (ii) holds as desired.

For (iii), we assume $V|_X$ is irreducible, and by (i), we suppose, without loss of generality, that $a_n \neq 0 = a_{n+1}$. Assume as well that λ is not as in (ii), and seeking a contradiction, take k as in (iii), and suppose that $p \nmid (a_k + a_n + n - k)$. Consider the T_X -weight $\omega'' = \omega - (\beta_k + \cdots + \beta_n) \in X^+(T_X)$. Then $\lambda - (\alpha_k + \cdots + \alpha_n)$ and $\lambda - (\alpha_k + \cdots + \alpha_{n-1} + \alpha_{n+1})$ are both T_Y -weights of V restricting to ω'' . Now $m_{L_X(\omega)}(\omega'') \leq m_{V_X(\omega)}(\omega'') = n - k + 1$, while an application of Proposition 2.2 yields $m_V(\lambda - (\alpha_k + \cdots + \alpha_n)) = n - k + 1$. (Recall that we assumed $a_n \neq 0 = a_{n+1}$ and $p \nmid (a_k + a_n + n - k)$.) Now $m_V(\lambda - (\alpha_k + \cdots + \alpha_{n-1} + \alpha_{n+1})) = 1$, as the latter weight is conjugate to $\lambda - \alpha_k$ under the action of the Weyl group for Y . Therefore $m_{V|_X}(\omega'') \geq n - k + 2 > n - k + 1 \geq m_{L_X(\omega)}(\omega'')$, yielding the existence of a second composition factor of V for X , a contradiction. The proof is complete. \square

In view of Proposition 3.6, we may and shall assume $\lambda = \sum_{r=1}^n a_r \lambda_r$, with $a_n \neq 0$ throughout the rest of the section, as well as the existence of $1 \leq k \leq n - 1$ maximal such that $a_k \neq 0$. For $1 \leq i \leq n$, set $P(i, i) = \emptyset$ and for $1 \leq i < j \leq n$, set

$$P(i, j) = \{(m_r)_{r=1}^s : 1 \leq s \leq j - i, i \leq m_1 < \cdots < m_s < j\}.$$

For any sequence $(m) = (m_r)_{r=1}^s \in P(i, j)$, write $f_{(m)} = f_{i, m_1} f_{m_1 + 1, m_2} \cdots f_{m_s + 1, j}$. By Lemma 2.1, we have that for every $1 \leq i < j \leq n$, the weight space $V_{\lambda - (\alpha_i + \cdots + \alpha_j)}$ is spanned by the vectors

$$\{f_{(m)} v^+ : (m) \in P(i, j)\} \cup \{f_{i, j} v^+\},$$

where $v^+ \in V$ is a maximal vector of weight λ for B_Y . We set

$$V_{i, j} = \langle f_{(m)} v^+ : (m) \in P(i, j) \rangle_K.$$

The following special case of [8, Theorem A.7], inspired by [13, Proposition 3.1], shall play a key role in the proof of Theorem 1.2.

Theorem 3.7. *Let Y be a simple algebraic group of type D_{n+1} over K , and consider an irreducible KY -module $V = L(\lambda)$ having p -restricted highest weight $\lambda = \sum_{r=1}^n a_r \lambda_r$, with $a_n \neq 0$. Then $f_{r,n} v^+ \in V_{r,n}$ for every $1 \leq r \leq n-1$ if and only if $p \mid (a_i + a_j + j - i)$ for every $1 \leq i < j \leq n$ such that $a_i a_j \neq 0$ and $a_s = 0$ for $i < s < j$.*

Proof. The result follows from an application of [8, Theorem A.7] to the A_n -Levi subgroup of Y corresponding to the simple roots $\alpha_1, \dots, \alpha_n$. \square

For $1 \leq i \leq n-1$, we let $\hat{V}_{i,n+1}$ denote the K -span of $\{f_{i,j} \hat{f}_{j+1,n+1} v^+ : i \leq j \leq n-1\}$ and $\{f_{(m)} \hat{f}_{j+1,n+1} v^+ : i < j \leq n-1, (m) \in P(i, j)\}$. The proof of the main result of this section (namely, Theorem 3.9) relies on the following preliminary result.

Lemma 3.8. *Adopt the notation introduced above and let $1 \leq r \leq k-1$ be such that $\hat{f}_{r,n+1} v^+ \in \hat{V}_{r,n+1}$. Then $f_{r,n} v^+ \in V_{r,n}$.*

Proof. Let $1 \leq r \leq k-1$ be as in the statement of the lemma. If $a_r = 0$, then $f_{\alpha_r} v^+ = 0$ and hence $f_{r,n} v^+ = \pm f_{\alpha_r} f_{r+1,n} v^+$, so that $f_{r,n} v^+ \in V_{r,n}$, as claimed. Therefore we assume $a_r \neq 0$ in the remainder of the proof. As $\hat{f}_{r,n+1} v^+ \in \hat{V}_{r,n+1}$ and since $\hat{f}_{s,n+1} v^+ = 0$ for $k < s \leq n$, we get the existence of $(\xi_s, \xi_{(m)}^s) : r \leq s \leq k-1, (m) \in P(r, s) \subset K$ such that

$$(3) \quad \hat{f}_{r,n+1} v^+ = \sum_{s=r}^{k-1} \left(\sum_{(m) \in P(r,s)} \xi_{(m)}^s f_{(m)} \hat{f}_{s+1,n+1} v^+ + \xi_s f_{r,s} \hat{f}_{s+1,n+1} v^+ \right).$$

Now observe that $f_{\alpha_n} e_{\alpha_{n+1}} \hat{f}_{t,n+1} v^+ = N_1 f_{t,n} v^+ + N_2 f_{t,n-1} f_{\alpha_n} v^+$ for every $r \leq t \leq k$, where $N_1, N_2 \in \{\pm 1\}$. Also, for every $r \leq s \leq k-1$ and every $(m) \in P(r, s)$, we have $[e_{\alpha_{n+1}}, f_{r,s}] = [e_{\alpha_{n+1}}, f_{(m)}] = 0$, as well as $[f_{\alpha_n}, f_{r,s}] = [f_{\alpha_n}, f_{(m)}] = 0$. Consequently successively applying $e_{\alpha_{n+1}}, f_{\alpha_n}$ to each side of (3) yields $N_1 f_{r,n} v^+ + N_2 f_{r,n-1} f_{\alpha_n} v^+ \in V_{r,n}$ for some $N_1 \neq 0, N_2 \in \{\pm 1\}$, from which the desired result follows. \square

In the next result, we show that in order to determine whether $V|_X$ is irreducible or not, it is enough to determine whether $f_{r,n} v^+ \in V_{r,n}$ or not, this for every $1 \leq r \leq n-1$.

Theorem 3.9. *Let $\lambda = \sum_{r=1}^n a_r \lambda_r$ be a p -restricted dominant weight such that $a_k a_n \neq 0$. Let $V = L_Y(\lambda)$ be an irreducible KY -module having highest weight λ . Then $V|_X$ is irreducible if and only if $f_{r,n} v^+ \in V_{r,n}$ for every $1 \leq r \leq n-1$.*

Proof. First assume $f_{r,n} v^+ \in V_{r,n}$ for every $1 \leq r \leq n-1$. By Proposition 3.5, in order to show that $V|_X$ is irreducible, it suffices to show that $f_{s,n} v^+ \in \text{Lie}(X)v^+$ for every $1 \leq s \leq n$. We proceed by induction on $0 \leq n-s \leq n-1$, starting by assuming $s = n$. Now since $a_{n+1} = 0$, we immediately get that $f_{\alpha_{n+1}} v^+ = 0$ and hence $f_{\beta_n} v^+ = f_{\alpha_n} v^+$ by (2), that is, $f_{\alpha_n} v^+ \in \text{Lie}(X)v^+$. Next assume $1 \leq s \leq n-1$. Since $f_{s,n} v^+ \in V_{s,n}$ by assumption, there exists $(\xi_i, \xi_{(m)}^i) : s \leq i \leq n-1, (m) \in P(s, i) \subset K$ such that

$$f_{s,n} v^+ = \sum_{i=s}^{n-1} \left(\xi_i f_{s,i} f_{i+1,n} v^+ + \sum_{(m) \in P(s,i)} \xi_{(m)}^i f_{(m)} f_{i+1,n} v^+ \right).$$

By Lemma (3.2), $f_{s,i} \in \text{Lie}(X)$ and $f_{(m)} \in \text{Lie}(X)$ for every $s \leq i \leq n-1$ and every $(m) \in P(s,i)$. Furthermore, we also have $f_{i+1,n}v^+ \in \text{Lie}(X)v^+$ for $s \leq i \leq n-1$ by induction. Therefore $f_{s,n}v^+ \in \text{Lie}(X)v^+$ as desired.

Conversely, assume $V|_X$ is irreducible, and let $1 \leq r \leq n-1$ be fixed. If $a_r = 0$, then $f_{\alpha_r}v^+ = 0$ and hence $f_{r,n}v^+ = \pm f_{\alpha_r}f_{r+1,n}v^+$, so that $f_{r,n}v^+ \in V_{r,n}$ in this situation. In addition, observe that $p \mid (a_k + a_n + n - k)$ by Proposition 3.6(iii), thus yielding $f_{k,n}v^+ \in V_{k,n}$ by Proposition 2.2. Therefore we assume $a_r \neq 0$ in the remainder of the proof, and $r < k$. By Lemma 3.3, we have $V = \text{Lie}(X)v^+$, in which case Proposition 3.5 applies, thus yielding $f_{r,n}v^+ \in \text{Lie}(X)v^+$. Since $f_{r,n}v^+ \in (\text{Lie}(X)v^+)_{\omega - (\beta_r + \dots + \beta_n)}$, we get the existence of $\xi \in K$, $x \in V_{r,n}$, and $y \in \hat{V}_{r,n+1}$ such that $f_{r,n}v^+ = \xi f_{r,n}v^+ \pm \xi \hat{f}_{r,n+1}v^+ + x + y$. Comparing T_Y -weights yields

$$(\xi - 1)f_{r,n}v^+ \in V_{r,n} \text{ and } \pm \xi \hat{f}_{r,n+1}v^+ \in \hat{V}_{r,n+1}.$$

If $\xi \neq 1$, then the assertion is immediate, while if on the other hand $\xi = 1$, then an application of Lemma 3.8 yields the desired result. \square

We are now able to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. First assume X acts irreducibly on V . By Proposition 3.6(i), we then have that, up to a graph automorphism of Y , $a_n \neq 0 = a_{n+1}$, and if $\lambda = a\lambda_n$, then λ is as in the statement of the result. So assume the existence of $1 \leq k \leq n-1$ maximal with $a_k \neq 0$. Then Theorem 3.9 applies, yielding $f_{r,n}v^+ \in V_{r,n}$ for every $1 \leq r \leq n-1$. An application of Theorem 3.7 then implies the desired divisibility conditions.

Conversely, assume $\lambda \in X^+(T_Y)$ satisfies the conditions in Theorem 1.2. In the case where $\lambda = a\lambda_n$ for some $a \in \mathbb{Z}_{>0}$, then the result follows from Proposition 3.6(ii), hence we assume the existence of $1 \leq k \leq n-1$ maximal such that $a_k a_n \neq 0$, and assume moreover the divisibility conditions as in the theorem. Here an application of Theorem 3.7 yields $f_{r,n}v^+ \in V_{r,n}$ for every $1 \leq r \leq n-1$, and hence Theorem 3.9 then shows that X acts irreducibly on V as desired. \square

4. PROOF OF PROPOSITIONS 1.5 AND 1.7

We first prove Proposition 1.5. Let $Y = \text{Spin}_{2n+2}(K)$ be a simply connected, simple algebraic group of type D_{n+1} over K , with $n \geq 2$. Let $X \subset Y$ be a semisimple, connected, proper, closed subgroup of Y . Fix T_Y a maximal torus of Y and let $T_Y \subset B_Y$ denote a Borel subgroup of Y . Also let $\{\lambda_1, \dots, \lambda_{n+1}\}$ be the corresponding set of fundamental dominant weights for T_Y . In this section, we give a proof of Proposition 1.5, starting with three results, proven in a more general setting in [23, Section 5]. For the convenience of the reader, and in order to render the current manuscript more self-contained, we include the proofs of these special cases here.

Proposition 4.1. *Let Y be a simply connected, simple algebraic group of type D_{n+1} with natural module W and let X be a semisimple, connected, proper, closed subgroup of Y acting irreducibly on a non-trivial, p -restricted, irreducible KY -module $V = L_Y(\lambda)$. Then one of the following holds:*

- (i) $W|_X$ is irreducible.
- (ii) $W|_X$ is reducible and $X \subset B_k \times B_{n-k}$ for some $0 \leq k < n$.

Proof. Suppose $W|_X$ is reducible. Let W_1 be a minimal non-zero X -invariant subspace of W . Then $W_1 \cap W_1^\perp = W_1$ or $\{0\}$. Suppose $W_1 \cap W_1^\perp = W_1$. Note that W_1

is not totally singular, else X lies in a proper parabolic subgroup of Y and so cannot act irreducibly on V . Therefore, we have $p = 2$; the set of singular vectors in W_1 being an X -invariant subspace of W_1 forces W_1 to be generated by a non-singular vector, and so (ii) holds with $k = 0$. In case $W_1 \cap W_1^\perp = \{0\}$, set $W_2 = W_1^\perp$, so that $W = W_1 \oplus W_2$, an orthogonal direct sum and the image of X in $\text{Isom}(W)$ lies in $\text{Isom}(W_1)' \times \text{Isom}(W_2)'$. If $\dim W_1$ is even, then $X \subset D_s \times D_{n+1-s}$, a maximal rank subgroup of Y and we may invoke [23, Theorem 4.1] to see that X acts reducibly on V . So $\dim W_1$ is odd and $X \subset B_k \times B_{n-k}$ as in (ii). \square

Proposition 4.2. *Let Y and V be as in Proposition 4.1(ii) for some $0 < k < n$, and let V be of highest weight λ . Then the closed, connected subgroup $H \subset Y$ of type $B_k \times B_{n-k}$ acts irreducibly on V if and only if $\lambda = \lambda_n$ or λ_{n+1} .*

Proof. To see that H acts irreducibly on the two half-spin modules for Y , we simply compute the restriction of the highest weight to a maximal torus of H , note that the restriction induces the tensor product of the spin modules for the two simple factors of H , and then a dimension comparison completes the proof.

We now assume H acts irreducibly on V , the irreducible KY -module of highest weight λ . We proceed by induction on n and first consider the case $n = 2$, so that Y is of type $D_3 = A_3$ and the image of H in SO_6 lies in the subgroup $\text{SO}_3 \times \text{SO}_3 \subset \text{SO}_6$. Since H acts irreducibly on V , H does not lie in a parabolic subgroup of Y and so acts irreducibly on the 4-dimensional p -restricted KY -modules, which correspond to the highest weights as in the statement of the result. Thus the preimage of $\text{SO}_3 \times \text{SO}_3$ in SL_4 acts on the 4-dimensional module via the tensor product representation of $A_1 \times A_1$ on $E \otimes E$, where E is the natural 2-dimensional representation of SL_2 . To see that $\text{SL}_2 \times \text{SL}_2$ acts reducibly (and hence H as well) on all other non-trivial, p -restricted, irreducibles for Y , we require a further argument (and some additional notation).

Let T be a maximal torus of $\text{SL}_2 \times \text{SL}_2$ with $T \subset T_Y$. Moreover, choose a base $\Pi = \{\alpha, \beta\}$ of the root system of $\text{SL}_2 \times \text{SL}_2$, and a base $\Pi(Y) = \{\alpha_1, \alpha_2, \alpha_3\}$ of the root system of Y , viewing Y as D_3 . Now it is straightforward to see that up to conjugation we may assume that $\alpha_1|_T = \beta - \alpha$, $\alpha_j|_T = \alpha$ for $j = 2, 3$. Now we apply [23, 6.1] to see that $\lambda \in \{c\lambda_j, b\lambda_1 + a\lambda_j, c \geq 1, b > 0, a + b = p - 1, j = 2, 3\}$. In case $\lambda = c\lambda_j$, $V|_{\text{SL}_2 \times \text{SL}_2} = S^c(E \otimes E)$ which is easily seen to be reducible if $c > 1$. In any case, $V|_{\text{SL}_2 \times \text{SL}_2}$ has a composition factor with highest weight $\lambda|_T$. In the cases $\lambda = b\lambda_1 + a\lambda_j$, we see that the weight $\lambda - \alpha_1$ restricts to T as $\lambda|_T - \beta + \alpha$ and hence lies in a second composition factor of $V|_{\text{SL}_2 \times \text{SL}_2}$. The result then holds for $n = 2$.

Assume now that $n \geq 3$ and that the result holds for $Y = D_\ell$ with $\ell < n + 1$. Assume as well that $p > 2$; we will treat the case $p = 2$ at the end of the proof. Let W_1 and W_2 be as in the previous proof so that B_k acts on W_1 and B_{n-k} acts on W_2 . Let U_1 , respectively U_2 , be maximal totally isotropic subspaces of W_1 , respectively W_2 . Then $U_0 = U_1 \oplus U_2$ is an n -dimensional totally isotropic subspace of W . Let $P \subset Y$ be the preimage of the stabilizer in $\text{Isom}(W_1)' \times \text{Isom}(W_2)'$ of U_0 and R the preimage in Y of the stabilizer in $\text{Isom}(W)$ of U_0 . Then $P \subset R$, $R_u(P) \subset R_u(R)$, and $(P/R_u(P))' \cong \text{SL}(U_1) \times \text{SL}(U_2)$, while $(R/R_u(R))' \cong \text{SL}(U_0) = \text{SL}_n$. In particular, the image of the Levi factor of P in $R/R_u(R)$ stabilizes U_1 and so lies in a proper parabolic subgroup of $R/R_u(R)$, and hence can act irreducibly on no non-trivial $(R/R_u(R))'$ -module. Then the above remarks and an application of the main proposition of [24] shows that $\lambda = a\lambda_n + b\lambda_{n+1}$ for some a, b .

Assume $\dim W_1 \geq \dim W_2$, so $\dim W_1 \geq 5$. (Recall that $\dim V \geq 8$.) Let P_1 be the stabilizer in $\mathrm{SO}(W_1)'$ of a singular 1-space. Then $P_1 \times \mathrm{SO}(W_2)'$ is a proper parabolic subgroup of $\mathrm{SO}(W_1)' \times \mathrm{SO}(W_2)'$ and is contained in the image (under the natural projection $Y \rightarrow \mathrm{SO}(W)$) of the stabilizer in Y of this 1-space. As $P_1 = B_{k-1}R_u(P_1)$, we have the Levi factor $B_{k-1}B_{n-k}$ projecting into the Levi factor of type D_n . Another application of [24] and the induction hypothesis yield the result.

Turn now to the case $n > 2$ and $p = 2$. In this case we have that the subgroup $B_k \times B_{n-k}$ stabilizes a non-singular 1-space and so lies in a subgroup of type B_n . Now the subgroup $B_k \times B_{n-k}$ is a maximal rank subgroup of B_n and we can appeal to [23, Theorem 4.1] applied to the pair $(B_k \times B_{n-k}, B_n)$ to see that the irreducible KB_n -module $V|_{B_n}$ is a twist of the spin module and hence we deduce that $\lambda = \lambda_n + \lambda_{n+1}$, λ_n , or λ_{n+1} . However, $L_Y(\lambda_n + \lambda_{n+1})|_{B_n}$ is not irreducible, as the highest weight affords a twist of the spin module for B_n and this is of dimension strictly less than the dimension of $L_Y(\lambda_n + \lambda_{n+1})$. So $\lambda = \lambda_n$ or λ_{n+1} as claimed. \square

Propositions 4.1 and 4.2 show that under the hypotheses of Proposition 1.5, either Proposition 1.5(a) holds, or X acts irreducibly on the natural KY -module W , or $X \subset B_n$ is as in Proposition 4.1(ii) with $k = 0$. Theorem 1.2 handles the latter situation in case $X = B_n$. The resolution of this case will follow from induction; see the end of this section. Now for the case where X acts irreducibly on W , we first show that X must act tensor indecomposably.

Proposition 4.3. *Let Y, X, V be as in Proposition 1.5, and let W be the natural $(2n + 2)$ -dimensional KY -module. If $W|_X$ is irreducible, then $W|_X$ is tensor indecomposable.*

Proof. Suppose the contrary. Then write $W|_X = W_1^{(f_1)} \otimes \cdots \otimes W_t^{(f_t)}$, where W_i is a p -restricted KX -module, f_1, \dots, f_t are distinct p -powers, and $t \geq 2$. Then the criterion on the highest weights of self-dual modules shows that each W_i carries an X -invariant non-degenerate bilinear form. Write $W = D \otimes F$, where each of D and F is a KX -module and the image of X in $\mathrm{Isom}(W)'$ lies in $\mathrm{Isom}(D)' \circ \mathrm{Isom}(F)'$, where this latter denotes the image of the natural morphism $\mathrm{Isom}(D)' \times \mathrm{Isom}(F)' \rightarrow \mathrm{Isom}(W)$, a commuting product of subgroups isomorphic to $\mathrm{Isom}(D)'$ and $\mathrm{Isom}(F)'$. Set $\dim D = d$ and $\dim F = f$ and assume $d \geq f \geq 2$ (and so at least one of d and f is even).

Extracting part of the argument given on page 76 of [23], we will now show that Y contains a semisimple group inducing $\mathrm{Isom}(D)' \circ \mathrm{Isom}(F)'$. As pointed out above, X stabilizes the product bilinear form on W . But then the irreducibility of X on W forces this to be a scalar multiple of the form defining Y . Adjusting the form on D if necessary, we may assume that the product form is precisely the form defining Y . This establishes the claim unless $p = 2$. In this case, we have $\mathrm{Isom}(D)' \circ \mathrm{Isom}(F)' \subset \mathrm{Sp}(D) \circ \mathrm{Sp}(F)$. The latter group preserves a quadratic form \mathcal{Q} on W , such that \mathcal{Q} has the same polarization as the bilinear form on W and also such that $\mathcal{Q}(x \otimes y) = 0$ for all $x \in D$ and $y \in F$ (see [17, Section 4.4]). By [2, 4.9], X fixes a unique quadratic form with prescribed bilinear form, and so this must necessarily be the form \mathcal{Q} and we again have the claim. Hence we now have that X lies in a closed subgroup J of Y which is the preimage in Y of the group $\mathrm{Isom}(D)' \circ \mathrm{Isom}(F)'$. In particular J acts irreducibly on V as well. Let D_1 be an

isotropic 1-space in D ; then $D = D_1 \oplus D_2 \oplus D_3$, where $D_1^\perp = D_1 \oplus D_2$, D_2 is non-degenerate, and D_3 is an isotropic 1-space. Similarly decompose $F = F_1 \oplus F_2 \oplus F_3$. Now consider the flag of subspaces in V :

$$0 = V_0 \subset V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subset V_5 = V,$$

where $V_1 = D_1 \otimes F_1$, $V_2 = (D_1 \otimes F_1^\perp) + (D_1^\perp \otimes F_1)$, $V_3 = V_2^\perp$, and $V_4 = V_1^\perp$. Then V_1 and V_2 are totally isotropic (totally singular if $p = 2$), $\dim V_1 = 1 = \dim(V/V_4)$, $\dim(V_2/V_1) = \dim(V_4/V_3) = d + f - 4$, and $\dim(V_3/V_2) = (d - 2)(f - 2) + 2$.

Set P_J to be the stabilizer in J of the above flag, and P_Y the stabilizer in Y of this flag. Then P_J is the product of the preimages of the parabolic subgroups P_D and P_F of $\text{Isom}(D)$ and $\text{Isom}(F)$ which are the stabilizers of the isotropic 1-spaces D_1 and F_1 , respectively. So $R_u(P_D)$ acts trivially on D_1 , D_1^\perp/D_1 , and D/D_1^\perp , and similarly for $R_u(P_F)$. On the other hand, $R_u(P_Y)$ is precisely the subgroup of Y which acts trivially on V_{j+1}/V_j for $0 \leq j \leq 4$. One then sees that $R_u(P_J) \subset R_u(P_Y)$. Now $(P_Y/R_u(P_Y))' \cong (\text{Isom}(V_3/V_2))' \circ \text{SL}(V_2/V_1)$ and $(P_J/R_u(P_J))' \cong (\text{Isom}(D_1^\perp/D_1))' \circ (\text{Isom}(F_1^\perp/F_1))'$.

Assume for the moment that $f > 2$; then V_3/V_2 is an orthogonal space of dimension at least 4. We note that the subspace $((D_1 \otimes F_3) + V_2)/V_2$ is a non-zero singular subspace in V_3/V_2 left invariant by $(\text{Isom}(D_1^\perp/D_1))' \circ (\text{Isom}(F_1^\perp/F_1))'$ and so the projection of this latter group in $\text{Isom}(V_3/V_2)$ is contained in a proper parabolic. It then follows from an application of [24] to the irreducible KY -module V and the subgroup J that the portion of the Dynkin diagram for Y corresponding to the subgroup $\text{Isom}(V_3/V_2)$ has zero labels, when representing λ by a labeled diagram. But this contradicts our assumption on λ .

Consider now the case where $f = 2$, so $\dim W = 2n + 2$ and $d = n + 1$. Since we are assuming $Y = D_{n+1}$ with $n \geq 4$, we have $d \geq 5$. Now $\text{Isom}(D_2) \circ \text{Isom}(F_2)$ stabilizes the image of $D_1 \otimes F_1^\perp$ in V_2/V_1 and so lies in a proper parabolic subgroup of $\text{Isom}(V_2/V_1)$. Note that $\dim(V_2/V_1) = n - 1$, and arguing as above we deduce that the nodes corresponding to $\text{SL}(V_2/V_1)$ in the Dynkin diagram are labelled zero. So now we have $\lambda = a_1\lambda_1 + a_n\lambda_n$, with $a_n \neq 0$. Moreover, the image of X in $\text{Isom}(W)'$ lies in $\text{Isom}(D)' \circ \text{Isom}(F)' = \text{Isom}(D)' \circ \text{Sp}_2$, which then implies that D is even-dimensional and the latter group is $\text{Sp}(D) \circ \text{Sp}_2$, and acts irreducibly on V . The factor $\text{Sp}(D)$ lies in the derived subgroup of an A_n -type Levi factor L of Y , indeed is the naturally embedded Sp_{n+1} subgroup of A_n . Moreover, $\text{Sp}(D)$ acts homogeneously on V . Now λ and $\lambda - \alpha_n$, or $\lambda - \alpha_{n-1} - \alpha_n - \alpha_{n+1}$, depending on whether the root system of L contains α_{n+1} or α_n , afford the highest weights of irreducible summands of $V|_L$. It is then straightforward to see that the restrictions of these weights to the subgroup $\text{Sp}(D)$ provide non-isomorphic composition factors. This provides the final contradiction in case $f = 2$ and completes the proof of the result. \square

For the proof of Proposition 1.5, we continue with our consideration of the case where X acts irreducibly and, by the previous result, tensor-indecomposably on the natural KY -module W . In particular, we may now assume that X is a simple, proper, closed subgroup of Y . The hypotheses of Proposition 1.5 then imply that X is of type B_m or of type F_4 . Moreover, we have the full set of hypotheses on the embedding of a parabolic subgroup of X in a parabolic subgroup of Y , in particular, with respect to the restriction of the highest weight λ to a Levi factor. We will in

fact show that the irreducibility of $W|_X$ and the hypotheses of Proposition 1.5 are incompatible.

Taking X a simple counterexample of minimal rank, we see that we may assume P_X is a maximal parabolic subgroup of X . In each case, we will require some detailed information about the commutator series of an irreducible KX -module with respect to a fixed maximal parabolic subgroup. We start by considering:

Case 1: X of type B_m , $m \geq 4$. Fix a maximal torus T_X of X . Let $\Pi(X) = \{\beta_1, \dots, \beta_m\}$ be a base of the root system of X , B_X the corresponding Borel subgroup containing T_X , and $\{\omega_1, \dots, \omega_m\}$ a set of fundamental dominant weights chosen with respect to the fixed base $\Pi(X)$. Set s_i to be the reflection corresponding to the root β_i for $1 \leq i \leq m$. For a torus S of X , write $X(S)$ for the group of rational characters of S . Let P_X be a maximal parabolic subgroup of X corresponding to the subset $\{\beta_i : 2 \leq i \leq m\}$, and containing the opposite Borel subgroup B_X^- . Let $Q_X = R_u(P_X)$ and $P_X = L_X Q_X$ for a Levi factor L_X of P_X . Let $\omega = \sum_{i=1}^m d_i \omega_i$ be a p -restricted dominant weight in $X(T_X)$, set $M = L_X(\omega)$ and assume X preserves a non-degenerate quadratic form on M , and let us denote this by \mathcal{Q} and the associated bilinear form by $(\ , \) : M \times M \rightarrow K$.

For a unipotent group J and a KJ -module N , we recall the standard notation $[N, J]$ for the subspace spanned by the set of vectors $n - xn$, where $x \in J$ and $n \in N$. We introduce an additional notation: set $[N, J^0] = N$ and set $[N, J^{i+1}] = [[N, J^i], J]$ for $i \geq 0$, so $[N, J^1] = [N, J]$.

Definition 4.4. Let $M = L_X(\omega)$ be as above.

- (i) We will say a T_X -weight ν in M has *level* i if $\nu = \omega - i\beta_1 - \sum_{j=2}^m c_j \beta_j$ for some integers $i, c_j \geq 0$.
- (ii) Let $e(\omega)$ denote the maximum level of a weight. (When ω is fixed, we will simply write e .)

Lemma 4.5. *Let $M = L_X(\omega)$. Then the following assertions hold:*

- (i) For $i \geq 0$, $[M, Q_X^i] = \sum M_\nu$, where the sum ranges over all T_X -weights ν of level at least i .
- (ii) For $i \geq 0$, the L'_X -module $[M, Q_X^i]/[M, Q_X^{i+1}]$ is isomorphic to the module

$$\sum_{\nu \in X(T_X) \text{ of level } i} M_\nu.$$

- (iii) The maximum level of weights in M is $e = 2(\sum_{i=1}^{m-1} d_i) + d_m$. If a weight ν is of level j , then $-\nu$ is of level $e - j$.
- (iv) Let S be a subtorus of T_X . For weights $\eta, \chi \in X(S)$, with $\eta \neq -\chi$, we have $V_\eta \subseteq V_\chi^\perp$. Moreover, if $\eta \neq 0$, then $\mathcal{Q}(u) = 0$ for all $u \in V_\eta$.
- (v) The subspace $[M, Q_X^i]$ is totally singular for $i \geq (e + 1)/2$.
- (vi) If e is even, then the subspace

$$\sum_{\nu \in X(T_X) \text{ of level } e/2} M_\nu$$

is non-degenerate.

- (vii) For $i \geq (e + 1)/2$, we have $[M, Q_X^i]^\perp = [M, Q_X^{e-i+1}]$.
- (viii) For all $i \geq (e/2)$, the quadratic form on M induces an L'_X -invariant quadratic form on $[M, Q_X^i]/[M, Q_X^{i+1}]$. If the form is non-degenerate on this

quotient space, then there exists ν of level i such that $-\nu$ is also of level i . In particular, in this case, d_m is even.

Proof. We start by proving (i), arguing by induction on i , the case $i = 0$ being trivial by definition. Let $i \geq 1$, and fix an ordering on $\Phi^+(X)$ such that any root having a non-zero coefficient of β_1 when expressed in terms of the simple roots is smaller than the others. By [15, Section 27], we clearly have that $[M, Q_X^i] \subseteq \sum M_\nu$, where the sum ranges over all T_X -weights ν of level at least i , and so it remains to show that $M_\nu \subseteq [M, Q_X^i]$ for all such T_X -weights ν . It is enough to show the latter for T_X -weights of level exactly i . Let then ν be a T_X -weight of level i in M , and let $\gamma_1 \leq \dots \leq \gamma_k$ be roots in $\Phi^+(X)$ such that $\sum_{r=1}^k \gamma_r = \lambda - \nu$. We show that $f_{\gamma_1} \cdots f_{\gamma_k} v^+ \in [M, Q_X^i]$, where v^+ is a maximal vector in M . (This will be enough by Lemma 2.1.) Set $w = f_{\gamma_2} \cdots f_{\gamma_k} v^+$. Then w has level $i - 1$, as γ_1 involves β_1 thanks to our choice of ordering on Φ^+ , and hence $w \in [M, Q_X^{i-1}]$ by induction. Also, we have

$$x_{-\gamma_1}(1)w - w \in f_{\gamma_1} \cdots f_{\gamma_k} v^+ + \sum_{r=1}^{\infty} M_{\nu - r\gamma_1},$$

and hence $x_{-\gamma_1}(1)w - w$ has non-zero coefficient of $f_{\gamma_1} \cdots f_{\gamma_k} v^+$, and all other terms in the sum are weight vectors of weights different from ν . One then deduces that $[M, Q_X^i]$ must contain $f_{\gamma_1} \cdots f_{\gamma_k} v^+$ as desired, showing that (i) holds.

The statement of (ii) now follows by induction on i .

For (iii), set w_0 to be the longest word of the Weyl group of X . Writing the ω_i in terms of the simple roots β_j , we see that

$$w_0(\omega) = -\omega = \omega - 2(d_1 + d_2 + \dots + d_{m-1} + \frac{1}{2}d_m)\beta_1 - \sum_{i=2}^m c_i \beta_i$$

for some non-negative integers c_i , giving the result. For the final statement, suppose that $\nu = \omega - j\beta_1 - \sum_{i=2}^m c_i \beta_i$, then $-\nu = \omega - 2\omega + j\beta_1 + \sum_{i=2}^m c_i \beta_i = \omega - (e - j)\beta_1 - \sum_{i=2}^m b_i \beta_i$ for some non-negative integers b_i .

The statement of (iv) is standard, and (v) and (vi) follow from (iv).

For (vii), note that by (i) and (iv), we have $[M, Q_X^{e-i+1}] \subseteq [M, Q_X^i]^\perp$. Then the result follows from a dimension argument using (ii) and (iii).

For (viii), we note that for $i \geq (e/2)$, $[M, Q_X^{i+1}]$ is totally singular and so the given quadratic form induces an L'_X -invariant form on the quotient. Now if $p \neq 2$, the second statement follows directly from (iv). If $p = 2$ and $[M, Q_X^i]/[M, Q_X^{i+1}]$ is odd-dimensional, so that the bilinear form has a 1-dimensional non-singular radical, then (iv) implies that the radical is generated by a vector of weight 0. So the weight $\nu = 0$ satisfies the given condition. For the claim about the parity of d_m , we recall that for ν of level j , $-\nu$ is of level $e - j$. So we deduce that e is even, and so d_m as well. \square

In case e is even, we will consider a certain T_X -weight at level $(e/2)$, namely the weight

$$\mu := \omega - (e/2)\beta_1 - (e/2 - d_1)\beta_2 - \dots - (d_{m-1} + (d_m/2))\beta_{m-1} - (d_m/2)\beta_m.$$

Note that

$$(4) \quad \mu = -(e/2)\omega_1 + \sum_{i=2}^{m-1} d_{i-1}\omega_i + (2d_{m-1} + d_m)\omega_m.$$

Lemma 4.6. *Assume d_m is even. For each $0 \leq j \leq d_m$, there exists a unique weight $\nu_j \in X(T_X)$, of level $j + \sum_{i=1}^{m-1} d_i$, such that $\nu_j|_{T_X \cap L'_X} = \mu|_{T_X \cap L'_X}$. In addition, each ν_j is of multiplicity 1 and if $\nu \in X(T_X)$ with $\nu|_{T_X \cap L'_X} = \mu|_{T_X \cap L'_X}$, then $\nu = \nu_j$ for some j . In particular, there exists an odd number of weights ν such that $\nu|_{T_X \cap L'_X} = \mu|_{T_X \cap L'_X}$.*

Proof. Let $\nu \in X(T_X)$ such that $\nu|_{T_X \cap L'_X} = \mu|_{T_X \cap L'_X}$. Then $\nu = \omega - \sum_{i=1}^m c_i \beta_i$, and the c_i must satisfy the equations

$$d_{j+1} + c_j - 2c_{j+1} + c_{j+2} = d_j \text{ for } 1 \leq j \leq m-2, \text{ and}$$

$$d_m + 2c_{m-1} - 2c_m = 2d_{m-1} + d_m.$$

Solving these equations leads to the relations $c_j = c_m + \sum_{i=j}^{m-1} d_i$ for $1 \leq j \leq m-1$. Now note as well that for ν as given, $s_{m-1}s_{m-2} \cdots s_1 \nu = \omega - c_m \beta_m$. Hence ν has multiplicity at most 1 and has multiplicity 1 if and only if $c_m \leq d_m$. So for all $0 \leq j \leq d_m$, there exists a weight ν_j of multiplicity 1 and of level $j + \sum_{i=1}^{m-1} d_i$ whose restriction to $T_X \cap L'_X$ is equal to $\mu|_{T_X \cap L'_X}$. \square

Proposition 4.7. *Assume the hypotheses of Proposition 1.5 with $X = B_m$. Then X acts reducibly on the natural KY -module.*

Proof. Suppose the contrary, that is, let W be the natural KY -module and suppose that $W|_X$ is irreducible. Then by Proposition 4.3, $W|_X$ is tensor indecomposable. If $W|_X$ is not p -restricted, then we replace $W|_X$ by a p -restricted representation ρ of a simply-connected cover \tilde{X} of X , and consider $W|_{\rho(\tilde{X})}$. For the purposes of the argument, we may replace X by $\rho(\tilde{X})$. Let $W = L_X(\omega)$ for $\omega = \sum_{i=1}^m d_i \omega_i$, a p -restricted dominant weight.

As in the hypotheses of Proposition 1.5, let P_Y be a parabolic subgroup of Y containing P_X with $Q_X \subset R_u(P_Y) = Q_Y$. Fix a maximal torus T_Y of P_Y with $T_X \subset T_Y$. We choose a base $\Pi(Y) = \{\alpha_1, \dots, \alpha_{n+1}\}$ of the root system of Y so that P_Y contains the opposite Borel $B_{\bar{Y}}$ subgroup with respect to this base.

Now let $\{0\} = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{t-1}$ be a flag of totally singular subspaces, such that P_Y is the full stabilizer in Y of this flag; so P_Y is the stabilizer of the flag

$$\{0\} = W_0 \subsetneq \cdots \subsetneq W_{t-1} \subseteq W_{t-1}^\perp \subsetneq W_{t-2}^\perp \cdots \subsetneq W_0^\perp = W.$$

Setting $W_{t-1+j} = W_{t-j}^\perp$ for $0 \leq j \leq t$, we have the flag

$$\{0\} = W_0 \subsetneq \cdots \subsetneq W_{t-1} \subseteq W_t \subsetneq W_{t+1} \cdots \subsetneq W_{2t-1} = W,$$

of which P_Y is the full stabilizer. By our hypotheses on P_Y , we have that W_t/W_{t-1} is an orthogonal space of dimension at least 8 (so $W_{t-1} \neq W_t$) and we have $L_t = (\text{Isom}(W_t/W_{t-1}))'$, a group of type D_m for $m \geq 4$. Note that $W_j = [W, Q_Y^{2t-1-j}]$ for all $j \geq 0$.

Recall that by hypothesis, L'_X is of type B_{m-1} . If $m > 4$, L'_X is the stabilizer in L_t of a non-singular 1-space of W_t/W_{t-1} . Note that if $m = 4$, so L'_X is of type B_3 and $(\text{Isom}(W_t/W_{t-1}))' = D_4$, it may be that L'_X acts irreducibly on W_t/W_{t-1} as a spin module. We will take care to consider the latter possibility in what follows.

We need to see how the two flags which are stabilized by P_X are related, i.e., the flag of subspaces $\{[W, Q_X^j]\}_{j \geq 0}$ and the flag of subspaces $\{W_i\}_{i \geq 0}$. We first show that e (as defined in Lemma 4.5(iii)) is even. Indeed, suppose the contrary. Then P_X lies in the stabilizer in Y of $[W, Q_X^{\frac{e+1}{2}}]$, a totally singular subspace of

dimension $\frac{1}{2} \dim W$ (using Lemma 4.5(vii)). Set $P = QL$ to be this stabilizer, a maximal parabolic subgroup of Y with Levi factor L of type A_n and $Q = R_u(P)$. Consider now the action of L'_X and L' on the commutator quotient $V/[V, Q]$. By [24], each group acts irreducibly. But given that the restriction of λ to $T_Y \cap L'$ has at least two non-zero labels, we see that there are no compatible configurations coming from Table 1 of [23]. (Here we are invoking Hypothesis 1.4(ii).) Hence e is even as claimed. In particular, the weight μ of (4) exists and Lemma 4.6 applies.

Choose s minimal such that $[W, Q_X^{(e/2)}] \subseteq W_s$. Then $s \geq t$ because $[W, Q_X^{(e/2)}]$ is not totally singular by Lemma 4.5(vi). If $s = t$, then $[W, Q_X^{(e/2)}] \subseteq W_t$ and $[W, Q_X^{(e/2)}]$ is not contained in W_{t-1} . If $W_\mu \not\subseteq W_{t-1}$, then $\mu|_{L'_X \cap T_X}$ occurs as a weight in the quotient module $(W_t/W_{t-1})|_{L'_X}$. But the only dominant $T_X \cap L'_X$ weights in this module are either the zero weight or the first fundamental dominant weight, or $m = 4$ and W_t/W_{t-1} is the spin module for L'_X . But comparing each of these weights and $\mu|_{T_X \cap L'_X}$ (see (4)), we see that either $\omega = 0$, or $\omega = \omega_1$, or $m = 4$ and μ affords the spin module. The first two possibilities are not consistent with our assumption that X acts irreducibly on the natural module for Y . In the last case, when $m = 4$ and μ affords the spin module for L'_X , we deduce that $2d_{m-1} + d_m = 1$, contradicting the fact that d_m is even. Hence we have $W_\mu \subseteq W_{t-1}$. Choose r minimal such that $W_\mu \subseteq W_r$. Then μ occurs in the quotient module W_r/W_{r-1} , which is dual to W_{r-1}^\perp/W_r^\perp , and so the weight $\mu|_{T_X \cap L'_X}$ occurs here as well. But by Lemma 4.6, there are an odd number of weights, each of which is of multiplicity 1, whose restriction to $T_X \cap L'_X$ is μ and so there exists $\nu \in X(T_X)$ such that $\nu|_{T_X \cap L'_X} = \mu$ and $W_\nu \subseteq W_t$ with $W_\nu \not\subseteq W_{t-1}$. Now we conclude as above by comparing $\nu|_{T_X \cap L'_X}$ and the weights occurring in $(W_t/W_{t-1})|_{L'_X}$.

Now consider the case that $s > t$. Then $[W, Q_X^{(e/2)+j}] \subseteq W_{s-j}$ is an L_X -invariant subspace. In particular, taking $j = s - t$ we have a totally singular L'_X -invariant subspace of W_t . But since the image of L'_X in $\text{Isom}(W_t/W_{t-1})$ does not lie in a proper parabolic we see that in fact $[W, Q_X^{(e/2)+s-t}] \subseteq W_{t-1}$. Then we have $W_{t-1}^\perp \subseteq [W, Q_X^{(e/2)+s-t}]^\perp$ by Lemma 4.5(vii), which implies that $W_t \subseteq [W, Q_X^{(e/2)-s+t+1}]$.

Now consider the inclusions

$$[W, Q_X^{(e/2)+s-t}] \subseteq W_{t-1} \subseteq W_t \subseteq [W, Q_X^{(e/2)-s+t+1}].$$

We again ask where our weight μ occurs. If $W_\mu \subseteq W_t$ and $W_\mu \not\subseteq W_{t-1}$, we can argue as before. If $W_\mu \cap W_t = \{0\}$ (recall $\dim W_\mu = 1$), then μ occurs as a weight in the quotient module $[W, Q_X^{(e/2)-s+t+1}]/W_t$ and therefore must also occur in the dual, which is the quotient module $W_{t-1}/[W, Q_X^{(e/2)+s-t}]$ and now we argue as above to see that there exists ν whose restriction to $T_X \cap L'_X$ is μ and occurring in the quotient W_t/W_{t-1} , leading to a contradiction as above. This completes the proof of the proposition. \square

Case 2: $X = F_4$.

Proposition 4.8. *Assume the hypotheses of Proposition 1.5 with $X = F_4$. Then the natural KY -module W is a reducible KX -module.*

Proof. Fix a maximal torus T_X of X . Let $\Pi(X) = \{\beta_1, \dots, \beta_4\}$ be a base of the root system of X , B_X the corresponding Borel subgroup containing T_X , and $\{\omega_1, \dots, \omega_4\}$ a set of fundamental dominant weights chosen with respect to the fixed base $\Pi(X)$.

Suppose that $W|_X$ is irreducible. Then by Proposition 4.3, we have $W|_X$ tensor-indecomposable and so without loss of generality we will assume the highest weight ω is p -restricted, that is, $\omega = \sum_{i=1}^4 d_i \omega_i$, with $d_i < p$ for all i . We will treat the case $p = 2$ separately below.

Assume for now that $p > 2$.

Let P_X be the maximal parabolic subgroup of X corresponding to the simple root β_1 , containing the opposite Borel subgroup. We extend the notion of “level” to this setting and rely on [23, 2.3]. Note that the maximum level of weights with respect to this parabolic is $e := 2(2d_1 + 3d_2 + 2d_3 + d_4)$. Consider the flag of totally singular subspaces

$$\{0\} = [W, Q_X^{e+1}] \subseteq [W, Q_X^e] \subseteq \cdots \subseteq [W, Q_X^{(e/2)+1}].$$

Then P_X lies in the stabilizer of this flag, which is the stabilizer of the full flag

$$\begin{aligned} \{0\} &= [W, Q_X^{e+1}] \subseteq [W, Q_X^e] \subseteq \cdots \subseteq [W, Q_X^{(e/2)+1}] \\ &\subseteq [W, Q_X^{(e/2)}] \subseteq \cdots \subseteq [W, Q_X] \subseteq W. \end{aligned}$$

The quotient $[W, Q_X^{(e/2)}]/[W, Q_X^{(e/2)+1}]$ is a non-degenerate subspace and we claim that it is non-trivial of dimension at least 6, which then implies that the Levi factor L'_X of type C_3 projects non-trivially into $\text{Isom}([W, Q_X^{(e/2)}]/[W, Q_X^{(e/2)+1}])$, a group of type D_ℓ with $\ell \geq 4$ (as there is no non-trivial morphism from C_3 into D_ℓ for $\ell \leq 3$). Indeed, consider the weight

$$\begin{aligned} s_1 s_2 s_3 s_4 s_2 s_3 (\omega - d_1 \beta_1 - d_2 \beta_2) &= \omega - \frac{e}{2} \beta_1 - \frac{e}{2} \beta_2 - (2d_1 + 4d_2 + 3d_3 + d_4) \beta_3 \\ &\quad - (2d_2 + d_3 + d_4) \beta_4. \end{aligned}$$

This weight is of level $(e/2)$, occurs in W by [22], and affords the dominant $T_X \cap L'_X$ weight $(2d_2 + d_3)\omega_2 + d_4\omega_3 + (2d_1 + d_3)\omega_4$, where here we write ω_i for $\omega_i|_{T_X \cap L'_X}$. Hence $\dim([W, Q_X^{(e/2)}]/[W, Q_X^{(e/2)+1}]) \geq 6$ as claimed.

Now consider the action of X on the irreducible module $V = L_Y(\lambda)$, where λ satisfies the congruence relations. Since C_3 projects non-trivially into the D_ℓ factor of P_Y , we must have that C_3 acts irreducibly on the D_ℓ module with the highest weight as given. But there is no such example in [23, Table 1], giving the desired contradiction. (Here we invoked Hypothesis 1.4(iii).)

Finally, consider the case where $p = 2$. Since $W|_X$ is tensor-indecomposable, [23, 1.6] implies that $\omega = d_1\omega_1 + d_2\omega_2$ or $d_3\omega_3 + d_4\omega_4$. Moreover, using the graph automorphism of F_4 , it suffices to consider the second situation, in which case [23, 2.3] still applies. Using [19], we see that the weight lattice of $W|_X$ is as in characteristic 0, in particular, we can exhibit as above in each case a non-zero dominant weight of level $e/2$ and the above argument goes through. If $\omega = \omega_3$, when $e = 4$, we take $\omega - 2\beta_1 - 2\beta_2 - 3\beta_3 - \beta_4$; if $\omega = \omega_4$ and $e = 2$, take $\omega - \beta_1 - \beta_2 - \beta_3 - \beta_4$; if $\omega = \omega_3 + \omega_4$ and $e = 6$, take $\omega - 3\beta_1 - 3\beta_2 - 4\beta_3 - 2\beta_4$. \square

Proof of Proposition 1.5. We can now complete the proof of Proposition 1.5. Let X, Y , and V be as in the statement of the proposition and assume Hypothesis 1.4 for all embeddings $H \subset G$ such that $\text{rank } G < \text{rank } Y$. By Propositions 4.1, 4.2, 4.3, 4.7, and 4.8, we have that either Proposition 1.5(a) holds, or X acts reducibly on W and $X \subset B_n \subset Y$. If $X = B_n$, Proposition 1.5(b) holds; so assume $X \subsetneq B_n$. The restriction of V to B_n affords a p -restricted irreducible KB_n -module with highest

weight having at least three non-zero coefficients when expressed in terms of the fundamental dominant weights for B_n . Moreover, X has a Levi subgroup of type B_m , by hypothesis. We now refer to [23, Table 1], invoking Hypothesis 1.4(i), to see that there are no such irreducible triples $(X, B_n, \lambda|_{T_{B_n}})$, where T_{B_n} is a maximal torus of B_n lying in the maximal torus T_Y . This completes the proof of the result. \square

We conclude this section by showing that the one fixed-rank example $B_3 \subset C_4$ acting irreducibly on $V = L_{C_4}(\lambda_3)$ has no impact on the inductive proof of the main result of [23].

Proof of Proposition 1.7. Let X, Y , and V be as in the statement of the proposition and assume Hypothesis 1.4 for all embeddings $H \subset G$ such that $\text{rank}(G) < \text{rank}(Y)$. Suppose first that X is not simple. Then $V|_X = E \otimes F$ and X lies in a subgroup $\text{Sp}(F) \circ \text{Sp}(E) \subset Y$. As discussed in the proof of Proposition 4.3, X preserves a quadratic form on V as well, so X lies in a maximal rank subgroup D_n of Y . Moreover, $V|_{D_n}$ has highest weight with non-zero coefficients of the two fundamental dominant weights corresponding to the two half-spin modules (see [23, Table 1, MR₄]). Proposition 1.5 then gives the result.

Now assume X is simple and let P_X, P_Y , and V be as in the statement of the result. Taking X a minimal counterexample, that is, acting irreducibly on V , we may assume $\text{rank}(X) = 4$, so $X = B_4$ or F_4 . Let $\omega_i, 1 \leq i \leq 4$, be a set of fundamental dominant weights with respect to a fixed maximal torus and Borel subgroup of X . Then hypothesis (v) of the proposition implies that $\lambda|_X = \sum_{i=1}^4 a_i \omega_i$ with $a_2 a_4 \neq 0$, respectively $a_1 a_3 \neq 0$, if X is of type B_4 , respectively F_4 . Then [23, 1.6] implies that $V|_X$ preserves a tensor product decomposition on V , and we argue as in the previous paragraph to obtain a contradiction. \square

5. PROOF OF THEOREM 1.8

Let Y be a simply connected, simple algebraic group of type $E_n, n = 6, 7, 8$, defined over K . We start by considering a maximal, closed, connected semisimple subgroup X of Y , satisfying the hypotheses of Theorem 1.8, namely, X has a proper parabolic subgroup whose Levi factor is of type B_m for some $m \geq 3$. Referring to [18, Theorem 1], we see that we must consider the following pairs (X, Y) :

- (F_4, E_6) ,
- $(A_1 F_4, E_7)$,
- $(G_2 F_4, E_8)$.

We start by dealing with the two latter possibilities.

Proposition 5.1. *The maximal subgroup $A_1 F_4 \subset E_7$ acts reducibly on all non-trivial irreducible KE_7 -modules. The maximal subgroup $G_2 F_4 \subset E_8$ acts reducibly on all non-trivial irreducible KE_8 -modules.*

Proof. We assume the contrary. Let $X = M_1 F_4 \subset Y$ be a maximal subgroup of $Y = E_n$ for $M_1 = A_1$, respectively G_2 , and $n = 7$, respectively 8. The factor F_4 is embedded in the usual way as a maximal subgroup of an E_6 Levi factor of Y . Assume that X acts irreducibly on $V = L_Y(\lambda)$, where λ is a non-zero p -restricted dominant weight. Adopting the usual notation as in previous results, we set $\lambda = \sum_{i=1}^n a_i \lambda_i$, where $\{\lambda_1, \dots, \lambda_n\}$ are the fundamental dominant weights with respect to a fixed choice of base for the root system of Y . Since X acts irreducibly

on V , F_4 acts homogeneously. Let $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ be a set of fundamental dominant weights for F_4 again with respect to a fixed choice of base of the root system which is compatible with the given choice for Y . By [24] we have one F_4 composition factor of V with highest weight $\omega = a_2\omega_1 + a_4\omega_2 + (a_3 + a_5)\omega_3 + (a_1 + a_6)\omega_4$. The homogeneity of $V|_{F_4}$ implies that $a_7 = 0$, and $a_8 = 0$ if $n = 8$, as otherwise $\lambda - \alpha_7$ (resp. $\lambda - \alpha_7 - \alpha_8$) would afford the highest weight of a composition factor different from $L_X(\omega)$. Now let i_0 be maximal such that $a_{i_0} \neq 0$ (such exists since $\lambda \neq 0$). If $i_0 = 2$, then $\lambda - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7$ affords an F_4 composition factor of V not isomorphic to that already given. Hence $i_0 \neq 2$. If $i_0 = 1$, then $\lambda - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7$ has the same property. For all other cases, we take $\lambda - \sum_{i=i_0}^7 \alpha_i$, which again affords an F_4 -composition factor with highest weight different from ω . This is the final contradiction. \square

The remainder of this section is devoted to the proof of the following result, first proven by Testerman [26, Theorem 5.0 (i)] under the assumption that [23, Table 1, IV1, IV1'] formed a complete family of irreducible triples for the usual embedding $B_n \subset D_{n+1}$.

Theorem 5.2. *Let Y be a simply connected, simple algebraic group of type E_6 over K and let X be the subgroup of type F_4 , embedded in Y in the usual way. Also let $V = L_Y(\lambda)$ be a non-trivial, irreducible KY -module having p -restricted highest weight $\lambda \in X^+(T_Y)$. Assume Hypothesis 1.4 for all embeddings $H \subset G$ with $\text{rank}(G) < \text{rank}(Y)$. Then X acts irreducibly on V if and only if one of the following holds, where we give λ up to graph automorphisms:*

- (i) $\lambda = (p - 3)\lambda_1$, with $p > 3$.
- (ii) $\lambda = \lambda_1 + (p - 2)\lambda_3$, with $p > 2$.

The proof relies on some detailed knowledge of the structure of certain Weyl modules for E_6 and F_4 . Section 5.1 below provides some results based on the Jantzen p -sum formula. In Section 5.2, we apply the methods from Section 5.1 to various Weyl modules in order to obtain insight on their structure, as well as information on certain weight multiplicities in the corresponding irreducible quotient. These results shall then prove useful in Section 5.3, in which we conclude by showing that Theorem 5.2 holds. Finally, at the end of the section, we see how these results lead to a proof of Theorem 1.8.

5.1. Understanding Weyl modules. Let G be a semisimple algebraic group over K , and let B be a Borel subgroup of G containing a fixed maximal torus T . Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ denote the corresponding base for the root system $\Phi = \Phi^+ \sqcup \Phi^-$ of G , and let $\lambda_1, \dots, \lambda_n$ denote the corresponding fundamental dominant weights for T . Let ρ denote the half-sum of all positive roots in Φ , or equivalently, the sum of all fundamental dominant weights. Also for $\lambda, \mu \in X^+(T)$ such that $\mu \prec \lambda$, define

$$d(\lambda, \mu) = 2(\lambda + \rho, \lambda - \mu) - (\lambda - \mu, \lambda - \mu).$$

The following corollary to the strong linkage principle [1] gives a necessary condition for μ to afford the highest weight of a KG -composition factor of $V_G(\lambda)$, in the case where $p > 2$ and G is not of type G_2 . We refer the reader to [23, 6.2] for a proof.

Proposition 5.3. *Assume $p > 2$ and let G be a simple algebraic group of type different from G_2 . Also let λ and μ be as above, and assume the inner product on*

$\mathbb{Z}\Phi$ is normalized so that long roots have length 1. If μ affords the highest weight of a composition factor of $V_G(\lambda)$, then

$$2\mathrm{d}(\lambda, \mu) \in p\mathbb{Z}.$$

Let $\{e^\mu\}_{\mu \in X(T)}$ denote the standard basis of the group ring $\mathbb{Z}[X(T)]$ over \mathbb{Z} . The Weyl group \mathscr{W} of G acts on $\mathbb{Z}[X(T)]$ by $we^\mu = e^{w\mu}$, $w \in \mathscr{W}$, $\mu \in X(T)$, and we write $\mathbb{Z}[X(T)]^\mathscr{W}$ to denote the set of fixed points. The *formal character* of a given KG -module V is the linear polynomial $\mathrm{ch} V \in \mathbb{Z}[X(T)]^\mathscr{W}$ defined by

$$\mathrm{ch} V = \sum_{\mu \in X(T)} m_V(\mu) e^\mu.$$

Also, for $\lambda \in X^+(T)$, we write

$$\chi(\lambda) = \mathrm{ch} V_G(\lambda) = \mathrm{ch} H^0(\lambda)$$

(see [16, II, 2.13], for instance). The *Jantzen p -sum formula* [16, II, Proposition 8.19] yields the existence of a filtration $V_G(\lambda) = V^0 \supseteq V^1 \supseteq \dots \supseteq V^k \supseteq V^{k+1} = 0$ of $V_G(\lambda)$, such that $V^0/V^1 \cong L_G(\lambda)$ and such that the sum $\sum_{r=1}^k \mathrm{ch} V^r \in \mathbb{Z}[X(T)]^\mathscr{W}$, denoted $\nu^c(T_\lambda)$, satisfies certain properties (loc. cit.). Throughout this section, we call such a filtration the *Jantzen filtration* of $V_G(\lambda)$. Moreover, since $\{\chi(\lambda)\}_{\lambda \in X^+(T)}$ forms a \mathbb{Z} -basis of $\mathbb{Z}[X(T)]^\mathscr{W}$ (see [16, II, Remark 5.8]), there exists $(a_\nu)_{\nu \in X^+(T)} \subset \mathbb{Z}$ such that

$$(5) \quad \nu^c(T_\lambda) = \sum_{\nu \in X^+(T)} a_\nu \chi(\nu).$$

Consider a T -weight $\mu \in X(T)$ with $\mu \prec \lambda$. Following [7, Section 3.2], we now define a “truncated” version of $\nu^c(T_\lambda)$, which shall prove useful in computations, by setting

$$(6) \quad \nu_\mu^c(T_\lambda) = \sum_{\substack{\nu \in X^+(T) \\ \mu \preceq \nu \prec \lambda}} a_\nu \chi_\mu(\nu),$$

where the a_ν ($\nu \in X^+(T)$) are as in (5), and where for every $\nu \in X^+(T)$, we have $\chi_\mu(\nu) = \sum_{\eta \in X^+(T), \mu \preceq \eta \preceq \nu} [V_G(\nu), L_G(\eta)] \mathrm{ch} L_G(\eta)$. Finally, the latter decomposition yields

$$(7) \quad \nu_\mu^c(T_\lambda) = \sum_{\substack{\xi \in X^+(T) \\ \mu \preceq \xi \prec \lambda}} b_\xi \mathrm{ch} L_G(\xi)$$

for some $b_\xi \in \mathbb{Z}$. The following proposition shows how (7) can be used in order to determine the possible composition factors of $V_G(\lambda)$, together with an upper bound for their multiplicity. We refer the reader to [7, Proposition 3.6] for a proof.

Proposition 5.4. *Let $\lambda \in X^+(T)$ and consider a T -weight $\mu \prec \lambda$. Also let $\xi \in X^+(T)$ be a dominant weight such that $\mu \preceq \xi \prec \lambda$. Then ξ affords the highest weight of a composition factor of $V_G(\lambda)$ if and only if $b_\xi \neq 0$ in (7). Also $[V_G(\lambda), L_G(\xi)] \leq b_\xi$.*

For $\nu \in X^+(T)$, we call the coefficient a_ν in (6) the *contribution* of ν to $\nu_\mu^c(T_\lambda)$, and we say that ν *contributes* to $\nu_\mu^c(T_\lambda)$ if its contribution is non-zero. Now applying Proposition 5.4 for specific weights μ, ξ with $\mu \preceq \xi \prec \lambda$ requires the knowledge of the contribution of ν to $\nu_\mu^c(T_\lambda)$ for each dominant T -weight $\mu \preceq \nu \prec \lambda$. In certain cases, knowing whether or not a given T -weight contributes to $\nu_\mu^c(T_\lambda)$ can be easily

determined, as the following result shows. We refer the reader to [7, Lemma 3.7] for a proof.

Lemma 5.5. *Let λ , μ , and ν be as above, with ν maximal, with respect to the partial order \preceq , such that ν contributes to $\nu_\mu^c(T_\lambda)$. Then ν affords the highest weight of a composition factor of $V_G(\lambda)$.*

Fix $\nu \in X^+(T)$, and recall from [3, Planches I–IV] the description of the simple roots and fundamental dominant weights for T in terms of a basis $\{\varepsilon_1, \dots, \varepsilon_{d_\Phi}\}$ for a Euclidean space E of dimension d_Φ . For $\alpha \in \Phi^+$ and $r \in \mathbb{Z}_{\geq 0}$ such that $1 < r < \langle \lambda + \rho, \alpha \rangle$, we write $\lambda + \rho - r\alpha = a_1\varepsilon_1 + \dots + a_{d_\Phi}\varepsilon_{d_\Phi}$, as well as $\nu + \rho = b_1\varepsilon_1 + \dots + b_{d_\Phi}\varepsilon_{d_\Phi}$, following the ideas of [21]. Also following [7, Section 3.2], we set $A_{\alpha,r} = (a_j)_{j=1}^{d_\Phi} \in \mathbb{Q}^{d_\Phi}$ and $B_\nu = (b_j)_{j=1}^{d_\Phi} \in \mathbb{Q}^{d_\Phi}$. The action of the Weyl group \mathscr{W} of G on the basis $\{\varepsilon_1, \dots, \varepsilon_{d_\Phi}\}$, described in [3, Planches I–IV], extends to an action of \mathscr{W} on \mathbb{Q}^{d_Φ} in the obvious way. (We write $w \cdot A$ for $w \in \mathscr{W}$, $A \in \mathbb{Q}^{d_\Phi}$.) In addition, define the *support* of an element $z \in \mathbb{Z}\Phi$ to be the subset $\text{supp}(z)$ of Π consisting of those simple roots α such that $c_\alpha \neq 0$ in the decomposition $z = \sum_{\alpha \in \Pi} c_\alpha \alpha$. Finally, for $w \in \mathscr{W}$, we write $\det(w)$ for the determinant of w as an invertible linear transformation of $X(T)_\mathbb{R} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The following result is our main tool for determining the contribution of ν to $\nu_\mu^c(T_\lambda)$ for each weight $\nu \in X^+(T)$ with $\mu \preceq \nu \prec \lambda$. We refer the reader to [7, Theorem 3.8] for a proof.

Theorem 5.6. *Let $\lambda \in X^+(T)$, and consider a weight $\mu \in X(T)$ with $\mu \prec \lambda$. Let $\nu \in X^+(T)$ be such that $\mu \preceq \nu \prec \lambda$. Write $I_\nu = \{(\alpha, r) \in \Phi^+ \times [2, \langle \lambda + \rho, \alpha \rangle] : \text{supp}(\alpha) = \text{supp}(\lambda - \nu), B_\nu \in \mathscr{W} \cdot A_{\alpha,r}\}$, and for each pair $(\alpha, r) \in I_\nu$, choose $w_{\alpha,r} \in \mathscr{W}$ such that $w_{\alpha,r} \cdot A_{\alpha,r} = B_\nu$. Then the contribution of ν to $\nu_\mu^c(T_\lambda)$ is given by*

$$- \sum_{(\alpha,r) \in I_\nu} \nu_p(r) \det(w_{\alpha,r}),$$

where for ℓ a prime number and $m \in \mathbb{Z}$, we write $\nu_\ell(m)$ for the ℓ -adic valuation of m .

5.2. Computing certain weight multiplicities. We now use the results introduced in the previous section to investigate the structure of certain Weyl modules, as well as to compute various weight multiplicities in certain irreducible modules. For $\lambda \in X^+(T)$ and $c_1, \dots, c_n \in \mathbb{Z}_{\geq 0}$, we use the notation $\lambda - c_1c_2 \cdots c_n$ for the weight $\lambda - \sum_{r=1}^n c_r \alpha_r$.

Lemma 5.7. *Let G be of type B_3 over K , and let $a, b \in \mathbb{Z}_{>0}$ be such that $a + b + 1 = p$. Also let $\lambda = a\lambda_1 + b\lambda_2 + a\lambda_3 \in X^+(T)$ and $\mu = \lambda - 111 \in \Lambda^+(\lambda)$. Then μ affords the highest weight of a composition factor of $V_G(\lambda)$ if and only if $(a, b) = (p - 2, 1)$.*

Proof. We first deal with the situation where $p = 3$, $a = b = 1$, so that we have $p = a + 2$ in this case. Here the only dominant T -weights $\nu \in X^+(T)$ such that $\mu \preceq \nu \prec \lambda$ are $\lambda - 110$, $\lambda - 011$, and μ itself. Now an application of Proposition 5.3 shows that $\lambda - 011$ cannot afford the highest weight of a composition factor of $V_G(\lambda)$. On the other hand $[V_G(\lambda), L_G(\lambda - 110)] = 1$ by Proposition 2.2, but $m_{L_G(\lambda - 110)}(\mu) = 0$ (as $\lambda - 110 = 3\lambda_3$) and hence

$$m_{L_G(\lambda)}(\mu) = m_{V_G(\lambda)}(\mu) - [V_G(\lambda), L_G(\mu)].$$

Now $m_{L_G(\lambda)}(\mu) = 3 < 4 = m_{V_G(\lambda)}(\mu)$ by [20], from which we deduce the existence of a second composition factor of $V_G(\lambda)$. Now since no weight ν with $\mu \prec \nu \prec \lambda$

affords the highest weight of a composition factor of $V_G(\lambda)$, the desired result follows in this case.

We thus assume $p > 3$ in the remainder of the proof and consider the Jantzen filtration $V_G(\lambda) = V^0 \supseteq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ of $V_G(\lambda)$. We start by computing all contributions to $\nu_\mu^c(T_\lambda)$. Here the dominant T -weights $\nu \in X^+(T)$ such that $\mu \preccurlyeq \nu \prec \lambda$ are $\lambda - 100$ (if $a > 1$), $\lambda - 010$ (if $b > 1$), $\lambda - 001$ (if $a > 1$), $\lambda - 101$ (if $a > 1$), $\lambda - 110$, $\lambda - 011$, and μ itself. Now $\lambda - 100$, $\lambda - 010$, $\lambda - 001$, and $\lambda - 101$ all have multiplicity 1 in $V_G(\lambda)$ and hence none of them can afford the highest weight of a composition factor of $V_G(\lambda)$ by [22]. Recursively applying Lemma 5.5 then shows that those same weights cannot contribute to $\nu_\mu^c(T_\lambda)$. In addition, an application of Proposition 5.3 yields $[V_G(\lambda), L_G(\lambda - 011)] = 0$, and hence $\lambda - 011$ does not contribute to $\nu_\mu^c(T_\lambda)$ by Lemma 5.5 again.

We now compute the contribution of $\nu_1 = \lambda - 110$ to $\nu_\mu^c(T_\lambda)$ by first determining all pairs $(\alpha, r) \in I_{\nu_1}$ as in Theorem 5.6. A straightforward computation yields

$$B_{\nu_1} = \frac{1}{2}(3a + 2b + 3, a + 2b + 3, a + 3),$$

and since $\lambda - \nu_1$ has support $\{\alpha_1, \alpha_2\}$, we get that $\alpha = \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$ by definition of I_{ν_1} . For $r \in \mathbb{Z}$, we have

$$A_{\varepsilon_1 - \varepsilon_3, r} = \frac{1}{2}(3a + 2b + 5 - 2r, a + 2b + 3, a + 1 + 2r).$$

Recall from Bourbaki [3, Planche II] that \mathscr{W} acts by all permutations and sign changes of the ε_i , hence the orbit of any 3-tuple $C = (c_1, c_2, c_3) \in \mathbb{Q}^3$ is given by

$$\mathscr{W} \cdot C = \{\pm c_{\sigma(1)}, \pm c_{\sigma(2)}, \pm c_{\sigma(3)} : \sigma \in \text{Sym}_3\}.$$

We thus deduce that $A_{\varepsilon_1 - \varepsilon_3, r}$ and B_{ν_1} are conjugate under the action of \mathscr{W} if and only if $\{3a + 2b + 5 - 2r, a + 1 + 2r\} = \{|3a + 2b + 3|, |a + 3|\}$. By studying each possibility separately, one easily shows that $A_{\varepsilon_1 - \varepsilon_3, r}$ is \mathscr{W} -conjugate to B_{ν_1} if and only if $r = p$, in which case the chosen element

$$w_{\varepsilon_1 - \varepsilon_3, r} = s_{\varepsilon_1 - \varepsilon_3}$$

satisfies $w_{\varepsilon_1 - \varepsilon_3, r} \cdot A_{\alpha, r} = B_{\nu_1}$ as desired. Consequently an application of Theorem 5.6 shows that ν_1 contributes to $\nu_\mu^c(T_\lambda)$ by $\nu_p(p) = 1$. We next determine the contribution of μ to $\nu_\mu^c(T_\lambda)$. Here again, a computation yields

$$B_\mu = \frac{1}{2}(3a + 2b + 3, a + 2b + 3, a + 1),$$

and since $\lambda - \mu$ has support Π , we get that $\alpha \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3\}$. Dealing with each possibility separately, one then concludes that μ contributes to $\nu_\mu^c(T_\lambda)$ by $\nu_p(3a + 2b + 4)$ by Theorem 5.6, so

$$\nu_\mu^c(T_\lambda) = \chi_\mu(\lambda - 110) + \nu_p(3a + 2b + 4)\chi_\mu(\mu).$$

Now $\chi_\mu(\lambda - 110) = \text{ch } L_G(\lambda - 110) + \delta_{p, a+2} \text{ch } L_G(\mu)$ and $\chi_\mu(\mu) = \text{ch } L_G(\mu)$. An application of Proposition 5.4 then completes the proof. \square

Lemma 5.8. *Let G be of type B_n ($n \geq 2$) over K , and let $a, b \in \mathbb{Z}_{>0}$ be such that $b > 1$, and $p \mid (a + b + n - 1)$. Also let $\lambda = a\lambda_1 + b\lambda_n \in X^+(T_G)$ and $\mu = \lambda - 1 \cdots 12 \in \Lambda^+(\lambda)$. Then μ affords the highest weight of a composition factor of $V_G(\lambda)$. Furthermore if $n = 2$, then*

$$[V_G(\lambda), L_G(\mu)] = 1.$$

Proof. Consider the Jantzen filtration $V_G(\lambda) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ of $V_G(\lambda)$. We proceed as in the proof of Lemma 5.7, starting by computing all contributions to $\nu_\mu^c(T_\lambda)$ in the case where $n = 2$. Here the only dominant T -weights $\nu \in X^+(T)$ satisfying $\mu \preceq \nu \prec \lambda$ are $\lambda - 10$ (if $a > 1$), $\lambda - 01$, $\lambda - 02$ (if $b > 3$), $\lambda - 11$, and μ itself. Now each of $\lambda - 10$, $\lambda - 01$, and $\lambda - 02$ has multiplicity at most 1 in $V_G(\lambda)$ and hence

$$[V_G(\lambda), L_G(\lambda - 10)] = [V_G(\lambda), L_G(\lambda - 01)] = [V_G(\lambda), L_G(\lambda - 02)] = 0$$

by [22]. Applying Lemma 5.5 then shows that none of these can contribute to $\nu_\mu^c(T_\lambda)$. Also $[V_G(\lambda), L_G(\lambda - 11)] = 0$ by Proposition 5.3, hence again showing that $\lambda - 11$ does not contribute to $\nu_\mu^c(T_\lambda)$ by Lemma 5.5. Finally, we compute the contribution of μ , by first determining all pairs $(\alpha, r) \in I_\mu$ as in Theorem 5.6. A straightforward computation yields

$$B_\mu = \frac{1}{2}(2a + b + 1, b - 1),$$

and since $\lambda - \mu$ has support Π , we get that $\alpha \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_2\}$ by definition of I_μ . We claim that for every $r \in \mathbb{Z}_{\geq 0}$, the 2-tuple $A_{\varepsilon_1, r}$ is not conjugate to B_μ under the action of the Weyl group \mathscr{W} of G . Indeed, as in the proof of Lemma 5.7, recall from [3, Planche II] that \mathscr{W} acts by all permutations and sign changes of the ε_i . Now $A_{\varepsilon_1, r} = \frac{1}{2}(2a + b + 3 - 2r, b + 1)$, and since none of the two coordinates of B_μ is equal to $\pm \frac{1}{2}(b + 1)$, we immediately deduce that $A_{\varepsilon_1, r} \cap (\mathscr{W} \cdot B_\mu) = \emptyset$ for every $r \in \mathbb{Z}$. Now considering the positive root $\varepsilon_1 + \varepsilon_2$, we have

$$A_{\varepsilon_1 + \varepsilon_2, r} = \frac{1}{2}(2a + b + 3 - 2r, b + 1 - 2r),$$

from which we deduce that $A_{\varepsilon_1 + \varepsilon_2, r}$ and B_μ are conjugate under the action of \mathscr{W} if and only if $\{2a + b + 3 - 2r, b + 1 - 2r\} = \{|2a + b + 1|, |b - 1|\}$. By studying each possibility separately, one shows that $A_{\varepsilon_1 + \varepsilon_2, r}$ is \mathscr{W} -conjugate to B_μ if and only if $r = p$, in which case the element

$$w_{\varepsilon_1 + \varepsilon_2, r} = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1} s_{\varepsilon_2}$$

satisfies $w_{\varepsilon_1 + \varepsilon_2, r} \cdot (A_{\varepsilon_1 + \varepsilon_2, r}) = B_\mu$ as desired. Therefore

$$\nu_\mu^c(T_\lambda) = \chi_\mu(\mu)$$

by Theorem 5.6, and since $\chi_\mu(\mu) = \text{ch } L_G(\mu)$, we get that

$$\nu_\mu^c(T_\lambda) = \text{ch } L_G(\mu).$$

An application of Proposition 5.4 then completes the proof.

Next we assume $n > 2$, in which case the dominant T -weights $\nu \in X^+(T)$ satisfying $\mu \preceq \nu \prec \lambda$ are $\lambda - \alpha_1$ (if $a > 1$), $\lambda - \alpha_n$, $\lambda - 2\alpha_n$ (if $b > 3$), $\lambda - \alpha_{n-1} - 2\alpha_n$ (if $b > 2$), $\lambda - (\alpha_1 + \dots + \alpha_n)$, and μ itself. Now arguing as in the $n = 2$ case, one shows that neither of $\lambda - \alpha_1$, $\lambda - \alpha_n$, $\lambda - 2\alpha_n$, $\lambda - \alpha_1 - \alpha_n$, $\lambda - \alpha_{n-1} - 2\alpha_n$, nor $\lambda - (\alpha_1 + \dots + \alpha_n)$ can contribute to $\nu_\mu^c(T_\lambda)$. Finally, we compute the contribution of μ , by first determining all pairs $(\alpha, r) \in I_\mu$ as in Theorem 5.6. Again, a computation yields

$$B_\mu = \frac{1}{2}(2a + b + 2n - 3, b + 2n - 3, b + 2n - 5, \dots, b + 3, b - 1),$$

and since $\lambda - \mu$ has support Π , we get that $\alpha \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_l : 2 \leq l \leq n\}$. Now for $r \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \Phi^+$, we have $A_{\alpha, r} = \frac{1}{2}(2a + b + 2n - 1, b + 2n - 3, b + 2n - 5, \dots, b + 1) - r\alpha$. Now by Bourbaki [3, Planche II] again, \mathscr{W} acts by all permutations and sign changes

of the ε_i , from which one immediately deduces that $A_{\alpha,r}$ cannot be conjugate to B_μ if $\alpha \neq \varepsilon_1 + \varepsilon_n$. Also, a straightforward computation yields

$$A_{\varepsilon_1+\varepsilon_n,r} = \frac{1}{2}(2a+b+2n-1-2r, b+2n-3, b+2n-5, \dots, b+3, b+1-2r),$$

from which we deduce that $A_{\varepsilon_1+\varepsilon_n,r}$ and B_μ are conjugate under the action of \mathscr{W} if and only if $\{2a+b+3+2n-1-2r, b+1-2r\} = \{|2a+b+2n-3|, |b-1|\}$. By studying each possibility separately, one shows that $A_{\varepsilon_1+\varepsilon_n,r}$ is \mathscr{W} -conjugate to B_μ if and only if $r = a+b+n-1$, in which case the element

$$w_{\varepsilon_1+\varepsilon_n,r} = s_{\varepsilon_1-\varepsilon_n} s_{\varepsilon_1} s_{\varepsilon_n}$$

satisfies $w_{\varepsilon_1+\varepsilon_n,r} \cdot (A_{\varepsilon_1+\varepsilon_n,r}) = B_\mu$ as desired. Therefore $\nu_\mu^c(T_\lambda) = \nu_p(a+b+n-1)\chi_\mu(\mu)$ by Theorem 5.6, and since $\chi_\mu(\mu) = \text{ch } L_G(\mu)$, we get that $\nu_\mu^c(T_\lambda) = \nu_p(a+b+n-1) \text{ch } L_G(\mu)$. An application of Proposition 5.4 then completes the proof. \square

We will also require some knowledge about the structure of certain Weyl modules for a simple group of type A_n over K . However, due to the complexity of the description of fundamental dominant weights in terms of an orthonormal basis of a Euclidean space for such G , it is more convenient to work in a group of type B_{n+1} , and then deduce the desired result for A_n .

Lemma 5.9. *Let G be of type B_{n+1} ($n \geq 3$) over K , and let $0 < a, b, c < p$ be positive integers. Also let $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_n \in X^+(T)$, $\nu_1 = \lambda - \alpha_1 - \alpha_2$, $\nu_2 = \lambda - (\alpha_2 + \dots + \alpha_n)$, and $\mu = \lambda - (\alpha_1 + \dots + \alpha_n)$. Then μ affords the highest weight of a composition factor of $V_G(\lambda)$ if and only if $p \mid (a+b+c+n-1)$. Also if $n = 3$, then we have*

$$\chi_\mu(\lambda) = \text{ch } L_G(\lambda) + \delta_{p,a+b+1} \text{ch } L_G(\nu_1) + \delta_{p,b+c+1} \text{ch } L_G(\nu_2) + \delta_{p,a+b+c+2} \text{ch } L_G(\mu).$$

Proof. As in the proofs of Lemmas 5.7 and 5.8, fix $V_G(\lambda) = V^0 \supseteq V^1 \supseteq \dots \supseteq V^k \supseteq 0$, the Jantzen filtration for $V_G(\lambda)$. Arguing as in the aforementioned proofs, one first checks that

$$\nu_\mu^c(T_\lambda) = \nu_p(a+b+1)\chi_\mu(\nu_1) + \nu_p(b+c+1)\chi_\mu(\nu_2) + \nu_p(a+b+c+n-1)\chi_\mu(\mu).$$

Now observe that $\chi_\mu(\nu_1) = \text{ch } L_G(\nu_1) + \nu_p(c+n-2) \text{ch } L_G(\mu)$, $\chi_\mu(\nu_2) = \text{ch } L_G(\nu_2)$, and $\chi_\mu(\mu) = \text{ch } L_G(\mu)$. Therefore an application of Proposition 5.4 shows that μ affords the highest weight of a composition factor if and only if $p \mid (a+b+c+n-1)$ as desired. Finally, if $n = 3$, then we get that $\nu_p(a+b+1) = \delta_{p,a+b+1}$, $\nu_p(b+c+1) = \delta_{p,b+c+1}$, and $\nu_p(a+b+c+2) = \delta_{p,a+b+c+2}$. Also $\chi_\mu(\nu_1) = \text{ch } L_G(\nu_1)$ in this case, so that an application of Proposition 5.4 completes the proof. \square

Proposition 5.10. *Let G , λ , and μ be as in Table 1, with $a, b, c \in \mathbb{Z}_{>0}$ such that $a+b+1 = p$. Then the multiplicity of μ in $V_G(\lambda)$, respectively $L_G(\lambda)$, is given in the fourth, respectively fifth, column of the table.*

Proof. Let G , λ , and μ be as in the first row of Table 1, and start by observing that an application of [6, Proposition 3] yields $m_{V_G(\lambda)}(\mu) = 4$. Also, the weights $\lambda - 100$, $\lambda - 010$, $\lambda - 001$, and $\lambda - 101$ all have multiplicity one in $V_G(\lambda)$ and hence none of them can afford the highest weight of a composition factor of $V_G(\lambda)$ by [22]. Setting

TABLE 1. Some weight multiplicities in various irreducibles. Here $a, b, c \in \mathbb{Z}_{>0}$ are such that $a + b + 1 = p$.

G	λ	μ	$m_{V_G(\lambda)}(\mu)$	$m_{L_G(\lambda)}(\mu)$
A_3	$a\lambda_1 + b\lambda_2 + c\lambda_3$	$\lambda - 111$	4	$3 - \delta_{a,c}$
A_4	$a\lambda_1 + b\lambda_2 + a\lambda_3$	$\lambda - 1121$	$6 - 2\delta_{a,1}$	$3 - \delta_{a,1}$
B_2	$a\lambda_1 + b\lambda_2$	$\lambda - 12$	$3 - \delta_{b,1}$	2

$\mu_1 = \lambda - 110$ and $\mu_2 = \lambda - 011$, we have $[V_G(\lambda), L_G(\mu_1)] = 1$ by Proposition 2.2, so that

$$(8) \quad \begin{aligned} & m_{L_G(\lambda)}(\mu) \\ &= m_{V_G(\lambda)}(\mu) - m_{L_G(\mu_1)}(\mu) - [V_G(\lambda), L_G(\mu_2)] m_{L_G(\mu_2)}(\mu) - [V_G(\lambda), L_G(\mu)]. \end{aligned}$$

Now if $a = c$, then μ_2 also affords the highest weight of exactly one composition factor of $V_G(\lambda)$ by Proposition 2.2, and one easily sees that $m_{L_G(\mu_1)}(\mu) = m_{L_G(\mu_2)}(\mu) = 1$. Also, applying Proposition 5.3 yields $[V_G(\lambda), L_G(\mu)] = 0$, from which the desired result follows. If on the other hand $a \neq c$, then $[V_G(\lambda), L_G(\mu_2)] = 0$, and we must consider separately the cases $c = p - 1$ or $c \neq p - 1$. In the former case, we get that $[V_G(\lambda), L_G(\mu)] = 1$ by Lemma 5.9, while $m_{L_G(\mu_1)}(\mu) = 0$, as $\mu_1 = (a - 1)\lambda_1 + (b - 1)\lambda_2 + p\lambda_3$ in this case. The assertion then immediately follows from (8). Finally, if $c \neq p - 1$, then μ does not afford the highest weight of a composition factor of $V_G(\lambda)$ by Lemma 5.9, while $m_{L_G(\mu_1)}(\mu) = 1$, as μ_1 is p -restricted in this case. Again (8) then yields the desired result.

Next consider G, λ, μ as in the second row of Table 1 and observe that if $a = 1$, then the result is an immediate consequence of the previous case, as μ is conjugate to $\lambda - 1110$ in this situation. So assume $a > 1$ in the remainder of the argument. A recursive application of [6, Proposition 1, Theorem 2] yields $m_{V_G(\lambda)}(\mu) = 6$. Also neither $\lambda - 1110, \lambda - 0120, \lambda - 0121, \lambda - 1120$, nor μ can afford the highest weight of a composition factor of $V_G(\lambda)$ by Proposition 5.3. The remaining potential highest weights of composition factors, except for $\mu_1 = \lambda - 1100$ and $\mu_2 = \lambda - 0110$, all have multiplicity 1 in $V_G(\lambda)$, so that

$$m_{L_G(\lambda)}(\mu) = m_{V_G(\lambda)}(\mu) - [V_G(\lambda), L_G(\mu_1)] m_{L_G(\mu_1)}(\mu) - [V_G(\lambda), L_G(\mu_2)] m_{L_G(\mu_2)}(\mu),$$

by [22]. Finally, notice that $m_{L_G(\mu_1)}(\mu) = 1$, while $m_{L_G(\mu_2)}(\mu) = 2$ by Proposition 2.2, hence the result in this situation as well.

Finally, let G, λ, μ be as in the third row of the table. If $b = 1$, then μ is conjugate to $\lambda - 11$, whose multiplicity in $V_G(\lambda)$ equals 2. An application of Proposition 5.3 then yields the desired result in this case. If on the other hand $b > 1$, then μ is dominant and $m_{V_G(\lambda)}(\mu) = 3$ by [6, Proposition 1, Theorem 2]. By [22], $[V_G(\lambda), L_G(\lambda - 10)] = [V_G(\lambda), L_G(\lambda - 01)] = 0$, while applying Proposition 5.3 shows that $\lambda - 11$ does not afford the highest weight of a composition factor of $V_G(\lambda)$. Therefore $m_{L_G(\lambda)}(\mu) = m_{V_G(\lambda)}(\mu) - [V_G(\lambda), L_G(\mu)]$ and hence the assertion follows from Lemma 5.8. \square

5.3. Proof of Theorem 5.2 and conclusion. Let Y be a simple algebraic group of type E_6 over K , and throughout this section, assume Hypothesis 1.4 for all embeddings $H \subset G$ with $\text{rank } G < \text{rank } Y$. Fix T_Y a maximal torus of Y and let B_Y be a Borel subgroup of Y containing T_Y . Let $\Pi(Y) = \{\alpha_1, \dots, \alpha_6\}$ denote the corresponding base for the root system $\Phi(Y) = \Phi^+(Y) \sqcup \Phi^-(Y)$ of Y , where $\Phi^+(Y)$ and $\Phi^-(Y)$ denote the set of positive and negative roots, respectively. Also write $\lambda_1, \dots, \lambda_6$ for the associated fundamental dominant weights. Consider the subgroup X of type F_4 defined by

$$X = \langle x_{\pm\beta_j}(c) : 1 \leq j \leq 4, c \in K \rangle,$$

where $x_{\pm\beta_1}(c) = x_{\pm\alpha_2}(c)$, $x_{\pm\beta_2}(c) = x_{\pm\alpha_4}(c)$, $x_{\pm\beta_3}(c) = x_{\pm\alpha_3}(c)x_{\pm\alpha_5}(c)$, and $x_{\pm\beta_4}(c) = x_{\pm\alpha_1}(c)x_{\pm\alpha_6}(c)$ for $c \in K$. Let T_X be the maximal torus of X defined by the $x_{\pm\beta_i}(c)$ (see [5, Section 4.3]), and let B_X be the Borel subgroup of X generated by the $x_{\pm\beta_i}(c)$ and T_X , so that $\Pi(X) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a corresponding base for the root system $\Phi(X)$ of X . (Here again $\Phi(X) = \Phi^+(X) \sqcup \Phi^-(X)$ in the obvious way.) We first recall an argument from [26].

Lemma 5.11. *Let X and Y be as above and $\lambda \in X^+(T_Y)$ a non-trivial p -restricted weight such that $L_Y(\lambda)|_X$ is irreducible. Then up to graph automorphism, one of the following holds:*

- (i) $\lambda = a\lambda_2 + a\lambda_3 + b\lambda_4$ for some $a, b \in \mathbb{Z}_{>0}$, with $a + b = p - 1$.
- (ii) $\lambda = (p - 3)\lambda_1$ for $p > 3$.
- (iii) $\lambda = \lambda_1 + (p - 2)\lambda_3$ for $p > 2$.

Proof. First we note that Theorem 1.2 applied to the Levi embedding $B_3 \subset D_4$ implies that there exists $i \in \{1, 3, 4, 5, 6\}$ such that $\langle \lambda, \alpha_i \rangle \neq 0$. Let L'_X be the C_3 Levi factor of X , which lies in L'_Y , an A_5 -type Levi factor of Y . Then by Hypothesis 1.4(iv), Theorem 1.2, and up to taking graph automorphisms, $\lambda \in \{a\lambda_1, (p-1)\lambda_3 + x\lambda_2 \ (p \neq 2), c\lambda_1 + b\lambda_3 + x\lambda_2 \ (cb \neq 0, b+c = p-1), b\lambda_3 + a\lambda_4 + x\lambda_2 \ (ab \neq 0, a+b = p-1)\}$. We now refer the reader to the proof of [26, (5.4)], which establishes that if $\lambda = a\lambda_1$, then $p > 3$, $a = p-3$, and $\lambda \neq (p-1)\lambda_3 + x\lambda_2$. The same proof shows that if $\lambda = c\lambda_1 + b\lambda_3 + x\lambda_2$, with $cb \neq 0$ and $b+c = p-1$, then $p > 2$, $x = 0$, and $b = p-2$. Finally, in the last case, where $\lambda = b\lambda_3 + a\lambda_4 + x\lambda_2$, with $ab \neq 0$ and $a+b = p-1$, the same proof shows that $x \neq 0$, and then Theorem 1.2 shows that (i) holds. \square

We are now ready to give a proof of Theorem 5.2, and to conclude by giving a proof of Theorem 1.8.

Proof of Theorem 5.2. Let $\lambda \in X^+(T_Y)$ be a non-zero, p -restricted, dominant weight for T_Y , and write $V = L_Y(\lambda)$ for the corresponding irreducible KY -module. First observe that if λ and p are as in (i) or (ii) of Theorem 1.8, then X acts irreducibly on $V = L_Y(\lambda)$ by [26, 5.6, 5.7]. By Lemma 5.11, in order to complete the proof, it only remains to show that $V|_X$ is reducible in the situation where $\lambda = a\lambda_2 + a\lambda_3 + b\lambda_4$ for some $a, b \in \mathbb{Z}_{>0}$ such that $a + b + 1 = p$, which we shall assume holds in the remainder of the proof. Here the restriction of λ to T_X is given by $\omega = a\omega_1 + b\omega_2 + a\omega_3 \in X^+(T_X)$. Consider the dominant T_X -weight

$$\omega' = \omega - \beta_1 - \beta_2 - 2\beta_3 - \beta_4 \in X^+(T_X).$$

Then $\lambda - \alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$, $\lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$, and $\lambda - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$ are all T_Y -weights of V restricting to ω' . Also the latter two are \mathscr{H}_Y -conjugate to

$\lambda - \alpha_2 - \alpha_3 - \alpha_4$, yielding

$$m_{V|_X}(\omega') \geq m_V(\lambda - \alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4) + 2m_V(\lambda - \alpha_2 - \alpha_3 - \alpha_4).$$

By applying Proposition 5.10 to the Levi subgroups of Y corresponding to the simple roots $\alpha_1, \dots, \alpha_4$, respectively $\alpha_2, \alpha_3, \alpha_4$, we get

$$(9) \quad m_{V|_X}(\omega') \geq 7 - \delta_{a,1}.$$

Next observe that recursively applying [6, Theorem 2] yields $m_{V_X(\omega)}(\omega') = 8 - 2\delta_{a,1}$. Also, the T_X -weight $\mu = \omega - \beta_1 - \beta_2$ affords the highest weight of a composition factor of $V_X(\omega)$ by Proposition 2.2, namely

$$L_X(\mu) = (a-1)\omega_1 + (b-1)\omega_2 + (a+2)\omega_3.$$

We now compute an upper bound for $m_{L_X(\omega)}(\omega')$ by dealing with each of the following four possibilities separately:

- (i) If $a = 1$ and $p \neq 3$, then μ is p -restricted and so $m_{L_X(\mu)}(\omega') = 1$. Therefore we have $m_{L_X(\omega)}(\omega') \leq m_{V_X(\omega)}(\omega') - 1 = 5 < 6 = m_{V|_X}(\omega)$.
- (ii) If $a > 1$ and $a \neq p - 2$, then again μ is p -restricted and so $m_{L_X(\mu)}(\omega') = 1$. Also by Lemma 5.8, the weight $\nu = \omega - \beta_2 - 2\beta_3$ affords the highest weight of a composition factor of $V_X(\omega)$, namely

$$L_X(\nu) = (a+1)\omega_1 + b\omega_2 + (a-2)\omega_3 + 2\omega_4.$$

As $m_{L_X(\nu)}(\omega') = 1$, we have $m_{L_X(\omega)}(\omega') \leq 6 < 7 = m_{V|_X}(\omega')$.

- (iii) If $a = 1$ and $p = 3$, then μ is not p -restricted and $m_{L_X(\omega)}(\omega') = 0$. However, applying Lemma 5.7 shows that $\omega - \beta_1 - \beta_2 - \beta_3 = \omega_2 + \omega_3 + \omega_4$ affords the highest weight of a composition factor of $V_X(\omega)$. Consequently, as $m_{L_X(\omega_2+\omega_3+\omega_4)}(\omega') = 1$, we get that $m_{L_X(\omega)}(\omega') \leq 5 < 6 = m_{V|_X}(\omega')$.
- (iv) If $a > 1$ and $a = p - 2$, then again ω' is not a weight of $L_X(\mu)$. However, an application of Lemma 5.7 (resp. Lemma 5.8) to the Levi subgroup of X corresponding to the simple roots $\beta_1, \beta_2, \beta_3$ (resp. β_2, β_3) yields the existence of a composition factor of $V_X(\omega)$ having highest weight $\omega - \beta_1 - \beta_2 - \beta_3$ (resp. $\omega - \beta_2 - 2\beta_3$). Therefore, as μ is a weight of each of these irreducible, we get that $m_{L_X(\omega)}(\omega') \leq 6 < 7 = m_{V|_X}(\omega')$.

Consequently, in each case, we get that $m_{V|_X}(\omega') > m_{L_X(\omega)}(\omega')$, showing the existence of a second composition factor of V for X . In particular $V|_X$ is reducible. This completes the proof of Theorem 5.2. \square

Proof of Theorem 1.8. Let $X \subset Y = E_n$ be as in the statement of Theorem 1.8. We embed X in a maximal proper closed connected subgroup of Y and deduce by the remarks at the beginning of Section 5, Proposition 5.1, and Theorem 5.2, that X lies in the F_4 subgroup of $Y = E_6$. So we must determine the irreducible configurations $X \subset F_4$ acting irreducibly on the irreducible F_4 -module with highest weight $\lambda \in \{(p-3)\lambda_1 \ (p > 3), \lambda_1 + (p-2)\lambda_3 \ (p > 2)\}$; this is covered by [26, Main Theorem]. Then [26, Main Theorem (iii), (iv)] completes the proof of Theorem 1.8. \square

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