# Construction of a 4 GHz resonant cavity operating in $TE_{011}$ mode: Simulation and Experiments

Tarek Vennemann
Supervised by Prof. H.M. Rønnow, P.Babkevich and M. Chung
Laboratory for Quantum Magnetism
Ecole Polytechnique Fédérale de Lausanne (EPFL)

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**Abstract**

During this project we studied cavities as an alternative method to stripline resonators, to create an oscillating magnetic field. We were able to simulate different cavity designs and reproduce theoretical results as a numerical simulation and experimentally. Thus getting familiar with cavities and studying how their design influences the $TE_{011}$ mode in which we lay our interest. It was shown that with cavities one could reach a higher quality factor than with stripline resonators. We also found out that in order to make a cavity small enough to be useful it is indispensable to use dielectric materials and studied the behavior of a few possible cavity designs.

## 1 Introduction

The main goal of the report was to design a resonant cavity that resonates at a frequency of 4GHz. The aim is to create a magnetic field varying at this frequency that can be used to excite samples and study their response. Thus main interest lies within the magnetic field, that should be strong and uniform along a certain direction at the sample position. This can be done using stripline resonators, but to improve the quality factor of the resonance and thus of the measurements, an alternative method using cavities was considered.

The projects aim was not to create a fully functional cavity that can be used for measurements but to gather information on cavities, simulate their response on COMSOL, and reproduce the resonances experimentally. Finally come up with a more complex design for a cavity that could be used for measurements and study it through simulations. We only considered cylindrical cavities during this project as they have a nice symmetry and are easy to build.

## 2 Theory and Simulations

### 2.1 Methods

Maxwell equations can be solved exactly in the case of a cylindrical cavity made out of a perfect conductor. The solutions give two types of modes: transverse magnetic (TM) where the magnetic field is perpendicular to the axis of the cylinder, and transverse electric (TE) having this time the electric field perpendicular to the axis of the cylinder. The solution to Maxwell's equations gives following modes and resonant frequencies for a cylinder of length L and diameter D :

$$f_{nml} = c\left[\left(\frac{x_{nm}}{D\pi}\right)^2 + \left(\frac{l}{2L}\right)^2\right]^{1/2}, \quad n, l > 0 \quad m > 1 \tag{1}$$

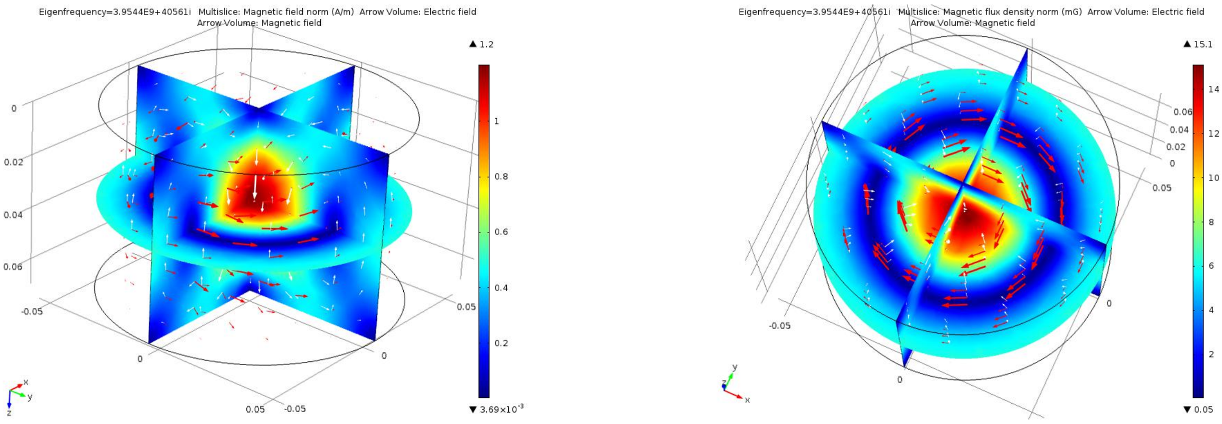
where $x_{nm}$ is the mth non zero root of the nth Bessel function for TM modes or the mth non zero root of the derivative of the nth Bessel function for TE modes (taken from Zangwill [2]). This also implies that

$f_{TE_{011}} = f_{TM_{111}}$. The field distribution of the $TE_{011}$, $TE_{111}$ and the $TM_{010}$ mode are shown in Figure 1. The $TE_{011}$ mode is particularly interesting as it has the magnetic field along the z-axis and presents the strongest amplitude compared to other modes, as the field is concentrated in one small region. This is why this mode is most suitable for our application and was mainly considered during this project. Moreover this mode has a high quality factor compared to other modes. We also considered several times the $TE_{111}$ mode as it has the lowest frequency of the TE modes and was thus easier to measure for the size and shape of our test cavity. Moreover the $TE_{111}$ mode requires a much smaller cavity than the $TE_{011}$ mode making it thus interesting if we have space problems. However it has a lower quality factor. For further information on choice of modes and what to pay attention to while building a cavity a good reference book is *EPR: Techniques and Applications by R. Alger* [1].

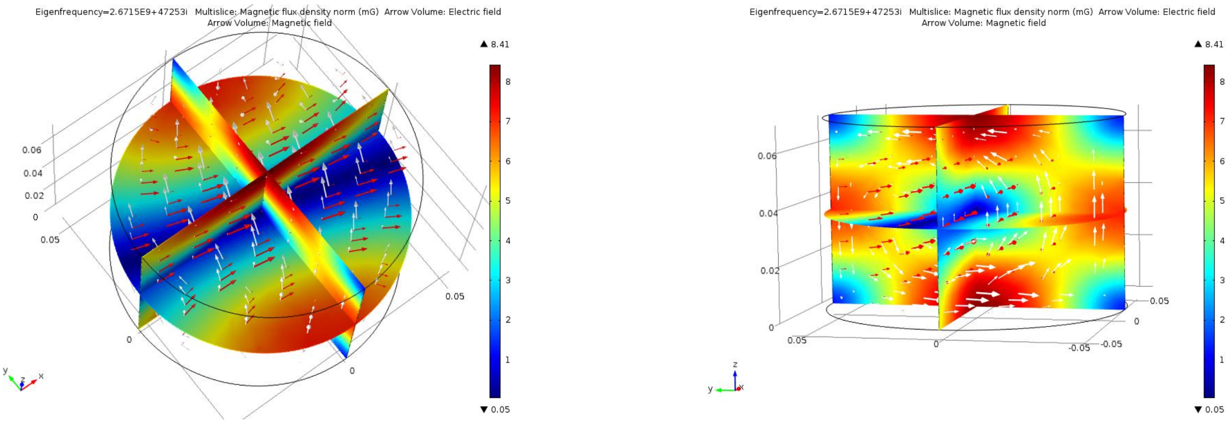
Figure 2 shows the current densities in the cavity walls for the three modes. When drilling a hole in the cavity wall, to introduce a sample or antenna for example, the currents should not be greatly disturbed. The symmetry of the $TE_{011}$ mode for example makes it less sensitive to circular centered holes in the top or bottom plates or to cuts perpendicular to its axis. As an example the resonance should not be affected if the top and bottom plates do not make a good uniform contact with the cylindrical part.

We simulated the eigenmodes of a cylindrical cavity on COMSOL using the RF-module and solving for eigenfrequencies. The cylinder was defined as having impedance boundary conditions, which we set to a conductivity of $59.6 \cdot 10^6$S/m (value of copper at room temperature). For the interior of the cylinder we could define the relative permittivity constant $\epsilon_r$ changing thus from vacuum to a dielectric filling, the relative magnetic permeability $\mu_r$ was always set to unity. The simulated eigenfrequencies correspond almost exactly to the theoretical calculations and can thus be considered reliable.

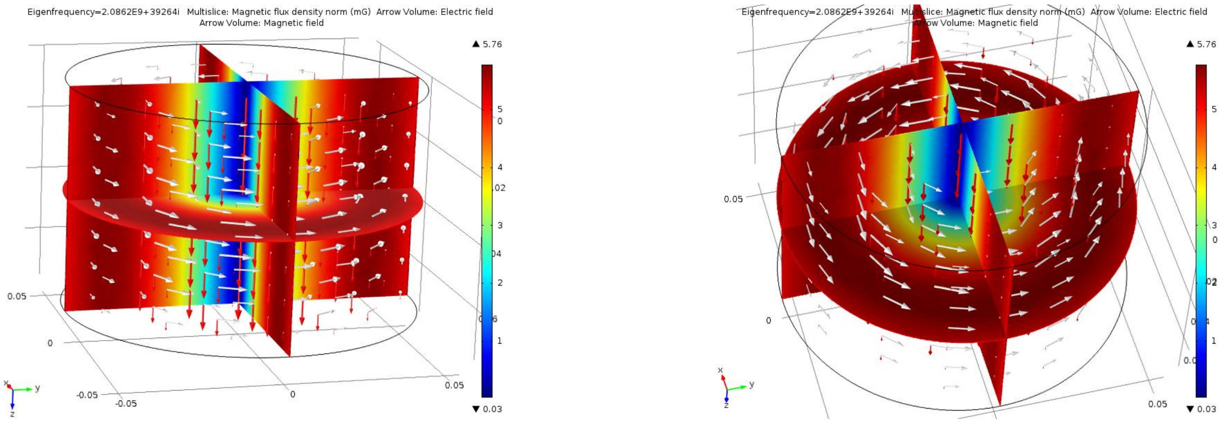
During the project we considered several designs of interest which are shown in Figure 3. Design 1 is just a uniform cylinder for which there is an analytical solution if one considers walls of infinite conductivity. Design 2 is a cylinder filled with dielectric but where a hole was drilled along its axis to make place for a sample that should be put at its center if one uses the $TE_{011}$ mode. Design 3 has in addition a hole a bit off center to make place for the coupling antenna. Finally Design 4 isn't completely filled with dielectric but only has a ring of dielectric. This leaves place above and below the ring for the antenna. The hole in the middle of the dielectric is, as before, for a sample. This design has the advantage that if one uses a stack of rings or rings of different heights one can tune the resonant frequency.



(a) $TE_{011}$ (b) $TE_{011}$

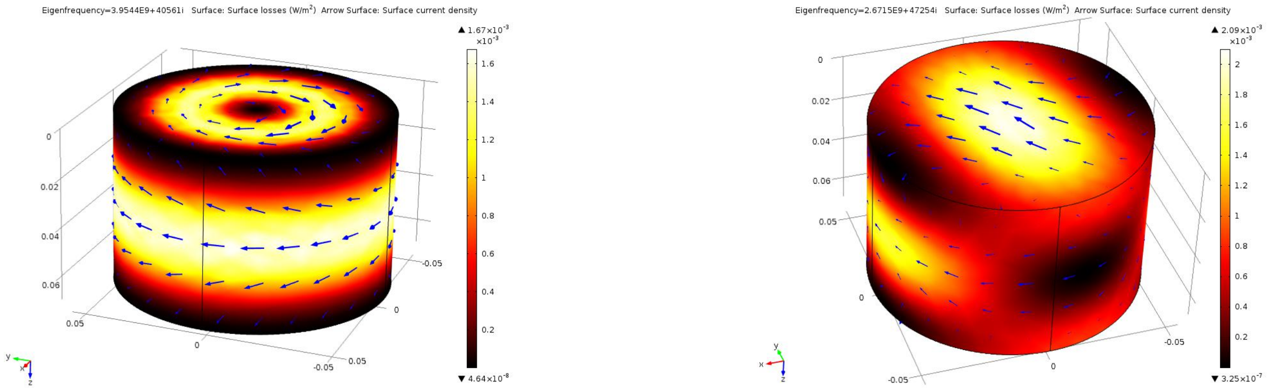


(c) $TE_{111}$ (d) $TE_{111}$

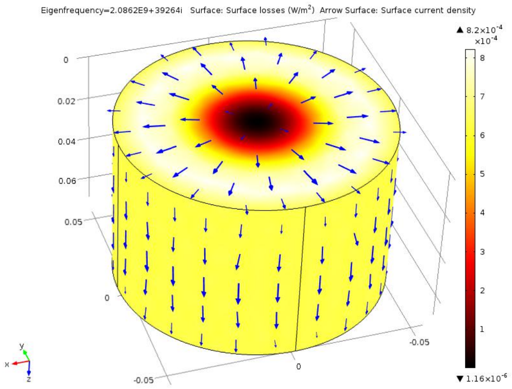


(e) $TM_{010}$ (f) $TM_{010}$

Figure 1: Simulation of the field distribution of the $TE_{011}$, $TE_{111}$ and $TM_{010}$. E-field represented by red arrows and B-field by white arrows. The color shows the amplitude of the B-field on the planes. We see that, in the $TE_{011}$ mode, the B-field is confined to only one region which results in a higher amplitude for the same input power. Moreover the field is along the z-axis making it easier to work with than the $TE_{111}$ mode for instance, where the coupling would decide where the B-field maximum lies due to the cylindrical symmetry. The $TM_{010}$ was put for the readers interest as it is a low frequency mode which one might come upon.

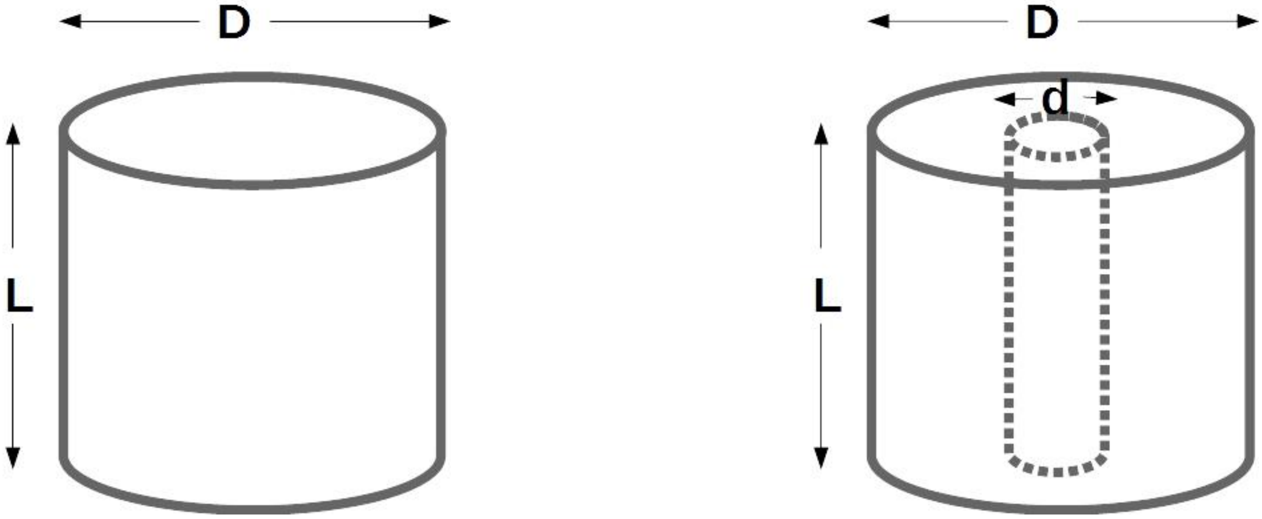


(a) $TE_{011}$ (b) $TE_{111}$



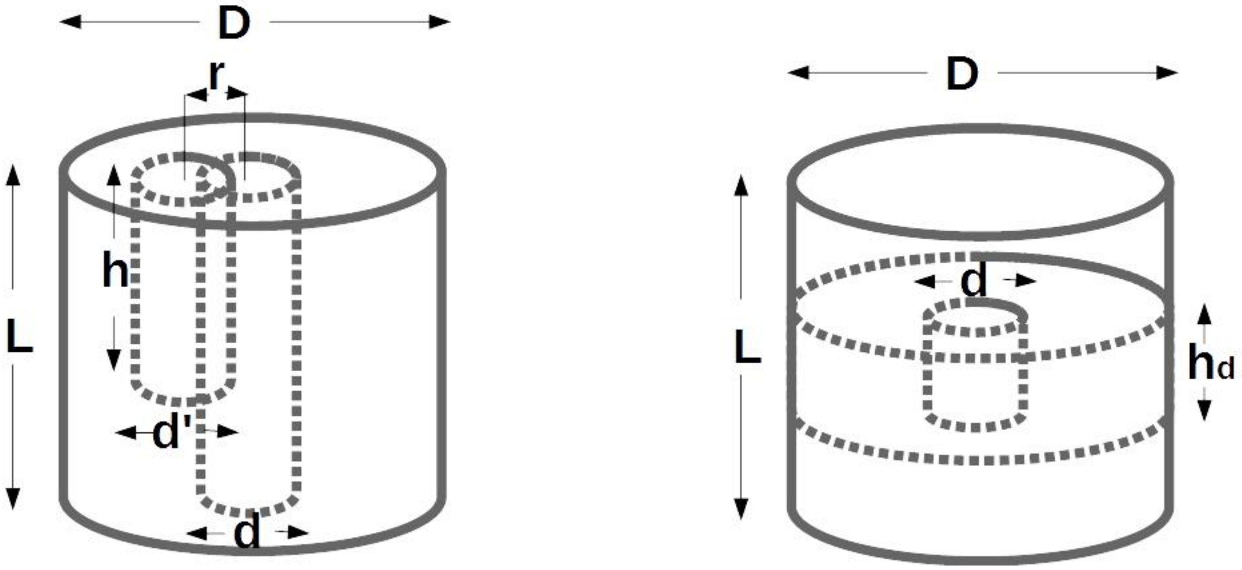
(c) $TM_{010}$

Figure 2: Simulation of the current density and direction in the walls of the cavity. This can be useful to predict if a discontinuity in the cavity, as a hole for instance, affects its resonance. For example the symmetry of the current in the $TE_{011}$ allows the cavity to be cut perpendicularly to its axis without disturbing the current, and thus the resonance will not be affected. This is also valid for circular, centered holes in the top and bottom plate which makes it easier to work with the $TE_{011}$ mode.



(a) Design 1: Cylinder filled with vacuum or a dielectric.

(b) Design 2: Cylinder filled with a dielectric in which a hole was drilled in the center along the axis of the cylinder.



(c) Design 3: Same as Design 2 but with a second hole of depth h drilled alongside the first one. During the whole project $r = D/6$.

(d) Design 4: A ring of dielectric of height h put inside a cylinder fille dwith vacuum.

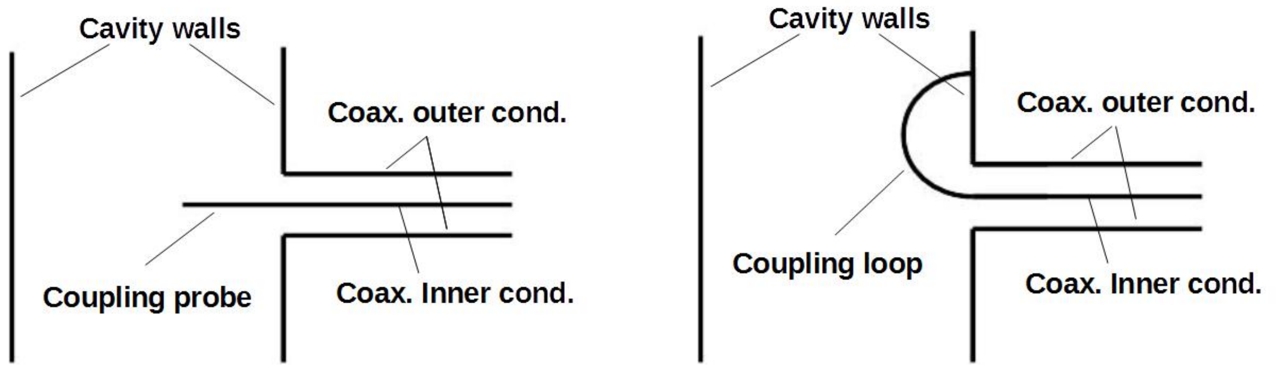
Figure 3: These four designs were considered during this project. The names for the designs and their defining parameters will be used during the whole report.

## 2.2 Coupling to the cavity

To excite a cavity we need to couple it to a RF-wave coming from a coaxial cable. The cavity can of course also be directly coupled to a waveguide but for practical reasons we only consider coaxial cables here. To achieve the coupling with a coaxial cable one can use a coupling probe or a coupling loop as shown in Figure 4. The coupling probe creates an electric field between itself and the wall and thus radiates energy into the resonator by coupling to the electric field. The closer the probe is to a maximum of electric field, the stronger the coupling gets, provided the electric field created by the probe is aligned with the field of the cavity.

The coupling loop on the other hand creates and thus will couple to the magnetic field. Again the orientation of the loop changes the couplling as the alignment between the field from the loop and from the resonant modes changes.

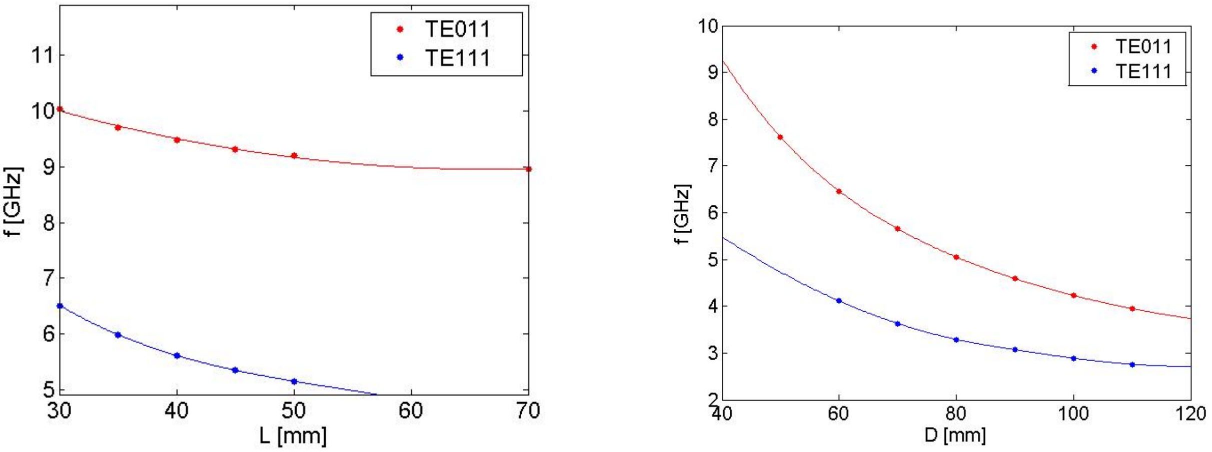
The coupling can be a useful tool as it can favour certain modes rather than others. Talking with A. Sienkiewicz we found out that a good coupling to the cavity for the $TE_{011}$ is a loop located at about a third of a radius from the center, near the top plate. For a fixed loop at the end of a coaxial cable we can then change the coupling by inserting the coax more or less into the cavity. This coupling method was used during the whole project.



(a) Coupling probe, couples to the electric field inside the cavity.

(b) Coupling loop, couples to the magnetic field inside the cavity.

Figure 4: Coupling methods between cavity and coaxial cable.



(a) L dependance of frequency for the $TE_{011}$ and $TE_{111}$ with $D = 42\text{mm}$.

(b) D dependance of frequency for the $TE_{011}$ and $TE_{111}$ modes with $L = 70\text{mm}$.

Figure 5: Shape dependance of frequency for the $TE_{011}$ and $TE_{111}$ modes in a cavity of Design 1 in vacuum.

# 3 Cavity dimension and design analysis by COMSOL simulations

As is shown in Eq. (1), the resonant frequencies depend on the size and shape of the cavity. The dependance of the frequency on the shape of the cylinder is shown in Figures 5a and 5b. This data was acquired through COMSOL simulations of cavities of Design 1 of different dimensions. The theoretical dependance is shown in Figure 6. We also see that to get a 4GHz cavity, with high quality ($L \approx D$) we need a length and diameter of around 100mm in vacuum (or air) which is too big for the experimental setup. Thus we need ways to reduce the size of the cavity without changing its resonant frequency.

A way to do this is to fill the cavity with dielectric. Indeed, the frequency depends, through $c = \frac{1}{\sqrt{\epsilon\mu}}$ as $\frac{1}{\sqrt{\epsilon_r}}$ on the relative permittivity of the cavity material. This means if we fill the cavity with a dielectric having a relative permittivity of $\epsilon_r$, the frequency will be reduced by a factor of $\sqrt{\epsilon}$. As an example we took $\epsilon_r = 40$ and could reduce the size of the cavity of Design 1 in the simulations to $L = 14\text{mm}$ and $D = 16\text{mm}$ for a frequency of 3.99 GHz. High permittivity materials with $\epsilon_r = 40$ are readily available. This size would be small enough for the experimental setup.

To have space for the sample at the middle of the cavity we simulated the change in frequency and field if we drill a hole of diameter d along the axis of the cylinder this results in Design 2. The change in frequency as a function of hole diameter $d$ is given in Figure 7a. An estimation of the maximum B-field amplitude is given in Figure 7b. This value is an estimate from COMSOL and it is not exactly clear to me how COMSOL computes this value, so one should be careful dealing with it but for similar cavity geometries I think it is safe to use these values qualitatively. We see that up to a diameter of $d = 3\text{mm}$ there is very little change in the resonant frequency. However the amplitude of the B-field seems to be affected right away.

Figure 9 shows how the field distribution is affected by a second hole of diameter $d' = 2\text{mm}$ off axis with different depths h. The second hole being off axis, it breaks the natural symmetry of the $TE_{011}$ mode and perturbs the uniformity of the magnetic field in the sample hole. However a hole going only a quarter through the cylinder ($h = L/4$) hardly perturbs the system. This depth could already be sufficient to place an antenna and keep a pretty uniform field distribution at the sample position.

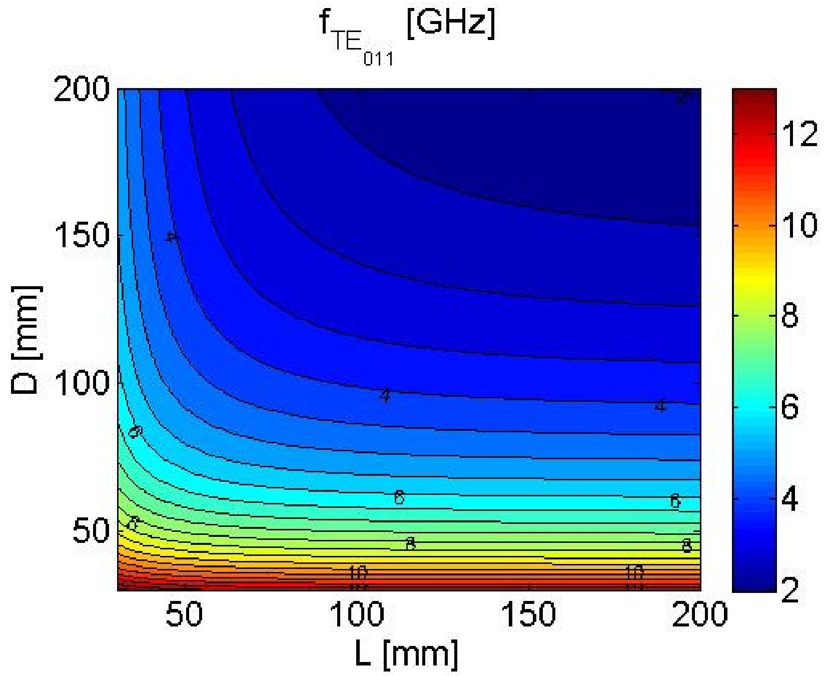
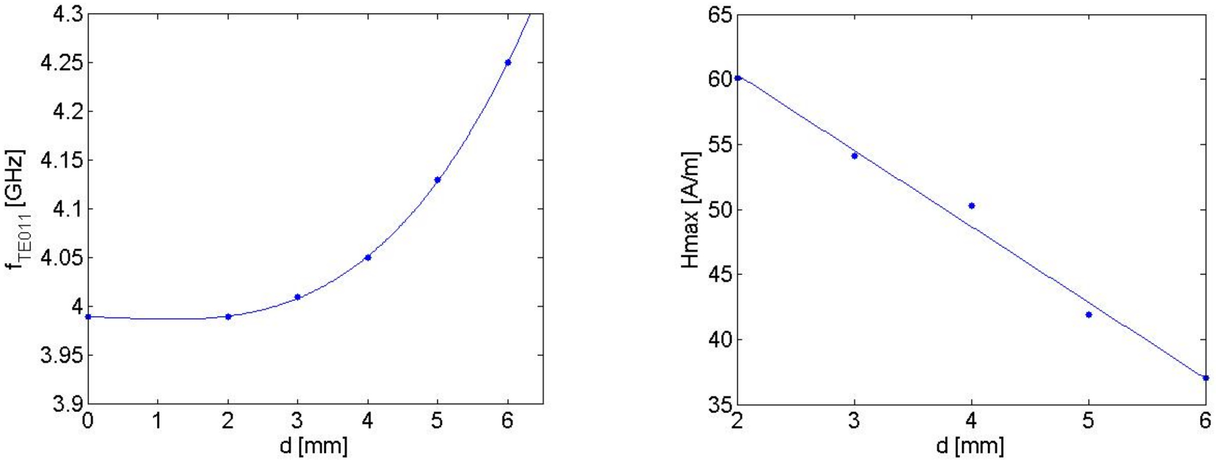


Figure 6: Shape dependance of frequency of $TE_{011}$ mode for a cylindrical cavity in vacuum.



(a) Frequency of the $TE_{011}$ mode

(b) Estimate of maximum magnetic field strength

Figure 7: Design 2 with $L = 14$mm and $D = 16$mm and $\epsilon_r = 40$. Variation of $TE_{011}$ mode frequency as a function of hole diameter $d$. We see that a small hole ($d = 3$mm so about $\frac{1}{5}D$) does not influence greatly the frequency of the $TE_{011}$ mode. However already a small hole decreases the magnitude of the B-field at the center of the cavity.

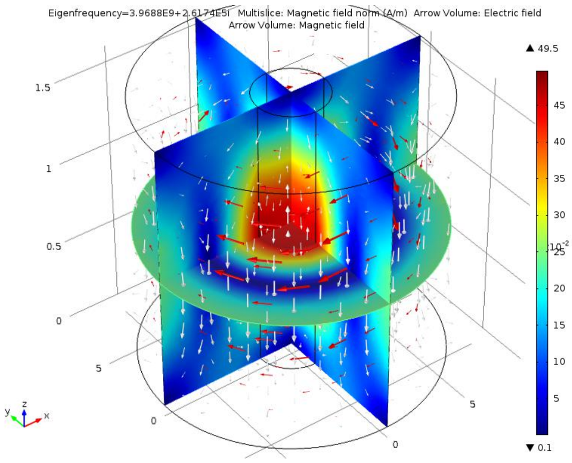
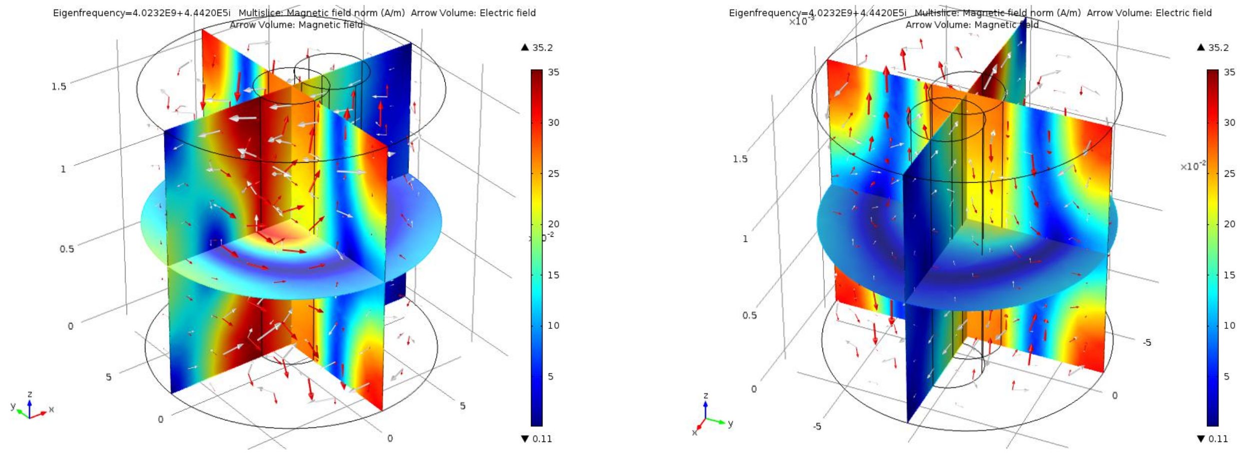
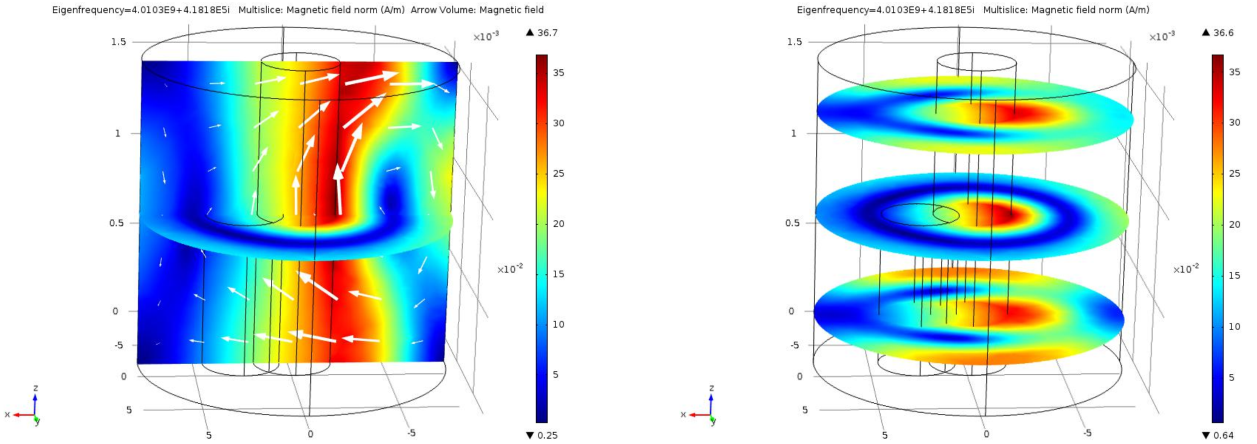


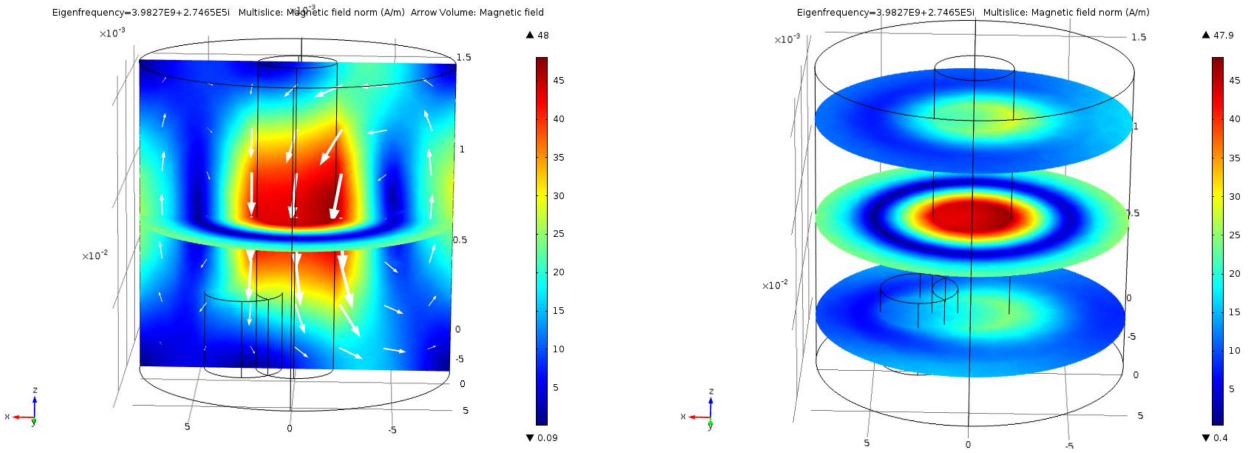
Figure 8: COMSOL simulation of the field distribution of the TE011 mode equivalent in a Design 2 cavity with dielectric $\epsilon_r = 40$, a hole of $d = 4$mm, $L = 16$mm and $D = 16$mm. E-field represented by red arrows and B-field by white arrows the color maps show the amplitude of the B-field on the considered planes. The shape is still pretty similar to the $TE_{011}$ mode in cavities of Design 1 as no symmetry was broken, but the field seems to be more uniform inside the hole.



(a) $h = L$ (b) $h = L$



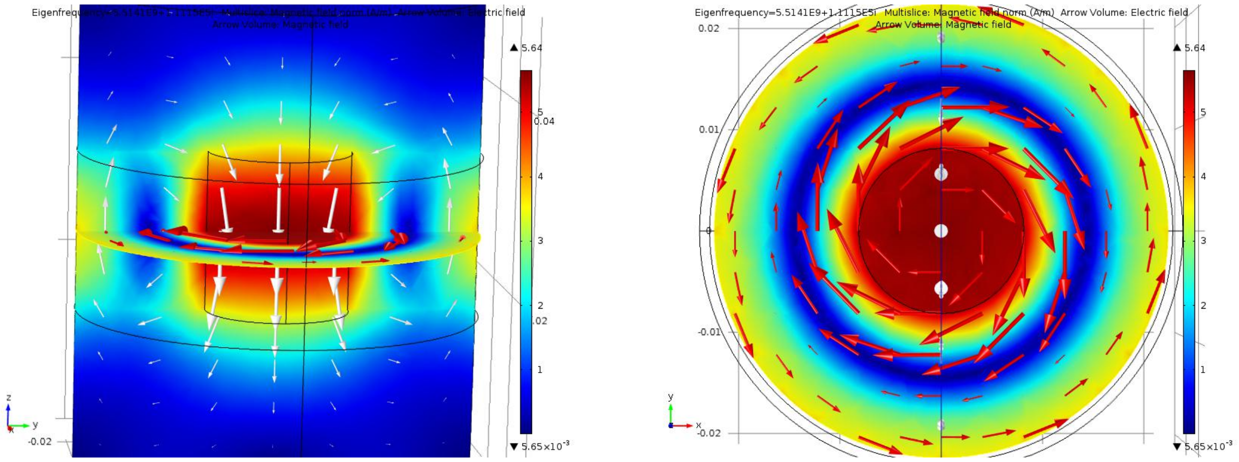
(c) $h = L/2$ (d) $h = L/2$



(e) $h = L/4$ (f) $h = L/4$

Figure 9: COMSOL simulation of the field distribution of the TE011 mode equivalent in a Design 3 cavity with dielectric $\epsilon_r = 40$ and a hole of $d = 4$mm, $d' = 4$mm, $L = 16$mm and $D = 16$mm for several $h$. E-field represented by red arrows and B-field by white arrows the color maps show the amplitude of the B-field on the considered planes. We can see that due to the symmetry breaking the $TE_{011}$ mode loses its uniformity and shape. The uniformity and shape can be recovered by making only a shallow hole with $h = L/4$ or smaller.

Finally, Design 4 has the advantage of already greatly reducing the size of the cavity, although less than Design 2 or 3, while keeping place for sample and antenna. By changing the height of the dielectric ring the frequency can be tuned. Moreover the dielectric only needs one hole for the sample and thus does not break any symmetry while leaving enough space for the antenna. The simulation of the electric and magnetic field of such a cavity is shown in Figure 10. This time the symmetry stays intact and the field is pretty uniformely distributed inside the hole. Figures 11a, 11b and 11c show how the frequency of the $TE_{011}$ mode varies when several ring parameters are changed. Most interesting for a future application is that by changing the height to up to half the cavity height we can tune the frequency by about 1 GHz. Moreover using a dielectric like DELRINE ($\epsilon_r \approx 3.7$, easily accessible and machinable) a small ring (a third the height of the cavity) already reduces significantly the frequency. This could be a good design for a cavity as it leaves enough space for the antenna and by stacking rings of dielectric one could tune the frequency to a wished value. Moreover the magnetic field seems concentrated in the area of the hole in the ring where the sample would be placed, maximising by this the effect of the cavity on the sample.



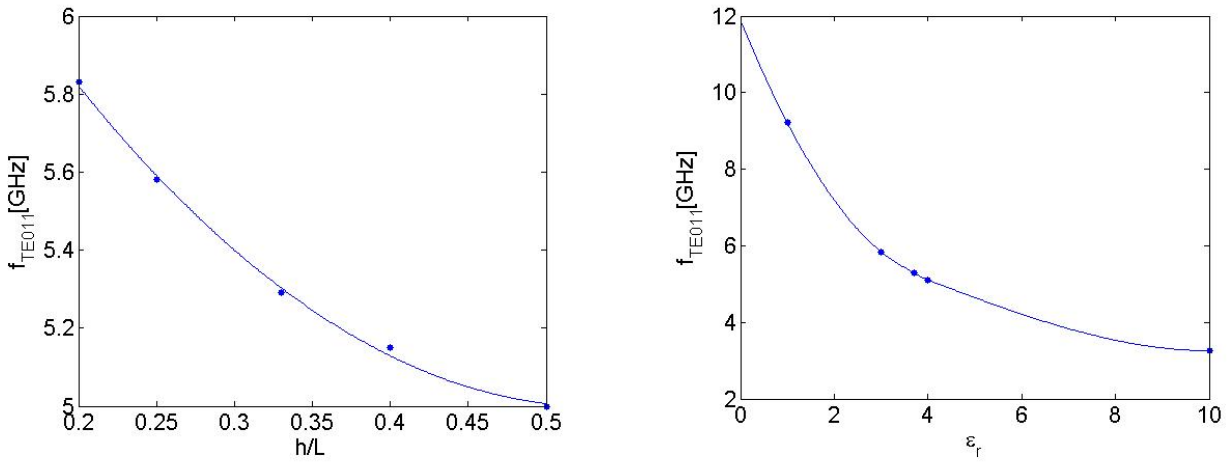
(a) (b)

Figure 10: COMSOL simulation of the field distribution of the TE011 mode equivalent in a cavity of Design 4 with $\epsilon_r = 3.7$, $d = 15$mm, $L = 50$mm, $D = 42$mm and $h = L/3$. E-field represented by red arrows and B-field by white arrows. The symmetry is conserved and the magnetic field-lines seem more concentrated in the hole of the ring. It also seems as though the magnetic field is pretty constant throughout the hole.

# 4 Experimental results

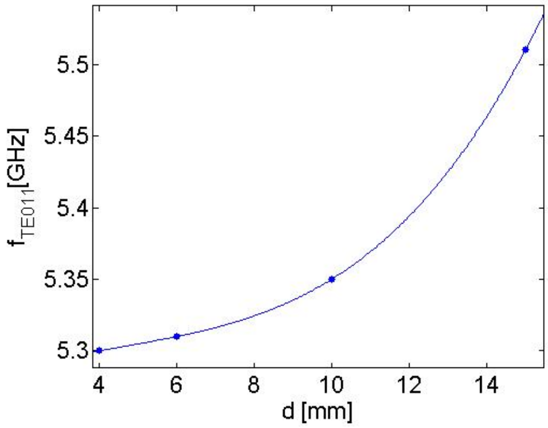
To test the compatibility of the simulation with real applications, a prototype cavity, shown in Figure 12, was analyzed. The cavity is of Design 1, has an inner diameter of $D = 42mm$ and was made by stacking several rings enabling thus a change in length $L$ going from 5mm to 65mm. Unfortunately the cavity was too small to be able to observe the $TE_{011}$ mode. However it was possible to observe the $TE_{111}$ mode. The antenna consisted of a loop at the end of a coaxial cable. The cable could be inserted more or less into the cavity as well as turned, to change the coupling. S11 reflection measurements for different cavity lengths done using a vector network analyzer are shown on Figures 13a, 13b, 13c and 13d. The sharp peak shows the $TE_{111}$ mode, the large broader peak changes its position when the antenna is moved and is thus probably due to a resonance of the antenna with the cavity walls and not of an eigenmode of the cavity. On Figure 14 we compare the theoretical values of the resonance frequencies with the ones obtained by the experiment. The experimental values correspond incredibly well with the theory.

At the LPMN lab at EPFL we were able to measure the $TE_{011}$ mode for $L = 50mm$ and $D = 42mm$ using their analyzer which could go to high enough frequencies. The results obtained were a resonant frequency of $f_{TE_{011}} = 9.207$GHz, a half width $\Delta f \approx 1.1 - 1.5$MHz. This gives a quality factor $Q = \frac{f}{\Delta f} \approx 9000$. Moreover



(a) $\epsilon_r = 3.7$ and $d = 3$mm. Frequency dependance of the ring-cavity for different ring heights.

(b) $\frac{h}{L} = \frac{1}{3}$ and $d = 3$mm.



(c) $\epsilon_r = 3.7$ and $\frac{h}{L} = \frac{1}{3}$.

Figure 11: Parameter dependance of $TE_{011}$ mode frequency in a cavity of Design 4. $L = 50$mm and $D = 42$mm. We see that just by changing the height of the dielectric ring we can tune the resonant frequency over a range of almost 1 GHz. Moreover the frequency does not seem to be sensitive to small hole diameters (less than 6mm).

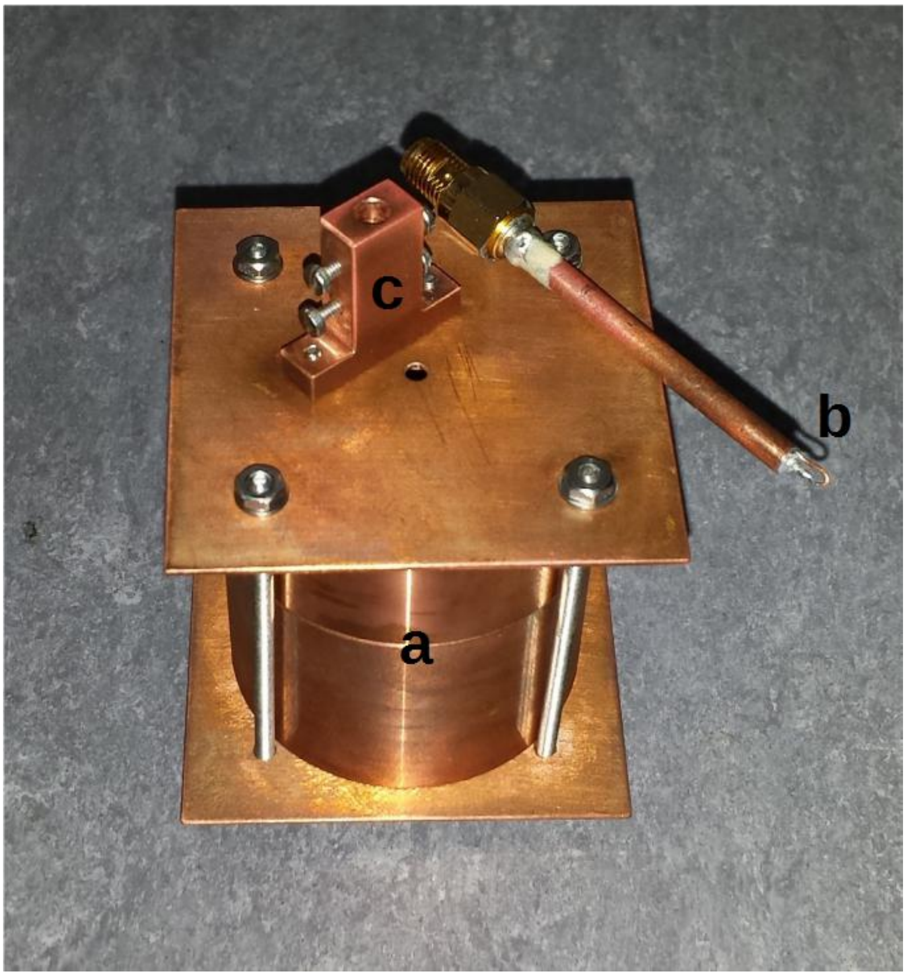
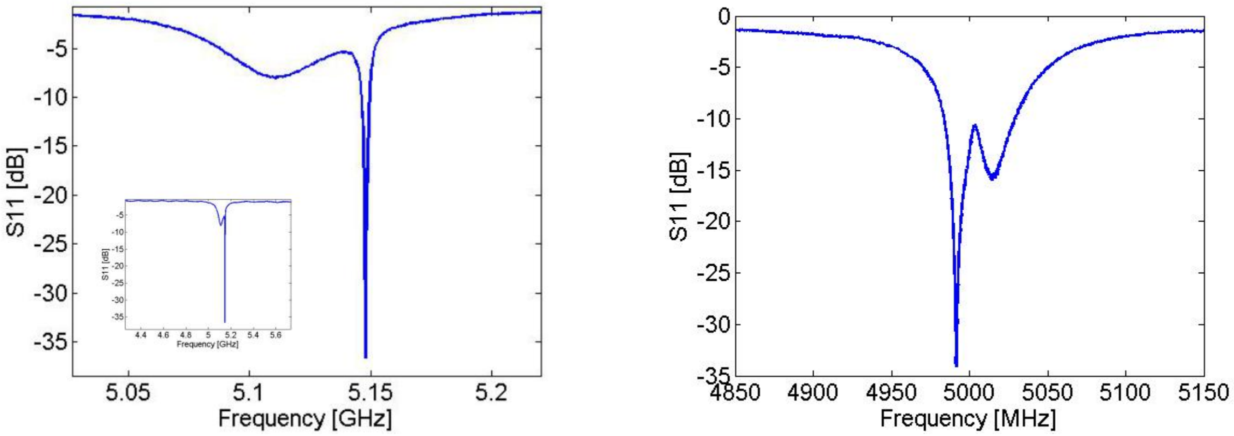
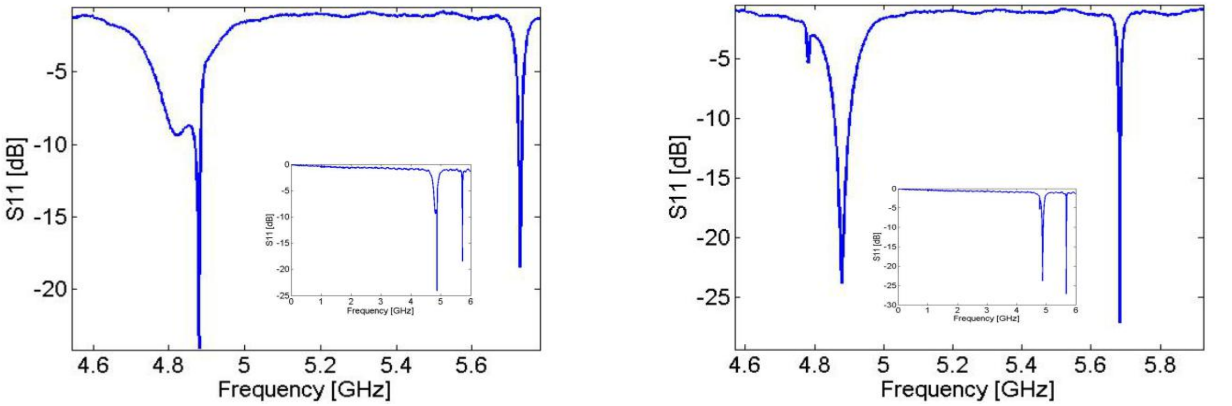


Figure 12: Cavity used for experiments(a). This cavity has $L = 50mm$ and $D = 42mm$ but its length could be varied by stacking more copper rings. One can also see the coupling antenna (b) and the antenna holder (c). The antenna consits of a loop at the end of a coaxial cable and can be inserted more or less into the cavity through a hole situated in the top plane at 1/3 of the radius.



(a) $L = 50$mm

(b) $L = 55$mm



(c) $L = 60$mm

(d) $L = 65$mm

Figure 13: Reflection spectrum of the cavity with $D = 42$mm for different lengths. The peak corresponding to the $TE_{111}$ mode is the left of the two narrow peaks for $L = 60$mm and $L = 65$mm and is the only narrow peak in the other two cases. It was hard to get a good coupling for the $L = 65$mm case as the peak is very shallow compared to the others. The broad peak is probably due to some resonance of the antenna and the other narrow one to some other mode.

the frequency again corresponds very well with the theory which predicts a frequency of 9.213 GHz and the simulations giving a frequency of 9.208 GHz.

To be able to have another look at the $TE_{011}$ resonance we built a cavity with $L = 100mm$ and $D = 100mm$ which gives a theoretical resonance frequency at $f_{TE_{011}} = 3.95$GHz. The S11 reflection measurement of this cavity is shown in Figure 15. One can see several peaks corresponding to the different modes of the cavity. The $TE_{011}$ mode is found as predicted by the theory at 3.95GHz and presents a high quality compared to most other modes. From the zoom on the resonance the quality factor can be estimated at around or even above 5000.

Let us compare the quality factors to those obtained by strip resonators. A spectrum of a strip resonator having a resonance frequency of about 3.1GHz is shown in Figure 16. The quality factor of this strip resonator can be estimated to 150. Compared to this the resonances obtained using cavities are of pretty high quality as their quality facor is an order of magnitude higher.

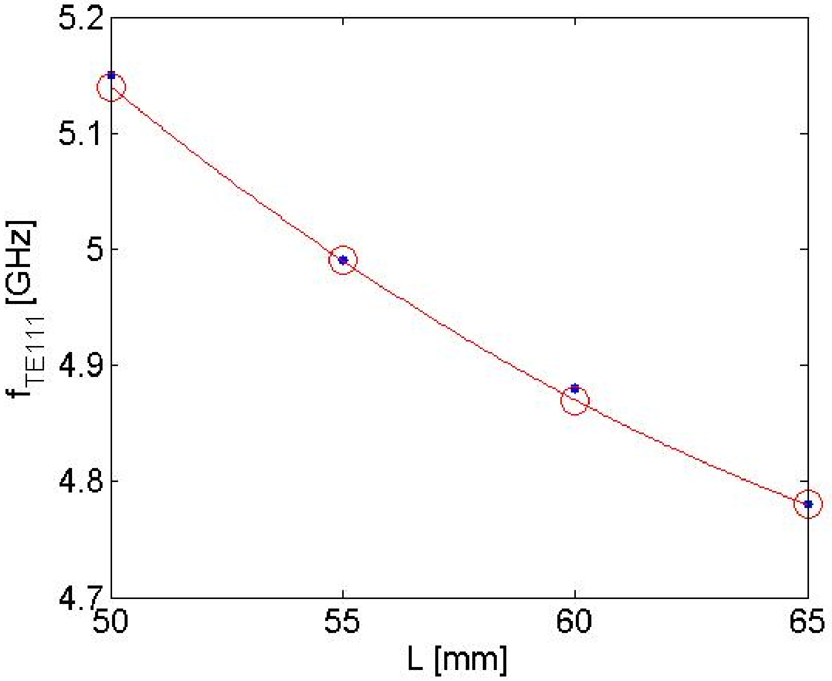
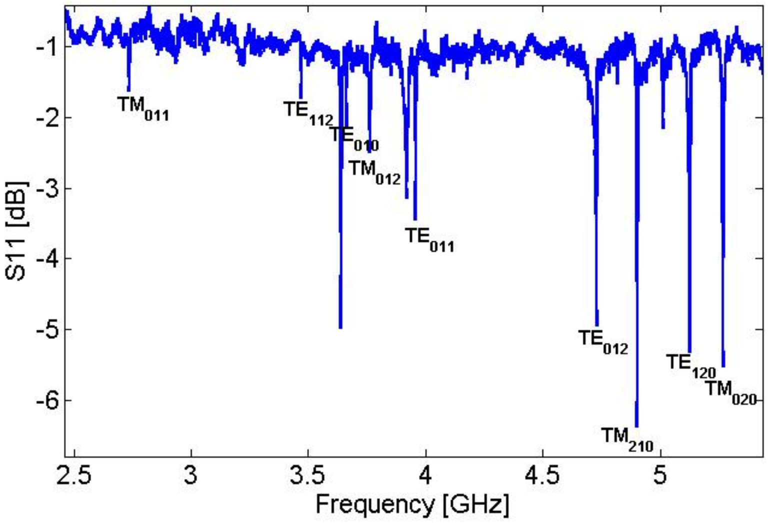
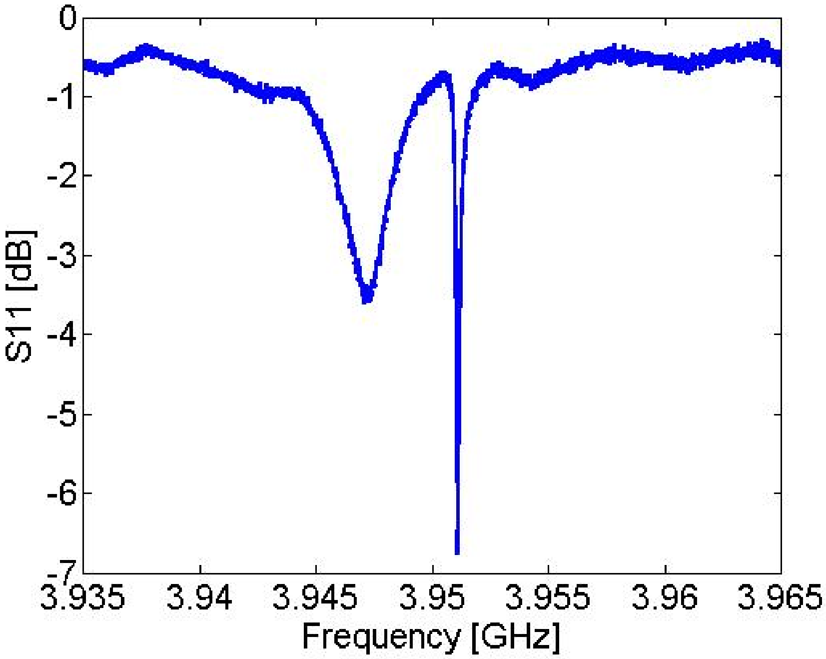


Figure 14: Measurement of the TE111 frequency for a cylinder of diameter 42mm. The hollow circles are the theoretical values.



(a)



(b) Zoom on the $TE_{011}$ resonance.

Figure 15: S11 absorption spectrum for cavity of $L = 100mm$, $D = 100mm$. We can recognize the $TE_{011}$ mode at a frequency of 3.95 GHz. The other peaks correspond to other frequencies or are as before due to the antenna as the one just next to the $TE_{011}$ mode in the zoom. The $TE_{011}$ mode presents a pretty high quality compared to most of the other modes.

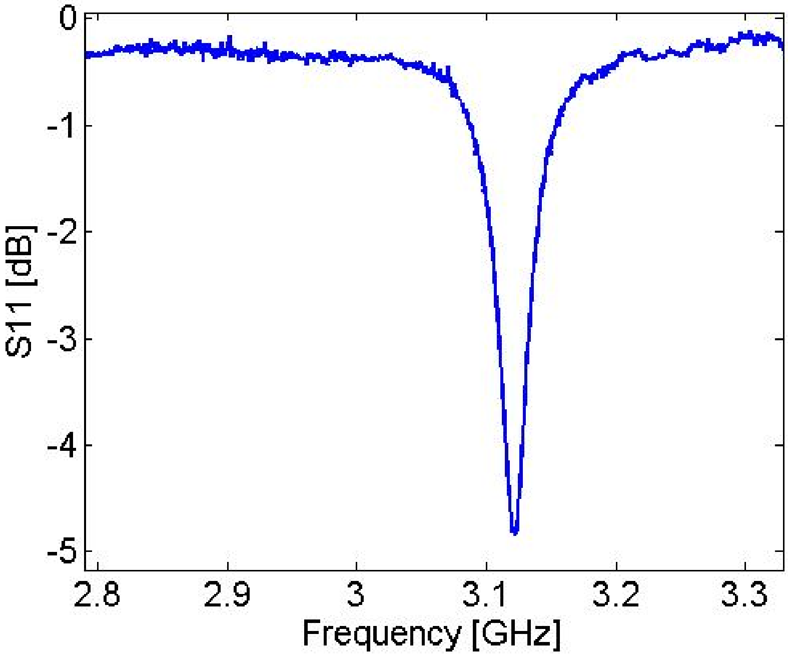


Figure 16: Reflection spectrum of a strip resonator. We can see $f_{\text{resonance}} = 3.12\text{GHz}$ and $\Delta f \approx 20\text{MHz}$ and thus we get a quality factor of $Q \approx 150$ which is one order of magnitude lower than typical values for the cavities we were able to measure.

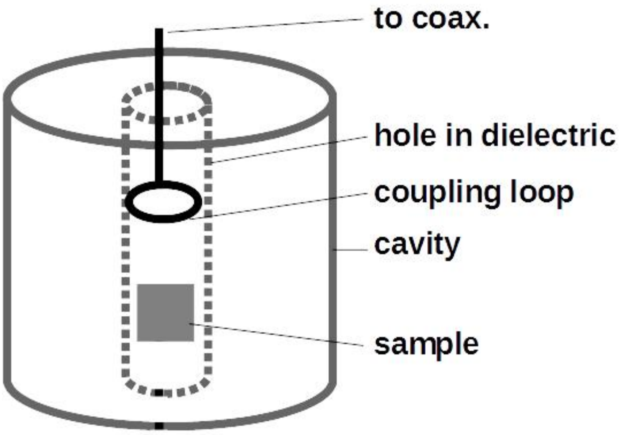


Figure 17: Example how one could accomodate sample and antenna, coming in from the top in here, in the same hole. The antenna would be a loop in a plane perpendicular to the axis of the cylinder and couple to the vertical magnetic field of the $TE_{011}$ mode.

## 5 Design idea

Considering the previous sections we can propose two designs of interest that would need to be tested further. One would be a cavity of Design 2 filled with a dielectric of $\epsilon_r = 40$ of dimensions $L = 16$mm and $D = 16$mm and a hole of diameter $d = 4$mm. To create the cavity one could use a dielectric bloc with the before mentioned dimensions and coat it with some high conductivity paint like silver. The hole would need to accomodate the sample and the antenna in order to not break the symmetry. To facilitate this the antenna could be a loop placed in a plane perpendicular to the axis of the cylinder as shown in Figure 17

The other would be a cavity of Design 4. The exact dimensions would need to be determined but a dielectric of about $\epsilon_r = 10$ would be needed in order to reduce sufficiently the cavity size. This cavity would be bigger than the previous one but it would have the advantage of having an easily tuneable frequency. Moreover there would be more freedom to the position and form of the antenna.

## 6 Conclusion

To conclude, I was able to acquire some knowledge and experience with cavities during this project. There is still a lot that could be done, for instance optimizing the antenna as well as its position in order to get better coupling. Nevertheless this project was able to show what sizes of cavities one needs to use in order to get a 4GHz resonance and that the use of dielectric is indispensable in order to make the cavity small enough to be useful. It also showed that cavities could provide a higher quality factor than stripline resonators, making them thus interesting for application.

The simulations also permitted to look into the response of more complex designs with holes in the dielectric or only partly filled cavities and it was shown that using rings of dielectric that only partly fill the cavity one could make a tuneable design that could be useable in a range of about 1GHz making its application more interesting. It would now be of interest to investigate more this direction mainly by trying to reproduce the resonance experimentally to see how well their quality can be made.

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## References

[1] Raymond S. Alger. *Electron Paramagnetic Resonance: Techniques and Applications.* John Wiley & Sons, Inc., 1968.

[2] Andrew Zangwill. *Modern Electrodynamics.* Cambridge University Press, 2013.