Maximal subgroups acting with two composition factors on irreducible representations of exceptional algebraic groups

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Résumé

Soit k un corps algébriquement clos de caractéristique p. Soient Y un groupe algébrique simple simplement connexe sur k et X un sous-groupe maximal parmi les sous-groupes fermés connexes simples de Y. A l'exclusion de certaines valeurs de p pour des plongements précis, nous classifions les représentations irréductibles p-restreintes de Y sur lesquelles X agit avec exactement deux facteurs de composition. Ce travail s'inscrit dans la continuité de la classification des sous-groupes irréductibles des groupes exceptionnels donnée par Testerman.

Mots-clés: Théorie des représentations, groupes algébriques, groupes exceptionnels, règles de branchement.

Abstract

Let Y be a simply connected simple algebraic group over an algebraically closed field k of characteristic p and let X be a maximal closed connected simple subgroup of Y. Excluding some small primes in specific cases, we classify the p-restricted irreducible representations of Y on which X acts with exactly two composition factors. This work follows on naturally from the classification of irreducible subgroups of exceptional algebraic groups given by Testerman.

Key words: Representation theory, algebraic groups, exceptional groups, branching rules.

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List of notations

For an explanation of the tables containing the data of the Jantzen p-sum formula see p. 19.

$\operatorname{rank}(G)$	rank of G, p. 1
$X(T)$	$= \operatorname{Hom}(T, \mathbb{G}_m)$, the group of characters of T, p. 1
Φ	root system of a group G, p. 1
Δ	set of simple roots of the root system Φ, p. 1
Φ^+	set of positive roots in Φ, p. 1
α_0	largest root in Φ, p. 1
$\tilde{\alpha}_0$	largest short root in Φ, p. 1
U_α	root subgroup corresponding to a root α, p. 1
$\alpha^\vee$	coroot corresponding to a root α, p. 1
$\Phi^\vee$	dual root system of Φ, p. 2
W	Weyl group of Φ, p. 2
w_0	longest element in W, p. 2
P_I	standard parabolic subgroup of G corresponding to $I \subseteq \Delta$, p. 2
L_I	standard Levi subgroup of P_I for $I \subseteq \Delta$, p. 2
λ_i	fundamental weight corresponding to the simple root α_i, p. 3
$X(T)^+$	set of dominant weights in $X(T)$, p. 3
$\succeq$	partial order on $X(T)$, p. 3
$\mathscr{L}(G)$	Lie algebra of G, p. 3
$\theta_1^{m_1}/\cdots/\theta_s^{m_s}$	set of composition factors of a kG-module V, where $\theta_i \in X(T)^+$ and m_i is the multiplicity of $L_G(\theta_i)$ in V, p. 4
V_λ	$= \{v \in V \mid tv = \lambda(t)v \text{ for all } t \in T\}$ for a kG-module V and a weight $\lambda \in X(T)$, p. 4
$\Lambda(V)$	$= \{\lambda \in X(T) \mid V_\lambda \neq 0\}$ for a kG-module V, p. 4
$\Lambda(V)^+$	$= \{\lambda \in X(T) \mid \lambda \in \Lambda(V) \cap X(T)^+\}$ for a kG-module V, p. 4
$m_V(\lambda)$	$= \dim V_\lambda$, the multiplicity of λ in V, p. 4
$\operatorname{rad} V$	radical of a kG-module V, p. 4
$V_G(\lambda)$	Weyl module of highest weight $\lambda \in X(T)^+$, p. 5

$\Lambda(\lambda)$	$= \Lambda(V_G(\lambda))$ for $\lambda \in X(T)^+$, p. 5
$m_\lambda(\mu)$	$= \dim V_G(\lambda)_\mu$, the multiplicity of μ in $V_G(\lambda)$, p. 5
$L_G(\lambda)$	irreducible module of highest weight $\lambda \in X(T)^+$, p. 5
$\operatorname{ch} V$	formal character of a kG-module V, p. 6
$\chi(\lambda)$	$= \sum_{i\geq 0}(-1)^i \operatorname{ch} H^i(\lambda) (= \operatorname{ch} V(\lambda)$ if $\lambda \in X(T)^+)$, p. 7
$[V_G(\lambda) : L_G(\theta)]$	coefficient $a_{\lambda,\theta}$ in $\operatorname{ch} V_G(\lambda) = \sum_{\theta\in X(T)^+} a_{\lambda,\theta} \operatorname{ch} L(\theta)$, p. 7
$[\lambda : \theta]$	$= [V_G(\lambda) : L_G(\theta)]$, p. 7
$(L_G(\lambda) : V_G(\theta))$	coefficient $b_{\lambda,\theta}$ in $\operatorname{ch} L_G(\lambda) = \sum_{\theta\in X(T)^+} b_{\lambda,\theta}\chi(\theta)$, p. 7
$(\lambda : \theta)$	$= (L_G(\lambda) : V_G(\theta))$, p. 7
$\operatorname{ch} V^S$	S-truncated character of V, p. 7
W_p	affine Weyl group, p. 8
$s_{\alpha,r}(\lambda)$	$= s_\alpha(\lambda) + rp\alpha$, p. 8
ρ	half-sum of positive roots, p. 8
$w \bullet \lambda$	$= w(\lambda + \rho) - \rho$, p. 8
$\widehat{C}$	upper closure of an alcove C, p. 8
$\overline{C}$	closure of an alcove C, p. 8
C_0	$= \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda + \rho, \alpha^\vee\rangle < p \ \forall\alpha \in \Phi^+\}$, p. 9
h	Coxeter number of Φ, p. 9
$\Sigma(C_0)$	$= \{s_{\alpha_i}, \alpha_i \in \Delta\} \cup \{s_{\tilde{\alpha}_0,1}\}$, p. 9
s_i	$= s_{\alpha_i}$ for $\alpha_i \in \Delta$ and $s_0 = s_{\tilde{\alpha}_0,1}$, p. 9
$\nu_p(z)$	p-adic valuation of z, p. 10
$\mathrm{JSF}(\lambda)$	$= \sum_{\alpha\in\Phi^+} \sum_{0<mp<\langle\lambda+\rho,\alpha^\vee\rangle} \nu_p(mp)\chi(s_{\alpha,mp} \bullet \lambda)$, p. 11
D	$= \{\lambda \in X(T) \mid \lambda + \rho \in X(T)^+\}$, p. 11
$\operatorname{support}(\alpha)$	$= \{i \mid a_i \neq 0\}$ for $\alpha \in \mathbb{Z}\Phi$, p. 12
$L_{L_I}(\lambda)$	simple kL_I-module of highest weight $\lambda \in X(T)^+$, p. 28
$\leq_V$	bound on a multiplicity or on a dimension by its value in a Weyl module, p. 37
$\leq_{BS}$	bound on a multiplicity in an irreducible module for which the character was not precisely determined, p. 42
$\overline{\delta_{i,j}}$	$= 1 - \delta_{i,j}$, p. 46
$\leq_{BC}$	bound on a multiplicity in a set of irreducible modules for which the character was precisely determined, p. 70
$\leq_{LB}$	bound on the dimension of a simple module from [Lüb07], p. 122
$\operatorname{lev}_\gamma(\theta)$	level of θ with respect to γ, p. 122

Introduction

Representation theory of algebraic groups. — In this thesis, we focus on the study of rational representations of connected reductive linear algebraic groups over an algebraically closed field k of characteristic $p \geq 0$, a prime or zero. We will assume henceforth all the groups to be linear and their representations to be finite-dimensional rational.

Let Y be a reductive algebraic group over k and let $\phi : Y \to \mathrm{GL}(V)$ be a representation of Y. By the Jordan-Hölder theorem, V admits a well-defined filtration by simple modules called a composition series. Since the simple modules constitute the building blocks of any kY-module, the first step towards understanding the representations of Y involves understanding the simple modules. In positive characteristic, as opposed to characteristic 0, the study of these modules is a difficult task and has constituted a vast area of research over the last forty years. Recall briefly that the set of simple kY-modules is in bijection with the set $X(T)^+$ of dominant weights of T. Under this bijection, a dominant weight λ is mapped to $L_Y(\lambda)$, the simple module of highest weight λ. By Steinberg's tensor product theorem, the task of understanding the simple kY-modules is reduced to that of understanding those with p-restricted highest weights. The good news is that for a fixed p, the set of p-restricted weights is finite.

The simple module $L_Y(\lambda)$ can be realized as the unique simple quotient of $V_Y(\lambda)$, the Weyl modules of highest weight λ. The dimension of $V_Y(\lambda)$ is given by Weyl's degree formula and the dimension of a given weight space in $V_Y(\lambda)$ can be recursively computed using Freudenthal's formula. However, $V_Y(\lambda)$ is an indecomposable kY-module and its structure is not known in general. Most of our efforts are focused at understanding the maximal submodule of $V_Y(\lambda)$ in order to gain an understanding of $L_Y(\lambda)$.

Over the course of the last years, Williamson and his collaborators made exciting progress in this area of research. Very recently, in [RW], Williamson and Riche discovered and proved a character formula for p-restricted simple kY-modules under the assumption that $p \geq 2h-2$, where h is the Coxeter number of Y. Even though it is now clear from a theoretical point of view which setting has to be adopted, the computational aspects related to this new character formula are at a very early stage. None of these new techniques will be discussed in this thesis, instead we will use and further develop well-established techniques in order to solve the following problem.

The problem considered. — The two first functors which are studied in representation theory are induction and restriction. These functors provide a way of building new representations and of deducing particular properties by taking advantage of the subgroup structure of a given group. Before stating the specific question we answer in this thesis, we will give a brief overview of a related question which led to the current work.

Question 1. *Let Y be a simply connected simple algebraic group over k and let X be a maximal closed subgroup of Y. For which p-restricted $\lambda \in X(T_Y)^+$ does X act irreducibly on $L_Y(\lambda)$, that is $L_Y(\lambda)$ remains irreducible as a kX-module?*

Fix a pair (X, Y) as in the previous question. In [Dyn52a, Dyn52b], Dynkin solves this question over an algebraically closed field of characteristic 0 with the additional assumptions that X is connected. He determines all the dominant weights λ such that X acts on $L_Y(\lambda)$ irreducibly. Later on, Seitz and Testerman extend this classification to fields of arbitrary characteristic, again with the connectedness hypothesis on X. In [Sei87], Seitz classifies all the triples (X, Y, λ), where Y is of classical type and in [Tes88], Testerman classifies the triples (X, Y, λ), where Y is of exceptional type. It should be noted that for Y of type A, the classification was obtained independently by Suprunenko in [Sup85]. In 2016, while working on this thesis, the author found a triple (X, Y, λ) which did not appear in the classification of Seitz. By a careful verification of Seitz's argument, Testerman spotted a gap in the proof of [Sei87, 8.7] and fixed it along with Cavallin in [CT19].

Question 1 has also been answered in the case of X disconnected. The corresponding classification, which will be less relevant for us in this thesis, can be found in the following series of papers [For96, For99, Gha10, BGMT15, BGT16].

A more detailed overview of Question 1 can be found in [BT17]. In this recent survey, Burness and Testerman summarize the different approaches and results which lead to the answer of Question 1. We now state a natural generalization of the previous question.

Question 2. *Let Y be a simply connected simple algebraic group over k and let X be a maximal closed connected simple subgroup of Y. For which p-restricted weight $\lambda \in X(T_Y)^+$ does X act on $L_Y(\lambda)$ with exactly two composition factors?*

Question 2 was first addressed by Cavallin for Y of classical type. In [Cav17b], he classifies the triples (X, Y, λ), where $X = \mathrm{Spin}_{2n}(k)$ and $Y = \mathrm{Spin}_{2n+1}(k)$, and in [Cav15, Conjecture 4], he gives a conjectural answer in the case of $X = SO_{2n}(k)$ and $Y = SL_{2n}(k)$.

In this work, we consider the case where Y is of exceptional type and answer the following question, excluding small primes for some specific embeddings.

Question 3. *Let Y be a simply connected simple algebraic group of exceptional type over k and let X be a maximal closed connected simple subgroup of Y. For which p-restricted weight $\lambda \in X(T_Y)^+$ does X act on $L_Y(\lambda)$ with exactly two composition factors?*

Tackling the problem. — The pairs (X, Y) as in Question 3 have been classified by Liebeck and Seitz in [LS04, Theorem 1]. Following [LS04, Theorem 1], we regroup these pairs into two families. The first one consists of pairs (X, Y) with X containing a maximal torus of Y, that is the ranks of X and Y are equal. Apart from known exceptions occurring when $(Y, p) \in \{(G_2, 3), (F_4, 2)\}$, such maximal subgroups X are in bijection with the set of closed subsystems of the root system of Y. They can easily be deduced using the Borel-de Siebenthal algorithm initially exposed in [BdS49, Section 7]. The second family of pairs are the ones which correspond to maximal subgroups of non-maximal rank and the construction of these maximal subgroups spans over the following papers [Sei91, Tes89, Tes92, LS04].

For completeness, we retranscribe the list of pairs of maximal rank in Table 2 from [LS04, Table 10.3] and the ones of non-maximal rank in Table 3 from [LS04, Table 1].

Y	X simple
G_2	A_2 (1 class if $p \neq 3$, 2 classes if $p = 3$)
F_4	B_4 $(p \geq 0)$, C_4 $(p = 2)$
E_7	A_7 $(p \geq 0)$
E_8	D_8 $(p \geq 0)$, A_8 $(p \geq 0)$

Table 2: Maximal closed connected simple subgroups of maximal rank

Y	X simple
G_2	A_1 $(p \neq 2, 3, 5)$
F_4	A_1 ($p = 0$ or $p \geq 13$), G_2 $(p = 7)$
E_6	A_2 $(p \neq 2, 3)$, G_2 $(p \neq 7)$, F_4 $(p \geq 0)$, C_4 $(p \neq 2)$
E_7	A_1 (2 classes, $p = 0$ or $p \geq 17, 19$, resp.), A_2 $(p \neq 2, 3)$
E_8	A_1 (3 classes, $p = 0$ or $p \geq 23, 29, 31$, resp.), B_2 $(p \neq 2, 3)$

Table 3: Maximal closed connected simple subgroups of non-maximal rank

Fix a pair (X, Y) as in Question 3. Let $B_X = U_X T_X$ denote a (positive) Borel subgroup of X and let $B_Y = U_Y T_Y$ denote a (positive) Borel subgroup of Y such that $B_X = B_Y \cap X$. Let $\Phi(Y), \Phi^+(Y), X(T_Y), X(T_Y)^+$ denote the set of roots, positive roots, characters and dominant weights corresponding to the choice of B_Y. We adopt similar notations for the root datum attached to X.

In order to solve Question 3, we should first manage to compute weight multiplicities occurring in simple kY and kX-modules. We achieve this by partially computing the characters of specific simple modules in terms of characters of Weyl modules. Our main tool to perform such a task is the Jantzen p-sum formula (JSF). Coupling the JSF with some results from the theory of translation functors and case-by-case arguments depending on the weights considered, we manage to gain enough information on the characters of the simple modules involved in order to compute the desired multiplicities. In particular, we are able to eliminate many p-restricted weights $\lambda \in X(T_Y)^+$ such that X acts on $L_Y(\lambda)$ with more than two composition factors. However, apart from specific cases for which the dimensions of the modules involved are known, we are not able to prove, at this point, whether for the remaining p-restricted weights, the subgroup X acts on the corresponding simple kY-modules with exactly two composition factors. This issue also constitutes the reason for the conjectural status of Cavallin's aforementioned result in the classical case for the embedding $SO_{2n}(k) \subseteq SL_{2n}(k)$.

The main theoretical result in this thesis yields, under some technical hypotheses, a method to solve the remaining cases. We shall provide here a brief idea of how it can be applied by ignoring the technical assumptions. For a precise statement of the result, we refer the reader to Corollary 1.4.7.

By [Tes88, Theorem (B)], we can select $\lambda \in X(T_Y)^+$ p-restricted such that X acts on $L_Y(\lambda)$ with at least two composition factors. Now $L_Y(\lambda)$ is generated by a maximal vector $v^+ \in L_Y(\lambda)$ for B_Y of weight λ. Since $B_X \subseteq B_Y$, we have that v^+ is a maximal vector for B_X of weight $\lambda|_{T_X}$. Hence $\lambda|_{T_X}$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Set $\mu = \lambda|_{T_X}$. Assume $\nu \in X(T_X)^+$ is maximal, with respect to the usual ordering on the weights, among the highest weights of the composition factors of $L_Y(\lambda)|_X/L_X(\mu)$. Then (under the omitted technical assumptions) X acts on $L_Y(\lambda)$ with more than two composition factors if and only if a weight of the form $\mu - \alpha|_{T_X}$ or $\nu - \alpha|_{T_X}$ for $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Here, α_0 denotes the largest root in $\Phi^+(Y)$.

In order to prove that X acts on $L_Y(\lambda)$ with exactly two composition factors, we prove that there is no weight of the form $\mu - \alpha|_{T_X}$ or $\nu - \alpha|_{T_X}$ for $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$ which affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. This method takes us back to a careful examination of the weights spaces of $L_Y(\lambda), L_X(\mu)$ and $L_X(\nu)$, but this time limited to a fixed range of weights.

Statement of results. — We now give an overview of the main results proved in this thesis, starting with the main theorem which settles Question 3, for large enough primes. The classification appears in Table A which can be found at the end of this thesis.

Theorem 1. *Let k be an algebraically closed field of characteristic $p > 0$. Let (X, Y, p) be as in Table 2 and Table 3. Assume in addition $(X, Y, p) \notin \{(A_2, G_2, \{2, 3\}), (B_4, F_4, \{2, 3, 5, 7, 11\}), (C_4, F_4, 2), (F_4, E_6, \{2, 3, 5, 7, 11\})\}$. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight. Then X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if λ is listed in Table A up to graph automorphism. Moreover, $L_Y(\lambda)|_X \cong L_X(\mu) \oplus L_X(\nu)$ with μ and ν given as in Table A.*

A few remarks are in order.

- This result is a combination of Proposition 2.5.1 and Proposition 3.0.1. In Proposition 2.5.1 we only consider the embedding $(X, Y) = (F_4, E_6)$ and in Proposition 3.0.1, all the other embeddings are treated. It is clear, when comparing the number of pages needed for the proof of both propositions, that the case of (F_4, E_6) is by far the most difficult one to solve.

- In the statement of Theorem 1, we exclude certain triples (X, Y, p). This is either because Theorem 1.1.10 does not hold for (Y, p) or because p is smaller than the Coxeter number of Y. Nevertheless, a complete answer can be obtained using a computer program. Indeed, for a fixed prime, it is in theory possible to compute the dimension of weight spaces in simple modules by calculating the rank modulo p of a bilinear form, the so-called contravariant form, introduced in [Ste16, Chapter 12] and in [Won72]. We have implemented such a program and by using it, we are able to deduce a complete answer for the excluded triples. However, we have decided not to include the result in this thesis.

- Even though Theorem 1 is obtained for an algebraically closed field of positive characteristic, it also holds for an algebraically closed field of characteristic 0. The cases that apply over an algebraically closed field of characteristic 0 are the ones in Table A which do not contain any dependence on p in the weight λ.

- In the course of proving Proposition 2.5.1, we compute precisely many "truncated" characters and weight multiplicities. The reader interested in this kind of data should have a look at Table 2.31 on Page 64 and Table 2.104 on Page 117.

We prove Proposition 2.5.1 by an inductive argument based on inclusions of Levi factors of parabolic subgroups. Solving two of these inductive steps yields the following propositions. Note that the same remarks as for Theorem 1 about the restrictions on p and the validity of the result in characteristic 0 also apply.

Proposition 2 (Proposition 2.2.1)**.** *Let k be an algebraically closed field of characteristic $p \geq 5$. Let Y be a simply connected simple algebraic group of type A_3 over k and let X be the maximal closed connected subgroup of type C_2 of Y given by the fixed points of a graph automorphism of Y. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight and set $\mu = \lambda|_{T_X}$. Then X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if λ is listed in Table* 2.3 *up to graph automorphism. Moreover, $L_Y(\lambda)|_X = L_X(\mu) \oplus L_X(\nu)$ with ν as in Table* 2.3.

Proposition 3 (Proposition 2.3.1)**.** *Let k be an algebraically closed field of characteristic $p \geq 7$. Let Y be a simply connected simple algebraic group of type A_5 over k and let X be the maximal closed connected subgroup of type C_3 of Y given by the fixed points of a graph automorphism of Y. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight and set $\mu = \lambda|_{T_X}$. Then X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if λ is listed in Table* 2.5 *up to graph automorphism. Moreover, $L_Y(\lambda)|_X = L_X(\mu) \oplus L_X(\nu)$ with ν as in Table* 2.5.

We decided to highlight these two propositions, since they represent the first steps towards solving Question 2 for the embedding of classical groups $Sp_{2n}(K) \subseteq SL_{2n}(K)$. Finally, we also prove a general result for this family of embeddings. The proof relies on the techniques we have introduced, illustrating how they can be used in a more general, i.e. unbounded rank, setting.

Proposition 4 (Proposition 2.3.2)**.** *Let $p > 2$ and $n \geq 2$. Let Y be of type A_{2n-1} and X be a subgroup of type C_n of Y given by the fixed points of a graph automorphism of Y. If $\lambda = a_n\lambda_n \in X(T_Y)^+$ with $a_n \in \{1, p-1\}$, then X acts on $L_Y(\lambda)$ with exactly two composition factors given by the highest weights $\lambda|_{T_X}$ and $\lambda|_{T_X} - \beta_{n-1} - \beta_n$, where $\{\beta_i\}_{1\leq i\leq n}$ denotes a basis of $\Phi(X)$. Moreover, $L_Y(\lambda)|_X \cong L_X(\lambda|_{T_X}) \oplus L_X(\lambda|_{T_X} - \beta_{n-1} - \beta_n)$.*

Use of the computer and external data. — This thesis contains many computations and we shall provide here an account of how these have been performed. All the computer assisted calculations were done using GAP [GAP18], Magma [BCP97] or in a negligible way Chevie.

In order to compute multiplicities in Weyl modules, we apply Proposition 1.1.12 on Page 8 along with a computer implementation in GAP of Freudenthal's formula or the function in Magma to compute multiplicities. Given two weights $\lambda, \mu \in X(T)^+$, these programs return the multiplicity of μ in the Weyl module of highest weight λ.

All the dimensions of Weyl modules are given by Weyl's degree formula and all the dimensions of irreducible modules are taken from the tables in [Lüb07].

We discuss now the calculations related to the JSF. For the notations, we refer to Section 1.3. We shall always work with the reformulation of the JSF given by Proposition 1.3.6 on Page 12. Computing the JSF by hand requires meticulous bookkeeping, a task which is prone to producing errors. We therefore implemented a computer program to keep track of all the intermediary computations. It works as follows. The user inputs the type of the group Y considered and a highest weight λ. The weight may be a function of the prime p and of some other parameters. For each positive root α in the root system of Y, the program prints the value of $\langle \lambda + \rho, \alpha^\vee \rangle$ and the user inputs the maximal integer which the summation index m can take (where m is as in (1.14) on Page 12). For each possible value of m, the user then reflects to a weight in D (see Page 11). Once such a weight is hit, the program outputs the weight in question, the difference in terms of simple roots between the dominant weight obtained and λ, and the determinant of the product of reflections which was applied. Note that each of these steps might depend on p and on the parameters involved in the definition of λ. It would be too long to detail these steps every time, however we include the final data obtained, so that the readers can verify the rest of the argument by themselves.

After Proposition 2.3.1 was established, the author's advisor received from Jantzen unpublished computations determining almost all of the characters of the p-restricted simple modules for types A_4, B_3 and C_3. These computations were not used by the author and all the methods used to compute the characters of simple modules appearing in this thesis come either from [Jan03, II. 8.20], from standard arguments or from case-by-case considerations.

Structure of this thesis. — **Chapter 1** focuses on developing the theory which will be used in order to solve Question 3. Section 1.1 contains an expository background about the representation theory of reductive algebraic group. Section 1.2 introduces the geometry of alcoves which constitute the framework in which we shall view the weights. It then discusses the Strong Linkage Principle and some of its consequences. In Section 1.3, we start by stating and reformulating Jantzen's p-sum formula (JSF) which is the main tool at our disposal to study the structure of Weyl modules. We explain precisely what are the limitations of the JSF and see how, in some cases, we manage to overcome them. In Subsection 1.3.2, we present a truncated version of the JSF along with an efficient way of algorithmically computing it. We conclude this lengthy and technical section by an example. Section 1.4 contains the core ideas to solve Question 3. We start by proving results about self-duality of modules. We then investigate in Proposition 1.4.4 which weights in the restriction of a simple kY-module can afford the weight of a maximal vector for B_X. In Proposition 1.4.6, we establish, under certain conditions, the existence of an additional maximal vector for B_X in the restriction of a simple KY-module. Finally, combining the previous results, we obtain in Corollary 1.4.7 the main method to solve Question 3. In Section 1.5, we introduce an inductive argument to answer Question 3 by considering Levi factors of parabolic subgroups.

In **Chapter 2**, we start by answering Question 2 for the pairs (C_2, A_3), (C_3, A_5) and partially investigate the pair (B_3, D_4). We then use the previous considerations to solve Question 3 for the pair (F_4, E_6) in Section 2.5.

In **Chapter 3**, we solve Question 3 for the remaining cases in Table 2 and Table 3. In Section 3.2 we consider the embeddings of maximal subgroups of non-maximal rank and in Section 3.3, the embeddings of maximal subgroups of maximal rank.

In **Appendix B**, we fix an ordering on the set of positive roots for a root system of type A_4, B_3 and C_3.

The classification obtained by solving Question 3 appears in Table A at the end of this thesis.

Chapter 1

Theoretical background

In this chapter, we start by recalling some well-known notions from the theory of algebraic groups which will be used in this thesis. For most of the concepts in Sections 1.1 and 1.2, we follow the exposition in [Jan03]. For Section 1.3, we follow the ideas presented in [McN98] and expanded further in [Cav15]. Based on these ideas, we develop our own approach in Subsection 1.3.2. In Section 1.4, we introduce the main theoretical methods we will repeatedly use in order to solve Question 3 and in Section 1.5, we present an inductive approach to solving Question 3 using inclusions of Levi factors of parabolic subgroups.

1.1 General notions

1.1.1 Characters and cocharacters. — Throughout this thesis, let k be an algebraically closed field of characteristic $p \geq 0$. Let $\mathbb{G}_m$ and $\mathbb{G}_a$ denote the multiplicative and additive group of k, respectively. Let G be a simple linear algebraic group over k. Henceforth, we assume all the groups to be linear. Fix a maximal torus $T \subseteq G$. By definition $T \cong \mathbb{G}_m^n$ for some n and we say G is of *rank* n. Set $\mathrm{rank}(G) = n$. Denote by $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, the group of characters of T and the group of cocharacters of T, respectively. The composition law on $X(T)$ and $Y(T)$ is written additively since both are abelian groups. Recall that for any $\lambda \in X(T)$ and $\phi \in Y(T)$, there is a unique integer $\langle \lambda, \phi \rangle$ such that the composition $\lambda \circ \phi$ is the map $\lambda \circ \phi : \mathbb{G}_m \longrightarrow \mathbb{G}_m$, where $\lambda \circ \phi(a) = a^{\langle \lambda, \phi \rangle}$. Moreover, recall that the pairing $\langle -, - \rangle : X(T) \times Y(T) \to \mathbb{Z}$ is bilinear and nondegenerate.

1.1.2 Root subgroups. — Let $\Phi \subseteq X(T)$ denote the root system of G with respect to T. Fix a set Δ of simple roots and recall that $|\Delta| = \mathrm{rank}(G)$. Let Φ^+ denote the set of positive roots with respect to Δ. Set $\Phi^- = -\Phi^+$. Let $\alpha_0 \in \Phi^+$ denote the largest root (i.e. the maximal long root) and let $\tilde{\alpha}_0$ denote the largest short root in Φ^+. Let $\alpha \in \Phi$. There exists, up to scalar, a unique morphism of algebraic groups $x_\alpha : \mathbb{G}_a \to G$ which induces an isomorphism onto its image, such that $t x_\alpha(a) t^{-1} = x_\alpha(\alpha(t)a)$ for all $t \in T,\ a \in \mathbb{G}_a$. We call $U_\alpha = \mathrm{im}(x_\alpha)$ the *root subgroup* corresponding to α. For $a \in \mathbb{G}_m$, set $n_\alpha(a) = x_\alpha(a) x_{-\alpha}(-a^{-1}) x_\alpha(a)$ and $\alpha^\vee(a) = n_\alpha(a) n_\alpha(1)^{-1}$. We have $n_\alpha(a) \in N_G(T)$ and $\alpha^\vee(a) \in T$. Note that $\alpha^\vee \in Y(T)$. We call $\alpha^\vee$ the *coroot* corresponding to

α and denote by $\Phi^\vee$ the set of coroots. It turns out that Φ along with the map $\alpha \longmapsto \alpha^\vee$ is an *abstract root system* in $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ in the sense of Bourbaki [Bou81, ch.VI, §1, n°1]. The set $\Phi^\vee$ along with the map $\alpha^\vee \longmapsto \alpha$ is also an abstract root system in $Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and for each $\alpha \in \Phi$, we have $\langle \alpha, \alpha^\vee \rangle = 2$.

1.1.3 Weyl group. — For each $\alpha \in \Phi$, denote by s_α the *reflection* corresponding to α on $X(T)$ given by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

We extend s_α linearly to a reflection on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Let the Weyl group of Φ, denoted W, be the subgroup of $GL(X(T) \otimes_{\mathbb{Z}} \mathbb{R})$ generated by the reflections s_α for $\alpha \in \Delta$. There is an explicit relationship between W and the group G we started with. The action by conjugation of $g \in N_G(T)$ on T induces an action on $X(T)$ and on $Y(T)$ as $g.\lambda(t) = \lambda(g^{-1}tg)$ for $\lambda \in X(T)$ and $t \in T$, and as $g.\phi(c) = g\phi(c)g^{-1}$ for $\phi \in Y(T)$ and $c \in \mathbb{G}_m$, respectively. The element $n_\alpha(a) \in N_G(T)$ defined previously acts on $X(T)$ in the same way as s_α, inducing the following isomorphism.

$$\begin{array}{ccccc} N_G(T) & \longrightarrow & N_G(T)/T & \cong & W \\ n_\alpha(1) & \longmapsto & \bar{n}_\alpha(1) & \longmapsto & s_\alpha \end{array}$$

Recall that for $w \in W$ and any representative $\dot{w} \in N_G(T)$, we have

$$\dot{w} U_\alpha \dot{w}^{-1} = U_{w\alpha}. \tag{1.1}$$

For $w \in W$, write $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_t}}$ with $\alpha_{i_r} \in \Delta$ and t minimal. We call t the *length* of w. There exists a unique element $w_0 \in W$ such that the length of w_0 is maximal. It is also the unique element in W satisfying $w_0(\Phi^+) = \Phi^-$.

1.1.4 Parabolic subgroups. — A subset $\Phi' \subseteq \Phi$ is called *closed* if $(\mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta) \cap \Phi \subseteq \Phi'$ for any $\alpha, \beta \in \Phi'$. It is called *unipotent* if $\Phi' \cap (-\Phi') = \emptyset$. For $\Phi' \subseteq \Phi$ closed and unipotent, denote by $U(\Phi')$ the closed subgroup generated by all U_α with $\alpha \in \Phi'$. For example, if $I \subseteq \Delta$ and $\Phi_I = \Phi \cap \mathbb{Z}I$, then $\Phi^+ \setminus \Phi_I$ is closed and unipotent. Let $U = U(\Phi^+)$, $U^- = U(\Phi^-)$ and set $B = UT$ to be a *(positive) Borel subgroup of* G and $B^- = U^-T$ to be the corresponding *negative Borel subgroup of* G.

Remark 1.1.1. In [Jan03], the opposite convention is used, that is B denotes a negative Borel subgroup.

For $I \subseteq \Delta$, set

$$L_I = \langle T,\, U_\alpha,\, U_{-\alpha} \,:\, \alpha \in I \rangle, \quad U_I = U(\Phi^+ \setminus \Phi_I).$$

We call $P_I = U_I L_I$ the *standard parabolic* subgroup of G corresponding to I, where U_I is the unipotent radical of P_I and the subgroup L_I is called the *standard Levi factor* of P_I. Note that L_I is a connected reductive subgroup of G with maximal torus T, Borel subgroup $B_I = L_I \cap B$ and root system Φ_I. For more details, we refer to [Spr98, 8.4.1]. Let us denote by L_I' the derived subgroup of L_I which is equal to $\langle U_\alpha, U_{-\alpha} \,:\, \alpha \in I \rangle$.

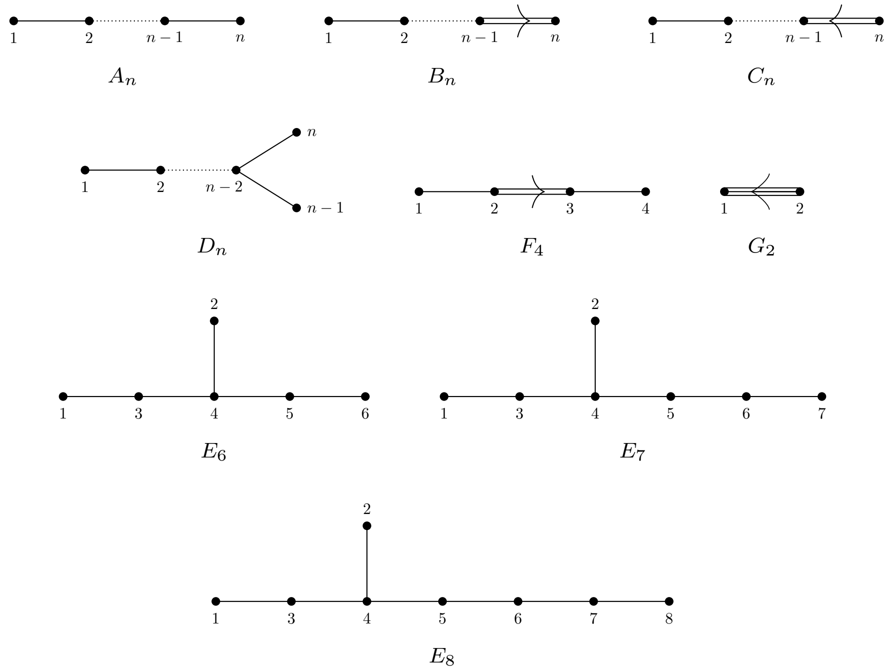


Figure 1.1: Labelled Dynkin diagrams

1.1.5 Dynkin Diagrams. — The labelling of the Dynkin diagrams in Fig. 1.1 fixes the ordering on the set of simple roots which we will adopt in this thesis.

1.1.6 Weight lattice. — Since $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by Φ, the set Δ is a basis of $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\{\alpha^\vee \mid \alpha \in \Delta\}$ is a basis of $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. To each $\alpha_i \in \Delta$, we associate an element $\lambda_i \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all j. We call $\{\lambda_i\}_{1 \leq i \leq n}$ the set of *fundamental weights.* We say G is *simply connected* if $\lambda_i \in X(T)$ for all i, in which case $\{\lambda_i\}_{1 \leq i \leq n}$ is a $\mathbb{Z}$-basis of $X(T)$.

From now on assume that G is simply connected.

The change of basis matrix from the fundamental weights to the simple roots is given by the Cartan matrix, an integer matrix. However, the inverse matrix has coefficients in $\mathbb{Q}$, therefore $X(T) \nsubseteq \mathbb{Z}\Phi$ in general. We call the elements of $X(T)$ *weights.* Let $\lambda = \sum_{i=1}^n a_i \lambda_i, a_i \in \mathbb{Z}$ be a weight. We call λ *dominant* if $a_i \geq 0$ for all i and denote the set of all dominant weights by $X(T)^+$. For $p > 0$, we call a dominant weight *p-restricted* if $a_i < p$ for all i and for $p = 0$, a weight is p-restricted if it is dominant. Moreover, for $\lambda, \mu \in X(T)$, we write $\lambda \succeq \mu$, if $\lambda - \mu = \sum_{i=1}^n b_i \alpha_i$ with $b_i \in \mathbb{Z}_{\geq 0}$ for all i. Note that $(X(T), \succeq)$ is a partially ordered set.

1.1.7 Lie algebra of G. — Let $\mathcal{D}$ denote the algebra of derivations of the affine k-algebra of G equipped with the natural G-action on the left and the right. Let $\mathscr{L}(G)$ be the subalgebra of $\mathcal{D}$ made up of the left invariant derivations. The algebra $\mathscr{L}(G)$ along with the Lie bracket

$[D, D'] = DD' - D'D$ is a Lie algebra (c.f. [Spr98, 4.4]). We call $\mathscr{L}(G)$ the *Lie algebra of* G and as in [Spr98, 4.8], we identify it with the tangent space of G at the identity.

1.1.8 Rational kG-modules. — A morphism ϕ of algebraic groups $\phi : G \longrightarrow \mathrm{GL}(V)$ with V a finite dimensional vector space over k is called a *rational representation* of G. Similarly, a kG-module is called *rational* if its corresponding representation is. Throughout this thesis, we assume all the representations of G and the kG-modules to be rational.

Let V be a kG-module and let $V = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = 0$ be a composition series of V. The simple quotients V_i/V_{i+1} for $0 \leq i \leq r-1$, which do not depend on the choice of the filtration by the Jordan-Hölder theorem, are called the *composition factors* of V. We denote by $L(\theta_1)^{m_1}/\cdots/L(\theta_s)^{m_s}$ or simply by $\theta_1^{m_1}/\cdots/\theta_s^{m_s}$, the set of composition factors of V, where $\theta_i \in X(T)^+$ and m_i is the number of times $L(\theta_i)$ appears as a composition factor of V.

Recall that V decomposes as a kT-module as follows

$$V = \bigoplus_{\lambda \in X(T)^+} V_\lambda,$$

where $V_\lambda = \{v \in V \mid tv = \lambda(t)v,\ \forall\, t \in T\}$. We say λ is a weight of V, if $V_\lambda \neq \{0\}$ and we call V_λ the *weight space* of λ. Denote the set of weights of V by $\Lambda(V)$ and the subset of $\Lambda(V)$ consisting of dominant weights by $\Lambda(V)^+$. The dimension $\dim V_\lambda$ for $\lambda \in \Lambda(V)$ is called the *multiplicity* of λ in V and is denoted by $m_V(\lambda)$. Recall that the radical of V, denoted $\operatorname{rad} V$, is the smallest submodule W of V such that V/W is semisimple.

The next classical results describes how root subgroups act on weight spaces. A proof of the lemma can be found in [MT11, Lemma 15.4].

Lemma 1.1.2. *Let V be a kG-module, $\alpha \in \Phi$ and $\gamma \in \Lambda(V)$. Then for all $v \in V_\gamma$, we have*

$$U_\alpha v \subseteq v + \sum_{m \in \mathbb{Z}_{>0}} V_{\gamma + m\alpha}.$$

1.1.9 Weyl modules. — In order to construct a class of kG-modules called Weyl modules, we temporarily assume $\operatorname{char} k = 0$. We follow the exposition in [Hum00, Chapter VII]. The decomposition into weight spaces for the adjoint representation of $\mathscr{L}(G)$ is given by

$$\mathscr{L}(G) = \bigoplus_{\alpha \in \Phi} \mathscr{L}(G)_\alpha \oplus \mathscr{L}(T).$$

Let $\mathcal{U}$ denote the *universal enveloping algebra* of $\mathscr{L}(G)$. Recall that the category of finite dimensional representations of $\mathscr{L}(G)$ is semisimple, with simple objects parametrized by the elements of $X(T)^+$. We denote by $V_{\mathscr{L}(G)}(\lambda)$ the irreducible representation of $\mathscr{L}(G)$ of highest weight $\lambda \in X(T)^+$. Fix $\lambda \in X(T)^+$ and set $V = V_{\mathscr{L}(G)}(\lambda)$. Recall that V is generated by a maximal vector v^+ of weight λ, that is $V = \mathcal{U}v^+$. Denote by κ the Killing form of $\mathscr{L}(G)$. Let $\alpha \in \Phi$ and denote by $t_\alpha \in \mathscr{L}(T)$ the unique elements in $\mathscr{L}(T)$ such that $\alpha(t) = \kappa(t_\alpha, t)$ for all $t \in \mathscr{L}(T)$. Set $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. It is possible to choose $(e_\alpha, e_{-\alpha}) \in \mathscr{L}(G)_\alpha \times \mathscr{L}(G)_{-\alpha}$ such that

- $[e_\alpha, e_{-\alpha}] = h_\alpha$.

- If $\beta \in \Phi$, $\alpha + \beta \in \Phi$, $[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}$, then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$.

By definition, the set $\{e_\alpha, e_{-\alpha}, h_\beta \mid \alpha \in \Phi^+ \text{ and } \beta \in \Delta\}$ is a *Chevalley basis* of $\mathscr{L}(G)$ [Hum00, 25.2]. For $\alpha \in \Phi^+$, we denote $e_{-\alpha}$ by f_α. Let $\mathscr{L}(G)_\mathbb{Z}$ denote the lattice in $\mathscr{L}(G)$ generated by this Chevalley basis. Fix an ordering $(\alpha_1, \dots, \alpha_m)$ of Φ^+, such that $\{\alpha_1, \dots, \alpha_n\} = \Delta$. For any sequences $A = (a_1, \dots, a_m), B = (b_1, \dots, b_m) \subseteq \mathbb{Z}^m_{\geq 0}$ and $C = (c_1, \dots, c_n) \subseteq \mathbb{Z}^n_{\geq 0}$, let E_A, F_B, H_C denote the following elements of $\mathcal{U}$

$$\begin{aligned} E_A &= \frac{e_{\alpha_1}^{a_1}}{a_1!} \cdots \frac{e_{\alpha_m}^{a_m}}{a_m!}, \\ F_B &= \frac{f_{\alpha_1}^{b_1}}{b_1!} \cdots \frac{f_{\alpha_m}^{b_m}}{b_m!}, \\ H_C &= \binom{h_{\alpha_1}}{c_1} \cdots \binom{h_{\alpha_n}}{c_n}, \end{aligned}$$

where $\binom{h_{\alpha_i}}{c_i} = \frac{h_{\alpha_i}(h_{\alpha_i}-1)\cdots(h_{\alpha_i}-c_i+1)}{c_i!}$. Using the PBW Theorem [Hum00, 17.3], we deduce that the set of all the products $E_A F_B H_C$ together with 1 form a basis of $\mathcal{U}$. Denote by $\mathcal{U}_\mathbb{Z}$ the lattice in $\mathcal{U}$ with this basis and by $\mathcal{U}_\mathbb{Z}^-$ the subring of $\mathcal{U}$ generated by all F_B together with 1. Then $\mathcal{U}_\mathbb{Z}^- v^+$ is a lattice in V which is invariant under $\mathcal{U}_\mathbb{Z}$. In other words, we view the $\mathcal{U}$-module V as a $\mathcal{U}_\mathbb{Z}$-module using this lattice. For an algebraically closed field K of any characteristic, let $M(K) = M \otimes_\mathbb{Z} K$ and $\mathscr{L}(G)_\mathbb{Z}(K) = \mathscr{L}(G)_\mathbb{Z} \otimes_\mathbb{Z} K$. It is clear that $M(K)$ is an $\mathscr{L}(G)_\mathbb{Z}(K)$-module.

Let again $\operatorname{char} k \geq 0$. The algebraic group G can be viewed as a Chevalley group as constructed in [Ste16, Chapter 3]. With this point of view, $M(k)$ becomes a kG-module called the *Weyl module of highest weight* λ, denoted $V_G(\lambda)$. We denote $\Lambda(V_G(\lambda))$ by $\Lambda(\lambda)$ and for $\mu \in X(T)$, the multiplicity $m_{V_G(\lambda)}(\mu)$ by $m_\lambda(\mu)$.

Definition 1.1.3. Let V be a kG-module. We say $v \in V$ is a *maximal vector* for B of weight $\lambda \in X(T)^+$, if $v \in V_\lambda \setminus \{0\}$ and B stabilizes $\langle v \rangle$.

We refer to [Jan03, II.2.13] for more details about the following universal property of Weyl modules.

Proposition 1.1.4. *Any kG-module generated by a maximal vector for B of weight $\lambda \in X(T)^+$ is a homomorphic image of $V_G(\lambda)$.*

1.1.10 Simple modules. — Let V be a simple kG-module. By the Lie-Kolchin theorem [Hum91, 17.6], there exists a maximal vector $v \in V$ for B of weight $\lambda \in X(T)^+$. By simplicity, V is generated by v as a kG-module, and by Lemma 1.1.2, the weight λ is maximal in $\Lambda(V)$. In fact, the set of dominant weights $X(T)^+$ is in bijection with the set of isomorphism classes of simple kG-modules. That is for $\lambda \in X(T)^+$, there exists up to isomorphism a unique simple kG-module generated by a maximal vector for B of weight λ. We call this module the *simple kG-module of highest weight* λ and denote it by $L_G(\lambda)$. The next result relates the previous construction with Proposition 1.1.4, for more details see [Jan03, II.2.13].

Proposition 1.1.5. *For $\lambda \in X(T)^+$,*

$$V_G(\lambda)/\operatorname{rad} V_G(\lambda) \cong L_G(\lambda).$$

Remark 1.1.6. When no confusion is possible, we drop the subscript indicating the group considered in the notation of the representations, e.g. we denote $V_G(\lambda)$ and $L_G(\lambda)$ by $V(\lambda)$ and $L(\lambda)$, respectively.

The next proposition describes the extensions between two simple kG-modules. We refer to [Jan03, II.2.12 and II.2.14] for more details about how this group is defined and for a proof of the proposition.

Proposition 1.1.7. *Let $\lambda, \mu \in X(T)^+$, then*

1) $\mathrm{Ext}^1_G(L(\lambda), L(\lambda)) = 0$.

2) $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\mu), L(\lambda))$.

3) If $\mu \not\succeq \lambda$, then $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(\mathrm{rad}_G V(\lambda), L(\mu))$.

In this thesis, we will mostly be considering simple modules with p-restricted highest weight. The next theorem due to Steinberg in [Ste63, Theorem 1.1] justifies this choice. In particular, it implies that if we know the dimensions of all the simple modules with p-restricted highest weight or the dimensions of the weight spaces of all the p-restricted simple modules, then we have this information for all the simple modules. For more details about the next theorem, see [MT11, Theorem 16.12].

Theorem 1.1.8 (Steinberg's tensor product theorem)**.** *Let $\lambda \in X(T)^+$ and write $\lambda = \sum_{i=0}^m p^i \varpi_i$ with $\varpi_i \in X(T)^+$, p-restricted. Then*

$$L(\lambda) \cong L(\varpi_0) \otimes L(\varpi_1)^{(p)} \otimes \cdots \otimes L(\varpi_m)^{(p^m)},$$

where $L(\varpi_i)^{(p^i)}$ stands for the kG-module obtained by precomposing the irreducible representation of highest weight λ_i with the i^{th} power of the Frobenius endomorphism.

The next result tells us that irreducible representations with p-restricted highest weight behave well with respect to taking their differential. Initially due to Curtis in [Cur60], we refer the reader to [Jan03, II.3.15], where the theorem is stated and proved in the more general framework of Frobenius kernels.

Theorem 1.1.9 (Curtis)**.** *Let $\lambda \in X(T)^+$. If λ is p-restricted, then the simple kG-module $L_G(\lambda)$ is simple as a $k\mathscr{L}(G)$-module.*

The following result due to Premet in [Pre88, Theorem 1] characterizes the weights of a simple module under certain weak assumptions on the characteristic.

Theorem 1.1.10 (Premet)**.** *Let G be simple and let $\lambda \in X(T)^+$. Assume that $(G, p) \notin \{(B_n, 2), (C_n, 2), (F_4, 2), (G_2, 2), (G_2, 3)\}$. If λ is p-restricted, then $\Lambda(L(\lambda)) = \Lambda(\lambda)$.*

1.1.11 Characters of modules. — Let $\mathbb{Z}[X(T)]$ denote the group ring of $X(T)$ with standard basis given by $\{e(\lambda)\}_{\lambda \in X(T)}$. Let V be a kG-module and define the *formal character* of V as

$$\mathrm{ch}\, V = \sum_{\theta \in \Lambda(V)} m_V(\theta) e(\theta).$$

The action of W on $X(T)$ induces an action of W on $\mathbb{Z}[X(T)]$. Recall that for $w \in W$ and $\theta \in \Lambda(V)$, we have $m_V(w\theta) = m_V(\theta)$, hence $\operatorname{ch} V \in Z[X(T)]^W$. For $\lambda \in X(T)$, denote by k_λ the one-dimensional kB^--module on which B^- acts via λ, i.e. U^- acts trivially and T acts via λ. Let $H^i(\lambda) = R^i \operatorname{ind}_{B^-}^G(k_\lambda)$, where $R^i \operatorname{ind}_{B^-}^G$ is the i^{th} right derived functor of the induction from B^- to G, see [Jan03, I.3.3] for more details. Set

$$\chi(\lambda) = \sum_{i\geq 0}(-1)^i \operatorname{ch} H^i(\lambda).$$

If λ is dominant, then by Kempf's vanishing theorem, we get that $\chi(\lambda) = \operatorname{ch} V(\lambda)$, the character of the Weyl module of highest weight λ. For a precise statement of Kempf's vanishing theorem, see [Jan03, II.4.5]. Recall that $\{\chi(\theta)\}_{\theta\in X(T)^+}$ and $\{\operatorname{ch} L(\theta)\}_{\theta\in X(T)^+}$ are two $\mathbb{Z}$-bases of $\mathbb{Z}[X(T)]^W$. In particular, for $\lambda \in X(T)^+$

$$\chi(\lambda) = \sum_{\theta\in X(T)^+} a_{\lambda,\theta} \operatorname{ch} L(\theta) \tag{1.2}$$

with $a_{\lambda,\theta} \in \mathbb{Z}_{\geq 0}$ and

$$\operatorname{ch} L(\lambda) = \sum_{\theta\in X(T)^+} b_{\lambda,\theta}\chi(\theta) \tag{1.3}$$

with $b_{\lambda,\theta} \in \mathbb{Z}$. Let $[V_G(\lambda) : L_G(\theta)]$ or $[\lambda : \theta]$ denote the coefficient $a_{\lambda,\theta}$ in (1.2) and similarly let $(L_G(\lambda) : V_G(\theta))$ or $(\lambda : \theta)$ denote the coefficient $b_{\lambda,\theta}$ in (1.3). Let $S \subseteq X(T)$ be a finite set. We define the *S-truncated character* of V as

$$\operatorname{ch} V^S = \sum_{\theta\in S} m_V(\theta)e(\theta).$$

Fix $\lambda \in X(T)^+$ and define the following condition for S and λ.

Condition 1.1.11.

- *The set S satisfies $S \subseteq X(T)^+$.*
- *For $\mu \in S$, if $\theta \in X(T)^+$ and $\mu \preceq \theta \preceq \lambda$, then $\theta \in S$.*

Let $S \subseteq X(T)$ and assume Condition 1.1.11 holds for S and λ, then

$$\chi(\lambda)^S = \sum_{\theta\in S} m_\lambda(\theta)e(\theta) = \sum_{\theta\in S} a_{\lambda,\theta} \operatorname{ch} L(\theta)^S \tag{1.4}$$

with $a_{\lambda,\theta}$ as in (1.2), where the first equality holds by definition of a truncated character and the second equality holds since S satisfies Condition 1.1.11 and since if $\mu, \theta \in X(T)^+$ with $\mu \not\preceq \theta$, then $m_\theta(\mu) = 0$. Similarly,

$$\operatorname{ch} L(\lambda)^S = \sum_{\theta\in S} m_{L(\lambda)}(\theta)e(\theta) = \sum_{\theta\in S} b_{\lambda,\theta}\chi(\theta)^S \tag{1.5}$$

with $b_{\lambda,\theta}$ as in (1.3). For notational simplicity, we will sometimes abbreviate (1.4) by

$$\chi(\lambda)^S = \sum_{\theta\in S} a_{\lambda,\theta}\theta, \tag{1.6}$$

and (1.5) by

$$\operatorname{ch} L(\lambda)^S = \sum_{\theta\in S} b_{\lambda,\theta}\theta. \tag{1.7}$$

1.1.11.1 Multiplicities in a Weyl module. — We use Freudenthal's formula to compute multiplicities in Weyl modules. Most of the weights we consider are given by a linear combination of fundamental weights whose coefficients are parameters which can be set to a wide range of values. We therefore need the following result from [Cav17a, Proposition A] which provides a bound on the values of the parameters for the multiplicity to become uniform. That is, we only need to apply Freudenthal's formula for given values of the parameters.

Proposition 1.1.12 (Cavallin). *Let $\lambda = \sum_{i=1}^{n} a_i\lambda_i \in X(T)^+$ be a dominant weight and let $\mu \in X(T)$ be such that $\mu = \lambda - \sum_{i=1}^{n} c_i\alpha_i$, for some $c_1, \cdots, c_n \in \mathbb{Z}_{\geq 0}$, so that $\mu \preceq \lambda$. Also assume the existence of a non-empty subset J of $\{1, \dots, n\}$ such that $0 \leq c_j \leq a_j$ for every $j \in J$ and set $\lambda' = \sum_{i \notin J} a_i\lambda_i + \sum_{i \in J} c_i\lambda_i$, $\mu' = \lambda' - \sum_{i=1}^{n} c_i\alpha_i$. Then*

$$m_\lambda(\mu) = m_{\lambda'}(\mu').$$

1.1.11.2 Multiplicities in an irreducible module. — Let $\lambda \in X(T)^+$ and $\mu \in \Lambda(\lambda)$. To compute $m_{L(\lambda)}(\mu)$, we need to compute (1.5) with $S = \{\theta \in X(T)^+ \mid \lambda \succeq \theta \succeq \mu\}$. Then

$$m_{L(\lambda)}(\mu) = \sum_{\theta \in S} b_{\lambda,\theta} m_\theta(\mu),$$

and we apply the previous paragraph to compute the multiplicities in the Weyl modules.

1.2 Alcove geometry

1.2.1 Affine Weyl group. — The *affine Weyl group* (associated to G and p), denoted by W_p, is the subgroup of $AGL(X(T) \otimes_{\mathbb{Z}} \mathbb{R})$ generated by all the affine reflections $s_{\alpha,r}$, for $\alpha \in \Phi$ and $r \in \mathbb{Z}$, where

$$s_{\alpha,r}(\lambda) = s_\alpha(\lambda) + rp\alpha,$$

for $\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Alternatively $W_p \cong p\mathbb{Z}\Phi \rtimes W$ with $p\mathbb{Z}\Phi$ acting on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ by translation. Let ρ denote the half-sum of positive roots or equivalently the sum of fundamental weights. From now on, we let the affine Weyl group act on $X(T)$ via the *dot action*, namely

$$w \bullet \lambda = w(\lambda + \rho) - \rho$$

for $w \in W_p$ and $\lambda \in X(T)$.

1.2.2 Alcoves. — For each tuple $(n_\alpha)_{\alpha \in \Phi^+} \in \mathbb{Z}^{|\Phi^+|}$, we associate a subset C of $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ defined by

$$C = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p,\ \forall \alpha \in \Phi^+\}.$$

If C is non-empty, we call C an *alcove*. The upper closure and the closure of C are given, respectively, by

$$\widehat{C} = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle \leq n_\alpha p,\ \forall \alpha \in \Phi^+\},$$
$$\overline{C} = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid (n_\alpha - 1)p \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq n_\alpha p,\ \forall \alpha \in \Phi^+\}.$$

We have that W_p acts simply transitively on the set of alcoves and the closure of any alcove is a fundamental domain for W_p acting on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by C_0 the alcove associated to the tuple $(1, 1, \dots, 1)$, that is

$$C_0 = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p, \ \forall \alpha \in \Phi^+\}.$$

We call C_0 the *fundamental alcove*. Define the Coxeter number of Φ, denoted by h, to be equal to $\max\{\langle \rho, \alpha^\vee \rangle + 1 \mid \alpha \in \Phi^+\}$, namely $h = \langle \rho, \tilde{\alpha}_0^\vee \rangle + 1$. Indeed, if $\alpha^\vee = \sum_{i=1}^n b_i \alpha_i^\vee$, then

$$\langle \rho, \alpha^\vee \rangle = \sum_{i=1}^n b_i \langle \rho, \alpha_i^\vee \rangle = \sum_{i=1}^n b_i, \tag{1.8}$$

and a case-by-case verification shows that the value of (1.8) is maximal when $\alpha^\vee$ is the largest root in the dual root system, that is when $\alpha = \tilde{\alpha}_0$. Therefore $C \cap X(T) \neq \emptyset$ for any alcove C if and only if $C_0 \cap X(T) \neq \emptyset$ if and only if $0 \in C_0$ if and only if $p \geq h$.

For $\alpha \in \Phi^+$ and $m \in \mathbb{Z}$, the set

$$F_{\alpha,m} = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \alpha^\vee \rangle = mp\},$$

is called *a wall*. For C an alcove, we say $F_{\alpha,m}$ is a wall of C if $\overline{C} \cap F_{\alpha,m} \neq \emptyset$. We associate to a wall $F_{\alpha,m}$ the reflection $s_F = s_{\alpha,m}$. For an alcove C, denote by $\Sigma(C)$ the set of all reflections s_F, where F is a wall of C. Observe that

$$\Sigma(C_0) = \{s_{\alpha_i}, \alpha_i \in \Delta\} \cup \{s_{\tilde{\alpha}_0,1}\}.$$

We denote s_{α_i} by s_i and $s_{\tilde{\alpha}_0,1}$ by s_0.

1.2.3 Linkage principle. — For $\lambda, \mu \in X(T)$, we say μ is *linked* to λ if there exists a sequence of affine reflections $s_{\beta_1,r_1}, \dots, s_{\beta_t,r_t} \in W_p$ with $\beta_i \in \Phi^+$ such that

$$\mu \preceq s_{\beta_1,r_1} \bullet \mu \preceq \cdots \preceq s_{\beta_t,r_t} \cdots s_{\beta_1,r_1} \bullet \mu = \lambda,$$

or if $\mu = \lambda$.

The next crucial proposition was proven by Andersen in [And80], for more details about the proof see [Jan03, II.6.13].

Proposition 1.2.1 (The Strong Linkage Principle)**.** *Let $\lambda, \mu \in X(T)^+$. If*

$$[V(\lambda) : L(\mu)] \neq 0,$$

then μ is linked to λ.

The next result gives a numerical condition to verify the linkage relation. It appears in [Sei87, (6.2)] in the framework of modules with 1-dimensional weight spaces, but it holds in our setting too. We normalize the inner product $(-,-)$ on $X(T) \otimes \mathbb{R}$, so that the long roots have length 1.

Proposition 1.2.2 (Seitz)**.** *Let G be simple and $\lambda, \mu \in X(T)^+$. Assume that $p > 2$ and that $p > 3$ if $G = G_2$. Write $\mu = \lambda - \sum_{i=1}^n c_i \beta_i$, where each $c_i \geq 0$. If μ is linked to λ, then*

$$2\left(\lambda + \rho, \sum_{i=1}^n c_i \beta_i\right) - \left(\sum_{i=1}^n c_i \beta_i, \sum_{i=1}^n c_i \beta_i\right) \in \begin{cases} (p/2)\mathbb{Z} & \text{if } G \neq G_2 \\ (p/6)\mathbb{Z} & \text{if } G = G_2. \end{cases}$$

The following two results are easy corollaries of Proposition 1.2.1. The first corollary follows from the fact that the change of basis between the bases $\{\operatorname{ch} L(\theta)\}$ and $\{\chi(\theta)\}$ of $\mathbb{Z}[X(T)^+]^W$ is unitriangular.

Corollary 1.2.3. *Let $\lambda, \mu \in X(T)^+$. If $(L(\lambda) : V(\mu)) \neq 0$, then μ is linked to λ.*

Proof. We prove the corollary by induction on $|\Lambda(\lambda) \cap X(T)^+|$. If $|\Lambda(\lambda) \cap X(T)^+| = 1$, then λ is a minuscule weight and the result is clear. Assume $|\Lambda(\lambda) \cap X(T)^+| \geq 2$. Note that the result holds by induction for any $\nu \in X(T)^+$ with $\nu \precneq \lambda$, since $\Lambda(\nu) \subsetneq \Lambda(\lambda)$. Let

$$\chi(\lambda) = \operatorname{ch} L(\lambda) + \sum_{\nu \in X(T)^+ \setminus \{\lambda\}} a_{\lambda,\nu} \operatorname{ch} L(\nu), \tag{1.9}$$

with $a_{\lambda,\nu} \in \mathbb{Z}_{\geq 0}$ and

$$\operatorname{ch} L(\nu) = \sum_{\mu \in X(T)^+} b_{\nu,\mu} \chi(\mu), \tag{1.10}$$

with $b_{\nu,\mu} \in \mathbb{Z}$. Substituting (1.10) in (1.9) and rearranging the terms, we get

$$\operatorname{ch} L(\lambda) = \chi(\lambda) - \sum_{\mu \in X(T)^+} \left(\sum_{\nu \in X(T)^+ \setminus \{\lambda\}} a_{\lambda,\nu} b_{\nu,\mu} \right) \chi(\mu).$$

By Proposition 1.2.1, if $a_{\lambda,\nu} \neq 0$, then ν is linked to λ. Moreover the induction hypothesis implies that if $b_{\nu,\mu} \neq 0$, then μ is linked to ν. Note that being linked is transitive. Therefore if $a_{\lambda,\nu} b_{\nu,\mu} \neq 0$, then μ is linked to λ, which finishes the proof. $\square$

Corollary 1.2.4. *Let $\lambda \in X(T)^+$. Assume $\lambda \in \widehat{C}_0$, then $V(\lambda)$ is irreducible.*

Proof. Let $\lambda \in X(T)^+ \cap \widehat{C}_0$. Assume $\mu \in X(T)^+$ is such that $[V(\lambda) : L(\mu)] \neq 0$. Then $\mu \preceq \lambda$ and so by the geometry of alcoves $\mu \in X(T)^+ \cap \widehat{C}_0$. Moreover, by Proposition 1.2.1, we have $\mu \in W_p \bullet \lambda$. Since W_p acts simply transitively on $\{\overline{C} \mid C \text{ is an alcove}\}$, we get $\mu = \lambda$ and $V(\lambda)$ is irreducible. $\square$

The proof of the following proposition can be found in [Jan03, II.6.24].

Proposition 1.2.5. *Let $\lambda \in X(T)^+$. Suppose that $\mu \in X(T)$ is maximal for the property of being linked to λ. If $\mu \in X(T)^+$ and $\mu \notin \{\lambda - p\alpha \mid \alpha \in \Phi^+\}$, then*

$$[V(\lambda) : L(\mu)] = 1.$$

1.3 The Jantzen p-sum formula

The Jantzen p-sum formula, abbreviated JSF, is a powerful tool for studying the composition factors of Weyl modules. As we will see its limitations can be mitigated by introducing a truncated version of the sum formula and whenever $p \geq h$ by using results from the theory of translation functors.

Let $\nu_p(z)$ for $z \in \mathbb{Z}_{>0}$ denote the *p-adic valuation* of z, that is the exponent of the highest power of p dividing z. See [Jan03, II.8] for a proof of the next theorem.

Theorem 1.3.1 (Jantzen p-sum formula)**.** *For each $\lambda \in X(T)^+$ there is a filtration of kG-modules*

$$V(\lambda) = V(\lambda)^0 \supseteq V(\lambda)^1 \supseteq V(\lambda)^2 \supseteq \cdots, \tag{1.11}$$

such that

$$\sum_{i>0} \operatorname{ch} V(\lambda)^i = \sum_{\alpha\in\Phi^+} \sum_{0<mp<\langle\lambda+\rho,\alpha^\vee\rangle} \nu_p(mp)\chi(s_{\alpha,mp}\bullet\lambda) \tag{1.12}$$

and

$$V(\lambda)/V(\lambda)^1 \cong L(\lambda). \tag{1.13}$$

For $\lambda \in X(T)^+$, denote the right-hand side of (1.12) by $\operatorname{JSF}(\lambda)$, that is

$$\operatorname{JSF}(\lambda) = \sum_{\alpha\in\Phi^+} \sum_{0<mp<\langle\lambda+\rho,\alpha^\vee\rangle} \nu_p(mp)\chi(s_{\alpha,mp}\bullet\lambda).$$

The reader should keep in mind the following two remarks. In the first one, we explain one of the benefits of assuming $p \geq h$ and in the second one, we deduce a straightforward formula for $\operatorname{ch} L(\lambda)$ in terms of the characters of the modules $V(\lambda)^i$ appearing in (1.11).

Remark 1.3.2. Assume $\lambda \in X(T)^+$ is a p-restricted weight. For $\alpha \in \Phi^+$, we have

$$\langle\lambda+\rho,\alpha^\vee\rangle \leq p\sum_{i=1}^{n}\langle\lambda_i,\alpha^\vee\rangle = p\langle\sum_{i=1}^{n}\lambda_i,\alpha^\vee\rangle = p\langle\rho,\alpha^\vee\rangle \leq p(h-1).$$

Therefore, if $p \geq h$, then the p-adic valuation in (1.12) is always equal to 1.

Remark 1.3.3. By (1.13), we have

$$\operatorname{ch} L(\lambda) = \chi(\lambda) - \operatorname{ch} V(\lambda)^1 = \chi(\lambda) - \operatorname{JSF}(\lambda) + \sum_{i>1} \operatorname{ch} V(\lambda)^i,$$

since $\operatorname{ch} V(\lambda)^1 = \operatorname{JSF}(\lambda) - \sum_{i>1} \operatorname{ch} V(\lambda)^i$.

Recall that $X(T)^+$ is a fundamental domain for the action of W on $X(T)$. Let $D = \{\lambda \in X(T) \mid \lambda+\rho \in X(T)^+\}$, then D is a fundamental domain for the dot action of W on $X(T)$. In the next lemma, we summarize some properties of the Weyl group and the affine Weyl group acting via the dot action on $X(T)$.

Lemma 1.3.4. *Let $\lambda \in X(T)$.*

1) $\chi(w\bullet\lambda) = \det(w)\chi(\lambda)$, $\forall w \in W$.

2) If $\lambda \in D \setminus X(T)^+$, then $\chi(\lambda) = 0$.

3) For $\alpha \in \Phi^+$ and $r \in \mathbb{Z}$, $s_{\alpha,r}\bullet\lambda = s_\alpha\bullet(\lambda - r\alpha)$.

Proof. The first assertion is proved in [Jan03, II.5.9]. If $\lambda \in D \setminus X(T)^+$, there exists $\alpha \in \Delta$ with $\langle\lambda+\rho,\alpha^\vee\rangle = 0$. Hence $s_\alpha\bullet\lambda = \lambda$ and by the first assertion $\chi(\lambda) = -\chi(\lambda)$, which implies the second assertion. The third assertion follows by developing each side of the equality. $\square$

Let $\alpha \in \mathbb{Z}\Phi$ and write $\alpha = \sum_{i=1}^{n} a_i\alpha_i$. Define the *support* of α in Δ to be

$$\operatorname{support}(\alpha) = \{i \mid a_i \neq 0\}.$$

We will often use the next lemma to avoid summing over all the roots in (1.12), whenever we are interested in computing the JSF up to a given weight. A proof of this lemma can be found in [McN98, Lemma 4.5.6].

Lemma 1.3.5. *Let $\lambda, \mu \in X(T)^+$, $\alpha \in \Phi^+$ and $2 \leq r \leq \langle \lambda + \rho, \alpha^\vee \rangle$. If $\lambda - r\alpha$ and μ are conjugate by W under the dot action, then α and $\lambda - \mu$ have equal support in Δ.*

1.3.1 Computing the Jantzen p-sum formula. — Some of the ideas in this subsection are taken from [McN98, 4.5]. Let $\lambda \in X(T)^+$. We wish to deduce some information about the character of $L(\lambda)$ using Remark 1.3.3. Observe that $s_{\alpha,mp} \bullet \lambda$ in (1.12) does not necessarily lie in D. Since D is a fundamental domain for W acting via the dot action on $X(T)$, there exists $w_{\alpha,mp} \in W$ such that $w_{\alpha,mp} \bullet (\lambda - mp\alpha) \in D$. Then, by 1) and 3) of Lemma 1.3.4, we have

$$\chi(s_{\alpha,mp} \bullet \lambda) = -\det(w_{\alpha,mp})\chi(w_{\alpha,mp} \bullet (\lambda - mp\alpha)),$$

which shows the following proposition.

Proposition 1.3.6 ([McN98, Remark 4.5.8])**.** *For $\lambda \in X(T)^+$, we have*

$$\operatorname{JSF}(\lambda) = -\sum_{\alpha \in \Phi^+} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} \nu_p(mp) \det(w_{\alpha,mp})\chi(w_{\alpha,mp} \bullet (\lambda - mp\alpha)), \tag{1.14}$$

with $w_{\alpha,mp} \in W$ satisfying $w_{\alpha,mp} \bullet (\lambda - mp\alpha) \in D$.

We can combine the coefficients in (1.14) and apply 2) of Lemma 1.3.4 to get

$$\operatorname{JSF}(\lambda) = \sum_{\theta \in X(T)^+} c_{\lambda,\theta}\chi(\theta) \tag{1.15}$$

with $c_{\lambda,\theta} \in \mathbb{Z}$. Since $\operatorname{JSF}(\lambda)$ is a sum of characters, namely $\sum_{i>0} \operatorname{ch} V(\lambda)^i$, we can write it as

$$\operatorname{JSF}(\lambda) = \sum_{\theta \in X(T)^+} d_{\lambda,\theta} \operatorname{ch} L(\theta), \tag{1.16}$$

where

$$d_{\lambda,\theta} = \sum_{i>0} [V(\lambda)^i : L(\theta)] \in \mathbb{Z}_{\geq 0}. \tag{1.17}$$

Note that obtaining (1.16) is not straightforward and depends upon knowing the decomposition of $\chi(\theta)$ in terms of irreducible characters for all $\theta \precneqq \lambda$ with $c_{\lambda,\theta} \neq 0$, i.e. knowing the coefficients in (1.2). We will see how to perform this in an effective way for a truncated version of the JSF in Subsection 1.3.2.

Using (1.13), (1.16) and (1.17), we deduce the following proposition.

Proposition 1.3.7. *Let $\mu \in X(T)^+ \setminus \{\lambda\}$. Then $L(\mu)$ is a composition factor of $V(\lambda)$ if and only if $d_{\lambda,\mu} \neq 0$ in* (1.16).

Remark 1.3.8. Let $\mu \in X(T)^+ \setminus \{\lambda\}$. By (1.13),

$$[V(\lambda) : L(\mu)] = [V(\lambda)^1 : L(\mu)]. \tag{1.18}$$

Hence if $d_{\lambda,\mu} = 1$, then (1.17) and (1.18) imply $[V(\lambda)^2 : L(\mu)] = 0$ and $[V(\lambda) : L(\mu)] = 1$. However, if $d_{\lambda,\mu} > 1$, then the JSF is not sufficient to determine the value of $[V(\lambda) : L(\mu)]$, since it might be that $[V(\lambda)^2 : L(\mu)] \neq 0$.

The issue raised in Remark 1.3.8 appears when we try to determine the character of simple modules using the JSF. We will see, in the remainder of this section, how to solve it in some specific cases. The first result in this direction comes from the theory of translation functors. For more details about the proof, see [Jan03, II.7.18]

Proposition 1.3.9. *Let $\lambda_0 \in C_0 \cap X(T)$ and $w \in W_p$ with $w \bullet \lambda_0 \in X(T)^+$. Let $s \in \Sigma(C_0)$ with $w \bullet \lambda_0 \preceq ws \bullet \lambda_0$. Then*

$$[V(w_1 \bullet \lambda_0) : L(w \bullet \lambda_0)] = [V(w_1 s \bullet \lambda_0) : L(w \bullet \lambda_0)]$$

for all $w_1 \in W_p$ such that $w_1 \bullet \lambda_0, w_1 s \bullet \lambda_0 \in X(T)^+$.

The previous proposition does not apply to $\lambda_0 \in (\overline{C}_0 \setminus C_0) \cap X(T)$, that is to λ_0 lying on a wall of the fundamental alcove. For such weights, we first need to apply the following proposition. For more details about the proof, see [Jan03, II.7.17].

Proposition 1.3.10. *Let $\lambda_0, \lambda_0' \in \overline{C}_0$ and $w \in W_p$ with $w \bullet \lambda_0 \in X(T)^+$. Suppose that $w \bullet \lambda_0'$ belongs to the upper closure of the alcove containing $w \bullet \lambda_0$.*

1) For all $w_1 \in W_p$ such that $w_1 \bullet \lambda_0, w_1 \bullet \lambda_0' \in X(T)^+$, we have

$$[V(w_1 \bullet \lambda_0) : L(w \bullet \lambda_0)] = [V(w_1 \bullet \lambda_0') : L(w \bullet \lambda_0')].$$

2) If $\operatorname{ch} L(w \bullet \lambda_0) = \sum_{w' \in W_p} a_{w,w'} \chi(w' \bullet \lambda_0)$ with almost all $a_{w,w'} = 0$, then

$$\operatorname{ch} L(w \bullet \lambda_0') = \sum_{w' \in W_p} a_{w,w'} \chi(w' \bullet \lambda_0').$$

We summarize the data required to apply Propositions 1.3.9 and 1.3.10 and their implications in tables like Table 2.9 on Page 47. In the next remarks, we explain how to read these tables and the results they contain. We refer to the different parts of the table by reading it from top to bottom and use the same notations as in the propositions.

Remark 1.3.11. The first part of the table gives the elements in W_p which reflect the weights γ, η or γ', η' we are considering, to the weight λ_0 or λ_0' which lies in the closure of the fundamental alcove. By looking at λ_0 or λ_0', it is straightforward to check if it lies in the interior or on a wall of the fundamental alcove.

Remark 1.3.12. The second part of the table only exists if the weights we are considering are linked to a weight which does not lie in the interior of the fundamental alcove, i.e. λ_0' lies on a wall. Whenever this occurs, we choose a weight λ_0 in the interior of the fundamental alcove in order to apply Proposition 1.3.10. A generic choice is to pick the trivial weight, but we sometimes need to pick a more specific weight in the interior of the fundamental alcove. In the notations of Proposition 1.3.10, we also exhibit the alcove which contains $w \bullet \lambda_0$ and $w \bullet \lambda_0'$ in its upper closure by giving the sequence of integers (n_α) as in Subsection 1.2.2.

Remark 1.3.13. In the last part of the table, we provide all the data required to check that the hypotheses of Proposition 1.3.9 hold and we state the implications of the proposition.

1.3.2 A truncated version of the Jantzen p-sum formula. — We now describe an inductive process which we will use repeatedly to compute a truncated version of the JSF, thus obtaining possible candidates for the linear expression of truncated Weyl characters in terms of truncated characters of simple modules. The idea of truncating the JSF first appeared in [Cav15]. Let $\lambda \in X(T)^+$. The first version of this process assumes that we have the truncated character of $\chi(\mu)$ for each μ appearing in the JSF of λ. We also describe a modified version of the process which does not assume that the truncated character of $\chi(\mu)$ is known for each μ appearing in the JSF of λ, but takes into account various candidates for it. We then prove a result which motivates the use of the inductive process. Finally, on Page 20, we give an example on how to apply the inductive process in order to compute a truncated character. The reader may want to follow the example which could help to understand the theoretical discussion.

Let $\lambda \in X(T)^+$. Recall Remark 1.3.3. For a set S satisfying Condition 1.1.11 for λ, define the S-truncated JSF(λ) denoted JSF$(\lambda)^S$ to be

$$\mathrm{JSF}(\lambda)^S = \chi(\lambda)^S - \operatorname{ch} L(\lambda)^S + \sum_{i>1} (\operatorname{ch} V^i)^S. \tag{1.19}$$

Note that

$$\mathrm{JSF}(\lambda)^S = \sum_{\theta \in S} c_{\lambda,\theta}\chi(\theta)^S = \sum_{\theta \in S} d_{\lambda,\theta} \operatorname{ch} L(\theta)^S, \tag{1.20}$$

where $c_{\lambda,\theta}$ and $d_{\lambda,\theta}$ are as in (1.15) and (1.16).

Remark 1.3.14. One of the advantages of the truncated version of the JSF is to reduce the number of roots over which we sum in (1.14) by using the description of the set S along with Proposition 1.2.2, Corollary 1.2.3, and Lemma 1.3.5.

Fix a set S satisfying Condition 1.1.11 for λ. We refer to Fig. 1.2 for a description of Algorithm 1. We apply this algorithm which returns a set $S_1 \subseteq S$ and for all $\mu \in S_1$ the JSF$(\mu)^S$,

$$\mathrm{JSF}(\mu)^S = \sum_{\theta \in S_1} c_{\mu,\theta}\chi(\theta)^S, \tag{1.21}$$

with $c_{\mu,\theta} \in \mathbb{Z}$ known. Note that Algorithm 1 ends, since the set S is finite.

Our goal is to find $(L(\lambda) : V(\theta))$ for all $\theta \in S_1$. We proceed inductively using the poset structure of $(S_1, \preceq)$. Note that λ is the maximal element in S_1. For a minimal element $\mu \in S_1$, we have JSF$(\mu)^S = 0$, hence $\chi(\mu)^S = \operatorname{ch} L(\mu)^S$ by (1.12) and (1.19).

Fix $\mu \in S_1$ not minimal. The induction hypothesis is the following.

Algorithm 1: Computing S_1 and some JSF.

Define $S_0 = \{\lambda\}, S_1 = \emptyset$;
while $S_0 \neq S_1$ **do**
    for $\mu \in S_0 \setminus S_1$ **do**
        Compute the (1.14) version of $\mathrm{JSF}(\mu)^S$ using Remark 1.3.14.;
        $S_0 = S_0 \cup \{\theta \mid c_{\mu,\theta} \neq 0 \text{ in (1.20) for } \mu\}$;
        $S_1 = S_1 \cup \{\mu\}$;
    end
end
Return S_1 and (1.21) for all $\mu \in S_1$;

Figure 1.2: Algorithm 1

Step 0
- For γ with $c_{\mu,\gamma} \neq 0$ in (1.21), we know the coefficients $e_{\gamma,\theta} \in \mathbb{Z}_{\geq 0}$ appearing in
$$\chi(\gamma)^S = \sum_{\theta \in S_1} e_{\gamma,\theta} \operatorname{ch} L(\theta)^S. \tag{1.22}$$
Note that we have relabelled the coefficients $a_{\gamma,\theta}$ of (1.4) by $e_{\gamma,\theta}$.
- For $\gamma \in S_1$ with $\gamma \precnsim \mu$, we know the coefficients $e_{b_{\gamma,\theta}} \in \mathbb{Z}$ appearing in
$$\operatorname{ch} L(\gamma)^S = \sum_{\theta \in S_1} e_{b_{\gamma,\theta}} \chi(\theta)^S. \tag{1.23}$$
Similarly as above, note that we have relabelled the coefficients $b_{\gamma,\theta}$ of (1.5) by $e_{b_{\gamma,\theta}}$.

The inductive step consists in applying Step 1 and Step 2 below.

Step 1 Substituting (1.22) in (1.21) for each γ such that $c_{\mu,\gamma} \neq 0$ yields
$$\mathrm{JSF}(\mu)^S = \sum_{\theta \in S_1} d_{\mu,\theta} \operatorname{ch} L(\theta)^S, \tag{1.24}$$
where $d_{\mu,\theta} = \sum_{\gamma \in S_1} c_{\mu,\gamma} e_{\gamma,\theta}$.

Step 2 Recall Remark 1.3.8 and Remark 1.3.3. Whenever $d_{\mu,\theta} > 1$ in (1.24), we deduce multiple possibilities for $\operatorname{ch} L(\mu)^S$ as follows. List all the sequences $(e_{\mu,\theta})_{\theta \in S_1}$, denoted by $(e_{\mu,\theta})$ for notational simplicity, with $\min(1, d_{\mu,\theta}) \leq e_{\mu,\theta} \leq d_{\mu,\theta}$ and $\theta \in S_1$. We do not understand $V(\mu)^1$ from (1.11) which is the key to determining $\operatorname{ch} L(\mu)^S$ by (1.13). Note that each sequence $(e_{\mu,\theta})$ yields a possibility for $(\operatorname{ch} V(\mu)^1)^S$ given by
$$(\operatorname{ch} V(\mu)^1)^S_{(e_{\mu,\theta})} = \chi(\mu)^S_{(e_{\mu,\theta})} - \operatorname{ch} L(\mu)^S_{(e_{\mu,\theta})} = \sum_{\theta \in S_1} e_{\mu,\theta} \operatorname{ch} L(\theta)^S. \tag{1.25}$$
Rewrite (1.25) as
$$\chi(\mu)^S_{(e_{\mu,\theta})} - \operatorname{ch} L(\mu)^S_{(e_{\mu,\theta})} = \sum_{\theta \in S_1} b_{e_{\mu,\theta}} \chi(\theta)^S, \tag{1.26}$$

where $b_{e_{\mu,\theta}} = \sum_{\gamma \in S_1} e_{\mu,\gamma} e_{b_{\gamma,\theta}}$. Formally, we have

$$\chi(\mu)^S_{(e_{\mu,\theta})} = \operatorname{ch} L(\mu)^S_{(e_{\mu,\theta})} + \sum_{\theta \in S_1} e_{\mu,\theta} \operatorname{ch} L(\theta)^S, \tag{1.27}$$

$$\operatorname{ch} L(\mu)^S_{(e_{\mu,\theta})} = \chi(\mu)^S_{(e_{\mu,\theta})} - \sum_{\theta \in S_1} b_{e_{\mu,\theta}} \chi(\theta)^S. \tag{1.28}$$

We refer to (1.27) and (1.28) as the Weyl character, respectively irreducible character, corresponding to the sequence $(e_{\mu,\theta})$.

Remark 1.3.15. One of the sequences this procedure returns is the correct sequence, namely a sequence $(e_{\mu,\theta})$ such that $[V(\mu) : L(\theta)] = e_{\mu,\theta}$ for all $\theta \in S_1$. In practice, we will not determine for each $\mu \in S_1$ the correct sequence, since there may exist more than one sequence $(e_{\mu,\theta})$ such that the coefficients $b_{e_{\mu,\theta}}$ in (1.28) satisfy $b_{e_{\mu,\theta}} = (L(\mu) : V(\theta))$ for all $\theta \in S_1$. Examples of sequences illustrating this situation appear in Lemma 1.3.17.

Let $\mu \in S_1$. Assume we have applied the previous steps to all $\theta \in S_1$ satisfying $\theta \precneqq \mu$. If for all $\theta \in S_1$ with $\theta \precneqq \mu$, the application of Step 2 to θ returned only one sequence $(e_{\theta,\nu})$, then we continue applying the steps above to μ. If for some $\theta \in S_1$ with $\theta \precneqq \mu$, the application of Step 2 to θ returned more than one sequence, then we apply the following modified version of the inductive process to μ.

Step 0' For each $\gamma \in S_1$ such that $c_{\mu,\gamma} \neq 0$ in (1.20), choose a sequence $(e_{\gamma,\theta})$ coming from Step 2 applied to γ or from Step 2' described below applied to γ, depending on whether we applied Step 2 or Step 2' to γ. Fixing these sequences, we inductively have a sequence $(e_{\nu,\theta})$ for any $\nu \in S_1$ satisfying $\nu \precneqq \mu$. That is, we have

$$\chi(\nu)^S_{(e_{\nu,\theta})} = \operatorname{ch} L(\nu)^S_{(e_{\nu,\theta})} + \sum_{\theta \in S_1} e_{\nu,\theta} \operatorname{ch} L(\theta)^S_{(e_{\theta,\eta})}, \tag{1.29}$$

with $e_{\nu,\theta} \in \mathbb{Z}_{\geq 0}$ known and

$$\operatorname{ch} L(\nu)^S_{(e_{\nu,\theta})} = \chi(\nu)^S_{(e_{\nu,\theta})} - \sum_{\theta \in S_1} b_{e_{\nu,\theta}} \chi(\theta)^S_{(e_{\theta,\eta})}, \tag{1.30}$$

with $b_{e_{\nu,\theta}} \in \mathbb{Z}$ known from (1.23), (1.26) or (1.33).

We repeat the next two steps for each combination of choices in Step 0'.

Step 1' Substituting (1.29) for each γ such that $c_{\mu,\gamma} \neq 0$ in (1.21) yields the following version of the JSF

$$\operatorname{JSF}(\mu)^S_{(e_{\mu,\theta})} = \sum_{\theta \in S_1} \overline{d_{\mu,\theta}} \operatorname{ch} L(\theta)^S_{(e_{\theta,\nu})}, \tag{1.31}$$

where $\overline{d_{\mu,\theta}} = \sum_{\gamma \in S_1} c_{\mu,\gamma} e_{\gamma,\theta}$. We overline the coefficients $d_{\mu,\theta}$ in (1.31) to emphasize their dependence on the choice of the sequences in Step 0' compared to the coefficients $d_{\mu,\theta}$ appearing in (1.24) which do not depend on any choice.

Step 2' Similarly as in Step 2, we index by $(e_{\mu,\theta})$ all the possible sequences given by letting $e_{\mu,\theta}$ range between $\min(1, d_{\mu,\theta}) \leq e_{\mu,\theta} \leq \overline{d_{\mu,\theta}}$ for $\theta \in S_1$. We get formally

$$\chi(\mu)^S_{(e_{\mu,\theta})} = \operatorname{ch} L(\mu)^S_{(e_{\mu,\theta})} + \sum_{\theta \in S_1} e_{\mu,\theta} \operatorname{ch} L(\theta)^S_{(e_{\theta,\nu})}, \tag{1.32}$$

which we rewrite formally as

$$\operatorname{ch} L(\mu)^S_{(e_{\mu,\theta})} = \chi(\mu)^S_{(e_{\mu,\theta})} - \sum_{\theta \in S_1} b_{e_{\mu,\theta}} \chi(\theta)^S_{(e_{\theta,\nu})} \tag{1.33}$$

where $b_{e_{\mu,\theta}} = \sum_{\gamma \in S_1} e_{\mu,\gamma} b_{e_{\gamma,\theta}}$.

We inductively apply this process to every element of S_1.

Remark 1.3.16. Let $(e_{\lambda,\theta})$ be a sequence coming from the inductive process applied to λ. To every such sequence is attached: An inductive choice of sequences $(e_{\gamma,\theta})$ from (1.22) or (1.29) along with the sequences $(b_{e_{\gamma,\theta}})$ from (1.23) or (1.30), a sequence $(d_{\lambda,\theta})$ or $(\overline{d_{\lambda,\theta}})$ from (1.24) or (1.31) from which $(e_{\lambda,\theta})$ is constructed, and a sequence $(b_{e_{\lambda,\theta}})$ from (1.26) or (1.33).

The next result, Lemma 1.3.17, justifies the introduction of the modified inductive process. It tells us that we do not always need to pinpoint the correct sequence $(e_{\lambda,\theta})$ in order to find $(L(\lambda) : V(\theta))$ for every $\theta \in S_1$. Before stating the lemma, we introduce some additional notations. For $\mu \in S$, set $S_\mu = \{\theta \in S \mid \theta \succeq \mu\}$. Note that S_μ also satisfies Condition 1.1.11 for λ. Similarly, let $(S_1)_\mu = \{\theta \in S_1 \mid \theta \succeq \mu\}$. Let $(e_{\lambda,\theta})$ be a sequence coming from the inductive process applied to λ and let $\mu \in S_1$. We define three conditions depending on μ for the sequence $(e_{\lambda,\theta})$.

(C1) If $\theta \in (S_1)_\mu \setminus \{\lambda\}$, then $b_{e_{\theta,\mu}} = (\theta : \mu)$.

(C2) If $\theta \in (S_1)_\mu \setminus \{\mu\}$ with $(\theta : \mu) \neq 0$, then $e_{\lambda,\theta} = [\lambda : \theta]$.

(C3) If $\theta \in (S_1)_\mu \setminus \{\lambda\}$ and $(\theta : \mu) \neq 0$, then $e_{\gamma,\theta} = [\gamma : \theta]$ for $\gamma \in (S_1)_\mu$.

The conditions (C1) and (C3) are satisfied if $(e_{\lambda,\theta})$ comes from the non-modified inductive process applied to λ, since (C1) and (C3) correspond to the choices made in Step 0 and Step 0'. Denote by $\mathcal{S}_\mu$ the following set

$$\mathcal{S}_\mu = \{(e_{\lambda,\theta}) \mid (e_{\lambda,\theta}) \text{ comes from the inductive process and satisfies (C1) – (C3) for } \mu\}.$$

Note that the set $\mathcal{S}_\mu$ is nonempty, since it contains the correct sequence as mentioned in Remark 1.3.15.

Lemma 1.3.17. *Let $\lambda \in X(T)^+$ and let S be a set satisfying Condition* 1.1.11 *for λ. Assume the output of the inductive process applied to S is available, that is we have the sequences $(e_{\gamma,\nu})$ and $(b_{e_{\gamma,\nu}})$ and $(d_{\gamma,\nu})$ or $(\overline{d_{\gamma,\nu}})$ for every $\gamma \in S_1$. Let $\mu \in S_1$. If for every sequence $(e_{\lambda,\theta}) \in \mathcal{S}_\mu$ the sequence $(d_{\lambda,\theta})$ or $(\overline{d_{\lambda,\theta}})$ which is attached to it satisfies $d_{\lambda,\mu} \in \{0,1\}$ or $\overline{d_{\lambda,\mu}} \in \{0,1\}$, then for every sequence $(e_{\lambda,\theta}) \in \mathcal{S}_\mu$, we have $b_{e_{\lambda,\mu}} = (L(\lambda) : V(\mu))$.*

Proof. Note that the S_μ-truncated version of (1.19) is given by

$$\operatorname{ch} L(\lambda)^{S_\mu} = \chi(\lambda)^{S_\mu} - \operatorname{JSF}(\lambda)^{S_\mu} + \sum_{i>1} (\operatorname{ch} V(\lambda)^i)^{S_\mu}. \tag{1.34}$$

Since the non-modified version of the inductive process can be viewed as a special case of the modified version, i.e. applying Step 0' with only one possibility for the inductive choice of the sequence $(e_{\gamma,\theta})$, assume the modified version was applied to λ. Let $(e_{\lambda,\theta}) \in \mathcal{S}_\mu$ with all the sequences attached to it as explained in Remark 1.3.16. Note that by (1.31) and (1.32), we can write a formal version of (1.34) which depends on the choices made in Step 0' for $(e_{\lambda,\theta})$ as follows

$$\operatorname{ch} L(\lambda)^{S_\mu}_{(e_{\lambda,\theta})} = \chi(\lambda)^{S_\mu}_{(e_{\lambda,\theta})} - \operatorname{JSF}(\lambda)^{S_\mu}_{(e_{\lambda,\theta})} + \left(\sum_{i>1} \operatorname{ch} V(\lambda)^i\right)^{S_\mu}_{(e_{\lambda,\theta})}, \tag{1.35}$$

where

$$\left(\sum_{i>1} \operatorname{ch} V(\lambda)^i\right)^{S_\mu}_{(e_{\lambda,\theta})} = \sum_{\theta \in (S_1)_\mu \setminus \{\lambda\}} (\overline{d_{\lambda,\theta}} - e_{\lambda,\theta}) \operatorname{ch} L(\theta)^{S_\mu}_{(e_{\theta,\nu})}. \tag{1.36}$$

Substitute (1.30) in (1.36) to obtain

$$\left(\sum_{i>1} \operatorname{ch} V^i\right)^{S_\mu}_{(e_{\lambda,\theta})} = \sum_{\theta,\gamma \in (S_1)_\mu \setminus \{\lambda\}} (\overline{d_{\lambda,\theta}} - e_{\lambda,\theta}) b_{e_{\theta,\gamma}} \chi(\gamma)^{S_\mu}_{(e_{\gamma,\nu})}. \tag{1.37}$$

Using (1.21) and (1.37) to expand (1.35), we get

$$\operatorname{ch} L(\lambda)^{S_\mu}_{(e_{\lambda,\theta})} = \chi(\lambda)^{S_\mu}_{(e_{\lambda,\theta})} - \sum_{\gamma \in (S_1)_\mu} c_{\lambda,\gamma} \chi(\gamma)^{S_\mu}_{(e_{\gamma,\nu})} + \sum_{\gamma,\theta \in (S_1)_\mu \setminus \{\lambda\}} (\overline{d_{\lambda,\theta}} - e_{\lambda,\theta}) b_{e_{\theta,\gamma}} \chi(\gamma)^{S_\mu}_{(e_{\gamma,\nu})}. \tag{1.38}$$

The coefficient of $\chi(\mu)$ on the left-hand side of (1.38) is equal to $b_{e_{\lambda,\mu}}$, where $b_{e_{\lambda,\mu}}$ is as in (1.30) or (1.33). Isolating the coefficient corresponding to $\chi(\mu)$ on the right-hand side of (1.38) yields

$$b_{e_{\lambda,\mu}} = -c_{\lambda,\mu} + \sum_{\theta \in (S_1)_\mu \setminus \{\lambda\}} (\overline{d_{\lambda,\theta}} - e_{\lambda,\theta}) b_{e_{\theta,\mu}}. \tag{1.39}$$

Recall that for $\theta \in S_\mu$, if $\overline{d_{\lambda,\theta}} \in \{0,1\}$, then by definition of $e_{\lambda,\theta}$, we have $\overline{d_{\lambda,\theta}} = e_{\lambda,\theta}$. We assumed that $\overline{d_{\lambda,\mu}} \in \{0,1\}$, hence $\overline{d_{\lambda,\mu}} - e_{\lambda,\mu} = 0$. Thus (1.39) becomes

$$b_{e_{\lambda,\mu}} = -c_{\lambda,\mu} + \sum_{\theta \in (S_1)_\mu \setminus \{\lambda,\mu\}} (\overline{d_{\lambda,\theta}} - e_{\lambda,\theta}) b_{e_{\theta,\mu}}.$$

By conditions (C1) and (C2), we get

$$b_{e_{\lambda,\mu}} = -c_{\lambda,\mu} + \sum_{\theta \in (S_1)_\mu \setminus \{\lambda,\mu\}} (\overline{d_{\lambda,\theta}} - [V(\lambda) : L(\theta)])(\theta : \mu).$$

Moreover, by definition of the sequence $(d_{\lambda,\theta})$ and $(\overline{d_{\lambda,\theta}})$ in (1.24) and (1.31), condition (C3) implies that for $\theta \in (S_1)_\mu \setminus \{\lambda\}$ with $(\theta : \mu) \neq 0$, we can replace $\overline{d_{\lambda,\theta}}$ by $d_{\lambda,\theta}$. Hence

$$b_{e_{\lambda,\mu}} = -c_{\lambda,\mu} + \sum_{\theta \in (S_1)_\mu \setminus \{\lambda,\mu\}} (d_{\lambda,\theta} - [V(\lambda) : L(\theta)])(\theta : \mu). \tag{1.40}$$

Note that (1.40) implies that the coefficient $b_{e_{\lambda,\mu}}$ does not depend on the choice of the sequence $(e_{\lambda,\theta}) \in \mathcal{S}_\mu$. Moreover, the set $\mathcal{S}_\mu$ contains the correct sequence $(e'_{\lambda,\theta})$ satisfying $b_{e'_{\lambda,\mu}} = (\lambda : \mu)$. Therefore, any sequence $(e_{\lambda,\theta}) \in \mathcal{S}_\mu$ satisfies $b_{e_{\lambda,\mu}} = b_{e'_{\lambda,\mu}} = (\lambda : \mu)$, which proves the lemma. □

Remark 1.3.18. If every sequence $(e_{\lambda,\theta})$ obtained from the inductive process satisfies $d_{\lambda,\mu} \in \{0, 1\}$ or $\overline{d_{\lambda,\mu}} \in \{0, 1\}$ then it is in particular true for all the elements in $\mathcal{S}_\mu$. Whenever this is the case, it is enough to pick any sequence lying in $\mathcal{S}_\mu$ in order to compute $(L(\lambda) : V(\mu))$.

Definition 1.3.19. We say $\mu \in S_1$ is a *problematic case* for λ, if there exists a sequence $(e_{\lambda,\theta})$ coming from the inductive process for λ verifying conditions (C1), (C2) and (C3), with $d_{\lambda,\mu} > 1$.

Remark 1.3.20. Let $\lambda \in X(T)^+$. Observe that once we have applied the inductive process to λ, determining $\operatorname{ch} L(\lambda)^S$ is the same as solving the problematic cases for λ.

In the upcoming chapters, we repeatedly use the inductive process in order to compute truncated characters of simple modules. Whenever the set S_1 contains weights different from λ which are non minimal in S_1, we summarize the inductive process in tables like Table 1.3. We explain how these tables should be read in the following remarks and then give a detailed example.

The table is divided into four clear parts separated by horizontal lines, we refer to its different parts by reading it from top to bottom.

Remark 1.3.21. The first part of the table contains the information about the group considered and the weight to which we are applying the inductive process.

Remark 1.3.22. For a summary of the output of the inductive process, one should start by reading the first column of the third part of the table. It contains the *nontrivial* outputs (1.21) of Algorithm 1, that is the truncated JSF of the non minimal elements of S_1 in terms of characters of Weyl modules. The output (1.21) for the minimal elements of S_1 is equal to 0, hence omitted. The elements of S_1 are listed in the fourth part of the table.

Remark 1.3.23. Recall that in order to express the JSF in terms of irreducible characters, we start with minimal elements and apply the steps Step 0 to Step 2 or Step 0' to Step 2'. The second column of the third part of the table contains the expressions (1.24) or (1.31) depending on whether we applied Step 1 or Step 1'. One should be able to reconstruct the process starting towards the bottom of the column with the second to minimal elements of S_1, since again the minimal elements are omitted. Superscripts and subscripts appear in the expressions whenever we apply Step 1'. They respectively correspond to the maximal and minimal entries ranging over the sequences $(\overline{d})$ defined in (1.31).

Remark 1.3.24. The second part of the table contains information about the irreducible character of the weight considered in terms of Weyl modules. The precision of the information depends on the output of the inductive process, therefore, this part of the table is the most variable. It sometimes contains more than one possibility with the correct character specified in order to help the reader follow an argument appearing in the text and sometimes the correct character is not determined, but we use the different possibilities to bound the multiplicity of a weight. In any case, the corresponding argument appearing in the text should clarify any confusion.

$\mu = (0, 0, p-1, 0)_{F_4}$	
$\mathrm{ch}\, L(\mu)_{2473} = \mu - A + B - D + E - F$	
See argument in Subsection 2.5.1.3	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2473} = A - B + C + D - E + F$	$\mathrm{JSF}(\mu)_{2473} = A + 2C + D +_0^1 F$
$\mathrm{JSF}(A)_{2473} = B + C + E$	$\mathrm{JSF}(A)_{2473} = B + C + E + 2F$
$\mathrm{JSF}(B)_{2473} = F$	$\mathrm{JSF}(B)_{2473} = F$
$\mathrm{JSF}(E)_{2473} = F$	$\mathrm{JSF}(E)_{2473} = F$
$A = \mu - 0131 = (1, 1, p-4, 1)$	$D = \mu - 2460 = (0, 0, p-5, 6)$,
$B = \mu - 1251 = (0, 2, p-6, 3)$	$E = \mu - 0241 = (2, 0, p-4, 2)$,
$C = \mu - 0363 = (3, 0, p-4, 0)$	$F = \mu - 1361 = (1, 1, p-6, 4)$

Table 1.3: An example of the inductive process for the group F_4

Example 1.3.25. Assume $p \geq 13$. Let us consider in detail the example covered in Table 1.3, that is let us show how to apply the inductive process in order to compute the truncated character $\mathrm{ch}\, L(\mu)^S$ for $\mu = (0, 0, p-1, 0) = (p-1)\lambda_3 \in X(T_{F_4})^+$ and $S = \{\theta \in X(T_{F_4})^+ \mid \mu \succeq \theta \succeq \mu - 2473\}$. Here, $\mu - 2473$ is a shorthand for $\mu - 2\alpha_1 - 4\alpha_2 - 7\alpha_3 - 3\alpha_4$, where $\{\alpha_i\}_{i=1}^4$ is a set of simple roots of a root system of type F_4. We use the notations defined in (1.6) and (1.7). Moreover, we denote $\mathrm{ch}\, L(\mu)^S$ by $\mathrm{ch}\, L(\mu)_{2473}$ and for $\gamma \in S$, we denote $\mathrm{JSF}(\gamma)^S$ by $\mathrm{JSF}(\gamma)_{2473}$.

We start by applying Algorithm 1 to $S_0 = \{\mu\}$ and S. The output consists in the set $S_1 = \{\mu, A, B, C, D, E, F\}$ described in the last part of Table 1.3 and the truncated character $\mathrm{JSF}(\theta)_{2473}$ in terms of characters of Weyl modules, for each $\theta \in S_1$. These truncated characters are listed in the first column of the third part of Table 1.3. We have that $\mathrm{JSF}(\theta)_{2473} = 0$ for $\theta \in \{C, D, F\}$, hence we omit them in the table and we deduce that the weights C, D, F are minimal in S_1. Thus, we get $\chi(\theta)_{2473} = \mathrm{ch}\, L(\theta)_{2473}$ for $\theta \in \{C, D, F\}$ and we can apply the non-modified inductive process to the weights B and E.

Let $\theta \in \{B, E\}$. Note that the sequence $(d_{\theta,\nu})$ can be read off from the second column of the third part of Table 1.3. Since $d_{\theta,\nu} \in \{0, 1\}$ for $\nu \in S_1$, we get only one possibility for the sequence $(e_{\theta,\nu})$ which yields $\mathrm{ch}\, L(\theta)_{2473} = \chi(\theta)_{2473} - \chi(F)_{2473}$. We now have $\mathrm{ch}\, L(\theta)_{2473}$ for $\theta \in \{B, C, D, E, F\}$, hence we can apply the non-modified inductive process to A. We get $d_{A,\nu} \in \{0, 1\}$ for $\nu \in S_1 \setminus \{F\}$ and $d_{A,F} = 2$ which implies that we get two sequences $(e_{A,\nu})$, one with $e_{A,F} = 1$ and the other one with $e_{A,F} = 2$. These two sequences yield two possibilities for $\mathrm{ch}\, L(A)_{2473}$ in terms of Weyl characters. Indeed, recall that by Remark 1.3.3, we have

$$\mathrm{ch}\, L(A)_{2473} = \chi(A)_{2473} - \mathrm{ch}\, V(A)^1_{2473} = \chi(A)_{2473} - \mathrm{JSF}(A)_{2473} + \sum_{i>1} \mathrm{ch}\, V(A)^i_{2473}, \tag{1.41}$$

where the $V(A)^i$ are the submodules occurring in the filtration of $V(A)$ given in Theorem 1.3.1. Moreover, by (1.17), we have

$$d_{A,F} = \sum_{i>0} [V(A)^i : L(F)].$$

Our current understanding is not enough to establish the value of $[V(A)^1 : L(F)]$, nor consequently the value of $(L(A) : V(F)) = d_{A,F} - [V(A)^1 : L(F)]$. Since $V(A)^0 \supseteq V(A)^1 \supseteq V(A)^2 \supseteq \cdots$ and $d_{A,F} = 2$, we have either $[V(A)^1 : L(F)] = 1$ and $\sum_{i>1}[V(A)^i : L(F)] = 1$, or $[V(A)^1 : L(F)] = 2$ and $\sum_{i>1}[V(A)^i : L(F)] = 0$. The sequence with $e_{A,F} = 1$ corresponds to the case $[V(A)^1 : L(F)] = 1$ and the sequence with $e_{A,F} = 2$ corresponds to the case $[V(A)^1 : L(F)] = 2$. Considering each case separately in (1.41), we get the following possibilities for the partial irreducible character $\operatorname{ch} L(A)_{2473}$.

$$\operatorname{ch} L(A)_{2473} = \begin{cases} A - B - C - E + F & \text{for the sequence with } e_{A,F} = 1 \\ A - B - C - E & \text{for the sequence with } e_{A,F} = 2 \end{cases}$$

We thus need to apply twice the modified inductive process to μ, taking into account the two possibilities we have obtained for $\operatorname{ch} L(A)_{2473}$ when applying Step 0'. Make S_1 into an ordered set $S_1 = (\mu, A, B, C, D, E, F)$. We deduce from Table 1.3 that

$$(d_{\mu,\theta})_{\theta \in S_1} = \begin{cases} (0,1,0,2,1,0,0) & \text{for the sequence with } e_{A,F} = 1 \\ (0,1,0,2,1,0,1) & \text{for the sequence with } e_{A,F} = 2, \end{cases}$$

which yields the following four possibilities for the sequences $(e_{\mu,\theta})_{\theta \in S_1}$ as in (1.32).

$$(e_{\mu,\theta})_{\theta \in S_1} = \begin{cases} (0,1,0,1,1,0,0) & \text{for the sequence with } e_{\mu,C} = 1 \text{ and } e_{A,F} = 1 \\ (0,1,0,2,1,0,0) & \text{for the sequence with } e_{\mu,C} = 2 \text{ and } e_{A,F} = 1 \\ (0,1,0,1,1,0,1) & \text{for the sequence with } e_{\mu,C} = 1 \text{ and } e_{A,F} = 2, \\ (0,1,0,2,1,0,1) & \text{for the sequence with } e_{\mu,C} = 2 \text{ and } e_{A,F} = 2, \end{cases}$$

Note that $F \npreceq C$, hence we do not need to determine $[\mu : C]$ in order to determine $(\mu : F)$. By Remark 1.3.18, the value of $(\mu : F)$ does not depend on the choice of the sequence $(e_{A,\theta})$ and we can deduce the value of $(\mu : F)$ by taking the opposite of the coefficient of F in the $\mathrm{JSF}(\mu)_{2473}$ in Table 1.3, thus $(\mu : F) = -1$. Therefore, the inductive process yields two possibilities for $\operatorname{ch} L(\mu)_{2473}$ depending on the value of $e_{\mu,C}$, namely

$$\operatorname{ch} L(\mu)_{2473} = \begin{cases} \mu - A + B - D + E - F & \text{if } [\mu : C] = 1 \\ \mu - A + B - C - D + E - F & \text{if } [\mu : C] = 2. \end{cases}$$

The problematic case C for μ is solved in Subsection 2.5.1.3 on Page 73.

1.4 Composition factors for the restriction

Let Y be a simply connected simple algebraic group over k. Fix a Borel subgroup $B_Y = U_Y T_Y$ with U_Y the unipotent radical of B_Y and T_Y a maximal torus of Y. Let $\Delta(Y) = \{\alpha_1, \ldots, \alpha_n\}$ denote a base of $\Phi(Y)$ compatible with the choice of B_Y. Recall that $\{e_\alpha, f_\alpha, h_{\alpha_i} \mid \alpha \in \Phi^+(Y), \alpha_i \in \Delta(Y)\}$ denotes a Chevalley basis of $\mathscr{L}(Y)$. For $\alpha, \alpha' \in \Phi(Y)$, let $N_{(\alpha,\alpha')}$ be the structure constant of $\mathscr{L}(Y)$ corresponding to e_α and e'_α. It satisfies $[e_\alpha, e_{\alpha'}] = N_{(\alpha,\alpha')} e_{\alpha+\alpha'}$, if $\alpha + \alpha' \in \Phi(Y)$ and $N_{(\alpha,\alpha')} = 0$ otherwise.

The next two lemmas are important, as they will imply, whenever needed, self-duality for the modules we will be considering. The first lemma is a well-known fact about restrictions and the second one is a more general version of [CT19, Lemma 3.1].

Lemma 1.4.1. *Let $X \leq Y$ be a reductive subgroup of Y. Let V be a kY-module. If V is self-dual as a kY-module, then V is self-dual as a kX-module.*

Lemma 1.4.2. *Let $X \leq Y$ be the subgroup of fixed points of a graph automorphism σ of order two stabilizing T_Y. If V is an irreducible kY-module, then V is self-dual as a module for X.*

Proof. Let w_0 denote the longest element in W_Y, the Weyl group of Y. If $w_0 = -1$, then $V \cong V^*$ and the result holds by Lemma 1.4.1. If not, we have ${}^{\sigma}V \cong V^*$ and

$$(V|_X)^* = (V^*)|_X \cong ({}^{\sigma}V)|_X = V|_X,$$

which proves the lemma. □

Lemma 1.4.3. *Let V be a self-dual kX-module such that V has two composition factors, that is $V \cong L_X(\mu)/L_X(\nu)$ for some $\mu, \nu \in X(T_X)^+$, then*

$$V \cong L_X(\mu) \oplus L_X(\nu). \tag{1.42}$$

Proof. If $\mu = \nu$, then by Proposition 1.1.7, we have $\mathrm{Ext}^1_X(L_X(\mu), L_X(\mu)) = 0$, which implies (1.42).

Assume $\mu \neq \nu$ and without loss of generality that $\nu \not\succeq \mu$. Denote by w_0 the longest element in the Weyl group of X. If $-w_0\nu = \mu$, a case-by-case verification depending on the type of $\Phi(X)$ implies that $\nu \notin \Lambda(\mu)^+$ and so $[V_X(\mu) : L_X(\nu)] = 0$. By Proposition 1.1.7, we get that $\mathrm{Ext}^1_X(L_X(\mu), L_X(\nu)) = 0$ and the result follows. If $-w_0\nu \neq \mu$, then both $L_X(\mu)$ and $L_X(\nu)$ are submodules of V, since $V \cong V^*$. The lemma follows. □

Let $X \leq Y$ be a maximal closed connected simple subgroup of Y. Let $B_X = U_X T_X$ be a Borel subgroup of X with $U_X = U_Y \cap X$ and $T_X = T_Y \cap X$, so that $B_X = B_Y \cap X$. Let $\Delta(X) = \{\beta_1, \dots, \beta_m\}$ denote a base of $\Phi(X)$ compatible with the choice of B_X. Let $\{e_\beta, f_\beta, h_{\beta_i} \mid \beta \in \Phi^+(X), \beta_i \in \Delta(X)\}$ denote a Chevalley basis of $\mathscr{L}(X)$. Note that $\mathscr{L}(X) \subseteq \mathscr{L}(Y)$.

We solve Question 3 in two steps. The first one consists in eliminating all the cases for which we can establish the existence of a third composition factor for X acting on $L_Y(\lambda)$. The second one consists in proving that X acts on the remaining cases with exactly two composition factors. The second step turns out to be more difficult than the first one. Under some assumptions on the embedding of X in Y, the next proposition provides a framework to solve the general problem of determining the composition factors of the restriction to X of a simple kY-module. It combines two ideas which have already been applied in [Sei87, Tes88, For96, Cav15, Cav17b], under specific assumptions on the embedding of X into Y, in order to determine if the restriction to X of an irreducible kY-module stays irreducible or has exactly two composition factors.

The first idea from [Sei87, §8] and [Tes88, (5.4)] is that if one is looking for the highest weight of an additional composition factor for the restriction of an irreducible module, then it does not lie "too far" away from the highest weights of the other composition factors. In fact, both weights will be at most separated by the restriction to T_X of the highest root in $\Phi^+(Y)$.

The second idea from [For96, Section 3] and developed in [Cav15, Cav17b] is that the possible candidates for the highest weight of an additional composition factor are in close relationship with the weights which are separated from the highest weights of other composition factors by the restriction to T_X of a positive root in $\Phi^+(Y)$.

Proposition 1.4.4. *Let X, Y be as above. Let α_0 and β_0 denote the largest root of $\Phi(Y)$ and $\Phi(X)$, respectively. Assume $e_{\beta_0} \in \langle e_{\alpha_0} \rangle$. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight and let v^+ be a maximal vector of weight λ in $L_Y(\lambda)$ for B_Y. If $w^+ \in L_Y(\lambda) \setminus \langle v^+ \rangle$ is a maximal vector for $\mathscr{L}(B_X)$ of weight $\theta \in X(T_X)^+$, then there exists a maximal vector in $L_Y(\lambda)$ for $\mathscr{L}(B_X)$ of weight $\nu \in X(T_X)^+$ and a positive root $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$ satisfying $\theta = \nu - \alpha|_{T_X}$.*

Proof. By Theorem 1.1.9, since λ is p-restricted, the simple kY-module $L_Y(\lambda)$ remains irreducible when viewed as a module for $\mathscr{L}(Y)$. Moreover, there exists $\alpha_j \in \Delta(Y)$ such that $e_{\alpha_j} w^+ \neq 0$, since $w^+ \notin \langle v^+ \rangle$. Consider the non-empty set

$$S = \left\{ s = e_{\beta_{b_1}} \cdots e_{\beta_{b_\ell}} e_{\alpha_j} w^+ \mid \ell \in \mathbb{Z}_{\geq 0},\ \beta_{b_i} \in \Delta(X), 1 \leq i \leq \ell \text{ and } s \neq 0 \right\}.$$

We first establish the following claim.

Claim: *Any $s \in S$ is a weight vector for T_X. Moreover, if $\gamma \in X(T_X)$ denotes the weight of $s \in S$, then $\theta = \gamma - \beta|_{T_X}$ for some $\beta \in \Phi^+(Y) \setminus \{\alpha_0\}$.*

Proof of the claim. Let $s \in S$ with $s = e_{\beta_{b_1}} \cdots e_{\beta_{b_\ell}} e_{\alpha_j} w^+$ and $\ell \in \mathbb{Z}_{\geq 0}$. It is clear that s is a weight vector for T_X. For the second part of the claim, we prove a more technical statement, namely that

$$s = \sum_{\alpha \in \Phi^+(Y)} r_\alpha e_\alpha w^+, \text{ where } r_\alpha \in k \text{ and } r_{\alpha_0} = 0. \tag{1.43}$$

Note that if (1.43) holds, then the claims follows. Indeed, recall that w^+ is of T_X-weight θ. Now, we have that s is a T_X-weight vector of weight $\alpha|_{T_X} + \theta$ for any $\alpha \in \Phi^+(Y)$ with $r_\alpha \neq 0$ in (1.43). In particular, if $r_\alpha r_{\alpha'} \neq 0$, then $\alpha|_{T_X} = \alpha'|_{T_X}$.

Recall that $s = e_{\beta_{b_1}} \cdots e_{\beta_{b_\ell}} e_{\alpha_j} w^+$ for $\ell \in \mathbb{Z}_{\geq 0}$. We prove that (1.43) holds by induction on ℓ. The case $\ell = 0$ is straightforward. Assume the result holds for a fixed $\ell \geq 0$ and let us prove it for $\ell + 1$. Let $s = e_{\beta_{b_1}} \cdots e_{\beta_{b_{\ell+1}}} e_{\alpha_j} w^+ \in S$. By the induction hypothesis we have

$$e_{\beta_{b_2}} \cdots e_{\beta_{b_{\ell+1}}} e_{\alpha_j} w^+ = \sum_{\alpha \in \Phi^+(Y)} r_\alpha e_\alpha w^+$$

with $r_{\alpha_0} = 0$. Let $e_{\beta_{b_1}} = \sum_{\gamma \in \Phi^+(Y)} d_\gamma e_\gamma$ with $d_\gamma \in k$, then

$$\begin{aligned} s = e_{\beta_{b_1}} \sum_{\alpha \in \Phi^+(Y)} r_\alpha e_\alpha w^+ &= \sum_{\gamma \in \Phi^+(Y)} \sum_{\alpha \in \Phi^+(Y)} r_\alpha d_\gamma e_\gamma e_\alpha w^+ \\ &= \sum_{\gamma, \alpha \in \Phi^+(Y)} r_\alpha d_\gamma [e_\gamma, e_\alpha] w^+ + \sum_{\gamma, \alpha \in \Phi^+(Y)} r_\alpha d_\gamma e_\alpha e_\gamma w^+ \\ &= \sum_{\gamma, \alpha \in \Phi^+(Y)} r_\alpha d_\gamma N_{(\gamma,\alpha)} e_{\gamma+\alpha} w^+ + \sum_{\alpha \in \Phi^+(Y)} r_\alpha e_\alpha \underbrace{e_{\beta_{b_1}} w^+}_{=0} \end{aligned}$$

$$= \sum_{\gamma,\alpha\in\Phi^+(Y)} r_\alpha d_\gamma N_{(\gamma,\alpha)} e_{\gamma+\alpha} w^+.$$

Note that if $\alpha + \gamma \notin \Phi^+(Y)$, then $N_{(\gamma,\alpha)} = 0$. Moreover, if $\alpha + \gamma = \alpha_0$, then $e_{\alpha_0} w^+ = \frac{1}{d_0} e_{\beta_0} w^+ = 0$ for some $d_0 \in k^*$. Therefore (1.43) holds and the claim follows. ■

Consider the following set.

$$\Lambda(S) = \{\gamma \in X(T_X) \mid \exists\, s \in S, \text{ a } T_X\text{-weight vector, with } T_X\text{-weight}(s) = \gamma\}$$

It is a poset with the partial order inherited from $X(T_X)$. The claim implies that

$$\Lambda(S) \subseteq \{\theta + \alpha|_{T_X} \mid \alpha \in \Phi^+(Y) \setminus \{\alpha_0\}\}. \tag{1.44}$$

Recall that $e_{\alpha_j} w^+ \in S$, hence $(\Lambda(S), \preceq)$ is a non-empty finite poset. Therefore, the set $\Lambda(S)$ admits a maximal element $\nu \in \Lambda(S)$ and there exists $s \in S$ such that the T_X-weight of s equals ν. By definition of S, we have that $s \neq 0$ and by maximality of ν, we have $e_{\beta_i} s = 0$ for all $\beta_i \in \Delta(X)$. Thus s is a maximal vector for $\mathscr{L}(B_X)$. Since $k\mathscr{L}(X)s \subseteq L_Y(\lambda)$ is finite dimensional, the weight $\nu \in X(T_X)^+$ and by (1.44), it is of the desired form. The proposition follows. □

The next corollary reformulates Proposition 1.4.4 in the setting we will be considering, by relating the action of $\mathscr{L}(X)$ and the action of X on a simple kY-module.

Corollary 1.4.5. *Let X, Y be as above. Let $\lambda \in X(T_Y)^+$ be p-restricted and let $\theta \in X(T_X)^+$. Let $\nu_1, \dots, \nu_r$ denote the highest weights of the composition factors (with possible repetitions) of $L_Y(\lambda)|_X$ with $\nu_i \succneq \theta$. Assume $\nu_1, \dots, \nu_r$ are p-restricted. Then $L_X(\theta)$ is a composition factor for $\mathscr{L}(X)$ acting on $L_Y(\lambda)$ if and only if $L_X(\theta)$ is a composition factor for X acting on $L_Y(\lambda)$. Moreover, if $e_{\beta_0} \in \langle e_{\alpha_0} \rangle$ and θ affords the weight of a maximal vector for $\mathscr{L}(B_X)$, then there exists ν_i with $1 \leq i \leq r$ which affords the weight of a maximal vector for $\mathscr{L}(B_X)$ and $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$ such that $\theta = \nu_i - \alpha|_{T_X}$.*

Proof. Clearly, the weight θ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$ if and only if $m_{L(\lambda)|_X}(\theta) > \sum_{i=1}^r m_{L(\nu_i)}(\theta)$. By Theorem 1.1.9, since $\nu_1, \dots, \nu_r$ are p-restricted, the irreducible kX-module $L_X(\nu_i)$ is irreducible as a module for $\mathscr{L}(X)$. That is, the weights ν_i also yield the highest weights of the composition factors for the action of $\mathscr{L}(X)$ on $L_Y(\lambda)$ which are strictly greater than θ . Therefore, θ affords the highest weight of a composition factor for $\mathscr{L}(X)$ acting on $L_Y(\lambda)$ if and only if $m_{L(\lambda)|_X}(\theta) > \sum_{i=1}^r m_{L(\nu_i)}(\theta)$, which finishes the proof of the first part of the corollary. The second part of the corollary follows directly from Proposition 1.4.4. □

One of the hypotheses of Proposition 1.4.4 is the existence of a maximal vector for $\mathscr{L}(B_X)$. Recall that a maximal vector for B_X is also a maximal vector for $\mathscr{L}(B_X)$. The following proposition ensures the existence of a maximal vector for B_X. Its statement and its proof generalize [Tes88, (5.5)] and [Cav15, 7.7.18].

Proposition 1.4.6. *Let X, Y be as above and let $V = L_Y(\lambda)|_X$ for $\lambda \in X(T_Y)^+$. Assume V admits r composition factors generated by maximal vectors $w_1^+, \dots, w_r^+ \in V$ for B_X with highest weights $\mu_1, \mu_2, \dots, \mu_r \in X(T_X)^+$ satisfying*

1) $[V_X(\mu_j) : L_X(\mu_i)] = 0$ *for* $i \neq j$.

2) For all $\gamma \in \Lambda(V)^+$, *if* $\mu_i \preceq \gamma$ *for some* $1 \leq i \leq r$, *then* $\dim V_\gamma = \sum_{j=1}^r \dim L_X(\mu_j)_\gamma$.

If V *is self-dual and has more than* r *composition factors, then there exists an additional maximal vector* $w_{r+1}^+ \in V \setminus \langle w_1^+, \dots, w_r^+ \rangle$ *for* B_X.

Proof. First observe that $\mu_1, \dots, \mu_r$ are distinct by hypothesis 1). Assume $\langle X w_i^+ \rangle$ is reducible for some $i = 1, \dots, r$. Observe that $\langle X w_i^+ \rangle$ is an image of $V_X(\mu_i)$ by Proposition 1.1.4. Moreover, any irreducible proper submodule of $\langle X w_i^+ \rangle$ is generated by a maximal vector $w_{r+1}^+ \in \langle X w_i^+ \rangle \setminus \langle w_i^+ \rangle$ and $w_{r+1}^+ \notin \langle w_1^+, \dots, w_{i-1}^+, w_{i+1}^+, \dots, w_r^+ \rangle$, since otherwise $w_{r+1}^+ \in \langle w_j^+ \rangle$ for some $j \neq i$ and $[V_X(\mu_i) : L_X(\mu_j)] \neq 0$ contradicting hypothesis 1). Hence the proposition holds if $\langle X w_i^+ \rangle$ is reducible for some $1 \leq i \leq r$. We can therefore assume $\langle X w_i^+ \rangle \cong L_X(\mu_i)$ for $i = 1, \dots, r$.

Set
$$W = \langle X w_1^+ \rangle + \cdots + \langle X w_r^+ \rangle.$$
Since all the modules in the sum are simple, it is a direct sum of kX-modules. Therefore, we can assume up to isomorphism that $W = \bigoplus_{i=1}^r L_X(\mu_i)$. Since V has more than r composition factors, the submodule W is proper in V. Let w_0 denote the longest element of the Weyl group of X and choose a coset representative $\dot{w}_0$ of w_0 in $N_X(T_X)$. Write V as a direct sum of T_X-modules as follows.
$$V = \bigoplus_{j=1}^r \langle w_j^+ \rangle \oplus V_0$$
Define r vectors $f_1, \dots, f_r$ in V^* by
$$f_i(v) = \begin{cases} 1 & \text{if } v = w_i^+ \\ 0 & \text{if } v \in \bigoplus_{j \neq i} \langle w_j^+ \rangle \oplus V_0. \end{cases} \tag{1.45}$$
Recall that the kX-module structure of V^* is given by $(xf)(v) = f(x^{-1}v)$ for $x \in X$, $f \in V^*$ and $v \in V$.

Claim: $\dot{w}_0^{-1} f_i \in V^*$ *is a maximal vector for* B_X *of* T_X*-weight* $-w_0\mu_i$ *for* $1 \leq i \leq r$.

Proof of the claim. Fix $i \in \{1, \dots, r\}$, a root $\beta \in \Phi^+(X)$ and $x_\beta(c) \in U_\beta \subseteq B_X$. Let $\gamma \in \Lambda(V)$ and $v \in V$ be a T_X-weight vector of weight γ. Note that the T_X-weight of $\dot{w}_0 v$ is given by $w_0\gamma$. Moreover, by (1.1) there is $c' \in k$ and $\beta' \in \Phi^+(X)$ such that $x_\beta(-c) = \dot{w}_0^{-1} x_{-\beta'}(c') \dot{w}_0$. By Lemma 1.1.2, we have
$$\begin{aligned} (x_\beta(c)\dot{w}_0^{-1} f_i)(v) &= f_i\left(\dot{w}_0 x_\beta(-c) v\right) \\ &= f_i\left(x_{-\beta'}(c')\dot{w}_0 v\right) \\ &= f_i\left(\dot{w}_0 v + \sum_{j \in \mathbb{Z}_{>0}} v_{w_0\gamma - j\beta'}\right), \end{aligned}$$
for some T_X-weight vectors $v_{w_0\gamma - j\beta'}$ of weight $w_0\gamma - j\beta'$ with $j \in \mathbb{Z}_{>0}$. Note that $w_0\gamma - j\beta' \precneqq w_0\gamma$. On the one hand, if $w_0\gamma \not\succeq \mu_i$ or $w_0\gamma = \mu_i$, then we have $f_i\left(\sum_{j \in \mathbb{Z}_{>0}} v_{w_0\gamma - j\beta'}\right) = 0$

by (1.45). On the other hand if $w_0\gamma \succneqq \mu_i$, then $\dot{w}_0 v \in \bigoplus_{j\neq i} L_X(\mu_j)$ by hypothesis 2) and so $f_i\left(\sum_{j\in\mathbb{Z}_{>0}} v_{w_0\gamma - j\beta'}\right) = 0$. We thus get $f_i\left(\sum_{j\in\mathbb{Z}_{>0}} v_{w_0\gamma - j\beta'}\right) = 0$ and $(u_\beta(c)\dot{w}_0^{-1}f_i)(v) = (\dot{w}_0^{-1}f_i)(v)$, which proves that $\dot{w}_0^{-1}f_i$ is fixed by U_X. Let us compute the T_X-weight of $\dot{w}_0^{-1}f_i$. For $t \in T_X$, we have

$$(tf_i)(v) = f_i(t^{-1}v) = \begin{cases} -\mu_i(t)f_i(v) & \text{if } v = w_i^+ \\ 0 & \text{if } v \in \bigoplus_{j\neq i}\langle w_j^+\rangle \oplus V_0 \end{cases},$$

so f_i is of weight $-\mu_i$. Therefore, $\dot{w}_0^{-1}f_i$ is a maximal vector of weight $-w_0\mu_i$ and the claim follows. ■

We can now complete the proof of the proposition. Denote the annihilator in V^* of W by $\mathrm{Ann}(W)$. Note that $(\dot{w}_0^{-1}f_i)(\dot{w}_0^{-1}w_i^+) \neq 0$ and $\dot{w}_0^{-1}w_i^+ \in W$ for all $i \in \{1,\ldots,r\}$. Hence $\dot{w}_0^{-1}f_i \notin \mathrm{Ann}(W)$ for all $i \in \{1,\ldots,r\}$ and $\mathrm{Ann}(W)$ is a non-zero submodule of V^*. Therefore, there is an additional maximal vector for B_X in $\mathrm{Ann}(W)$, that is V^* contains $r+1$ maximal vectors. By the self-duality of V, we get the desired result. □

The next corollary constitutes the main tool we will use in order to solve Question 3.

Corollary 1.4.7. *Let $\lambda \in X(T_Y)^+$ be p-restricted. Assume $L_Y(\lambda)|_X$ is not irreducible. Let μ be a maximal element in $\Lambda(L_Y(\lambda)|_X)^+$ and let $\nu \in X(T_X)^+$ be such that for all $\gamma \in \Lambda(L_Y(\lambda)|_X)^+$ with $\gamma \succneqq \nu$, $m_{L_Y(\lambda)|_X}(\gamma) = m_{L_X(\mu)}(\gamma)$ and $m_{L_Y(\lambda)|_X}(\nu) > m_{L_X(\mu)}(\nu)$. Let $\nu_1, \nu_2, \ldots, \nu_r$ denote all the highest weights of composition factors (with possible repetitions) of $L_Y(\lambda)|_X$ which are greater than or equal to $\mu, \nu, \mu - \alpha|_{T_X}$ or $\nu - \alpha|_{T_X}$ for some $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$. Then $\mu, \nu \in \{\nu_1, \ldots, \nu_r\}$. Without loss of generality, set $\nu_1 = \mu$ and $\nu_2 = \nu$. Assume the following.*

1) $L_Y(\lambda)|_X$ is self-dual,

2) $\nu \neq \nu_i$ for $i \in \{3,\ldots,r\}$,

3) $[V_X(\mu) : L_X(\nu)] = 0$,

4) $e_{\beta_0} \in \langle e_{\alpha_0}\rangle$,

5) ν_i is p-restricted for $1 \leq i \leq r$.

If $L_Y(\lambda)|_X$ has more than two composition factors, then $r \geq 3$ and there is $i \in \{3,\ldots,r\}$ such that $\nu_i = \mu - \alpha|_{T_X}$ or $\nu_i = \nu - \alpha|_{T_X}$ for some $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$.

Proof. Note that the choice of μ implies that μ affords the weight of a maximal vector for B_X. Moreover, it also implies that either $\nu = \mu$ or $\nu \not\succeq \mu$. By hypothesis 3), the latter holds. By hypotheses 1) to 3), and the choice of μ and ν, we can apply Proposition 1.4.6 in order to establish the existence of a maximal vector $w^+ \in L_Y(\lambda)$ for B_X of T_X-weight different from the weights μ and ν. Indeed, either ν affords the weight of a maximal vector for B_X and Proposition 1.4.6, applied to $\{\mu,\nu\}$, establishes the existence of a third maximal vector for B_X or ν does not afford the weight of a maximal vector for B_X and Proposition 1.4.6, applied only to μ, implies the existence of a second maximal vector for B_X in $L_Y(\lambda)|_X$.

Let $\theta \in X(T_X)^+$ be maximal among the weights $\gamma \in X(T_X)^+$ for which there exists a maximal vector for $\mathscr{L}(B_X)$ in $L_Y(\lambda)$ of T_X-weight γ different from μ and ν. By the previous paragraph, such a weight θ exists since a maximal vector for B_X is also a maximal vector for $\mathscr{L}(B_X)$. By hypothesis 4) and the considerations so far, the hypotheses of Proposition 1.4.4 hold. Therefore, by maximality of θ and Proposition 1.4.4, there exists $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$ such that $\theta = \mu - \alpha|_{T_X}$ or $\theta = \nu - \alpha|_{T_X}$. Applying Corollary 1.4.5 using hypotheses 4) and 5), we deduce that $\theta = \nu_i$ for some $i \in \{3, \dots, r\}$. The result follows. □

Remark 1.4.8. The contrapositive of Corollary 1.4.7 gives us a way to prove that X acts on a given simple kY-module with exactly two composition factors. Recall the notations of Corollary 1.4.7 and assume hypothesis 1) to 5) hold. If for all $i \in \{3, \dots, r\}$ and for all $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$, we have $\nu_i \neq \mu - \alpha|_{T_X}$ and $\nu_i \neq \nu - \alpha|_{T_X}$, then by Corollary 1.4.7, $r = 2$ and X acts on $L_Y(\lambda)$ with exactly two composition factors. Now, in order to check that for all $i \in \{3, \dots, r\}$ we have $\nu_i \neq \mu - \alpha|_{T_X}$ and $\nu_i \neq \nu - \alpha|_{T_X}$ for all $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$, let $\nu_1, \nu_2, \nu_3, \dots, \nu_s$ with $2 \leq s \leq r$ denote the highest weights of the composition factors which are not of the form $\mu - \alpha|_{T_X}$ or $\nu - \alpha|_{T_X}$ for $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$. We then show

$$m_{L_Y(\lambda)|_X}(\mu - \alpha|_{T_X}) = \sum_{i=1}^{2} m_{L_X(\nu_i)}(\mu - \alpha|_{T_X}) \tag{1.46}$$

and

$$m_{L_Y(\lambda)|_X}(\nu - \alpha|_{T_X}) = \sum_{i=1}^{2} m_{L_X(\nu_i)}(\nu - \alpha|_{T_X}) \tag{1.47}$$

for all $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$. Therefore, $s = r = 2$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

By the next lemma, we can usually considerably reduce the number of weights for which we have to compute (1.46) and (1.47).

Lemma 1.4.9. *Let $\lambda \in X(T_Y)^+$, let μ be a maximal element in $\Lambda(L_Y(\lambda)|_X)^+$ and let $\nu \in X(T_X)^+$ be such that for all $\gamma \in \Lambda(L_Y(\lambda)|_X)^+$ with $\gamma \succnsim \nu$, $m_{L_Y(\lambda)|_X}(\gamma) = m_{L_X(\mu)}(\gamma)$ and $m_{L_Y(\lambda)|_X}(\nu) > m_{L_X(\mu)}(\nu)$. Let $\nu_1, \nu_2, \dots, \nu_s$ denote all the highest weights of composition factors (with possible repetitions) for X acting on $L_Y(\lambda)$ which are different from $\mu - \alpha|_{T_X}$ and $\nu - \alpha|_{T_X}$ for all $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$. Let*

$$S = X(T_X)^+ \cap \{\mu - \alpha|_{T_X}, \nu - \alpha|_{T_X} \text{ for } \alpha \in \Phi^+(Y) \setminus \{\alpha_0\}\}.$$

Assume all the weights in S are p-restricted and $(X, p) \notin \{(B_2, 2), (C_2, 2), (F_4, 2), (G_2, 2), (G_2, 3)\}$. Then the equality

$$m_{L_Y(\lambda)|_X}(\theta) = \sum_{i=1}^{s} m_{L_X(\nu_i)}(\theta), \tag{1.48}$$

holds for all $\theta \in S$ if it holds for all the minimal elements $\theta \in S$ with respect to $\succeq$.

Proof. By Theorem 1.1.10 and the general theory of weights, if $\gamma_1, \gamma_2 \in S$ satisfies $\gamma_1 \succeq \gamma_2$, then $\gamma_2 \in \Lambda(L_X(\gamma_1))$. Therefore, if (1.48) holds for $\theta = \gamma_2$, then it also holds for $\theta = \gamma_1$. The result follows. □

1.5 Restriction to Levi subgroups

It this section, we study the restriction of an irreducible representation to a Levi factor of a parabolic subgroup. We will see how these restrictions relate to restrictions to maximal subgroups in order to solve Question 3 more efficiently.

Let $I \subseteq \Delta$ and recall from Subsection 1.1.4 the definition of the Levi factor L_I of P_I. The next proposition tells us how to obtain a simple kL_I-module from a simple kG-module. For a proof of the proposition, see [Jan03, II.2.11].

Proposition 1.5.1. *Let $I \subseteq \Delta$ and $\lambda \in X(T)^+$. Then*

$$\bigoplus_{\nu \in \mathbb{Z}I} L_G(\lambda)_{\lambda-\nu},$$

is the simple L_I-module with highest weight λ.

For $\lambda \in X(T)^+$, we denote by $L_{L_I}(\lambda)$ the *simple kL_I-module* of highest weight λ given by Proposition 1.5.1. Set $T_{L'_I} = T \cap L'_I$. In the next proposition, we state without proof a few properties relating simple modules for L'_I, L_I and G. The second and the third assertions follow from Proposition 1.5.1.

Proposition 1.5.2. *Let $I \subseteq \Delta$ and $\lambda \in X(T)^+$ with $\lambda = \sum_{i=1}^n a_i\lambda_i$. The following holds.*

1) The restriction of the simple kL_{L_I}-module $L_{L_I}(\lambda)$ to L'_I is given by

$$L_{L_I}(\lambda)|_{L'_I} = L_{L'_I}(\lambda|_{T_{L'_I}}),$$

where $\lambda|_{T_{L'_I}} = \sum_{i\in I} a_i\lambda_i$.

2) The weights of $L_{L'_I}(\lambda|_{T_{L'_I}})$ are given by

$$\Lambda(L_{L'_I}(\lambda|_{T_{L'_I}})) = \{(\lambda-\nu)|_{T_{L'_I}} \mid \nu \in \mathbb{Z}I \text{ and } \lambda-\nu \in \Lambda(L(\lambda))\}.$$

3) For $\nu \in \mathbb{Z}I$, we have $m_{L_G(\lambda)}(\lambda-\nu) = m_{L_{L'_I}(\lambda|_{T_{L'_I}})}((\lambda-\nu)|_{T_{L'_I}})$.

We will use the next proposition in order to recursively answer Question 3 using inclusions of Levi factors of parabolic subgroups. The setup is as follows. Let Y be a simply connected simple algebraic group and let X be a closed connected simple subgroup of Y. Let $B_Y = U_YT_Y$ be a Borel subgroup of Y, with U_Y the unipotent radical of B_Y and T_Y a maximal torus of Y. Let $B_X = U_XT_X$ be a Borel subgroup of X with $U_X = U_Y \cap X$ and $T_X = T_Y \cap X$.

Proposition 1.5.3. *Let $I \subseteq \Delta(Y)$, $J \subseteq \Delta(X)$. Assume*

1) $L'_J = L'_I \cap X$,

2) for $\alpha \in \Phi(Y)$, $\alpha \in \mathbb{Z}I$ if and only if $\alpha|_{T_X} \in \mathbb{Z}J$.

Let $\lambda \in X(T_Y)^+$. If

$$L_Y(\lambda)|_X = L_X(\nu_1)^{m_{\nu_1}}/\cdots/L_X(\nu_k)^{m_{\nu_k}}$$

for $\nu_i \in X(T_X)^+$ distinct, then

$$L_{L'_I}(\lambda|_{T_{L'_I}})|_{L'_J} = L_{L'_J}(\tilde{\nu_1}|_{T_{L'_J}})^{m_{\tilde{\nu_1}}}/\cdots/L_{L'_J}(\tilde{\nu_\ell}|_{T_{L'_J}})^{m_{\tilde{\nu_\ell}}},$$

where $\{\tilde{\nu_1},\dots,\tilde{\nu_\ell}\} = \{\nu_i$ such that $\lambda|_{T_X} - \nu_i \in \mathbb{Z}J\}$. In particular, if L'_J acts with r composition factors on $L_{L'_I}(\lambda|_{T_{L'_I}})$, then X acts with at least r composition factors on $L_Y(\lambda)$.

Proof. Let V be a kX-module. Let V_0 denote the T_X-module

$$V_0 = \bigoplus_{\beta\in\mathbb{Z}J} V_{\lambda|_{T_X}-\beta},$$

and let V_1 be a T_X-complement of V_0 in V. Note that V_0 and V_1 are L_J modules, hence

$$V|_{L_J} = V_0 \oplus V_1.$$

Claim: *Assume $V = L_X(\gamma_1)^{m_{\gamma_1}}/\cdots/L_X(\gamma_r)^{m_{\gamma_r}}$ with $\gamma_i \in X(T_X)^+$ distinct, then*

$$V_0|_{L_J} = L_{L_J}(\tilde{\gamma_1})^{m_{\tilde{\gamma_1}}}/\cdots/L_{L_J}(\tilde{\gamma_s})^{m_{\tilde{\gamma_s}}}$$

with $\{\tilde{\gamma_1},\dots,\tilde{\gamma_s}\} = \{\gamma_i$ such that $\lambda|_{T_X} - \gamma_i \in \mathbb{Z}J\}$.

Proof of the claim. We prove the claim by induction on r. If $r = 1$, then V is irreducible and so $V = L_X(\gamma)$ for some $\gamma \in X(T_X)^+$. Note that

$$\{\lambda|_{T_X} - \beta \mid \beta \in \mathbb{Z}J\} \cap \{\gamma - \beta \mid \beta \in \mathbb{Z}J\} \neq \emptyset$$

if and only if $\lambda|_{T_X} - \gamma \in \mathbb{Z}J$ if and only if $\{\lambda|_{T_X} - \beta \mid \beta \in \mathbb{Z}J\} = \{\gamma - \beta \mid \beta \in \mathbb{Z}J\}$. By definition of V_0 and $L_{L_J}(\gamma)$, we have

$$V_0|_{L_J} = \begin{cases} L_{L_J}(\gamma) & \text{if } \lambda|_{T_X} - \gamma \in \mathbb{Z}J \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the base case of the induction holds. Let $r > 1$ and assume the result holds for any kX-module with at most $r-1$ composition factors. Let W be a maximal submodule of V. Assume without loss of generality that $V/W \cong L_X(\gamma_1)$. By the induction hypothesis, we have $W|_{L_J} = W_0|_{L_J} \oplus W_1|_{L_J}$, where $W_0|_{L_J} = L_{L_J}(\tilde{\gamma_1})^{m_{\tilde{\gamma_1}}-\delta_{\tilde{\gamma_1},\gamma_1}}/\dots/L_{L_J}(\tilde{\gamma_s})^{m_{\tilde{\gamma_s}}-\delta_{\tilde{\gamma_s},\gamma_1}}$. We have

$$W_i \subseteq V_i \text{ and } V_i/W_i \cong L_X(\gamma_1)_i. \tag{1.49}$$

Since

$$L_X(\gamma_1)_0|_{L_J} = \begin{cases} L_{L_J}(\gamma_1) & \text{if } \lambda|_{T_X} - \gamma_1 \in \mathbb{Z}J \\ 0 & \text{otherwise,} \end{cases}$$

restricting (1.49) to L_J implies the claim. ■

We apply the claim to $V = L_Y(\lambda)|_X$ and get

$$(L_Y(\lambda)|_X)_0|_{L_J} = L_{L_J}(\tilde{\nu_1})^{m_{\tilde{\nu_1}}}/\cdots/L_{L_J}(\tilde{\nu_\ell})^{m_{\tilde{\nu_\ell}}},$$

hence by 1) of Proposition 1.5.2,

$$(L_Y(\lambda)|_X)_0|_{L'_J} = L_{L'_J}(\tilde{\nu_1}|_{T_{L'_J}})^{m_{\tilde{\nu_1}}}/\cdots/L_{L'_J}(\tilde{\nu_\ell}|_{T_{L'_J}})^{m_{\tilde{\nu_\ell}}}. \tag{1.50}$$

Now, since for $\alpha \in \Phi(Y)$, $\alpha \in \mathbb{Z}I$ if and only if $\alpha|_{T_X} \in \mathbb{Z}J$, we have

$$\left.\left(\bigoplus_{\alpha\in\mathbb{Z}I} L_Y(\lambda)_{\lambda-\alpha}\right)\right|_{T_X} = \bigoplus_{\beta\in\mathbb{Z}J} L_Y(\lambda)_{\lambda|_{T_X}-\beta}.$$

Moreover, $L_{L_I}(\lambda) = \bigoplus_{\alpha\in\mathbb{Z}I} L_Y(\lambda)_{\lambda-\alpha}$ and $(L_Y(\lambda)|_X)_0 = \bigoplus_{\beta\in\mathbb{Z}J} L_Y(\lambda)_{\lambda|_{T_X}-\beta}$. Since additionally, $L'_J = L'_I \cap X$, we get

$$L_{L_I}(\lambda)|_{L'_J} = (L_Y(\lambda)|_X)_0|_{L'_J}. \tag{1.51}$$

Applying 1) of Proposition 1.5.2, to the left-hand side (1.51) and expanding the right-hand side of (1.51) using (1.50), we obtain

$$L_{L'_I}(\lambda|_{T_{L'_I}})|_{L'_J} = L_{L'_J}(\tilde{\nu_1}|_{T_{L'_J}})^{m_{\tilde{\nu_1}}}/\cdots/L_{L'_J}(\tilde{\nu_\ell}|_{T_{L'_J}})^{m_{\tilde{\nu_\ell}}}$$

which proves the lemma. □

Remark 1.5.4. Recall Definition 1.3.19 of a problematic case. Suppose that L'_J acts on $L_{L'_I}(\lambda|_{T_{L'_I}})$ with exactly r composition factors known by the induction step, that is X acts on $L_Y(\lambda)$ with at least r composition factors by Proposition 1.5.3, let us say $\nu_1, \nu_2, \ldots, \nu_r \in X(T_X)^+$ with $\lambda|_{T_X} - \nu_i \in \mathbb{Z}J$. Let $\theta \in X(T_X)^+$ be a problematic case for ν_i for some $i \in \{1, \ldots, r\}$. If $\lambda|_{T_X} - \theta \in \mathbb{Z}J$, then we can use the following equality in order to solve the problematic case.

$$m_{L_Y(\lambda)|_X}(\theta) = \sum_{i=1}^{r} m_{L_X(\nu_i)}(\theta).$$

A similar reasoning holds if $\theta \in X(T_Y)^+$ is a problematic case for λ by taking the restriction of θ to T_X. For an example of how this remark is applied, we refer to Subsection 2.5.1.3 on Page 73.

Chapter 2

The embedding $(\boldsymbol{F_4}, \boldsymbol{E_6})$

The goal of this chapter is to prove the main result of this thesis, namely to answer Question 3 for the embedding $F_4 < E_6$. The answer can be found in Proposition 2.5.1.

2.1 Preliminaries

Let Y be a simply connected simple algebraic group of type E_6. Let $B_Y = U_Y T_Y$ be a Borel subgroup of Y. Following [Sei91, Theorem (15.1)], let X be the closed connected simple subgroup of type F_4 given by the fixed point subgroup of a graph automorphism of Y. Let $B_X = X \cap B_Y$ be a Borel subgroup of X, with $B_X = U_X T_X$, where $T_X = X \cap T_Y$ is a maximal torus of X and $U_X = U_Y \cap X$ is the unipotent radical of B_X. Let $\Delta(Y) = \{\alpha_1, \alpha_2, \dots, \alpha_6\}$ be a base of $\Phi(Y)$ corresponding to B_Y and $\Delta(X) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a base of $\Phi(X)$ corresponding to B_X, where we label the Dynkin diagrams as in Subsection 1.1.5. Then $\mathscr{L}(X)$ embeds into $\mathscr{L}(Y)$ as follows

$$e_{\beta_1} = e_{\alpha_2},\ \ e_{\beta_2} = e_{\alpha_4},\ \ e_{\beta_3} = e_{\alpha_3} + e_{\alpha_5},\ \ e_{\beta_4} = e_{\alpha_1} + e_{\alpha_6},$$

where $e_{\alpha_i} \in \mathscr{L}(Y)_{\alpha_i}$ and $e_{\beta_i} \in \mathscr{L}(X)_{\beta_i}$. We thus get the following restriction to T_X of the simple roots in $\Phi(Y)$

$$\beta_1 = \alpha_2|_{T_X}, \quad \beta_2 = \alpha_4|_{T_X}, \quad \beta_3 = \alpha_3|_{T_X} = \alpha_5|_{T_X}, \quad \beta_4 = \alpha_1|_{T_X} = \alpha_6|_{T_X}. \tag{2.1}$$

Denote by $\{\lambda_1, \lambda_2, \dots, \lambda_6\}$ the set of fundamental weights in $X(T_Y)$ corresponding to $\Delta(Y)$ and by $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ the set of fundamental weights in $X(T_X)$ corresponding to $\Delta(X)$. The change of basis from simple roots to fundamental weights and (2.1) imply that

$$\mu_1 = \lambda_2|_{T_X}, \quad \mu_2 = \lambda_4|_{T_X}, \quad \mu_3 = \lambda_3|_{T_X} = \lambda_5|_{T_X}, \quad \mu_4 = \lambda_1|_{T_X} = \lambda_6|_{T_X}.$$

Consider a pair (I, J) with $(I, J) = (\{\alpha_3, \alpha_4, \alpha_5\}, \{\beta_2, \beta_3\}), (\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \{\beta_1, \beta_2, \beta_3\})$ or $(\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}, \{\beta_2, \beta_3, \beta_4\})$. For each pair (I, J), note that $L_J' = L_I' \cap X$ and for $\alpha \in \mathbb{Z}\Phi(Y)$, $\alpha \in \mathbb{Z}I$ if and only if $\alpha|_{T_X} \in \mathbb{Z}J$. Therefore, we can solve Question 3 by recursively applying Proposition 1.5.3 to the Levi factors L_I and L_J of the standard parabolic subgroups P_I of Y and P_J of X. The inclusions of pairs of subgroups are drawn in Fig. 2.1, where $Y_{\mathrm{type}(L_I)} = L_I'$ and

$$\begin{tikzcd}
 & (X_{C_3}, Y_{A_5}) \arrow[dr, hook] & \\
(X_{C_2}, Y_{A_3}) \arrow[ur, hook] \arrow[dr, hook] & & (X, Y) = (X_{F_4}, Y_{E_6}) \\
 & (X_{B_3}, Y_{D_4}) \arrow[ur, hook] &
\end{tikzcd}$$

Figure 2.1: Inclusions of pairs of subgroups

$X_{\mathrm{type}(L_J)} = L'_J$ denote the derived subgroup of L_I and L_J, respectively. For a pair (H, G) as in Fig. 2.1, let $B_G = T_G U_G = B_Y \cap G$ and $B_H = T_H U_H = B_X \cap H$.

For a pair (H, G) as in Fig. 2.1, the next result, from [Sei87, Tes88, CT19] and independently from [Sup85] for G of type A, tells us when H acts irreducibly on an irreducible kG-module.

Theorem 2.1.1. *Let (H, G) be as in Fig. 2.1 and $\lambda \in X(T_G)^+$ be a p-restricted non trivial weight. Then H acts irreducibly on $L_G(\lambda)$ if and only if λ is listed in Table 2.2 up to graph automorphism.*

Type of (H, G)	$\lambda \in X(T_G)^+$	Conditions
(C_2, A_3)	$a\lambda_1$	$1 \leq a \leq p-1$
	$a\lambda_1 + b\lambda_2$	$a + b = p - 1,\ ab \neq 0$
(C_3, A_5)	$a\lambda_1$	$1 \leq a \leq p-1$
	$a\lambda_i + b\lambda_{i+1},\ i \in \{1, 2\}$	$a + b = p - 1,\ a \neq 0$ if $i = 2$
(B_3, D_4)	$c\lambda_3$	$c \neq 0$
	$a\lambda_1 + c\lambda_3$	$a + c + 2 \equiv 0 \mod p, ac \neq 0$
	$b\lambda_2 + c\lambda_3$	$b + c = p - 1, bc \neq 0$
	$a\lambda_1 + b\lambda_2 + c\lambda_3$	$a = c, a + b = p - 1, ab \neq 0$
(F_4, E_6)	$\lambda_1 + (p-2)\lambda_3$	
	$(p-3)\lambda_1$	

Table 2.2: Irreducible cases

Let (H, G) be as in Fig. 2.1. Let $\lambda \in X(T_G)^+$ be p-restricted and assume H does not act irreducibly on $L_G(\lambda)$, that is λ does not appear in Table 2.2 up to graph automorphism. Set $\mu = \lambda|_{T_H}$. The T_H-weight μ affords the highest weight of a composition factor for H acting on $L_G(\lambda)$ since $B_H \subseteq B_G$. We choose the highest weight ν of a second composition factor for H acting on $L_G(\lambda)$ as follows. By (2.1), any weight $\gamma \in \Lambda(L_G(\lambda)|_H))^+$ different from μ satisfies $\gamma \not\succeq \mu$ and μ appears in $L_G(\lambda)|_H$ with multiplicity 1. We will thus select ν as in Corollary 1.4.7. That is, in the context of Corollary 1.4.7, $\mu = \mu_1$ and ν is such that for all $\gamma \in \Lambda(L_G(\lambda)|_H)^+$ with $\gamma \succneqq \nu$, we have

$$m_{L(\lambda)|_H}(\gamma) = m_{L(\mu)}(\gamma)$$

and

$$m_{L(\lambda)|_H}(\nu) > m_{L(\mu)}(\nu).$$

Note that in the computations previously performed to find ν, we calculate $\operatorname{ch} L(\mu)^S$, where

$$S = \{\gamma \in X(T_H)^+ \mid \gamma \in \Lambda(L_G(\lambda)|_H)^+ \text{ and } \gamma \succeq \nu\}.$$

In most cases, we also compute $[\mu : \nu]$ along the way, which allows us to check if hypothesis 3) of Corollary 1.4.7 is satisfied. Assume it is, that is $[\mu : \nu] = 0$. Additionally, we check that for the given embeddings, we have $e_{\beta_0} \in \langle e_{\alpha_0} \rangle$ for α_0 and β_0 the largest root in $\Phi^+(G), \Phi^+(H)$ respectively. The next remark implies that $L_G(\lambda)|_H$ is self-dual and so we can apply Corollary 1.4.7 whenever it is needed.

Remark 2.1.2. Let (H, G) be as Fig. 2.1. The restriction to H of a simple kG-module is self-dual by Lemma 1.4.2, since H is obtained from G as the fixed-point subgroup of a graph automorphism of G.

Let us introduce some notations for the rest of this thesis.

Notation 2.1.3. Let (H, G) be as in Fig. 2.1. Let $\lambda \in X(T_G)^+$. Set $\mu = \lambda|_{T_H}$ and let $\beta \in \mathbb{Z}_{\geq 0}\Phi^+(H)$. Consider the set $S_{\lambda,\beta} \subseteq X(T_G)^+$ given by

$$S_{\lambda,\beta} = \{\gamma \in \Lambda(\lambda) \cap X(T_G)^+ \mid \gamma|_{T_H} \succeq \mu - \beta\}.$$

Note that the set $S_{\lambda,\beta}$ satisfies Condition 1.1.11 for λ by (2.1). We denote $\operatorname{ch} L(\lambda)^{S_{\lambda,\beta}}$ by $\operatorname{ch} L(\lambda)_\beta$. For $\theta \in X(T_H)^+$, consider the set $S_{\theta,\beta} \subseteq X(T_H)^+$ given by

$$S_{\theta,\beta} = \{\gamma \in \Lambda(\theta) \cap X(T_H)^+ \mid \gamma \succeq \mu - \beta\}.$$

Note that $S_{\theta,\beta}$ satisfies Condition 1.1.11 for θ and we denote $\operatorname{ch} L(\theta)^{S_{\theta,\beta}}$ by $\operatorname{ch} L(\theta)_\beta$.

For $\lambda = \sum_{i=1}^n a_i\lambda_i \in X(T_G)$, we also denote $\lambda = (a_1, \dots, a_n)$. If $\alpha = \sum_{i=1}^n r_i\alpha_i \in \mathbb{Z}_{\geq 0}\Phi^+(G)$, we denote $\lambda - \alpha$ by $\lambda - r_1r_2\cdots r_n$. Finally, set $\operatorname{ch} L(\lambda) = 0$ if $\lambda + \rho$ is dominant but λ is not. We adopt the same notations and conventions for the weights in $X(T_H)$.

We now present general results which we will apply repeatedly in this chapter. We start with an easy lemma describing for a group of type A_2 the structure of p-restricted Weyl modules.

Lemma 2.1.4. *Let G be of type A_2. Let $\lambda = (a, b) \in X(T)^+$ be a p-restricted weight. Then*

- *$V(\lambda)$ is irreducible if $a + b + 1 < p$, or $a = p - 1$, or $b = p - 1$,*
- *$V(\lambda) = L(\lambda)/L(\lambda - r(\alpha_1 + \alpha_2))$ with $r = a + b + 2 - p$ otherwise.*

Proof. Let $\lambda \in X(T)^+$ be a p-restricted weight. In type A_2, there are only two p-restricted alcoves given by $(1, 1, 1)$ and $(1, 1, 2)$, where the positive roots are ordered as in Appendix B. Note that the intersection of these two alcoves corresponds to the wall $F_{\alpha_1+\alpha_2,1}$. By Corollary 1.2.4, if λ belongs to the upper closure of the fundamental alcove, that is $a + b + 2 \leq p$, then $V(\lambda)$ is irreducible. Assume λ belongs to the upper closure of the alcove given by $(1, 1, 2)$, that is $p + 1 \leq a + b + 2 \leq 2p$.

By Proposition 1.2.1 the only weight apart from λ which could afford the highest weight of a composition factor for $V(\lambda)$ is

$$s_{11,1} \bullet \lambda = \lambda - (a+b+2-p)11.$$

If $a = p-1$ or $b = p-1$, then $s_{11,1} \bullet \lambda \in X(T) \setminus X(T)^+$, which implies that $V(\lambda)$ is irreducible. Assume $a, b \neq p-1$. We show that if $\mu = s_{11,1} \bullet \lambda$, then the hypotheses of Proposition 1.2.5 are satisfied. The only hypothesis which is not clear is that μ is maximal with the property of being linked to λ. Any weight $\nu \in X(T)$ linked to λ has to be linked to one of the following weights

$$\begin{aligned}
s_{11,r_1} \bullet \lambda &= \lambda - (a+b+2-r_1 p)11,\\
s_{10,r_2} \bullet \lambda &= \lambda - (a+1-r_2 p)10,\\
s_{01,r_3} \bullet \lambda &= \lambda - (b+1-r_3 p)01,
\end{aligned}$$

with $r_1, r_2, r_3 \in \mathbb{Z}$. Note that there is no value of r_2 such that $s_{10,r_2} \bullet \lambda \succeq s_{11,1} \bullet \lambda$ and similarly for r_3. The condition $\nu \preceq \lambda$ forces $a+b+2 \geq r_1 p$, hence $r_1 \leq 1$. Moreover, if $\mu \preceq \nu$, then $r_1 \geq 1$. Thus $r_1 = 1$ and μ is maximal among the weights linked to λ. Therefore, we get $[V(\lambda) : L(\mu)] = 1$ and the lemma follows. □

The next lemma is a basic result and a proof of it can be found in [Tes88, (1.30)].

Lemma 2.1.5. *Let $\lambda = \sum_{i=1}^{n} a_i \lambda_i \in X(T)^+$ be p-restricted. For $1 \leq i \leq n$,*

$$m_{L(\lambda)}(\lambda - r\alpha_i) = \begin{cases} 1 & \textit{if } 0 \leq r \leq a_i, \\ 0 & \textit{otherwise.} \end{cases}$$

The next two lemmas give some information about a specific weight space in irreducible modules for a simple group G of rank 2 or of type B_3. For a proof of Lemma 2.1.6, see [Tes88, (1.35)]. Note that for G of type A_2, the next result is a particular case of Lemma 2.1.4.

Lemma 2.1.6. *Let G be a simple group of rank two. Let $\lambda = (a, b) \in X(T)^+$ be p-restricted, with $a, b > 0$.*

1) If G is of type A_2, then

$$m_{L(\lambda)}(\lambda - 11) = \begin{cases} 1 & \textit{if } a+b+1 = p \\ 2 & \textit{otherwise.} \end{cases}$$

2) If G is of type B_2, then

$$m_{L(\lambda)}(\lambda - 11) = \begin{cases} 1 & \textit{if } 2a+b+2 \equiv 0 \mod p \\ 2 & \textit{otherwise.} \end{cases}$$

3) If G is of type G_2, then

$$m_{L(\lambda)}(\lambda - 11) = \begin{cases} 1 & \textit{if } a+3b+3 \equiv 0 \mod p \\ 2 & \textit{otherwise.} \end{cases}$$

Lemma 2.1.7. *Let G be of type B_3. If $\mu = (a, 0, c) \in X(T)^+$ satisfies $2a + c + 4 \equiv 0 \mod p$, then $[V(\mu) : L(\mu - 111)] = 1$.*

Proof. Assume $2a+c+4 \equiv 0 \mod p$. Let $2a+c+4 = rp$ for some $r > 0$, then $s_{111,r} \bullet (\mu - 111) = \mu$, hence $\mu - 111$ is linked to μ. Moreover, if $a \neq p - 1$, then $\mu - 111$ is maximal with respect to the weights linked to μ. Therefore by Proposition 1.2.5, $[V(\mu) : L(\mu - 111)] = 1$. If $a = p - 1$, then $c = p - 2$ and both $\mu - 110$, $\mu - 011$ are linked to μ. Using the JSF, we deduce that $[V(\mu) : L(\mu - 111)] = 1$ in this case too. $\square$

The following result will be used extensively. It is a consequence of the description for A_3 of the characters of p-restricted Weyl modules.

Lemma 2.1.8. *Assume $p > 3$. Let G be of type A_3. Let $\lambda \in X(T)^+$ be a p-restricted weight. For $\theta \in \Lambda(\lambda)$, if $[V(\lambda) : L(\theta)] \neq 0$, then $[V(\lambda) : L(\theta)] = 1$.*

Proof. If $\lambda \not\succeq \theta$, then $[V(\lambda) : L(\theta)] = 0$. Assume $\lambda \succeq \theta$ and $\lambda \in W_p \bullet \theta$, since otherwise $[V(\lambda) : L(\theta)] = 0$ by Proposition 1.2.1. Let $\lambda = w \bullet \theta$ for $w \in W_p$. Let $\theta' \in X(T_{A_3})^+$ be a weight lying in the interior of the alcove containing θ in its upper closure and replace λ by $\lambda' = w \bullet \theta'$. Since W_p acts simply transitively on the set of closure of alcoves, we get that λ' lies in the interior of an alcove. By Proposition 1.3.10 we have $[V(\lambda) : L(\theta)] = [V(\lambda') : L(\theta')]$ and we can assume without loss of generality that λ, θ lie in the interior of alcoves. Using the description of the composition factors for $V(\lambda)$ in [Jan03, II. 8.20], we deduce that $[V(\lambda) : L(\theta)] \leq 1$, which proves the result. $\square$

The next lemma gives a first exposition to reasonings used in the proof of Proposition 2.2.1.

Lemma 2.1.9. *Let G be of type A_{2n-1} and H be a subgroup of type C_n of G given by the fixed points of a graph automorphism of G. Let $\{\alpha_i\}$ denote the set of simple roots in $\Phi(G)$. Let $\lambda = \sum_{i=1}^{n} a_i \lambda_i \in X(T_G)^+$ be p-restricted. If there is $i \in \{1, \ldots, n-1\}$ such that $a_i a_{2n-i} \neq 0$, then $(\lambda - \alpha_i)|_{T_H}$ affords the highest weight of composition factor for H acting on G. Moreover, if H acts on $L_G(\lambda)$ with exactly two composition factors, then either $a_i = 1$ or $a_{2n-i} = 1$.*

Proof. Let $\{\beta_i\}$ denote the simple roots in $\Phi(H)$. Set $\mu = \lambda|_{T_H}$. Note that $\alpha_i|_{T_H} = \alpha_{2n-i}|_{T_H} = \beta_i$. Therefore by Lemma 2.1.5, we have

$$m_{L_G(\lambda)|_H}(\mu - \beta_i) = m_{L_G(\lambda)}(\lambda - \alpha_i) + m_{L_G(\lambda)}(\lambda - \alpha_{2n-i}) = 2,$$

and $m_{L_H(\mu)}(\mu - \beta_i) \leq 1$. Hence a second composition factor for H on $L_G(\lambda)$ is given by the highest weight $\nu = \mu - \beta_i$. Similarly, if both $a_i, a_{2n-i} > 1$, then

$$m_{L_G(\lambda)|_H}(\mu - 2\beta_i) = m_{L_G(\lambda)}(\lambda - 2\alpha_i) + m_{L_G(\lambda)}(\lambda - 2\alpha_{2n-i}) + m_{L_G(\lambda)}(\lambda - \alpha_i - \alpha_{2n-i}) = 3,$$

whereas

$$m_\mu(\mu - 2\beta_i) + m_\nu(\mu - 2\beta_i) \leq 1 + 1 = 2.$$

Hence X acts on $L_Y(\lambda)$ with more than two composition factors. $\square$

The following powerful result proved by Jantzen in his thesis [Jan73, p. 113], tells us precisely when a p-restricted Weyl module in type A is simple.

Proposition 2.1.10 (Jantzen). *Let G be of type A_n. Let $\lambda \in X(T)^+$ be a p-restricted weight. Then $V(\lambda)$ is simple if and only if for each positive root $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ $(1 \leq i \leq j \leq n)$ with $\langle \lambda + \rho, \alpha^\vee \rangle = ap^s + bp^{s+1}$ where $a, b, s \in \mathbb{Z}_{\geq 0}$ and $0 < a < p$, there exist integers $i_0 = i < i_1 < i_2 < \cdots < i_b \leq j < i_{b+1} = j + 1$ such that*

$$\langle \lambda + \rho, (\alpha_{i_\nu} + \alpha_{i_\nu+1} + \cdots + \alpha_{i_{\nu+1}-1})^\vee \rangle = \begin{cases} p^{s+1} & \text{for } 1 \leq \nu \leq b-1 \text{ and for one } \nu \in \{0, b\} \\ ap^s & \text{for the other } \nu \in \{0, b\}. \end{cases}$$

Corollary 2.1.11. *Assume $p > 2$ and let G be of type A_n with $n \geq 2$. Let $\lambda \in X(T)^+$ be such that $\lambda = c\lambda_1 + \lambda_2$ for $1 \leq c \leq p-1$. Then $V(\lambda)$ is simple if and only if $c \neq p-2$.*

Proof. Let $\lambda = c\lambda_1 + \lambda_2$. If $c = p-2$, then $V_Y(\lambda)$ is reducible by Lemma 2.1.4 and Proposition 1.5.1. Assume $c \neq p-2$. Recall the notations of Proposition 2.1.10. Let $\alpha \in \Phi^+(A_n)$ with $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, where $1 \leq i \leq j \leq n$. Assume $i \geq 3$, then the condition of Proposition 2.1.10 holds for λ and α, since the trivial Weyl module for A_{j-i+1} is irreducible. Similarly, the condition holds for $i = 2$, since the Weyl module with highest weight given by the first fundamental weight for A_{j-i+1} is irreducible. If $i = 1$ and $j = 1$, then the Weyl module with highest weight $c\lambda_1$ for A_1 is irreducible, hence the condition of Proposition 2.1.10 holds. Let $i = 1$ and $j \geq 2$. Write $j = qp + r$ with $0 \leq r < p$ and $q \in \mathbb{Z}_{\geq 0}$. Note that $\langle \lambda + \rho, \alpha^\vee \rangle = j + c + 1 = qp + r + c + 1$. The tuple (a, b, s) of Proposition 2.1.10 is given as follows.

$$(a, b, s) = \begin{cases} ((q+1)p, 0, 0) & \text{if } r + c + 1 = p \\ (r + c + 1, q, 0) & \text{if } r + c + 1 < p \\ (r + c + 1 - p, q + 1, 0) & \text{if } r + c + 1 > p \end{cases}$$

If $c = p - 1$, let

$$(i_0, i_1, i_2, i_3, \ldots, i_b, i_{b+1}) = (1, 2, p+1, 2p+1, \ldots, bp+1, bp+r+1),$$

If $c \neq p - 1$, let

$$(i_0, i_1, \ldots, i_{b+1}) = \begin{cases} (1, p-c, 2p-c, \ldots, bp-c, (b-1)p+r+1) & \text{if } r + c + 1 > p \\ (1, p-c, 2p-c, \ldots, bp-c, bp+r+1) & \text{if } r + c + 1 < p \\ (1, j+1) & \text{if } r + c + 1 = p. \end{cases}$$

The previous sequences of integers satisfy the condition of Proposition 2.1.10 and so $V(\lambda)$ is irreducible. □

The last result of this section is taken from [McN98, Proposition 4.2.2. (h)].

Lemma 2.1.12. *Let G be of type C_n, then $V(a\lambda_1)$ is simple for $0 \leq a \leq p-1$.*

Proof. Combining [Sei87, (1.14)] with [Sei87, (8.1) (c)] proves the lemma. □

The rest of this chapter is devoted to solving Question 3 for the embedding (F_4, E_6). The reader can see the methods we have presented so far applied in detail in Subsection 2.3.1.4.1 on Page 44. The rest of the time, we may omit some details, since the arguments are repetitive.

2.2 $(X,Y)=(C_2,A_3)$

The goal of this section is to prove the following proposition.

Proposition 2.2.1. *Let k be an algebraically closed field of characteristic $p\geq 5$. Let Y be a simply connected simple algebraic group of type A_3 over k and let X be the maximal closed connected subgroup of type C_2 of Y given by the fixed points of a graph automorphism of Y. Let $\lambda\in X(T_Y)^+$ be a p-restricted weight and set $\mu=\lambda|_{T_X}$. Then X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if λ is listed in Table 2.3 up to graph automorphism. Moreover, $L_Y(\lambda)|_X=L_X(\mu)\oplus L_X(\nu)$ with ν as in Table 2.3.*

λ (a,b,c)	Conditions $abc\neq 0$	ν
$(0,b,0)$	$b=1,p-1$	$\mu-11=(0,b-1)$
$(a,b,0)$	$a\neq p-2,p-4,b=1$	$\mu-11=(a,0)$
	$a+b=p,3\leq a\leq p-2$	$\mu-11=(a,b-1)$
$(a,0,c)$	$a=1,c\neq p-1$	$\mu-10=(c-1,1)$
(a,b,c)	$a=1,b+c=p-1,c\neq 1$	$\mu-10=(c-1,b+1)$

Table 2.3: The case $C_2\leq A_3$

Note that the decomposition into a direct sum follows directly from Lemmas 1.4.2 and 1.4.3. The rest of the proof of Proposition 2.2.1 can be found in Subsection 2.2.1.

2.2.1 Proof of proposition 2.2.1. — Recall the notations introduced in Notation 2.1.3, (1.6) and (1.7) related to the truncated characters of simple or Weyl modules. Let $\{\alpha_1,\alpha_2,\alpha_3\}$ be a set of simple roots in $\Phi(Y)$ and $\{\lambda_1,\lambda_2,\lambda_3\}$ be the corresponding set of fundamental weights in $X(T_Y)$. Similarly, let $\{\beta_1,\beta_2\}$ be a set of simple roots in $\Phi(X)$ and $\{\mu_1,\mu_2\}$ be the corresponding set of fundamental weights in $X(T_X)$. Let $\lambda=a\lambda_1+b\lambda_2+c\lambda_3=(a,b,c)\in X(T_Y)^+$ be a p-restricted weight. Set $\lambda|_X=\mu\in X(T_X)^+$, then $\mu=(a+c)\mu_1+b\mu_2=(a+c,b)$. We record important information about the composition factors and the multiplicities in Table 2.4. It should be read as follows. Under the column ν lies the linear combination of simple roots in $\Phi(X)$ which we need to subtract from $\lambda|_{T_X}$ in order to get the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Similarly, the column under θ gives the linear combination of simple roots in $\Phi(X)$ we need to subtract from $\lambda|_{T_X}$ in order to get a weight θ satisfying

$$m_{L_Y(\lambda)|_X}(\theta)>m_{L_X(\mu)}(\theta)+m_{L_X(\nu)}(\theta).$$

The multiplicities appear in the columns labelled as follows.

$$\lambda(-)=m_{L_Y(\lambda)|_X}(-)\quad \mu(-)=m_{L_X(\mu)}(-)\quad \nu(-)=m_{L_X(\nu)}(-)$$

Whenever a multiplicity is preceded by an inequality indexed by V (i.e. $\leq_V$), we indicate the multiplicity in the corresponding Weyl module instead of the irreducible module. This provides a bound on the multiplicity in the irreducible module, since the multiplicity of a weight in the Weyl module is always greater than or equal to its multiplicity in the corresponding irreducible module. The last column of the table refers to the subsection in which the corresponding case is solved. Note that (C_2, A_3) corresponds to the first pair of subgroups in Fig. 2.1, hence to the base case of the inductive argument relying on Proposition 1.5.3. We therefore need to consider all the p-restricted weights $\lambda \in X(T_Y)^+$ up to graph automorphism. In what follows, whenever λ is written as a linear combination of parameters (i.e. a, b, c), we assume these parameters to be nonzero.

2.2.1.1 $\boldsymbol{\lambda = a\lambda_1}$. — Since λ appears in Table 2.2, X acts irreducibly on $L_Y(\lambda)$.

2.2.1.2 $\boldsymbol{\lambda = b\lambda_2}$. — Note that neither $\mu - 10$ nor $\mu - 01$ afford the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Moreover, we have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 11) = (2, 1)$, hence a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 11 = (0, b-1)$.

Let $b \neq 1, p-1$. Applying the JSF to λ yields $\operatorname{ch} L(\lambda)_{22} = \lambda$, hence we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 22) = (4, \leq_V 2, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

If $b = 1$, we have $\dim(L(\lambda), L(\mu), L(\nu)) = (6, 5, 1)$, which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

If $b = p-1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. That is, we have to show that none of the following weights affords the highest weight of an additional composition factor for X acting on $L_Y(\lambda)$.

$$\mu - 11 = (0, p-2) \qquad \mu - 12 = (2, p-4) \qquad \mu - 22 = (0, p-3)$$

By Lemma 1.4.9, it is enough to consider $\mu - 22$. Applying the JSF to λ, μ and ν yields $\operatorname{ch} L(\lambda)_{22} = \lambda - (\lambda - 121), \operatorname{ch} L(\mu)_{22} = \mu$ and $\operatorname{ch} L(\nu)_{11} = \nu$, respectively. Therefore $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 22) = (3, 2, 1)$, which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.2.1.3 $\boldsymbol{a\lambda_1 + b\lambda_2}$. — If $a + b = p - 1$, then λ appears in Table 2.2 and so X acts irreducibly on $L_Y(\lambda)$. Henceforth assume $a + b \neq p - 1$. Note that $\nu = \mu - 11 = (a, b-1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. We solve this case by separating it into two subcases depending on whether $b = 1$ or $b \neq 1$.

2.2.1.3.1 $\boldsymbol{b = 1}$. — If $a = p - 4$, Lemma 2.1.6 implies that ν affords the highest weight of a composition factor for $V_X(\mu)$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 11) = (3, 1)$ and ν also affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$.

Assume $a \neq p - 4$. We prove that X acts on $L_Y(\lambda)$ with exactly two composition factors. Let $a = 1$ (so that $p \neq 5$), then $\dim(L(\lambda), L(\mu), L(\nu)) = (20, 16, 4)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors. Let $a \neq 1$. We apply Corollary 1.4.7 and show that none of the following weights affords the highest weight of an additional composition factor for X acting on $L_Y(\lambda)$.

$$\mu - 11 = (a, 0) \qquad \mu - 21 = (a-2, 1)$$

By Lemma 1.4.9, it is enough to consider the weight $\mu - 21$. Moreover by Lemma 2.1.4, we deduce that $\operatorname{ch} L(\lambda)_{21} = \lambda$ and by Proposition 1.2.2, that $\operatorname{ch} L(\mu)_{21} = \mu$ and $\operatorname{ch} L(\nu)_{10} = \nu$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 21) = (4, 3, 1)$, which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.2.1.3.2 $\boldsymbol{b > 1}$. — Let $a + 2b + 2 \not\equiv 0 \mod p$ and additionally $a + b \neq p$ if $a > 1$. By Proposition 1.2.2, we get that $\operatorname{ch} L(\lambda)_{22} = \lambda$. Therefore $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 22) = (7 - \delta_{a,1}, \leq_V 4 - \delta_{a,1}, \leq_V 2)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a+2b+2 \equiv 0 \mod p$, then by Lemma 2.1.6 the weight ν affords the highest of a composition factor for $V_X(\mu)$. Moreover, by Lemma 2.1.4 we have $\operatorname{ch} L(\lambda)_{11} = \lambda$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)})$ $(\mu - 11) = (3, 1)$ and ν also affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$.

Let $a + b = p$ and $a \geq 3$. Note that we have excluded $a = 2$, since $a + b = p$ and $a = 2$ implies that $a + 2b + 2 \equiv 0 \mod p$. We prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. That is, we prove that none of the following weights affords the highest weight of an additional composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 11 &= (a, p - a - 1)\\
\mu - 21 &= (a - 2, p - a)\\
\mu - 12 &= (a + 2, p - a - 3)\\
\mu - 22 &= (a, p - a - 2)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider the weight $\mu - 22$. Note that the JSF applied to λ, μ and ν yields the truncated characters $\operatorname{ch} L(\lambda)_{22} = \lambda - (\lambda - 220)$, $\operatorname{ch} L(\mu)_{22} = \mu$ and $\operatorname{ch} L(\nu)_{11} = \nu$, respectively. Therefore $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 22) = (6, 4, 2)$, which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.2.1.4 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3}$. — By Lemma 2.1.9, the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 10 = (a + c - 2, 1)$ and either $a = 1$ or $c = 1$. Henceforth, assume without loss of generality that $a = 1$.

If $c = p - 1$, the weight μ is not p-restricted and ν affords the highest weight of a composition factor for $V_X(\mu)$. Therefore $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 10) = (2, 0, \leq_V 1)$ and ν also affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$.

If $c \neq p - 1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. That is, we prove that none of the following weights affords the highest weight of an additional composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 10 &= (c - 1, 1)\\
\mu - 20 &= (c - 3, 2)\\
\mu - 21 &= (c - 1, 0)
\end{aligned}$$

By Lemma 1.4.9 it is enough to consider the weight $\mu - 21$. Applying the JSF to λ, μ and ν yields $\operatorname{ch} L(\lambda)_{21} = \lambda - \delta_{c,p-3}(\lambda - 111)$, $\operatorname{ch} L(\mu)_{21} = \mu$ and $\operatorname{ch} L(\nu)_{11} = \nu - \delta_{c,p-3}(\nu - 11)$. Therefore

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 21) = (4 - \delta_{c,p-3} - \delta_{c,1}, 2, 2 - \delta_{c,p-3} - \delta_{c,1})$$

and X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.2.1.5 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3}$. — Repeating the argument appearing at the beginning of Subsection 2.2.1.4, a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 10 = (a+c-2, b+1)$ and we can assume without loss of generality that $a = 1$ and $c < p-1$.

If $b \neq p-2$ and $b \neq p-c-1$, then by Proposition 1.2.2, we have $\operatorname{ch} L(\lambda)_{11} = \lambda$. Thus $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 11) = (4, \leq_V 2, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

If $b = p-2$, the JSF applied to λ and μ yields $\operatorname{ch} L(\lambda)_{21} = \lambda - (\lambda - 110) - \delta_{c,1}(\lambda - 011)$ and $\operatorname{ch} L(\mu)_{21} = \mu - \delta_{c,1}(\mu - 11)$, respectively. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 21) = (6 - 2\delta_{c,1}, 3 - \delta_{c,1}, \leq_V 2 - \delta_{c,1})$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

If $b + c = p-1$ and $c \neq 1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. That is, we prove that none of the following weights affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 01 &= (c+3, p-c-3)\\
\mu - 10 &= (c-1, p-c)\\
\mu - 11 &= (c+1, p-c-2)\\
\mu - 20 &= (c-3, p-c+1)\\
\mu - 21 &= (c-1, p-c-1)
\end{aligned}$$

Note that $\mu - 01$ is not always p-restricted. Hence, we cannot include it in the usual argument which consists in applying Lemma 1.4.9. By comparing multiplicities, we get that $\mu - 01$ does not afford the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. For the remaining weights, it is enough to consider the weight $\mu - 21$ by Lemma 1.4.9. The JSF applied to λ, μ and ν yields $\operatorname{ch} L(\lambda)_{21} = \lambda - (\lambda - 011)$, $\operatorname{ch} L(\mu)_{21} = \mu$ and $\operatorname{ch} L(\nu)_{11} = \nu$, respectively. Therefore, $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 21) = (5, 3, 2)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

Table 2.4: Multiplicities for the proof of Proposition 2.2.1

λ	Conditions		ν	θ	Multiplicities			Details
$abc \neq 0$					$\lambda(\theta)$	$\mu(\theta)$	$\nu(\theta)$	
$(a,0,0)$					irreducible			Table 2.2
$(0,b,0)$	$b \neq 1, p-1$		11	22	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.2.1.2
	$b = 1$		11		two composition factors			Subsection 2.2.1.2
	$b = p-1$		11		two composition factors			Subsection 2.2.1.2
$(a,b,0)$	$b = 1$	$a = p-4$	11	11	3	1	$\leq_V 1$	Subsection 2.2.1.3.1
		$a \neq p-4, p-2$	11		two composition factors			Subsection 2.2.1.3.1
		$a = p-2$			irreducible			Table 2.2
	$b > 1$	$a+b = p, a \geq 3$	11		two composition factors			Subsection 2.2.1.3.2
		$a + 2b + 2 \equiv 0 \mod p$	11	11	3	1	$\leq_V 1$	Subsection 2.2.1.3.2
		$a + b = p-1$			irreducible			Table 2.2
		otherwise	11	22	$7 - \delta_{a,1}$	$\leq_V 4 - \delta_{a,1}$	$\leq_V 2$	Subsection 2.2.1.3.2
$(a,0,c)$	$a, c > 1$		10	20	3	$\leq_V 1$	$\leq_V 1$	Subsection 2.2.1.4
	$a = 1$	$c \neq p-1$	10		two composition factors			Subsection 2.2.1.4
	$a = 1$	$c = p-1$	10	10	2	0	$\leq_V 1$	Subsection 2.2.1.4
(a,b,c)	$a, c > 1$		10	20	3	$\leq_V 1$	$\leq_V 1$	Subsection 2.2.1.5
	$a = 1$	$c = p-1$	10	10	2	0	$\leq_V 1$	Subsection 2.2.1.5
	$a = 1, b = p-2$	$c \neq 1, p-1$	10	21	6	$\leq_V 3$	$\leq_V 2$	Subsection 2.2.1.5
		$c = 1$	10	21	4	2	$\leq_V 1$	Subsection 2.2.1.5
	$a = 1, b = p-c-1$	$c \neq 1$	10		two composition factors			Subsection 2.2.1.5
	otherwise		10	11	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.2.1.5

2.3 $(X, Y) = (C_3, A_5)$

The goal of this section is to prove the following proposition.

Proposition 2.3.1. *Let k be an algebraically closed field of characteristic $p \geq 7$. Let Y be a simply connected simple algebraic group of type A_5 over k and let X be the maximal closed connected subgroup of type C_3 of Y given by the fixed points of a graph automorphism of Y. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight and set $\mu = \lambda|_{T_X}$. Then X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if λ is listed in Table 2.5 up to graph automorphism. Moreover, $L_Y(\lambda)|_X = L_X(\mu) \oplus L_X(\nu)$ with ν as in Table 2.5.*

λ (a,b,c,d,e)	Conditions $abcde \neq 0$	ν
$(0,b,0,0,0)$	$b = 1$	$\mu - 121$
$(0,0,c,0,0)$	$c = 1, p-1$	$\mu - 011$
$(a,b,0,0,0)$	$a \neq p-2, p-6, b = 1$	$\mu - 121$
	$a + b = p,\ a \neq 4,\ b > 1$	$\mu - 121$
$(a,0,c,0,0)$	$a = p-3, c = 1$	$\mu - 011$
$(a,0,0,d,0)$	$a = 1, d = p-1$	$\mu - 110$
	$a = p-4, d = 1$	$\mu - 110$
$(a,0,0,0,e)$	$a \neq p-1, e = 1$	$\mu - 100$
$(a,b,c,0,0)$	$a = 1, b + c = p-1, b \neq 2$	$\mu - 111$
$(a,b,0,d,0)$	$a = 2, b = p-3, d = 1$	$\mu - 010$
$(a,b,0,0,e)$	$a + b = p-1, e = 1$	$\mu - 100$
$(a,0,c,d,0)$	$a = 1, c + d = p-1, d \neq 1, p-2$	$\mu - 110$

Table 2.5: The case $C_3 \leq A_5$

Note that the decomposition into a direct sum follows directly from Lemmas 1.4.2 and 1.4.3. The rest of the proof of the proposition can be found in Subsection 2.3.1.

2.3.1 Proof of Proposition 2.3.1. — Recall the notations introduced in Notation 2.1.3 and (1.6) and (1.7) related to the truncated characters of simple or Weyl modules. Let $\{\alpha_1, \ldots, \alpha_5\}$ be a set of simple roots in $\Phi(Y)$ and $\{\lambda_1, \ldots, \lambda_5\}$ be the corresponding set of fundamental weights in $X(T_Y)$. Similarly, let $\{\beta_1, \beta_2, \beta_3\}$ be a set of simple roots in $\Phi(X)$ and $\{\mu_1, \mu_2, \mu_3\}$ be the corresponding set of fundamental weights in $X(T_X)$. Let $\lambda = (a,b,c,d,e) \in X(T_Y)^+$ be a p-restricted weight. Set $\lambda|_{T_X} = \mu$, then $\mu = (a+e, b+d, c)$. As for the proof of Proposition 2.2.1, we record information about the multiplicities and the composition factor in Table 2.31. Whenever a multiplicity in Table 2.31 appears with an inequality with subscript BS (i.e $\leq_{BS}$), it means we did not determine the exact character of the corresponding irreducible module, but we can bound the multiplicity in the irreducible module. In this proof, we use for the first time the tables which summarize the inductive process described in Subsection 1.3.2 and the application of Propositions 1.3.9 and 1.3.10. Recall Remarks 1.3.11 to 1.3.13 and Remarks 1.3.21 to 1.3.24 which explain how they should be read.

By Propositions 1.5.3 and 2.2.1 and Theorem 2.1.1, if X acts on $L_Y(\lambda)$ with exactly two composition factors, then up to graph automorphism, λ appears in Table 2.6. Note that the coefficients a and e in Table 2.6 satisfy $0 \leq a, e \leq p-1$ and by Lemma 2.1.9, $a = 1$ or $e = 1$ if $ae \neq 0$. In what follows, whenever λ is written as a linear combination of parameters (i.e. a, b, c, d, e), we assume these parameters to be nonzero.

λ (a,b,c,d,e)	Conditions $bcd \neq 0$	2^{nd} factor for $L_Y(\lambda)\|_X$
$(a,0,0,0,e)$		not determined
$(a,b,0,0,e)$	$1 \leq b \leq p-1$	not determined
$(a,0,c,0,e)$	$c = 1, p-1$	$\mu - 011$
$(a,b,c,0,e)$	$b \neq p-2, p-4, c = 1$	$\mu - 011$
	$b+c = p, 3 \leq b \leq p-2$	$\mu - 011$
	$b+c = p-1$	not determined
$(a,b,0,d,e)$	$b = 1, d \neq p-1$	$\mu - 010$
	$b \neq p-1, d = 1$	$\mu - 010$
(a,b,c,d,e)	$b = 1, c+d = p-1, d \neq 1$	$\mu - 010$
	$b \neq 1, b+c = p-1, d = 1$	$\mu - 010$

Table 2.6: The weights to consider inductively

2.3.1.1 $\boldsymbol{\lambda = a\lambda_1}$. — The weight λ appears in Table 2.2, hence X acts on $L_Y(\lambda)$ irreducibly.

2.3.1.2 $\boldsymbol{\lambda = b\lambda_2}$. — If $b = p-1$, then λ appears in Table 2.2 and so X acts irreducibly on $L_Y(\lambda)$. Henceforth assume $b \neq p-1$. It is not hard to check that the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 121 = (0, b-1, 0)$.

If $b = 1$, then $(\dim L_Y(\lambda), \dim L_X(\mu), \dim L_X(\nu)) = (15, 14, 1)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors. If $b \neq 1$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{242} = \lambda$ and the multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.3 $\boldsymbol{\lambda = c\lambda_3}$. — By Table 2.6, either $c = 1$ or $c = p-1$ and a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 011 = (1, 0, c-1)$. By Proposition 2.3.2, we have that X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.3.1.4 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2}$. — The case $a+b=p-1$ appears in Table 2.2, hence assume $a+b \neq p-1$. It is easy to see that a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 121 = (a, b-1, 0)$.

Assume $b=1$. Using Weyl's degree formula, we get the following.

$$\dim V_Y(\lambda) = \frac{1}{24}a^5 + \frac{19}{24}a^4 + \frac{137}{24}a^3 + \frac{461}{24}a^2 + \frac{117}{4}a + 15$$
$$\dim V_X(\mu) = \frac{1}{30}a^5 + \frac{2}{3}a^4 + 5a^3 + \frac{52}{3}a^2 + \frac{809}{30}a + 14$$
$$\dim V_X(\nu) = \frac{1}{120}a^5 + \frac{1}{8}a^4 + \frac{17}{24}a^3 + \frac{15}{8}a^2 + \frac{137}{60}a + 1$$

Hence $\dim V_Y(\lambda) = \dim V_X(\mu) + \dim V_X(\nu)$. Moreover, by Corollary 2.1.11, we have that $V_Y(\lambda)$ is irreducible and by Lemma 2.1.12, we have that $V_X(\nu)$ is irreducible. Thus X acts on $L_Y(\lambda)$ with exactly two composition factor if and only if $V_X(\mu)$ is irreducible. Applying the JSF to μ we get that $\operatorname{ch} L(\mu) = \mu$ if $a \neq p-6$ and $\operatorname{ch} L(\mu)_{121} = \mu - (\mu - 121)$ if $a = p-6$. Therefore, X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if $a \neq p-6$.

Assume $b>1$ and $a+b \neq p$. Let $\theta \in X(T_Y)^+$ be a weight such that $\mu \succeq \theta|_{T_X} \succeq \mu - 242$. If θ affords the highest weight of a composition factor for $V_Y(\lambda)$, then θ has to be dominant and thus of the form $\lambda - r_1r_2r_300$, with $r_i \in \mathbb{Z}_{\geq 0}$. Computing the truncated JSF of λ up to $\mu - 242$ using Lemma 1.3.5, we get that $\operatorname{ch} L(\lambda)_{242} = \lambda$. Comparing the multiplicities listed in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.4.1 $\boldsymbol{a+b=p,\ b>1}$. — We give a detailed explanations for this case. It is interesting, since it includes a subcase appearing in Table 2.5.

The weight λ we are considering comes from an irreducible case in Table 2.6. Hence by Proposition 1.5.2,

$$m_{L(\lambda)|_X}(\mu - \beta) = m_{L(\mu)}(\mu - \beta), \tag{2.2}$$

for all $\beta \in \{\mathbb{Z}_{\geq 0}\beta_2 + \mathbb{Z}_{\geq 0}\beta_3\}$ or using our abbreviated notations for roots, for all β of the form $\beta = 0xy$ with $x, y \in \mathbb{Z}_{\geq 0}$. Let $v^+ \in L_Y(\lambda)_\lambda$ be a maximal vector for B_Y of weight λ. Since $B_X \subseteq B_Y$, the maximal vector v^+ is also a maximal vector for B_X of weight $\mu = \lambda|_{T_X}$. Hence $L_X(\mu)$ is a composition factor of $L_Y(\lambda)|_{T_X}$. Note that λ does not appear in Table 2.2, so X does not act irreducibly on $L_Y(\lambda)$. We start by finding a second composition factor for the action of X on $L_Y(\lambda)$. By (2.2), the highest weight of a second composition factor has to be of the form $\mu - xyz$ with $x, y, z \in \mathbb{Z}_{\geq 0}$ and $x \neq 0$. We show that $\mu - 121$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The dominant weights of the form $\mu \succeq \mu - xyz \succeq \mu - 121$ with $x \neq 0$ are as follows.

$$\mu - 100 = (a-2, p-a+1, 0)$$
$$\mu - 110 = (a-1, p-a-1, 1)$$
$$\mu - 120 = (a, p-a-3, 2)$$
$$\mu - 111 = (a-1, p-a+1, -1)$$
$$\mu - 121 = (a, p-a-1, 0)$$

We compute the multiplicity of each of these weights in $L_Y(\lambda)|_X$ and $L_X(\mu)$ in order to determine which one of them affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

Recall the following formula which gives the multiplicity of $\theta \in X(T_X)^+$ in $L_Y(\lambda)|_X$.

$$m_{L(\lambda)|_X}(\theta) = \sum_{\substack{\gamma\in\Lambda(\lambda)\\ \gamma|_{T_X}=\theta}} m_{L(\lambda)}(\gamma)$$

$\boldsymbol{\mu-100}$ There is only one weight in $\Lambda(\lambda)$ which restricts to $\mu-100$, namely $\lambda-10000$. By Lemma 2.1.5, we have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu-\mu-100)=(1,1)$, hence $\mu-100$ does not afford the highest weight of a second composition factor.

$\boldsymbol{\mu-110, \mu-120}$ Both cases are similar. We apply Lemma 2.1.4 to prove that $\operatorname{ch} L(\lambda)_{120}=\lambda$ and $\operatorname{ch} L(\mu)_{120}=\mu$. Moreover, there is only one weight in $\Lambda(\lambda)$ which restricts to $\mu-110$ and similarly for $\mu-120$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu-110)=(2,2)$ and $(m_{L(\lambda)|_X}, m_{L(\mu)})$ $(\mu-120)=(2,2)$, which implies that $\mu-110$ and $\mu-120$ do not afford the highest weight of composition factors for X acting on $L_Y(\lambda)$.

$\boldsymbol{\mu-111}$ The only weight in $\Lambda(\lambda)$ which restricts to $\mu-111$ is $\lambda-11100$. We proceed as follows in order to compute the multiplicity of $\lambda-11100$ in $L_Y(\lambda)$. The weights in $\theta\in\Lambda(\lambda)^+$ with $\lambda-11100\in\Lambda(\theta)$ are the dominant weights satisfying $\lambda-xyz00$ with $0\leq x,y,z\leq 1$. We want to find out if θ affords the highest weight of a composition factor for $V(\lambda)$. We apply Proposition 1.2.2 for each θ and find that none of them is linked to λ. Hence $m_{L(\lambda)|_X)}(\mu-111)=m_{V(\lambda)|_X}(\mu-111)=2$. Applying a similar argument to the weights $\theta\in\Lambda(\mu)^+$ such that $\mu-111\in\Lambda(\theta)$ implies that $m_{L(\mu)}(\mu-111)=m_\mu(\mu-111)=2$. Therefore $\mu-111$ does not afford the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

$\boldsymbol{\mu-121}$ We apply the JSF to λ and μ with the goal of computing the truncated character of $L_Y(\lambda)$ and $L_X(\mu)$ for the set $\{\theta\in\Lambda(\lambda)\cap X(T_Y)^+ \mid \theta|_{T_X}\succeq \mu-121\}$ and the set $\{\theta\in\Lambda(\mu)\cap X(T_X)^+ \mid \theta\succeq\mu-121\}$, respectively. We apply the process explained in Subsection 1.3.2 to λ and μ with the sets above. We get the following truncated characters $\operatorname{ch} L(\lambda)_{121}=\lambda$ and

$$\operatorname{ch} L(\mu)_{121}=\mu-\delta_{a,4}(\mu-121).$$

The weights in $\Lambda(\lambda)$ which restrict to $\mu-121$ are $\lambda-11110$, $\lambda-12100$ and $\lambda-01111$. Using Proposition 1.1.12 and the truncated character formula just computed, we get

$$\begin{aligned} m_{L(\lambda)|_X}(\mu-121) &= m_{V(\lambda)|_X}(\mu-121)\\ &= m_\lambda(\lambda-12100)+m_\lambda(\lambda-11110)+m_\lambda(\lambda-01111)\\ &= 3-\delta_{b,1}+2+1\\ &= 6-\delta_{b,1}. \end{aligned}$$

Similarly,

$$m_{L(\mu)}(\mu-121)=m_\mu(\mu-121)-\delta_{a,4}m_A(\mu-121)=5-\delta_{b,1}-\delta_{a,4}.$$

Hence $\nu=\mu-121=(a,p-a-1,0)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

If $a = 4$, then ν also affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. Assume from now on that $a \neq 4$. We are going to prove that X acts on $L(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. That is, if no weight of the form $\mu - \alpha|_{T_X}$ or $\nu - \alpha|_{T_X}$ with $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$, then X acts on $L_Y(\lambda)$ with exactly two composition factors. We have already shown that the weights $\mu - \alpha|_{T_X}$ with $\alpha \in \Phi^+(Y) \setminus \{\alpha_0\}$ different from $\mu - 121$ do not afford the highest weight of a third composition factor. The remaining possibilities are listed below.

$$\begin{aligned}
\mu - 131 &= (a+1, p-a-3, 1)\\
\mu - 221 &= (a-2, p-a, 0)\\
\mu - 142 &= (a+2, p-a-3, 0)\\
\mu - 231 &= (a-1, p-a-2, 1)\\
\mu - 242 &= (a, p-a-2, 0)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider $\mu - 242$. Let us compute the multiplicity of $\mu - 242$ in $L_Y(\lambda)|_X$, $L_X(\mu)$ and $L_X(\nu)$. The JSF applied to λ yields the following truncated character

$$\operatorname{ch} L(\lambda)_{242} = \lambda - \overline{\delta_{a,1}}(\lambda - 22000) - \delta_{a,1}(\lambda - 23100),$$

where $\overline{\delta_{i,j}} = (1 - \delta_{i,j})$. The computations of the JSF applied to μ and ν are summarized in Tables 2.7 and 2.8, respectively. They yield the truncated character of $L(\mu)$ up to $\mu - 242$ for $a \neq 3$ and of $L(\nu)$ up to $\nu - 121$ for $a \neq 2$. We have two problematic cases, namely $[\mu(3) : C]$ and $[\nu(2) : E]$, where $\mu(3)$ stands for μ with $a = 3$ and $\nu(2)$ for ν with $a = 2$.

$[\boldsymbol{\mu(3) : C}]$ Set $[V(\mu(3)) : L(C)] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. By Table 2.9, we have

$$[V(\mu(3)) : L(C)] = [V((4, p-5, 0)) : L((3, p-6, 1))]_{C_3}.$$

Note that in C_3, we have $(4, p-5, 0) - (3, p-6, 1) = 110$. Therefore by Proposition 1.5.1 and Lemma 2.1.4, we get $\zeta = 1$.

$[\boldsymbol{\nu(2) : E}]$ Note that $\nu(2) = s_0 s_2 s_3 s_2 s_1 \bullet (0, -1, 0)$ and $E = s_0 s_2 s_3 \bullet (0, -1, 0)$. Moreover, the weight E belongs to the upper closure of the alcove defined by $(1, 1, 1, 1, 1, 2, 1, 2, 2)$. Note that C, the weight in the previous problematic case, also belongs to the upper closure of the alcove $(1, 1, 1, 1, 1, 2, 1, 2, 2)$. We have $\mu(3) = s_0 s_2 s_3 s_2 s_1 \bullet (1, -1, 0)$ and $C = s_0 s_2 s_3 \bullet (1, -1, 0)$. Therefore, by Proposition 1.3.10 and the resolution of the previous problematic case we have that $[V(\nu(2)) : L(E)] = 1$.

We compute the multiplicities using Proposition 1.1.12 and obtain

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 242) = (17 - 2\delta_{b,2}, 14 - 2\delta_{b,2}, 3).$$

Thus X acts on $L_Y(\lambda)$ with exactly two composition factors if $a \neq 4$.

$\mu=(a,p-a,0)_{C_3}$	
ch $L(\mu)_{242}=\mu-\overline{\delta_{a,1}}A-\delta_{a,1}B$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{242}=\overline{\delta_{a,1}}A+\delta_{a,1}B+\delta_{a,3}C$	$\mathrm{JSF}(\mu)_{242}=\overline{\delta_{a,1}}A+\delta_{a,1}B+2\delta_{a,3}C$
$\mathrm{JSF}(A)_{242}=\delta_{a,3}C$	$\mathrm{JSF}(A)_{242}=\delta_{a,3}C$
$A=\mu-220=(a-2,p-a-2,2)$	$C=\mu-242=(a,p-a-2,0)$
$B=\mu-241=(a,p-a-4,2)$	

Table 2.7: JSF of μ up to $\mu-242$

$\nu=(a,p-a-1,0)_{C_3}$	
ch $L(\nu)_{121}=\nu-D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{121}=D+\delta_{a,2}E$	$\mathrm{JSF}(\nu)_{121}=D+2\delta_{a,2}E$
$\mathrm{JSF}(D)_{121}=\delta_{a,2}E$	$\mathrm{JSF}(D)_{121}=\delta_{a,2}E$
$D=\nu-110=(a-1,p-a-2,1)$	$E=\nu-121=(a,p-a-2,0)$

Table 2.8: JSF of ν up to $\nu-121$

$\lambda_0'=(1,-1,0)\notin C_0$	
$\gamma'=w_1\bullet\lambda_0'=(3,p-3,0)$	$\eta'=w\bullet\lambda_0'=(3,p-5,0)$
$w_1=s_0s_2s_3s_2s_1$	$w=s_0s_2s_3$
$\lambda_0=(0,0,0)\in C_0$	
$\gamma=w_1\bullet\lambda_0=(4,p-4,0)$	$\eta=w\bullet\lambda_0=(3,p-6,1)$
$C_{\eta'}=(1,1,1,1,1,2,1,2,2)$	$C_\eta=(1,1,1,1,1,2,1,2,2)$
Proposition 1.3.10 $\implies$ $[\gamma':\eta']=[\gamma:\eta]$	
$s=s_1$	
$ws\bullet\lambda_0=(2,p-6,2)$	
$ws\bullet\lambda_0-w\bullet\lambda_0=011$	
Proposition 1.3.9 $\implies$ $[w_1s\bullet\lambda_0:\eta]=[\gamma:\eta]$, where $w_1s\bullet\lambda_0=(4,p-5,0)$	

Table 2.9: Computing $[(3,p-3,0):(3,p-5,0)]_{C_3}$

2.3.1.5 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3}$**.** — By Table 2.6, either $c = 1$ or $c = p-1$ and the T_X-weight $\nu = \mu - 011 = (a+1, 0, c-1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. If $a+c+2 \not\equiv 0 \mod p$, then $\operatorname{ch} L(\lambda)_{111} = \lambda$ and $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 111) = (5, \leq_V 3, \leq_V 1)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth, assume $a + c + 2 \equiv 0 \mod p$.

If $a, c = p-1$, then applying the JSF we get $\operatorname{ch} L(\lambda)_{111} = \lambda - (\lambda - 11100)$ and $\operatorname{ch} L(\mu)_{111} = \mu - (\mu - 111)$. The multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a = p-3$ and $c = 1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. We prove that none of the following weights affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\mu - 100 = (p-5, 1, 1) \qquad\qquad \mu - 111 = (p-4, 1, 0)$$

By Lemma 1.4.9, it is enough to consider $\mu - 111$. Applying the JSF, we get $\operatorname{ch} L(\lambda)_{111} = \lambda - (\lambda - 11100)$ and $\operatorname{ch} L(\mu)_{111} = \mu$. Moreover, the Weyl module $V_X(\nu)$ is irreducible by Lemma 2.1.12. Therefore, we have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 111) = (4, 3, 1)$$

and X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.3.1.6 $\boldsymbol{\lambda = a\lambda_1 + d\lambda_4}$**.** — It is easy to check that a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 110 = (a-1, d-1, 1)$.

If $a, d > 1$, then $(\lambda - \alpha)|_{T_X} = \mu - 220$ and $\lambda - \alpha \in \Lambda(\lambda)$ for $\alpha \in \{22000, 00022, 21010, 10021, 11011, 20020\}$. Hence by Theorem 1.1.10 and Lemma 2.1.5, we get $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 220) = (6, \leq_V 3, \leq_V 2)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth, assume $a = 1$ or $d = 1$.

If $a + d \neq p-3$ or $d \neq p-1$, then $\operatorname{ch} L(\lambda)_{121} = \lambda$ and comparing the multiplicities appearing in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $d = p-1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. We prove that none of the weights below affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned} \mu - 120 &= (1, p-4, 2) \\ \mu - 121 &= (1, p-2, 0) \\ \mu - 131 &= (2, p-4, 1) \\ \mu - 231 &= (0, p-3, 1) \end{aligned}$$

By Lemma 1.4.9, it is enough to consider $\mu - 231$. Applying the JSF, we get $\operatorname{ch} L(\lambda)_{231} = \lambda - (\lambda - 00121)$, $\operatorname{ch} L(\mu)_{231} = \mu$ and $\operatorname{ch} L(\nu)_{121} = \nu - (\nu - 021)$. Therefore, we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 231) = (11, 8, 3)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

If $d = p-4$, then by the JSF, we have $\operatorname{ch} L(\lambda)_{231} = \lambda - (\lambda - 11110)$. Comparing the multiplicities appearing in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a = p-4$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors. The weights which could afford the highest weight of a third composition factor generated by a maximal vector for B_X are given as follows.

$$\mu - 210 = (p-7, 1, 1)$$
$$\mu - 121 = (p-4, 0, 0)$$
$$\mu - 221 = (p-6, 1, 0)$$

By Lemma 1.4.9, it is enough to consider $\mu - 221$. Applying the JSF to λ, we obtain $\operatorname{ch} L(\lambda)_{221} = \lambda - (\lambda - 11110)$. Applying the JSF to μ and ν, yields $\operatorname{ch} L(\mu)_{221} = \mu$ and $\operatorname{ch} L(\nu)_{111} = \nu - (\nu - 111)$, respectively. Hence,

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 221) = (8, 6, 2)$$

and X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.3.1.7 $\boldsymbol{\lambda = a\lambda_1 + e\lambda_5}$. — By Lemma 2.1.9, we have that the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 100 = (a+e-2, 1, 0)$ and that either $a = 1$ or $e = 1$. Assume without loss of generality that $e = 1$.

If $a = p-1$, we have $m_{L_Y(\lambda)|_X}(\mu - 100) = 2$ by Theorem 1.1.10 and Lemma 2.1.5. Moreover, we have $m_{L_X(\mu)}(\mu - 100) = 0$ by Theorem 1.1.8. Thus ν also affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$.

If $a \neq p-1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. We prove that none of the weights listed below affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\mu - 200 = (a-3, 2, 0)$$
$$\mu - 210 = (a-2, 0, 1)$$
$$\mu - 221 = (a-1, 0, 0)$$

By Lemma 1.4.9, it is enough to consider $\mu - 221$. Applying the JSF to λ, we get $\operatorname{ch} L(\lambda)_{221} = \lambda - \delta_{a,p-5}(\lambda - 11111)$. Moreover, by Lemma 2.1.12, the Weyl module $V_X(\mu)$ is irreducible and the JSF applied to ν yields $\operatorname{ch} L(\nu)_{121} = \nu - \delta_{a,p-5}(\nu - 121)$. Therefore

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 221) = (7 - \delta_{a,p-5} - 2\delta_{a,1}, 3, 4 - \delta_{a,p-5} - 2\delta_{a,1}),$$

which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.3.1.8 $\boldsymbol{\lambda = b\lambda_2 + c\lambda_3}$. — If $b + c = p-1$, then λ appears in Table 2.2 and so X acts irreducibly on $L_Y(\lambda)$. Assume $b + c \neq p-1$ and λ is as in Table 2.6. Then by Table 2.6, a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 011 = (1, b, c-1)$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{121} = \lambda$. Comparing the multiplicities appearing in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.9 $\boldsymbol{\lambda = b\lambda_2 + d\lambda_4}$. — By Table 2.6 we can assume $b \neq p-1$ and $d = 1$ and a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 010 = (1, b-1, 1)$.

If $b \neq p-3$, the JSF yields $\operatorname{ch} L(\lambda)_{121} = \lambda$. If $b = p-3$, then $\operatorname{ch} L(\lambda)_{121} = \lambda - (\lambda - 01110)$ and $\operatorname{ch} L(\mu)_{121} = \mu - (\mu - 121)$. In both cases, the multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.10 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3}$. — We have three cases to consider by Table 2.6. We solve all three of them in different subsections.

2.3.1.10.1 $\boldsymbol{b \neq p-2, p-4,\ c = 1}$. — By Table 2.6, a second composition factor is given by the highest weight $\nu = \mu - 011 = (a+1, b, 0)$.

If $a+b+3 \not\equiv 0 \mod p$ and $a+b \neq p-1$, the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{111} = \lambda$. If $a+b = p-1$, the JSF applied to λ and μ yields $\operatorname{ch} L(\lambda)_{111} = \lambda - (\lambda - 11000)$ and $\operatorname{ch} L(\mu)_{111} = \mu - (\mu - 110)$, respectively. If $a+b+3 \equiv 0 \mod p$, then JSF applied to λ yields $\operatorname{ch} L(\lambda)_{132} = \lambda - (\lambda - 11100)$. In all three cases, the multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.10.2 $\boldsymbol{b + c = p,\ 3 \leq b \leq p-2}$. — By Table 2.6, a second composition factor is given by the highest weight $\mu - 011 = (a+1, b, p-b-1)$.

Let $a \neq p-2$ and $a+b \neq p-1$, the JSF yields $\operatorname{ch} L(\lambda)_{111} = \lambda$. Let $a+b = p-1$, the JSF yields $\operatorname{ch} L(\lambda)_{111} = \lambda - (\lambda - 11000)$ and $\operatorname{ch} L(\mu)_{111} = \mu - (\mu - 110)$. Let $a = p-2$, the truncated character $\operatorname{ch} L(\lambda)_{122}$ is computed in Table 2.10, where we apply Lemma 2.1.8 in order to determine $[\lambda : C]$. In all these cases, comparing the multiplicities appearing in Table 2.31, we get that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (p-2, b, p-b, 0, 0)_{A_5}$	
$\operatorname{ch} L(\lambda)_{122} = \lambda - A - B + C$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\operatorname{JSF}(\lambda)_{122} = A + B$	$\operatorname{JSF}(\lambda)_{122} = A + B + 2C$
$\operatorname{JSF}(A)_{122} = C$	$\operatorname{JSF}(A)_{122} = C$
$\operatorname{JSF}(B)_{122} = C$	$\operatorname{JSF}(B)_{122} = C$
$A = \lambda - 02200 = (p, b-2, p-b-2, 2, 0)$ 022	$C = \lambda - 12200 = (p-2, b-1, p-b-2, 2, 0)$ 122
$B = \lambda - 11100 = (p-3, b, p-b-1, 1, 0)$ 111	

Table 2.10: JSF of λ up to $\mu - 122$

2.3.1.10.3 $\boldsymbol{b + c = p-1}$. — We first find the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The truncated character $\operatorname{ch} L(\lambda)_{111}$ is computed in Table 2.11 and the JSF applied to μ yields

$$\operatorname{ch} L(\mu)_{111} = \mu - \delta_{a,p-b-1}(\mu - 110) - \delta_{a,b-1}(\mu - 111).$$

We deduce the following multiplicities.

$$(m_{L_Y(\lambda)|_X}, m_{L(\mu)})(\mu - 110) = \begin{cases} (1,1) & \text{if } a+b = p-1 \\ (2,2) & \text{otherwise} \end{cases}$$

$$(m_{L_Y(\lambda)|_X}, m_{L(\mu)})(\mu-111) = \begin{cases} (4,3) & \text{if } a+b=p-1 \\ (5,3) & \text{if } a=b-1 \\ (5,4) & \text{otherwise} \end{cases}$$

Therefore, a second composition factor is given by the highest weight $\nu=\mu-111=(a-1,p-c,c-1)$. Note that if $a=b-1$, then ν also affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. Henceforth, assume $a\neq b-1$.

This case is complicated. In order to solve it, we split it into several subcases.

$\lambda=(a,b,p-b-1,0,0)_{A_5}$	
$\operatorname{ch} L(\lambda)_{111}=\lambda-A-\delta_{a,p-b-1}B$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{111}=A+\delta_{a,p-b-1}B+\delta_{a,p-1}C$	$\mathrm{JSF}(\lambda)_{111}=A+\delta_{a,p-b-1}B+2\delta_{a,p-1}C$
$\mathrm{JSF}(A)_{111}=\delta_{a,p-1}C$	$\mathrm{JSF}(A)_{111}=\delta_{a,p-1}C$
$A=\lambda-01100=(a+1,b-1,p-b-2,1,0)\ 011$	$C=\lambda-11100=(a-1,b,p-b-2,1,0)\ 111$
$B=\lambda-11000=(a-1,b-1,p-b,0,0)\ 110$	

Table 2.11: JSF of λ up to $\mu-111$

$a\neq 1,\ c\neq 1$

Recall that we assume $a\neq b-1$. If $a\neq c,c+1$, the truncated character $\operatorname{ch} L(\lambda)_{222}$ is computed in Table 2.12, where the value of $[\lambda:B]$ is determined by applying Lemma 2.1.8. The output of JSF applied to μ appears in Table 2.13. Note that it yields multiple possibilities for the truncated character $\operatorname{ch} L(\mu)_{222}$. We select the first one, since it maximizes the multiplicity of $\mu-222$ in $L_X(\mu)$. Comparing the multiplicities in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a=c$, the truncated character $\operatorname{ch} L(\lambda)_{222}$ is computed in Table 2.14, where we apply Lemma 2.1.8 in order to determine $[\lambda:C]$. Applying the JSF to μ and ν yields the following truncated characters.

$$\begin{aligned} \operatorname{ch} L(\mu)_{222} &= \mu-(\mu-110)-\overline{\delta_{a,p-2}}(\mu-021)-\delta_{a,p-2}(\mu-121)-\delta_{a,b}(\mu-222) \\ \operatorname{ch} L(\nu)_{111} &= \nu-(\nu-110) \end{aligned}$$

The multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a=c+1$, the truncated character $\operatorname{ch} L(\lambda)_{222}$ is computed in Table 2.15, where we apply Lemma 2.1.8 in order to determine $[\lambda:B]$. The JSF applied to μ for $a=p-2$ is computed in Table 2.16. We obtain two possibilities for the truncated character $\operatorname{ch} L(\mu)_{222}$ and we use the first one to bound the multiplicity of $\mu - 222$ in $\operatorname{ch} L_X(\mu)$. For $a \neq p-2$, the JSF yields the following truncated character.

$$\operatorname{ch} L(\mu)_{222} = \mu - \overline{\delta_{a,p-1}}(\mu - 220) - \overline{\delta_{a,p-1}}(\mu - 021) - \delta_{a,p-1}(\mu - 221).$$

The multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (a, p-c-1, c, 0, 0)_{A_5}$	
$\operatorname{ch} L(\lambda)_{222} = \lambda - A$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{222} = A + \delta_{a,p-1} B$	$\mathrm{JSF}(\lambda)_{222} = A + 2\delta_{a,p-1} B$
$\mathrm{JSF}(A)_{222} = \delta_{a,p-1} B$	$\mathrm{JSF}(A)_{222} = \delta_{a,p-1} B$
$A = \lambda - 01100 = (a+1, p-c-2, c-1, 1, 0)\ 011$	$B = \lambda - 11100 = (a-1, p-c-1, c-1, 1, 0)\ 111$

Table 2.12: JSF of λ up to $\mu - 222$

$\mu = (a, p-c-1, c)_{C_3}$	
Possibilities	
$\operatorname{ch} L(\mu)_{222} = \mu - A - \delta_{a,b} D$	
$\operatorname{ch} L(\mu)_{222} = \mu - A - \delta_{a,p-2} B - \delta_{a,b} D$	
$\operatorname{ch} L(\mu)_{222} = \mu - A - \delta_{a,p-1} C - \delta_{a,b} D$	
$\operatorname{ch} L(\mu)_{222} = \mu - A - \delta_{a,p-2} B - \delta_{a,p-1} C - \delta_{a,b} D$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{222} = A + \delta_{a,p-2} B + \delta_{a,p-1} C + \delta_{a,b} D$	$\mathrm{JSF}(\mu)_{222} = A + 2\delta_{a,p-2} B + 2\delta_{a,p-1} C + \delta_{a,b} D$
$\mathrm{JSF}(A)_{222} = \delta_{a,p-2} B + \delta_{a,p-1} C$	$\mathrm{JSF}(A)_{222} = \delta_{a,p-2} B + \delta_{a,p-1} C$
$A = \mu - 021 = (a+2, p-c-3, c)$	$C = \mu - 221 = (a-2, p-c-1, c)$,
$B = \mu - 121 = (a, p-c-2, c)$	$D = \mu - 222 = (a-2, p-c+1, c-2)$

Table 2.13: JSF of μ up to $\mu - 222$

$c = 1, a \neq 1$

Note that $a \neq p-3$, since $a \neq b-1$. If $a \neq p-2, p-1$, the truncated characters $\operatorname{ch} L(\lambda)_{232}$ and $\operatorname{ch} L(\mu)_{232}$ are computed in Tables 2.17 and 2.18 for $a \neq 2$ and in Tables 2.19 and 2.20 for $a = 2$.

$\lambda = (a, p-a-1, a, 0, 0)_{A_5}$	
$\operatorname{ch} L(\lambda)_{222} = \lambda - A - B + \overline{\delta_{a,p-2}}C$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{222} = A + B$	$\mathrm{JSF}(\lambda)_{222} = A + B + 2\overline{\delta_{a,p-2}}C$
$\mathrm{JSF}(A)_{222} = \overline{\delta_{a,p-2}}C$	$\mathrm{JSF}(A)_{222} = \overline{\delta_{a,p-2}}C$
$\mathrm{JSF}(B)_{222} = \overline{\delta_{a,p-2}}C$	$\mathrm{JSF}(B)_{222} = \overline{\delta_{a,p-2}}C$
$A = \lambda - 01100 = (a+1, p-a-2, a-1, 1, 0)$ 011	$C = \lambda - 12100 = (a, p-a-3, a, 1, 0)$ 121
$B = \lambda - 11000 = (a-1, p-a-2, a+1, 0, 0)$ 110	

Table 2.14: JSF of λ up to $\mu - 222$

$\lambda = (a, p-a, a-1, 0, 0)_{A_5}$	
$\operatorname{ch} L(\lambda)_{222} = \lambda - A - \overline{\delta_{a,p-1}}C - \delta_{a,p-1}D$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{222} = A + \delta_{a,p-1}B + \overline{\delta_{a,p-1}}C$	$\mathrm{JSF}(\lambda)_{222} = A + 2\delta_{a,p-1}B + \overline{\delta_{a,p-1}}C + \delta_{a,p-1}D$
$\mathrm{JSF}(A)_{222} = \delta_{a,p-1}B - \delta_{a,p-1}D$	$\mathrm{JSF}(A)_{222} = \delta_{a,p-1}B$
$\mathrm{JSF}(B)_{222} = \delta_{a,p-1}D$	$\mathrm{JSF}(B)_{222} = \delta_{a,p-1}D$
$A = \lambda - 01100 = (a+1, p-a-1, a-2, 1, 0)$ 011	$C = \lambda - 22000 = (a-2, p-a-2, a+1, 0, 0)$ 220
$B = \lambda - 11100 = (a-1, p-a, a-2, 1, 0)$ 111	$D = \lambda - 22100 = (a-2, p-a-1, a-1, 1, 0)$ 221

Table 2.15: JSF of λ up to $\mu - 222$

In both cases, the multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a = p-2$, the truncated character $\operatorname{ch} L(\lambda)_{232}$ is computed in Table 2.21 using Remark 1.3.18. The JSF of μ is computed in Table 2.22. We have two problematic cases, namely C and D. We do not determine $[\mu : C]$ nor $[\mu : D]$, but we use the first possibility for $\operatorname{ch} L(\mu)_{232}$ to bound the multiplicity of $\mu - 232$ in $L_X(\mu)$. The multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a = p-1$, the truncated character $\operatorname{ch} L(\lambda)_{232}$ is computed in Table 2.23 by applying Lemma 2.1.8 and Remark 1.3.18. We bound the possibilities for the truncated character $\operatorname{ch} L(\mu)_{232}$ appearing in Table 2.24 by the possibility maximizing the multiplicity of $\mu - 232$ in $L_X(\mu)$. Comparing the multiplicities in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\mu = (p-2, 2, p-3)_{C_3}$	
Possibilities	
ch $L(\mu)_{222} = \mu - A - B$	
ch $L(\mu)_{222} = \mu - A - B - C$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{222} = A + B + C$	$\mathrm{JSF}(\mu)_{222} = A + B + 2C$
$\mathrm{JSF}(B)_{222} = C$	$\mathrm{JSF}(B)_{222} = C$
$A = \mu - 220 = (p-4, 0, p-1)$	$C = \mu - 121 = (p-2, 1, p-3)$
$B = \mu - 021 = (p, 0, p-3)$	

Table 2.16: JSF of μ up to $\mu - 222$

$\lambda = (a, p-2, 1, 0, 0)_{A_5}$	
ch $L(\lambda)_{232} = \lambda - A + B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{232} = A - B$	$\mathrm{JSF}(\lambda)_{232} = A$
$\mathrm{JSF}(A)_{232} = B$	$\mathrm{JSF}(A)_{232} = B$
$A = \lambda - 01100 = (a+1, p-3, 0, 1, 0)$ 011	$B = \lambda - 02210 = (a+2, p-4, 0, 0, 1)$ 032

Table 2.17: JSF of λ up to $\mu - 232$

$\mu = (a, p-2, 1)_{C_3}$	
ch $L(\mu)_{232} = \mu - A + B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{232} = A - B$	$\mathrm{JSF}(\mu)_{232} = A$
$\mathrm{JSF}(A)_{232} = B$	$\mathrm{JSF}(A)_{232} = B$
$A = \mu - 021 = (a+2, p-4, 1)$	$B = \mu - 032 = (a+3, p-4, 0)$

Table 2.18: JSF of μ up to $\mu - 232$

$a = 1$

Note that $b \neq 2$, since $a \neq b - 1$. We prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. Let us distinguish between the cases $c = 1$ and $c \neq 1$.

Let $c = 1$. We prove that none of the following weights affords the highest weight of a composition

$\lambda = (2, p-2, 1, 0, 0)_{A_5}$	
ch $L(\lambda)_{232} = \lambda - A - B + C + D$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{232} = A + B - C$	$\mathrm{JSF}(\lambda)_{232} = A + B + 2D$
$\mathrm{JSF}(A)_{232} = D$	$\mathrm{JSF}(A)_{232} = D$
$\mathrm{JSF}(B)_{232} = C + D$	$\mathrm{JSF}(B)_{232} = C + D$
$A = \lambda - 22000 = (0, p-4, 3, 0, 0)$ 220	$C = \lambda - 02210 = (4, p-4, 0, 0, 1)$ 032
$B = \lambda - 01100 = (3, p-3, 0, 1, 0)$ 011	$D = \lambda - 23100 = (1, p-5, 2, 1, 0)$ 231

Table 2.19: JSF of λ up to $\mu - 232$

$\mu = (2, p-2, 1)_{C_3}$	
ch $L(\mu)_{232} = \mu - A + B - C$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{232} = A - B + C$	$\mathrm{JSF}(\mu)_{232} = A + C$
$\mathrm{JSF}(C)_{232} = B$	$\mathrm{JSF}(C)_{232} = B$
$A = \mu - 220 = (0, p-4, 3)$	$C = \mu - 021 = (4, p-4, 1)$
$B = \mu - 032 = (5, p-4, 0)$	

Table 2.20: JSF of μ up to $\mu - 232$

$\lambda = (p-2, p-2, 1, 0, 0)_{A_5}$	
ch $L(\lambda)_{232} = \lambda - A + B$	
Remark 1.3.18	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{232} = A - B$	$\mathrm{JSF}(\lambda)_{232} = A +_0^1 C$
$\mathrm{JSF}(A)_{232} = B + C$	$\mathrm{JSF}(A)_{232} = B + 2C$
$\mathrm{JSF}(B)_{232} = C$	$\mathrm{JSF}(B)_{232} = C$
$A = \lambda - 01100 = (p-1, p-3, 0, 1, 0)$ 011	$C = \lambda - 12210 = (p-2, p-3, 0, 0, 1)$ 132
$B = \lambda - 02210 = (p, p-4, 0, 0, 1)$ 032	

Table 2.21: JSF of λ up to $\mu - 232$

factor for X acting on $L_Y(\lambda)$.

$$\mu - 121 = (1, p-3, 1)$$
$$\mu - 132 = (2, p-3, 0)$$
$$\mu - 232 = (0, p-2, 0)$$

$\mu = (p-2, p-2, 1)_{C_3}$	
Possibilities	
$\mathrm{ch}\, L(\mu)_{232} = \mu + A - B$	
$\mathrm{ch}\, L(\mu)_{232} = \mu + A - B - C$	
$\mathrm{ch}\, L(\mu)_{232} = \mu + A - B - D$	
$\mathrm{ch}\, L(\mu)_{232} = \mu + A - B - C + D$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{232} = -A + B + C$	$\mathrm{JSF}(\mu)_{232} = B + 2C +_1^2 D$
$\mathrm{JSF}(A)_{232} = D$	$\mathrm{JSF}(A)_{232} = D$
$\mathrm{JSF}(B)_{232} = A + C$	$\mathrm{JSF}(B)_{232} = A + C + 2D$
$\mathrm{JSF}(C)_{232} = D$	$\mathrm{JSF}(C)_{232} = D$
$A = \mu - 032 = (p+1, p-4, 0)$	$C = \mu - 121 = (p-2, p-3, 1)$,
$B = \mu - 021 = (p, p-4, 1)$	$D = \mu - 232 = (p-3, p-2, 0)$

Table 2.22: JSF of μ up to $\mu - 232$

$\lambda = (p-1, p-2, 1, 0, 0)_{A_5}$	
$\mathrm{ch}\, L(\lambda)_{232} = \lambda - A + C$	
Lemma 2.1.8 and Remark 1.3.18	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{232} = A + B - C - D$	$\mathrm{JSF}(\lambda)_{232} = A + 2B +_0^1 D$
$\mathrm{JSF}(A)_{232} = B + C$	$\mathrm{JSF}(A)_{232} = B + C + 2D$
$\mathrm{JSF}(B)_{232} = D$	$\mathrm{JSF}(B)_{232} = D$
$\mathrm{JSF}(C)_{232} = D$	$\mathrm{JSF}(C)_{232} = D$
$A = \lambda - 01100 = (p, p-3, 0, 1, 0)\ 011$	$C = \lambda - 02210 = (p+1, p-4, 0, 0, 1)\ 032$
$B = \lambda - 11100 = (p-2, p-2, 0, 1, 0)\ 111$	$D = \lambda - 22210 = (p-3, p-2, 0, 0, 1)\ 232$

Table 2.23: JSF of λ up to $\mu - 232$

By Lemma 1.4.9, it is enough to consider $\mu - 232$. The truncated character $\mathrm{ch}\, L(\lambda)_{232}$ is computed in Table 2.25 using Lemma 2.1.8. The JSF of μ is computed in Table 2.26 and yields two possibilities for the truncated character $\mathrm{ch}\, L(\mu)_{232}$ depending on the value of $[\mu : D]$. By Table 2.27, we have that $[\mu : D] = [A : D]$ and by Table 2.26, we have $[A : D] = 1$. The JSF applied to ν yields $\mathrm{ch}\, L(\nu)_{121} = \nu$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 232) = (12, 9, 3)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

Let $c \neq 1$. The weights which could afford the highest weight of a third composition factor

$\mu=(p-1,p-2,1)_{C_3}$	
Possibilities	
$\operatorname{ch} L(\mu)_{232}=\mu+A-B$	
$\operatorname{ch} L(\mu)_{232}=\mu+A-B-C$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\operatorname{JSF}(\mu)_{232}=-A+B+C$	$\operatorname{JSF}(\mu)_{232}=B+2C$
$\operatorname{JSF}(B)_{232}=A+C$	$\operatorname{JSF}(B)_{232}=A+C$
$A=\mu-032=(p+2,p-4,0)$	$C=\mu-221=(p-3,p-2,1)$
$B=\mu-021=(p+1,p-4,1)$	

Table 2.24: JSF of μ up to $\mu-232$

generated by a maximal vector for B_X are given by the following weights.

$$\begin{aligned}\mu-112&=(0,p-c+2,c-3)\\ \mu-121&=(1,p-c-2,c)\\ \mu-122&=(1,p-c,c-2)\\ \mu-132&=(2,p-c-2,c-1)\\ \mu-232&=(0,p-c-1,c-1)\end{aligned}$$

It is enough to consider $\mu-232$. The truncated character $\operatorname{ch} L(\lambda)_{232}$ is computed in Table 2.28 and the JSF applied to μ and ν yields $\operatorname{ch} L(\mu)_{232}=\mu-(\mu-021)$ and $\operatorname{ch} L(\nu)_{121}=\nu-(\nu-021)$, respectively. Therefore, $(m_{L(\lambda)|_X},m_{L(\mu)},m_{L(\nu)})(\mu-232)=(17,14,3)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.3.1.11 $\boldsymbol{\lambda=a\lambda_1+b\lambda_2+d\lambda_4}$. — By Table 2.6, we have that either $b=1$ and $d\neq p-1$ or $b\neq p-1$ and $d=1$, and a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu=\mu-010=(a+1,b+d-2,1)$.

Note that for $\alpha\in\{11000,10010,00011\}$, we have $\alpha|_{T_X}=110$. If $a+b\neq p-1$, then by Lemma 2.1.4 and Theorem 1.1.10 $(m_{L(\lambda)|_X},m_{L(\mu)},m_{L(\nu)})(\mu-110)=(4,2,1)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth assume $a+b=p-1$.

If $b=1$ and $d\neq 1$ (recall $d\neq p-1$ too), then JSF yields $\operatorname{ch} L(\lambda)_{120}=\lambda-(\lambda-11000)$. The multiplicities listed in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $d=1$ and $a\neq 2$, then the JSF yields $\operatorname{ch} L(\lambda)_{121}=\lambda-(\lambda-11000)$ and $\operatorname{ch} L(\nu)_{111}=\nu-(\nu-110)$. Comparing the multiplicities in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $d=1$ and $a=2$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. We prove that none of the weights listed below affords the highest weight

$\lambda = (1, p-2, 1, 0, 0)_{A_5}$	
ch $L(\lambda)_{232} = \lambda - A - B - 2C + D + E$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{232} = A + B + C - D$	$\mathrm{JSF}(\lambda)_{232} = A + B + C + 2E$
$\mathrm{JSF}(A)_{232} = -C + E$	$\mathrm{JSF}(A)_{232} = E$
$\mathrm{JSF}(B)_{232} = -C + D + E$	$\mathrm{JSF}(B)_{232} = D + E$
$\mathrm{JSF}(E)_{232} = C$	$\mathrm{JSF}(E)_{232} = C$
$A = \lambda - 11000 = (0, p-3, 2, 0, 0)$ 110	$D = \lambda - 02210 = (3, p-4, 0, 0, 1)$ 032
$B = \lambda - 01100 = (2, p-3, 0, 1, 0)$ 011	$E = \lambda - 12100 = (1, p-4, 1, 1, 0)$ 121
$C = \lambda - 23200 = (0, p-4, 0, 2, 0)$ 232	

Table 2.25: JSF of λ up to $\mu - 232$

$\mu = (1, p-c-1, c)_{C_3}$	
ch $L(\mu)_{232} = \mu - A + B - C + D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{232} = A - B + C$	$\mathrm{JSF}(\mu)_{232} = A + C + 2D$
$\mathrm{JSF}(A)_{232} = D$	$\mathrm{JSF}(A)_{232} = D$
$\mathrm{JSF}(C)_{232} = B + D$	$\mathrm{JSF}(C)_{232} = B + D$
$A = \mu - 110 = (0, p-c-2, c+1)$	$C = \mu - 021 = (3, p-c-3, c)$,
$B = \mu - 032 = (4, p-c-3, c-1)$	$D = \mu - 131 = (2, p-c-4, c+1)$

Table 2.26: JSF of μ up to $\mu - 232$

$\lambda_0 = (0, 0, 0) \in C_0$	
$\gamma = w_1 \bullet \lambda_0 = (1, p-2, 1)$	$\eta = w \bullet \lambda_0 = (2, p-5, 2)$
$w_1 = s_0 s_2 s_3 s_2 s_1 s_2 s_3$	$w = s_0 s_2 s_3 s_1 s_2$
$s = s_1$	
$ws \bullet \lambda_0 = (3, p-4, 1)$	
$ws \bullet \lambda_0 - w \bullet \lambda_0 = 110$	
Proposition 1.3.9 $\implies$ $[w_1 s \bullet \lambda_0 : \eta] = [\gamma : \eta]$, where $w_1 s \bullet \lambda_0 = (0, p-3, 2)$	

Table 2.27: Computing $[(1, p-2, 1) : (2, p-5, 2)]_{C_3}$

$\lambda = (1, p-c-1, c, 0, 0)_{A_5}$	
ch $L(\lambda)_{232} = \lambda - A + B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{232} = A - B$	$\mathrm{JSF}(\lambda)_{232} = A$
$\mathrm{JSF}(A)_{232} = B$	$\mathrm{JSF}(A)_{232} = B$
$A = \lambda - 01100 = (2, p-c-2, c-1, 1, 0)$ 011	$B = \lambda - 02210 = (3, p-c-3, c-1, 0, 1)$ 032

Table 2.28: JSF of λ up to $\mu - 232$

of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 100 &= (0, p-1, 0) \\
\mu - 110 &= (1, p-3, 1) \\
\mu - 020 &= (4, p-6, 2) \\
\mu - 120 &= (2, p-5, 2) \\
\mu - 121 &= (2, p-3, 0) \\
\mu - 131 &= (3, p-5, 1)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider $\mu - 131$. The JSF applied to μ and ν yields $\mathrm{ch}\, L(\mu)_{131} = \mu$ and $\mathrm{ch}\, L(\nu)_{121} = \nu - (\nu - 110) - (\nu - 011)$, respectively. Moreover, the output of the JSF applied to λ appears in Table 2.29. There are two possibilities for the truncated character $\mathrm{ch}\, L(\lambda)_{131}$ depending on the value of $[\lambda : C]$. Let $[\lambda : C] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. Note that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 131) = (8 + \zeta, 6, 3)$. Thus $\zeta = 1$, since the multiplicity of $\mu - 131$ in the direct sum $L_X(\mu) \oplus L_X(\nu)$ cannot be greater than the multiplicity of $\mu - 131$ in $L_Y(\lambda)|_X$. Therefore X acts on $L_Y(\lambda)$ with exactly two composition factors.

$\lambda = (2, p-3, 0, 1, 0)_{A_5}$	
ch $L(\lambda)_{131} = \lambda - A - B + C$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{131} = A + B$	$\mathrm{JSF}(\lambda)_{131} = A + B + 2C$
$\mathrm{JSF}(A)_{131} = C$	$\mathrm{JSF}(A)_{131} = C$
$\mathrm{JSF}(B)_{131} = C$	$\mathrm{JSF}(B)_{131} = C$
$A = \lambda - 11000 = (1, p-4, 1, 1, 0)$ 110	$C = \lambda - 12110 = (2, p-5, 1, 0, 1)$ 131
$B = \lambda - 01110 = (3, p-4, 0, 0, 1)$ 021	

Table 2.29: JSF of λ up to $\mu - 131$

2.3.1.12 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + e\lambda_5}$. — By Lemma 2.1.9, a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 100 = (a+e-2, b+1, 0)$ and either $a = 1$ or $e = 1$. If $a + b \neq p - 1$, then a similar argument as in Subsection 2.3.1.11 implies that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 110) = (4, \leq_V 2, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth assume $a + b = p - 1$ and $a = 1$ or $e = 1$.

If $a = 1$ and $e \neq 1$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{210} = \lambda - (\lambda - 11000)$ and comparing the multiplicities in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $e = 1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. We prove that none of the weights below affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 110 &= (a, p-a-2, 1)\\
\mu - 121 &= (a+1, p-a-2, 0)\\
\mu - 200 &= (a-3, p-a+1, 0)\\
\mu - 210 &= (a-2, p-a-1, 1)\\
\mu - 221 &= (a-1, p-a-1, 0)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider $\mu - 221$. We consider two separate cases depending on whether $a \neq 3$ or $a = 3$.

Assume $a \neq 3$. The truncated character $\operatorname{ch} L(\lambda)_{221}$ is computed in Table 2.30. Moreover the JSF applied to μ and ν yields $\operatorname{ch} L(\mu)_{221} = \mu - (\mu - 220)$ and $\operatorname{ch} L(\nu)_{121} = \nu - \overline{\delta_{a,1}}(\nu - 110)$, respectively. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 221) = (9, 6, 3)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

Assume $a = 3$. The truncated character $\operatorname{ch} L(\lambda)_{221}$ is computed in Table 2.30. The partial character $\operatorname{ch} L(\nu)_{121}$ appears in Table 2.8. The JSF applied to μ yields $\operatorname{ch} L(\mu)_{221} = \mu - (\mu - 220) - (\mu - 121)$. Therefore

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 221) = (8, 5, 3),$$

which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

$\lambda = (a, p-a-1, 0, 0, 1)_{A_5}$	
$\operatorname{ch} L(\lambda)_{221} = \lambda - A + \overline{\delta_{a,1}}B - \delta_{a,3}C$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{221} = A - \overline{\delta_{a,1}}B + \delta_{a,3}C$	$\mathrm{JSF}(\lambda)_{221} = A + \delta_{a,3}C$
$\mathrm{JSF}(A)_{221} = \overline{\delta_{a,1}}B$	$\mathrm{JSF}(A)_{221} = \overline{\delta_{a,1}}B$
$A = \lambda - 11000 = (a-1, p-a-2, 1, 0, 1)$ 110	$C = \lambda - 01111 = (a+1, p-a-2, 0, 0, 0)$ 121
$B = \lambda - 22100 = (a-2, p-a-2, 0, 1, 1)$ 221	

Table 2.30: JSF of λ up to $\mu - 221$

2.3.1.13 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3 + d\lambda_4}$. — It is easy to check that $\mu - 110 = (a-1, d-1, c+1)$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. We can assume that λ corresponds to an irreducible case in Table 2.2, since otherwise X acts on $L_Y(\lambda)$ with more than two composition factors by Proposition 1.5.3 and Table 2.6. Therefore, assume from now on that $c + d = p - 1$ and set $\nu = \mu - 110$.

If $a, d > 1$, then applying the same argument as in Subsection 2.3.1.6 implies that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 220) = (6, \leq_V 3, \leq_V 2)$ and X acts on $L_Y(\lambda)$ with more than two factors. Henceforth assume either $a = 1$ or $d = 1$.

If $a+d = p-1$, the JSF applied to λ and to μ yields $\operatorname{ch} L(\lambda)_{110} = \lambda$ and $\operatorname{ch} L(\mu)_{110} = \mu - (\mu - 110)$, respectively. The multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $d = 1$ and $a \neq p-2$, the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{121} = \lambda - (\lambda - 00110)$ and comparing the multiplicities in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a = 1$ and $d \neq 1, p-2$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. We prove that none of the following weights affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 111 &= (0, p-c, c-1) \\
\mu - 120 &= (1, p-c-4, c+2) \\
\mu - 121 &= (1, p-c-2, c) \\
\mu - 131 &= (2, p-c-4, c+1) \\
\mu - 231 &= (0, p-c-3, c+1)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider $\mu - 231$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{231} = \lambda - (\lambda - 00110) - \delta_{c,p-3}(\lambda - 11100)$. Moreover the JSF applied to μ and ν yields $\operatorname{ch} L(\mu)_{231} = \mu - (\mu - 021) - \delta_{c,p-3}(\mu - 111)$ and $\operatorname{ch} L(\nu)_{121} = \nu - (\nu - 021)$, respectively. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 231) = (11 - \delta_{c,p-3}, 8 - \delta_{c,p-3}, 3)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.3.1.14 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3 + e\lambda_5}$. — By Table 2.6, we get that X acts on $L_Y(\lambda)$ with at least two composition factors. Moreover, Lemma 2.1.9 implies the existence of a third composition factor for X acting on $L_Y(\lambda)$.

2.3.1.15 $\boldsymbol{\lambda = b\lambda_2 + c\lambda_3 + d\lambda_4}$. — By Table 2.6, we must have $b + c = p - 1$, $b \neq 1$ and $d = 1$ and a second composition factor for X acting on $L_Y(\lambda)$ is given by the highest T_X-weight $\nu = \mu - 010 = (1, b + d - 2, c + 1)$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{121} = \lambda - (\lambda - 01100)$ and comparing the multiplicities in Table 2.31 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.16 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4}$. — By Table 2.6, we must have $b + c = p - 1, b \neq 1$ and $d = 1$ or $c + d = p - 1, d \neq 1$ and $b = 1$ and the weight $\nu = \mu - 010 = (a + 1, b + d - 2, c + 1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

If $a+b \neq p-1$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-110) = (4, \leq_V 2, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Assume from now on that $a+b=p-1$. If $b+c=p-1, b\neq 1$ and $d=1$, the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{111} = \lambda-(\lambda-11000)-(\lambda-01100)$. If $c+d=p-1, d\neq 1$ and $b=1$, the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{120} = \lambda-(\lambda-11000)$. In both cases, the multiplicities listed in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.17 $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + e\lambda_5$. — By Lemma 2.1.9, the T_X-weights $\mu - 100 = (a+e-2, p-c, c)$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$ and either $a=1$ or $e=1$. Therefore, if λ comes inductively from a case with two composition factors, then X acts on $L_Y(\lambda)$ with more than two composition factors. Assume λ comes from an irreducible case appearing in Table 2.6, that is $b+c=p-1$ and set $\nu=\mu-100$.

If $a\neq c$, then a similar argument as in Subsection 2.3.1.11 implies that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})$ $(\mu-110) = (4, \leq_V 2, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth, assume $a=c$.

If $a=1$ and $e=p-3$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{210} = \lambda-(\lambda-11000)$. If $a=p-3$ and $e=1$, then the JSF applied to λ and ν yields $\operatorname{ch} L(\lambda)_{211} = \lambda-(\lambda-11000)-(\lambda-01100)-(\lambda-00111)$ and $\operatorname{ch} L(\nu)_{111} = \nu-(\nu-110)$, respectively. If $a\neq p-3$ and $e\neq p-3$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{111} = \lambda-(\lambda-11000)-(\lambda-01100)$. In all three cases, the multiplicities in Table 2.31 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.1.18 $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4 + e\lambda_5$. — It is clear by Lemma 2.1.9 that $\mu-100$ and $\mu-010$ afford the highest weight of a composition factor. Hence X acts on $L_Y(\lambda)$ with more than two composition factors.

2.3.2 An additional result. —

Proposition 2.3.2. *Let $p>2$ and $n\geq 2$. Let G be of type A_{2n-1} and H be a subgroup of type C_n of G given by the fixed points of a graph automorphism of G. If $\lambda = a_n\lambda_n \in X(T_G)^+$ with $a_n \in \{1, p-1\}$, then H acts on $L_G(\lambda)$ with exactly two composition factors given by the highest weights $\lambda|_{T_H}$ and $\lambda|_{T_H} - \beta_{n-1} - \beta_n$.*

Proof. Let $\lambda = a_n\lambda_n$ for $a_n \in \{1, p-1\}$ and set $\mu = \lambda|_{T_H}$. Let $n=2$. By Table 2.3 for $p>3$ and by the tables in [Lüb07] for $p=3$, we have that H acts on $L_G(\lambda)$ with exactly two composition factors and the second composition factor is given by $\nu = \mu-\beta_2-\beta_3$. Let $n\geq 2$. By Proposition 1.5.3, we have that $\nu = \mu-\beta_{n-1}-\beta_n$ affords the highest weight of a second composition factor for H acting on $L_G(\lambda)$. We prove that H acts on $L_G(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. That is, we prove that the weights of the form $\mu-\alpha|_{T_H}$ or $\nu-\alpha|_{T_H}$ for $\alpha\in\Phi^+(G)\setminus\{\alpha_0\}$ do not afford the highest weight of a composition factor for H acting on $L_G(\lambda)$.

Let $a_n = p-1$. The weights of the form $\mu-\alpha|_{T_H}$ or $\nu-\alpha|_{T_H}$ for $\alpha\in\Phi^+(G)\setminus\{\alpha_0\}$ which are dominant are given by $\mu - b_{n-2}\beta_{n-2} - b_{n-1}\beta_{n-1} - b_n\beta_n$ with either $b_{n-2}=0$, $b_{n-1}\leq b_n$, $b_{n-1}\in\{0,1,2\}$ and $b_n\in\{0,1,2\}$ or $b_{n-2}=1$, $b_{n-1}=2$ and $b_n=2$. By Proposition 1.5.3 and the case $n=2$, the weights with $b_{n-2}=0$ do not afford the highest weight of a third composition factor. We verify that the multiplicity of the weights with $b_{n-2}\neq 0$ are equal in $L_G(\lambda)|_H$ and

$L_H(\mu) \oplus L_H(\nu)$. Note that by Proposition 1.5.2, we can set $n = 3$ and compute these multiplicities assuming H is of type C_3 and G of type A_5. By [Sei87, (6.1)], all the weight spaces of $L_G(\lambda)$ are of dimension 1, therefore $m_{L_G(\lambda)|_H}(\mu - 122) = 4$. The JSF applied to μ and ν yields $\operatorname{ch} L(\mu)_{122} = \mu$ and $\operatorname{ch} L(\nu)_{111} = \nu - (\nu - 111)$, respectively. We have $(m_{L(\mu)}, m_{L(\nu)})(\mu - 122) = (2, 2)$, hence the result follows for $a_n = p - 1$.

Let $a_n = 1$. Reasoning as in the case $a_n = p - 1$, the only candidate for the highest weight of a third composition factor is $\mu - 011$ and by Proposition 1.5.3 and the case $n = 2$ it does not afford the highest weight of a third composition factor for H acting on $L_G(\lambda)$. The proposition follows. $\square$

Table 2.31: Multiplicities for the proof of Proposition 2.3.1

λ	Conditions				Multiplicities			
$abcde \neq 0$			ν	θ	$\lambda(\theta)$	$\mu(\theta)$	$\nu(\theta)$	Details
$(a,0,0,0,0)$						irreducible		Table 2.2
$(0,b,0,0,0)$		$b=1$	121			two composition factors		Subsection 2.3.1.2
		$b=p-1$				irreducible		Table 2.2
		$b\neq 1,p-1$	121	242	$13-\delta_{b,3}-4\delta_{b,2}$	$\leq_V 9-\delta_{b,3}-3\delta_{b,2}$	$\leq_V 3-\delta_{b,2}$	Subsection 2.3.1.2
$(0,0,c,0,0)$	$c=1$		011			two composition factors		Subsection 2.3.1.3
	$c=p-1$		011			two composition factors		Subsection 2.3.1.3
$(a,b,0,0,0)$	$a+b=p-1$					irreducible		Table 2.2
	$b=1$	$a\neq p-2,p-6$	121			two composition factors		Subsection 2.3.1.4
		$a=p-6$	121	121	5	3	$\leq_V 1$	Subsection 2.3.1.4
	$b>1$	$a+b\neq p-1,p$	121	242	$24-\delta_{b,3}-3\delta_{a,1}$	$\leq_V 18-\delta_{b,3}-3\delta_{a,1}$	$\leq_V 5$	Subsection 2.3.1.4
					$24-5\delta_{b,2}-3\delta_{a,1}$	$\leq_V 18-4\delta_{b,2}-3\delta_{a,1}$	$\leq_V 5-\delta_{b,2}$	Subsection 2.3.1.4
		$a=4, b=p-4$	121	121	6	4	$\leq_V 1$	Subsection 2.3.1.4
		$a+b=p,\ a\neq 4$	121			two composition factors		Subsection 2.3.1.4
$(a,0,c,0,0)$	$c=1,p-1$	$a+c+2\not\equiv 0 \mod p$	011	111	5	$\leq_V 3$	$\leq_V 1$	Subsection 2.3.1.5
	$c=p-1$	$a=p-1$	011	111	4	2	$\leq_V 1$	Subsection 2.3.1.5
	$c=1$	$a=p-3$	011			two composition factors		Subsection 2.3.1.5
$(a,0,0,d,0)$	$a,d>1$		110	220	6	$\leq_V 3$	$\leq_V 2$	Subsection 2.3.1.6
	$a+d\neq p-3, d\neq p-1$	$a=1$ or $d=1$	110	121	$8-2\delta_{d,1}$	$\leq_V 5-\delta_{d,1}$	$\leq_V 2-\delta_{d,1}$	Subsection 2.3.1.6
	$a=1, d=p-1$		110			two composition factors		Subsection 2.3.1.6
	$a+d=p-3$	$a=1$	110	231	13	$\leq_V 8$	$\leq_V 4$	Subsection 2.3.1.6
		$d=1$	110			two composition factors		Subsection 2.3.1.6
$(a,*,*,*,e)$	$a,e>1$		100	200	3	$\leq_V 1$	$\leq_V 1$	Lemma 2.1.9
$(a,0,0,0,e)$	$a=p-1, e=1$		100	100	2	0	$\leq_V 1$	Subsection 2.3.1.7
	$a\neq p-1, e=1$		100			two composition factors		Subsection 2.3.1.7
$(0,b,c,0,0)$	$b+c=p-1$					irreducible		Table 2.2

	$b \neq p-1, p-2, p-4$	$c = 1$	011	121	$7 - \delta_{b,1}$	$\leq_V 4 - \delta_{b,1}$	$\leq_V 2$	Subsection 2.3.1.8
	$b + c = p,\ b \geq 3$		011	121	7	$\leq_V 4$	$\leq_V 2$	Subsection 2.3.1.8
$(0, b, 0, d, 0)$	$b \neq p-3, p-1, d = 1$		010	121	$9 - \delta_{b,1}$	$\leq_V 3$	$\leq_V 4 - \delta_{b,1}$	Subsection 2.3.1.9
	$b = p-3, d = 1$		010	121	7	2	$\leq_V 4$	Subsection 2.3.1.9
$(a, b, c, 0, 0)$	$b \neq p-2, p-4, c = 1$	$a + b \neq p-1, p-3, 2p-3$	011	111	6	$\leq_V 4$	$\leq_V 1$	Subsection 2.3.1.10
		$a + b = p-1$	011	111	5	3	$\leq_V 1$	Subsection 2.3.1.10
		$a + b + 3 \equiv 0 \mod p$	011	132	$18 - \delta_{b,2} - 5\delta_{b,1}$	$\leq_V 12 - \delta_{b,2} - 4\delta_{b,1}$	$\leq_V 5 - \delta_{b,1}$	Subsection 2.3.1.10
	$b + c = p$ and $3 \leq b \leq p-2$	$a + b \neq p-1, a \neq p-2$	011	111	6	$\leq_V 4$	$\leq_V 1$	Subsection 2.3.1.10
		$a + b = p-1$	011	111	5	3	$\leq_V 1$	Subsection 2.3.1.10
		$a = p-2$	011	122	14	$\leq_V 9$	$\leq_V 4$	Subsection 2.3.1.10
	$b + c = p-1, a = b-1$		111	111	5	3	$\leq_V 1$	Subsection 2.3.1.10
	$b + c = p-1, a \neq b-1$	$a = c, a \neq 1$	111	222	12	$8 - \delta_{a,b}$	3	Subsection 2.3.1.10
		$a = c+1, a, c \neq 1$	111	222	16	$\leq_{BS} 11$	$\leq_V 4$	Subsection 2.3.1.10
		$a \neq 1, c, c+1$ and $c \neq 1$	111	222	17	$\leq_{BS} 12 - \delta_{a,b}$	$\leq_V 4$	Subsection 2.3.1.10
		$c = 1, a \neq 1, p-2, p-1$	111	232	$23 - 2\delta_{a,2}$	$17 - 2\delta_{a,2}$	$\leq_V 5$	Subsection 2.3.1.10
		$c = 1, a = p-2, p-1$	111	232	23	$\leq_{BS} 17$	$\leq_V 5$	Subsection 2.3.1.10
		$a = 1$	111			two composition factors		Subsection 2.3.1.10
$(a, b, 0, d, 0)$	$a + b \neq p-1$		010	110	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.3.1.11
	$a + b = p-1$	$a \neq 2, d = 1$	010	121	9	$\leq_V 5$	3	Subsection 2.3.1.11
		$a = 2, d = 1$	010			two composition factors		Subsection 2.3.1.11
		$b = 1, d \neq 1, p-1$	010	120	5	$\leq_V 2$	$\leq_V 2$	Subsection 2.3.1.11
$(a, b, 0, 0, e)$	$a + b \neq p-1$		100	110	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.3.1.12
	$a + b = p-1$	$e = 1$	100			two composition factors		Subsection 2.3.1.12
		$a = 1, e \neq 1$	100	210	5	$\leq_V 2$	$\leq_V 2$	Subsection 2.3.1.12
$(a, 0, c, d, 0)$	$c + d = p-1$	$a, d > 1$	110	220	6	$\leq_V 3$	≤ 2	Subsection 2.3.1.13
		$a + d = p-1, a = 1$ or $d = 1$	110	110	3	1	$\leq_V 1$	Subsection 2.3.1.13
		$a = 1, d \neq 1, p-2$	110			two composition factors		Subsection 2.3.1.13
		$a \neq p-2, d = 1$	110	121	8	$\leq_V 6$	$\leq_V 1$	Subsection 2.3.1.13
	$c + d \neq p-1$		011	110	3	$\leq_V 2$	$\leq_V 0$	Subsection 2.3.1.13

$(a, 0, c, 0, e)$	$c = 1, p-1$		011	100	2	$\leq_V 1$	$\leq_V 0$	Subsection 2.3.1.14
$(0, b, c, d, 0)$	$b + c = p-1, b \neq 1, d = 1$		010	121	9	$\leq_V 4$	$\leq_V 4$	Subsection 2.3.1.15
$(a, b, c, d, 0)$	$b + c = p-1, b \neq 1, d = 1$	$a + b \neq p-1$	010	110	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.3.1.16
		$a + b = p-1$	010	111	6	$\leq_V 4$	$\leq_V 1$	Subsection 2.3.1.16
	$b = 1, c + d = p-1, d \neq 1$	$a \neq p-2$	010	110	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.3.1.16
		$a = p-2$	010	120	5	$\leq_V 2$	$\leq_V 2$	Subsection 2.3.1.16
$(a, b, c, 0, e)$	$b + c = p-1$	$a \neq c$ and $a = 1$ or $e = 1$	100	110	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.3.1.17
		$a = c, e \neq p-3, a = 1$	100	111	7	$\leq_V 4$	$\leq_V 2$	Subsection 2.3.1.17
		$a = c, a \neq p-3, e = 1$	100	111	7	$\leq_V 4$	$\leq_V 2$	Subsection 2.3.1.17
		$a = c, a = p-3, e = 1$	100	211	8	$\leq_V 4$	3	Subsection 2.3.1.17
		$a = c = 1, e = p-3$	100	210	5	$\leq_V 2$	$\leq_V 2$	Subsection 2.3.1.17
	$b + c \neq p-1$		011	100	2	$\leq_V 1$	0	Subsection 2.3.1.17
(a, b, c, d, e)			010	100	2	$\leq_V 1$	0	Subsection 2.3.1.18

2.4 $(X,Y)=(B_3,D_4)$

Recall the notations of Notation 2.1.3 and (1.6) and (1.7). In this section, we consider the pair of groups $(X,Y)=(B_3,D_4)$ with the embedding of X into Y fixed in the beginning of this chapter. It constitutes the last step in the inductive argument we have to carry out before considering the embedding (F_4,E_6). Let $\{\alpha_i\}_{i=1}^4$ be a set of simple roots in $\Phi(Y)$ and $\{\lambda_i\}_{i=1}^4$ be the corresponding set of fundamental weights in $X(T_Y)$. Similarly, let $\{\beta_i\}_{i=1}^3$ be a set of simple roots in $\Phi(X)$ and $\{\mu_i\}_{i=1}^3$ be the corresponding set of fundamental weights in $X(T_X)$. Note that for $\lambda=(a,b,c,d)\in X(T_Y)^+$, we have $\lambda|_{T_X}=(a,b,c+d)$.

2.4.1 Deducing information. — The goal of this section is to deduce the information listed in Table 2.32 about the action of X on specific irreducible representations of Y. Let $\lambda\in X(T_Y)^+$. We write "yes" in the last column of Table 2.32 to indicate that X acts on $L_Y(\lambda)$ with more than two composition factors. In case we do not determine whether or not X acts on $L_Y(\lambda)$ with more than two composition factors, we write n.d. for not determined. In what follows, whenever λ is written as a linear combination of parameters (i.e. a,b,c,d), we assume these parameters to be nonzero.

λ (a,b,c,d)	Conditions $abcd\neq 0$	2^{nd} factor for $L_Y(\lambda)\|_X$	3^{rd}
$(a,0,0,0)$	$a\neq 1,p-2$	$\lambda\|_{T_X}-111$	yes
	$a=1,p-2$	$\lambda\|_{T_X}-111$	n.d.
$(a,b,0,0)$	$b=1,p-1$	$\lambda\|_{T_X}-011$	yes
$(a,0,c,0)$	$a\neq 1,p-1,c=p-1$	$\lambda\|_{T_X}-111$	yes
	$a=1,c=p-1$	$\lambda\|_{T_X}-111$	n.d.
	$a+c+2\not\equiv 0 \mod p,\ 2a+c+4\equiv 0 \mod p$	$\lambda\|_{T_X}-111$	yes
	$a+c+2\not\equiv 0 \mod p,\ 2a+c+4\not\equiv 0 \mod p$	$\lambda\|_{T_X}-111$	n.d.
$(a,b,c,0)$	$b+c=p-1,\ 2a+b+3\equiv 0 \mod p,\ a\neq c$	$\lambda\|_{T_X}-111$	yes
	$b+c=p-1,\ 2a+b+3\not\equiv 0 \mod p,\ a\neq c$	$\lambda\|_{T_X}-111$	n.d.
$(a,0,c,d)$	$a=1,c=p-3,d=1$	$\lambda\|_{T_X}-001$	n.d.
	$a\neq 1,c=p-3,d=1$	$\lambda\|_{T_X}-001$	yes

Table 2.32: Action of X on $L_Y(\lambda)$

2.4.1.1 $\lambda=a\lambda_1$. — We have $(m_{L(\lambda)|_X},m_{L(\mu)})(\mu-100)=(1,1)$, $(m_{L(\lambda)|_X},m_{L(\mu)},m_{L(\nu)})(\mu-110)=(1,1,0)$ and $(m_{L(\lambda)|_X},m_{L(\mu)})(\mu-111)=(2,\leq_V 1,)$. Hence $\nu=\mu-111=(a-1,0,0)$ affords the highest weight of a second composition factor for X acting on $V_Y(\lambda)$.

Let $a\neq 1,p-2$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{222}=\lambda$. Comparing the multiplicities in Table 2.33 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.4.1.2 $\boldsymbol{a\lambda_1 + b\lambda_2}$. — By Table 2.3, a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 011 = (a+1, b-1, 0)$. If $b = 1$ and $a \neq p-2$ or $b = p-1$ and $1 \leq a \leq p-1$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{111} = \lambda$. If $a = p-2$ and $b = 1$, then the JSF applied to λ and μ yields $\operatorname{ch} L(\lambda)_{122} = \lambda - (\lambda - 1100)$ and $\operatorname{ch} L(\mu)_{122} = \mu - (\mu - 110)$. In both cases, comparing the multiplicities appearing in Table 2.33 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.4.1.3 $\boldsymbol{a\lambda_1 + c\lambda_3}$. — Assume $a+c+2 \not\equiv 0 \mod p$, since otherwise X acts irreducibly on $L_Y(\lambda)$ by Table 2.2. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{111} = \lambda$. We get the following multiplicities $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 100) = (1, 1)$, $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 110) = (1, 1)$ and $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 111) = (4, \leq_V 3)$. Hence a second composition factor for X acting on $L_Y(\lambda)$ is given by the weight $\nu = \mu - 111 = (a-1, 0, c)$. If $2a + c + 4 \equiv 0 \mod p$, then we have $m_{L(\mu)}(\mu - 111) = 2$ by Lemma 2.1.7 and ν also affords the highest weight of a third composition factor. If $c = p-1$ and $a \geq 2$, then the JSF to λ yields $\operatorname{ch} L(\lambda)_{222} = \lambda$ and comparing the multiplicities appearing in Table 2.33 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.4.1.4 $\boldsymbol{a\lambda_1 + b\lambda_2 + c\lambda_3}$. — Assume $b+c = p-1$ and $a \neq c$. Applying the JSF to λ along with Lemma 2.1.8 yields $\operatorname{ch} L(\lambda)_{111} = \lambda - (\lambda - 0110)$. We get $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 111) = (5, \leq_V 4)$ and $\nu = \mu - 111 = (a-1, b, p-b-1)$ affords the highest weight of second composition factor for X acting on $L_Y(\lambda)$. If $2a+b+3 \equiv 0 \mod p$, then the JSF applied to μ implies that $\operatorname{ch} L(\mu)_{111} = \mu - (\mu - 111)$ and ν affords the highest weight of a third composition factor for X acting on $L(\lambda)$.

2.4.1.5 $\boldsymbol{a\lambda_1 + c\lambda_3 + d\lambda_4}$. — Assume $c = p-3$ and $d = 1$. By Table 2.3, we have that $\nu = \mu - 001 = (a, 1, p-4)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. If $a \neq 1, p-3$, then by Proposition 1.2.2 and the JSF applied to λ, we get $\operatorname{ch} L(\lambda)_{111} = \lambda$. The multiplicities listed in Table 2.33 imply that X acts on $L_Y(\lambda)$ with more than two composition factors. If $a = p-3$, then the JSF applied to λ yields

$$\operatorname{ch} L(\lambda)_{112} = \lambda - (\lambda - 1101) - (\lambda - 0111)$$

and the JSF applied to ν yields $\operatorname{ch} L(\nu)_{111} = \nu - (\nu - 011)$. Comparing the dimensions in Table 2.33 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

λ	Conditions				Multiplicities			
$a\ b\ {}^{c}_{d}$ $abcd \neq 0$			ν	θ	$\lambda(\theta)$	$\mu(\theta)$	$\nu(\theta)$	Details
$(a, 0, 0, 0)$		$a \neq 1, p-2$	111	222	5	$\leq_V 3$	$\leq_V 1$	Subsection 2.4.1.1
		$a = 1, p-2$	111					Subsection 2.4.1.1
$(a, b, 0, 0)$	$b = 1$	$a \neq p-2$	011	111	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.4.1.2
		$a = p-2$	011	122	5	3	$\leq_V 1$	Subsection 2.4.1.2
	$b = p-1$		011	111	4	$\leq_V 2$	$\leq_V 1$	Subsection 2.4.1.2
$(a, 0, c, 0)$	$a \neq 1, p-1, c = p-1$		111	222	13	$\leq_V 9$	$\leq_V 3$	Subsection 2.4.1.3
	$a = 1, c = p-1$		111					Subsection 2.4.1.3
	$2a + c + 4 \equiv 0 \mod p, a \neq p-2$		111	111	4	2	$\leq_V 1$	Subsection 2.4.1.3
	$a + c + 2 \not\equiv 0 \mod p, 2a + c + 4 \not\equiv 0 \mod p$		111					Subsection 2.4.1.3
$(a, b, c, 0)$	$b + c = p-1, a \neq c$	$2a + b + 3 \equiv 0 \mod p$	111	111	5	3	$\leq_V 1$	Subsection 2.4.1.4
		otherwise	111					Subsection 2.4.1.4
$(a, 0, c, d)$	$c = p-3, d = 1$	$a = 1$	001					Subsection 2.4.1.5
		$a = p-3$	001	112	9	$\leq_V 5$	3	Subsection 2.4.1.5
		$a \neq 1, p-3$	001	111	6	$\leq_V 3$	$\leq_V 2$	Subsection 2.4.1.5

Table 2.33: Multiplicities in order to deduce Table 2.32

2.5 $(X, Y) = (F_4, E_6)$

In this section, we solve Question 3 for the case $(X, Y) = (F_4, E_6)$. We assume $p \geq 13$.

Proposition 2.5.1. *Let k be an algebraically closed field of characteristic $p \geq 13$. Let Y be a simply connected simple algebraic group of type E_6 over k and let X be the maximal closed connected subgroup of type F_4 of Y given by the fixed points of a graph automorphism of Y. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight and set $\mu = \lambda|_{T_X}$. Then X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if λ is listed in Table 2.34 up to graph automorphism. Moreover, $L_Y(\lambda)|_X \cong L_X(\mu) \oplus L_X(\nu)$ with ν given as in Table 2.34.*

λ (a,b,c,d,e,f)	Conditions $abcdef \neq 0$	ν
$(a,0,0,0,0,0)$	$a = 1, p-2$	$\mu - 1232$
$(0,b,0,0,0,0)$	$b = 1, p-2$	$\mu - 1110$
$(0,0,c,0,0,0)$	$c = p-1$	$\mu - 1231$
$(0,0,0,d,0,0)$	$d = p-1$	$\mu - 0110$
$(a,b,0,0,0,0)$	$a = p-4, b = 1$	$\mu - 1110$
$(a,0,0,0,e,0)$	$a = p-4, e = 1$	$\mu - 0011$
$(a,b,c,0,0,0)$	$a = 2, b = 1, c = p-3$	$\mu - 1111$
$(a,0,c,0,e,0)$	$a = 2, c = p-3, e = 1$	$\mu - 0010$

Table 2.34: Two-composition factors weights

Note that the statement about the decomposition into a direct sum follows directly from Lemmas 1.4.2 and 1.4.3. The rest of the proof is given in Subsection 2.5.1.

2.5.1 Proof of Proposition 2.5.1. — Recall the notations introduced in Notation 2.1.3 and (1.6) and (1.7) related to the truncated characters of simple or Weyl modules. Let $\{\alpha_i\}_{i=1}^6$ be a set of simple roots in $\Phi(Y)$ and $\{\lambda_i\}_{i=1}^6$ be the corresponding set of fundamental weights in $X(T_Y)$. Similarly, let $\{\beta_i\}_{i=1}^4$ be a set of simple roots in $\Phi(X)$ and $\{\mu_i\}_{i=1}^4$ be the corresponding set of fundamental weights in $X(T_X)$. Let $\lambda = (a,b,c,d,e,f) \in X(T_Y)^+$ be a p-restricted weight. Set $\mu = \lambda|_{T_X}$, then $\mu = (b, d, c+e, a+f)$. As for the proofs of the previous cases, we record information about the multiplicities and the composition factors in Table 2.104. Recall the notations $\leq_V$ and $\leq_{BS}$ which we introduced at the beginning of Sections 2.2 and 2.3. We define an additional notation. It sometimes happens that we know the precise decomposition of the partial irreducible character for a set of weights, but we only need a bound on a multiplicity in our argument. Whenever this is the case, we bound the multiplicity by the irreducible character which gives the greatest multiplicity and indicate it in Table 2.104 by a subscript BC (i.e $\leq_{BC}$). For example, in Table 2.51 on Page 85, we obtain the truncated irreducible character in the case $a = \frac{p-5}{2}$, but in Table 2.104, we include this case in the case $a \leq p-4, a \neq 1,3,7$, where we bound the multiplicity by the truncated irreducible character obtained for $a \leq p-4, a \neq 1,3,7, \frac{p-5}{2}$.

The argument to prove that X acts on $L_Y(\lambda)$ with more than two composition factors sometimes relies on computing the multiplicities for more than just one weight. In these cases, we indicate the different weights considered along with their multiplicities in Table 2.104. For example in the case $\lambda = a\lambda_1 + c\lambda_3, a+c = p-1$ and $a = 7$.

We start the proof by combining the results we have obtained in Sections 2.3 and 2.4 and Theorem 2.1.1 using Proposition 1.5.3. Let $\lambda \in X(T_Y)^+$ be p-restricted. Recall the notations of the pairs (X_{C_3}, Y_{A_5}) and (X_{B_3}, Y_{D_4}) from the beginning of Chapter 2. Assume X_{C_3} does not act irreducibly on $L_{Y_{A_5}}(\lambda|_{T_{Y_{A_5}}})$ and denote by $\mu_1 \in X(T_X)^+ \setminus \{\mu\}$ the highest weight of a second composition factor for X acting of $L_Y(\lambda)$ given by Proposition 1.5.3. Similarly, assume X_{B_3} does not act irreducibly on $L_{Y_{D_4}}(\lambda|_{T_{Y_{D_4}}})$ and denote by $\mu_2 \in X(T_X)^+ \setminus \{\mu\}$ the highest weight of a second composition factor for X acting of $L_Y(\lambda)$ given by Proposition 1.5.3. If $\mu_1 \neq \mu_2$, then X acts on $L_Y(\lambda)$ with more than two composition factors.

With the previous considerations in mind, by carefully combining Tables 2.2, 2.5 and 2.32, we get that if X acts on $L_Y(\lambda)$ with at most two composition factors, then λ appears up to graph automorphism in Table 2.35. In what follows, whenever λ is written as a linear combination of parameters (i.e. a, b, c, d, e, f), we assume these parameters to be nonzero.

Remark 2.5.2. We will apply Lemma 1.3.17 and Remark 1.3.18 without any explicit reference.

2.5.1.1 $\boldsymbol{\lambda = a\lambda_1}$. — Assume $a \neq p-3$, since otherwise X acts irreducibly on $L_Y(\lambda)$ by Theorem 2.1.1. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1232} = \lambda$. Applying the JSF to μ along with Proposition 1.2.2, we get $\operatorname{ch} L(\mu)_{1232} = \mu$. Computing multiplicities in $L_Y(\lambda)$ and $L_X(\mu)$, we get that the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\mu - 1232$. Set $\nu = \mu - 1232 = (0,0,0,a-1)$.

If $a = 1,2,3$, then

$$\dim(L(\lambda), L(\mu), L(\nu)) = \begin{cases} (27, 26, 1) & \text{if } a = 1 \\ (351, 324 - \delta_{p,13}, 26) & \text{if } a = 2 \\ (3003, 2652, 324) & \text{if } a = 3 \end{cases}$$

and X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if $a = 1$. Assume now $a \neq 1,2,3$.

If $a \neq p-2$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{2464} = \lambda$ and the multiplicities listed in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a = p-2$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. By Propositions 1.5.3 and 2.3.1, it is sufficient to prove that none of the weights listed below affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned} \mu - 1232 &= (0,0,0,p-3) \\ \mu - 1233 &= (0,0,1,p-5) \\ \mu - 1354 &= (1,0,0,p-5) \\ \mu - 2464 &= (0,0,0,p-4) \end{aligned}$$

$\lambda = (a,b,c,d,e,f)$	Conditions with $abcdef \neq 0$	2^{nd} factor for $L_Y(\lambda)\|_X$
$(a,0,0,0,0,0)$	$1 \leq a \leq p-1$	undetermined
$(0,b,0,0,0,0)$	$b = 1, p-2$	$\mu - 1110$
$(0,0,c,0,0,0)$	$c = 1$	$\mu - 0121$
	$c = p-1$	undetermined
$(0,0,0,d,0,0)$	$d = 1, p-1$	$\mu - 0110$
$(a,b,0,0,0,0)$	$b \in \{1, p-2\}, a \in \{1, \dots, p-1\}$	$\mu - 1110$
$(a,0,c,0,0,0)$	$a \neq p-2, p-6, c = 1$	$\mu - 0121$
	$a + c = p,\ a \neq 4, p-1$	$\mu - 0121$
	$a + c = p-1$	undetermined
$(a,0,0,d,0,0)$	$a = p-3, d = 1$	$\mu - 0110$
$(a,0,0,0,e,0)$	$a = 1, e = p-1$	$\mu - 0011$
	$a = p-4, e = 1$	$\mu - 0011$
$(a,0,0,0,0,f)$	$a \neq p-1, f = 1$	$\mu - 0001$
$(0,b,c,0,0,0)$	$b = 1, c = p-1$	$\mu - 1110$
	$b = p-1, c = p-1$	undetermined
	$b = p-3, c = 1$	$\mu - 0121$
$(0,0,c,d,0,0)$	$c + d = p-1$	undetermined
$(a,b,c,0,0,0)$	$a + c = p-1, b = a-1$	undetermined
	$a + c = p-1, b \neq a-1, 2b + c + 4 \not\equiv 0 \mod p$	$\mu - 1110$
	$a \neq p-2, p-6, b = p-3, c = 1$	$\mu - 0121$
	$a \neq 4, p-1,\ a + c = p,\ b + c + 2 \equiv 0 \mod p$	$\mu - 0121$
$(a,b,0,0,e,0)$	$a = 1, b = p-1, e = p-1$	$\mu - 0011$
	$a = p-4, b = p-3, e = 1$	$\mu - 0011$
$(a,0,c,d,0,0)$	$a = 1, c + d = p-1, c \neq 2$	$\mu - 0111$
$(a,0,c,0,e,0)$	$a = 2, c = p-3, e = 1$	$\mu - 0010$
$(a,0,c,0,0,f)$	$a + c = p-1, f = 1$	$\mu - 0001$
$(a,0,0,d,e,0)$	$a = 1, d + e = p-1, e \neq 1, p-2$	$\mu - 0011$
$(0,b,c,d,0,0)$	$b = c$ and $c + d = p-1$	undetermined
	$c + d = p-1, b \neq c, 2b + d + 3 \neq 0 \mod p$	$\mu - 1110$
$(a,b,c,d,0,0)$	$a = 1, c + d = p-1, c \neq 2, b = c$	$\mu - 0111$
$(a,b,c,0,e,0)$	$a = 2, b = 1, c = p-3, e = 1$	$\mu - 0010$
$(a,b,0,d,e,0)$	$a = 1, b = e, d + e = p-1$ and $e \neq 1, p-2$	$\mu - 0011$
$(a,b,c,0,0,f)$	$a + c = p-1, b = a-1, f = 1$	$\mu - 0001$

Table 2.35: The weights to consider.

By Lemma 1.4.9, it is enough to consider $\mu - 2464$. The JSF applied to λ, μ and ν yields $\operatorname{ch} L(\lambda)_{2464} = \lambda - (\lambda - 424420)$, $\operatorname{ch} L(\mu)_{2464} = \mu$ and

$$\operatorname{ch} L(\nu)_{1232} = \nu, \tag{2.3}$$

respectively. Hence, we have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2464) = (37, 32, 5),$$

which implies that X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.5.1.2 $\boldsymbol{\lambda = b\lambda_2}$. — By Table 2.35, we have $b = 1, p-2$ and $\nu = \mu - 1110 = (b-1, 0, 0, 1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

If $b = 1$, then $\dim(L(\lambda), L(\mu), L(\nu)) = (78, 52, 26)$ and X acts on $L_Y(\lambda)$ with exactly two composition factors.

If $b = p-2$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. By Propositions 1.5.3 and 2.3.1, it is sufficient to prove that none of the weights listed below affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 2110 &= (p-5, 1, 0, 1)\\
\mu - 2220 &= (p-4, 0, 0, 2)\\
\mu - 2221 &= (p-4, 0, 1, 0)\\
\mu - 2342 &= (p-3, 0, 0, 0)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider $\mu - 2342$. Applying the JSF to λ, μ and ν yields $\operatorname{ch} L(\lambda)_{2342} = \lambda - (\lambda - 021210)$, $\operatorname{ch} L(\mu)_{2342} = \mu$ and $\operatorname{ch} L(\nu)_{1232} = \nu - (\nu - 1111)$, respectively. We have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2342) = (12, 8, 4),$$

which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

2.5.1.3 $\boldsymbol{\lambda = c\lambda_3}$. — By Table 2.35, we have $c = 1$ or $c = p-1$. Let $c = 1$. By Table 2.35, a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 0121 = (1, 0, c-1, 0)$. Note that $V_Y(\lambda)$ is irreducible by [Lüb07] and the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $c = p-1$. We first find the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Computing the multiplicity of $\mu - 1231$ in $L_Y(\lambda)|_X$ and $L_X(\mu)$ using Tables 2.36 and 2.37, we get that $\mu - 1231$ affords the highest weight of a second composition factor. Set $\nu = \mu - 1231 = (0, 0, p-2, 1)$. We prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. By Propositions 1.5.3 and 2.3.1, it is sufficient to prove that none of the weights listed below affords the highest weight of an additional composition factor for

X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 1231 &= (0,0,p-2,1)\\
\mu - 1241 &= (0,1,p-4,2)\\
\mu - 1351 &= (1,0,p-4,3)\\
\mu - 1242 &= (0,1,p-3,0)\\
\mu - 1352 &= (1,0,p-3,1)\\
\mu - 2462 &= (0,0,p-3,2)\\
\mu - 2463 &= (0,0,p-2,0)\\
\mu - 2473 &= (0,1,p-4,1)
\end{aligned} \tag{2.4}$$

By Lemma 1.4.9, it is enough to consider $\mu - 2473$. The computation of $\operatorname{ch} L(\lambda)_{2473}, \operatorname{ch} L(\mu)_{2473}$ and $\operatorname{ch} L(\nu)_{1242}$ are summarized in Tables 2.36 to 2.38. Note that we need to determine the value of $[\mu : C]$, $[\lambda : G]$ and $[\nu : J]$.

$[\boldsymbol{\mu : C}]$ Let $[\mu : C] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 0363) = (10, 9+\zeta)$. By Remark 1.5.4 and Theorem 2.1.1, we have $\zeta = 1$.

$[\boldsymbol{\lambda : G}]$ Let $[\lambda : G] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2471) = (38+\zeta, 35, 4)$, hence $\zeta = 1$.

$[\boldsymbol{\nu : J}]$ Let $[\nu : J] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1473) = (57, 51, 5+\zeta). \tag{2.5}$$

We compute the multiplicities of all the weights greater than $\mu - 1473$ in $L_Y(\lambda)|_X, L_X(\mu)$ and $L_X(\nu)$, and then deduce that none of the weights greater than $\mu - 1473$ affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. Let us consider the weight $J = \mu - 1473$. Recall that our goal is to determine the weights greater than or equal to $\mu - 2473$ which afford the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Note that from Table 2.38, we get

$$\operatorname{ch} L(\nu)_{1242} = \nu - G - H - I + \zeta J.$$

We have to study the following two cases depending on the value of ζ.

$\boldsymbol{\zeta = 1}$ We are in the case $[\nu : J] = 1$. By (2.5), the weight J does not afford the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. Then, we use the following partial character of $L_X(\nu)$ in order to compute the multiplicities in $L_X(\nu)$ of the weights greater than or equal to $\mu - 2473$.

$$\operatorname{ch} L(\nu)_{1242} = \nu - G - H - I + J \tag{2.6}$$

$\boldsymbol{\zeta = 0}$ We are in the case $[\nu : J] = 2$. Note that

$$\operatorname{ch} L(\nu)_{1242} = \nu - G - H - I.$$

By (2.5), the weight J affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. Therefore, in order to determine which weights between J and $\mu - 2473$ afford the highest weight of a composition factor for X acting on $L_Y(\lambda)$, we need to take into account the contribution of $L_X(\mu), L_X(\nu)$ and $L_X(J)$ when computing multiplicities. We combine the contribution of $L_X(\nu)$ and $L_X(J)$ in the same partial character formula as follows.

$$\operatorname{ch} L(\nu)_{1242} + \operatorname{ch} L(J)_{1000} = \nu - G - H - I + J \tag{2.7}$$

Since the right-hand side of (2.6) and the right-hand side of (2.7) are equal, we can assume without loss of generality, for the purpose of further character and multiplicity computations, that we are in the case $\zeta = 1$. In the case $\zeta = 0$, the weight J affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Since our goal is to prove that X acts on $L_Y(\lambda)$ with exactly two composition factors, we should also explain why the case $\zeta = 0$ does not conflict with our goal. Note that the weight J does not appear in (2.4), the list of weights which can afford the highest weight of a third composition factor generated by a maximal vector for $\mathscr{L}(B_X)$. By Corollary 1.4.7, if X acts on $L_Y(\lambda)$ with more than two composition factors, then there exists a weight in (2.4) which affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Thus, even if $\zeta = 0$ and J affords the highest weight of a composition factor, another weight apart from μ, ν and J also affords the highest weight of a composition factor if X acts on $L_Y(\lambda)$ with more than two composition factors, so we can assume $\zeta = 1$ for this purpose too.

Using the partial character formulas we have computed, we get

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2473) = (95, 83, 12),$$

which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors and implies that $[\nu : J] = 1$.

2.5.1.4 $\boldsymbol{\lambda = d\lambda_4}$. — By Table 2.35, we have $d = 1$ or $p - 1$ and $\nu = \mu - 0110 = (1, d-1, 0, 1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

If $d = 1$, then $V_Y(\lambda)$ is irreducible by [Lüb07] and comparing the multiplicities appearing in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $d = p - 1$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. By Propositions 1.5.3 and 2.3.1, it is sufficient to prove that none of the weights listed below affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 1210 &= (0, p-3, 2, 1)\\
\mu - 1220 &= (0, p-2, 0, 2)\\
\mu - 1330 &= (1, p-3, 0, 3)\\
\mu - 1221 &= (0, p-2, 1, 0)\\
\mu - 1331 &= (1, p-3, 1, 1)\\
\mu - 1342 &= (1, p-2, 0, 0)\\
\mu - 1452 &= (2, p-3, 0, 1)
\end{aligned} \tag{2.8}$$

$\lambda = (0,0,p-1,0,0,0)_{E_6}$	
ch $L(\lambda)_{2473} = \lambda - A + B + C - D - E - F + 2G + H$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2473} = A - B - C + D + E + F - G - H$	$\mathrm{JSF}(\lambda)_{2473} = A +_0^1 D + F + 2G +_0^1 H$
$\mathrm{JSF}(A)_{2473} = B + C - E + G$	$\mathrm{JSF}(A)_{2473} = B + C + 2D + G +_0^1 H$
$\mathrm{JSF}(B)_{2473} = D - G - H$	$\mathrm{JSF}(B)_{2473} = D$
$\mathrm{JSF}(C)_{2473} = D + E - G$	$\mathrm{JSF}(C)_{2473} = D + E + 2H$
$\mathrm{JSF}(D)_{2473} = G + H$	$\mathrm{JSF}(D)_{2473} = G + H$
$\mathrm{JSF}(E)_{2473} = H$	$\mathrm{JSF}(E)_{2473} = H$
$\mathrm{JSF}(F)_{2473} = G$	$\mathrm{JSF}(F)_{2473} = G$
$A = \lambda - 102100 = (0,1,p-3,0,1,0)$ 0121	$E = \lambda - 104321 = (2,3,p-5,0,0,0)$ 0362
$B = \lambda - 113200 = (1,0,p-4,0,2,0)$ 1231	$F = \lambda - 024420 = (4,0,p-5,0,0,2)$ 2460
$C = \lambda - 103210 = (1,2,p-4,0,0,1)$ 0241	$G = \lambda - 125420 = (3,0,p-6,1,0,2)$ 2471
$D = \lambda - 114310 = (2,1,p-5,0,1,1)$ 1351	$H = \lambda - 115421 = (3,2,p-6,0,1,0)$ 1472

Table 2.36: JSF of λ up to $\mu - 2473$

$\mu = (0,0,p-1,0)_{F_4}$	
ch $L(\mu)_{2473} = \mu - A + B - D + E - F$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2473} = A - B + C + D - E + F$	$\mathrm{JSF}(\mu)_{2473} = A + 2C + D +_0^1 F$
$\mathrm{JSF}(A)_{2473} = B + C + E$	$\mathrm{JSF}(A)_{2473} = B + C + E + 2F$
$\mathrm{JSF}(B)_{2473} = F$	$\mathrm{JSF}(B)_{2473} = F$
$\mathrm{JSF}(E)_{2473} = F$	$\mathrm{JSF}(E)_{2473} = F$
$A = \mu - 0131 = (1,1,p-4,1)$	$D = \mu - 2460 = (0,0,p-5,6)$,
$B = \mu - 1251 = (0,2,p-6,3)$	$E = \mu - 0241 = (2,0,p-4,2)$,
$C = \mu - 0363 = (3,0,p-4,0)$	$F = \mu - 1361 = (1,1,p-6,4)$

Table 2.37: JSF of μ up to $\mu - 2473$

By Lemma 1.4.9, it is enough to consider $\mu - 1452$. The computations to determine ch $L(\lambda)_{1452}$, ch $L(\mu)_{1452}$ and ch $L(\nu)_{1342}$ are summarized in Tables 2.39 to 2.41, respectively. We still need to solve several problematic cases, treated successively below.

[$\boldsymbol{\lambda : F}$] Let $[\lambda : F] = 2-\zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1320) = (3+\zeta, 3, 1)$, hence $\zeta = 1$.

$\nu=(0,0,p-2,1)_{F_4}$	
ch $L(\nu)_{1242}=\nu-G-H-I+J$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1242}=G+H+I$	$\mathrm{JSF}(\nu)_{1242}=G+H+I+2J$
$\mathrm{JSF}(G)_{1242}=J$	$\mathrm{JSF}(G)_{1242}=J$
$\mathrm{JSF}(I)_{1242}=J$	$\mathrm{JSF}(I)_{1242}=J$
$G=\nu-0011=(0,1,p-3,0)$	$I=\nu-0132=(1,1,p-4,0)$,
$H=\nu-1230=(0,0,p-4,4)$	$J=\nu-0242=(2,0,p-4,1)$

Table 2.38: JSF of ν up to $\nu-1242$

[$\boldsymbol{\nu : H}$] By Proposition 1.5.2, we have $[\nu:H]=[(1,p-2,0):(1,p-3,0)]_{B_3}$. By Table 2.42, we have $[(1,p-2,0):(1,p-3,0)]_{B_3}=[(3,p-4,0):(2,p-5,2)]_{B_3}$. Moreover in B_3, we have $(3,p-4,0)-(2,p-5,2)=110$ and Lemma 2.1.4 implies that $[\nu:H]=1$.

[$\boldsymbol{\nu : I}$] Let $[\nu:I]=2-\zeta$ with $\zeta\in\{0,1\}$. We have $(m_{L(\lambda)|_X},m_{L(\mu)},m_{L(\nu)})(\mu-1321)=(6,3,2+\zeta)$. Noticing that $I=\mu-1321=\nu-1211$ does not appear in (2.8), the list of weights which afford the highest weight of a composition factor generated by a maximal vector for $\mathscr{L}(B_X)$, we apply the same reasoning as for the problematic case $[\nu:J]$ in Subsection 2.5.1.3. We check, by computing multiplicities, that there is no weight greater than I, apart from the weights μ and ν, which affords the highest weight of a composition factor. Then, either $\zeta=1$ and I does not afford the highest weight of a third composition factor for X acting on $L_Y(\lambda)$ or $\zeta=2$ and I affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. As in Subsection 2.5.1.3, for the purpose of further character computations, we can assume without loss of generality that $\zeta=1$, so that I does not afford the highest weight of composition factor for X acting on $L_Y(\lambda)$.

It remains to settle the problematic cases $[\lambda:I],[\lambda:K]$ and $[\nu:J]$. The upcoming argument is technical and relies on a repetitive application of the reasoning which was conducted in order to solve the problematic case $[\nu:I]$. The idea is as follows and relies on Corollary 1.4.7: Whenever we try to solve a problematic case for a weight which does not lie in (2.8), the list of weights which could afford the highest weight of a third composition factor generated by a maximal vector for $\mathscr{L}(B_X)$, we can add as many composition factors as we need in order to match the multiplicities and continue the argument until we reach a contradiction. Recall that by Propositions 1.5.3 and 2.3.1, there is no weight of the form $\mu-0xyz$ with $x,y,z\in\mathbb{Z}_{\geq 0}$ apart from μ and ν which affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

[$\boldsymbol{\lambda : I}$] & [$\boldsymbol{\lambda : K}$] Note that by symmetry $[\lambda:I]=[\lambda:K]$. Let $[\lambda:I]=2-\zeta$ with $\zeta\in\{0,1,2\}$. We compute the multiplicity of the weights greater than $\mu-1431$ in $L_Y(\lambda)|_X,L_X(\mu)$ and $L_X(\nu)$, and then deduce that there is no weight greater than $\mu-1431$ apart from μ and ν which affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1431) = (12+2\zeta, 6, 6)$, hence $\mu-1431$ affords the highest weight of 2ζ composition factors for X acting on $L_Y(\lambda)$. Note that $\mu-1431$ does not lie in (2.8), therefore if X acts on $L_Y(\lambda)$ with more than two composition factors, then, by Corollary 1.4.7, either $\mu-1342$ or $\mu-1452$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Let $W_{1431} = L_X(\mu-1431)^{2\zeta}$ and $W_{1431} = 0$ if $\zeta = 0$.

$[\boldsymbol{\nu : J}]$ Let $[\nu : J] = 5-\xi$ with $\xi \in \{0, \dots, 4\}$. Recall that $[\nu : H] = 1$ and $[\nu : I] = 1$, so using Table 2.41, we get

$$\operatorname{ch} L(\nu)_{1331} = \nu - F - G + I - (2-\xi)J.$$

We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1441) = (18+4\zeta, 9, 7+\xi)$ and $m_{W_{1431}}(\mu-1441) = 2\zeta$. Note that all the dominant weights greater than $\mu-1441$ and not of the form $\mu-0xyz$ with $x, y, z \in \mathbb{Z}_{\geq 0}$, are also greater than $\mu-1431$. Hence, we have already checked in the previous problematic case that there is no weight greater than $\mu-1441$ which affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$ apart from μ, ν and possibly $\mu-1431$. Let r be the number of composition factors for X acting on $L_Y(\lambda)$ of highest weight $\mu-1441$. Taking into account the multiplicities that we have computed above, we have $r = 18+4\zeta-9-7-\xi-2\zeta = 2+2\zeta-\xi$. Reasoning as in the problematic case $[\nu : I]$, we can assume without loss of generality that $\xi = 0$ and $r = 2+2\zeta$. Set $W_{1441} = L_X(\mu-1441)^r$.

We now proceed to show that neither $\mu-1342$ nor $\mu-1452$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. The only dominant weights greater than $\mu-1452$ and not greater than $\mu-1441$ which are not of the form $\mu-0xyz$ with $x, y, z \in \mathbb{Z}_{\geq 0}$ are $\mu-1342$ and $\mu-1442$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1342) = (18, 11, 7)$, hence $\mu-1342$ does not afford the highest weight of a composition factor for X acting on $L_Y(\lambda)$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1442) = (24+8\zeta, 14, 8)$, $m_{W_{1431}}(\mu-1442) = 4\zeta$ and $m_{W_{1441}}(\mu-1442) = r = 2+2\zeta$. Since $24+8\zeta-14-8-4\zeta-2-2\zeta = 2\zeta$, we get that $\mu-1442$ affords the highest weight of 2ζ composition factors for X acting on $L_Y(\lambda)$. Set $W_{1442} = L_Y(\mu-1442)^{2\zeta}$ and $W_{1442} = 0$ if $\zeta = 0$. We finally compute the multiplicity of $\mu-1452$ in $L_Y(\lambda)|_X$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1452) = (30+10\zeta, 15, 11)$, $m_{W_{1431}}(\mu-1452) = 4\zeta$, $m_{W_{1441}}(\mu-1452) = 2(2+2\zeta)$ and $m_{W_{1442}}(\mu-1452) = 2\zeta$. Note that $30+10\zeta-15-11-4\zeta-4-4\zeta-2\zeta = 0$. Thus, for any value of $\zeta \in \{0, 1, 2\}$, the weight $\mu-1452$ does not afford the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Therefore, X acts on $L_Y(\lambda)$ with exactly two composition factors. In turn, this implies $\zeta = 0$ and $\xi = 2$.

2.5.1.5 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2}$. — By Table 2.35, we have that $b = 1, p-2$ and the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu-1110 = (b-1, 0, 0, a+1)$.

If $b = 1$ and $a \neq p-4$, then the JSF applied to λ implies that $\operatorname{ch} L(\lambda)_{1111} = \lambda$ and the multiplicities listed in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $b = 1$ and $a = p-4$, we show that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. By Propositions 1.5.3 and 2.3.1, it is sufficient to prove that none of the

$\lambda = (0,0,0,p-1,0,0)_{E_6}$	
ch $L(\lambda)_{1452} = \lambda - A - B - C + D + E + 2F + G + H - 2I - J - 2K - L$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1452} = A + B + C - D - E - F - G - H + I + J + K$	$\mathrm{JSF}(\lambda)_{1452} = A + B + C + 2F +_0^2 I +_0^1 J +_0^2 K +_0^1 L$
$\mathrm{JSF}(A)_{1452} = D + F - K - L$	$\mathrm{JSF}(A)_{1452} = D + F + 2I$
$\mathrm{JSF}(B)_{1452} = F + G - I - L$	$\mathrm{JSF}(B)_{1452} = F + G + 2K$
$\mathrm{JSF}(C)_{1452} = E + F + H + L$	$\mathrm{JSF}(C)_{1452} = E + F + H + 2I + 2J + 2K + 2L$
$\mathrm{JSF}(D)_{1452} = I$	$\mathrm{JSF}(D)_{1452} = I$
$\mathrm{JSF}(E)_{1452} = I + J$	$\mathrm{JSF}(E)_{1452} = I + J$
$\mathrm{JSF}(F)_{1452} = I + K + L$	$\mathrm{JSF}(F)_{1452} = I + K + L$
$\mathrm{JSF}(G)_{1452} = K$	$\mathrm{JSF}(G)_{1452} = K$
$\mathrm{JSF}(H)_{1452} = J + K$	$\mathrm{JSF}(H)_{1452} = J + K$
$A = \lambda - 011200 = (1,0,0,p-3,2,0)$ 1210	$G = \lambda - 010321 = (0,1,3,p-4,0,0)$ 1321
$B = \lambda - 010210 = (0,0,2,p-3,0,1)$ 1210	$H = \lambda - 001321 = (1,3,1,p-4,0,0)$ 0331
$C = \lambda - 001210 = (1,2,0,p-3,0,1)$ 0220	$I = \lambda - 112410 = (0,2,1,p-5,2,1)$ 1431
$D = \lambda - 112300 = (0,1,0,p-4,3,0)$ 1321	$J = \lambda - 102421 = (0,4,1,p-5,1,0)$ 0442
$E = \lambda - 102310 = (0,3,0,p-4,1,1)$ 0331	$K = \lambda - 011421 = (1,2,2,p-5,1,0)$ 1431
$F = \lambda - 011310 = (1,1,1,p-4,1,1)$ 1320	$L = \lambda - 012420 = (2,2,0,p-4,0,2)$ 1440

Table 2.39: JSF of λ up to $\mu - 1452$

$\mu = (0,p-1,0,0)_{F_4}$	
ch $L(\mu)_{1452} = \mu + A - B - C - D + E$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1452} = -A + B + C + D - E$	$\mathrm{JSF}(\mu)_{1452} = B +_0^1 C + D$
$\mathrm{JSF}(A)_{1452} = C$	$\mathrm{JSF}(A)_{1452} = C$
$\mathrm{JSF}(B)_{1452} = A + E$	$\mathrm{JSF}(B)_{1452} = A + 2C + E$
$\mathrm{JSF}(E)_{1452} = C$	$\mathrm{JSF}(E)_{1452} = C$
$A = \mu - 1330 = (1,p-3,0,3)$	$D = \mu - 0342 = (3,p-3,0,0)$
$B = \mu - 1210 = (0,p-3,2,1)$	$E = \mu - 1321 = (1,p-4,3,0)$
$C = \mu - 1441 = (2,p-4,1,2)$	

Table 2.40: JSF of μ up to $\mu - 1452$

weights listed below affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\mu - 1111 = (0,0,1,p-5)$$
$$\mu - 1122 = (0,1,0,p-6)$$
$$\mu - 1232 = (1,0,0,p-5)$$
$$\mu - 2342 = (0,0,0,p-4)$$

$\nu = (1, p-2, 0, 1)_{F_4}$	
$\operatorname{ch} L(\nu)_{1342} = \nu - F - G + I$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1342} = F + G + H$	$\mathrm{JSF}(\nu)_{1342} = F + G + 2H + 2I +_1^5 J$
$\mathrm{JSF}(F)_{1342} = H + I$	$\mathrm{JSF}(F)_{1342} = H + I + 2J$
$\mathrm{JSF}(G)_{1342} = I + J$	$\mathrm{JSF}(G)_{1342} = I + 2J$
$\mathrm{JSF}(H)_{1342} = J$	$\mathrm{JSF}(H)_{1342} = J$
$\mathrm{JSF}(I)_{1342} = J$	$\mathrm{JSF}(I)_{1342} = J$
$F = \nu - 1100 = (0, p-3, 2, 1)$	$I = \nu - 1211 = (1, p-4, 3, 0)$
$G = \nu - 0111 = (2, p-3, 1, 0)$	$J = \nu - 1331 = (2, p-4, 1, 2)$
$H = \nu - 1220 = (1, p-3, 0, 3)$	

Table 2.41: JSF of ν up to $\nu - 1342$

$\lambda'_0 = (0, -1, 0) \notin C_0$	
$\gamma' = w_1 \bullet \lambda'_0 = (1, p-2, 0)$	$\eta' = w \bullet \lambda'_0 = (1, p-3, 0)$
$w_1 = s_0 s_1 s_2 s_3 s_2 s_0 s_1$	$w = s_0 s_1 s_2 s_3 s_2 s_0 s_2$
$\lambda_0 = (0, 0, 0) \in C_0$	
$\gamma = w_1 \bullet \lambda_0 = (3, p-3, 0)$	$\eta = w \bullet \lambda_0 = (2, p-5, 2)$
$C_{\eta'} = (1, 1, 1, 1, 2, 3, 1, 2, 2)$	$C_\eta = (1, 1, 1, 1, 2, 3, 1, 2, 2)$
Proposition 1.3.10 $\Longrightarrow$ $[\gamma' : \eta'] = [\gamma : \eta]$	
$s = s_1$	
$ws \bullet \lambda_0 = (1, p-5, 4)$	
$ws \bullet \lambda_0 - w \bullet \lambda_0 = 012$	
Proposition 1.3.9 $\Longrightarrow$ $[w_1 s \bullet \lambda_0 : \eta] = [\gamma : \eta]$, where $w_1 s \bullet \lambda_0 = (3, p-4, 0)$	

Table 2.42: Computing $[(1, p-2, 0) : (1, p-3, 0)]_{B_3}$

By Lemma 1.4.9, it is enough to consider the weight $\mu - 2342$. The JSF applied to λ and μ yields $\operatorname{ch} L(\lambda)_{2342} = \lambda - (\lambda - 111100)$ and $\operatorname{ch} L(\mu)_{2342} = \mu - (\mu - 1122)$, respectively. Moreover, by (2.3) on Page 73, we have $\operatorname{ch} L(\nu)_{1232} = \nu$. Hence, we get

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2342) = (25, 20, 5)$$

and X acts on $L_Y(\lambda)$ with exactly two composition factors.

If $b = p - 2$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1111} = \lambda - \delta_{a,p-1}(\lambda - 111100)$ and the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.5.1.6 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3}$. — By Table 2.35, we have to consider the three cases which we solve separately below.

$\boldsymbol{c = 1,\ a \neq p-2, p-6}$. — By Table 2.35, the weight $\nu = \mu - 0121 = (1,0,0,a)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. If $a \neq p-8$, the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1231} = \lambda$. If $a = p-8$, the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1353} = \lambda - (\lambda - 112210)$. In both cases, the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\boldsymbol{a + c = p,\ a \neq 4, p-1}$. — As in the previous case, by Table 2.35, the weight $\nu = \mu - 0121 = (1,0,p-a-1,a)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

If $a \neq 2, 6$, then the JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1231} = \lambda$. If $a = 2$, then the JSF applied to λ and μ yields

$$\operatorname{ch} L(\lambda)_{1242} = \lambda - (\lambda - 202000) - (\lambda - 012210)$$

and $\operatorname{ch} L(\mu)_{1242} = \mu - (\mu - 0022) - (\mu - 1230)$. The computations to determine the truncated character ν appear in Table 2.43. In both cases, the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $a = 6$. The JSF applied to μ yields $\operatorname{ch} L(\mu)_{1363} = \mu - (\mu - 0022)$. Moreover, the computations to determine $\operatorname{ch} L(\lambda)_{1363}$ and $\operatorname{ch} L(\nu)_{1242}$ appear in Tables 2.44 and 2.45. We determine $[\lambda : D]$ and show that we do not need to determine $[\nu : B]$ in order to prove that X acts on $L_Y(\lambda)$ with more than two composition factors. Let $[\lambda : D] = 2 - \zeta$ and $[\nu : B] = 2 - \xi$ with $\zeta, \xi \in \{0,1\}$. We have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1363) = (114 + 16\zeta, 100, 22 + 7\xi)$$

which implies that $\zeta = 1$. Moreover, for any value of ξ, we get that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\nu = (1,0,p-3,2)_{F_4}$	
$\operatorname{ch} L(\nu)_{1121} = \nu - D - E$	
To determine $[\nu : F]$, see Table 2.8 on Page 47	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1121} = D + E + F$	$\mathrm{JSF}(\nu)_{1121} = D + E + 2F$
$\mathrm{JSF}(D)_{1121} = F$	$\mathrm{JSF}(D)_{1121} = F$
$D = \nu - 0011 = (1,1,p-4,1)$	$F = \nu - 0121 = (2,0,p-4,2)$
$E = \nu - 1120 = (0,1,p-5,4)$	

Table 2.43: JSF of ν up to $\nu - 1121$

$\boldsymbol{a + c = p-1}$. — Let $a \neq 1$, since otherwise X acts irreducibly on $L_Y(\lambda)$ by Theorem 2.1.1. We first determine the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The

$\lambda = (6,0,p-6,0,0,0)_{E_6}$	
ch $L(\lambda)_{1363} = \lambda - A + B - C + D - E$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1363} = A - B + C$	$\mathrm{JSF}(\lambda)_{1363} = A + C + 2D +_0^1 E$
$\mathrm{JSF}(A)_{1363} = B + D$	$\mathrm{JSF}(A)_{1363} = B + D + 2E$
$\mathrm{JSF}(B)_{1363} = E$	$\mathrm{JSF}(B)_{1363} = E$
$\mathrm{JSF}(C)_{1363} = D - E$	$\mathrm{JSF}(C)_{1363} = D$
$\mathrm{JSF}(D)_{1363} = E$	$\mathrm{JSF}(D)_{1363} = E$
$A = \lambda - 202000 = (4,0,p-8,2,0,0)$ 0022	$D = \lambda - 213210 = (5,0,p-8,1,0,1)$ 1242
$B = \lambda - 303100 = (3,1,p-8,1,1,0)$ 0133	$E = \lambda - 314310 = (4,1,p-8,0,1,1)$ 1353
$C = \lambda - 112210 = (6,0,p-7,0,0,1)$ 1231	

Table 2.44: JSF of λ up to $\mu - 1363$

$\nu = (1,0,p-7,6)_{F_4}$	
Possibilities	
ch $L(\nu)_{1242} = \nu - A$	
ch $L(\nu)_{1242} = \nu - A - B$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1242} = A + B$	$\mathrm{JSF}(\nu)_{1242} = A + 2B$
$\mathrm{JSF}(A)_{1242} = B$	$\mathrm{JSF}(A)_{1242} = B$
$A = \nu - 0011 = (1,1,p-8,5)$	$B = \nu - 1121 = (0,1,p-8,6)$

Table 2.45: JSF of ν up to $\nu - 1242$

computation of $\mathrm{ch}\, L(\lambda)_{1231}$ and $\mathrm{ch}\, L(\mu)_{1231}$ is summarized in Tables 2.46 and 2.47, respectively. We solve the problematic cases as follows.

[$\mu : B$] Let $a = 2$ and $[\mu : B] = 2 - \zeta$ with $\zeta \in \{0,1\}$. Note that $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 0121) = (3, 2+\zeta)$. By Remark 1.5.4 and Theorem 2.1.1, we get $\zeta = 1$.

[$\lambda : B$] Let $a = 4$ and $[\lambda : B] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 1231) = (8+\zeta, 8)$. The truncated JSF of λ up to $\mu - 1232$ is computed in Table 2.50. We get

$$\mathrm{ch}\, L(\lambda)_{1232} = \lambda - (\lambda - 101000) - (\lambda - 202100) - (1-\zeta)(\lambda - 112210).$$

Moreover, $\mathrm{ch}\, L(\mu)_{1232}$ is computed in Table 2.51. Using these characters to calculate the multiplicity of $\mu-1232$ in $L_Y(\lambda)|_X$ and $L_X(\mu)$, we get $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu-1232) = (10-2\zeta, 9)$, which implies that $\zeta = 1$.

Computing multiplicities, we get that $\nu = \mu - 1231 = (0,0,p-a-2,a+1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ and that $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu-1231) = (9-3\delta_{a,p-2}, 8-3\delta_{a,p-2}-\delta_{a,3})$.

Note that $\mu = (0,0,p-a-1,a)$ and $\nu = (0,0,p-a-2,a+1)$. Hence if we know the truncated character of μ up to some weight $\theta \in X(T_X)^+$ for $a \in \{2,\dots,p-2\}$, then we know the truncated character of ν up to θ for $a \in \{1,\dots,p-3\}$.

If $a=3$, then ν also affords the highest weight of a third composition factor. Assume $a = p-2$, the computations to determine $\operatorname{ch} L(\lambda)_{2463}$ and $\operatorname{ch} L(\mu)_{2463}$ are summarized in Tables 2.48 and 2.49. Assume $a \neq 3, p-2$, the computations to determine $\operatorname{ch} L(\lambda)_{2462}$ and $\operatorname{ch} L(\mu)_{2462}$ are summarized in Tables 2.50 and 2.51. In both cases, the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (a,0,p-a-1,0,0,0)_{E_6}$	
$\operatorname{ch} L(\lambda)_{1231} = \lambda - A$ See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1231} = A + \delta_{a,4}B$	$\mathrm{JSF}(\lambda)_{1231} = A + 2\delta_{a,4}B$
$\mathrm{JSF}(A)_{1231} = \delta_{a,4}B$	$\mathrm{JSF}(A)_{1231} = \delta_{a,4}B$
$A = \lambda - 101000 = (a-1,0,p-a-2,1,0,0)$ 0011	$B = \lambda - 112210 = (a,0,p-a-2,0,0,1)$ 1231

Table 2.46: JSF of λ up to $\mu - 1231$

$\mu = (0,0,p-a-1,a)_{F_4}$	
$\operatorname{ch} L(\mu)_{1231} = \mu - A - \delta_{a,3}C$ See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1231} = A + \delta_{a,2}B + \delta_{a,3}C$	$\mathrm{JSF}(\mu)_{1231} = A + 2\delta_{a,2}B + \delta_{a,3}C$
$\mathrm{JSF}(A)_{1231} = \delta_{a,2}B$	$\mathrm{JSF}(A)_{1231} = \delta_{a,2}B$
$A = \mu - 0011 = (0,1,p-a-2,a-1)$	$C = \mu - 1231 = (0,0,p-a-2,a+1)$
$B = \mu - 0121 = (1,0,p-a-2,a)$	

Table 2.47: JSF of μ up to $\mu - 1231$

2.5.1.7 $\boldsymbol{\lambda = a\lambda_1 + d\lambda_4}$. — By Table 2.35, we have that $a = p-3, d = 1$ and $\nu = \mu - 0110 = (1,d-1,0,a+1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

$\lambda = (p-2, 0, 1, 0, 0, 0)_{E_6}$	
ch $L(\lambda)_{2463} = \lambda - A + B - C - D$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2463} = A - B + C + D$	$\mathrm{JSF}(\lambda)_{2463} = A$
$\mathrm{JSF}(A)_{2463} = B - C - D$	$\mathrm{JSF}(A)_{2463} = B$
$\mathrm{JSF}(B)_{2463} = C + D$	$\mathrm{JSF}(B)_{2463} = C + D$
$A = \lambda - 101000 = (p-3, 0, 0, 1, 0, 0)$ 0011	$C = \lambda - 313200 = (p-5, 0, 0, 0, 2, 0)$ 1233
$B = \lambda - 202100 = (p-4, 1, 0, 0, 1, 0)$ 0122	$D = \lambda - 303210 = (p-5, 2, 0, 0, 0, 1)$ 0243

Table 2.48: JSF of λ up to $\mu - 2463$

$\mu = (0, 0, 1, p-2)_{F_4}$	
ch $L(\mu)_{2463} = \mu - A + B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2463} = A - B$	$\mathrm{JSF}(\mu)_{2463} = A$
$\mathrm{JSF}(A)_{2463} = B$	$\mathrm{JSF}(A)_{2463} = B$
$A = \mu - 0011 = (0, 1, 0, p-3)$	$B = \mu - 0133 = (1, 1, 0, p-5)$

Table 2.49: JSF of μ up to $\mu - 2463$

$\lambda = (a, 0, p-a-1, 0, 0, 0)_{E_6}$	
ch $L(\lambda)_{2462} = \lambda - A + B$ See Table 2.46 on Page 83	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2462} = A - B + \delta_{a,4}C$	$\mathrm{JSF}(\lambda)_{2462} = A + 2\delta_{a,4}C$
$\mathrm{JSF}(A)_{2462} = B + \delta_{a,4}C$	$\mathrm{JSF}(A)_{2462} = B + \delta_{a,4}C$
$A = \lambda - 101000 = (a-1, 0, p-a-2, 1, 0, 0)$ 0011	$C = \lambda - 112210 = (a, 0, p-a-2, 0, 0, 1)$ 1231
$B = \lambda - 202100 = (a-2, 1, p-a-2, 0, 1, 0)$ 0122	

Table 2.50: JSF of λ up to $\mu - 2462$

The JSF applied to λ yields $\mathrm{ch}\, L(\lambda)_{1221} = \lambda - (\lambda - 101100)$ and the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.5.1.8 $\lambda = a\lambda_1 + e\lambda_5$. — By Table 2.35, either $a = 1, e = p-1$ or $a = p-4, e = 1$. Moreover, a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 0011 = (0, 1, e-1, a-1)$.

$\mu = (0, 0, p-a-1, a)_{F_4}$	
Possibilities	
$\operatorname{ch} L(\mu)_{2462} = \mu - A - \delta_{a,\frac{p+5}{2}} D$	
$\operatorname{ch} L(\mu)_{2462} = \mu - A - \delta_{a,7} C - \delta_{a,\frac{p+5}{2}} D$	
See Table 2.47 on Page 83 for $[\mu : B]$.	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2462} = A + \delta_{a,2} B + \delta_{a,7} C + \delta_{a,\frac{p+5}{2}} D$	$\mathrm{JSF}(\mu)_{2462} = A + 2\delta_{a,2} B + 2\delta_{a,7} C + \delta_{a,\frac{p+5}{2}} D$
$\mathrm{JSF}(A)_{2462} = \delta_{a,2} B + \delta_{a,7} C$	$\mathrm{JSF}(A)_{2462} = \delta_{a,2} B + \delta_{a,7} C$
$A = \mu - 0011 = (0, 1, p-a-2, a-1)$	$C = \mu - 1232 = (0, 0, p-a-1, a-1)$,
$B = \mu - 0121 = (1, 0, p-a-2, a)$	$D = \mu - 2462 = (0, 0, p-a-3, a+2)$

Table 2.51: JSF of μ up to $\mu - 2462$

Assume $a = 1$ and $e = p-1$. The truncated character $\operatorname{ch} L(\lambda)_{1231}$ is computed in Table 2.52. The JSF applied to ν yields $\operatorname{ch} L(\nu)_{1220} = \nu - (\nu - 0120)$. Comparing the multiplicities appearing in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

Assume $a = p-4$ and $e = 1$. Applying Corollary 1.4.7, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors. By Propositions 1.5.3 and 2.3.1, it is enough to prove that none of the following weights affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 1231 &= (0, 0, 0, p-3)\\
\mu - 1232 &= (0, 0, 1, p-5)\\
\mu - 1233 &= (0, 0, 2, p-7)\\
\mu - 1243 &= (0, 1, 0, p-6)\\
\mu - 1353 &= (1, 0, 0, p-5)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider the weight $\mu - 1353$. The truncated characters $\operatorname{ch} L(\lambda)_{1353}$ and $\operatorname{ch} L(\nu)_{1342}$ are computed in Tables 2.53 and 2.54, respectively. The JSF applied to μ yields $\operatorname{ch} L(\mu)_{1353} = \mu - (\mu - 1233)$. Therefore we have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1353) = (65, 46, 19),$$

which proves the result.

2.5.1.9 $\boldsymbol{\lambda = a\lambda_1 + f\lambda_6}$. — By Table 2.35, we have that $f = 1, a \neq p-1$ and $\nu = \mu - 0001 = (0, 0, 1, a-1)$ affords the highest weight of a second composition factor for X acting $L_Y(\lambda)$. If $a = 1$, then $L_Y(\lambda) = V_Y(\lambda)$ by the tables in [Lüb07]. Assume $a \neq 1$. Applying the JSF to λ yields

$$\operatorname{ch} L(\lambda)_{1232} = \lambda - \delta_{a,p-5}(\lambda - 101111) - \delta_{a,p-3}(\lambda - 212210).$$

$\lambda = (1,0,0,0,p-1,0)_{E_6}$	
$\operatorname{ch} L(\lambda)_{1231} = \lambda - A + B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1231} = A - B$	$\mathrm{JSF}(\lambda)_{1231} = A$
$\mathrm{JSF}(A)_{1231} = B$	$\mathrm{JSF}(A)_{1231} = B$
$A = \lambda - 000121 = (1,1,1,0,p-3,0)$ 0121	$B = \lambda - 010231 = (1,0,2,0,p-4,1)$ 1231

Table 2.52: JSF of λ up to $\mu - 1231$

$\lambda = (p-4,0,0,0,1,0)_{E_6}$	
$\operatorname{ch} L(\lambda)_{1353} = \lambda - A + B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1353} = A - B$	$\mathrm{JSF}(\lambda)_{1353} = A$
$\mathrm{JSF}(A)_{1353} = B$	$\mathrm{JSF}(A)_{1353} = B$
$A = \lambda - 101110 = (p-5,1,0,0,0,1)$ 0121	$B = \lambda - 202221 = (p-6,2,0,0,0,0)$ 0243

Table 2.53: JSF of λ up to $\mu - 1353$

$\nu = (0,1,0,p-5)_{F_4}$	
$\operatorname{ch} L(\nu)_{1342} = \nu - A + B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1342} = A - B$	$\mathrm{JSF}(\nu)_{1342} = A$
$\mathrm{JSF}(A)_{1342} = B$	$\mathrm{JSF}(A)_{1342} = B$
$A = \nu - 0111 = (1,0,1,p-6)$	$B = \nu - 0232 = (2,0,0,p-6)$

Table 2.54: JSF of ν up to $\nu - 1342$

If $a = p-5$, then the JSF applied to ν yields $\operatorname{ch} L(\nu)_{1231} = \nu - (\nu - 0121)$. In all cases, comparing the multiplicities listed in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors in all cases.

2.5.1.10 $\boldsymbol{\lambda = b\lambda_2 + c\lambda_3}$. — By Table 2.35, we have to consider the cases $(b,c) \in \{(p-3,1),(1,p-1),(p-1,p-1)\}$.

If $b = p-3, c = 1$, then a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 0121 = (p-2,0,0,0)$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1231} = \lambda - (\lambda - 011100)$ and comparing the multiplicities in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $b = 1, c = p-1$, then a second composition factor is given by $\nu = \mu - 1110 = (0,0,p-1,1)$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1121} = \lambda - (\lambda - 102100)$ and the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $b = p-1, c = p-1$. The computation of $\operatorname{ch} L(\lambda)_{2242}$ and $\operatorname{ch} L(\mu)_{2242}$ are summarized in Tables 2.55 and 2.56, respectively. We prove that $\nu = \mu - 1121 = (p-2, 1, p-2, 0)$ affords the highest weight of a second composition factor and that X acts on $L_Y(\lambda)$ with more than two composition factors by studying the different possibilities for the coefficients $[\lambda : F], [\mu : F], [\mu : A], (\lambda : G)$ and $(\mu : E)$.

$[\boldsymbol{\lambda} : \boldsymbol{F}]$ & $[\boldsymbol{\mu} : \boldsymbol{F}]$ Let $[\lambda : F] = 2 - \zeta$ with $\zeta \in \{0,1\}$ and $[\mu : F] = 2 - \xi$ with $\xi \in \{0,1\}$. We check by computing multiplicities in $L_Y(\lambda)|_X, L_X(\mu)$ and $L_X(\nu)$ that there is no weight greater than $\mu - 1121$ apart from μ which affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Moreover, we have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 1121) = (6+\zeta, 6)$ and $(m_{L(\lambda)|_X}, m_{L(\mu)})$ $(\mu - 1231) = (11 + 4\zeta, 12 + \xi)$. Therefore $\zeta = 1$ and ν affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. If $\xi = 0$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1131)$ $= (8,6,1)$ and X acts with more than two composition factors on $L_Y(\lambda)$. Assume from now on that $\xi = 1$.

$[\boldsymbol{\mu} : \boldsymbol{A}]$ Let $[\mu : A] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 2220) = (7, 6+\zeta)$. By Remark 1.5.4 and Theorem 2.1.1 we have $\zeta = 1$.

The computation of $\operatorname{ch} L(\nu)_{1121}$ appears in Table 2.57. We get an additional problematic case given by $[\nu : H]$.

$[\boldsymbol{\nu} : \boldsymbol{H}]$ Let $[\nu : H] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2231) = (22, 19, 2+\zeta).$$

If $\zeta = 0$, then the weight $\mu - 2231$ affords the highest weight of a third composition factor. Therefore assume $\zeta = 1$.

$(\boldsymbol{\lambda} : \boldsymbol{G})$ & $(\boldsymbol{\mu} : \boldsymbol{E})$ Let $[\lambda : G] = 2 - \zeta$ and $[\mu : E] = 2 - \xi$, with $\zeta, \xi \in \{0,1,2\}$. We show that we can exclude the case $\xi = 0$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2242) = (38 + 2\zeta, 31 + \xi, 5)$, thus if $\xi = 0$, then X acts on $L_Y(\lambda)$ with more than two composition factors. Hence we can assume that $[\mu : E] \in \{0,1\}$ and by Lemma 1.3.17, we get $(\mu : E) = 1$. Therefore, $(m_{L(\lambda)|_X}$, $m_{L(\mu)}, m_{L(\nu)})(\mu - 2242) = (38 + 2\zeta, 32, 5)$ and X acts with more than two composition factors on $L_Y(\lambda)$.

Therefore, X acts on $L_Y(\lambda)$ with more than two composition factors.

2.5.1.11 $\boldsymbol{\lambda = c\lambda_3 + d\lambda_4}$. — Note that by Table 2.35, we have $c + d = p - 1$. We first find the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. For convenience, we give here the truncated character formulas for the simple modules $L_Y(\lambda)$ and $L_X(\mu)$ up to the weight $\mu - 1221$ which will be computed over the next pages.

$$\operatorname{ch} L(\lambda)_{1221} = \lambda - (\lambda - 001100) - \overline{\delta_{d,1}}(\lambda - 102200) - \overline{\delta_{d,p-2}}(\lambda - 012200)$$

$\lambda = (0, p-1, p-1, 0, 0, 0)_{E_6}$	
Possibilities	
$\mathrm{ch}\, L(\lambda)_{2242} = \lambda - A - B + C + D + E$	
$\mathrm{ch}\, L(\lambda)_{2242} = \lambda - A - B + D + E + F - G$	
$\mathrm{ch}\, L(\lambda)_{2242} = \lambda - A - B + D + E + F$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2242} = A + B - C - D - E$	$\mathrm{JSF}(\lambda)_{2242} = A + B +_0^1 C + 2F +_0^2 G$
$\mathrm{JSF}(A)_{2242} = -C + D + F$	$\mathrm{JSF}(A)_{2242} = D + F + 2G$
$\mathrm{JSF}(B)_{2242} = C + E + F$	$\mathrm{JSF}(B)_{2242} = 2C + E + F + 2G$
$\mathrm{JSF}(D)_{2242} = G$	$\mathrm{JSF}(D)_{2242} = G$
$\mathrm{JSF}(E)_{2242} = G$	$\mathrm{JSF}(E)_{2242} = G$
$\mathrm{JSF}(F)_{2242} = C + G$	$\mathrm{JSF}(F)_{2242} = C + G$
$A = \lambda - 102100 = (0, p, p-3, 0, 1, 0)$ 0121	$E = \lambda - 022210 = (2, p-3, p-3, 1, 0, 1)$ 2230
$B = \lambda - 011100 = (1, p-2, p-2, 0, 1, 0)$ 1110	$F = \lambda - 112100 = (0, p-2, p-3, 1, 1, 0)$ 1121
$C = \lambda - 122200 = (0, p-3, p-2, 0, 2, 0)$ 2221	$G = \lambda - 123210 = (1, p-3, p-4, 2, 0, 1)$ 2241
$D = \lambda - 103210 = (1, p+1, p-4, 0, 0, 1)$ 0241	

Table 2.55: JSF of λ up to $\mu - 2242$

$$\mathrm{ch}\, L(\mu)_{1221} = \mu - \overline{\delta_{d,p-2}}(\mu - 0120) - \delta_{d,p-3}(\mu - 1221) - \delta_{d,p-2}(\mu - 1220)$$

Using these formulas, we get that $\nu = \mu - 1221 = (0, d-1, p-d, 0)$ affords the highest weight of a second factor for X acting on $L_Y(\lambda)$ and that

$$(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 1221) = (9 - \delta_{d,1}, 8 - \delta_{d,1} - \delta_{d,p-3}).$$

Note that $\mu = (0, d, p-d-1, 0)$ and $\nu = (0, d-1, p-d, 0)$. By comparing the coefficients of μ and ν we get that if we know truncated character of μ up to $\mu - 2442$ for $d \in \{1, \dots, p-2\}$, then we know the truncated character of ν up to $\mu - 2442$ for $d \in \{2, \dots, p-1\}$. We solve this case by considering the following subcases separately.

$\boldsymbol{d = 1.}$ — We summarize the computation of $\mathrm{ch}\, L(\lambda)_{2452}$ and $\mathrm{ch}\, L(\mu)_{2452}$ in Tables 2.58 and 2.59, respectively. Moreover, the JSF applied to ν yields $\mathrm{ch}\, L(\nu)_{1231} = \nu - (\nu - 0131)$. The multiplicities listed in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\boldsymbol{d \neq 1, p-4, p-3, p-2.}$ — The JSF are computed in Tables 2.60 and 2.61 and we prove that X acts on $L_Y(\lambda)$ by studying the problematic cases.

$[\boldsymbol{\mu : F}]$ Let $d = p - 6$ and $[\mu : F] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})$ $(\mu - 1231) = (12, 10 + \zeta, 1)$. If $\zeta = 0$, then X acts on $L_Y(\lambda)$ with more than two composition factors, hence assume $\zeta = 1$.

$\mu=(p-1,0,p-1,0)_{F_4}$	
Possibilities	
ch $L(\mu)_{2242}=\mu-B-C+D+E$	
ch $L(\mu)_{2242}=\mu-B-C+D+2E$	
ch $L(\mu)_{2242}=\mu-B-C+D+F$	
ch $L(\mu)_{2242}=\mu-B-C+D+E+F$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2242}=A+B+C-D-E$	$\mathrm{JSF}(\mu)_{2242}=2A+B+C+_0^2E+2F$
$\mathrm{JSF}(B)_{2242}=D+F$	$\mathrm{JSF}(B)_{2242}=D+2E+F$
$\mathrm{JSF}(C)_{2242}=A+E+F$	$\mathrm{JSF}(C)_{2242}=A+2E+F$
$\mathrm{JSF}(D)_{2242}=E$	$\mathrm{JSF}(D)_{2242}=E$
$\mathrm{JSF}(F)_{2242}=E$	$\mathrm{JSF}(F)_{2242}=E$
$A=\mu-2220=(p-3,0,p-1,2)$	$D=\mu-0241=(p+1,0,p-4,2)$,
$B=\mu-0131=(p,1,p-4,1)$	$E=\mu-2241=(p-3,2,p-4,2)$,
$C=\mu-1120=(p-2,1,p-3,2)$	$F=\mu-1131=(p-2,2,p-4,1)$

Table 2.56: JSF of μ up to $\mu-2242$

$\nu=(p-2,1,p-2,0)_{F_4}$	
Possibilities	
ch $L(\nu)_{1121}=\nu-G-H-I$	
ch $L(\nu)_{1121}=\nu-G-I-J$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1121}=G+H+I$	$\mathrm{JSF}(\nu)_{1121}=G+2H+I+J$
$\mathrm{JSF}(G)_{1121}=H-J$	$\mathrm{JSF}(G)_{1121}=H$
$\mathrm{JSF}(H)_{1121}=J$	$\mathrm{JSF}(H)_{1121}=J$
$G=\nu-1100=(p-3,0,p,0)$	$I=\nu-0120=(p-1,1,p-4,2)$,
$H=\nu-1110=(p-3,1,p-2,1)$	$J=\nu-1121=(p-3,2,p-3,0)$

Table 2.57: JSF of ν up to $\nu-1121$

$[\boldsymbol{\lambda : I}]$ Let $d=p-7$ and $[\lambda:I]=2-\zeta$ with $\zeta\in\{0,1\}$. We have $(m_{L(\lambda)|_X},m_{L(\mu)},m_{L(\nu)})(\mu-1342)$ $=(30+\zeta,28,3)$, hence $\zeta=1$.

$[\boldsymbol{\lambda : H}]$ By Proposition 1.5.2, we can solve this case in the Levi factor L_I of P_I, where $I=$

$\lambda = (0, 0, p-2, 1, 0, 0)_{E_6}$	
ch $L(\lambda)_{2452} = \lambda - A + B + C - D - E$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2452} = A - B - C + D + E$	$\mathrm{JSF}(\lambda)_{2452} = A +^1_0 D$
$\mathrm{JSF}(A)_{2452} = B + C - E$	$\mathrm{JSF}(A)_{2452} = B + C + 2D$
$\mathrm{JSF}(B)_{2452} = D$	$\mathrm{JSF}(B)_{2452} = D$
$\mathrm{JSF}(C)_{2452} = D + E$	$\mathrm{JSF}(C)_{2452} = D + E$
$A = \lambda - 001100 = (1, 1, p-3, 0, 1, 0)$ 0110	$D = \lambda - 013310 = (3, 1, p-5, 0, 1, 1)$ 1340
$B = \lambda - 012200 = (2, 0, p-4, 0, 2, 0)$ 1220	$E = \lambda - 003321 = (3, 3, p-5, 0, 0, 0)$ 0351
$C = \lambda - 002210 = (2, 2, p-4, 0, 0, 1)$ 0230	

Table 2.58: JSF of λ up to $\mu - 2452$

$\mu = (0, 1, p-2, 0)_{F_4}$	
ch $L(\mu)_{2452} = \mu + A - B - C + D$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2452} = -A + B + C - D$	$\mathrm{JSF}(\mu)_{2452} = {}^1_0B + C +^1_0 E$
$\mathrm{JSF}(A)_{2452} = B + E$	$\mathrm{JSF}(A)_{2452} = B + E$
$\mathrm{JSF}(C)_{2452} = A + D + E$	$\mathrm{JSF}(C)_{2452} = A + 2B + D + 2E$
$\mathrm{JSF}(D)_{2452} = B$	$\mathrm{JSF}(D)_{2452} = B$
$A = \mu - 0230 = (2, 0, p-4, 3)$	$D = \mu - 1240 = (0, 2, p-6, 4)$
$B = \mu - 1350 = (1, 1, p-6, 5)$	$E = \mu - 0241 = (2, 1, p-5, 2)$
$C = \mu - 0120 = (1, 1, p-4, 2)$	

Table 2.59: JSF of μ up to $\mu - 2452$

$\{\alpha_i\}_{\{1\leq i\leq 4\}}$. Note that L_I is of type A_4. Let $\theta, \theta_0 \in X(T_{A_4})^+$ be two weights given by $\theta = (0, p-d-1, d, 0)$ and $\theta_0 = (0, p-d-3, d-2, 0)$. Let $[\theta : \theta_0] = 2 - \zeta$, with $\zeta \in \{0, 1\}$. Note that for $d \in \{2, \dots, p-4\}$, the weight θ_0 lies in the upper closure of the fundamental alcove and

$$s_0 s_1 s_4 s_0 \bullet \theta_0 = \theta$$

By Proposition 1.3.10, the value of ζ is independent of $d \in \{2, \dots, p-4\}$. Let $d = 2$. The argument in Table 2.62 implies that $[\lambda : H] = [A : H]$ and by Table 2.60, we know that $[A : H] = 1$. For later use, note that this argument also holds if $d = p-4$.

Comparing the multiplicities in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors. Note that for $d = 2$, we have already computed $\mathrm{ch}\, L(\nu)_{1221}$ in the case $d = 1$.

$\lambda = (0,0,p-d-1,d,0,0)_{E_6}$	
$\mathrm{ch}\, L(\lambda)_{2442} = \lambda - A + B + C + D - E - F - G + H$	
See argument	
In order for the table to fit in the margins, we omitted $\mathrm{JSF}(-) =$ in the second column.	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2442} = A - B - C - D + E + F + G + \delta_{d,p-7} I$	$A +_0^1 E +_0^1 F +_0^1 G + 2H + 2\delta_{d,p-7} I +_0^1 \delta_{d,p-5} J$
$\mathrm{JSF}(A)_{2442} = B + C + D + H + \delta_{d,p-7} I + \delta_{d,p-5} J$	$B + C + D + 2E + 2F + 2G + H + \delta_{d,p-7} I + 2\delta_{d,p-5} J$
$\mathrm{JSF}(B)_{2442} = E + F - H$	$E + F$
$\mathrm{JSF}(C)_{2442} = E + G - H$	$E + G$
$\mathrm{JSF}(D)_{2442} = F + G + \delta_{d,p-5} J$	$F + G + \delta_{d,p-5} J$
$\mathrm{JSF}(E)_{2442} = H$	H
$A = \lambda - 001100 = (1,1,p-d-2,d-1,1,0)\ 0110$	$F = \lambda - 103310 = (1,3,p-d-3,d-2,1,1)\ 0341$
$B = \lambda - 102200 = (0,2,p-d-2,d-2,2,0)\ 0221$	$G = \lambda - 013310 = (3,1,p-d-4,d-1,1,1)\ 1340$
$C = \lambda - 012200 = (2,0,p-d-3,d-1,2,0)\ 1220$	$H = \lambda - 224400 = (0,0,p-d-3,d-2,4,0)\ 2442$
$D = \lambda - 002210 = (2,2,p-d-3,d-1,0,1)\ 0230$	$I = \lambda - 112321 = (0,1,p-d-1,d-1,0,0)\ 1342$
$E = \lambda - 113300 = (1,1,p-d-3,d-2,3,0)\ 1331$	$J = \lambda - 012321 = (2,1,p-d-2,d-1,0,0)\ 1341$

Table 2.60: JSF of λ up to $\mu - 2442$

$\mu = (0,d,p-d-1,0)_{F_4}$	
Possibilities	
$\mathrm{ch}\, L(\mu)_{2442} = \mu - A + B + C + \delta_{d,2} D - \delta_{d,\frac{p-5}{2}} E$	
$\mathrm{ch}\, L(\mu)_{2442} = \mu - A + B + C + \delta_{d,2} D - \delta_{d,\frac{p-5}{2}} E - \delta_{d,p-6} F$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2442} = A - B - C - \delta_{d,2} D + \delta_{d,\frac{p-5}{2}} E + \delta_{d,p-6} F$	$\mathrm{JSF}(\mu)_{2442} = A + \delta_{d,\frac{p-5}{2}} E + 2\delta_{d,p-6} F$
$\mathrm{JSF}(A)_{2442} = B + C + \delta_{d,2} D + \delta_{d,p-6} F$	$\mathrm{JSF}(A)_{2442} = B + C + \delta_{d,2} D + \delta_{d,p-6} F$
$A = \mu - 0120 = (1,d,p-d-3,2)$	$D = \mu - 0340 = (3,d-2,p-d-3,4)$,
$B = \mu - 1240 = (0,d+1,p-d-5,4)$	$E = \mu - 2442 = (0,d-2,p-d+1,0)$,
$C = \mu - 0241 = (2,d,p-d-4,2)$	$F = \mu - 1231 = (0,d,p-d-2,1)$

Table 2.61: JSF of μ up to $\mu - 2442$

$\boldsymbol{d = p-4.}$ — The computation of $\mathrm{ch}\, L(\lambda)_{2452}$ and $\mathrm{ch}\, L(\mu)_{2452}$ are summarized in Tables 2.63 and 2.64, respectively. We prove that X acts on $L_Y(\lambda)$ with more than two composition factors by studying the different possibilities for the values of $[\mu : D], [\lambda : K], (\lambda : M), (\lambda : J), (\lambda : I), (\lambda : N)$.

$\boldsymbol{[\mu : D]}$ Let $[\mu : D] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1342) =$

$\lambda_0 = (0, p-5, 0, 0) \in C_0$	
$\gamma = w_1 \bullet \lambda_0 = (0, p-3, 2, 0)$	$\eta = w \bullet \lambda_0 = (0, p-5, 0, 0)$
$w_1 = s_0 s_1 s_4 s_0$	$w = \mathrm{id}$
$s = s_0$	
$ws \bullet \lambda_0 = (1, p-5, 0, 1)$	
$ws \bullet \lambda_0 - w \bullet \lambda_0 = 1111$	
Proposition 1.3.9 $\implies$ $[w_1 s \bullet \lambda_0 : \eta] = [\gamma : \eta]$, where $w_1 s \bullet \lambda_0 = (1, p-4, 1, 1)$	

Table 2.62: Computing $[(0, p-3, 2, 0) : (0, p-5, 0, 0)]_{A_4}$

$(31, 27+\zeta, 3)$, hence assume $\zeta = 1$, since otherwise X acts on $L_Y(\lambda)$ with more than two composition factors.

$[\boldsymbol{\lambda} : \boldsymbol{K}]$ Arguing as in Subsection 2.5.1.11 for solving the case $[\lambda : H]$ implies that $[\lambda : K] = 1$.

We obtain the following possibilities for $\operatorname{ch} L(\lambda)_{2452}$

$$\operatorname{ch} L(\lambda)_{2452} = \lambda - A + B + C + D - E - F - G - H + (1+\zeta_I)I + (-1+\zeta_J)J + K + \zeta_N N + \zeta_M M,$$

with $\zeta_I, \zeta_J, \zeta_N, \zeta_M \in \mathbb{Z}_{\geq 0}$. We prove that X acts on $L_Y(\lambda)$ with more than two composition factors for any choice of the ζ's.

$(\boldsymbol{\lambda} : \boldsymbol{M})$ We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2441) = (40+\zeta_M, 35, 5)$, hence if $\zeta_M > 0$, we get a third composition factor. Hence assume that $\zeta_M = 0$.

$(\boldsymbol{\lambda} : \boldsymbol{J})$ We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2442) = (59+\zeta_J, 51, 8)$, hence if $\zeta_J > 0$, we get a third composition factor. Hence assume that $\zeta_J = 0$.

$(\boldsymbol{\lambda} : \boldsymbol{I})$ & $(\boldsymbol{\lambda} : \boldsymbol{N})$ We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1451) = (32+\zeta_I+\zeta_N, 29, 3)$, hence if either $\zeta_N > 0$ or $\zeta_I > 0$, then X acts with more than two factors on $L_Y(\lambda)$. We assume that $\zeta_I = \zeta_N = 0$.

Therefore, the last possibility for $\operatorname{ch} L(\lambda)_{2452}$ is

$$\operatorname{ch} L(\lambda)_{2452} = \lambda - A + B + C + D - E - F - G - H + I - J + K. \tag{2.9}$$

Using (2.9), we get $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2452) = (80, 70, 11)$ which is impossible. Hence (2.9) does not yield the correct truncated character, which proves that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\boldsymbol{d = p - 3}$. — The JSF applied to μ yields

$$\operatorname{ch} L(\mu)_{1221} = \mu - (\mu - 0120) - (\mu - 1221)$$

The result of the JSF applied to λ appears in Table 2.65 and we need to determine $[\lambda : D]$. Let $[\lambda : D] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 1220) = (4+\zeta, 5)$, hence $\zeta = 1$. The multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (0, 0, 3, p-4, 0, 0)_{E_6}$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:[1]
$\mathrm{JSF}(\lambda)_{2452} = A - B - C - D + E + F + G + H - I + J$	$A +_0^1 E +_0^1 F +_0^1 G +_0^5 I + 2J + 2K +_0^1 L +_0^4 M +_0^4 N$
$\mathrm{JSF}(A)_{2452} = B + C + D - H + K + L$	$B + C + D + 2E + 2F + 2G +_1^6 I + J + K + 2L +_1^5 M +_1^5 N$
$\mathrm{JSF}(B)_{2452} = E + F - J - K + L$	$E + F + 2I + L + 2M + 2N$
$\mathrm{JSF}(C)_{2452} = E + G - K + M$	$E + G + 2I + 2M$
$\mathrm{JSF}(D)_{2452} = F + G + H + N$	$F + G + H + 2I + 2N$
$\mathrm{JSF}(E)_{2452} = I + K + M$	$I + K + M$
$\mathrm{JSF}(F)_{2452} = I + N$	$I + N$
$\mathrm{JSF}(G)_{2452} = I$	I
$\mathrm{JSF}(L)_{2452} = J + M + N$	$J + M + N$
$A = \lambda - 001100 = (1, 1, 2, p-5, 1, 0)$ 0110	$H = \lambda - 003321 = (3, 3, 0, p-5, 0, 0)$ 0351
$B = \lambda - 102200 = (0, 2, 2, p-6, 2, 0)$ 0221	$I = \lambda - 114410 = (2, 2, 0, p-6, 2, 1)$ 1451
$C = \lambda - 012200 = (2, 0, 1, p-5, 2, 0)$ 1220	$J = \lambda - 122421 = (0, 0, 4, p-6, 1, 0)$ 2442
$D = \lambda - 002210 = (2, 2, 1, p-5, 0, 1)$ 0230	$K = \lambda - 224400 = (0, 0, 1, p-6, 4, 0)$ 2442
$E = \lambda - 113300 = (1, 1, 1, p-6, 3, 0)$ 1331	$L = \lambda - 112310 = (0, 1, 3, p-6, 1, 1)$ 1331
$F = \lambda - 103310 = (1, 3, 1, p-6, 1, 1)$ 0341	$M = \lambda - 123410 = (1, 0, 2, p-6, 2, 1)$ 2441
$G = \lambda - 013310 = (3, 1, 0, p-5, 1, 1)$ 1340	$N = \lambda - 113420 = (1, 2, 2, p-6, 0, 2)$ 1451

Table 2.63: JSF of λ up to $\mu - 2452$

$\mu = (0, p-4, 3, 0)_{F_4}$	
Possibilities	
$\mathrm{ch}\, L(\mu)_{2452} = \mu - A + B + C - D$	
$\mathrm{ch}\, L(\mu)_{2452} = \mu - A + B + C$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2452} = A - B - C + D$	$\mathrm{JSF}(\mu)_{2452} = A + 2D$
$\mathrm{JSF}(A)_{2452} = B + C + D$	$\mathrm{JSF}(A)_{2452} = B + C + D$
$A = \mu - 0120 = (1, p-4, 1, 2)$	$C = \mu - 1351 = (1, p-4, 0, 3)$,
$B = \mu - 0241 = (2, p-4, 0, 2)$	$D = \mu - 1342 = (1, p-5, 3, 0)$

Table 2.64: JSF of μ up to $\mu - 2452$

$d = p - 2$. — We compute $\mathrm{ch}\, L(\lambda)_{2442}$ and $\mathrm{ch}\, L(\mu)_{2442}$ in Tables 2.66 and 2.67. We show that X acts on $L_Y(\lambda)$ with more than two composition factors for both possible values of $[\mu : D]$. Indeed, if $[\mu : D] = 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1331) = (19, 16, 2)$ and if $[\mu : D] = 1$, then $(m_{L(\lambda)|_X},$

$\lambda = (0,0,2,p-3,0,0)_{E_6}$	
$\operatorname{ch} L(\lambda)_{1221} = \lambda - A + B + C$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1221} = A - B - C + D$	$\mathrm{JSF}(\lambda)_{1221} = A + 2D$
$\mathrm{JSF}(A)_{1221} = B + C + D$	$\mathrm{JSF}(A)_{1221} = B + C + D$
$A = \lambda - 001100 = (1,1,1,p-4,1,0)$ 0110	$C = \lambda - 012200 = (2,0,0,p-4,2,0)$ 1220
$B = \lambda - 102200 = (0,2,1,p-5,2,0)$ 0221	$D = \lambda - 011210 = (1,0,2,p-4,0,1)$ 1220

Table 2.65: JSF of λ up to $\mu - 1221$

$m_{L(\mu)}, m_{L(\nu)})(\mu - 2442) = (60, 51, 7)$.

$\lambda = (0,0,1,p-2,0,0)_{E_6}$	
$\operatorname{ch} L(\lambda)_{2442} = \lambda - A + B + C + D - E - F - G$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2442} = A - B - C - D + E + F + G$	$\mathrm{JSF}(\lambda)_{2442} = A +_0^1 E +_0^1 G$
$\mathrm{JSF}(A)_{2442} = B + C + D - F$	$\mathrm{JSF}(A)_{2442} = B + C + D + 2E + 2G$
$\mathrm{JSF}(B)_{2442} = E$	$\mathrm{JSF}(B)_{2442} = E$
$\mathrm{JSF}(C)_{2442} = E + F + G$	$\mathrm{JSF}(C)_{2442} = E + F + G$
$\mathrm{JSF}(D)_{2442} = G$	$\mathrm{JSF}(D)_{2442} = G$
$A = \lambda - 001100 = (1,1,0,p-3,1,0)$ 0110	$E = \lambda - 113410 = (1,2,0,p-5,2,1)$ 1441
$B = \lambda - 102200 = (0,2,0,p-4,2,0)$ 0221	$F = \lambda - 022420 = (2,0,1,p-4,0,2)$ 2440
$C = \lambda - 012310 = (2,1,0,p-4,1,1)$ 1330	$G = \lambda - 012421 = (2,2,1,p-5,1,0)$ 1441
$D = \lambda - 002321 = (2,3,0,p-4,0,0)$ 0341	

Table 2.66: JSF of λ up to $\mu - 2442$

2.5.1.12 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3}$. — By Table 2.35, we have to consider the following cases.

$\boldsymbol{b \neq a-1}$ **or** $\boldsymbol{a + c \neq p-1}$. — By Table 2.35, either $a \neq p-2, p-6, b = p-3$ and $c = 1$ or $a \neq 4, p-1, a+c = p$ and $b + c + 2 \equiv 0 \mod p$ or $a + c = p-1$ and $b \neq a-1$.

If $a \neq p-2, p-6, b = p-3$ and $c = 1$, then again by Table 2.35, $\nu = \mu - 0121 = (p-2, 0, 0, a)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The computation of $\operatorname{ch} L(\lambda)_{1121}$ is summarized in Table 2.68. Let us determine $[\lambda : B]$ for $a = p-1$. By Proposition 1.2.2, we have that $\operatorname{ch} L(\mu)_{1121} = \mu$. Set $[\lambda : B] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})$

$\mu = (0, p-2, 1, 0)_{F_4}$	
Possibilities	
$\operatorname{ch} L(\mu)_{2442} = \mu - A - B + C$	
$\operatorname{ch} L(\mu)_{2442} = \mu - A - B + D$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2442} = A + B - C$	$\mathrm{JSF}(\mu)_{2442} = A + B +_0^1 C + 2D$
$\mathrm{JSF}(A)_{2442} = C + D$	$\mathrm{JSF}(A)_{2442} = 2C + D$
$\mathrm{JSF}(B)_{2442} = -C + D$	$\mathrm{JSF}(B)_{2442} = D$
$\mathrm{JSF}(D)_{2442} = C$	$\mathrm{JSF}(D)_{2442} = C$
$A = \mu - 1220 = (0, p-3, 1, 2)$	$C = \mu - 2441 = (0, p-4, 2, 2)$,
$B = \mu - 0231 = (2, p-3, 0, 1)$	$D = \mu - 1331 = (1, p-4, 2, 1)$

Table 2.67: JSF of μ up to $\mu - 2442$

$(\mu - 1121) = (10 + \zeta, 10, 1)$, hence $\zeta = 1$ and $[\lambda : B] = 1$. Comparing the multiplicities appearing in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a \neq 4, p-1, a+c = p$ and $b + c + 2 \equiv 0 \mod p$, then by Table 2.35, the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 0121 = (b+1, 0, c-1, a)$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1111} = \lambda - (\lambda - 011100)$. The multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a + c = p - 1$ and $b \neq a - 1$, then by Table 2.35, the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 1110 = (b-1, 0, p-a-1, a+1)$. The computation of $\operatorname{ch} L(\lambda)_{1111}$ and $\operatorname{ch} L(\mu)_{1111}$ is summarized in Tables 2.69 and 2.70, respectively. If $b \neq p-2$, then $\operatorname{ch} L(\lambda)_{1111}$ and $\operatorname{ch} L(\mu)_{1111}$ are known and the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors. Assume $b = p-2$. Let $[\lambda : B] = 2 - \zeta$ and $[\mu : B] = 2 - \xi$ with $\zeta, \xi \in \{0, 1\}$. We will show that X acts on $L_Y(\lambda)$ with more than two composition factors for any choice of the values of ζ, ξ. Note that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1111) = (5 + \zeta, 3 + \xi, 1)$. Therefore, we are done if $\zeta \geq \xi$. Assume by contradiction that $\zeta = 0$ and $\xi = 1$. Note that by Proposition 1.5.2, we can work in the Levi factor L_I of P_I, where $I = \{\alpha_i,\ 1 \leq i \leq 4\}$ for the computation of $[\lambda : B]$. Note that L_I is of type A_4. Let $\theta = (a, p-a-1, 0, p-2)$, then for any value of $1 \leq a \leq p-2$, the weight $\theta - 1111 = (a-1, p-a-1, 0, p-3)$ lies in the upper closure of the alcove $(1,1,1,1,1,1,1,2,2,2)$. Moreover, if $\theta_0 = (p-a-2, 0, -1, 0)$, then

$$s_0 s_1 s_2 s_4 s_3 s_2 \bullet \theta_0 = \theta, \quad s_0 s_1 s_4 s_3 \bullet \theta_0 = \theta.$$

By Proposition 1.5.2, we have $[\theta : \theta - 1111]_{A_4} = 2 - \zeta$ and by Proposition 1.3.10, the value of ζ does not depend on a. Now set $a = 2$.

We compute $\operatorname{ch} L(\lambda)_{1121}$ and $\operatorname{ch} L(\mu)_{1121}$ and we get the same linear combination of truncated characters as for $\operatorname{ch} L(\lambda)_{1111}$ and $\operatorname{ch} L(\mu)_{1111}$, that is

$$\operatorname{ch} L(\lambda)_{1121} = \lambda - (\lambda - 101000) - (\lambda - 111100)$$

and

$$\operatorname{ch} L(\mu)_{1121} = \mu - (\mu - 0011).$$

Moreover, by Proposition 1.2.2 we have $\operatorname{ch} L(\nu)_{0011} = \nu$. Computing the multiplicities yields $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1121) = (9, 8, 2)$, which contradicts the choice of ζ and ξ. Therefore, X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (p-1, p-3, 1, 0, 0, 0)_{E_6}$	
$\operatorname{ch} L(\lambda)_{1121} = \lambda - A$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1121} = A + \delta_{a,p-1} B$	$\mathrm{JSF}(\lambda)_{1121} = A + 2\delta_{a,p-1} B$
$\mathrm{JSF}(A)_{1121} = \delta_{a,p-1} B$	$\mathrm{JSF}(A)_{1121} = \delta_{a,p-1} B$
$A = \lambda - 011100 = (p, p-4, 0, 0, 1, 0)$ 1110	$B = \lambda - 111100 = (p-2, p-4, 1, 0, 1, 0)$ 1111

Table 2.68: JSF of λ up to $\mu - 1121$

$\lambda = (a, b, p-a-1, 0, 0, 0)_{E_6}$	
Possibilities	
$\operatorname{ch} L(\lambda)_{1111} = \lambda - A$	
$\operatorname{ch} L(\lambda)_{1111} = \lambda - A - \delta_{b,p-2} B$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1111} = A + \delta_{b,p-2} B$	$\mathrm{JSF}(\lambda)_{1111} = A + 2\delta_{b,p-2} B$
$\mathrm{JSF}(A)_{1111} = \delta_{b,p-2} B$	$\mathrm{JSF}(A)_{1111} = \delta_{b,p-2} B$
$A = \lambda - 101000 = (a-1, b, p-a-2, 1, 0, 0)$ 0011	$B = \lambda - 111100 = (a-1, b-1, p-a-1, 0, 1, 0)$ 1111

Table 2.69: JSF of λ up to $\mu - 1111$

$\boldsymbol{b = a - 1, a + c = p - 1.}$ — We first determine the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The JSF applied to λ and μ yields

$$\operatorname{ch} L(\lambda)_{1111} = \lambda - (\lambda - 101000) - (\lambda - 011100)$$

and $\operatorname{ch} L(\mu)_{1111} = \mu - (\mu - 0011)$, respectively. Computing multiplicities, we get that $\nu = \mu - 1111 = (a-2, 0, p-a, a-1)$ affords the highest weight of a second composition factor. We will distinguish between the case $a \neq 2$ and $a = 2$.

Assume $a \neq 2$. The computations to determine $\operatorname{ch} L(\lambda)_{2222}$ and $\operatorname{ch} L(\mu)_{2222}$ are summarized in Tables 2.71 and 2.72. Moreover, the JSF applied to ν yields $\operatorname{ch} L(\nu)_{1111} = \nu - (\nu - 0011)$. Let

$\mu = (b, 0, p-a-1, a)_{F_4}$	
Possibilities	
ch $L(\mu)_{1111} = \mu - A$	
ch $L(\mu)_{1111} = \mu - A - \delta_{b,p-2} B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1111} = A + \delta_{b,p-2} B$	$\mathrm{JSF}(\mu)_{1111} = A + 2\delta_{b,p-2} B$
$\mathrm{JSF}(A)_{1111} = \delta_{b,p-2} B$	$\mathrm{JSF}(A)_{1111} = \delta_{b,p-2} B$
$A = \mu - 0011 = (b, 1, p-a-2, a-1)$	$B = \mu - 1111 = (b-1, 0, p-a, a-1)$

Table 2.70: JSF of μ up to $\mu - 1111$

us determine $[\lambda : D]$ for $a \neq p-2$. Set $[\lambda : D] = 2 - \zeta$ with $\zeta \in \{0, 1\}$, we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1121) = (7 + \zeta, 1, 7)$. Therefore $\zeta = 1$ and comparing the multiplicities in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (a, a-1, p-a-1, 0, 0, 0)_{E_6}$	
ch $L(\lambda)_{2222} = \lambda - A + B - C + \overline{\delta_{a,p-2}} D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2222} = A - B + C$	$\mathrm{JSF}(\lambda)_{2222} = A + C + 2\overline{\delta_{a,p-2}} D$
$\mathrm{JSF}(A)_{2222} = B + \overline{\delta_{a,p-2}} D$	$\mathrm{JSF}(A)_{2222} = B + \overline{\delta_{a,p-2}} D$
$\mathrm{JSF}(C)_{2222} = \overline{\delta_{a,p-2}} D$	$\mathrm{JSF}(C)_{2222} = \overline{\delta_{a,p-2}} D$
$A = \lambda - 101000 = (a-1, a-1, p-a-2, 1, 0, 0)$ 0011	$C = \lambda - 011100 = (a+1, a-2, p-a-2, 0, 1, 0)$ 1110
$B = \lambda - 202100 = (a-2, a, p-a-2, 0, 1, 0)$ 0122	$D = \lambda - 112100 = (a, a-2, p-a-3, 1, 1, 0)$ 1121

Table 2.71: JSF of λ up to $\mu - 2222$

If $a = 2$, we prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. By Propositions 1.5.3 and 2.3.1, it is sufficient to prove that none of the following

$\mu = (a-1, 0, p-a-1, a)_{F_4}$	
Possibilities	
$\operatorname{ch} L(\mu)_{2222} = \mu - A - \overline{\delta_{a,p-2}}B - \delta_{a,p-2}C$	
$\operatorname{ch} L(\mu)_{2222} = \mu - A - \overline{\delta_{a,p-2}}B - \delta_{a,p-2}C - \delta_{a,\frac{p-1}{2}}D$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2222} = A + \overline{\delta_{a,p-2}}B + \delta_{a,p-2}C + \delta_{a,\frac{p-1}{2}}D$	$\mathrm{JSF}(\mu)_{2222} = A + \overline{\delta_{a,p-2}}B + \delta_{a,p-2}C + 2\delta_{a,\frac{p-1}{2}}D$
$\mathrm{JSF}(A)_{2222} = \delta_{a,\frac{p-1}{2}}D$	$\mathrm{JSF}(A)_{2222} = \delta_{a,\frac{p-1}{2}}D$
$A = \mu - 0011 = (a-1, 1, p-a-2, a-1)$	$C = \mu - 1121 = (a-2, 1, p-a-2, a),$
$B = \mu - 1120 = (a-2, 1, p-a-3, a+2)$	$D = \mu - 2222 = (a-3, 0, p-a+1, a-2)$

Table 2.72: JSF of μ up to $\mu - 2222$

weights affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned} \mu - 1121 &= (0, 1, p-4, 2) \\ \mu - 1231 &= (1, 0, p-4, 3) \\ \mu - 1122 &= (0, 1, p-3, 0) \\ \mu - 1232 &= (1, 0, p-3, 1) \\ \mu - 1242 &= (1, 1, p-5, 2) \\ \mu - 2342 &= (0, 0, p-3, 2) \\ \mu - 2343 &= (0, 0, p-2, 0) \\ \mu - 2353 &= (0, 1, p-4, 1) \end{aligned} \tag{2.10}$$

By Lemma 1.4.9, it is enough to consider the weight $\mu - 2353$. We summarize the computations to determine $\operatorname{ch} L(\lambda)_{2353}$ and $\operatorname{ch} L(\mu)_{2353}$ in Tables 2.73 and 2.74, respectively. Note that $\operatorname{ch} L(\nu)_{1242}$ has already been computed in Table 2.38 on Page 77 and is equal to

$$\operatorname{ch} L_{1242}(\nu) = \nu - (\nu - 0011) - (\nu - 1230) - (\nu - 0132) + (\nu - 0242).$$

Let us solve the problematic cases for λ and μ.

[$\mu : C$] Let $[\mu : C] = 2 - \zeta$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 0121) = (3, 2+\zeta, 0)$ and Remark 1.5.4 and Theorem 2.1.1 implies that $\zeta = 1$.

[$\lambda : F$] Let $[\lambda : F] = 2 - \zeta$. Since $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1121) = (7+\zeta, 7, 1)$, we have $\zeta = 1$.

[$\mu : E$] Let $[\mu : E] = 2-\zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1131) = (8, 6+\zeta, 1)$. We check that there is no weight greater than $\mu - 1131$ apart from μ and ν which affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Note that $\mu - 1131$ does not appear in (2.10), the list of weights which can afford the highest weight of a third composition factor generated by a maximal vector for B_X. Therefore either we assume by contradiction that $\zeta = 0$ and that $\mu - 1131$ affords the highest weight of a third composition factor or we assume that $\zeta = 1$ and that $\mu - 1131$ does not afford the highest weight of a third composition factor. Arguing as in Subsection 2.5.1.3, we can assume, without loss of generality, that $\zeta = 1$.

$(\mu : F)$ The possibilities for $\operatorname{ch} L(\mu)_{1241}$ are given by

$$\operatorname{ch} L(\mu)_{1241} = \mu - A - B + D + E + (\zeta - 2)F,$$

with $\zeta \in \mathbb{Z}_{\geq 0}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1241) = (18, 14 + \zeta, 2)$. Since $\mu - 1241$ does not appear in (2.10), arguing as in the previous case we may assume that $\zeta = 2$.

Therefore,

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2353) = (78, 66, 12),$$

and X acts on $L_Y(\lambda)$ with exactly two composition factors.

$\lambda = (2, 1, p-3, 0, 0, 0)_{E_6}$	
ch $L(\lambda)_{2353} = \lambda - A + B - C + 2D + E + F - G$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{2353} = A - B + C - D - E$	$\mathrm{JSF}(\lambda)_{2353} = A + C + 2F +_0^1 G$
$\mathrm{JSF}(A)_{2353} = B + D + F$	$\mathrm{JSF}(A)_{2353} = B + D + F + 2G$
$\mathrm{JSF}(B)_{2353} = -D + G$	$\mathrm{JSF}(B)_{2353} = G$
$\mathrm{JSF}(C)_{2353} = D + E + F - G$	$\mathrm{JSF}(C)_{2353} = E + F$
$\mathrm{JSF}(F)_{2353} = -D + G$	$\mathrm{JSF}(F)_{2353} = G$
$\mathrm{JSF}(G)_{2353} = D$	$\mathrm{JSF}(G)_{2353} = D$
$A = \lambda - 101000 = (1, 1, p-4, 1, 0, 0)$ 0011	$E = \lambda - 023321 = (5, 0, p-6, 1, 0, 0)$ 2351
$B = \lambda - 202100 = (0, 2, p-4, 0, 1, 0)$ 0122	$F = \lambda - 112100 = (2, 0, p-5, 1, 1, 0)$ 1121
$C = \lambda - 011100 = (3, 0, p-4, 0, 1, 0)$ 1110	$G = \lambda - 213200 = (1, 1, p-5, 0, 2, 0)$ 1232
$D = \lambda - 324300 = (0, 0, p-5, 0, 3, 0)$ 2343	

Table 2.73: JSF of λ up to $\mu - 2353$

2.5.1.13 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + e\lambda_5}$. — By Table 2.35, we have that $a = 1, b = p-1, e = p-1$ or $a = p-4, b = p-3, e = 1$. Additionally, we have that $\nu = \mu - 0011 = (b, 1, e-1, a-1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

If $a = 1, b = p-1$ and $e = p-1$, we summarize the computations to determine $\operatorname{ch} L(\lambda)_{1231}$ and $\operatorname{ch} L(\nu)_{1220}$ in Tables 2.75 and 2.76, respectively. Moreover, the JSF applied to μ yields $\operatorname{ch} L(\mu)_{1231} = \mu - (\mu - 1120)$. The following argument proves that X acts on $L_Y(\lambda)$ with more than two composition factors. Let $[\lambda : C] = 2 - \zeta$ and $[\nu : D] = 2 - \xi$ with $\zeta, \xi \in \{0, 1\}$. We have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1121) = (15 + \zeta, 11, 4) \tag{2.11}$$

$\mu = (1,0,p-3,2)_{F_4}$	
ch $L(\mu)_{2353} = \mu - A - B + D + E$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{2353} = A + B + C - D$	$\mathrm{JSF}(\mu)_{2353} = A + B + 2C + 2E +_1^5 F$
$\mathrm{JSF}(A)_{2353} = C + D + E$	$\mathrm{JSF}(A)_{2353} = C + D + E + 2F$
$\mathrm{JSF}(B)_{2353} = E + F$	$\mathrm{JSF}(B)_{2353} = E + 2F$
$\mathrm{JSF}(C)_{2353} = F$	$\mathrm{JSF}(C)_{2353} = F$
$\mathrm{JSF}(E)_{2353} = F$	$\mathrm{JSF}(E)_{2353} = F$
$A = \mu - 0011 = (1,1,p-4,1)$	$D = \mu - 0243 = (3,0,p-4,0)$,
$B = \mu - 1120 = (0,1,p-5,4)$	$E = \mu - 1131 = (0,2,p-6,3)$,
$C = \mu - 0121 = (2,0,p-4,2)$	$F = \mu - 1241 = (1,1,p-6,4)$

Table 2.74: JSF of μ up to $\mu - 2353$

and

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1231) = (28 + 4\zeta, 22, 7 + \xi). \tag{2.12}$$

By (2.12), we have $\zeta = 1$ and we do not determine the value of ξ, since (2.11) implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

If $a = p-4, b = p-3$ and $e = 1$, the JSF applied to λ yields

$$\mathrm{ch}\, L(\lambda)_{1121} = \lambda - (\lambda - 010110) - (\lambda - 101110)$$

and the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (1,p-1,0,0,p-1,0)_{E_6}$	
ch $L(\lambda)_{1231} = \lambda - A - B + C$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1231} = A + B$	$\mathrm{JSF}(\lambda)_{1231} = A + B + 2C$
$\mathrm{JSF}(A)_{1231} = C$	$\mathrm{JSF}(A)_{1231} = C$
$\mathrm{JSF}(B)_{1231} = C$	$\mathrm{JSF}(B)_{1231} = C$
$A = \lambda - 010110 = (1,p-2,1,0,p-2,1)$ 1110	$C = \lambda - 010121 = (1,p-2,1,1,p-3,0)$ 1121
$B = \lambda - 000121 = (1,p,1,0,p-3,0)$ 0121	

Table 2.75: JSF of λ up to $\mu - 1231$

$\nu = (p-1, 1, p-2, 0)_{F_4}$	
Possibilities	
ch $L(\nu)_{1220} = \nu - C - D$	
ch $L(\nu)_{1220} = \nu - C$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1220} = C + D$	$\mathrm{JSF}(\nu)_{1220} = C + 2D$
$\mathrm{JSF}(C)_{1220} = D$	$\mathrm{JSF}(C)_{1220} = D$
$C = \nu - 0120 = (p, 1, p-4, 2)$	$D = \nu - 1120 = (p-2, 2, p-4, 2)$

Table 2.76: JSF of ν up to $\nu - 1220$

2.5.1.14 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3 + d\lambda_4}$. — By Table 2.35, we have that $a = 1, c + d = p - 1, c \neq 2$ and $\nu = \mu - 0111 = (1, p-c-2, c+1, a-1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. If $c = 4$, then the JSF applied to μ and ν yields $\operatorname{ch} L(\mu)_{1332} = \mu - (\mu - 0120)$ and $\operatorname{ch} L(\nu)_{1221} = \nu - (\nu - 0120)$, respectively. Moreover, the computations to determine $\operatorname{ch} L(\lambda)_{1332}$ are summarized in Table 2.77. Let us determine $[\lambda : F]$. Set $[\lambda : F] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1332) = (56 + 2\zeta, 44, 13)$, hence $\zeta = 1$. If $c \neq 4$, then the computation of $\operatorname{ch} L(\lambda)_{1221}$ and $\operatorname{ch} L(\mu)_{1221}$ is summarized in Tables 2.78 and 2.79, respectively. In both cases, the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (1, 0, 4, p-5, 0, 0)_{E_6}$	
ch $L(\lambda)_{1332} = \lambda - A + B + C + D - E + F$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1332} = A - B - C - D + E$	$\mathrm{JSF}(\lambda)_{1332} = A + E + 2F$
$\mathrm{JSF}(A)_{1332} = B + C + D + F$	$\mathrm{JSF}(A)_{1332} = B + C + D + F$
$\mathrm{JSF}(E)_{1332} = F$	$\mathrm{JSF}(E)_{1332} = F$
$A = \lambda - 001100 = (2, 1, 3, p-6, 1, 0)$ 0110	$D = \lambda - 002210 = (3, 2, 2, p-6, 0, 1)$ 0230
$B = \lambda - 203300 = (0, 3, 3, p-8, 3, 0)$ 0332	$E = \lambda - 111210 = (0, 0, 5, p-6, 0, 1)$ 1221
$C = \lambda - 012200 = (3, 0, 2, p-6, 2, 0)$ 1220	$F = \lambda - 112310 = (1, 1, 4, p-7, 1, 1)$ 1331

Table 2.77: JSF of λ up to $\mu - 1332$

2.5.1.15 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3 + e\lambda_5}$. — By Table 2.35, we have $a = 2, c = p - 3, e = 1$ and $\nu = \mu - 0010 = (0, 1, p-4, 3)$ affords the highest weight of a second composition factor for X

$\lambda = (1, 0, c, p-c-1, 0, 0)_{E_6}$	
$\operatorname{ch} L(\lambda)_{1221} = \lambda - \delta_{c,p-2}A - B + \overline{\delta_{c,1}}C + \delta_{c,p-2}D$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1221} = \delta_{c,p-2}A + B - \overline{\delta_{c,1}}C$	$\mathrm{JSF}(\lambda)_{1221} = \delta_{c,p-2}A + B +^{2}_{1} \delta_{c,p-2}D$
$\mathrm{JSF}(A)_{1221} = \delta_{c,p-2}D$	$\mathrm{JSF}(A)_{1221} = \delta_{c,p-2}D$
$\mathrm{JSF}(B)_{1221} = \overline{\delta_{c,1}}C + \delta_{c,p-2}D$	$\mathrm{JSF}(B)_{1221} = \overline{\delta_{c,1}}C + \delta_{c,p-2}D$
$A = \lambda - 101000 = (0, 0, c-1, p-c, 0, 0)$ 0011	$C = \lambda - 012200 = (3, 0, c-2, p-c-2, 2, 0)$ 1220
$B = \lambda - 001100 = (2, 1, c-1, p-c-2, 1, 0)$ 0110	$D = \lambda - 102100 = (1, 1, c-2, p-c-1, 1, 0)$ 0121

Table 2.78: JSF of λ up to $\mu - 1221$

$\mu = (0, p-c-1, c, 1)_{F_4}$	
$\operatorname{ch} L(\mu)_{1221} = \mu - \overline{\delta_{c,1}}A - \delta_{c,p-2}B - \delta_{c,1}C - \delta_{c,\frac{p+5}{2}}D$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1221} = \overline{\delta_{c,1}}A + \delta_{c,p-2}B + \delta_{c,1}C + \delta_{c,\frac{p+5}{2}}D$	$\mathrm{JSF}(\mu)_{1221} = \overline{\delta_{c,1}}A + \delta_{c,p-2}B + \delta_{c,1}C + \delta_{c,\frac{p+5}{2}}D$
$A = \mu - 0120 = (1, p-c-1, c-2, 3)$	$C = \mu - 1220 = (0, p-c-2, c, 3)$,
$B = \mu - 0011 = (0, p-c, c-1, 0)$	$D = \mu - 1221 = (0, p-c-2, c+1, 1)$

Table 2.79: JSF of μ up to $\mu - 1221$

acting on $L_Y(\lambda)$. We prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. By Propositions 1.5.3 and 2.3.1, it is enough to prove that none the weights listed below affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$.

$$\begin{aligned}
\mu - 1230 &= (0, 0, p-4, 5)\\
\mu - 1231 &= (0, 0, p-3, 3)\\
\mu - 1241 &= (0, 1, p-5, 4)\\
\mu - 1232 &= (0, 0, p-2, 1)\\
\mu - 1242 &= (0, 1, p-4, 2)\\
\mu - 1252 &= (0, 2, p-6, 3)\\
\mu - 1352 &= (1, 0, p-4, 3)
\end{aligned}$$

By Lemma 1.4.9, it is enough to consider $\mu - 1352$. We summarize the computation of $\operatorname{ch} L(\lambda)_{1352}$ and $\operatorname{ch} L(\nu)_{1342}$ in Tables 2.80 and 2.81, respectively. Moreover, the JSF applied to μ yields $\operatorname{ch} L(\mu)_{1352} = \mu - (\mu - 0022) - (\mu - 1230)$. Note that we can apply Remark 1.5.4 and Proposition 2.3.1 in order to solve all the problematic cases, that is we solve the problematic cases by equalizing the multiplicities.

[$\boldsymbol{\lambda : E}$] Let $[\lambda : E] = 2-\zeta$ with $\zeta \in \{0,1\}$, we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-0131) = (8+\zeta, 6, 3)$, hence $\zeta = 1$.

[$\boldsymbol{\nu : E}$] Let $[\nu : E] = 2-\zeta$ with $\zeta \in \{0,1\}$, we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-0241) = (13, 9, 3+\zeta)$, hence $\zeta = 1$.

[$\boldsymbol{\nu : D}$] Let $[\nu : D] = 2-\zeta$ with $\zeta \in \{0,1\}$, we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-0142) = (12, 8, 3+\zeta)$, hence $\zeta = 1$.

[$\boldsymbol{\nu : C}$] Let $[\nu : C] = 5 - \zeta$ with $0 \leq \zeta \leq 4$. We have

$$\mathrm{ch}\, L(\nu)_{0252} = \nu - A - B - (3-\zeta)C + D + E.$$

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 0252) = (24, 16, 6+\zeta) \text{ with } \zeta \in \mathbb{Z}_{\geq 0}, \text{ thus } \zeta = 2.$$

Therefore,

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1352) = (78, 53, 25),$$

which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

$\lambda = (2, 0, p-3, 0, 1, 0)_{E_6}$	
ch $L(\lambda)_{1352} = \lambda - A + B - C + D + E - F - G$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1352} = A - B + C - D$	$\mathrm{JSF}(\lambda)_{1352} = A + C + 2E +_0^1 F +_0^1 G$
$\mathrm{JSF}(A)_{1352} = B + E - F$	$\mathrm{JSF}(A)_{1352} = B + E + 2G$
$\mathrm{JSF}(B)_{1352} = G$	$\mathrm{JSF}(B)_{1352} = G$
$\mathrm{JSF}(C)_{1352} = D + E - G$	$\mathrm{JSF}(C)_{1352} = D + E + 2F$
$\mathrm{JSF}(D)_{1352} = F$	$\mathrm{JSF}(D)_{1352} = F$
$\mathrm{JSF}(E)_{1352} = F + G$	$\mathrm{JSF}(E)_{1352} = F + G$
$A = \lambda - 101000 = (1, 0, p-4, 1, 1, 0)\ 0011$	$E = \lambda - 102110 = (2, 1, p-5, 1, 0, 1)\ 0131$
$B = \lambda - 202100 = (0, 1, p-4, 0, 2, 0)\ 0122$	$F = \lambda - 103221 = (3, 2, p-6, 1, 0, 0)\ 0252$
$C = \lambda - 001110 = (3, 1, p-4, 0, 0, 1)\ 0120$	$G = \lambda - 203210 = (1, 2, p-5, 0, 1, 1)\ 0242$
$D = \lambda - 002221 = (4, 2, p-5, 0, 0, 0)\ 0241$	

Table 2.80: JSF of λ up to $\mu - 1352$

2.5.1.16 $\boldsymbol{\lambda = a\lambda_1 + c\lambda_3 + f\lambda_6}$. — By Table 2.35, we have that $a + c = p-1, f = 1$ and $\nu = \mu - 0001 = (0, 0, p-a, a-1)$ affords the highest weight of a second composition factor.

Let $a = 1$. The computation of $\mathrm{ch}\, L(\lambda)_{1232}$ is summarized in Table 2.82. Moreover, the JSF applied to μ and ν yields $\mathrm{ch}\, L(\mu)_{1232} = \mu - (\mu - 0022) - (\mu - 1230)$ and $\mathrm{ch}\, L(\nu)_{1231} = \nu - (\nu - 0131)$, respectively. The multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\nu = (0,1,p-4,3)_{F_4}$	
$\operatorname{ch} L(\nu)_{1342} = \nu - A - B - C + D + E$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1342} = A + B + C$	$\mathrm{JSF}(\nu)_{1342} = A + B +_1^5 C + 2D + 2E$
$\mathrm{JSF}(A)_{1342} = D + E$	$\mathrm{JSF}(A)_{1342} = 2C + D + E$
$\mathrm{JSF}(B)_{1342} = D + E$	$\mathrm{JSF}(B)_{1342} = 2C + D + E$
$\mathrm{JSF}(D)_{1342} = C$	$\mathrm{JSF}(D)_{1342} = C$
$\mathrm{JSF}(E)_{1342} = C$	$\mathrm{JSF}(E)_{1342} = C$
$A = \nu - 0110 = (1,0,p-4,4)$	$D = \nu - 0132 = (1,2,p-6,2)$
$B = \nu - 0011 = (0,2,p-5,2)$	$E = \nu - 0231 = (2,0,p-5,4)$
$C = \nu - 0242 = (2,1,p-6,3)$	

Table 2.81: JSF of ν up to $\nu - 1342$

Let $a = 3$. The computation of $\operatorname{ch} L(\lambda)_{1232}$, $\operatorname{ch} L(\mu)_{1232}$ and $\operatorname{ch} L(\nu)_{1231}$ is summarized in Tables 2.83 to 2.85, respectively. We will now deduce enough information about the problematic cases for λ, μ and ν to prove that X acts on $L_Y(\lambda)$ with more than two composition factors.

$[\nu : E]$ By Table 2.8 on Page 47, we have $[\nu : E] = 1$.

$[\lambda : D]$ & $[\mu : C]$ Let $[\lambda : D] = 2 - \zeta$ and $[\mu : C] = 2 - \xi$ with $\zeta, \xi \in \{0,1\}$. We have,

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 0132) = (9 + \zeta, 6 + \xi, 3).$$

By Remark 1.5.4 and Proposition 2.3.1, we have $\zeta = \xi$. If $(\zeta, \xi) = (1,1)$, then

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1232) = (26, 17, 8).$$

If $(\zeta, \xi) = (0,0)$, then

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1232) = (24, 15, 8).$$

Therefore, X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $a = 4$. The JSF applied to μ and ν yields $\operatorname{ch} L(\mu)_{1232} = \mu - (\mu - 0022)$ and $\operatorname{ch} L(\nu)_{1231} = \nu - (\nu - 0011) - (\nu - 1231)$. Moreover, the computation of $\operatorname{ch} L(\lambda)_{1232}$ is summarized in Table 2.86. We prove that X acts on $L_Y(\lambda)$ with more than two composition factors without determining the value of $[\lambda : C]$. Let $[\lambda : C] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have

$$(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1232) = (30 + 2\zeta, 21, 7),$$

which implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $a \neq 1,3,4$. The JSF applied to λ yields $\operatorname{ch} L(\lambda)_{1231} = \lambda - (\lambda - 101000)$ and the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (1,0,p-2,0,0,1)_{E_6}$	
$\operatorname{ch} L(\lambda)_{1232} = \lambda - A + B - C$	
JSF in Weyl characters:	JSF in irreducible characters:
$\operatorname{JSF}(\lambda)_{1232} = A - B + C$	$\operatorname{JSF}(\lambda)_{1232} = A + C$
$\operatorname{JSF}(A)_{1232} = B$	$\operatorname{JSF}(A)_{1232} = B$
$A = \lambda - 101000 = (0,0,p-3,1,0,1)$ 0011	$C = \lambda - 012210 = (3,0,p-4,0,0,2)$ 1230
$B = \lambda - 213200 = (0,0,p-4,0,2,1)$ 1232	

Table 2.82: JSF of λ up to $\mu - 1232$

$\lambda = (3,0,p-4,0,0,1)_{E_6}$	
Possibilities	
$\operatorname{ch} L(\lambda)_{1232} = \lambda - A + B - C$	
$\operatorname{ch} L(\lambda)_{1232} = \lambda - A + B - C + D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\operatorname{JSF}(\lambda)_{1232} = A - B + C$	$\operatorname{JSF}(\lambda)_{1232} = A + C + 2D$
$\operatorname{JSF}(A)_{1232} = B + D$	$\operatorname{JSF}(A)_{1232} = B + D$
$\operatorname{JSF}(C)_{1232} = D$	$\operatorname{JSF}(C)_{1232} = D$
$A = \lambda - 101000 = (2,0,p-5,1,0,1)$ 0011	$C = \lambda - 001111 = (4,1,p-5,0,0,0)$ 0121
$B = \lambda - 202100 = (1,1,p-5,0,1,1)$ 0122	$D = \lambda - 102111 = (3,1,p-6,1,0,0)$ 0132

Table 2.83: JSF of λ up to $\mu - 1232$

2.5.1.17 $\boldsymbol{\lambda = a\lambda_1 + d\lambda_4 + e\lambda_5}$. — By Table 2.35, we have that $a = 1, d+e = p-1, e \neq 1, p-2$ and $\nu = \mu - 0011 = (0, p-e, e-1, 0)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

Let $e \neq 2, 4$. We summarize the computation of $\operatorname{ch} L(\lambda)_{1221}$ in Table 2.87. Moreover, the JSF applied to μ yields $\operatorname{ch} L(\mu)_{1221} = \mu - (\mu - 0120) - \delta_{e,\frac{p-5}{2}}(\mu - 1221)$.

Let $e = 2$. The truncated characters $\operatorname{ch} L(\lambda)_{1232}$ and $\operatorname{ch} L(\mu)_{1232}$ are computed in Tables 2.88 and 2.89. Moreover, the JSF applied to ν yields $\operatorname{ch} L(\nu)_{1221} = \nu - (\nu - 1220)$. We solve the problematic cases for λ and μ as follows.

[$\boldsymbol{\lambda : F}$] Let $[\lambda : F] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1220) = (4+\zeta, 5, 0)$, hence $\zeta = 1$.

[$\boldsymbol{\lambda : G}$] Let $[\lambda : G] = 2 - \zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 0221) = (7+\zeta, 6, 2)$, hence $\zeta = 1$.

$\mu = (0,0,p-4,4)_{F_4}$	
Possibilities	
$\operatorname{ch} L(\mu)_{1232} = \mu - A - B$	
$\operatorname{ch} L(\mu)_{1232} = \mu - A - B + C$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1232} = A + B$	$\mathrm{JSF}(\mu)_{1232} = A + B + 2C$
$\mathrm{JSF}(A)_{1232} = C$	$\mathrm{JSF}(A)_{1232} = C$
$\mathrm{JSF}(B)_{1232} = C$	$\mathrm{JSF}(B)_{1232} = C$
$A = \mu - 0022 = (0,2,p-6,2)$	$C = \mu - 0132 = (1,1,p-6,3)$
$B = \mu - 0121 = (1,0,p-5,4)$	

Table 2.84: JSF of μ up to $\mu - 1232$

$\nu = (0,0,p-3,2)_{F_4}$	
$\operatorname{ch} L(\nu)_{1231} = \nu - D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1231} = D + E$	$\mathrm{JSF}(\nu)_{1231} = D + 2E$
$\mathrm{JSF}(D)_{1231} = E$	$\mathrm{JSF}(D)_{1231} = E$
$D = \nu - 0011 = (0,1,p-4,1)$	$E = \nu - 0121 = (1,0,p-4,2)$

Table 2.85: JSF of ν up to $\nu - 1231$

$[\boldsymbol{\mu : D}]$ If $[\mu : D] = 2-\zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-0231) = (11, 7+\zeta, 3)$. By Remark 1.5.4 and Proposition 2.3.1, we get that $\zeta = 1$.

Let $e = 4$. The JSF applied to μ and ν yields $\operatorname{ch} L(\mu)_{1231} = \mu - (\mu - 0120)$ and $\operatorname{ch} L(\nu)_{1220} = \nu - (\nu - 0120)$, respectively. Moreover, the computation of $\operatorname{ch} L(\lambda)_{1231}$ is summarized in Table 2.90. Let us determine $[\lambda : E]$. Set $[\lambda : E] = 2 - \zeta$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1231) = (24 + 2\zeta, 21, 5)$, hence $\zeta = 1$.

In all three cases, the multiplicities listed in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.5.1.18 $\boldsymbol{\lambda = b\lambda_2 + c\lambda_3 + d\lambda_4}$. — By Table 2.35, we have that $c + d = p - 1, b \neq c$ and $2b + d + 3 \not\equiv 0 \mod p$, or $b = c$ and $c + d = p - 1$. We solve both of these cases separately.

$\lambda=(4,0,p-5,0,0,1)_{E_6}$	
Possibilities	
ch $L(\lambda)_{1232}=\lambda-A+B-C$	
ch $L(\lambda)_{1232}=\lambda-A+B$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1232}=A-B+C$	$\mathrm{JSF}(\lambda)_{1232}=A+2C$
$\mathrm{JSF}(A)_{1232}=B+C$	$\mathrm{JSF}(A)_{1232}=B+C$
$A=\lambda-101000=(3,0,p-6,1,0,1)$ 0011	$C=\lambda-112210=(4,0,p-6,0,0,2)$ 1231
$B=\lambda-202100=(2,1,p-6,0,1,1)$ 0122	

Table 2.86: JSF of λ up to $\mu-1232$

$\lambda=(1,0,0,p-e-1,e,0)_{E_6}$	
ch $L(\lambda)_{1221}=\lambda-A+B+C$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1221}=A-B-C$	$\mathrm{JSF}(\lambda)_{1221}=A$
$\mathrm{JSF}(A)_{1221}=B+C$	$\mathrm{JSF}(A)_{1221}=B+C$
$A=\lambda-000110=(1,1,1,p-e-2,e-1,1)$ 0110	$C=\lambda-000221=(1,2,2,p-e-3,e-1,0)$ 0221
$B=\lambda-010220=(1,0,2,p-e-2,e-2,2)$ 1220	

Table 2.87: JSF of λ up to $\mu-1221$

$\boldsymbol{c+d=p-1, b\neq c, 2b+d+3\not\equiv 0 \mod p.}$ — By Table 2.35, the weight $\nu=\mu-1110=(b-1,d,c,1)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The computation of $\operatorname{ch} L(\lambda)_{1121}$ and $\operatorname{ch} L(\mu)_{1121}$ is summarized in Tables 2.91 and 2.92, respectively. Comparing the multiplicities in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\boldsymbol{b=c, c+d=p-1.}$ — Note that by Theorem 2.1.1 this case comes from an irreducible case for both embeddings (C_3,A_5) (B_3,D_4). Let us find a second composition factor for X acting on $L_Y(\lambda)$. The JSF applied to λ yields

$$\operatorname{ch} L(\lambda)_{1121}=\lambda-(\lambda-010100)-(\lambda-001100).$$

Moreover, the computation of $\operatorname{ch} L(\mu)_{1121}$ is summarized in Table 2.93. In order to determine $[\mu:B]$ for $c=p-2$, let $[\mu:B]=2-\zeta$ with $\zeta\in\{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1110)$ $=(3,2+\zeta,0)$, which implies that $\zeta=1$ by Theorem 2.1.1 and Remark 1.5.4. Now, computing multiplicities for all the values of c, we get

$$(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu-1121)=(7-\delta_{c,1}, 6-\delta_{c,1}-\delta_{c,p-2}).$$

$\lambda = (1,0,0,p-3,2,0)_{E_6}$	
ch $L(\lambda)_{1232} = \lambda - A - B + C + D + E + G$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1232} = A + B - C - D - E + F$	$\mathrm{JSF}(\lambda)_{1232} = A + B + 2F + 2G$
$\mathrm{JSF}(A)_{1232} = C + D + E + F + G$	$\mathrm{JSF}(A)_{1232} = C + D + E + F + G$
$\mathrm{JSF}(B)_{1232} = G$	$\mathrm{JSF}(B)_{1232} = G$
$A = \lambda - 000110 = (1,1,1,p-4,1,1)$ 0110	$E = \lambda - 000221 = (1,2,2,p-5,1,0)$ 0221
$B = \lambda - 101100 = (0,1,0,p-4,3,0)$ 0111	$F = \lambda - 011210 = (2,0,0,p-4,2,1)$ 1220
$C = \lambda - 010220 = (1,0,2,p-4,0,2)$ 1220	$G = \lambda - 101210 = (0,2,1,p-5,2,1)$ 0221
$D = \lambda - 001220 = (2,2,0,p-4,0,2)$ 0230	

Table 2.88: JSF of λ up to $\mu - 1232$

$\mu = (0,p-3,2,1)_{F_4}$	
ch $L(\mu)_{1232} = \mu - A - B - C + D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1232} = A + B + C$	$\mathrm{JSF}(\mu)_{1232} = A + B + C + 2D$
$\mathrm{JSF}(A)_{1232} = D$	$\mathrm{JSF}(A)_{1232} = D$
$\mathrm{JSF}(B)_{1232} = D$	$\mathrm{JSF}(B)_{1232} = D$
$A = \mu - 0120 = (1,p-3,0,3)$	$C = \mu - 1230 = (0,p-3,0,4)$,
$B = \mu - 0111 = (1,p-4,3,0)$	$D = \mu - 0231 = (2,p-4,1,2)$

Table 2.89: JSF of μ up to $\mu - 1232$

Therefore, $\nu = \mu - 1121 = (c-1, p-c, c-1, 0)$ affords the highest weight of a second composition factor and if $c = p-2$, then ν also affords the highest weight of a third composition factor.

Let $c = 1$. The computation of $\operatorname{ch} L(\lambda)_{1231}$ is summarized in Table 2.94. Moreover, the JSF applied to μ yields $\operatorname{ch} L(\mu)_{1231} = \mu - (\mu - 1100) - (\mu - 0231)$. The multiplicities listed in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $2 \leq c \leq p-3$. Recall that $\mu = (c, p-c-1, c, 0)$ and $\nu = (c-1, p-c, c-1, 0)$. By comparing the coefficients of μ and ν, we deduce that if we know the truncated character of μ up to a weight $\theta \in X(T_X)^+$ for $c \in \{1, \ldots, p-3\}$, then we know the truncated character of ν up to θ for $c \in \{2, \ldots, p-2\}$. The computation of $\operatorname{ch} L(\lambda)_{2242}$ and $\operatorname{ch} L(\mu)_{2242}$ is summarized in Tables 2.95 and 2.96, respectively. We deduce enough information about the problematic cases in order to prove that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (1, 0, 0, p-5, 4, 0)_{E_6}$	
ch $L(\lambda)_{1231} = \lambda - A + B + C + D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1231} = A - B - C - D + E$	$\mathrm{JSF}(\lambda)_{1231} = A + 2E$
$\mathrm{JSF}(A)_{1231} = B + C + D + E$	$\mathrm{JSF}(A)_{1231} = B + C + D + E$
$A = \lambda - 000110 = (1, 1, 1, p-6, 3, 1)$ 0110	$D = \lambda - 000221 = (1, 2, 2, p-7, 3, 0)$ 0221
$B = \lambda - 010220 = (1, 0, 2, p-6, 2, 2)$ 1220	$E = \lambda - 111210 = (0, 0, 1, p-6, 4, 1)$ 1221
$C = \lambda - 001220 = (2, 2, 0, p-6, 2, 2)$ 0230	

Table 2.90: JSF of λ up to $\mu - 1231$

$\lambda = (0, b, p-d-1, d, 0, 0)_{E_6}$	
ch $L(\lambda)_{1121} = \lambda - A$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1121} = A + \delta_{b,p-1}B$	$\mathrm{JSF}(\lambda)_{1121} = A + 2\delta_{b,p-1}B$
$\mathrm{JSF}(A)_{1121} = \delta_{b,p-1}B$	$\mathrm{JSF}(A)_{1121} = \delta_{b,p-1}B$
$A = \lambda - 001100 = (1, b+1, p-d-2, d-1, 1, 0)$ 0110	$B = \lambda - 011100 = (1, b-1, p-d-2, d, 1, 0)$ 1110

Table 2.91: JSF of λ up to $\mu - 1121$

$\mu = (b, d, p-d-1, 0)_{F_4}$	
Possibilities	
ch $L(\mu)_{1121} = \mu - A$	
ch $L(\mu)_{1121} = \mu - A - \delta_{b,p-1}B$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1121} = A + \delta_{b,p-1}B$	$\mathrm{JSF}(\mu)_{1121} = A + 2\delta_{b,p-1}B$
$\mathrm{JSF}(A)_{1121} = \delta_{b,p-1}B$	$\mathrm{JSF}(A)_{1121} = \delta_{b,p-1}B$
$A = \mu - 0120 = (b+1, d, p-d-3, 2)$	$B = \mu - 1120 = (b-1, d+1, p-d-3, 2)$

Table 2.92: JSF of μ up to $\mu - 1121$

$[\boldsymbol{\mu : D}]$ Let $c = p - 3$ and $[\mu : D] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})$

$(\mu - 1242) = (25, 21 + \zeta, 3)$. Hence assume $\zeta = 1$, since otherwise X acts on $L_Y(\lambda)$ with more than two composition factors.

$[\boldsymbol{\lambda} : \boldsymbol{F}]$ By Lemma 2.1.8, we have $[\lambda : F] = 1$.

$[\boldsymbol{\lambda} : \boldsymbol{G}]$ Let $c = p-4$ and $[\lambda : G] = 2-\zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu-1231) = (16 + \zeta, 15, 2)$, hence $\zeta = 1$.

$[\boldsymbol{\mu} : \boldsymbol{E}]$ Let $c = p - 5$ and $[\mu : E] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})$ $(\mu - 1231) = (17, 14 + \zeta, 2)$ and X acts on $L_Y(\lambda)$ with more than two composition factors if $\zeta = 0$. Assume $\zeta = 1$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2242) = (38, 31, 6)$, which also implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\mu = (c, p-c-1, c, 0)_{F_4}$	
ch $L(\mu)_{1121} = \mu - A - \overline{\delta_{c,1}}C - \delta_{c,p-2}D$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1121} = A + \delta_{c,p-2}B + \overline{\delta_{c,1}}C$	$\mathrm{JSF}(\mu)_{1121} = A + 2\delta_{c,p-2}B + \overline{\delta_{c,1}}C + \delta_{c,p-2}D$
$\mathrm{JSF}(A)_{1121} = \delta_{c,p-2}B - \delta_{c,p-2}D$	$\mathrm{JSF}(A)_{1121} = \delta_{c,p-2}B$
$\mathrm{JSF}(B)_{1121} = \delta_{c,p-2}D$	$\mathrm{JSF}(B)_{1121} = \delta_{c,p-2}D$
$A = \mu - 1100 = (c-1, p-c-2, c+2, 0)$	$C = \mu - 0120 = (c+1, p-c-1, c-2, 2)$,
$B = \mu - 1110 = (c-1, p-c-1, c, 1)$	$D = \mu - 1121 = (c-1, p-c, c-1, 0)$

Table 2.93: JSF of μ up to $\mu - 1121$

$\lambda = (0, 1, 1, p-2, 0, 0)_{E_6}$	
ch $L(\lambda)_{1231} = \lambda - A - B + C + D$	
Lemma 2.1.8	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1231} = A + B - C$	$\mathrm{JSF}(\lambda)_{1231} = A + B + 2D$
$\mathrm{JSF}(A)_{1231} = D$	$\mathrm{JSF}(A)_{1231} = D$
$\mathrm{JSF}(B)_{1231} = C + D$	$\mathrm{JSF}(B)_{1231} = C + D$
$A = \lambda - 010100 = (0, 0, 2, p-3, 1, 0)$ 1100	$C = \lambda - 102200 = (0, 3, 0, p-4, 2, 0)$ 0221
$B = \lambda - 001100 = (1, 2, 0, p-3, 1, 0)$ 0110	$D = \lambda - 011200 = (1, 1, 1, p-4, 2, 0)$ 1210

Table 2.94: JSF of λ up to $\mu - 1231$

$\lambda = (0, c, c, p-c-1, 0, 0)_{E_6}$	
$\text{ch } L(\lambda)_{2242} = \lambda - A - B + C + D + E + F$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\text{JSF}(\lambda)_{2242} = A + B - C - D - E + \delta_{c,p-4}G$	$\text{JSF}(\lambda)_{2242} = A + B + 2F + 2\delta_{c,p-4}G$
$\text{JSF}(A)_{2242} = D + F$	$\text{JSF}(A)_{2242} = D + F$
$\text{JSF}(B)_{2242} = C + E + F + \delta_{c,p-4}G$	$\text{JSF}(B)_{2242} = C + E + F + \delta_{c,p-4}G$
$A = \lambda - 010100 = (0, c-1, c+1, p-c-2, 1, 0)$ 1100	$E = \lambda - 002210 = (2, c+2, c-2, p-c-2, 0, 1)$ 0230
$B = \lambda - 001100 = (1, c+1, c-1, p-c-2, 1, 0)$ 0110	$F = \lambda - 011200 = (1, c, c, p-c-3, 2, 0)$ 1210
$C = \lambda - 102200 = (0, c+2, c-1, p-c-3, 2, 0)$ 0221	$G = \lambda - 112210 = (0, c, c-1, p-c-1, 0, 1)$ 1231
$D = \lambda - 020210 = (0, c-2, c+2, p-c-2, 0, 1)$ 2210	

Table 2.95: JSF of λ up to $\mu - 2242$

$\mu = (c, p-c-1, c, 0)_{F_4}$	
Possibilities	
$\text{ch } L(\mu)_{2242} = \mu - A - B + \overline{\delta_{c,2}}C - \delta_{c,\frac{p-3}{2}}F$	
$\text{ch } L(\mu)_{2242} = \mu - A - B + \overline{\delta_{c,2}}C - \delta_{c,p-3}D - \delta_{c,\frac{p-3}{2}}F$	
$\text{ch } L(\mu)_{2242} = \mu - A - B + \overline{\delta_{c,2}}C - \delta_{c,p-5}E - \delta_{c,\frac{p-3}{2}}F$	
$\text{ch } L(\mu)_{2242} = \mu - A - B + \overline{\delta_{c,2}}C - \delta_{c,p-3}D - \delta_{c,p-5}E - \delta_{c,\frac{p-3}{2}}F$	
See argument	
In order for the table to fit in the margins, we omitted $\text{JSF}(-) =$ in the second column.	
JSF in Weyl characters:	JSF in irreducible characters:
$\text{JSF}(\mu)_{2242} = A + B - \overline{\delta_{c,2}}C + \delta_{c,p-3}D + \delta_{c,p-5}E + \delta_{c,\frac{p-3}{2}}F$	$A + B +_1^2 \delta_{c,p-3}D +_1^2 \delta_{c,p-5}E + \delta_{c,\frac{p-3}{2}}F$
$\text{JSF}(B)_{2242} = \overline{\delta_{c,2}}C + \delta_{c,p-3}D + \delta_{c,p-5}E$	$\overline{\delta_{c,2}}C + \delta_{c,p-3}D + \delta_{c,p-5}E$
$A = \mu - 1100 = (c-1, p-c-2, c+2, 0)$	$D = \mu - 1242 = (c, p-c, c-2, 0)$,
$B = \mu - 0120 = (c+1, p-c-1, c-2, 2)$	$E = \mu - 1231 = (c, p-c-1, c-1, 1)$,
$C = \mu - 0241 = (c+2, p-c-1, c-3, 2)$	$F = \mu - 2242 = (c-2, p-c+1, c-2, 0)$

Table 2.96: JSF of μ up to $\mu - 2242$

2.5.1.19 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4}$. — By Table 2.35, we have that $a = 1, b = c, c+d = p-1$ and $c \neq 2$. Moreover, we get that $\nu = \mu - 0111 = (c+1, p-c-2, c+1, 0)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

Let $c \neq p-2$. The computation of $\text{ch } L(\lambda)_{1121}$ and $\text{ch } L(\mu)_{1121}$ are summarized in Tables 2.97 and 2.98. Let us determine $[\lambda : C]$ for $c = p-3$. Set $[\lambda : C] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have

$(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 1111) = (6+\zeta, 6)$. Since $c \neq p-1$, ν is p-restricted which implies that $m_{L(\nu)}(\mu - 1111) = 1$ and thus $\zeta = 1$ The multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $c = p-2$. The computation of $\operatorname{ch} L(\lambda)_{1232}$ and $\operatorname{ch} L(\mu)_{1232}$ are summarized in Tables 2.99 and 2.100. Moreover, the JSF applied to ν yields $\operatorname{ch} L(\nu)_{1121} = \nu - (\nu - 1120)$. Denote by L_I the Levi factor of P_I, where $I = \{\alpha_i\}_{\{1\leq i\leq 4\}}$. Note that L_I is of type A_4. We solve the problematic cases for λ and μ as follows.

[$\boldsymbol{\lambda : F}$] By Lemma 2.1.8, we have $[\lambda : F] = 1$.

[$\boldsymbol{F : G}$] Note that $[F : G] = 1 + [A : G]$. We solve the problematic case $[A : G]$ to get the value of $[F : G]$. We work in L_I. By Proposition 1.5.2 and Table 2.101,

$$[A : G] = [(0, p-4, 3, p-4) : (0, p-5, 1, p-2)]_{A_4}$$

and $(0, p-4, 3, p-4) - (0, p-5, 1, p-2) = 1220$ in A_4. The JSF applied to $(0, p-4, 3, p-4)$ in A_4 shows that $[(0, p-4, 3, p-4) : (0, p-5, 1, p-2)]_{A_4} = 0$, which implies that $[A : G] = 0$ and $[F : G] = 1$.

[$\boldsymbol{\mu : P}$] Let $[\mu : P] = 2-\zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 0131) = (5, 3+\zeta, 1)$. By Remark 1.5.4 and Proposition 2.3.1, we get that $\zeta = 1$.

[$\boldsymbol{\mu : L}$] Let $[\mu : L] = 2-\zeta$ with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1110) = (3, 2+\zeta, 0)$. By Remark 1.5.4 and Theorem 2.1.1, we get that $\zeta = 1$.

[$\boldsymbol{\mu : N}$] Let $[\mu : N] = 2 - \zeta$, with $\zeta \in \{0,1\}$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 1230) = (7, 7+\zeta, 0)$, hence $\zeta = 0$.

[$\boldsymbol{\lambda : G}$] We work in L_I. By Proposition 1.5.2 and Table 2.102,

$$[\lambda : G] = [(2, p-2, 1, p-4) : (0, p-5, 1, p-2)]_{A_4}$$

and $(2, p-2, 1, p-4) - (0, p-5, 1, p-2) = 3420$ in A_4. By Proposition 1.3.7, we have that $[\lambda : G] \neq 0$. We can apply again Proposition 1.5.3 along with Lemma 2.1.8 in order to deduce that $[\lambda : G] = 1$.

Comparing the multiplicities appearing in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

2.5.1.20 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + e\lambda_5}$. — By Table 2.35, we have $a = 2, b = 1, c = p-3, e = 1$ and $\nu = \mu - 0010 = (1, 1, p-4, 3)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The JSF applied to λ yields

$$\operatorname{ch} L(\lambda)_{1111} = \lambda - (\lambda - 101000) - (\lambda - 011100).$$

Comparing the multiplicities listed in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (1, c, c, p-c-1, 0, 0)_{E_6}$	
ch $L(\lambda)_{1121} = \lambda - A - B$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1121} = A + B + \delta_{c,p-3}C$	$\mathrm{JSF}(\lambda)_{1121} = A + B + 2\delta_{c,p-3}C$
$\mathrm{JSF}(A)_{1121} = \delta_{c,p-3}C$	$\mathrm{JSF}(A)_{1121} = \delta_{c,p-3}C$
$A = \lambda - 010100 = (1, c-1, c+1, p-c-2, 1, 0)$ 1100	$C = \lambda - 111100 = (0, c-1, c, p-c-1, 1, 0)$ 1111
$B = \lambda - 001100 = (2, c+1, c-1, p-c-2, 1, 0)$ 0110	

Table 2.97: JSF of λ up to $\mu - 1121$

$\mu = (c, p-c-1, c, 1)_{F_4}$	
Possibilities	
ch $L(\mu)_{1121} = \mu - A - B - \delta_{c,\frac{p-5}{2}}D$	
ch $L(\mu)_{1121} = \mu - A - B - \delta_{c,p-4}C - \delta_{c,\frac{p-5}{2}}D$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1121} = A + B + \delta_{c,p-4}C + \delta_{c,\frac{p-5}{2}}D$	$\mathrm{JSF}(\mu)_{1121} = A + B + 2\delta_{c,p-4}C + \delta_{c,\frac{p-5}{2}}D$
$\mathrm{JSF}(A)_{1121} = \delta_{c,p-4}C$	$\mathrm{JSF}(A)_{1121} = \delta_{c,p-4}C$
$A = \mu - 1100 = (c-1, p-c-2, c+2, 1)$	$C = \mu - 1111 = (c-1, p-c-1, c+1, 0)$,
$B = \mu - 0120 = (c+1, p-c-1, c-2, 3)$	$D = \mu - 1121 = (c-1, p-c, c-1, 1)$

Table 2.98: JSF of μ up to $\mu - 1121$

2.5.1.21 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + d\lambda_4 + e\lambda_5}$. — By Table 2.35, we have $a = 1, b = e, e + d = p - 1, e \neq 1, p - 2$ and $\nu = \mu - 0011 = (b, p - b, b - 1, 0)$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Assume $b \neq p - 4$. The JSF applied to λ and μ yields

$$\operatorname{ch} L(\lambda)_{1121} = \lambda - (\lambda - 010100) - (\lambda - 000110) - \delta_{b,2}(\lambda - 101100)$$

and

$$\operatorname{ch} L(\mu)_{1121} = \mu - (\mu - 1100) - (\mu - 0120) - \delta_{b,2}(\mu - 0111) - \delta_{b,\frac{p-5}{2}}(\mu - 1121),$$

respectively. Assume $b = p - 4$. The JSF applied to λ and ν yields

$$\operatorname{ch} L(\lambda)_{1132} = \lambda - (\lambda - 010100) - (\lambda - 000110) - (\lambda - 111110)$$

and $\operatorname{ch} L(\nu)_{1121} = \nu - (\nu - 0120)$, respectively. Moreover, in Table 2.103, we determine two possibilities for $\operatorname{ch} L(\mu)_{1132}$. In both cases, comparing the multiplicities appearing in Table 2.104 implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\lambda = (1, p-2, p-2, 1, 0, 0)_{E_6}$	
ch $L(\lambda)_{1232} = \lambda - A - B - C - 2D + E + F + G$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1232} = A + B + C + D - E$	$\mathrm{JSF}(\lambda)_{1232} = A + B + C + D + 2F +_1^4 G +_0^1 H$
$\mathrm{JSF}(A)_{1232} = -D + F$	$\mathrm{JSF}(A)_{1232} = F +_0^1 G$
$\mathrm{JSF}(B)_{1232} = G$	$\mathrm{JSF}(B)_{1232} = G$
$\mathrm{JSF}(C)_{1232} = -D + E + F + H$	$\mathrm{JSF}(C)_{1232} = E + F +_0^1 G + 2H$
$\mathrm{JSF}(D)_{1232} = G$	$\mathrm{JSF}(D)_{1232} = G$
$\mathrm{JSF}(E)_{1232} = H$	$\mathrm{JSF}(E)_{1232} = H$
$\mathrm{JSF}(F)_{1232} = D + G$	$\mathrm{JSF}(F)_{1232} = D + 2G$
$A = \lambda - 101000 = (0, p-2, p-3, 2, 0, 0)$ 0011	$E = \lambda - 002210 = (3, p, p-4, 0, 0, 1)$ 0230
$B = \lambda - 010100 = (1, p-3, p-1, 0, 1, 0)$ 1100	$F = \lambda - 102100 = (1, p-1, p-4, 1, 1, 0)$ 0121
$C = \lambda - 001100 = (2, p-1, p-3, 0, 1, 0)$ 0110	$G = \lambda - 213200 = (0, p-2, p-4, 1, 2, 0)$ 1232
$D = \lambda - 203200 = (0, p, p-4, 0, 2, 0)$ 0232	$H = \lambda - 012210 = (3, p-2, p-4, 1, 0, 1)$ 1230

Table 2.99: JSF of λ up to $\mu - 1232$

$\mu = (p-2, 1, p-2, 1)_{F_4}$	
ch $L(\mu)_{1232} = \mu - I + J - K - M - N - O + P$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1232} = I - J + K + L + M$	$\mathrm{JSF}(\mu)_{1232} = I + K + 2L + M +_1^2 N + O + 2P$
$\mathrm{JSF}(I)_{1232} = L - N - O$	$\mathrm{JSF}(I)_{1232} = L$
$\mathrm{JSF}(J)_{1232} = N$	$\mathrm{JSF}(J)_{1232} = N$
$\mathrm{JSF}(K)_{1232} = P$	$\mathrm{JSF}(K)_{1232} = P$
$\mathrm{JSF}(L)_{1232} = N + O$	$\mathrm{JSF}(L)_{1232} = N + O$
$\mathrm{JSF}(M)_{1232} = J + N + P$	$\mathrm{JSF}(M)_{1232} = J + 2N + P$
$I = \mu - 1100 = (p-3, 0, p, 1)$	$M = \mu - 0120 = (p-1, 1, p-4, 3)$,
$J = \mu - 0230 = (p, 0, p-4, 4)$	$N = \mu - 1230 = (p-2, 1, p-4, 4)$,
$K = \mu - 0011 = (p-2, 2, p-3, 0)$	$O = \mu - 1132 = (p-3, 3, p-4, 0)$,
$L = \mu - 1110 = (p-3, 1, p-2, 2)$	$P = \mu - 0131 = (p-1, 2, p-5, 2)$

Table 2.100: JSF of μ up to $\mu - 1232$

2.5.1.22 $\boldsymbol{\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + f\lambda_6}$**.** — By Table 2.35, we have that $b = a-1, a+c = p-1, f = 1$ and $\nu = \mu - 0001 = (a-1, 0, p-a, a-1)$ affords the highest weight of a second composition factor

$\lambda_0' = (p-4, 0, 0, -1) \notin C_0$	
$\gamma' = w_1 \bullet \lambda_0' = (0, p-3, 2, p-2)$	$\eta' = w \bullet \lambda_0' = (0, p-4, 1, p-2)$
$w_1 = s_0 s_1 s_2 s_4 s_3 s_0$	$w = s_0 s_1 s_2$
$\lambda_0 = (p-5, 0, 0, 0) \in C_0$	
$\gamma = w_1 \bullet \lambda_0 = (1, p-4, 3, p-3)$	$\eta = w \bullet \lambda_0 = (0, p-5, 1, p-2)$
$C_{\eta'} = (1, 1, 1, 1, 1, 1, 2, 1, 2, 2)$	$C_\eta = (1, 1, 1, 1, 1, 1, 2, 1, 2, 2)$
Proposition 1.3.10 $\implies [\gamma' : \eta'] = [\gamma : \eta]$	
$s = s_3$	
$ws \bullet \lambda_0 = (0, p-5, 0, p)$	
$ws \bullet \lambda_0 - w \bullet \lambda_0 = 0001$	
Proposition 1.3.9 $\implies [w_1 s \bullet \lambda_0 : \eta] = [\gamma : \eta]$, where $w_1 s \bullet \lambda_0 = (0, p-4, 3, p-4)$	

Table 2.101: Computing $[(0, p-3, 2, p-2) : (0, p-4, 1, p-2)]_{A_4}$

$\lambda_0' = (p-4, 0, 0, -1) \notin C_0$	
$\gamma' = w_1 \bullet \lambda_0' = (1, p-2, 1, p-2)$	$\eta' = w \bullet \lambda_0' = (0, p-4, 1, p-2)$
$w_1 = s_0 s_1 s_2 s_4 s_3 s_2 s_0$	$w = s_0 s_1 s_2$
$\lambda_0 = (p-5, 0, 0, 0) \in C_0$	
$\gamma = w_1 \bullet \lambda_0 = (2, p-3, 2, p-3)$	$\eta = w \bullet \lambda_0 = (0, p-5, 1, p-2)$
$C_{\eta'} = (1, 1, 1, 1, 1, 1, 2, 1, 2, 2)$	$C_\eta = (1, 1, 1, 1, 1, 1, 2, 1, 2, 2)$
Proposition 1.3.10 $\implies [\gamma' : \eta'] = [\gamma : \eta]$	
$s = s_3$	
$ws \bullet \lambda_0 = (0, p-5, 0, p)$	
$ws \bullet \lambda_0 - w \bullet \lambda_0 = 0001$	
Proposition 1.3.9 $\implies [w_1 s \bullet \lambda_0 : \eta] = [\gamma : \eta]$, where $w_1 s \bullet \lambda_0 = (2, p-2, 1, p-4)$	

Table 2.102: Computing $[(1, p-2, 1, p-2) : (0, p-4, 1, p-2)]_{A_4}$

for X acting on $L_Y(\lambda)$. If $a \neq p-3$, then the JSF applied to λ yields

$$\operatorname{ch} L(\lambda)_{1111} = \lambda - (\lambda - 101000) - (\lambda - 011100).$$

If $a = p-3$, then the JSF applied to λ and ν yields

$$\operatorname{ch} L(\lambda)_{1112} = \lambda - (\lambda - 101000) - (\lambda - 011100) - (\lambda - 010111)$$

and $\operatorname{ch} L(\nu)_{1111} = \nu - (\nu - 0011)$, respectively. In both cases, the multiplicities in Table 2.104 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

$\mu = (p-4, 3, p-4, 1)_{F_4}$	
Possibilities	
ch $L(\mu)_{1132} = \mu - A - B$	
ch $L(\mu)_{1132} = \mu - A - B - C$	
Multiplicity bounded above by the first possibility	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\mu)_{1132} = A + B + C$	$\mathrm{JSF}(\mu)_{1132} = A + B + 2C$
$\mathrm{JSF}(A)_{1132} = C$	$\mathrm{JSF}(A)_{1132} = C$
$A = \mu - 1100 = (p-5, 2, p-2, 1)$	$C = \mu - 1111 = (p-5, 3, p-3, 0)$
$B = \mu - 0120 = (p-3, 3, p-6, 3)$	

Table 2.103: JSF of μ up to $\mu - 1132$

Table 2.104: Multiplicities for the proof of Proposition 2.5.1

λ b $a\ c\ d\ e\ f$	Conditions		ν	θ	Multiplicities $\lambda(\theta)$	 $\mu(\theta)$	 $\nu(\theta)$	Details
$(a,0,0,0,0,0)$	$a=1$		1232		two composition factors			Subsection 2.5.1.1
	$a=2,3$		1232	more than two composition factors, compare dimensions				Subsection 2.5.1.1
	$a=p-2$		1232		two composition factors			Subsection 2.5.1.1
	$a\geq 4, a\neq p-3,p-2$		1232	2464	38	$\leq_V 32$	$\leq_V 5$	Subsection 2.5.1.1
	$a=p-3$				irreducible			Table 2.2
$(0,b,0,0,0,0)$	$b=1$		1110		two composition factors			Subsection 2.5.1.2
	$b=p-2$		1110		two composition factors			Subsection 2.5.1.2
$(0,0,c,0,0,0)$	$c=1$		0121	1231	7	$\leq_V 5$	$\leq_V 1$	Subsection 2.5.1.3
	$c=p-1$		1231		two composition factors			Subsection 2.5.1.3
$(0,0,0,d,0,0)$	$d=1$		0110	1221	10	$\leq_V 4$	$\leq_V 4$	Subsection 2.5.1.4
	$d=p-1$		0110		two composition factors			Subsection 2.5.1.4
$(a,b,0,0,0,0)$	$b=1$	$a\neq p-4$	1110	1111	6	$\leq_V 4$	$\leq_V 1$	Subsection 2.5.1.5
		$a=p-4$	1110		two composition factors			Subsection 2.5.1.5
	$b=p-2$	$a=p-1$	1110	1111	5	$\leq_V 4$	0	Subsection 2.5.1.5
		$a\neq p-1$	1110	1111	6	$\leq_V 4$	$\leq_V 1$	Subsection 2.5.1.5
$(a,0,c,0,0,0)$	$c=1,\ a\neq p-2,p-6$	$a\neq p-8$	0121	1231	10	$\leq_V 8$	$\leq_V 1$	Subsection 2.5.1.6
		$a=p-8$	0121	1353	61	$\leq_V 48$	$\leq_V 12$	Subsection 2.5.1.6
	$a+c=p,\ a\neq 4,p-1$	$a\neq 2,6$	0121	1231	$18-\delta_{a,p-2}$	$\leq_V 14-\delta_{a,p-2}$	$\leq_V 3$	Subsection 2.5.1.6
		$a=2$	0121	1242	39	31	7	Subsection 2.5.1.6
		$a=6$	0121	1363	130	100	$\leq_{BS} 29$	Subsection 2.5.1.6
	$a+c=p-1$	$a=1$			irreducible			Table 2.2
		$a=3$	1231	1231	9	7	1	Subsection 2.5.1.6
		$a=7$	1231	1232/2462	10/62	8/53	1/8	Subsection 2.5.1.6
		$a=p-4$	1231	2462	55	46	8	Subsection 2.5.1.6

		$a = p-3$	1231	2462	37	31	5	Subsection 2.5.1.6
		$a = p-2$	1231	2463	38	32	$\leq_V 5$	Subsection 2.5.1.6
		$a \leq p-5, a \neq 1, 3, 7$	1231	2462	62	$\leq_{BC} 53$	$\leq_{BC} 8$	Subsection 2.5.1.6
$(a, 0, 0, d, 0, 0)$	$a = p-3,\ d = 1$		0110	1221	14	$\leq_V 9$	$\leq_V 4$	Subsection 2.5.1.7
$(a, 0, 0, 0, e, 0)$	$a = 1,\ e = p-1$		0011	1231	19	$\leq_V 14$	4	Subsection 2.5.1.8
	$a = p-4,\ e = 1$		0011	two composition factors				Subsection 2.5.1.8
$(a, 0, 0, 0, 0, f)$	$a \neq p-1,\ f = 1$	$a = 1$	0001	1232	12	$\leq_V 5$	$\leq_V 5$	Subsection 2.5.1.9
		$a \neq 1, p-3, p-5$	0001	1232	15	$\leq_V 5$	$\leq_V 8$	Subsection 2.5.1.9
		$a = p-3$	0001	1232	14	$\leq_V 5$	$\leq_V 8$	Subsection 2.5.1.9
		$a = p-5$	0001	1232	13	$\leq_V 5$	7	Subsection 2.5.1.9
$(0, b, c, 0, 0, 0)$	$b = p-3,\ c = 1$		0121	1231	12	$\leq_V 10$	$\leq_V 1$	Subsection 2.5.1.10
	$b = 1,\ c = p-1$		1110	1121	10	$\leq_V 7$	$\leq_V 2$	Subsection 2.5.1.10
	$b = p-1, c = p-1$		1121	1131/2231	8/22	6/19	1/2	Subsection 2.5.1.10
				2242	≥ 38	≤ 32	5	
$(0, 0, c, d, 0, 0)$	$c + d = p-1$	$d = 1$	1221	2452	61	52	8	Subsection 2.5.1.11
		$d = 2$	1221	2442	52	44	7	Subsection 2.5.1.11
		$d = 3$	1221	2442	59	50	8	Subsection 2.5.1.11
		$4 \leq d \leq p-7, d = p-5$	1221	2442	60	$\leq_{BC} 51$	8	Subsection 2.5.1.11
		$d = p-6$	1221	1231/2442	12/60	10/51	1/8	Subsection 2.5.1.11
		$d = p-4$	1221	more than two composition factors, see argument				Subsection 2.5.1.11
		$d = p-3$	1221	1221	9	7	1	Subsection 2.5.1.11
		$d = p-2$	1221	1331/2442	19/60	16/51	2/7	Subsection 2.5.1.11
$(a, b, c, 0, 0, 0)$	$a + c = p-1,\ b \neq a-1$	$2b + c + 4 \not\equiv 0 \mod p$	1110	1111	6	$\leq_{BC} 4$	$\leq_V 1$	Subsection 2.5.1.12
	$a + c = p-1,\ b = a-1$	$a \neq 2, p-2$	1111	2222	21	$\leq_{BS} 16$	4	Subsection 2.5.1.12
		$a = p-2$	1111	2222	21	16	4	Subsection 2.5.1.12
		$a = 2$	1111	two composition factors				Subsection 2.5.1.12
	$b = p-3, c = 1$	$a \neq p-2, p-6$	0121	1111	7	$\leq_V 6$	$\leq_V 0$	Subsection 2.5.1.12
	$b + c + 2 \equiv 0 \mod p$	$a + c = p,\ a \neq 4, p-1$	0121	1111	7	$\leq_V 6$	$\leq_V 0$	Subsection 2.5.1.12
$(a, b, 0, 0, e, 0)$	$a = 1,\ b = p-1,\ e = p-1$		0011	1121	16	11	4	Subsection 2.5.1.13

	$a=p-4,\ b=p-3,\ e=1$		0011	1121	13	$\leq_V 10$	$\leq_V 2$	Subsection 2.5.1.13
$(a,0,c,d,0,0)$	$a=1, c+d=p-1,\ c\neq 2$	$c=4$	0111	1332	58	44	13	Subsection 2.5.1.14
		$c=p-2$	0111	1221	14	10	$\leq_V 3$	Subsection 2.5.1.14
		otherwise	0111	1221	20	$\leq_{BC} 15$	$\leq_V 4$	Subsection 2.5.1.14
$(a,0,c,0,e,0)$	$a=2,\ c=p-3,\ e=1$		0010	two composition factors				Subsection 2.5.1.15
$(a,0,c,0,0,f)$	$a+c=p-1,\ f=1$	$a=1$	0001	1232	29	20	8	Subsection 2.5.1.16
		$a=3$	0001	1232	$\{24,26\}$	$\{15,17\}$	8	Subsection 2.5.1.16
		$a=4$	0001	1231/1232	19/30	14/21	4/7	Subsection 2.5.1.16
		$a=2$ or $5\leq a\leq p-3$	0001	1231	$19-\delta_{a,p-3}$	$\leq_V 14-\delta_{a,p-3}$	$\leq_V 4$	Subsection 2.5.1.16
		$a=p-2$	0001	1231	12	$\leq_V 8$	$\leq_V 3$	Subsection 2.5.1.16
$(a,0,0,d,e,0)$	$a=1, d+e=p-1,\ e\neq 1,p-2$	$e\neq 2,4$	0011	1221	19	$\leq_{BC} 15$	$\leq_V 3$	Subsection 2.5.1.17
		$e=2$	0011	1232	32	23	8	Subsection 2.5.1.17
		$e=4$	0011	1221	19	15	$\leq_V 3$	Subsection 2.5.1.17
$(0,b,c,d,0,0)$	$c+d=p-1, b\neq c,\ 2b+d+3\not\equiv 0 \mod p$		1110	1121	$10-\delta_{c,1}$	$\leq_{BS} 7-\delta_{c,1}$	$\leq_V 2$	Subsection 2.5.1.18
	$b=c, c+d=p-1$	$c=1$	1121	1231	16	14	$\leq_V 1$	Subsection 2.5.1.18
		$c=2$	1121	2242	33	27	5	Subsection 2.5.1.18
		$c=3$	1121	2242	37	30	6	Subsection 2.5.1.18
		$c=p-2$	1121	1121	7	5	$\leq_V 1$	Subsection 2.5.1.18
		$c=p-3$	1121	1242/2242	25/38	21/31	3/6	Subsection 2.5.1.18
		$c=p-5$	1121	1231/2242	17/38	14/31	2/6	Subsection 2.5.1.18
		otherwise	1121	2242	38	$\leq_{BC} 31$	6	Subsection 2.5.1.18
$(a,b,c,d,0,0)$	$a=1, b=c,\ c+d=p-1,\ c\neq 2,p-2$		0111	1121	$13-\delta_{c,1}$	$\leq_{BC} 11-\delta_{c,1}$	$\leq_V 1$	Subsection 2.5.1.19
	$a=1, b=c,\ c+d=p-1,\ c=p-2$		0111	1232	31	24	6	Subsection 2.5.1.19
$(a,b,c,0,e,0)$	$a=2,\ b=1,\ c=p-3,\ e=1$		0010	1111	9	$\leq_V 6$	$\leq_V 2$	Subsection 2.5.1.20
$(a,b,0,d,e,0)$	$a=1, b=e,\ d+e=p-1,\ e\neq 1,p-2, b\neq p-4$		0011	1121	$16-\delta_{b,2}$	$\leq_{BC} 11-\delta_{b,2}$	$\leq_V 4$	Subsection 2.5.1.21
	$a=1, b=e,\ d+e=p-1,\ e\neq 1,p-2, b=p-4$		0011	1132	24	$\leq_{BS} 16$	7	Subsection 2.5.1.21
$(a,b,c,0,0,f)$	$a+c=p-1,\ b=a-1,\ f=1,\ a\neq p-3$		0001	1111	10	$\leq_V 6$	$\leq_V 3$	Subsection 2.5.1.22
	$a+c=p-1,\ b=a-1,\ f=1,\ a=p-3$		0001	1112	11	$\leq_V 6$	4	Subsection 2.5.1.22

Chapter 3

The other embeddings

The goal of this chapter is to answer Question 3 for (X, Y) as in Tables 3.1 and 3.2, excluding the case $(X, Y) = (F_4, E_6)$ which has already been solved in Chapter 2 and the cases $(X, Y, p) \in \{(A_2, G_2, \{2, 3\}), (B_4, F_4, \{2, 3, 5, 7, 11\}), (C_4, F_4, 2)\}$ which we do not consider in this thesis.

G	X simple
G_2	A_2 (1 class if $p \neq 3$, 2 classes if $p = 3$)
F_4	B_4 ($p \geq 0$), C_4 ($p = 2$)
E_7	A_7 ($p \geq 0$)
E_8	D_8 ($p \geq 0$), A_8 ($p \geq 0$)

Table 3.1: Maximal closed connected simple subgroups of maximal rank

G	X simple
G_2	A_1 ($p \neq 2, 3, 5$)
F_4	A_1 ($p = 0$ or $p \geq 13$), G_2 ($p = 7$)
E_6	A_2 ($p \neq 2, 3$), G_2 ($p \neq 7$), F_4 ($p \geq 0$), C_4 ($p \neq 2$)
E_7	A_1 (2 classes, $p = 0$ or $p \geq 17, 19$, resp.), A_2 ($p \neq 2, 3$)
E_8	A_1 (3 classes, $p = 0$ or $p \geq 23, 29, 31$, resp.), B_2 ($p \neq 2, 3$)

Table 3.2: Maximal closed connected simple subgroups of non-maximal rank

Proposition 3.0.1. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let (X, Y, p) be as in Table 3.2 and Table 3.1. Assume in addition $(X, Y) \neq (F_4, E_6)$ and $(X, Y, p) \notin \{(A_2, G_2, \{2, 3\}), (B_4, F_4, \{2, 3, 5, 7, 11\}), (C_4, F_4, 2)\}$. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight. Then X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if λ is listed in Table A up to graph automorphism. Moreover, $L_Y(\lambda)|_X \cong L_X(\mu) \oplus L_X(\nu)$ with μ and ν given as in Table A.*

Throughout this chapter, let Y be a simply connected simple algebraic group of exceptional type and let X be a maximal closed connected simple subgroup of Y as in the statement of Proposition 3.0.1. Let $B_Y = U_Y T_Y$ be a Borel subgroup of Y and let $B_X = B_Y \cap X$ be a Borel subgroup of X with $U_X = U_Y \cap X$ and $T_X = T_Y \cap X$.

For λ, μ and ν as in Table A, we first show that if $L_Y(\lambda)|_X \cong L_X(\mu)/L_X(\nu)$, then $L_Y(\lambda) \cong L_X(\mu) \oplus L_X(\nu)$. If $Y \neq E_6$, then $L_Y(\lambda)$ is self-dual, since $-1 \in W_Y$, the Weyl groups of Y. If $(X,Y) = (G_2, E_6)$, then $-w_0\lambda = \lambda$ with w_0 the longest element of W_Y and so $L_Y(\lambda)$ is self-dual. Now, by Lemma 1.4.1, we have that $L_Y(\lambda)|_X$ is self-dual and by Lemma 1.4.3, we get $L_Y(\lambda) \cong L_X(\mu) \oplus L_X(\nu)$. If $(X,Y) = (C_4, E_6)$, then using the tables in [Lüb07] and Proposition 1.2.1, we check that $[V_X(\mu) : L_X(\nu)] = 0$, hence $\mathrm{Ext}^1(L_X(\mu), L_X(\nu)) = 0$ by Proposition 1.1.7 and so $L_Y(\lambda) \cong L_X(\mu) \oplus L_X(\nu)$. The second part of Proposition 3.0.1 follows.

From now on, we focus on proving the first part of the statement. Let $\lambda \in X(T_Y)^+$ be a p-restricted weight. In most cases, to prove that X acts on $L_Y(\lambda)$ with more than two composition factors, we bound the multiplicity of a weight in $L_Y(\lambda)|_X$ by the number of weights in $\Lambda(V_Y(\lambda))$ which restrict to it. When it is not straightforward, we list these weights. By Theorem 1.1.10, this argument does not depend on p, unless $(Y,p) \in \{(F_4,2),(G_2,2),(G_2,3)\}$. Note that we have excluded the latter cases in Proposition 3.0.1.

In other cases, we compare the dimensions of the irreducible modules using the tables from [Lüb07] or Weyl's degree formula. When it is not necessary to provide the exact dimension, we bound the dimension by an inequality. An inequality with subscript V (i.e. $\leq_V$) indicates the dimension of the corresponding Weyl module, an inequality with subscript LB (i.e. $\leq_{LB}$ or $\geq_{LB}$) gives a bound using the data in [Lüb07]. It has two meanings: Either the dimension of the corresponding module does not appear in [Lüb07] and we bound it below by the bound stated in [Lüb07], or the dimension of the module depends on the characteristic and we bound the dimension below by the smallest possibility or above by the greatest possibility. This type of argument implicitly takes into account the characteristic of k.

Finally, we need to keep the choice of p in mind when we apply Proposition 1.2.2 and the JSF. In order to apply Proposition 1.2.2, we need to assume $p \neq 2$ and additionally $p \neq 3$ for $Y = G_2$. For the JSF, in view of Remark 1.3.2, we assume $p \geq h$, which is the reason for excluding $p = 2,3,5,7,11$ for $(X,Y) = (C_4, F_4)$ in Proposition 3.0.1.

Notation 3.0.2. Let $\Delta(Y) = \{\alpha_i\}$ be a base of $\Phi(Y)$ corresponding to B_Y and $\Delta(X) = \{\beta_i\}$ be a base of $\Phi(X)$ corresponding to B_X. Let $\{\lambda_i\}$, $\{\mu_i\}$, denote a set of fundamental weights with respect to $\Delta(Y), \Delta(X)$, respectively. As in Notation 2.1.3, let a sequence of digits abbreviate a linear combination of simple roots, where each digit corresponds to a coefficient in the linear combination. From now on, we allow the coefficients to be negative which yields signed sequences. For example, if $|\Delta(Y)| = 4$, we abbreviate $\alpha_1 - 2\alpha_3 = \alpha_1 + 0\alpha_2 - 2\alpha_3 + 0\alpha_4 \in \mathbb{Z}\Delta(Y)$ by $10(-2)0$.

3.1 Preliminary lemmas

Let $\gamma, \theta \in X(T_X)$. Assume $\gamma - \theta = \sum_{i=1}^{|\Delta(X)|} n_i\beta_i \in \mathbb{Z}\Delta(X)$. Define the *level* of θ with respect to γ, denoted $\mathrm{lev}_\gamma(\theta)$, as $\sum_{i=1}^{|\Delta(X)|} n_i$. Let $\lambda \in X(T_Y)^+$ and set $\mu = \lambda|_{T_X}$. Let $\nu_1, \ldots, \nu_r \in X(T_X)^+$ be distinct weights such that $L_Y(\lambda)|_X = L_X(\nu_1)^{i_1}/\ldots/L_X(\nu_r)^{i_r}$ for some $i_1, \ldots, i_r \in \mathbb{Z}_{\geq 0}$. Let

$\theta \in \Lambda(L_Y(\lambda)|_X)^+$. Set

$$m_{\lambda,\theta} = \sum_{\substack{\zeta \in X(T_Y) \\ \zeta|_{T_X}=\theta}} m_{L_Y(\lambda)}(\zeta) - \sum_{\substack{\nu_s \text{ s.t.} \\ \nu_s \succneq \theta}} i_s m_{L_X(\nu_s)}(\theta).$$

Clearly, $m_{\lambda,\theta} \neq 0$ if and only if $\theta = \nu_j$ for some $1 \leq j \leq r$, in which case $m_{\lambda,\theta} = i_j$. Let $\ell = \operatorname{lev}_\mu(\theta)$. Note that we can rewrite $m_{\lambda,\theta}$ as

$$m_{\lambda,\theta} = \sum_{\substack{\zeta \in X(T_Y) \\ \zeta|_{T_X}=\theta}} m_{L_Y(\lambda)}(\zeta) - \sum_{\substack{\nu_s \text{ s.t.} \\ \operatorname{lev}_\mu(\nu_s)<\ell}} i_s m_{L_X(\nu_s)}(\theta).$$

Remark 3.1.1. For $\alpha \in \Delta(Y)$, write $\alpha|_{T_X} = \sum_{j=1}^{|\Delta(X)|} n_j\beta_j$ with $n_j \in \mathbb{Z}$. Assume $\sum_{j=1}^{|\Delta(X)|} n_j \geq 0$ for all $\alpha \in \Delta(Y)$, that is $\operatorname{lev}_{\alpha|_{T_X}}(0) \geq 0$. Then it is clear that the following holds.

1) For $\gamma \in \Lambda(L_Y(\lambda)|_X)^+$, we have $\operatorname{lev}_\mu \gamma \geq 0$.

2) If $\gamma \in \Lambda(L_Y(\lambda)|_X)^+$ and $\operatorname{lev}_\mu \gamma = 0$, then $\gamma = \nu_j$ for some $1 \leq j \leq r$.

The next two lemmas are easy and tell us in specific cases when a weight appears with nonzero multiplicity in a Weyl module.

Lemma 3.1.2. *Let Y be of type A_n for $n \geq 1$. Let $\lambda \in X(T_{A_n})^+$ and write λ as $\lambda = \sum_{i=1}^n a_i\lambda_i$.*

1) Let $1 \leq i \leq n$, then $\lambda - r\alpha \in \Lambda(V_Y(\lambda))$ for $0 \leq r \leq a_i$, where $\alpha = \sum_{j=i}^n \alpha_j$.

2) If $a_{i_1}a_{i_2} \neq 0$ for some $1 \leq i_1 < i_2 \leq n$, then $\lambda - 2\alpha \in \Lambda(V_Y(\lambda))$, where $\alpha = \sum_{k=j_1}^{j_2} \alpha_k$ and $1 \leq j_1 \leq i_1 < i_2 \leq j_2 \leq n$.

Lemma 3.1.3. *Let Y be of type E_n for $n \in \{6,7,8\}$. Let $\lambda \in X(T_Y)^+$ and write λ as $\lambda = \sum_{i=1}^n a_i\lambda_i$. If $a_ia_j \neq 0$ for some $1 \leq i < j \leq n$, then $\lambda - 2\sum_{r=1}^n \alpha_r \in \Lambda(V_Y(\lambda))$.*

Proof. We prove the lemma using Proposition 1.1.12 and a computer program. □

3.2 Maximal subgroups of non-maximal rank

The goal of this section is to prove Proposition 3.0.1 for the pairs (X,Y) with $\operatorname{rank}(X) < \operatorname{rank}(Y)$.

3.2.1 (G_2, E_6). — Let $(X,Y) = (G_2,E_6)$ and note that by Table 3.2, we have $p \neq 7$. This embedding was first constructed in [Tes89, (G.1) and (G.2)] with the additional assumption that $p \neq 2,3$. By [LS04, Theorem (6.1)], this assumption on p can be lifted. Moreover, by comparing [Sei91, Theorem (7.1)] and [LS04, Lemma 6.2.2 and Lemma 6.3.7], we can assume up to conjugacy that the simple roots of $\Phi(Y)$ restrict to T_X as

$$\alpha_i|_{T_X} = \beta_1 \text{ for } i \in \{1,2,3,5,6\} \quad \alpha_4|_{T_X} = \beta_2 - \beta_1. \tag{3.1}$$

Note that these restrictions imply that the hypothesis on the levels in Remark 3.1.1 is satisfied. Performing a change of basis by multiplying the restrictions in (3.1) by the Cartan matrix, we obtain that the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows.

$$\lambda_1|_{T_X} = \lambda_6|_{T_X} = 2\mu_1 \qquad\qquad \lambda_2|_{T_X} = \mu_1 + \mu_2$$

$$\lambda_3|_{T_X} = \lambda_5|_{T_X} = 2\mu_1 + \mu_2 \qquad\qquad \lambda_4|_{T_X} = 3\mu_2$$

Let $\lambda \in X(T_Y)^+$ and write $\lambda = \sum_{i=1}^{6} a_i\lambda_i$. Set $\mu = \lambda|_{T_X}$. We have

$$\mu = (2(a_1 + a_3 + a_5 + a_6) + a_2, a_2 + a_3 + a_5 + 3a_4).$$

We start by giving a general argument and then solve the remaining cases by comparing dimensions.

Assume $a_4 \neq 0$, then $\nu = (\lambda - \alpha_4)|_{T_X} = \mu - (-1)1$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ by Remark 3.1.1, since $\mathrm{lev}_\mu(\nu) = 0$. Note that for $\alpha = \alpha_4 + \alpha_i$, $i \in \{2,3,5\}$, we have $(\alpha)|_{T_X} = 01$ and $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 3, \leq_V 1, \leq_V 1)$. Hence X acts on $L_Y(\lambda)$ with more than two composition factors.

Assume $a_4 = 0$. Let $i,j,k \in \{1,2,3,5,6\}$ distinct and assume $a_ia_ja_k \neq 0$. Then $(\lambda - \alpha_\ell)|_{T_X} = \mu - 10$ for $\ell \in \{i,j,k\}$. Thus $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 10) = (\geq 3, \leq_V 1)$ and $\mu - 10$ affords the highest weight of a second and a third composition factor for X acting on $L_Y(\lambda)$. By symmetry, we are left to consider the following two cases:

1) $\lambda = a_i\lambda_i$, with for $i \in \{1,2,3\}$,

2) $\lambda = a_i\lambda_i + a_j\lambda_j$, with $a_ia_j \neq 0$ and $(i,j) \in \{(1,2),(1,3),(1,5),(1,6),(2,3),(3,5)\}$.

Consider first case 2). Note that $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 10) = (\geq 2, \leq_V 1)$, hence a second composition factor is given by $\nu = \mu - 10$. Let $(i,j) \in \{(1,5),(1,6),(3,5)\}$. Note that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 3, \leq_V 1, \leq_V 1)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors. Let $(i,j) \in \{(1,2),(2,3)\}$. Note that for $\alpha \in \{110100, 101100, 011100, 0101100, 001110\}$, we have $\alpha|_{T_X} = 11$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 11) = (\geq 4, \leq_V 2, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Let $(i,j) = (1,3)$, then for $\alpha \in \{201100, 102100, 101110, 111100, 011110, 001111\}$, we have $\alpha|_{T_X} = 21$. Therefore, we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 21) = (\geq 6, \leq_V 3, \leq_V 2)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

Consider now case 1), that is $\lambda = a_i\lambda_i$ with $i \in \{1,2,3\}$. Let $\lambda = a_1\lambda_1$. If $a_1 = 1$ and $p \neq 2$, then X acts irreducibly on $L_Y(\lambda)$ by [Tes88, Table 1]. If $a_1 = 1$ and $p = 2$, using the tables in [Lüb07], we see that X acts on $L_Y(\lambda)$ with more than two composition factors. If $a_1 \geq 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 20) = (\geq 2, \leq_V 1)$ and $\mu - 20$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The cases $a_1 = 2, 3$ are solved by comparing the dimensions appearing in Table 3.3. Assume $a_1 \geq 4$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 40) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors on $L_Y(\lambda)$.

Let $\lambda = a_2\lambda_2$. Assume $a_2 = 1$, so $\mu = (1,1)$. The only weights in $\Lambda(L_Y(\lambda)|_X)$ which are dominant and greater than $\mu - 21$ are $\mu, \mu - 11$. Moreover, $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 11) = (2,2)$ and $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 21) = (3,2)$. Hence $\mu - 21$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Comparing dimensions in Table 3.3 implies that X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if $p \neq 3$. Assume $a_2 \geq 2$. We have $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 11) = (\geq 3, \leq_V 2)$, hence $\mu - 11$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. If $a_2 = 2$, we compare dimensions in Table 3.3. Assume $a_2 \geq 3$. We have $(\alpha)|_{T_X} = 21$ for $\alpha \in \{030100, 021100, 020110, 111100, 010111\}$, hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 21) = (\geq 5, \leq_V 3, \leq_V 1)$ and X acts with more than two composition factors.

Let $\lambda = a_3\lambda_3$. Assume $a_3 = 1$, then $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 11) = (\geq 3, \leq_V 2)$ and $\mu - 11$ affords the highest weight of a second composition factor and we compare dimensions using Table 3.3.

Assume $a_3 \geq 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 20) = (\geq 2, \leq_V 1)$ and $\nu = \mu - 20$ affords the highest weight of a second composition factor. Moreover, $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 11) = (\geq 4, \leq_V 2, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. This completes the argument for the pair (G_2, E_6).

λ	$\mu = \lambda\vert_{T_X}$	ν	$\dim(L_Y(\lambda), L_X(\mu), L_X(\nu))$
$2\lambda_1$	$4\mu_1$	$2\mu_2$	$(\geq_{LB} 324, \leq_V 182, \leq_V 77)$
$3\lambda_1$	$6\mu_1$	$2\mu_1 + 2\mu_2$	$(\geq_{LB} 3002, \leq_V 714, \leq_V 729)$
λ_2	$\mu_1 + \mu_2$	μ_2	$(78 - \delta_{p,3}, 64 - 15\delta_{p,3}, 14 - 7\delta_{p,3})$
$2\lambda_2$	$2\mu_1 + 2\mu_2$	$3\mu_1 + \mu_2$	$(\geq_{LB} 2429, \leq_V 729, \leq_V 448)$
λ_3	$2\mu_1 + \mu_2$	$3\mu_1$	$(\geq_{LB} 324, \leq_V 189, \leq_V 77)$

Table 3.3: Some dimensions for (G_2, E_6)

3.2.2 (A_2, E_6). — Let $(X, Y) = (A_2, E_6)$ and note that by Table 3.2, we have $p \neq 2, 3$. The construction in [Tes89, (A1) and (A2)] implies that up to conjugacy the restriction to T_X of the simple roots in $\Phi(Y)$ is given by

$$\alpha_i|_{T_X} = \beta_1 \text{ for } i \in \{1,2,3,5,6\} \quad \alpha_4|_{T_X} = \beta_2 - 2\beta_1,$$

which implies the following restriction to T_X for the fundamental weights in $X(T_Y)^+$.

$$\lambda_1|_{T_X} = \lambda_6|_{T_X} = 2\mu_1 + 2\mu_2 \qquad \lambda_2|_{T_X} = \mu_1 + 4\mu_2$$
$$\lambda_3|_{T_X} = \lambda_5|_{T_X} = 2\mu_1 + 5\mu_2 \qquad \lambda_4|_{T_X} = 9\mu_2$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = \sum_{i=1}^{6} a_i\lambda_i$ and set $\mu = \lambda|_{T_X}$. Assume $a_4 \neq 0$, then $\nu = \mu - (-2)1$ affords the highest weight of second composition factor for X acting on $L_Y(\lambda)$. Moreover, we have $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - (-1)1) = (\geq 3, 0, \leq_V 1)$, hence establishing the existence of a third composition factor for X acting on $L_Y(\lambda)$. Henceforth assume $a_4 = 0$.

Let $a_3 \neq 0$ or $a_2 \neq 0$, then $\nu = \mu - (-1)1$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. If $a_2 = 1$ and $a_3 = 0$, then we have $\dim(L_Y(\lambda), L_X(\mu), L_X(\nu)) = (78, \leq_V 35, \leq_V 35)$ where $\mu = (1,4)$ and $\nu = (4,1)$. If $a_3 \neq 0$ or $a_2 \geq 2$ or $a_2a_1 \neq 0$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 3, \leq_V 1, \leq_V 1)$. Hence, by symmetry, X acts on $L_Y(\lambda)$ with more than two composition factors if either $a_2 \neq 0$ or $a_3 \neq 0$. By symmetry, this argument also holds if $a_5 \neq 0$. Henceforth, assume $a_3 = a_5 = a_2 = 0$.

Assume $a_1a_6 \neq 0$, then $\nu = \mu - 10$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Moreover, $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

By symmetry, the last case to consider is $\lambda = a_1\lambda_1$. If $a_1 = 1$, then by [Tes88, Table 1], we have that X acts irreducibly on $L_Y(\lambda)$. Assume $a_1 \geq 2$. Then $\nu = \mu - 20$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Moreover, $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 11) = (\geq 3, \leq_V 2, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. This completes the argument for the pair (A_2, E_6).

3.2.3 (C_4, E_6). — Let $(X, Y) = (C_4, E_6)$, then by Table 3.2, we have $p \neq 2$. Let $\tilde{X}$ be the maximal closed connected simple subgroup of type C_4 in Y given in the proof of [Tes88, Theorem (5.0)] as follows[1].

$$\tilde{X} = \langle u_{\pm(\alpha_2+\alpha_4+\alpha_5)}(-t)u_{\pm(\alpha_2+\alpha_3+\alpha_4)}(t), u_{\pm\alpha_1}(t)u_{\pm\alpha_6}(t), u_{\pm\alpha_3}(t)u_{\pm\alpha_5}(t), u_{\pm\alpha_4}(t) \mid t \in k\rangle$$

Let $B_{\tilde{X}} = B_Y \cap \tilde{X}$ be a Borel subgroup of $\tilde{X}$. The simple roots with respect to a positive set of roots corresponding to $B_{\tilde{X}}$ generate a root system of type C_4 and are given by the following restriction of simple roots in $\Phi^+(Y)$.

$$\begin{aligned} &\beta_1' = (\alpha_2+\alpha_3+\alpha_4)|_{T_{\tilde{X}}} = (\alpha_2+\alpha_4+\alpha_5)|_{T_{\tilde{X}}} && \beta_2' = \alpha_1|_{T_{\tilde{X}}} = \alpha_6|_{T_{\tilde{X}}} \\ &\beta_3' = \alpha_3|_{T_{\tilde{X}}} = \alpha_5|_{T_{\tilde{X}}} && \beta_4' = \alpha_4|_{T_{\tilde{X}}} \end{aligned}$$

Let X denote the subgroup of Y obtained by conjugating $\tilde{X}$ by a coset representative of $s_{\alpha_4}s_{\alpha_2}$ in $N_Y(T_Y)$. The simple roots of X corresponding to the Borel subgroup $B_X = X \cap B_Y$ are given by the following restrictions.

$$\begin{aligned} &\alpha_1|_{T_X} = \alpha_6|_{T_X} = \beta_2 && \alpha_2|_{T_X} = \beta_4 \\ &\alpha_3|_{T_X} = \alpha_5|_{T_X} = \beta_1 && \alpha_4|_{T_X} = \beta_3 - \beta_1 \end{aligned}$$

We deduce the following restriction of the fundamental weights in $X(T_Y)^+$.

$$\begin{aligned} &\lambda_1|_{T_X} = \lambda_6|_{T_X} = \mu_2 && \lambda_2|_{T_X} = \mu_4 \\ &\lambda_3|_{T_X} = \lambda_5|_{T_X} = \mu_1 + \mu_3 && \lambda_4|_{T_X} = 2\mu_3 \end{aligned}$$

Using Proposition 1.5.3, we can reduce the number of candidates $\lambda \in X(T_Y)^+$ on which X acts with exactly two composition factors. Consider the Levi factor L_I of the parabolic subgroup P_I of Y given by $I = \{\alpha_i\}_{i\neq 2}$ and note that $L_I \cap X = L_J$, where L_J is the Levi factor of the parabolic subgroup P_J of X given by $J = \{\beta_j\}_{j\neq 4}$. We have that L_I is of type A_5 and L_J of type A_3 (or equivalently D_3 up to relabelling of the Dynkin diagram). Since $L_J' = L_I' \cap X$ and for $\alpha \in \Phi(Y)$, $\alpha \in \mathbb{Z}I$ if and only if $\alpha|_{T_X} \in \mathbb{Z}J$, we can apply Proposition 1.5.3. The irreducible kL_I'-modules on which L_J' acts irreducibly are given by $L(\lambda_i|_{T_{L_J'}})$ for $i \in \{1,3,5,6\}$ by [Sei87, Theorem 1]. Moreover, the following result classifies the kL_I'-irreducible modules on which L_J' acts with exactly two composition factors.

Proposition 3.2.1 ([Cav15, Theorem 5.1]). *Let I, J be as above and consider an irreducible kL_I'-module $L_{L_I'}(\lambda|_{T_{L_I'}})$ having p-restricted highest weight $\lambda \in X(T_{L_I})^+$. Then L_J' has exactly two composition factors on $L_{L_I'}(\lambda|_{T_{L_I'}})$ if and only if λ and p are as in Table 3.4, where λ is given up to graph automorphism of L_I'. Moreover, $L_{L_I'}(\lambda|_{T_{L_I'}}) = L_{L_J'}(\mu|_{T_{L_J'}})/L_{L_J'}(\nu|_{T_{L_J'}})$ with μ and ν as in Table 3.4.*

Recall we have assumed $p \neq 2$. Combining the irreducible action and those with two composition factors, we only need to consider $\lambda \in X(T_Y)^+$, with λ appearing in the first column of Table 3.5, up to graph automorphism.

[1] Compared to [Tes88, Theorem (5.0)], we have reordered the generators, so that they match our labelling of the Dynkin diagram of type C_4.

λ	$\mu = \lambda$	ν	$\mu - \nu$	p
$\lambda_1 + \lambda_3$	$\mu_1 + \mu_2 + \mu_3$	μ_2	111	$p \neq 5$
$\lambda_1 + \lambda_6$	$2\mu_2$	$\mu_1 + \mu_3$	010	$p \neq 2$
$2\lambda_1$	$2\mu_2$	0	121	$p \neq 2, 3$
$2\lambda_1 + \lambda_6$	$3\mu_2$	$\mu_1 + \mu_2 + \mu_3$	010	$p = 7$
$3\lambda_1$	$3\mu_2$	μ_2	121	$p \neq 2, 3$
λ_3	$\mu_1 + \mu_3$	0	111	$p = 2$
λ_4	$2\mu_3$	$2\mu_1$	$(-1)01$	$p \neq 2$

Table 3.4: Two composition factors for the pair (A_3, A_5)

$\lambda = \sum_{i=1}^{6} a_i\lambda_i$	$\mu = \lambda\|_{T_X}$	ν	$\mu - \nu$	p
$\lambda_1 + a_2\lambda_2 + \lambda_3$	$\mu_1 + \mu_2 + \mu_3 + a_2\mu_4$	$\mu_2 + (a_2 + 1)\mu_4$	1110	$p \neq 2, 5$
$\lambda_1 + a_2\lambda_2 + \lambda_6$	$2\mu_2 + a_2\mu_4$	$\mu_1 + \mu_3 + a_2\mu_4$	0100	$p \neq 2$
$2\lambda_1 + a_2\lambda_2$	$2\mu_2 + a_2\mu_4$	$(a_2 + 1)\mu_4$	1210	$p \neq 2, 3$
$2\lambda_1 + a_2\lambda_2 + \lambda_6$	$3\mu_2 + a_2\mu_4$	$\mu_1 + \mu_2 + \mu_3 + a_2\mu_4$	0100	$p = 7$
$3\lambda_1 + a_2\lambda_2$	$3\mu_2 + a_2\mu_4$	$\mu_2 + (a_2 + 1)\mu_4$	1210	$p \neq 2, 3$
$a_2\lambda_2 + \lambda_4$	$2\mu_3 + a_2\mu_4$	$2\mu_1 + (a_2 + 1)\mu_4$	$(-1)010$	$p \neq 2$
$\lambda_1 + a_2\lambda_2$	$\mu_2 + a_2\mu_4$	$-$	$-$	$p \neq 2$
$a_2\lambda_2 + \lambda_3$	$\mu_1 + \mu_3 + a_2\mu_4$	$-$	$-$	$p \neq 2$

Table 3.5: Cases from (A_3, A_5)

Let $\lambda \in X(T_Y)^+$ and $\mu = \lambda|_{T_X}$ be as listed in Table 3.5. Let ν be as in Table 3.5 if its entry is nonempty in the line corresponding to λ. Assume $\lambda \notin \{a_2\lambda_2 + \lambda_4, \lambda_1 + a_2\lambda_2, a_2\lambda_2 + \lambda_3\}$. If $a_2 = 0$, then we argue by comparing dimensions, these are listed in Table 3.6. Note that in this case, X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if $\lambda = \lambda_1 + \lambda_3$ and $p = 3$. If $a_2 \neq 0$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - (010100)|_{T_X}) = (\geq 1, 0, 0)$, where $(010100)|_{T_X} = (-1)011$. Hence X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $\lambda = a_2\lambda_2 + \lambda_4$. If $a_2 = 0$, we compare dimensions using Table 3.6. If $a_2 \neq 0$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - (010200)|_{T_X}) = (\geq 1, 0, 0)$, where $(010200)|_{T_X} = (-2)021$. Hence X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $\lambda \in \{\lambda_1 + a_2\lambda_2, a_2\lambda_2 + \lambda_3\}$. Assume $a_2 = 0$. If $\lambda = \lambda_1$, then X acts irreducibly on $L_Y(\lambda)$ by [Tes88, Table 1]. If $\lambda = \lambda_3$, then $\mu = \mu_1 + \mu_3$. We have that $\nu = \mu - 0121 = 2\mu_1$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Indeed, by Proposition 1.5.3, the weights which could afford the highest weight of a second composition factor between μ and $\mu - 0121$ are the dominant weights of the form $\mu - 0xy1$ with $x \in \{0, 1\}$ and $y \in \{0, 1, 2\}$. However, none of these weights are dominant apart from μ and $\mu - 0121$. Moreover, for $\alpha \in \{111210, 011211, 112200\}$, we have $\alpha|_{T_X} = 0121$ and $\alpha \in \Lambda(\lambda)$. Combining this with the tables in [Lüb07], we get $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 0121) = (3, 2 - \delta_{p,3})$. Therefore, $\mu - 0121$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Note that it also affords the highest weight of a third composition factor for $p = 3$. Comparing dimensions, we get $\dim(L(\lambda), L(\mu), L(\nu)) = (351, 315, 36)$ if $p \neq 3$. Hence X acts on $L_Y(\lambda_3)$ with exactly two composition factors if and only if $p \neq 3$. Assume $a_2 \neq 0$. Set $\nu = (\lambda - 010100)_{T_X} = \mu - (-1)011 \in X(T_X)^+$. Note that ν affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The cases with $a_2 = 1$ are solved by comparing the dimensions listed in Table 3.6. Assume finally that $a_2 \geq 2$. Then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - (020200)|_{T_X}) = (\geq 1, 0, 0)$, where $(020200)|_{T_X} = (-2)022$ and so X acts on $L_Y(\lambda)$ with more than two composition factors. This completes the argument for the pair (C_4, E_6).

λ	μ	ν	char.	$\dim(L(\lambda), L(\mu), L(\nu))$
$\lambda_1 + \lambda_3$	$\mu_1 + \mu_2 + \mu_3$	$\mu_2 + \mu_4$	$p = 3$	$(2404, 1891, 513)$
			$p \neq 2, 3, 5$	$(5824, \leq_V 4096, \leq_V 792)$
$\lambda_1 + \lambda_6$	$2\mu_2$	$\mu_1 + \mu_3$	$p = 3$	$(572, 266, 279)$
			$p \neq 2, 3$	$(650, \leq_V 308, \leq_V 315)$
$2\lambda_1$	$2\mu_2$	μ_4	$p = 5$	$(324, 281, \leq_V 42)$
			$p \neq 2, 3, 5$	$(351, \leq_V 308, \leq_V 42)$
$2\lambda_1 + \lambda_6$	$3\mu_2$	$\mu_1 + \mu_2 + \mu_3$	$p = 7$	$(5994, \leq_V 2184, 3502)$
$3\lambda_1$	$3\mu_2$	$\mu_2 + \mu_4$	$p \neq 2, 3$	$(\geq_{LB} 3002, \leq_V 2184, \leq_V 792)$
λ_4	$2\mu_3$	$2\mu_1 + \mu_4$	$p \neq 2$	$(\geq_{LB} 2771, \leq_V 825, \leq_V 1155)$
$\lambda_1 + \lambda_2$	$\mu_2 + \mu_4$	$2\mu_1 + \mu_2$	$p = 5$	$(1377, \leq_V 792, 558)$
			$p \neq 2, 5$	$(\geq_{LB} 1701, \leq_V 792, \leq_V 594)$
$\lambda_2 + \lambda_3$	$\mu_1 + \mu_3 + \mu_4$	$3\mu_1 + \mu_3$	$p \neq 2$	$(\geq_{LB} 15822, \leq_V 6237, \leq_V 3696)$

Table 3.6: Some dimensions for (C_4, E_6)

3.2.4 (A_2, E_7). — Let $(X, Y) = (A_2, E_7)$, then by Table 3.2, we have $p \neq 2, 3$. Using the description of the embedding of the Lie algebra of X into the Lie algebra of Y given in [Sei91, p. 89, (5.8)] for $p \neq 5$ and the argument in [LS04, Lemma 4.1.3] for $p = 5$, we deduce that up to conjugacy the simple roots of $\Phi(Y)$ restrict to T_X as

$$\begin{aligned} &\alpha_1|_{T_X} = \alpha_4|_{T_X} = \beta_2 \quad \alpha_2|_{T_X} = \alpha_3|_{T_X} = \beta_1 - \beta_2 \quad \alpha_5|_{T_X} = \beta_2 - \beta_1 \\ &\alpha_6|_{T_X} = 2\beta_1 - \beta_2 \qquad \alpha_7|_{T_X} = 2\beta_2 - 2\beta_1. \end{aligned} \tag{3.2}$$

Whence the following restriction to T_X of the fundamental weights in $X(T_Y)^+$.

$$\lambda_1|_{T_X} = 4(\mu_1 + \mu_2) \qquad \lambda_2|_{T_X} = 7\mu_1 + 4\mu_2 \qquad \lambda_3|_{T_X} = 9\mu_1 + 6\mu_2 \qquad \lambda_4|_{T_X} = 11(\mu_1 + \mu_2)$$

$\lambda_5|_{T_X} = 7\mu_1 + 10\mu_2 \qquad \lambda_6|_{T_X} = 6(\mu_1 + \mu_2) \qquad \lambda_7|_{T_X} = 6\mu_2$

Write $\sum_{j=1}^{7} a_j\lambda_j$ and set $\mu = \lambda|_{T_X}$. Recall the definition of the level of a weight and Remark 3.1.1. Note that (3.2) implies that the hypothesis of Remark 3.1.1 on the level is satisfied. Henceforth, we assume the level of a T_X-weight to be with respect to μ. Let $i, k \in \{2, 3, 5, 7\}$ with $i \neq k$. By Remark 3.1.1, if $a_i \neq 0$, then $\mu - m\alpha_i$ affords the highest weight of a composition factor for $m = 1, \ldots, a_i$. Thus if $a_i a_k \neq 0$ or $a_i \geq 2$, then X acts on $L_Y(\lambda)$ with more than two composition factors. We assume from now on that $0 \leq a_i \leq 1$ and $a_i a_k = 0$ for $i, k \in \{2, 3, 5, 7\}$ with $i \neq k$.

Let $a_6 \neq 0$. Let $i \in \{2, 3\}$ and assume $a_i \neq 0$. A second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 1(-1)$. Note that $(0000010)|_{T_X}, (0111000)|_{T_X} = \mu - 2(-1)$, hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 2(-1)) = (\geq 2, 0, \leq_V 1)$ and X acts with more than two composition factors on $L_Y(\lambda)$. Let $i \in \{5, 7\}$ and suppose $a_i \neq 0$, then both $\mu - \alpha_i|_{T_X}$ and $\mu - \alpha_6|_{T_X}$ afford a highest weight of a composition factor for X acting on $L_Y(\lambda)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors. Therefore, assume $a_i = 0$ for $i \in \{2, 3, 5, 7\}$. We check that there is no composition factor with highest weight of level 0 apart from μ. Indeed, any weight in $\Lambda(L_Y(\lambda))$ different from λ is of the form $\lambda - \sum_{j=1}^{7} r_j\alpha_j \in \mathbb{Z}_{\geq 0}\Delta(Y)$ and the assumption on the coefficients of λ forces one of r_1, r_4, r_6 to be nonzero. Hence the level of any weight different from μ in $\Lambda(L_Y(\lambda)|_X)$ is strictly positive. By Section 3.1, we deduce that $\nu = \mu - 2(-1)$ affords the highest weight of second composition factor. Moreover, $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - (\alpha_5 + \alpha_6 + \alpha_7)|_{T_X}) = (\geq 1, 0, 0)$, where $(\alpha_5 + \alpha_6 + \alpha_7)|_{T_X} = (-1)2$ and so X acts on $L_Y(\lambda)$ with more than two composition factors. Note that this argument does not depend on the value of a_i for $i \in \{1, 4\}$. Therefore, X acts on $L_Y(\lambda)$ with more than two composition factors if $a_6 \neq 0$. Henceforth assume $a_6 = 0$.

Let $a_5 \neq 0$, so $a_2, a_3, a_7 = 0$. The highest weight of a second composition factor is given by $\nu = \mu - (-1)1$. Moreover, $(\alpha_4 + \alpha_5)|_{T_X}, (\alpha_5 + \alpha_6 + \alpha_7)|_{T_X} = (-1)2$ and $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})$ $(\mu - (-1)2) = (\geq 2, 0, \leq_V 1)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth assume $a_5 = 0$.

Let $a_7 \neq 0$, so $a_2, a_3, a_5 = 0$. The highest weight of a second composition factor is given by $\nu = \mu - (-2)2$. Assume additionally $a_1 \neq 0$ or $a_4 \neq 0$. Note that $(\alpha_1)|_{T_X}, (\alpha_4)|_{T_X}, (\alpha_6 + \alpha_7)|_{T_X} = 01$, therefore $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 2, \leq_V 1, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Assume $\lambda = a_7\lambda_7$. We have already shown that $0 \leq a_7 \leq 1$. By comparing dimensions, we get $\dim(L_Y(\lambda), L_X(\mu), L_X(\nu)) = (56, 28, 28)$ if $p \neq 5$ and $\dim(L_Y(\lambda), L_X(\mu), L_X(\nu)) = (56, < 28, < 28)$ if $p = 5$. Therefore, X acts on $L_Y(\lambda_7)$ with exactly two composition factors if and only if $p \neq 5$. Henceforth assume $a_7 = 0$.

Let $i \in \{1, 4\}$ and $j \in \{2, 3\}$ and assume $a_i a_j \neq 0$. We have that a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 1(-1)$. Note that $\alpha_1|_{T_X}, \alpha_4|_{T_X}, (\alpha_2 + \alpha_4 + \alpha_5)|_{T_X}, (\alpha_3 + \alpha_4 + \alpha_5)|_{T_X} = 01$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 2, \leq_V 1, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

We are left to consider the cases $\lambda = a_1\lambda_1 + a_4\lambda_4$ and $\lambda = \lambda_i$ for $i \in \{2, 3\}$. Assume $\lambda = \lambda_2$. A second composition factor is given by $\nu = \mu - 1(-1) = 4\mu_1 + 7\mu_2$. Comparing dimensions yields $\dim(L_Y(\lambda_2), L_X(\mu), L_X(\nu)) = (912, \leq_V 260, \leq_V 260)$, which implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

Suppose $\lambda = \lambda_3$. A second composition factor is given by $\nu = \mu - 1(-1)$. Note that $(\alpha_1 + \alpha_3)|_{T_X}, (\alpha_3 + \alpha_4)|_{T_X}, (\sum_{i=2}^{5} \alpha_i)|_{T_X} = 10$, therefore $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 10) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

Assume $\lambda = a_1\lambda_1 + a_4\lambda_4$. Reasoning as in the case $a_6 \neq 0$, we obtain that there are no weights in $\Lambda(L_Y(\lambda)|_X)$ of level 0 apart from μ. Assume $a_1a_4 \neq 0$. Since $(1010000)|_{T_X}, (0101000)|_{T_X}, (0011000)|_{T_X} = 10$, we have that $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 10) = (\geq 3, \leq_V 1)$. Hence by Section 3.1, $\mu - 10$ affords the highest weight of a second and third composition factor for X acting on $L_Y(\lambda)$. Assume $\lambda = a_4\lambda_4$. Note that $(0101000)|_{T_X}, (0011000)|_{T_X}, (0111100)|_{T_X} = 10$, hence $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 10) = (\geq 3, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Assume $\lambda = a_1\lambda_1$, so $\mu = 4a_1\mu_1 + 4a_1\mu_2$. By [Sei91, Theorem (5.1)] and [LS04, Lemma 4.1.3], if $a_1 = 1$, then X acts with exactly two composition factors on $L_Y(\lambda)$ and the second composition factor is given by $\mu - 33$. Therefore, assume $a_1 \geq 2$. It is a quick check to see that we do not have any composition factors of level 1 either. Note that $(2000000)|_{T_X}, (1011100)|_{T_X} = 02$, hence $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 02) = (\geq 2, \leq_V 1)$ and $\nu = \mu - 02$ affords the highest weight of a second composition factor. Moreover, $(2020000)|_{T_X}, (1111000)|_{T_X} = 20$. Therefore, we get that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 2, \leq_V 1, 0)$, which implies that X acts on $L_Y(\lambda)$ with more than two composition factors. This completes the argument for the pair (A_2, E_7).

3.2.5 (B_2, E_8). — Let $(X, Y) = (B_2, E_8)$, then by Table 3.2, we have $p \neq 2, 3$. Using the description of the embedding of the Lie algebra of X into the Lie algebra of Y given in [Sei91, eq. (10) on p. 111] for $p \neq 5$ and the proof of [LS04, Lemma 5.1.6] for $p = 5$, we get that up to conjugacy the restriction to T_X of the simple roots in $\Phi(Y)$ is given by

$$\alpha_1|_{T_X} = \alpha_6|_{T_X} = 0 \qquad \alpha_2|_{T_X} = \alpha_5|_{T_X} = \alpha_8|_{T_X} = \beta_2 - \beta_1$$
$$\alpha_3|_{T_X} = \beta_1 - \beta_2 \qquad \alpha_4|_{T_X} = \alpha_7|_{T_X} = \beta_1.$$

From these restrictions, we deduce that the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows.

$$\lambda_1|_{T_X} = 4(\mu_1 + \mu_2) \quad \lambda_2|_{T_X} = 3\mu_1 + 10\mu_2 \quad \lambda_3|_{T_X} = 8(\mu_1 + \mu_2) \quad \lambda_4|_{T_X} = 9\mu_1 + 16\mu_2$$
$$\lambda_5|_{T_X} = 5\mu_1 + 16\mu_2 \quad \lambda_6|_{T_X} = 4\mu_1 + 12\mu_2 \quad \lambda_7|_{T_X} = 3\mu_1 + 8\mu_2 \quad \lambda_8|_{T_X} = 6\mu_2$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = \sum_{i=1}^{8} a_i\lambda_i$ and set $\mu = \lambda|_{T_X}$. Note that the hypotheses of Remark 3.1.1 hold for this embedding. We assume the level of all the weights to be with respect to μ.

Consider first the cases where $a_i \neq 0$ for some $i \in \{1, 2, 3, 5, 6, 8\}$. Note that $\nu = \mu - (\alpha_i)|_{T_X}$ affords the highest weight of a second composition factor. Let $i \in \{1, 3, 5, 6\}$, then there is $1 \leq j \leq 8$ with $\langle \alpha_i, \alpha_j \rangle \neq 0$, such that $(\lambda - \alpha_i - \alpha_j)|_{T_X}$ is of level 0. By Remark 3.1.1, we have that $(\lambda - \alpha_i - \alpha_j)|_{T_X}$ affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$. Let $i = 2$. Note that $\alpha|_{T_X} = 01$ for $\alpha \in \{01010000, 01111000, 01111100\}$. Hence, $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth assume $a_j = 0$ for $j \in \{1, 2, 3, 5, 6\}$. Let $i = 8$ and assume $a_8 \geq 2$, then $\mu - (-2)2$ affords the highest weight of a third composition factor. If $a_7a_8 \neq 0$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 10) = (\geq 2, \leq_V 1, 0)$ and X acts with more than two composition factors on $L_Y(\lambda)$. If $a_4a_8 \neq 0$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors. If $a_8 = 1$ and $a_i = 0$ for $1 \leq i \leq 7$, then by [Sei91, Theorem (6.1)] and [LS04, Lemma 5.1.6], we have that $L_Y(\lambda)|_X = L_X((0,2))^{1+\delta_{p,5}}/L_X((0,6))/L_X((3,2))$, which implies that X acts on $L_Y(\lambda)$ with more than two composition factors.

We are left to consider the cases $\lambda = a_4\lambda_4 + a_7\lambda_7$. We reason as in Section 3.1 in order to show that X acts on $L_Y(\lambda)$ with more than two composition factors. We check that there is no weight in $\Lambda(L_Y(\lambda)|_X)$ apart from μ of level 0. Note that $(\alpha)|_{T_X} = 01$ for $\alpha \in \{01010000, 00011000, 00011100\}$ and if $a_4 \neq 0$, then $(\lambda - \alpha) \in \Lambda(L_Y(\lambda))$. Thus $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 01) = (\geq 3, \leq_V 1)$. Note that $(\alpha)|_{T_X} = 01$ for $\alpha \in \{00000011, 00000111, 00001110\}$ and if $a_7 \neq 0$, then $(\lambda - \alpha) \in \Lambda(L_Y(\lambda))$. Thus $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 01) = (\geq 3, \leq_V 1)$. Therefore, we get that X acts on $L_Y(\lambda)$ with more than two composition factors. This completes the argument for the pair (B_2, E_8).

3.2.6 (G_2, F_4). — Let $(X, Y) = (G_2, F_4)$, then by Table 3.2, we have $p = 7$. By the construction of this embedding in [Tes89, Proposition (F1)], we have that up to conjugacy the restriction to T_X of the simple roots in $\Phi(Y)$ is given by

$$\alpha_i|_{T_X} = \beta_1 \text{ for } i \in \{1, 3, 4\} \quad \alpha_2|_{T_X} = \beta_2 - \beta_1.$$

We deduce that the restriction to T_X of the fundamental weights in $X(T_Y)^+$ is as follows.

$$\lambda_1|_{T_X} = \mu_1 + \mu_2 \qquad \lambda_2|_{T_X} = 3\mu_2 \qquad \lambda_3|_{T_X} = 2\mu_1 + \mu_2 \qquad \lambda_4|_{T_X} = 2\mu_1$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = \sum_{i=1}^4 a_i\lambda_i$ and set $\mu = \lambda|_{T_X}$. Note that the hypotheses of Remark 3.1.1 hold for this embedding and we assume the level of a weight to be with respect to μ. If $a_2 \neq 0$, then $\mu - (-1)1$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. If $a_2 \geq 2$, then $\mu - (-2)2$ affords the highest of a composition factor as well and X acts with more than two composition factors. Henceforth assume $0 \leq a_2 \leq 1$.

Let $i, j \in \{1, 3, 4\}$ distinct and assume $a_ia_j \neq 0$, then $\mu - 10$ affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Let $1 \leq i, j, k \leq 4$ distinct with $a_ia_ja_k \neq 0$. If $i, j, k \in \{1, 3, 4\}$, then $\mu - 10$ affords the highest weight of a third composition factor for X acting on $L_Y(\lambda)$ and if $i, j, k \notin \{1, 3, 4\}$, then $\mu - (-1)1$ does. Hence X acts on $L_Y(\lambda)$ with more than two composition factors. Therefore, assume $\lambda = a_i\lambda_i + a_j\lambda_j$, $1 \leq i < j \leq 4$ with $a_ia_j \neq 0$. Set

$$\nu = \begin{cases} \mu - (-1)1 & \text{if } a_2 \neq 0 \\ \mu - 10 & \text{otherwise .} \end{cases}$$

By the previous considerations, ν affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

Let $(i, j) = (1, 2)$. Recall that $a_2 = 1$. If $a_1 \neq p - 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors. If $a_1 = p-2$, then $\alpha|_{T_X} = 11$ for $\alpha \in \{2100, 1110, 0111, 0120\}$, hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 11) = (\geq 4, \leq_V 2, \leq_V 1)$ and X acts with more than two composition factors.

Let $(i, j) = (1, 3)$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 2, \leq_V 1, 0)$ and X acts with more than two composition factors.

Let $(i, j) = (1, 4)$. If $a_1 \geq 2$ or $a_4 \geq 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors. If $a_1 = a_4 = 1$, then comparing dimensions yields $\dim(L(\lambda), L(\mu), L(\nu)) = (1053, \leq_V 448, \leq_V 286)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $(i,j) = (2,3)$. Recall that $a_2 = 1$. If $a_3 \geq 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 2, \leq_V 1, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Note that $\alpha|_{T_X} = 11$ for $\alpha \in \{1110, 0120, 0111\}$. If $a_3 = 1$, then by applying Proposition 1.2.2, we can show that $m_{L(\lambda)}(\lambda - 0120) = m_\lambda(\lambda - 0120) = 2$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 11) = (\geq 4, \leq_V 2, \leq_V 1)$ and X acts with more than two composition factors.

If $(i,j) = (2,4)$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 01) = (\geq 3, \leq_V 1, \leq_V 1)$, hence more than two composition factors.

Let $(i,j) = (3,4)$. If $a_3, a_4 \geq 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors. If $a_3 \geq 2$ or $a_4 \geq 2$, and $a_3 + a_4 \neq p-1$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors. If $(a_3, a_4) = (1,5)$ or $(a_3, a_4) = (5,1)$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 30) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors. If $(a_3, a_4) = (1,1)$, then $\mu = 4\mu_1 + \mu_2$ and $\nu = 2(\mu_1 + \mu_2)$. Comparing dimensions yields $\dim(L(\lambda), L(\mu), L(\nu)) = (2991, \leq_V 924, \leq_V 729)$, and so X acts with more than two composition factors on $L_Y(\lambda)$.

The last cases to consider are $\lambda = a_i\lambda_i$. Let $\lambda = a_1\lambda_1$, so that $\mu = a_1(\mu_1 + \mu_2)$. We check by direct computations that there is no weight of level 0 or 1 which affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. The only weights of level 2 in $\Lambda(L_Y(\lambda)|_X)$ are $\mu - 20$, $\mu - 02$ and $\mu - 11$. It is straightforward that $\mu - 20$ and $\mu - 02$ appear with multiplicity 1 in both $L_Y(\lambda)|_X$ and $L_X(\mu)$. Moreover, $\mu - 11$ appears with multiplicity $2 - \delta_{a,1}$ in $L_Y(\lambda)|_X$ and we check that it also the case in $L_X(\mu)$ by applying Lemma 2.1.6. Therefore, there is no weight of level 2 either which affords the highest weight of a second composition factor. Note that $\alpha|_{T_X} = 12$ for $\alpha \in \{3200, 2210, 1220\}$. Hence, if $a_1 \geq 3$, then $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 12) = (\geq 3, \leq_V 2)$ and a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 12$. Moreover, we have $\alpha|_{T_X} = 21$ for $\alpha \in \{3100, 2110, 1120, 1111\}$. Thus $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 21) = (\geq 4, \leq_V 3, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. If $a_1 = 1, 2$, then $(\alpha)|_{T_X} = 21$ for $\alpha \in \{1111, 2110, 1120\}$ and using the tables in [Lüb07], we get $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 21) = (\geq 3 - \delta_{a_1,1}, \leq_V 2 - \delta_{a_1,1})$. Thus a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 21 = (a_1 - 1)\mu_1 + a_1\mu_2$. Comparing dimensions yields

$$\dim(L(\lambda), L(\mu), L(\nu)) = \begin{cases} (52, 38, 14) & \text{if } a_1 = 1 \\ (755, 481, 248) & \text{if } a_1 = 2 \end{cases}$$

which implies that X acts on $L_Y(\lambda)$ with exactly two composition factors if $a_1 = 1$ and with more than two composition factors if $a_1 = 2$.

Let $\lambda = a_2\lambda_2$. We have already proven that $a_2 = 1$ and $\nu = \mu - (-1)1$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Note that $\mu = 3\mu_2$ and $\nu = 5\mu_1$. Moreover, $(\lambda - \alpha)|_{T_X} = \mu - 11$ for $\alpha \in \{1110, 0120, 0111\}$. Hence $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 11) = (\geq 3, \leq_V 1, \leq_V 1)$ and X acts with more than two composition factors.

If $\lambda = a_3\lambda_3$, then by a similar reasoning as in the case $\lambda = a_1\lambda_1$, we prove that there is no weight of level 0 and 1 apart from μ which affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. For $\alpha \in \{0111, 1110, 0120\}$, we have $\alpha|_{T_X} = 11$ and so $(m_{L(\lambda)|_X}, m_{L(\mu)})(\mu - 11) = (\geq 3, \leq_V 2,)$. Hence $\nu = \mu - 11$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Note that if $a_3 \geq 2$, then $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 20) = (\geq 2, \leq_V 1, 0)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors. If $a_3 = 1$, then $\mu = (2,1)$, $\nu = (3,0)$

and we get by comparing dimensions $\dim(L(\lambda), L(\mu), L(\nu)) = (273, 189, 77)$, thus establishing the existence of a third composition factor.

Let $\lambda = a_4\lambda_4$. If $a_4 = 1$, then X acts irreducibly on $L_Y(\lambda)$ by [Tes88, Table 1]. If $a_4 \geq 2$, then a second composition factor is given by $\nu = \mu - 20$. Note that $\mu = (2a_4, 0)$ and $\nu = (2a_4 - 4, 2)$. If $a_4 = 2, 3$, then comparing dimensions yields

$$\dim(L(\lambda), L(\mu), L(\nu)) = \begin{cases} (298, \leq_V 182, \leq_V 77) & \text{if } a = 2 \\ (2651, \leq_V 714, \leq_V 729) & \text{if } a = 3, \end{cases}$$

and X acts on $L_Y(\lambda)$ with more than two composition factors. If $a_4 \geq 4$, then we get that $(m_{L(\lambda)|_X}, m_{L(\mu)}, m_{L(\nu)})(\mu - 40) = (\geq 3, \leq_V 1, \leq_V 1)$, and X acts with more than two composition factors on $L_Y(\lambda)$. This completes the argument for the pair (G_2, F_4).

3.2.7 Maximal A_1's. — We now solve the cases where $X = A_1$. The construction of the maximal subgroups of type A_1 can be found in [Tes92]. We will use [Tes92, Lemma 4] which describes the embedding of the Lie algebra of X into the Lie algebra of Y, in order to find the restriction to T_X of the simple roots of $\Phi(Y)$.

Recall from the representation theory of A_1 that the dimension of a weight space of an irreducible representation of X is at most 1. Consequently, if there is a weight $\theta \in \Lambda(L_Y(\lambda))|_{T_X}$ of multiplicity greater or equal to 3 in $L_Y(\lambda)|_X$, then X acts on $L_Y(\lambda)$ with more than two composition factors. In this subsection, our notation for $\mu - m\beta_1$ becomes $\mu - m$. Additionally, since $X(T_X)^+$ is a one-dimensional lattice, for a weight $\mu \in X(T_X)^+$, there is $a \in \mathbb{Z}$ such that $\mu = a\mu_1$ and we denote μ by (a).

3.2.7.1 (A_1, G_2). — Let $(X, Y) = (A_1, G_2)$, then by Table 3.2, we have $p \neq 2, 3, 5$. The simple roots in $\Phi(Y)$ restrict to T_X as follows.

$$\alpha_i|_{T_X} = \beta_1 \text{ for } i \in \{1, 2\}$$

Thus the fundamental weights in $X(T_Y)^+$ restrict to T_X as

$$\lambda_1|_{T_X} = (6) \quad \lambda_2|_{T_X} = (10).$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = a_1\lambda_1 + a_2\lambda_2$ and set $\mu = \lambda|_{T_X}$. Assume $a_1 \geq 4$ or $a_2 \geq 4$, then $\mu - 4$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. Henceforth assume $0 \leq a_1, a_2 \leq 3$.

Assume $a_1 a_2 \neq 0$. If $a_1 \geq 2$ and $a_2 \geq 2$, then $\mu - 2$ is of multiplicity at least 3 in $L_Y(\lambda)|_{T_X}$. If $a_1 \geq 3$ or $a_2 \geq 3$, then $\mu - 3$ is of multiplicity at least 3 in $L_Y(\lambda)|_{T_X}$. If $(a_1, a_2) \in \{(1, 2), (2, 1)\}$, then Lemma 2.1.6 implies that the multiplicity of $\mu - 2$ is at least 3 in $L_Y(\lambda)|_X$. If $(a_1, a_2) = (1, 1)$, then the highest weight of a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \mu - 1$. The dimensions listed in Table 3.7 imply that X acts on $L_Y(\lambda)$ with more than two composition factors.

Let us consider the cases $\lambda = a_i\lambda_i$ for $i \in \{1, 2\}$. If $a_1 = 1$, then X acts irreducibly on $L_Y(\lambda)$ by [Tes88, Table 1]. If $a_2 = 1$, then we check that $\mu - 4$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Comparing dimensions yields $\dim(L(\lambda), L(\mu), L(\nu)) =$

$(14, \leq_V 11, \leq_V 3)$. Therefore, X acts with exactly two composition factors on $L_Y(\lambda)$ if and only if $p \neq 7$.

If $2 \leq a_1 \leq 3$ or $2 \leq a_2 \leq 3$, a second composition factor is given by $\nu = \mu - 2$. We solve these cases by comparing the dimensions listed in Table 3.7.

λ	μ	ν	$\dim(L(\lambda), L(\mu), L(\nu))$
$2\lambda_1$	(12)	(8)	$(\geq_{LB} 26, \leq_V 13, \leq_V 9)$
$3\lambda_1$	(18)	(14)	$(77, \leq_V 19, \leq_V 15)$
$2\lambda_2$	(20)	(16)	$(77, \leq_V 21, \leq_V 17)$
$3\lambda_2$	(30)	(26)	$(\geq_{LB} 148, \leq_V 31, \leq_V 27)$
$\lambda_1 + \lambda_2$	(16)	(14)	$(\geq_{LB} 38, \leq_V 17, \leq_V 15)$

Table 3.7: Some dimensions for (A_1, G_2)

3.2.7.2 (A_1, F_4). — Let $(X, Y) = (A_1, F_4)$, then by Table 3.2, we have $p = 0$ or $p \geq 13$. The restriction to T_X of the simple roots of $\Phi(Y)$ is given by $\alpha_i|_{T_X} = \beta_1$ for $1 \leq i \leq 4$. We get that the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows.

$$\lambda_1|_{T_X} = (22) \qquad \lambda_2|_{T_X} = (42) \qquad \lambda_3|_{T_X} = (30) \qquad \lambda_4|_{T_X} = (16)$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = \sum_{i=1}^4 a_i \lambda_i$ and set $\mu = \lambda|_{T_X}$. Let $1 \leq i, j, k \leq 4$ be pairwise distinct with $a_i a_j a_k \neq 0$, then the multiplicity of $\mu - 1$ is at least 3 in $L_Y(\lambda)|_X$ and so X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $(i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ with $a_i a_j \neq 0$, then the multiplicity of $\mu - 2$ in $L_Y(\lambda)|_X$ is at least 3. Let $(i, j) \in \{(1, 2), (3, 4)\}$ with $a_i a_j \neq 0$. If $a_i \geq 2$ or $a_j \geq 2$, then $\mu - 2$ is of multiplicity at least 3 in $L_Y(\lambda)|_X$. If $a_i = a_j = 1$, then by Lemma 2.1.6 we get that $\mu - 2$ is of multiplicity at least 3 in $L_Y(\lambda)|_X$. In all these cases, we have that X acts on $L_Y(\lambda)$ with more than two composition factors.

The last cases to consider are $\lambda = a_i \lambda_i$ for $1 \leq i \leq 4$. If $a_i \geq 2$, then $\mu - 4$ is of multiplicity at least 3 in $L_Y(\lambda)|_X$. Therefore, assume $a_i = 1$. If $\lambda = \lambda_4$, then we check that $\mu - 4$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Comparing dimensions yields $\dim(L(\lambda), L(\mu), L(\nu)) = (26, 17, 9)$ if $p \geq 17$ and $\dim(L(\lambda), L(\mu), L(\nu)) = (26, < 17, 9)$ if $p = 13$. Therefore X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if $p \geq 17$.

Assume $\lambda = \lambda_i$ for $1 \leq i \leq 3$. Note that the highest weight of a composition factor for X acting on $L_Y(\lambda)$ has to be strictly smaller than μ. Therefore, we can bound above the dimension of a second composition factor by the dimension of $V_X(\mu)$. We solve the remaining cases in Table 3.8 by noticing that $\dim L_Y(\lambda) > 2 \dim V_X(\mu)$.

3.2.7.3 (A_1, E_7). — Let $(X, Y) = (A_1, E_7)$. By Table 3.2, there are two conjugacy classes of maximal A_1 subgroups in E_7. For the first conjugacy class, we have $p = 0$ or $p \geq 19$. The restriction

λ	μ	$\dim(L_Y(\lambda), V_X(\mu))$
λ_1	(22)	(52,23)
λ_2	(42)	(1274,43)
λ_3	(30)	(273,31)

Table 3.8: Some dimensions for (A_1, F_4)

to T_X of the simple roots of $\Phi(Y)$ in the first class is given by $\alpha_i|_{T_X} = \beta_1$ for all $1 \leq i \leq 7$. This implies that the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows.

$$\lambda_1|_{T_X} = (34) \qquad \lambda_2|_{T_X} = (49) \qquad \lambda_3|_{T_X} = (66) \qquad \lambda_4|_{T_X} = (96)$$
$$\lambda_5|_{T_X} = (75) \qquad \lambda_6|_{T_X} = (52) \qquad \lambda_7|_{T_X} = (27)$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = \sum_{i=1}^{7} a_i\lambda_i$ and set $\mu = \lambda|_{T_X}$. If there is $i \in \{3, 4, 5\}$ with $a_i \neq 0$, then $\mu - 3$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. If there is $i \in \{2, 6\}$ with $a_i \neq 0$, then $\mu - 4$ has multiplicity at least 3 in $L_Y(\lambda)|_X$.

We are left with the cases $\lambda = a_1\lambda_1 + a_7\lambda_7$. If $a_1a_7 \neq 0$, then $\mu - 2$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. Assume $\lambda = a_1\lambda_1$. If $a_1 \geq 2$, then $\mu - 5$ has multiplicity at least 3 in $L_Y(\lambda)$. If $a_1 = 1$, then using the tables in [Lüb07], we get that $L_Y(\lambda)$ has a weight with multiplicity at least 3, hence X acts with more than two composition factors of $L_Y(\lambda)$. Assume $\lambda = a_7\lambda_7$. If $a_7 \geq 2$, then $\mu - 6$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. If $a_7 = 1$, then a second composition factor is given $\nu = \mu - 5$. Since $(\dim(L(\lambda), L(\mu), L(\nu)) = (56, \leq_V 28, \leq_V 18)$, we get that X acts on $L_Y(\lambda)$ with more than two composition factors.

For the second conjugacy class, we have by Table 3.2 that $p = 0$ or $p \geq 17$. The simple roots of $\Phi(Y)$ restrict to T_X as $\alpha_i|_{T_X} = \beta_1$ for $1 \leq i \leq 7$ and $i \neq 4$, and $\alpha_4|_{T_X} = 0$. This implies that the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows.

$$\lambda_1|_{T_X} = (26) \qquad \lambda_2|_{T_X} = (37) \qquad \lambda_3|_{T_X} = (50) \qquad \lambda_4|_{T_X} = (72)$$
$$\lambda_5|_{T_X} = (57), \qquad \lambda_6|_{T_X} = (40), \qquad \lambda_7|_{T_X} = (21).$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = \sum_{i=1}^{7} a_i\lambda_i$ and set $\mu = \lambda|_{T_X}$. Assume $a_4 \neq 0$, then $\mu - 1$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. Let $i \in \{3, 5\}$ and assume $a_i \neq 0$, then $\mu - 2$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. Let $i \in \{2, 6\}$ and assume $a_i \neq 0$, then $\mu - 3$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. Assume $a_1 \neq 0$, then $\mu - 4$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. Assume $a_7 \neq 0$, then $\mu - 5$ has multiplicity at least 3 in $L_Y(\lambda)|_X$. Therefore, X acts on $L_Y(\lambda)$ with more that two composition factors.

3.2.7.4 (A_1, E_8). — Assume $(X, Y) = (A_1, E_8)$. By Table 3.2, there are three classes of maximal A_1-subgroups in E_8. For the first, we have $p = 0$ or $p \geq 31$. The simple roots of $\Phi(Y)$ restrict to T_X as $\alpha_i|_{T_X} = \beta_1$ for $1 \leq i \leq 8$ and the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows

$$\lambda_1|_{T_X} = (92) \qquad \lambda_2|_{T_X} = (136) \qquad \lambda_3|_{T_X} = (182) \qquad \lambda_4|_{T_X} = (270)$$

$$\lambda_5|_{T_X} = (220) \qquad \lambda_6|_{T_X} = (168) \qquad \lambda_7|_{T_X} = (114) \qquad \lambda_8|_{T_X} = (58)$$

Note that the restriction to T_X of α_i for $1 \leq i \leq 7$ is the same as in the first case of Subsection 3.2.7.3. By Proposition 1.5.2 and the reasoning in Subsection 3.2.7.3, we can assume that $\lambda \in X(T_Y)^+$ is of the form $\lambda = a_1\lambda_1 + a_7\lambda_7 + a_8\lambda_8$, with $0 \leq a_1, a_7 \leq 1$, if X acts on $L_Y(\lambda)$ with at most two composition factors.

If $a_7 = 1$, then $\mu - 4$ has multiplicity at least 3 in $L_Y(\lambda)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth assume $a_7 = 0$. If $a_1a_8 \neq 0$, then $\mu - 2$ has multiplicity at least 3 in $L_Y(\lambda)$. Let $\lambda = a_8\lambda_8$. If $a_8 \geq 2$, then $\mu - 5$ has multiplicity at least 3 in $L_Y(\lambda)$. Assume $\lambda = \lambda_i$ for $i \in \{1, 8\}$. Using the tables in [Lüb07], we get that in both of these cases $L_Y(\lambda)$ has a weight of multiplicity at least 3. Hence X acts on $L_Y(\lambda)$ with more than two composition factors.

For the second class, we have $p = 0$ or $p \geq 29$. The restriction to T_X of the simple roots of $\Phi(Y)$ is given by $\alpha_i|_{T_X} = \beta_1$ for $1 \leq i \leq 8$ and $i \neq 4$, and $\alpha_4|_{T_X} = 0$. We thus get that the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows.

$$\lambda_1|_{T_X} = (72) \qquad \lambda_2|_{T_X} = (106) \qquad \lambda_3|_{T_X} = (142) \qquad \lambda_4|_{T_X} = (210)$$
$$\lambda_5|_{T_X} = (172) \qquad \lambda_6|_{T_X} = (132) \qquad \lambda_7|_{T_X} = (90) \qquad \lambda_8|_{T_X} = (46)$$

Let $\lambda \in X(T_Y)^+$, write $\lambda = \sum_{i=1}^{8} a_i\lambda_i$ and set $\mu = \lambda|_{T_X}$. Notice that as for the first class, the restriction to T_X of α_i for $1 \leq i \leq 7$ is the same as in the second class of Subsection 3.2.7.3. By Proposition 1.5.2 and the reasoning in Subsection 3.2.7.3, if X acts on $L_Y(\lambda)$ with exactly two composition factors, then $\lambda = a_8\lambda_8$. Note that if $a_8 \neq 0$, then $\mu - 6$ has multiplicity at least 3 in $L_Y(\lambda)|_X$, hence X acts on $L_Y(\lambda)$ with more than two composition factors.

For the third class, we have $p = 0$ or $p \geq 23$. The restriction to T_X of the simple roots of $\Phi(Y)$ is given by $\alpha_i|_{T_X} = \beta_1$ for $1 \leq i \leq 8$ and $i \neq 4, 6$, and $\alpha_4|_{T_X} = \alpha_6|_{T_X} = 0$. This implies that the fundamental weights in $X(T_Y)^+$ restrict to T_X as follows.

$$\lambda_1|_{T_X} = (60) \qquad \lambda_2|_{T_X} = (88) \qquad \lambda_3|_{T_X} = (118) \qquad \lambda_4|_{T_X} = (174)$$
$$\lambda_5|_{T_X} = (142) \qquad \lambda_6|_{T_X} = (108) \qquad \lambda_7|_{T_X} = (74) \qquad \lambda_8|_{T_X} = (38)$$

Let $a_4 \neq 0$ or $a_6 \neq 0$, then $\mu - 1$ has at least multiplicity 3 in $L_Y(\lambda)|_X$. Let $i \in \{1, 2, 3, 5, 7\}$ and assume $a_i \neq 0$, then $\mu - 3$ has at least multiplicity 3 in $L_Y(\lambda)|_X$. Let $a_8 \neq 0$, then $\mu - 5$ has at least multiplicity 3 in $L_Y(\lambda)|_X$. Therefore, X acts on $L_Y(\lambda)$ with more than two composition factors.

3.3 Maximal subgroups of maximal rank

The goal of this section is to prove Proposition 3.0.1 for the pairs (X, Y) as in Table 3.1, excluding the cases $(Y, p) \in \{(F_4, \{2, 3, 5, 7, 11\}), (G_2, \{2, 3\})\}$. Let $B_Y = U_YT_Y$ be a Borel subgroup of Y. We fix $B_X = U_XT_Y$ to be the Borel subgroup of X, where $U_X = U_Y \cap X$.

The pairs (X, Y) are in correspondence with the maximal closed subsystems of $\Phi(Y)$. Recall that they can be deduced using a theorem from Borel-de Siebenthal (c.f. [MT11, Theorem 13.12]).

For each embedding (X, Y), we start by applying the theorem from Borel-de Siebenthal to find a base $\Delta'(X)$ of $\Phi(X) \subseteq \Phi(Y)$. However, if $\Delta'(X) \not\subseteq \Phi^+(Y)$, then $\Delta'(X)$ is not equal to the base $\Delta(X)$ corresponding to our choice of B_X. Nevertheless, there is an element $w \in W_X$, the Weyl group of X, which conjugates $\Delta'(X)$ to $\Delta(X) \subseteq \Phi^+(Y)$. In order to distinguish between a linear expression in terms of simple roots in $\Delta(Y)$ and one in $\Delta(X)$, we add a subscript Y or X. For example, if $\mathrm{rank}(Y) = 4$, then $(1111)_X = \sum_{i=1}^4 \beta_i$ denotes a linear expression in terms of the simple roots in $\Delta(X)$ and $(1111)_Y = \sum_{i=1}^4 \alpha_i$ denotes a linear expression in terms of the simple roots in $\Delta(Y)$.

Let $\lambda \in X(T_Y)^+$ and notice that since T_Y is a maximal torus of X, it makes sense to consider $L_X(\lambda)$ and $V_X(\lambda)$. Denote the longest element of W_X by $(w_0)_X$. Recall that by Lemma 1.4.1, the restriction commutes with taking the dual. Therefore, if $L_Y(\lambda)$ is self-dual, then $L_Y(\lambda)|_X$ is also self-dual. The next lemma provides in certain cases a method to establish the existence of third composition factor for X acting on $L_Y(\lambda)$.

Lemma 3.3.1. *Let $\lambda \in X(T_Y)^+$ and assume $\nu \in X(T_X)^+$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Assume $L_Y(\lambda)|_X$ is self-dual. If $-(w_0)_X \cdot \lambda \neq \nu$ and either $-(w_0)_X \cdot \lambda \neq \lambda$ or $-(w_0)_X \cdot \nu \neq \nu$, then X acts on $L_Y(\lambda)$ with more than two composition factors.*

Proof. The hypotheses imply that a module with composition factors $L_X(\lambda)/L_X(\nu)$ is not self-dual. Since $L_Y(\lambda)|_X$ is self-dual, X acts on $L_Y(\lambda)$ with more than two composition factors. □

3.3.1 (A_8, E_8). — Assume $(X, Y) = (A_8, E_8)$. Up to conjugacy, the simple roots of a root system of type A_8 in $\Phi(Y)$ are given by

$$\begin{array}{llll} \beta_1 = \alpha_8 & \beta_2 = \alpha_7 & \beta_3 = \alpha_6 & \beta_4 = \alpha_5 \\ \beta_5 = \alpha_4 & \beta_6 = \alpha_3 & \beta_7 = \alpha_1 & \beta_8 = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \end{array}$$

which can be rewritten as

$$\begin{array}{llll} \alpha_1 = \beta_7 & \alpha_3 = \beta_6 & \alpha_4 = \beta_5 & \alpha_5 = \beta_4 \\ \alpha_6 = \beta_3 & \alpha_7 = \beta_2 & \alpha_8 = \beta_1 & \end{array}$$

and
$$\alpha_2 = \frac{1}{3}(\beta_8 - \beta_1 - 2\beta_2 - 3\beta_3 - 4\beta_4 - 5\beta_5 - 3\beta_6 - \beta_7).$$

We deduce the following linear expressions relating the fundamental weights with respect to $\Delta(Y)$ to those with respect to $\Delta(X)$.

$$\begin{array}{llll} \lambda_1 = \mu_7 + \mu_8 & \lambda_2 = 3\mu_8 & \lambda_3 = \mu_6 + 3\mu_8 & \lambda_4 = \mu_5 + 5\mu_8 \\ \lambda_5 = \mu_4 + 4\mu_8 & \lambda_6 = \mu_3 + 3\mu_8 & \lambda_7 = \mu_2 + 2\mu_8 & \lambda_8 = \mu_1 + \mu_8 \end{array}$$

Let $\lambda \in X(T_Y)^+$ and write $\lambda = \sum_{i=1}^8 a_i\lambda_i$. Note that

$$(\lambda - r_1\alpha_2 + \mathbb{Z}\Phi(X)) \cap (\lambda - r_2\alpha_2 + \mathbb{Z}\Phi(X)) = \emptyset \tag{3.3}$$

for $r_1, r_2 \in \{0,1,2\}$ distinct. Hence by Lemma 3.1.2, if $a_i \geq 2$ for some $1 \leq i \leq 8$, then X acts on $L_Y(\lambda)$ with more than two composition factors. Henceforth assume $a_i \leq 1$ for all $1 \leq i \leq 8$.

By Lemma 3.1.2, the following set is nonempty.

$$\left\{\gamma = \sum_{i=1}^{8} b_i\alpha_i \mid 0 \leq b_i \leq 1,\ b_2 = 1,\ \lambda - \gamma \in \Lambda(V_Y(\lambda)),\ \text{support}(\gamma) \text{ is connected}\right\}$$

Let α be an element in this set such that $|\,\text{support}(\alpha)|$ is minimal. By (3.3), we have that $\nu = \lambda - \alpha$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. By Lemma 3.1.3, if there are $1 \leq i < j \leq 8$ such that $a_i a_j \neq 0$, then $\lambda - (22222222)_Y \in \Lambda(L_Y(\lambda))$ and by (3.3), X acts on $L_Y(\lambda)$ with more than two composition factors. Assume $\lambda = \lambda_i$ for $1 \leq i \leq 8$. Since $-1 \in W_Y$, we have that $L_Y(\lambda)$ is self-dual and so by Lemma 1.4.1, we have that $L_Y(\lambda)|_X$ is too. Now, we apply Lemma 3.3.1 in order to show that X acts on $L_Y(\lambda)$ with more than two composition factors. If $L_X(\lambda_i)$ is self-dual, then is $i = 8$. Note that $\lambda_8 = \mu_1 + \mu_8$ and $\nu = \mu_6$, hence $\lambda_8 \neq (w_0)_X \cdot \nu \neq \nu$. Therefore, X acts on $L_Y(\lambda)$ with more than two composition factors. Assume $L_X(\lambda)$ is not self-dual, that is $(w_0)_X \cdot \lambda \neq \lambda$. If $-(w_0)_X \cdot \lambda = \nu$, then $\lambda + (w_0)_X \cdot \lambda = \alpha$, that is $(w_0)_X\alpha = \alpha$. A case-by-case verification implies that $(w_0)_X \cdot \alpha = \alpha$ never holds. Therefore, we have $(w_0)_X \cdot \lambda \neq \nu$ and X acts with more than two composition factors on $L_Y(\lambda)$.

3.3.2 (D_8, E_8). — Assume $(X,Y) = (D_8, E_8)$. Up to conjugacy, the simple roots of a root system of type D_8 in $\Phi(Y)$ are given by

$$\begin{array}{llll}
\beta_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 & \beta_2 = \alpha_8 & \beta_3 = \alpha_7 & \beta_4 = \alpha_6 \\
\beta_5 = \alpha_5 & \beta_6 = \alpha_4 & \beta_7 = \alpha_2 & \beta_8 = \alpha_3
\end{array}$$

which can be written as

$$\alpha_1 = \frac{1}{2}(\beta_1 - \beta_3 - 2\beta_4 - 3\beta_5 - 4\beta_6 - 2\beta_7 - 3\beta_8)$$

and

$$\begin{array}{llll}
\alpha_2 = \beta_7 & \alpha_3 = \beta_8 & \alpha_4 = \beta_6 & \alpha_5 = \beta_5 \\
\alpha_6 = \beta_4 & \alpha_7 = \beta_3 & \alpha_8 = \beta_2. &
\end{array}$$

We deduce the following linear expressions relating the fundamental weights with respect to $\Delta(Y)$ to those with respect to $\Delta(X)$.

$$\begin{array}{llll}
\lambda_1 = 2\mu_1 & \lambda_2 = 2\mu_1 + \mu_7 & \lambda_3 = 3\mu_1 + \mu_8 & \lambda_4 = 4\mu_1 + \mu_6 \\
\lambda_5 = 3\mu_1 + \mu_5 & \lambda_6 = 2\mu_1 + \mu_4 & \lambda_7 = \mu_1 + \mu_3 & \lambda_8 = \mu_2
\end{array}$$

Let $\lambda \in X(T_Y)^+$ with $\lambda \neq 0$ and write $\lambda = \sum_{i=1}^{8} a_i\lambda_i$. Note that for $r \in \{0, 2\}$

$$(\lambda - r\alpha_2 + \mathbb{Z}\Phi(X)) \cap (\lambda - \alpha_2 + \mathbb{Z}\Phi(X)) = \emptyset. \tag{3.4}$$

By Lemma 3.1.2, the following set is nonempty.

$$\left\{\gamma = \sum_{i=1}^{8} b_i\alpha_i \mid 0 \leq b_i \leq 1,\ b_1 = 1,\ \lambda - \gamma \in \Lambda(V_Y(\lambda)),\ \text{support}(\gamma) \text{ is connected}\right\}$$

Let α be an element in this set with $|\text{support}(\alpha)|$ minimal. It is clear that $\nu = \lambda - \alpha$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

If $a_i \geq 2$ for some $1 \leq i \leq 8$ or $a_ia_j \neq 0$ for $1 \leq i < j \leq 8$, then by Lemma 3.1.2 or Lemma 3.1.3, respectively, the following set is nonempty.

$$\left\{\gamma = \sum_{i=1}^{8} 2b_i\alpha_i \mid 0 \leq b_i \leq 1,\ b_1 = 1,\ \lambda - \gamma \in \Lambda(V_Y(\lambda)),\ \text{support}(\gamma) \text{ is connected}\right\}$$

Let α' be an element in this set. Then $\alpha' \notin \Lambda(L_X(\lambda))$ and by (3.4), $\alpha' \notin \Lambda(L_X(\nu))$. Hence if $a_i \geq 2$ for some $1 \leq i \leq 8$ or $a_ia_j \neq 0$ for $1 \leq i < j \leq 8$, then X acts on $L_Y(\lambda)$ with more than composition factors. Henceforth, we assume that $\lambda = \lambda_i$ for some $1 \leq i \leq 8$.

Consider the following linear combinations of simple roots in terms of $\Delta(Y)$ and in terms of $\Delta(X)$.

$$\begin{aligned}
&(21221000)_Y = (10(-1)(-2)(-2)(-2)(-1)(-1))_X && (22442222)_Y = (1210(-1)001)_X\\
&(22242222)_Y = (1210(-1)00(-1))_X && (22244222)_Y = (1210100(-1))_X\\
&(22244422)_Y = (1212100(-1))_X && (22244442)_Y = (1232100(-1))_X
\end{aligned}$$

For $\gamma \in ((21221000)_Y, (22442222)_Y, (22242222)_Y, (22244222)_Y, (22244422)_Y, (22244442)_Y)$ and for $i \in (1, 3, 4, 5, 6, 7)$, respectively, a computer verification implies that setting $\lambda = \lambda_i$, we have $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - \gamma) = (\geq 1, 0, 0)$. Hence X acts on $L_Y(\lambda)$ with more than two composition factors for $i \in \{1, 3, 4, 5, 6, 7\}$. If $\lambda = \lambda_i$ for $i \in \{2, 8\}$, comparing dimensions yields

$$\dim(L_Y(\lambda), L_X(\lambda), L_X(\nu)) = \begin{cases} (\geq_{LB} 113243, \leq_V 15360, \leq_V 60060) & \text{if } i = 2\\ (248, 120 - 2\delta_{p,2}, 128) & \text{if } i = 8. \end{cases}$$

Therefore, we get that X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if $i = 8$ and $p \neq 2$. This completes the argument for the pair (A_2, E_6).

3.3.3 (A_7, E_7). — Assume $(X, Y) = (A_7, E_7)$. Up to conjugacy, the simple roots of a root system of type A_7 in $\Phi(Y)$ are given by

$$\begin{aligned}
&\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 && \beta_2 = \alpha_7 && \beta_3 = \alpha_6 && \beta_4 = \alpha_5\\
&\beta_5 = \alpha_4 && \beta_6 = \alpha_3 && \beta_7 = \alpha_1 &&
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\alpha_1 = \beta_7 && \alpha_3 = \beta_6 && \alpha_4 = \beta_5\\
&\alpha_5 = \beta_4 && \alpha_6 = \beta_3 && \alpha_7 = \beta_2
\end{aligned}$$

and

$$\alpha_2 = \frac{1}{2}(\beta_1 - \beta_3 - 2\beta_4 - 3\beta_5 - 2\beta_6 - \beta_7).$$

We deduce the following linear expressions relating the fundamental weights with respect to $\Delta(Y)$ to those with respect to $\Delta(X)$.

$$\lambda_1 = \mu_1 + \mu_7 \qquad \lambda_2 = 2\mu_1 \qquad \lambda_3 = 2\mu_1 + \mu_6 \qquad \lambda_4 = 3\mu_1 + \mu_5$$
$$\lambda_5 = 2\mu_1 + \mu_4 \qquad \lambda_6 = \mu_1 + \mu_3 \qquad \lambda_7 = \mu_2$$

Let $\lambda \in X(T)^+$ and write $\lambda = \sum_{i=1}^{7} a_i\lambda_i$. By a similar reasoning as in Subsection 3.3.2, if X acts on $L_Y(\lambda)$ with two composition factors, then we can assume that $\lambda = \lambda_i$ for $1 \leq i \leq 7$ and that a second composition factor for X acting on $L_Y(\lambda)$ is given by $\nu = \lambda - \alpha$, where α is an element of the following set with $|\operatorname{support}(\alpha)|$ minimal.

$$\left\{ \gamma = \sum_{i=1}^{7} b_i\alpha_i \mid 0 \leq b_i \leq 1,\ b_2 = 1,\ \lambda - \gamma \in \Lambda(V_Y(\lambda)),\ \operatorname{support}(\gamma) \text{ is connected} \right\}$$

Assume $i \in \{1, 7\}$. Comparing dimensions, we get

$$\dim(L_Y(\lambda_i), L_X(\lambda_i), L_X(\nu)) = \begin{cases} (133 - \delta_{p,2}, 63 - \delta_{p,2}, 70) & \text{if } i = 1 \\ (56, 28, 28) & \text{if } i = 7. \end{cases}$$

Hence X acts on $L_Y(\lambda_1)$ with exactly two composition factors if $i \in \{1, 7\}$. Assume $i \neq 1, 7$. Note that $-1 \in W_Y$, hence $L_Y(\lambda)$ is self-dual. Applying Lemma 3.3.1, a simple case-by-case calculation shows that $L_X(\lambda)$ is not self-dual and that $-(w_0)_X \cdot \lambda \neq \nu$, hence establishing the existence of a third composition factor for X acting on $L_Y(\lambda)$.

3.3.4 (B_4, F_4). — Assume $(X, Y) = (B_4, F_4)$. Recall that we have assumed $p \neq 2, 3, 5, 7, 11$. Up to conjugacy, the simple roots of a root system of type B_4 in $\Phi(Y)$ are given by

$$\beta_1 = \alpha_2 + 2\alpha_3 + 2\alpha_4 \qquad \beta_2 = \alpha_1 \qquad \beta_3 = \alpha_2 \qquad \beta_4 = \alpha_3$$

which can be written as

$$\alpha_1 = \beta_2 \qquad \alpha_2 = \beta_3 \qquad \alpha_3 = \beta_4 \qquad \alpha_4 = \frac{1}{2}(\beta_1 - \beta_3) - \beta_4.$$

We deduce the following linear relation between the fundamental weights with respect to $\Delta(Y)$ and those with respect to $\Delta(X)$.

$$\lambda_1 = \mu_2 \qquad \lambda_2 = \mu_1 + \mu_3 \qquad \lambda_3 = \mu_1 + \mu_4 \qquad \lambda_4 = \mu_1$$

Remark 3.3.2. Let $\lambda \in X(T_Y)^+$. Note that hypotheses 1) and 4) of Corollary 1.4.7 are satisfied. Indeed, since -1 belongs to the Weyl group of Y, we have that $L_Y(\lambda)$ is self-dual, hence $L_Y(\lambda)|_X$ is too by Lemma 1.4.1. Moreover, we check using the embedding of $\Phi(X) \subseteq \Phi(Y)$ that $e_{\beta_0} = d_0 e_{\alpha_0}$ with $d_0 \in k^*$, where β_0, α_0 denote the largest root in $\Phi(X), \Phi(Y)$, respectively.

Recall the notations introduced in (1.6), (1.7) and Notation 2.1.3. Let $\lambda \in X(T_Y)^+$ and write $\lambda = \sum_{i=1}^{4} a_i\lambda_i$. Set $\nu = \lambda - \sum_{\max\{i \mid a_i \neq 0\}}^{4} \alpha_i$ and note that $\nu \in \Lambda(V_Y(\lambda))$. Since

$$(\lambda + \mathbb{Z}\Phi(X)) \cap (\lambda - \alpha_4 + \mathbb{Z}\Phi(X)) = \emptyset,$$

we get that ν affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$.

Now, if $a_3 \geq 2$ or $a_4 \geq 2$, then $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (0022)_Y) = (\geq 1, 0, 0)$ or $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (0002)_Y) = (\geq 1, 0, 0)$, respectively. Hence X acts on $L_Y(\lambda)$ with more than two composition factors and we assume from now on that $0 \leq a_3, a_4 \leq 1$.

Let $i \in \{1,2,3\}$. If $a_i a_4 \neq 0$, then $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1112)_Y) = (\geq 1, 0, 0)$, which implies that X acts with more than two composition factors. Henceforth assume $a_i a_4 = 0$ for $1 \leq i \leq 3$.

Let $a_2 a_3 \neq 0$ (and $0 \leq a_1$). Recall that $a_3 = 1$ and $a_4 = 0$, so $\nu = \mu - \frac{1}{2}(\beta_1 - \beta_3)$. Note that $(0111)_Y = \frac{1}{2}(\beta_1 + \beta_3)$. If $a_2 \neq \frac{p-3}{2}$, then by Proposition 1.2.2 and Lemma 2.1.6, we have that $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (0111)_Y) = (2, 0, 1)$, which establishes the existence of a third composition factor for X acting on $L_Y(\lambda)$. The case $a_1 = 0$ and $a_2 = \frac{p-3}{2}$ is solved in Subsection 3.3.4.2.

Assume $a_1 a_2 a_3 \neq 0$. As in the previous paragraph, $a_3 = 1, a_4 = 0, \nu = \mu - (0011)_Y$ and moreover, we may assume $a_2 = \frac{p-3}{2}$ if X acts on $L_Y(\lambda)$ with at most two composition factors.

Let $a_1 = \frac{p+1}{2}$. By Proposition 1.2.2 and Lemma 2.1.6, we have

$$\operatorname{ch} L_Y(\lambda)_{(1121)_Y} = \lambda - (\lambda - (0110)_Y) - (\lambda - (1100)_Y).$$

Note that $\nu = \frac{p-3}{2}\mu_1 + \frac{p+1}{2}\mu_2 + \frac{p-1}{2}\mu_3$. By Proposition 1.2.2, we have $\operatorname{ch} L_X(\nu)_{(0111)_X} = \nu$, hence $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1121)_Y) = (3, 0, 2)$ and X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $a_1 \neq \frac{p+1}{2}$ and set $a = a_1$. Note that $\lambda - (1111)_Y = \nu - (0110)_X$. The computations related to the JSF of $V_Y(\lambda)$ up to $\lambda - (1111)_Y$ appear in Table 3.9. Let us determine $[V_Y(\lambda) : L_Y(B)]$ for $a = p-1$, where $B = \lambda - (1110)_Y$.

$\lambda = (a, -\frac{3}{2} + \frac{p}{2}, 1, 0)_{F_4}$	
$\operatorname{ch} L(\lambda)_{1111} = \lambda - A$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1111} = A + \delta_{a,p-1}B$	$\mathrm{JSF}(\lambda)_{1111} = A + 2\delta_{a,p-1}B$
$\mathrm{JSF}(A)_{1111} = \delta_{a,p-1}B$	$\mathrm{JSF}(A)_{1111} = \delta_{a,p-1}B$
$A = \lambda - 0110 = (a+1, -\frac{5}{2} + \frac{p}{2}, 1, 1)$	$B = \lambda - 1110 = (a-1, -\frac{3}{2} + \frac{p}{2}, 1, 1)$

Table 3.9: JSF of λ up to $\mu - 1111$

Let $a = p-1$ and $[V_Y(\lambda) : L_Y(B)] = 2-\zeta$, with $\zeta \in \{0,1\}$. We have $m_{L_Y(\lambda)}(\lambda-(1110)_Y) = 2+\zeta$. In order to compute ζ, we show that the dimension of the weight space $L_Y(\lambda)_B$ is at least 3, which will imply that it is equal to 3. Recall Theorem 1.1.9 and let us use the $k\mathscr{L}(Y)$-module structure of

$L_Y(\lambda)$. Let v^+ be a maximal vector for B_Y of weight λ and let $\{e_\alpha, f_\alpha, h_{\alpha_i}\}$ be a Chevalley basis of $\mathscr{L}(Y)$. One checks that $m_\lambda(\lambda - (1110)_Y) = 4$ and

$$\{f_1 = f_{\alpha_1} f_{\alpha_2} f_{\alpha_3} v^+, f_2 = f_{\alpha_1+\alpha_2} f_{\alpha_3} v^+, f_3 = f_{\alpha_1+\alpha_2+\alpha_3} v^+, f_4 = f_{\alpha_2+\alpha_3} f_{\alpha_1} v^+\}$$

is a basis of $V_Y(\lambda)_B$. We prove that $\{f_1, f_2, f_3\}$ is linearly independent in $L_Y(\lambda)$, which in turn implies that $\zeta = 1$. Note that we do not need to specify structure constants for the upcoming computation. Let $c_i \in k$ be such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{3.5}$$

Applying e_{α_2} to (3.5), we deduce that $c_1 = r' c_2$ with $r' \in k^*$. Note that $\{f_{\alpha_1} f_{\alpha_2} v^+, f_{\alpha_1+\alpha_2} v^+\}$ is linearly independent since $m_{L_Y(\lambda)}(\lambda - 1100) = 2$. Applying e_{α_3} to (3.5) and using the linear independence of $\{f_{\alpha_1} f_{\alpha_2} v^+, f_{\alpha_1+\alpha_2} v^+\}$, we deduce that $c_1 = 0$ and $c_2 = r'' c_3$ with $r'' \in k^*$. Therefore, the trivial solution is the only solution to (3.5) and so $\zeta = 1$. Hence $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})$ $(\lambda - (1111)_Y) = (3, 0, \leq_V 2)$, and X acts on $L_Y(\lambda)$ with more than two composition factors.

Let $a_1 a_3 \neq 0$. We may assume $a_2 = a_4 = 0$ and $a_3 = 1$. Note that $\mu = \mu_1 + a_1\mu_2 + \mu_4$, $\nu = a_1\mu_2 + \mu_3$ and that $\lambda - (0121)_Y = \nu - (0011)_X$. By Proposition 1.2.2, we have $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})$ $(\lambda - (0121)_Y) = (2, 0, 1)$ and X acts with more than two composition factors.

Assume $a_1 a_2 \neq 0$. Let $a_1 + a_2 \neq p - 1$. Note that $\lambda - (1111)_Y = \nu - \beta_2$. By Proposition 1.2.2 and the JSF, we have $\operatorname{ch} L_Y(\lambda)_{(1111)_Y} = \lambda$, hence $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1111)_Y) = (2, 0, 1)$ and X acts with more than two composition factors.

Let $a_1 + a_2 = p - 1$. Assume $a_2 \geq 2$. Note that $(0222)_Y = (1010)_X$. Using Proposition 1.2.2 and the JSF, we get that if $a_1 \neq \frac{p-1}{2}$, then $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1010)_X) = (2, \leq_V 1, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. The case $a_1 = \frac{p-1}{2}$ is solved in Subsection 3.3.4.3.

Now, assume $a_2 = 1$ and $a_1 = p - 2$. Note that $(1222)_Y = (1110)_X$ and $\lambda = \mu_1 + (p-2)\mu_2 + \mu_3$. The weight $\lambda - (1222)_Y$ is conjugate to the dominant weight $\lambda - (1220)_Y$. Using Proposition 1.2.2 and the JSF, we have $\operatorname{ch} L_Y(\lambda)_{(1220)_Y} = \lambda - (\lambda - (1100)_Y)$, hence $m_{L_Y(\lambda)}(\lambda - (1220)_Y) = 3$. Moreover, using the JSF and Lemma 2.1.8, we get that $\operatorname{ch} L_X(\lambda)_{(1110)_X} = \lambda - (\lambda - (1100)_X) - (\lambda - (0110)_X)$. Therefore, we get that $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1110)_X) = (3, 2, 0)$ and X acts with more than two composition factors.

We now consider the cases $\lambda = a_i \lambda_i$ for $1 \leq i \leq 4$ and $0 \leq a_i \leq 1$ if $i \in \{3, 4\}$. Assume $\lambda = \lambda_i$ for $1 \leq i \leq 4$. Comparing the dimensions appearing below implies that X acts on $L_Y(\lambda)$ with exactly two composition factors if and only if $i = 1$.

$$\dim(L_Y(\lambda_i), L_X(\lambda_i), L_X(\nu)) = \begin{cases} (52, 36, 16) & \text{if } i = 1 \\ (\geq_{LB} 1222, \leq_V 594, \leq_V 432) & \text{if } i = 2 \\ (273, 128, 84) & \text{if } i = 3 \\ (26, 9, 16) & \text{if } i = 4 \end{cases}$$

Let $a_2 \neq 0$ and $a_2 \geq 2$. Recall $\nu = (a_2 - 1)(\mu_1 + \mu_3) + \mu_2 + \mu_4$. Assume $a_2 \neq \frac{p-1}{2}$. Note that $(0222)_Y = (1010)_X$. The weights in $\Lambda(V_Y(\lambda))^+$ greater than $\lambda - (0222)_Y$ are given by $\lambda - \alpha$ for $\alpha \in \{(0100), (0200), (0110)_Y, (0210)_Y, (0220)_Y, (0221)_Y\}$. Since for $\alpha \neq (0220)_Y, (0221)_Y$, the multiplicity of $\lambda - \alpha$ in $V_Y(\lambda)$ is 1, the weight $\lambda - \alpha$ does not afford the highest weight of a composition factor for $V_Y(\lambda)$. For $\alpha = (0220)_Y$, we apply Proposition 1.2.2 in order to deduce that $\lambda - \alpha$ does not afford the highest weight of a composition factor for $V_Y(\lambda)$. For $\alpha = (0221)_Y$, we apply Proposition 1.2.2 along with the JSF in order to deduce that $\lambda - \alpha$ does not afford the highest weight of a composition factor for $V_Y(\lambda)$ either. Hence, $\operatorname{ch} L_Y(\lambda)_{(0222)_Y} = \lambda$. Therefore, $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (0222)_Y) = (2, \leq_V 1, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. Assume $a_2 = \frac{p-1}{2}$. Note that $\lambda - (1221)_Y = \nu - (0111)_X$. Applying Propositions 1.2.2 and 1.2.5, we get $\operatorname{ch} L_Y(\lambda)_{(1221)_Y} = \lambda - (\lambda - (0220)_Y)$ and $\operatorname{ch} L_X(\nu)_{(0111)_X} = \nu - (\nu - (0011)_X)$. Hence $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1221)_Y) = (4, 0, 3)$ which establishes the existence of a third composition factor for the action of X on $L_Y(\lambda)$.

Let $\lambda = a_1\lambda_1$ and $a_1 \geq 2$. Recall that $\nu = (a_1 - 1)\mu_2 + \mu_4$. The case $a_1 = \frac{p-3}{2}$ requires more work and is solved in Subsection 3.3.4.1. Assume $a_1 \neq \frac{p-3}{2}$. Note that $(2222)_Y = (1210)_X$ and that $\lambda - (2222)_Y$ is conjugate to the dominant weight $\lambda - (2220)_Y$. The weights in $\Lambda(V_Y(\lambda))^+$ which are greater than $\lambda - (2220)_Y$ are $\lambda - \alpha$ for $\alpha \in \{(1000)_Y, (2000)_Y, (1100)_Y, (2100)_Y, (1110)_Y, (2110)_Y\}$. Since for any α in the previous list, the multiplicity of $\lambda - \alpha$ in $V_Y(\lambda)$ is 1, the weight $\lambda - \alpha$ does not afford the highest weight of a composition factor for $V_Y(\lambda)$ by Theorem 1.1.10. Moreover, since we have assumed $a_1 \neq \frac{p-3}{2}$, we have by Proposition 1.2.2 that $\lambda - (2220)_Y$ does not afford the highest weight of a composition factor for $V_Y(\lambda)$. Hence $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1210)_X) = (3, \leq_V 2, 0)$ and X acts with more than two composition factors on $L_Y(\lambda)$.

To complete the case $B_4 < F_4$, it remains to consider the three special cases left aside in the above arguments. These are treated one by one below.

3.3.4.1 $\boldsymbol{\lambda = \frac{p-3}{2}\lambda_1}$. — Let $a = \frac{p-3}{2}$. Note that $\lambda = a\lambda_1 = a\mu_2$. Recall Remark 3.3.2 and that $\nu = \lambda - (1111)_Y = (a-1)\mu_2 + \mu_4$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. We prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. We prove that none of the following weights afford the highest weight of a composition factor for X acting on $L_Y(\lambda)$. The right-hand side of the equalities correspond to the coefficients appearing in the linear combination of the weights in terms of the fundamental weights μ_1, μ_2, μ_3 and μ_4.

$$
\begin{aligned}
\lambda - (1000)_Y &= (1, a-2, 1, 0) & \qquad \lambda - (1110)_Y &= (1, a-1, 0, 0)\\
\lambda - (1111)_Y &= (0, a-1, 0, 1) & \qquad \nu - (1000)_Y &= (1, a-3, 1, 1)\\
\nu - (1110)_Y &= (1, a-2, 0, 1) & \qquad \nu - (1111)_Y &= (0, a-2, 0, 2)\\
\nu - (1121)_Y &= (0, a-2, 1, 0) & \qquad \nu - (1231)_Y &= (0, a-1, 0, 0)
\end{aligned}
$$

We check that for each weight θ appearing in the list above, we have either $\nu - (1231)_Y \in \Lambda(L_X(\theta))$ or $\nu - (1110)_Y \in \Lambda(L_X(\theta))$. Hence by Lemma 1.4.9, it is enough to consider $\nu - (1110)_Y$ and $\nu - (1231)_Y$. In fact, the reader should keep in mind that $\nu - (1231)_Y = \lambda - (2342)_Y \npreceq_X \lambda$ and that $\nu - (1110)_Y \npreceq_X \nu$. Therefore by Corollary 1.4.7, we have that X acts on $L_Y(\lambda)$ with exactly two composition factors if

$$m_{L_Y(\lambda)}(\nu - (1231)_Y) = m_{L_X(\lambda)}(\lambda - (2342)_Y)$$

and

$$m_{L_Y(\lambda)}(\nu - (1110)_Y) = m_{L_X(\nu)}(\nu - (1110)_Y).$$

Applying the JSF to $V_Y(\lambda)$ yields $\operatorname{ch} L_Y(\lambda)_{(2342)_Y} = \lambda - (\lambda - (2220)_Y)$. Observe that $(1110)_Y = (0111)_X$ and $(2342)_Y = (1222)_X$. Applying the JSF to $V_X(\mu)$ and $V_X(\nu)$ yields $\operatorname{ch} L_X(\lambda)_{(1222)_X} = \lambda - (\lambda - (0222)_X)$ and $\operatorname{ch} L_X(\nu)_{(0111)_X} = \nu - (\nu - (0111)_X)$. Hence $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (2221)_Y) = (2, 0, 2)$ and $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (2342)_Y) = (5, 5, 0)$, which proves the result.

3.3.4.2 $\boldsymbol{\lambda = \frac{p-3}{2}\lambda_2 + \lambda_3}$. — Let $a = \frac{p-3}{2}$. Note that $\lambda = a\lambda_2 + \lambda_3 = (a+1)\mu_1 + a\mu_3 + \mu_4$ Recall that $\nu = \lambda - (0011)_Y = a\mu_1 + (a+1)\mu_3$. We prove that X acts on $L_Y(\lambda)$ with exactly two composition factors by applying Corollary 1.4.7. The hypotheses of Corollary 1.4.7 hold by Remark 3.3.2. We prove that none of the following weights afford the highest weight of a composition factor for X acting on $L_Y(\lambda)$. The right-hand side of the equalities correspond to the coefficients appearing in the linear combination of the weights in terms of the fundamental weights μ_1, μ_2, μ_3 and μ_4.

$$\begin{array}{ll}
\lambda - (0100)_Y = (a+1, 1, a-2, 3) & \lambda - (0110)_Y = (a+1, 1, a-1, 1) \\
\lambda - (0011)_Y = (a, 0, a+1, 0) & \lambda - (0111)_Y = (a, 1, a-1, 2) \\
\lambda - (0121)_Y = (a, 1, a, 0) & \lambda - (1220)_Y = (a+2, 0, a-1, 1) \\
\lambda - (0122)_Y = (a-1, 1, a, 1) & \lambda - (1221)_Y = (a+1, 0, a-1, 2) \\
\lambda - (1231)_Y = (a+1, 0, a, 0) & \lambda - (1222)_Y = (a, 0, a-1, 3) \\
\lambda - (1232)_Y = (a, 0, a, 1) & \lambda - (1342)_Y = (a, 1, a-1, 1) \\
\nu - (0122)_Y = (a-2, 1, a+1, 0) & \nu - (1222)_Y = (a-1, 0, a, 2) \\
\nu - (1232)_Y = (a-1, 0, a+1, 0) & \nu - (1342)_Y = (a-1, 1, a, 0)
\end{array} \tag{3.6}$$

We check that for each weight θ appearing in the list above, we have either $\lambda - (1342)_Y \in \Lambda(L_X(\theta))$ or $\nu - (1342)_Y \in \Lambda(L_X(\theta))$. By Lemma 1.4.9, it is enough to consider $\lambda - (1342)_Y$ and $\nu - (1342)_Y$. The reader should keep in mind that $\lambda - (1342)_Y \not\preceq_X \lambda$ and $\nu - (1342)_Y \not\preceq_X \nu$. Now, we have that X acts on $L_Y(\lambda)$ with exactly two composition factors if

$$m_{L_Y(\lambda)}(\lambda - (1342)_Y) = m_{L_X(\lambda)}(\lambda - (1342)_Y)$$

and

$$m_{L_Y(\lambda)}(\nu - (1342)_Y) = m_{L_X(\nu)}(\nu - (1342)_Y).$$

The partial characters $\operatorname{ch} L_Y(\lambda)_{(1353)_Y}$ and $\operatorname{ch} L_X(\lambda)_{(1122)_X}$ are computed in Tables 3.10 and 3.11.

Observe that $\lambda - (1342)_Y = \lambda - (1122)_X$ and $\nu - (1342)_Y = \nu - (1122)_X$. In order to obtain $\operatorname{ch} L_X(\mu)_{(1122)_X}$, we need to solve the problematic case $[V_X(\lambda) : L_X(C)]$ appearing in Table 3.11. Note that $\lambda - (1332)_Y = \lambda - (1121)_X$ and $\lambda - (1332)_Y$ cannot afford the highest weight of a third composition factor generated by a maximal vector for $\mathscr{L}(B_X)$, since it is not listed in (3.6). We check by computing multiplicities in $L_Y(\lambda)$ and $L_X(\lambda)$ that there is no weight apart from λ in $\Lambda(V_Y(\lambda))^+$ greater than $\lambda - (1121)_X$ (with respect to the order involving the roots in $\Delta(X)$), which affords the highest weight of a composition factor for X acting on $L_Y(\lambda)$. Reasoning as in Subsection 2.5.1.3, we can assume that $[V_X(\lambda) : L_X(C)] = 1$. Computing the multiplicity of $\lambda - (1342)_Y$ in $L_Y(\lambda)$ and $L_X(\lambda)$ yields $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1342)_Y) = (5, 5, 0)$.

$\lambda = (0, a, 1, 0)_{F_4}$	
ch $L(\lambda)_{1353} = \lambda - A + B + C$	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1353} = A - B - C$	$\mathrm{JSF}(\lambda)_{1353} = A$
$\mathrm{JSF}(A)_{1353} = B + C$	$\mathrm{JSF}(A)_{1353} = B + C$
$A = \lambda - 0110 = (1, a-1, 1, 1)$	$C = \lambda - 0221 = (2, a-2, 2, 0)$
$B = \lambda - 1330 = (1, a-2, 1, 3)$	

Table 3.10: JSF of λ up to $\mu - 1353$

$\lambda = (a+1, 0, a, 1)_{B_4}$	
ch $L(\lambda)_{1122} = \lambda - A - B + C$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1122} = A + B$	$\mathrm{JSF}(\lambda)_{1122} = A + B + 2C$
$\mathrm{JSF}(A)_{1122} = C$	$\mathrm{JSF}(A)_{1122} = C$
$\mathrm{JSF}(B)_{1122} = C$	$\mathrm{JSF}(B)_{1122} = C$
$A = \lambda - 1110 = (a, 0, a-1, 3)$	$C = \lambda - 1121 = (a, 1, a-2, 3)$
$B = \lambda - 0011 = (a+1, 1, a-1, 1)$	

Table 3.11: JSF of λ up to $\mu - 1122$

Applying the JSF to $V_X(\nu)$ yields $\operatorname{ch} L_X(\nu)_{(1122)_X} = \nu - (\nu - (0022)_X) - (\nu - (1110)_X)$, hence $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (1353)_Y) = (5, 0, 5)$. Therefore, we have that X acts on $L_Y(\lambda)$ with exactly two composition factors.

3.3.4.3 $\boldsymbol{\lambda = \frac{p-1}{2}(\lambda_1 + \lambda_2)}$. — Let $a = \frac{p-1}{2}$ and note that $\lambda = a(\lambda_1 + \lambda_2) = a(\mu_1 + \mu_2 + \mu_3)$. We apply Corollary 1.4.7 in order to prove that X acts on $L_Y(\lambda)$ with exactly two composition factors. Recall that $\nu = \lambda - (0111)_Y = (a-1)\mu_1 + (a+1)\mu_2 + (a-1)\mu_3 + \mu_4$. We prove that none of the weights listed below afford the highest weight of a composition factor for X acting on $L_Y(\lambda)$. The right-hand side of the equalities correspond to the coefficients appearing in the linear combination of the weights in terms of the fundamental weights μ_1, μ_2, μ_3 and μ_4.

$$\begin{array}{ll}
\lambda - (1000)_Y = (a+1, a-2, a+1, 0) & \lambda - (0100)_Y = (a, a+1, a-2, 2) \\
\lambda - (0110)_Y = (a, a+1, a-1, 0) & \lambda - (1100)_Y = (a+1, a-1, a-1, 2) \\
\lambda - (1110)_Y = (a+1, a-1, a, 0) & \lambda - (0111)_Y = (a-1, a+1, a-1, 1) \\
\lambda - (1111)_Y = (a, a-1, a, 1) & \lambda - (1220)_Y = (a+1, a, a-1, 0) \\
\lambda - (0122)_Y = (a-2, a+1, a, 0) & \lambda - (1221)_Y = (a, a, a-1, 1)
\end{array}$$

$$\lambda-(1122)_Y=(a-1,a-1,a+1,0) \qquad \lambda-(1222)_Y=(a-1,a,a-1,2)$$
$$\lambda-(1232)_Y=(a-1,a,a,0) \qquad \lambda-(1342)_Y=(a-1,a+1,a-1,0)$$
$$\nu-(0100)_Y=(a-1,a+2,a-3,3) \qquad \nu-(0001)_Y=(a-2,a+1,a-1,2)$$
$$\nu-(1100)_Y=(a,a,a-2,3) \qquad \nu-(0110)_Y=(a-1,a+2,a-2,1)$$
$$\nu-(0111)_Y=(a-2,a+2,a-2,2) \qquad \nu-(0121)_Y=(a-2,a+2,a-1,0)$$
$$\nu-(1220)_Y=(a,a+1,a-2,1) \qquad \nu-(0122)_Y=(a-3,a+2,a-1,1)$$
$$\nu-(1221)_Y=(a-1,a+1,a-2,2) \qquad \nu-(1122)_Y=(a-2,a,a,1)$$
$$\nu-(1222)_Y=(a-2,a+1,a-2,3) \qquad \nu-(1232)_Y=(a-2,a+1,a-1,1)$$
$$\nu-(1342)_Y=(a-2,a+2,a-2,1)$$

We check that each weight θ in the list above, we have either $\lambda-(1342)_Y\in\Lambda(L_X(\theta))$ or $\nu-(1342)_Y=\lambda-(1453)_Y\in\Lambda(L_X(\theta))$. Hence, by Lemma 1.4.9, in order to prove that X acts on $L_Y(\lambda)$ with exactly two composition factors, we need to show that

$$m_{L_Y(\lambda)}(\lambda-(1342)_Y)=m_{L_X(\lambda)}(\lambda-(1342)_Y),$$

and

$$m_{L_Y(\lambda)}(\lambda-(1453)_Y)=m_{L_X(\nu)}(\lambda-(1453)_Y).$$

Note that $(1342)_Y=(1122)_X$. The partial JSF applied to $V_X(\lambda)$ up to $\lambda-(1122)_X$ yields

$$\operatorname{ch}L_X(\lambda)_{(1122)_X}=\lambda-(\lambda-(1100)_X)-(\lambda-(0110)_X)-(\lambda-(0022)_X).$$

The partial JSF of ν up to $\nu-(1122)_X$ is computed in Table 3.12. There is a problematic case,

$\nu=(a-1,a+1,a-1,1)_{B_4}$	
$\operatorname{ch}L(\nu)_{1122}=\nu-A-B-C$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\nu)_{1122}=A+B+C$	$\mathrm{JSF}(\nu)_{1122}=A+B+C+2D$
$\mathrm{JSF}(B)_{1122}=D$	$\mathrm{JSF}(B)_{1122}=D$
$\mathrm{JSF}(C)_{1122}=D$	$\mathrm{JSF}(C)_{1122}=D$
$A=\nu-1100=(a-2,a,a,1)$	$C=\nu-0011=(a-1,a+2,a-2,1)$,
$B=\nu-0110=(a,a,a-2,3)$	$D=\nu-0121=(a,a+1,a-3,3)$

Table 3.12: JSF of ν up to $\nu-1122$

namely $\nu-(0121)_X$. We solve it using Proposition 1.3.9 and working in the Levi factor L_I of the parabolic subgroup P_I of X of type B_3, where $I=\{\beta_2,\beta_3,\beta_4\}$. For a weight $\theta\in X(T_Y)^+$, we denote by θ_{B_3} the restriction of θ to $T_{L_I'}$. Set $w=s_0s_1s_2s_3$ and $\nu_0=(\frac{p-7}{2},0,1)$. Note that ν_0 lies in the interior of the fundamental alcove. We have $\nu=ws_2s_0\bullet\nu_0$, $B_{B_3}=ws_0\bullet\nu_0$, $C_{B_3}=$

$ws_2 \bullet \nu_0$, $D_{B_3} = w \bullet \nu_0$. We have $B_{B_3} - D_{B_3} = 011$ and Proposition 1.3.9 implies $[V_{L_I'}(\nu_{B_3}) : L_{L_I'}(D_{B_3})] = [V_{L_I'}(C_{B_3}) : L_{L_I'}(D_{B_3})]$. By Table 3.12, we have $[V_{L_I'}(C_{B_3}) : L_{L_I'}(D_{B_3})] = 1$, which settles the problematic case.

The result of the partial JSF applied to $V_Y(\lambda)$ appears in Table 3.13. We need to solve

$\lambda = (a, a, 0, 0)_{F_4}$	
$\mathrm{ch}\, L(\lambda)_{1453} = \lambda - A - B + C - D + E - F$	
See argument	
JSF in Weyl characters:	JSF in irreducible characters:
$\mathrm{JSF}(\lambda)_{1453} = A + B - C + D$	$\mathrm{JSF}(\lambda)_{1453} = A + B + 2E +_0^1 F$
$\mathrm{JSF}(A)_{1453} = E - F$	$\mathrm{JSF}(A)_{1453} = E$
$\mathrm{JSF}(B)_{1453} = C - D + E$	$\mathrm{JSF}(B)_{1453} = C + E + 2F$
$\mathrm{JSF}(C)_{1453} = D + F$	$\mathrm{JSF}(C)_{1453} = D + F$
$\mathrm{JSF}(E)_{1453} = F$	$\mathrm{JSF}(E)_{1453} = F$
$A = \lambda - 1100 = (a-1, a-1, 2, 0)$	$D = \lambda - 0452 = (a+4, a-3, 0, 1)$,
$B = \lambda - 0220 = (a+2, a-2, 0, 2)$	$E = \lambda - 1320 = (a+1, a-3, 2, 2)$,
$C = \lambda - 0331 = (a+3, a-3, 1, 1)$	$F = \lambda - 1431 = (a+2, a-4, 3, 1)$

Table 3.13: JSF of λ up to $\mu - 1453$

the problematic case $[V_Y(\lambda) : L_Y(E)]$. Let $[V_Y(\lambda) : L_Y(E)] = 2 - \zeta$ with $\zeta \in \{0, 1\}$. We have $m_{L_Y(\lambda)}(\lambda - (1342)_Y) = 2 + 3\zeta$ and $m_{L_X(\lambda)}(\lambda - (1342)_Y) = 5$. Hence $m_{L_Y(\lambda)}(\lambda - (1342)_Y)$ is at least 5, which implies that $\zeta = 1$ and both multiplicities agree.

Moreover, $m_{L_Y(\lambda)}(\lambda - (1453)_Y) = 5$ and $m_{L_X(\lambda)}(\lambda - (1453)_Y) = 5$, which proves that X acts on $L_Y(\lambda)$ with exactly two composition factors.

3.3.5 (A_2, G_2). — Let $(X, Y) = (A_2, G_2)$ and recall we have assumed $p \geq 5$. Up to conjugacy, the simple roots of a root system of type A_2 in $\Phi(Y)$ are given by

$$\beta_1 = 3\alpha_1 + \alpha_2 \quad \beta_2 = \alpha_2$$

which can be written as

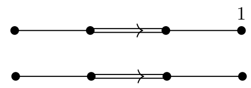
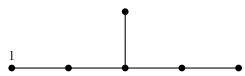
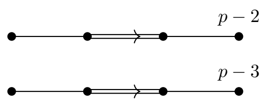
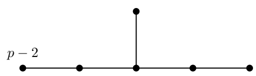
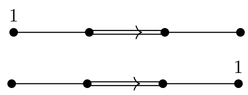
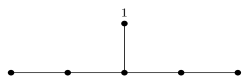
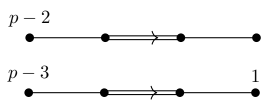
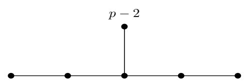
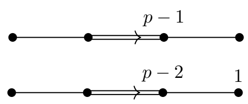
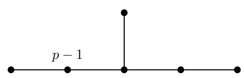
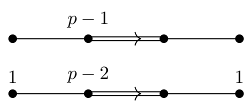
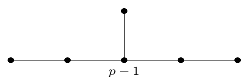
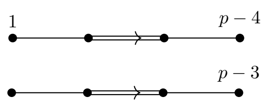
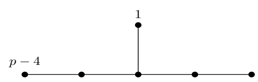
$$\alpha_1 = \frac{1}{3}(\beta_1 - \beta_2) \quad \alpha_2 = \beta_2.$$

We deduce the following linear expressions relating the fundamental weights with respect to $\Delta(Y)$ to those with respect to $\Delta(X)$.

$$\lambda_1 = \mu_1 \quad \lambda_2 = \mu_1 + \mu_2$$

Let $\lambda \in X(T_Y)^+$ with $\lambda = a_1\lambda_1 + a_2\lambda_2 = (a_1 + a_2)\mu_1 + a_2\mu_2$. If $a_1 \neq 0$, then $\nu = \lambda - (10)_Y$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Moreover, $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})(\lambda - (21)_Y) = (\geq 1, 0, 0)$ and X acts on $L_Y(\lambda)$ with more than two composition factors. So assume $a_1 = 0$ and $a_2 \neq 0$. Then $\nu = \lambda - (11)_Y$ affords the highest weight of a second composition factor for X acting on $L_Y(\lambda)$. Moreover, $(m_{L_Y(\lambda)}, m_{L_X(\lambda)}, m_{L_X(\nu)})$ $(\lambda - (21)_Y) = (\geq 1, 0, 0)$, hence X acts on $L_Y(\lambda)$ with more than two composition factors.

Table A

(X, Y)	$L(\lambda)\|_X = L_X(\mu)/L_X(\nu)$	$\lambda \in X(T_Y)^+$	p
(F_4, E_6)			
(F_4, E_6)			
(F_4, E_6)			
(F_4, E_6)			
(F_4, E_6)			
(F_4, E_6)			
(F_4, E_6)			

(F_4, E_6)

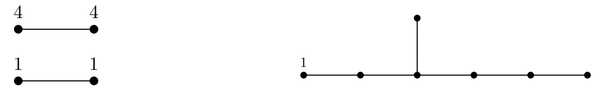
(F_4, E_6)

(F_4, E_6)

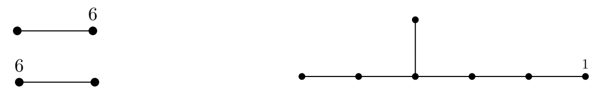
(G_2, E_6)

$p \neq 3$

(A_2, E_7)



(A_2, E_7)



$p \neq 5$

(G_2, F_4)

(C_4, E_6)

$p \neq 3$

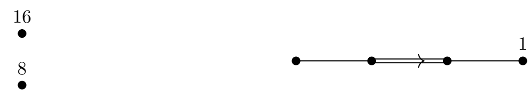
(C_4, E_6)

$p = 3$

(A_1, G_2)

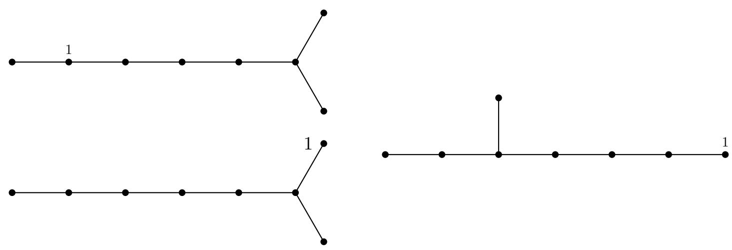
$p \neq 7$

(A_1, F_4)



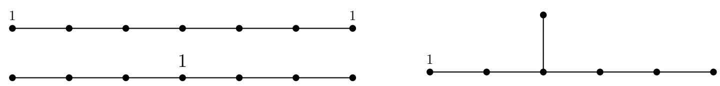
$p \neq 13$

(D_8, E_8)

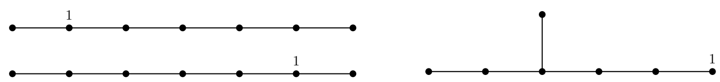


$p \neq 2$

(A_7, E_7)



(A_7, E_7)



(B_4, F_4)

(B_4, F_4)

(B_4, F_4)

(B_4, F_4)

Chapter B

Root system data

B.1 Fixing an ordering on the set of positive roots

Type	Roots
A_4	$(1000, 0100, 0010, 0001, 1100, 0110, 0011, 1110, 0111, 1111)$
B_3	$(100, 010, 001, 110, 011, 111, 012, 112, 122)$
C_3	$(100, 010, 001, 110, 011, 111, 021, 121, 221)$

Table B.1: Fixing an ordering on the set of positive roots

References

[And80] H. H. Andersen. The strong linkage principle. *Journal für die reine und angewandte Mathematik*, 315:53–59, 1980. doi:10.1515/crll.1980.315.53.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[BdS49] A. Borel and J. de Siebenthal. Les sous-groupes fermés de rang maximum des groupes de Lie clos. *Commentarii Mathematici Helvetici*, 23:200–221, 1949. doi:10.1007/bf02565599.

[BGMT15] T. C. Burness, S. Ghandour, C. Marion, and D. M. Testerman. *Irreducible almost simple subgroups of classical algebraic groups*, volume 236 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2015. doi:10.1090/memo/1114.

[BGT16] T. C. Burness, S. Ghandour, and D. M. Testerman. *Irreducible geometric subgroups of classical algebraic groups*, volume 1113 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2016. doi:10.1090/memo/1130.

[Bou81] N. Bourbaki. *Groupes et algèbres de Lie*. Masson, 1981.

[BT17] T. C. Burness and D. M. Testerman. Irreducible subgroups of simple algebraic groups - a survey. In *Proceedings of Groups St Andrews*, 2017.

[Cav15] M. Cavallin. *Restricting representations of classical algebraic groups to maximal subroups*. PhD thesis, Ecole Polytechnique Fédérale de Lausanne, 2015. `https://infoscience.epfl.ch/record/211110`.

[Cav17a] M. Cavallin. An algorithm for computing weight multiplicities in irreducible modules for complex semisimple Lie algebras. *Journal of Algebra*, 471:492–510, 2017. doi:10.1016/j.jalgebra.2016.08.044.

[Cav17b] M. Cavallin. Restriction of irreducible modules for $\mathrm{Spin}_{2n+1}(K)$ to $\mathrm{Spin}_{2n}(K)$. *Journal of Algebra*, 482:159–203, 2017. doi:10.1016/j.jalgebra.2017.03.019.

[CT19] M. Cavallin and D. M. Testerman. A new family of irreducible subgroups of the orthogonal algebraic groups. *Trans. Amer. Math. Soc.*, 6:45–79, 2019. doi:10.1090/btran/28.

[Cur60] C. W. Curtis. Representations of Lie algebras of classical type with applications to linear groups. *Indiana Univ. Math. J.*, 9(2):307–326, 1960. doi:10.1512/iumj.1960.9.59018.

[Dyn52a] E. B. Dynkin. Maximal subgroups of the classical groups. *Tr. Mosk. Mat. Obs.*, 1:39–166, 1952.

[Dyn52b] E. B. Dynkin. Semisimple subalgebras of semisimple Lie algebras. *Mat. Sbornik*, 32:349–462, 1952.

[For96] B. Ford. Overgroups of irreducible linear groups, I. *Journal of Algebra*, 181:26–69, 1996. doi:10.1006/jabr.1996.0108.

[For99] B. Ford. Overgroups of irreducible linear groups, II. *Trans. Amer. Math. Soc.*, 351:3869–3913, 1999. doi:10.1090/S0002-9947-99-02138-8.

[GAP18] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10.0*, 2018.

[Gha10] S. Ghandour. Irreducible disconnected subgroups of exceptional algebraic groups. *Journal of Algebra*, 323:2671–2709, 2010. doi:10.1016/j.jalgebra.2010.02.018.

[Hum91] J. E. Humphreys. *Linear Algebraic Groups.* Graduate Texts in Mathematics. Springer-Verlag, 1991.

[Hum00] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Graduate Texts in Mathematics. Springer, 2000.

[Jan73] J. C. Jantzen. *Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formem.* PhD thesis, Universität Bonn, 1973. Volume 67. Bonner math. Schr.

[Jan03] J. C. Jantzen. *Representations of Algebraic Groups*, volume 107 of *Mathematical Surveys and Monographs.* American Mathematical Society, 2nd edition, 2003.

[LS04] M. W. Liebeck and G. M. Seitz. *The maximal subgroups of positive dimension in exceptional algebraic groups*, volume 169 of *Memoirs of the American Mathematical Society.* American Mathematical Society, 2004. doi:10.1090/memo/0802.

[Lüb07] F. Lübeck. `http://www.math.rwth-aachen.de/Frank.Luebeck/chev/WMSmall/index.html`, 2007. accessed January 12, 2019.

[McN98] G. J. McNinch. Dimensional criteria for semisimplicity of representations. *Proc. London Math. Soc.*, 76(1):95–149, 1998. doi:10.1112/S0024611598000045.

[MT11] G. Malle and D. M. Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type.* Cambridge studies in advanced mathematics. Cambridge University Press, 1st edition, 2011.

[Pre88] A. A. Premet. Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic. *Math. USSR Sbornik*, 61(1):167–183, 1988. doi:10.1070/sm1988v061n01abeh003200.

[RW] S. Riche and G. Williamson. A simple character formula. In preparation.

[Sei87] G. M. Seitz. *The maximal subgroups of classical algebraic groups*, volume 67 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1st edition, 1987. doi:10.1090/memo/0365.

[Sei91] G. M. Seitz. *Maximal subgroups of exceptional algebraic groups*, volume 90 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1991. doi:10.1090/memo/0441.

[Spr98] T. A. Springer. *Linear Algebraic Groups*. Modern Birkhäuser Classics, 2nd edition, 1998.

[Ste63] R. Steinberg. Representations of algebraic groups. *Nagoya Math. J.*, 22:33–56, 1963. doi:10.1017/s0027763000011016.

[Ste16] R. Steinberg. *Lectures on Chevalley Groups*, volume 66 of *University Lecture Series*. AMS, 2016.

[Sup85] I. Suprunenko. Constraints of irreducible representations of the group $SL(m, K)$ on $SO(m, K)$, 1985. Russian.

[Tes88] D. M. Testerman. *Irreducible subgroups of exceptional algebraic groups*, volume 75 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1st edition, 1988. doi:10.1090/memo/0390.

[Tes89] D. M. Testerman. A construction of certain maximal subgroups of the algebraic groups E_6 and F_4. *Journal of Algebra*, 122(2):299–322, 1989. doi:10.1016/0021-8693(89)90218-4.

[Tes92] D. M. Testerman. The construction of the maximal A_1's in the exceptional algebraic groups. *Proceedings of the American Mathematical Society*, 116(3):635–644, November 1992. doi:10.1090/s0002-9939-1992-1100666-2.

[Won72] W. J. Wong. Irreducible modular representations of finite Chevalley groups. *Journal of Algebra*, 20:355–367, 1972. doi:10.1016/0021-8693(72)90063-4.

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