Extensive amenability and a Tits alternative for topological full groups

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Abstract

This dissertation investigates the amenability of topological full groups using a property of group actions called extensive amenability. Extensive amenability is a core concept of several amenability results for groups of dynamical origin. We study its properties and present some applications.

The main result of the thesis is such an application, a Tits alternative for topological full groups of minimal actions of finitely generated groups. On the one hand, we show that topological full groups of minimal actions of virtually cyclic groups are amenable. On the other hand, if G is a finitely generated not virtually cyclic group, we construct a minimal free action of G on a Cantor space such that the topological full group contains a non-abelian free group.

Keywords: *group action, amenability, extensive amenability, topological full group, Tits alternative.*

Résumé

Cette thèse étudie la moyennabilité des groupes pleins-topologiques à l'aide d'une propriété des actions de groupes, la moyennabilité extensive. La moyennabilité extensive est un concept qui se trouve au coeur de plusieurs résultats de moyennabilité pour les groupes d'origine dynamique. Nous étudions ses propriétés et en présentons des applications.

Le résultat principal de cette thèse est une alternative de Tits pour les groupes pleins-topologiques associés aux actions minimales des groupes de type fini. D'une part nous montrons que les groupes pleins-topologiques des actions minimales des groupes virtuellement cycliques sont moyennables. D'autre part, si G est un groupe de type fini non virtuellement cyclique, nous construisons une action libre et minimal de G sur un espace de Cantor dont le groupe plein-topologique contient un sous-groupe libre non abélien.

Mots-clefs: *action de groupe, moyennabilité, moyennabilité extensive, groupe plein-topologique, alternative de Tits.*

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Contents

Introduction

The subject of amenability began with the study of finitely additive measures in the early twentieth century. One of the most fundamental questions was the following: is there a good notion of *volume* on the 3-dimensional, or more generally, n-dimensional, space? Based on Hausdorff's 1914 work ([17]), Banach and Tarski proved an astonishing theorem: it is possible to cut a unit ball into five pieces that can be rearranged to form two copies of the unit ball ([2]). In other words, the unit ball admits a *paradoxical decomposition.* This result shows that there is no way to define a volume that is invariant under rotations and translations.

In 1929 John von Neumann discovered the underlying cause of the Banach-Tarski paradox. He defined a property of groups called *amenability* ([44]). A group is amenable if there exists a finitely additive measure on the group that is invariant under translation by group elements. Later Tarski proved that amenability is the only obstruction to the existence of paradoxical decompositions ([40], [41]).

The early examples of non-amenable groups all rely on the free group admitting a paradoxical decomposition. In 1957 Day formulated the conjecture, usually attributed to von Neumann, that a group is amenable if and only if it contains the free group of rank 2 as a subgroup ([9]). The conjecture was disproved decades later by Ol'shanskii ([34], [35], [36]). Today we know numerous counterexamples. To mention a few milestones: Ol'shanskii and Sapir constructed the first finitely presented ones ([37]). In 2013 Monod found a counterexample given by piecewise projective homeomorphisms of the line ([31]). Lodha and Moore found a finitely presented subgroup of Monod's group, providing the first torsion-free finitely presented counterexample ([26]).

However, there are some classes of groups for which the von Neumann-Day conjecture is true. In 1972 Jacques Tits proved that a finitely generated linear group is either virtually solvable, and therefore amenable, or it contains the free group on two generators ([42]). Since then, we say that a group G satisfies a *Tits alternative* if a similar statement holds for G: either it admits a free subgroup, or has some restrictive property, for example virtual solvability or amenability. The classes of groups satisfying a Tits alternative include hyperbolic groups ([16], [14]), mapping class groups of compact surfaces ([18], [30]), $\mathrm{Out}(F_n)$ ([4], [5]), and large subclasses of CAT(0) groups ([1], [38], [46]). In [39] the author of this dissertation established a similar result about the topological full groups of minimal Cantor group actions.

Let G be a group and consider an action of G on a compact topological space C by homeomorphisms. This action is called *minimal* if C has no proper G-invariant closed subset. The *topological full group* $[[G \curvearrowright C]]$ is the group of all homeomorphisms of C that are piecewise given by elements of G, where each piece is open in C.

Topological full groups were introduced by Giordano, Putnam and Skau [15] for $\mathbb{Z}$-actions. Matui investigated these groups in a series of papers ([27], [28], [29]). He showed that the derived subgroup of the topological full group is often simple, and in some cases, e.g. for minimal subshifts, it is also finitely generated. Nekrashevych further generalized these results in [33].

Juschenko and Monod proved that for any minimal action of $\mathbb{Z}$ on a Cantor space Σ, the topological full group $[[\mathbb{Z} \curvearrowright \Sigma]]$ is amenable ([22]). Relying on the results of Matui, they provided the first examples of finitely generated infinite simple amenable groups.

It is natural to ask whether the Juschenko-Monod theorem holds for minimal actions of other (necessarily amenable) groups as well. The result of Elek and Monod ([12]) answers this question, showing that even in the case of $\mathbb{Z}^2$ there exists a counterexample. Their result is even stronger: they construct a minimal action of $\mathbb{Z}^2$ on a Cantor space such that the topological full group contains a non-abelian free group.

The goal of this thesis is to generalize the results of [22] and [12]. We show that the Juschenko-Monod theorem holds for virtually cyclic groups and any compact space. On the other hand, if the group G is infinite but not virtually $\mathbb{Z}$, we construct a minimal action of G on a Cantor space such that the topological full group contains a non-abelian free group. These two statements together provide a Tits alternative for all finitely generated groups.

Theorem A (See Theorem 3.21.)**.** *Let G be a virtually cyclic group. Then for any minimal action of G on a compact Hausdorff topological space C by homeomorphisms, the topological full group $[[G \curvearrowright C]]$ is amenable.*

Theorem B (See Theorem 4.12.)**.** *Let G be a finitely generated group that is not virtually cyclic. Then there exists a minimal free action of G on a Cantor space Σ by homeomorphisms such that the topological full group $[[G \curvearrowright \Sigma]]$ contains a non-abelian free group.*

One of the main ingredients of the proof of Theorem A is a property of group actions called *extensive amenability*. This property was first used without an explicit definition in [22]. Later Juschenko, Nekrashevych and de la Salle applied similar methods to show the amenability of certain groups of dynamical origin ([23]). Finally, Juschenko, Matte Bon, Monod and de la Salle coined the term 'extensive amenability'. They studied this property and used it to prove the amenability of subgroups of low rank of the group of interval exchange transformations ([21]). The secondary objective of this thesis is to introduce the reader to the topic of extensively amenable group actions, and present some applications.

The structure is as follows: The first chapter introduces the basic notations, conventions, as well as some well-known results in various topics that will be necessary in subsequent chapters. In the second chapter, we define extensive amenability and study its properties in detail. This chapter is based on [21] and [23]. The third chapter deals with some applications of extensive amenability, including the proof of Theorem A. Finally, in the fourth chapter we present the proof of Theorem B.

Chapter 1

General tools

1.1 Conventions

Definition 1.1. Let (P) be a property of groups. Let us recall the following two notions.

A group G is *virtually* (P), if it has a finite index subgroup with property (P).

We call the group G *locally* (P), if all finitely generated subgroups of G have the property (P).

Notation 1.2. We will denote the set of integers by $\mathbb{Z}$, the set of non-negative integers by $\mathbb{N}$.

Notation 1.3. For integers $a, b \in \mathbb{Z}$ we will denote the interval of integers between a and b by $[a, b]$.

$$[a, b] := \{k \in \mathbb{Z} : \min(a, b) \leq k \leq \max(a, b)\}.$$

Definition 1.4. A topological space X is a *Cantor space* if X is a non-empty, compact, metrizable, totally disconnected space without isolated points.

Recall that this is equivalent to being homeomorphic to the Cantor set $2^{\mathbb{N}}$, where 2 denotes the 2-element set $\{0, 1\}$ with the discrete topology.

Definition 1.5. Let G be a finitely generated group with a finite symmetric generating set S. The *Cayley graph* of G is a graph denoted by $\mathrm{Cay}(G, S) = (V, E)$, where $V = G$ and $E = \{(g, sg) : g \in G, s \in S\}$.

We usually denote the distance in the Cayley graph by d. Then (G, d) is a metric space, we call this distance function the *word metric with respect to S*.

1.2 Group actions

An action of the group G on a set X is a homomorphism ϕ from G to $\mathrm{Sym}(X)$, the group of bijections from X to itself. We usually consider the action map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto \phi(g)(x). \end{aligned}$$

When we are only dealing with one action we will usually denote $\phi(g)(x)$ by $g \cdot x$ or simply gx. The notation for the group action will be $G \curvearrowright X$.

If X has some additional structure then usually the image of ϕ is contained in the automorphism group of X, i.e., those bijections $X \to X$ that preserve the structure of the space. We will mostly work with the following three kinds of actions.

- The general case is when X is a set and there is no additional structure on it. The image of ϕ lies in $\mathrm{Sym}(X)$.
- Many times our groups will act on topological spaces by homeomorphisms. If X is a topological space then $\mathrm{im}\phi \subseteq \mathrm{Homeo}(X)$.
- Sometimes when a group G acts on another group H, then the image of ϕ lies in the automorphism group $\mathrm{Aut}(H)$.

Definition 1.6. Let G be a group acting on two sets X, Y. Recall that a map $f\colon X \to Y$ is called a *G-map* if for every $g \in G$ and every $x \in X$, we have $f(g \cdot x) = g \cdot f(x)$.

1.2.1 Piecewise groups and topological full groups

Consider a group G acting on a set X. Let us define the *full group* of this action denoted by $[G \curvearrowright X]$. The elements of the full group are bijections on X that are piecewise given by elements of G. Formally, $\varphi \in [G \curvearrowright X]$ if $\varphi\colon X \to X$ is a bijection, and for every $x \in X$ there exists $g \in G$ such that $\varphi(x) = g \cdot x$.

We will work with an important subgroup of the full group, namely the *piecewise group* of an action, denoted by $\mathrm{PW}(G \curvearrowright X)$.

The difference from the full group is that the elements of $\mathrm{PW}(G \curvearrowright X)$ are piecewise given by finitely many elements of G. In other words, $\varphi \in \mathrm{PW}(G \curvearrowright X)$ if

$\varphi\colon X \to X$ is a bijection, and there exists a natural number $n \in \mathbb{N}$, a finite partition $X = \sqcup_{i=1}^{n} X_i$ and group elements $g_1, \ldots, g_n \in G$, such that for each $i \in [1, n]$ and $x \in X_i$, we have $\varphi(x) = g_i \cdot x$. If this holds then we call φ a *piecewise map.*

Remark 1.7. There is an equivalent characterization of piecewise maps that we will use. The bijection φ from X to itself is a piecewise G map if and only if there exists a finite set $S \subset G$, such that $\varphi(x) \in S \cdot x$, for every $x \in X$.

This is indeed equivalent, since if φ is a piecewise map, then let $S = \{g_1, \ldots, g_n\}$, where the g_i's are the group elements from the definition. Conversely, if there exists S as before, then let $S = \{s_1, \ldots, s_m\}$, and $\bar{X}_i = \{x \in X : \varphi(x) = s_i \cdot x\}$. The problem is that the $\bar{X}_i$'s might not be disjoint, so let

$$X_i = \bar{X}_i \setminus \left(\bigcup_{j=1}^{i-1} \bar{X}_j \right).$$

This way the X_i's are pairwise disjoint and $\varphi(x) = s_i \times x$ for $x \in X_i$, so φ is a piecewise G map.

This characterization can be useful at times, for example it immediately shows that the piecewise G maps form a group.

If G is a group acting on a topological space X, then we are able to define another subgroup of the full group called the *topological full group* of the action, denoted by $[[G \curvearrowright X]]$.

The elements of $[[G \curvearrowright X]]$ are piecewise given by elements of G, where each piece is open in X. This means that if $\varphi \in [[G \curvearrowright X]]$ then $\varphi\colon X \to X$ is again a bijection, and there exists an index set Ω, group elements $\{g_i\}_{i \in \Omega}$ ($g_i \in G$ for every i), and a partition $X = \sqcup_{i \in \Omega} X_i$ with each X_i open in X, such that for all $i \in \Omega$, we have $\varphi(x) = g_i \cdot x$ for every $x \in X_i$.

Notice that if G acts on a compact topological space X by homeomorphisms, then the topological full group of this action is a subgroup of the piecewise group. Indeed, due to the compactness of X, a partition to open subsets is necessarily finite. This is the reason why we usually consider actions on compact spaces.

Definition 1.8. Let (X, d) be a metric space. For a bijection $\varphi\colon X \to X$ let

$$|\varphi| = \sup_{x \in X} \{d(x, \varphi(x))\} \in \mathbb{R} \cup \{\infty\}.$$

The *wobbling group* $\mathrm{W}(X)$ is defined as the set of all bijections $\varphi\colon X \to X$ such that $|\varphi|$ is finite.

The wobbling bijections and wobbling groups served as tools to prove various non-amenability results (see [25], [11], [6]). The wobbling group of $\mathbb{Z}$ was central in the study of the topological full group of a minimal Cantor $\mathbb{Z}$-action ([22]). In [20], the authors investigated the relationship between certain properties of the metric space X, and the group structure of the wobbling group $\mathrm{W}(X)$.

Due to the following observation, in the case of group actions, we can focus our attention on piecewise groups.

Proposition 1.9. *Let G be a finitely generated group, and $S \subseteq G$ any finite symmetric generating set. Consider G as a metric space with the word metric with respect to S. Then $\mathrm{W}(G) = \mathrm{PW}(G \curvearrowright G)$, where we consider the G-action on itself by left multiplication. In particular, $\mathrm{W}(G)$ does not depend on the choice of the generating set S.*

Proof. Let $\varphi\colon G \to G$ be a piecewise G map. Then there exists $n \in \mathbb{N}$, a partition $G = \sqcup_{i=1}^{n} X_i$, and group elements $g_1, \dots, g_n \in G$, such that for $x \in X_i$ we have $\varphi(x) = g_i \cdot x$. Write each g_i as a product of generators, let L be the maximum of their lengths. Then $|\varphi| = L < \infty$, so $\varphi \in \mathrm{W}(G)$.

For the other direction, take $\varphi \in \mathrm{W}(G)$. Let $|\varphi| = D < \infty$. For an arbitrary $x \in G$, we know that $\mathrm{d}(x, \varphi(x)) \leq D$, where d denotes the word metric. Therefore, we have $\varphi(x) \in S^D \cdot x$. The set S^D is a finite subset of G, so by Remark 1.7, φ is a piecewise G map. $\square$

1.2.2 Invariant sets under group actions

Recall that the action of a group G on a topological space X is *minimal* if the only closed G-invariant subsets of X are the emptyset and X itself. Equivalently, all G-orbits are dense in X.

We will need the following lemmas about group actions.

Lemma 1.10. *Let G be a group, $H \lhd G$ a finite index normal subgroup and C a compact space. Suppose that G acts minimally on C by homeomorphisms. Then there exists a closed set $C_0 \subseteq C$, a natural number $n \in \mathbb{N}$, and group elements $g_1 = e_G, g_2, \dots, g_n \in G$, such that for every i, the set $g_i C_0$ is a minimal closed H-invariant set and*

$$C = \bigsqcup_{i=1}^{n} g_i C_0.$$

Proof. If $C = \varnothing$, then the statement holds for $C_0 = \varnothing$, so we can assume that C is non-empty.

Since C is compact, there exists a minimal, non-empty H-invariant closed subset $C_0 \subseteq C$. Now take any $g \in G$ and consider $gC_0 \subseteq C$. The subgroup H is normal in G, so gC_0 is also a minimal H-invariant closed set. Hence, either it is equal to C_0, or they are disjoint. For group elements $g, h \in G$ let us say that $g \sim h$ iff $gC_0 = hC_0$. This is an equivalence relation on G, furthermore if $g \nsim h$ then $gC_0 \cap hC_0 = \varnothing$. Since the index of H is finite, there are finitely many equivalence classes. Let n be the number of such classes. Let us choose one element from each equivalence class ($g_1 = e_G$ from its own class), denote them by $g_1, g_2, \ldots, g_n \in G$. Hence, the set

$$\bigcup_{g \in G} gC_0 = \bigsqcup_{i=1}^{n} g_i C_0$$

is non-empty, closed and G-invariant. By the minimality of the G-action, we have

$$\bigsqcup_{i=1}^{n} g_i C_0 = C.$$

This finishes the proof of the lemma. □

Lemma 1.11. *Let H be a group acting on a topological space X by homeomorphisms. Assume that we can divide the space into finitely many minimal closed H-invariant subsets, say $X = \bigsqcup_{i=1}^{n} X_i$. Now let $Y \subseteq X$ be an open or a closed H-invariant set. Then there exists $I \subseteq [1, n]$ such that $Y = \bigsqcup_{i \in I} X_i$.*

Proof. First assume that $Y \subseteq X$ is a closed H-invariant set. For $i \in [1, n]$ the subset $Y \cap X_i$ is also a closed H-invariant set that is contained in X_i. By the minimality of X_i either $Y \cap X_i = \varnothing$ or $X_i \subseteq Y$. This is true for every $i \in [1, n]$, so Y is the union of some of the X_i's as stated in the lemma.

If Y is open, then $X \setminus Y$ is a closed H-invariant set, so we can use the previous part for the complement of Y. This finishes the proof. □

1.3 Recurrence of random walks

Let $\mathcal{G} = (V, E)$ be a locally finite graph and $v_0 \in V$ a selected vertex. A *random walk* on $\mathcal{G}$ from the vertex v_0 is as follows: we start at v_0, and if at the t-th step we are at the vertex v_t, we select a neighbor of v_t at random and move to that vertex. Usually we consider a graph with edge weights (i.e., a network) and the transition probabilities from v_t are proportional to the edge weights around v_t. A special case is when all weights are equal, in other words we move to a neighbor of v_t with probability $1/\deg(v_t)$, where $\deg(v_t)$ is the degree of v_t. In this case we call our random walk the *simple random walk* on the graph $\mathcal{G}$. The sequence of random vertices $(v_t)_{t\in\mathbb{N}}$ is a Markov chain. In the case when $\mathcal{G}$ is a directed graph, we are only allowed to move along the edges in one direction.

Definition 1.12. We say that a random walk is *recurrent* if the probability of returning to the starting point is 1. Otherwise, it is said to be *transient*.

Let $\mathcal{G} = (V, E)$ be a locally finite connected graph. The *capacity* of a vertex $v \in V$ is defined by

$$c(v) = \inf \left\{ \left(\sum_{(x,y)\in E} |\theta(x) - \theta(y)|^2 \right)^{1/2} : \quad \theta\colon V \to \mathbb{R}_+ \text{ fin. supp.}, \theta(v) = 1 \right\}.$$

The following theorem characterizes recurrent random walks in terms of electrical networks.

Theorem 1.13 (Theorem 2.12. in [45]). *The simple random walk starting from the point $v \in V$ on a locally finite connected graph $\mathcal{G} = (V, E)$ is recurrent if and only if $c(v) = 0$.*

Furthermore, if there exists $v \in V$ with $c(v) = 0$, then $c(u) = 0$ for all $u \in V$. In particular, the recurrence of the simple random walk on $\mathcal{G}$ does not depend on the starting point.

Definition 1.14. Let $\mathcal{G}$ be a locally finite connected graph. Then $\mathcal{G}$ is said to be *recurrent* if the simple random walk on $\mathcal{G}$ is recurrent for one (and hence for every) starting point.

The following statement is a corollary of Theorem 1.13. (See [45], Corollary 2.15.)

Proposition 1.15. *Let $\mathcal{G}$ be a recurrent graph. Then any connected subgraph of $\mathcal{G}$ is also recurrent.*

1.3.1 Random walks on groups

Let G be a finitely generated group acting on a set X. Let μ be a symmetric measure on G, i.e., $\mu(g) = \mu(g^{-1}$ for every $g \in G$. Consider the random walk on X with transition probabilities

$$p(x, y) = \sum_{\substack{g \in G \\ gx=y}} \mu(g).$$

The recurrence of the described Markov chain does not depend on the choice of μ, as long as the measure is symmetric and its support generates the group. (See for example [45], Corollary 3.5.)

We can describe this random walk on X defined via a group action as a random walk on a graph.

Definition 1.16. Let G be a finitely generated group with a finite symmetric generating set S and consider an action of G on a set X. The *Schreier graph* $\mathrm{Sch}(X, G, S)$ is defined as follows. The vertices are points of X and the edge set is $\{(x, sx) : x \in X, s \in S\}$.

Note that if μ is a symmetric measure on G with support in S, then the above described random walk on X is identical to the random walk on the Schreier graph $\mathrm{Sch}(X, G, S)$ with edge weight $\mu(s)$ on the edges (x, sx) (for all $x \in X$). Since the recurrence of this random walk does not depend on the choice of μ, the recurrence of the graph $\mathrm{Sch}(X, G, S)$ does not depend on the choice of the finite symmetric generating set S.

Definition 1.17. Let G be a group acting transitively on a set X. The action $G \curvearrowright X$ is said to be *recurrent*, if the Schreier graph $\mathrm{Sch}(X, G, S)$ is recurrent for one (and hence for any) finite symmetric generating set S.

If the action of G on X is not necessarily transitive, we will say that $G \curvearrowright X$ is *recurrent* if the G-action is recurrent on every orbit.

Definition 1.18. Let G be a finitely generated group. We say that G is *recurrent* if the G-action on itself by left multiplication is recurrent.

The following theorem gives a complete description of recurrent groups.

Theorem 1.19 (Varopoulos, [43]). *A finitely generated group is recurrent if and only if it is finite, or virtually $\mathbb{Z}$, or virtually $\mathbb{Z}^2$.*

The following proposition is another corollary of Theorem 2.12 from [45].

Proposition 1.20. *Let G be a finitely generated recurrent group. Then for any set X and for any action of G on X the action $G \curvearrowright X$ is recurrent.*

1.4 Amenability

Definition 1.21. Let X be a set. A linear functional $\mathfrak{m}\colon \ell^\infty(X) \to \mathbb{R}$ is a *mean* if $\mathfrak{m}(\chi_X) = 1$ and it is positive, i.e., if $f \in \ell^\infty(X)$, $f \geq 0$, then $\mathfrak{m}(f) \geq 0$.

Let G be a group acting on the set X. For $f \in \ell^\infty(X)$ and $g \in G$ let $(g \cdot f)(x) = f(g^{-1}x)$ for all $x \in X$. This defines an action of G on $\ell^p(X)$ for $1 \leq p \leq \infty$.

Definition 1.22. Let G be a group acting on a set X. The group action $G \curvearrowright X$ is *amenable* if there exists a mean $\mathfrak{m}\colon \ell^\infty(X) \to \mathbb{R}$ that is invariant under the G-action, i.e., $\mathfrak{m}(g \cdot f) = \mathfrak{m}(f)$ for all $f \in \ell^\infty(X)$ and every $g \in G$.

A group G is *amenable* if the action of G on itself by left multiplication is amenable.

Amenability has various equivalent characterizations, we will use the following.

Finitely additive invariant measure. The action $G \curvearrowright X$ is amenable if and only if there exists a finitely additive probability measure $\mu\colon \mathscr{P}(X) \to [0,1]$ such that $\mu(g \cdot E) = \mu(E)$ for every $E \subseteq X$ and $g \in G$.

Notice that for a given mean $\mathfrak{m}\colon \ell^\infty(X) \to \mathbb{R}$, its restriction to characteristic functions of subsets of X is a finitely additive probability measure, that is invariant under the action of G if the mean $\mathfrak{m}$ is G-invariant. Conversely, we can linearly extend a finitely additive probability measure $\mu\colon \mathscr{P}(X) \to [0,1]$ so that it becomes a linear functional. Hence, we identify μ with its extension, and we will call both functions a *mean on X*.

Notation 1.23. If $\mathfrak{m}\colon \ell^\infty(X) \to \mathbb{R}$ is a mean, for $E \subseteq X$ we will write $\mathfrak{m}(E)$ instead of $\mathfrak{m}(\chi_E)$.
Furthermore, we will use the notation $\int_X f(x)\,\mathrm{d}\mathfrak{m}(x) = \mathfrak{m}(f)$.

Reiter's condition. The group action $G \curvearrowright X$ is amenable if and only if for every $\varepsilon > 0$ and for every finite subset $E \subset G$ there exists $f \in \ell^1(X)$ with $\|f\|_1 = 1$ and $f \geq 0$ such that for all $g \in E$ we have $\|g \cdot f - f\|_1 < \varepsilon$.

Equivalently, $G \curvearrowright X$ is amenable iff there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \ell^1(X)$ of positive unit vectors such that for every $g \in G$, we have

$$\lim_{n\to 0} \|g \cdot f_n - f_n\|_1 \longrightarrow 0.$$

The following lemmas present well-known properties of amenable group actions.

Lemma 1.24. *Every action of an amenable group is amenable.*

Proof. Let G be an amenable group acting on the set X. Take a point $x \in X$, and define $\varphi\colon G \to X$, $\varphi(g) = g \cdot x$. Let $\mathfrak{m}$ be a G-invariant mean on G, then the push-forward of $\mathfrak{m}$ by φ is a G-invariant mean on X. Hence, $G \curvearrowright X$ is amenable. □

Lemma 1.25. *Let G be a group acting on a set X. Assume that the action $G \curvearrowright X$ is amenable and that the stabilizer G_x is amenable for every $x \in X$. Then G itself is also amenable.*

Proof. For $y \in X$, the subgroup G_y acts on G by left multiplication. This action is amenable, so there exists a G_y-invariant mean on G. Let us fix a set of orbit representatives $Y \subseteq X$, and for all $y \in Y$, fix a G_y-invariant mean μ_y on G. For an arbitrary $x \in X$, there exists $y \in Y$ and $g \in G$ such that $g \cdot y = x$, let $\mu_x = g \cdot \mu_y$. This is well-defined since μ_y is G_y-invariant.

Since $G \curvearrowright X$ is amenable, there is a G-invariant mean μ on X. Let us define $\mathfrak{m}\colon \mathscr{P}(G) \to [0,1]$ as follows. For $E \subseteq X$ let

$$\mathfrak{m}(E) = \int_X \mu_x(E)\, \mathrm{d}\mu(x).$$

This map is finitely additive, since the μ_x's are finitely addditive. It is also G-invariant, since

$$\mathfrak{m}(gE) = \int_X \mu_x(gE)\, \mathrm{d}\mu(x) = \int_X \mu_{g^{-1}x}(E)\, \mathrm{d}\mu(x) = \mathfrak{m}(E).$$

(The last equality is due to the G-invariance of μ.) Hence, G is amenable. □

1.5 Group extensions and virtually $\mathbb{Z}$ groups

In Section 3.1 we will work with virtually $\mathbb{Z}$ groups.

Lemma 1.26. *Let G be a virtually $\mathbb{Z}$ group. Then there exists a finite index normal subgroup $N \triangleleft G$ that is isomorphic to $\mathbb{Z}$.*

Proof. Since G is virtually $\mathbb{Z}$, we can find a finite index subgroup $H \leq G$ that is isomorphic to $\mathbb{Z}$. Let

$$N = \bigcap_{g \in G} g^{-1} H g.$$

Since H has finitely many conjugates in G, N is a finite index subgroup of H. Therefore, N is isomorphic to $\mathbb{Z}$ and it has finite index in G. The fact that N is a normal subgroup follows from its definition. □

Thus, a virtually $\mathbb{Z}$ group is always an extension of a finite group by $\mathbb{Z}$. Consider the short exact sequence

$$\{0\} \longrightarrow \mathbb{Z} \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow \{1\},$$

where Q is a finite group. We will denote the identity element of Q by e_Q and the identity element of G by e_G. Choose a section $x \mapsto g_x$ from Q to G such that $g_{e_Q} = e_G$, so we have that $\pi(g_x) = x$ for every $x \in Q$. This defines the cocycle $f\colon Q \times Q \to \mathbb{Z}$ by the equality

$$g_x g_y = \iota(f(x,y)) g_{xy} \quad \text{for } x, y \in Q.$$

The map $\alpha\colon Q \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$, $x \mapsto \alpha_x$ defined by

$$g_x \iota(n) g_x^{-1} = \iota(n^{\alpha_x}) \quad \text{for } x \in Q \text{ and } n \in \mathbb{Z}$$

is a homomorphism. The maps f and α determine the extension.

Definition 1.27. A virtually $\mathbb{Z}$ group G is defined by a finite group Q, a cocycle $f\colon Q \times Q \to \mathbb{Z}$, and a homomorphism $\alpha\colon Q \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$. We have that $G = \mathbb{Z} \times Q$ as a set, and the multiplication is defined as follows.

$$(a, x)(b, y) = (f(x, y) + a + b^{\alpha_x}, xy) \quad \text{for } a, b \in \mathbb{Z}, x, y \in Q$$

Remark 1.28. The identity element of G is $e_G = (0, e_Q)$. Also note that the choice $g_{e_Q} = e_G$ implies that $f(e_Q, x) = f(x, e_Q) = 0$ for all $x \in Q$.

Lemma 1.29. *Let H be a subgroup of the virtually $\mathbb{Z}$ group G. Then one of the following two statements holds.*

- *H is finite and it has infinite index in G,*
- *H is infinite and it has finite index in G.*

Proof. If H is finite, then it must have infinite index, since G itself is infinite. Now assume that H is infinite. Using the notations in Definition 1.27, there exists at least one $x \in Q$ such that $H \cap (\mathbb{Z} \times \{x\})$ is infinite. Suppose that $(a, x), (b, x) \in H$ for $a \neq b$, then $(a, x)^{-1}(b, x) \in H \cap (\mathbb{Z} \times \{e_Q\})$. Since $a \neq b$, this is not the identity element. Therefore $H \cap (\mathbb{Z} \times \{e_Q\})$ is a nontrivial subgroup, say it is equal to $k\mathbb{Z} \times \{e_Q\}$. Hence $|G : H| \leq |G : (k\mathbb{Z} \times \{e_Q\})| = k|Q|$, so the index of H is finite. □

Chapter 2

Extensive amenability

In 2013 Juschenko and Monod proved the amenability of the topological full group of a minimal Cantor $\mathbb{Z}$-action ([22]). They essentially developed a new method for proving the amenability of groups. Extensive amenability of group actions is a fundamental concept in this new approach. Since then several papers investigated or used extensive amenability ([20], [23], [21], [13]).

This chapter reviews the definition and properties of extensively amenable actions, all the results we discuss here were obtained in [22], [21] and [23]. We also draw some ideas of proofs from the books [3] (Chapter 11) and [19] (Chapters 5-8).

2.1 Definition

Let G be a group acting on a set X. The finite subsets of X form an abelian group with the symmetric difference. Let us denote this group by $\mathscr{P}_f(X)$. The G-action on X gives rise to an action of G on $\mathscr{P}_f(X)$. (Note that the group $\mathscr{P}_f(X)$ is isomorphic to the direct sum $\oplus_X(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}^{(X)}$.)

Notation 2.1. For $x \in X$ we introduce the following notation for the collection of subsets containing x.

$$\mathcal{E}_x = \{E \subseteq X : x \in E, |E| < \infty\} \subset \mathscr{P}_f(X).$$

Definition 2.2. The action $G \curvearrowright X$ is *extensively amenable* if there exists a G-invariant mean $\mathfrak{m}$ on $\mathscr{P}_f(X)$ such that $\mathfrak{m}(\mathcal{E}_x) = 1$ for every $x \in X$.

Remark 2.3. Let $A \in \mathscr{P}_f(X)$. Let us define

$$\mathcal{E}_A = \{E \subseteq X : A \subseteq E, |E| < \infty\} \subset \mathscr{P}_f(X).$$

Notice that

$$\mathcal{E}_A = \bigcap_{a \in A} \mathcal{E}_a.$$

Therefore, if $\mathfrak{m}$ is a mean on $\mathscr{P}_f(X)$ that gives full weight to $\mathcal{E}_x$ for all $x \in X$, then we also have $\mathfrak{m}(\mathcal{E}_A) = 1$ for all $A \in \mathscr{P}_f(X)$.

Definition 2.4. The group $\mathscr{P}_f(X) \rtimes G$ acts on $\mathscr{P}_f(X)$ given by the formula

$$(E, g)(F) = E \triangle g(F),$$

where $E, F \in \mathscr{P}_f(X)$ and $g \in G$.

2.2 Equivalent characterizations and properties

The first proposition provides some equivalent characterizations for the extensive amenability of a transitive action.

Proposition 2.5 (Lemma 3.1 in [22].)**.** *Let G be a group acting transitively on a set X, and $x \in X$ a point. The following statements are equivalent.*

1. *$G \curvearrowright X$ is extensively amenable.*

2. *There exists a G-invariant mean on $\mathscr{P}_f(X)$ giving full weight to the set $\mathcal{E}_x = \{E \subseteq X : x \in E, |E| < \infty\}$.*

3. *There exists a G-invariant mean on $\mathscr{P}_f(X)$ giving non-zero weight to the set $\mathcal{E}_x = \{E \subseteq X : x \in E, |E| < \infty\}$.*

4. *The action $\mathscr{P}_f(X) \rtimes G \curvearrowright \mathscr{P}_f(X)$ is amenable (i.e., it admits an invariant mean).*

Proof. 2. $\Rightarrow$ 1. This direction follows from the transitivity of the action. If a G-invariant mean gives full weight to $\mathcal{E}_x$, then it also gives full weight to $\mathcal{E}_y$ for all $y \in X$.

1. $\Rightarrow$ 4. For every $E \in \mathscr{P}_f(X)$ let us define a mean μ_E on $\mathscr{P}_f(X)$ as follows. For $\mathcal{E} \subset \mathscr{P}_f(X)$ let

$$\mu_E(\mathcal{E}) = \frac{|\mathscr{P}_f(E) \cap \mathcal{E}|}{|\mathscr{P}_f(E)|} = \frac{1}{2^{|E|}} |\{F \in \mathcal{E} : F \subseteq E\}|,$$

in other words, μ_E counts the ratio of subsets of E that are contained in $\mathcal{E}$.

Since the G-action on X is extensively amenable, there exists a mean μ on $\mathscr{P}_f(X)$ giving full weight to sets of the form $\mathcal{E}_A$ for $A \in \mathscr{P}_f(X)$. We define another mean $\mathfrak{m}$ on $\mathscr{P}_f(X)$, for $\mathcal{E} \subset \mathscr{P}_f(X)$ let

$$\mathfrak{m}(\mathcal{E}) = \int_{\mathscr{P}_f(X)} \mu_E(\mathcal{E}) \, \mathrm{d}\mu(E).$$

Since $E \mapsto \mu_E$ is a G-map and μ is G-invariant, the mean $\mathfrak{m}$ is also G-invariant. For $A \in \mathscr{P}_f(X)$, let $A \triangle \mathcal{E} = \{A \triangle F : F \in \mathcal{E}\}$. Note that if $A \subseteq E$, then μ_E is invariant under the action of A, since if $F = A \triangle F'$ with $F' \in \mathcal{E}$, then $F \subseteq E$ if and only if $F' \subseteq E$. Hence, for $A \subseteq E$, we have

$$\begin{aligned}\mu_E(A \triangle \mathcal{E}) &= \frac{1}{2^{|E|}} |\{F \in A \triangle \mathcal{E} : F \subseteq E\}| \\ &= \frac{1}{2^{|E|}} |\{F' \in \mathcal{E} : F' \subseteq E\}| = \mu_E(\mathcal{E}).\end{aligned}$$

Therefore, $\mathfrak{m}$ is invariant under the action of any $A \in \mathscr{P}_f(X)$:

$$\begin{aligned}\mathfrak{m}(A \triangle \mathcal{E}) &= \int_{\mathscr{P}_f(X)} \mu_E(A \triangle \mathcal{E}) \, \mathrm{d}\mu(E) \\ &= \int_{\mathcal{E}_A} \mu_E(A \triangle \mathcal{E}) \, \mathrm{d}\mu(E) \\ &= \int_{\mathcal{E}_A} \mu_E(\mathcal{E}) \, \mathrm{d}\mu(E) \\ &= \int_{\mathscr{P}_f(X)} \mu_E(\mathcal{E}) \, \mathrm{d}\mu(E) = \mathfrak{m}(\mathcal{E}),\end{aligned}$$

since $\mu(\mathcal{E}_A) = 1$.

We proved that the mean $\mathfrak{m}$ on $\mathscr{P}_f(X)$ is G-invariant, and also $\mathscr{P}_f(X)$-invariant. This implies that $\mathfrak{m}$ is also invariant under the action of $\mathscr{P}_f(X) \rtimes G$, so the action $\mathscr{P}_f(X) \rtimes G \curvearrowright \mathscr{P}_f(X)$ is amenable.

4. $\Rightarrow$ 3. Let $\mathfrak{m}$ be a $\mathscr{P}_f(X) \rtimes G$-invariant mean on $\mathscr{P}_f(X)$. Then $\mathfrak{m}$ is G-invariant, and since it is invariant under the action of $\{x\} \in \mathscr{P}_f(X)$, we have that $\mathfrak{m}(\mathcal{E}_x) = \mathfrak{m}(\{x\} \triangle \mathcal{E}_x) = \mathfrak{m}(\mathscr{P}_f(X) \setminus \mathcal{E}_x)$. Hence, $\mathfrak{m}(\mathcal{E}_x) = 1/2$.

3. $\Rightarrow$ 2. Let μ be a G-invariant mean on $\mathscr{P}_f(X)$ such that $\mu(\mathcal{E}_x) = c > 0$. For $n \in \mathbb{N}$ let us define

$$\begin{aligned} \varphi_n \colon \mathscr{P}_f(X)^n &\longrightarrow \mathscr{P}_f(X) \\ (E_1, \ldots, E_n) &\longmapsto \bigcup_{i=1}^n E_i. \end{aligned}$$

Let $\mu^{\otimes n}$ be the nth power of the mean μ. This is a mean on $\mathscr{P}_f(X)^n$ that is invariant under the diagonal G-action. Let μ_n be the push-forward of the mean $\mu^{\otimes n}$ by the map φ_n. Since the φ_n's are G-maps, the means μ_n are all G-invariant, and we also have

$$\mu_n(\mathcal{E}_x) = 1 - \mu_n(\mathcal{E}_x^c) = 1 - \mu^{\otimes n}(\varphi_n^{-1}(\mathcal{E}_x^c)) = 1 - \mu(\mathcal{E}_x^c)^n = 1 - (1-c)^n.$$

Therefore, if $\mathfrak{m}$ is an accumulation point of the sequence $(\mu_n)_{n\in\mathbb{N}}$, then $\mathfrak{m}$ is a G-invariant mean on $\mathscr{P}_f(X)$ that gives full weight to $\mathcal{E}_x$. □

The next statement shows that, from the viewpoint of extensive amenability, it is enough to consider finitely generated groups and transitive actions.

Proposition 2.6 (Lemma 2.2 in [21].). *Let G be a group acting on a set X, then the following two statements are equivalent.*

1. *$G \curvearrowright X$ is extensively amenable.*
2. *For every finitely generated subgroup $H \leq G$ and every H-orbit $Y \subseteq X$, the action $H \curvearrowright Y$ is extensively amenable.*

Proof. 1. $\Rightarrow$ 2. Consider a finitely generated subgroup $H \leq G$ and an H-orbit $Y \subseteq X$. Let μ be a G-invariant mean on $\mathscr{P}_f(X)$ such that $\mu(\mathcal{E}_x) = 1$ for every $x \in Y$. For $\mathcal{A} \in \mathscr{P}_f(X)$ define $\mathcal{A}' = \{E \cap Y : E \in \mathcal{A}\} \in \mathscr{P}_f(Y)$, then $\mathcal{A} \mapsto \mathcal{A}'$ is an H-map. The pushforward $\mathfrak{m}$ of the mean μ to $\mathscr{P}_f(Y)$ is H-invariant, and we also have $\mathfrak{m}(\mathcal{E}_x') = 1$ for every $x \in Y$. Therefore, the action $H \curvearrowright Y$ is extensively amenable.

2. $\Rightarrow$ 1. For a finitely generated subgroup $H \leq G$ and a union $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_k$ of H-orbits let us define a mean $\mathfrak{m}_{H,Y}$ on $\mathscr{P}_f(X)$ as follows. For every $i \in [1,k]$ choose an H-invariant mean $\mathfrak{m}_i$ on $\mathscr{P}_f(Y_i)$ that gives full weight to $\mathcal{E}_y$ for all $y \in Y_i$. For $\mathcal{E} \subseteq \mathscr{P}_f(X)$ let

$$\mathfrak{m}_{H,Y}(\mathcal{E}) = \prod_{i=1}^k \mathfrak{m}_i\big(\{E \cap Y_i : E \in \mathcal{E}\}\big).$$

The mean $\mathfrak{m}_{H,Y}$ is H-invariant and gives full weight to $\mathcal{E}_x$ for every $x \in Y$.

Let us order the pairs (H, Y) by inclusion and let $\mathfrak{m}$ be a cluster point of the net $(\mathfrak{m}_{H,Y})$. Then $\mathfrak{m}$ is a G-invariant mean on $\mathscr{P}_f(X)$ that gives full weight to $\mathcal{E}_x$ for all $x \in X$. Hence, $G \curvearrowright X$ is extensively amenable. □

Proposition 2.7 (Lemma 2.1 in [21].). *Every action of an amenable group is extensively amenable.*
If X is non-empty, then every extensively amenable action on X is amenable.

Proof. First, let G be an amenable group and consider an action of G on a space X. By Proposition 2.6, we may assume that G is finitely generated and that the G-action on X is transitive. Note that $\mathscr{P}_f(X)$ is an abelian, hence also amenable group. Therefore, the semidirect product $\mathscr{P}_f(X) \rtimes G$ is also amenable, since the class of amenable groups is closed under extensions. Every action of an amenable group is also amenable, so $\mathscr{P}_f(X) \rtimes G \curvearrowright \mathscr{P}_f(X)$ is amenable. Hence, by Proposition 2.5, $G \curvearrowright X$ is extensively amenable.

For the second statement, assume that $G \curvearrowright X$ is an extensively amenable action. Let μ be a G-invariant mean on $\mathscr{P}_f(X)$ that gives full weight to the set $\mathcal{E}_x$ for each $x \in X$. Let us define $\mathfrak{m}\colon \mathscr{P}(X) \to [0, 1]$ in the following way. For $E \subseteq X$ let

$$\mathfrak{m}(E) = \int_{\mathscr{P}_f(X)\setminus\{\varnothing\}} \frac{|E \cap F|}{|F|} \,\mathrm{d}\mu(F).$$

We have $\mathfrak{m}(X) = 1$, since $X \neq \varnothing$. Furthermore, $\mathfrak{m}$ is finitely additive, so $\mathfrak{m}$ is a mean on X. Due to the G-invariance of μ, the mean $\mathfrak{m}$ is also G-invariant, so the G-action on X is amenable. □

Remark 2.8. Note that there exist amenable actions that are not extensively amenable, and non-amenable groups can act extensively amenably (see Example 2.11).

The following proposition states that extensive amenability is preserved by extensions of actions.

Proposition 2.9 (Proposition 2.4 in [21].). *Let G be a group acting on the sets X and Y, and let $q\colon X \to Y$ be a G-map. If the action $G \curvearrowright Y$ is extensively amenable and for every $y \in Y$ the stabilizer G_y acts extensively amenably on $q^{-1}(y)$, then $G \curvearrowright X$ is also extensively amenable.*
If q is surjective, then the converse also holds.

Proof. First assume that the actions $G \curvearrowright Y$ and $G_y \curvearrowright q^{-1}(y)$ are extensively amenable for every $y \in Y$. Let $T \subset Y$ be a transversal for the action of G on Y, i.e., T contains exactly one element of each G-orbit in Y. For every $y \in T$ let μ_y be a G_y-invariant mean giving full weight to the collection of sets containing any given element of $q^{-1}(y)$. For any $g \in G$ and $y \in T$, let us define μ_{gy} to be the push-forward of μ_y by g. If $gy = g'y$, then the push-forward measures by g and g' are the same by the G_y-invariance of μ_y, so μ_{gy} is well-defined.

For every finite subset $F = \{y_1, \dots, y_n\} \subseteq Y$ let us define a mean μ_F on $\mathscr{P}_f(X)$ as follows.

$$\mu_F(\mathcal{E}) = \prod_{i=1}^{n} \mu_{y_i}\big(\{E \cap q^{-1}(y_i) : E \in \mathcal{E}\}\big)$$

for $\mathcal{E} \subset \mathscr{P}_f(X)$. Note that for all $g \in G$, the push-forward of μ_F by g is exactly μ_{gF} and that μ_F gives full weight to the set $\mathcal{E}_x \subset \mathscr{P}_f(X)$ for every $x \in q^{-1}(F)$.

Now consider a G-invariant mean μ on $\mathscr{P}_f(Y)$ that gives full weight to the collection of sets containing any given element of Y. For $\mathcal{E} \subset \mathscr{P}_f(X)$ let

$$\mathfrak{m}(\mathcal{E}) = \int_{\mathscr{P}_f(Y)} \mu_F(\mathcal{E})\, \mathrm{d}\mu(F).$$

The mean $\mathfrak{m}$ on $\mathscr{P}_f(X)$ is G-invariant due to the properties of the μ_F's and μ. On the other hand, for any $x \in X$ we have

$$\mathfrak{m}(\mathcal{E}_x) = \int_{\mathscr{P}_f(Y)} \mu_F(\mathcal{E}_x)\, \mathrm{d}\mu(F) = \int_{\mathcal{E}_x} \mu_F(\mathcal{E}_x)\, \mathrm{d}\mu(F) = \int_{\mathcal{E}_x} 1\, \mathrm{d}\mu(F) = 1.$$

Therefore, the action $G \curvearrowright X$ is extensively amenable.

For the converse, assume that q is onto and that $G \curvearrowright X$ is extensively amenable. By Proposition 2.6, the action $G_y \curvearrowright q^{-1}(y)$ is also extensively amenable for every $y \in Y$. Let μ be a mean on $\mathscr{P}_f(X)$ witnessing the extensive amenability of the G-action on X. For $\mathcal{E} \subseteq \mathscr{P}_f(Y)$ define

$$\mathfrak{m}(\mathcal{E}) = \mu\big(\{E \in \mathscr{P}_f(X) : q(E) \in \mathcal{E}\}\big).$$

Then $\mathfrak{m}$ is a G-invariant mean on $\mathscr{P}_f(Y)$, and for every $y \in Y$ choose $x \in q^{-1}(y)$, then we have

$$\mathfrak{m}(\mathcal{E}_y) = \mu\big(\{E \in \mathscr{P}_f(X) : q(E) \in \mathcal{E}_y\}\big) \geq \mu(\mathcal{E}_x) = 1.$$

Hence, the action $G \curvearrowright Y$ is also extensively amenable. □

Corollary 2.10 (Corollary 2.5 in [21].). *Let $K \leq H \leq G$ be groups. Then the action $G \curvearrowright G/K$ is extensively amenable if and only if both of the actions $G \curvearrowright G/H$ and $H \curvearrowright H/K$ are extensively amenable.*

Proof. We apply Proposition 2.9 for G and $X = G/K$, $Y = G/H$ and the map $q\colon G/K \to G/H$, $gK \mapsto gH$. Then the stabilizer of $H \in G/H$ is precisely $H \leq G$, and $q^{-1}(H) = \{hK : h \in H\} = H/K$. □

Example 2.11 (Monod-Popa, [32].). The corresponding statement for amenable actions does not hold. Let Q be a non-amenable group. Define

$$G = Q \wr \mathbb{Z} = \left(\bigoplus_{n\in\mathbb{Z}} Q\right) \rtimes \mathbb{Z}$$
$$H = \bigoplus_{n\in\mathbb{Z}} Q$$
$$K = \bigoplus_{n\in\mathbb{N}} Q.$$

We claim that the actions $G \curvearrowright G/H$ and $G \curvearrowright G/K$ are amenable, but the action $H \curvearrowright H/K$ is not.

- The action $G \curvearrowright G/H$ factors through an action of $\mathbb{Z}$, so it is amenable, and also extensively amenable.
- Consider the action $G \curvearrowright G/K$. Let t denote the positive generator of $\mathbb{Z}$ in G. For $f \in \ell^\infty(G/K)$, set $\mathfrak{m}_r(f) = f(t^r K)$. Then $\mathfrak{m}_r$ is a mean that is invariant under the action of $t^{-r}Kt^r = \oplus_{n\geq -r}Q$. Since $H = \cup_{r\in\mathbb{N}}(\oplus_{n\geq -r}Q)$, a weak-* limit of the $\mathfrak{m}_r$'s is an H-invariant mean $\mathfrak{m}$ on G/K.

  Let μ be a G-invariant mean on G/H (we know that $G \curvearrowright G/H$ is amenable). For $f \in \ell^\infty(G/K)$ define $\bar{\mathfrak{m}}(f) = \mu(gH \mapsto \mathfrak{m}(g \cdot f))$. Note that the function $gH \mapsto \mathfrak{m}(g \cdot f)$ is well-defined due to the H-invariance of $\mathfrak{m}$. Then $\bar{\mathfrak{m}}$ is a G-invariant mean on G/K. Therefore, the action $G \curvearrowright G/K$ is amenable.
- Observe that the amenability of the action $H \curvearrowright H/K$ would imply that the group $\oplus_{n<0}Q$ is amenable, which contradicts the non-amenability of Q. Hence, the action $H \curvearrowright H/K$ is not amenable.

Note that $G \curvearrowright G/K$ cannot be extensively amenable due to Corollary 2.10. Therefore, this is an example of an amenable, but not extensively amenable action.

On the other hand, $G \curvearrowright G/H$ is an extensively amenable action of a non-amenable group.

Let G be a group acting on a set X. We have seen in Proposition 2.5 that the extensive amenability of the G-action on X is equivalent to the amenability of the affine action $\mathscr{P}_f(X) \rtimes G \curvearrowright \mathscr{P}_f(X)$. The next theorem (that was obtained in [21]) generalizes this equivalence by replacing the 'lamp functor' $X \mapsto \mathscr{P}_f(X) = (\mathbb{Z}/2\mathbb{Z})^{(X)}$ by other functors applied to the space X.

Let us denote the category of group actions by $\mathbf{A}$. The objects are triples (X, G, φ), where X is a set, G is a group, and $\varphi\colon G \times X \to X$ is a group action. We will omit the action from the notation and write only (X, G). A morphism $f\colon (X, G) \to (Y, H)$ is a pair of maps $f_0\colon X \to Y$, $f_1\colon G \to H$ such that $f_1(g)f_0(x) = f_0(gx)$ holds for all $x \in X$ and $g \in G$. The subcategories of amenable and extensively amenable actions will be denoted by $\mathbf{AA}$ and $\mathbf{EA}$ respectively.

Let $\mathbf{I}$ denote the category whose objects are sets and morphisms are injections, and let $\mathbf{FI}$ denote the subcategory {finite sets, injections}. We will consider functors $F\colon \mathbf{FI} \to \mathbf{AA}$, $X \mapsto (F_0(X), F_1(X))$. Since the direct limits of amenable actions are also amenable, we can extend F to a functor $\mathbf{I} \to \mathbf{AA}$ that we denote by the same letter. If F takes values in $\mathbf{EA}$, we will call it an *extensively amenable functor*.

Definition 2.12. The functor $F\colon \mathbf{I} \to \mathbf{AA}$ is called *tight* on the set X if, for every $x \in X$, no $F_1(X)$-invariant mean on $F_0(X)$ gives weight 1 to the image of $F_0(X \setminus \{x\}) \to F_0(X)$.

Consider a functor $F\colon \mathbf{I} \to \mathbf{AA}$. If G is a group acting on the set X, then the set $F_0(X)$ and the group $F_1(X)$ inherit G-actions. Hence, we can define the action $F_1(X) \rtimes G \curvearrowright F_0(X)$ as follows. For $g \in G$, $\gamma \in F_1(X)$ and $x \in F_0(X)$ let

$$(\gamma, g)(x) = \gamma(gx).$$

Let $F\colon \mathbf{I} \to \mathbf{AA}$ such that F is the extension of a functor $\mathbf{FI} \to \mathbf{AA}$, and assume that F is tight on the set X. In the next theorem, we will see that the extensive amenability of an action $G \curvearrowright X$ is equivalent to the amenability of the affine action $F_1(X) \rtimes G \curvearrowright F_0(X)$.

Theorem 2.13. *[Theorem 3.14 in [21].] Let G be a group acting on a set X, and let $F\colon \mathbf{I} \to \mathbf{AA}$ be extended from a functor $\mathbf{FI} \to \mathbf{AA}$ as described above.*

If $G \curvearrowright X$ is extensively amenable, then the action $F_1(X) \rtimes G \curvearrowright F_0(X)$ is amenable. If furthermore F is an extensively amenable functor, then $F_1(X) \rtimes G \curvearrowright F_0(X)$ is extensively amenable.

On the other hand, if $F_1(X) \rtimes G \curvearrowright F_0(X)$ is amenable and F is tight on X, then the action $G \curvearrowright X$ is extensively amenable.

Proof. First assume that $G \curvearrowright X$ is extensively amenable. Since F is a functor with image in **AA**, for every $E \in \mathscr{P}_f(X)$, the action $F_1(E) \curvearrowright F_0(E)$ is amenable. Thus, there exists a mean μ_E on $F_0(E)$ that is invariant under the action of $F_1(E)$. We extend μ_E to a mean on F_0. We can choose the means μ_E in such a way that the map $\mathscr{P}_f(X) \to \mathcal{M}(F_0(X))$, $E \mapsto \mu_E$ becomes a G-map. (This can be done by choosing μ_E to be a $F_1(X) \rtimes \mathrm{Sym}(E)$-invariant mean for one E of a fixed cardinality, then translating this to all other finite subsets of X of the same cardinality. The $\mathrm{Sym}(E)$-invariance ensures that the means do not depend on the choice of the bijections.)

Let μ be a G-invariant mean on $\mathscr{P}_f(X)$ that gives full weight to $\mathcal{E}_A$ for all $A \in \mathscr{P}_f(X)$. For every $Y \subseteq F_0(X)$ let us define

$$\mathfrak{m}(Y) = \int_{\mathscr{P}_f(X)} \mu_E(Y)\, \mathrm{d}\mu(E).$$

This $\mathfrak{m}$ is a G-invariant mean on $F_0(X)$. Let $A \in \mathscr{P}_f(X)$, then for $Y \subset F_0(X)$, we have

$$\begin{aligned}
\mathfrak{m}(F_1(A)(Y)) &= \int_{\mathscr{P}_f(X)} \mu_E(F_1(A)(Y))\, \mathrm{d}\mu(E) \\
&= \int_{\mathcal{E}_A} \mu_E(F_1(A)(Y))\, \mathrm{d}\mu(E) \\
&= \int_{\mathcal{E}_A} \mu_E(Y)\, \mathrm{d}\mu(E) \\
&= \int_{\mathscr{P}_f(X)} \mu_E(Y)\, \mathrm{d}\mu(E) = \mathfrak{m}(Y),
\end{aligned}$$

since $\mu(\mathcal{E}_A) = 1$ and μ_E is $F_1(E)$-invariant, hence it is also $F_1(A)$-invariant for $A \subseteq E$. Therefore, $\mathfrak{m}$ is $F_1(A)$-invariant for all $A \in \mathscr{P}_f(X)$, so it is $F_1(X)$-invariant. This proves that the action $F_1(X) \rtimes G \curvearrowright F_0(X)$ is amenable.

For the converse, assume that the $F_1(X) \rtimes G$-action on $F_0(X)$ is extensively amenable and F is tight on X. For $y \in F_0(X)$ let us define the support of y as follows.

$$\mathrm{supp}(y) = \cap\{E \in \mathscr{P}_f(X) : y \in \mathrm{im}(F_0(E) \to F_0(X))\}.$$

The map $\mathrm{supp}\colon F_0(X) \to \mathscr{P}_f(X)$ is a G-map. Let μ be a $F_1(X) \rtimes G$-invariant mean on $F_0(X)$, and define $\mathfrak{m}$ as the pushforward of μ by supp. Then $\mathfrak{m}$ is a G-invariant mean on $\mathscr{P}_f(X)$. Let us compute $\mathfrak{m}(\mathcal{E}_x)$ for some $x \in X$. Note that for $y \in F_0(X)$ we have $x \in \mathrm{supp}(y)$ if and only if for every $E \in \mathscr{P}_f(X)$, $y \in \mathrm{im}(F_0(E) \to F_0(X))$ implies $x \in E$. Therefore, $x \notin \mathrm{supp}(y)$ if and only if there exists $E \in \mathscr{P}_f(X)$ such that $x \notin E$ and $y \in \mathrm{im}(F_0(E) \to F_0(X))$, i.e., we have

$$\begin{aligned}\mathrm{supp}^{-1}(\mathcal{E}_x) = \{y \in F_0(X) : x \in \mathrm{supp}(y)\} =\\ = F_0(X) \setminus (\cup\{\mathrm{im}(F_0(E) \to F_0(X)) : E \in \mathscr{P}_f(X), x \notin E\}) =\\ = F_0(X) \setminus F_0(X \setminus \{x\}).\end{aligned}$$

Since F is tight on X, the set $F_0(X)\setminus F_0(X\setminus\{x\})$ has positive measure. Hence, since μ is $F_1(X)$-invariant, we have $\mathfrak{m}(\mathcal{E}_x) = \mu(\mathrm{supp}^{-1}(\mathcal{E}_x)) = \mu(F_0(X)\setminus F_0(X\setminus\{x\})) > 0$. We conclude that the action $G \curvearrowright X$ is extensively amenable.

It remains to be proved that for an extensively amenable functor F and an extensively amenable action $G \curvearrowright X$, the action $F_1(X) \rtimes G \curvearrowright F_0(X)$ is also extensively amenable.

Consider the functor $\tilde{F}_0(Y) = \mathscr{P}_f(Y)$ and $\tilde{F}_1(Y) = \mathscr{P}_f(Y)$ for a set Y. Note that we can use the first statement of the theorem for the functor $\tilde{F}$ and the action $F_1(Y) \curvearrowright F_0(Y)$, since $\tilde{F}\colon \mathbf{I} \to \mathbf{AA}$ is the extension of a functor $\mathbf{FI} \to \mathbf{AA}$, and F is an extensively amenable functor. We get that the action

$$\mathscr{P}_f(F_0(Y)) \rtimes F_1(Y) \curvearrowright \mathscr{P}_f(F_0(Y))$$

is amenable. Hence, by setting $H_0(Y) = \mathscr{P}_f(F_0(Y))$ and $H_1(Y) = \mathscr{P}_f(F_0(Y)) \rtimes F_1(Y)$ we get a functor $H\colon \mathbf{I} \to \mathbf{AA}$ that is the extension of a functor $\mathbf{FI} \to \mathbf{AA}$. Let us use the first statement of the theorem for the action $G \curvearrowright X$ and the functor H. We get that the action

$$(\mathscr{P}_f(F_0(X)) \rtimes F_1(X)) \rtimes G \curvearrowright \mathscr{P}_f(F_0(X))$$

is amenable. Notice that

$$\begin{aligned}(\mathscr{P}_f(F_0(X)) \rtimes F_1(X)) \rtimes G = \mathscr{P}_f(F_0(X)) \rtimes (F_1(X) \rtimes G) =\\ = \tilde{F}_1(F_0(X)) \rtimes (F_1(X) \rtimes G)\,.\end{aligned}$$

Now apply the converse statement to the tight functor $\tilde{F}$ and the amenable action

$$\tilde{F}_1(F_0(X)) \rtimes (F_1(X) \rtimes G) \curvearrowright \tilde{F}_0(F_0(X)).$$

This implies that the action $F_1(X) \rtimes G \curvearrowright F_0(X)$ is extensively amenable, finishing the proof of the theorem. □

As mentioned earlier, a specific example of such a functor is

$$X \mapsto \mathscr{P}_f(X) \curvearrowright \mathscr{P}_f(X) = (\mathbb{Z}/2\mathbb{Z})^{(X)} \curvearrowright (\mathbb{Z}/2\mathbb{Z})^{(X)}.$$

Let us mention a few other examples of functors to which the above result can be applied.

Let **AG** denote the category of amenable groups whose morphisms are group homomorphisms. Note that **AG** can be realized as a subcategory of AA: for an amenable group G we consider the action $G \curvearrowright G$ by left multiplication. This way the discussed result also holds for any functor $F\colon \mathbf{I} \to \mathbf{AG}$ that is the extension of a functor $\mathbf{FI} \to \mathbf{AG}$.

Example 2.14. Fix an amenable group A. Let $F\colon \mathbf{FI} \to \mathbf{AG}$ be the functor that maps a finite set X to A^X. Then the extension of F maps an arbitrary set X to $A^{(X)}$, the restricted product, since $A^{(X)}$ is the directed union of subgroups of the form A^Y, where $Y \subseteq X$ is a finite subset.

Example 2.15. For a finite set X consider the group $\mathrm{Sym}(X)$. The extension of this functor is $X \mapsto \mathrm{Sym}_f(X)$, where $\mathrm{Sym}_f(X)$ is the group of finitely supported permutations on X. We will see this example in an application to interval exchange transformations in Section 3.2.

Example 2.16. The functor $X \mapsto \mathrm{Sym}_f(X) \curvearrowright X$, $\mathbf{I} \to \mathbf{AA}$ is the extension of the functor that maps a finite set X to the action $\mathrm{Sym}(X) \curvearrowright X$.

2.3 Recurrence

In general, it can be quite difficult to determine whether a given action is extensively amenable. In this section we show that recurrence of the Schreier graph implies the extensive amenability of an action. Since recurrent actions are well-understood, this method remains the only efficient way to establish extensive amenability of an action.

Theorem 2.17 (Theorem 2 in [23].)**.** *Let G be a group acting transitively on the set X. If the graph of the action $G \curvearrowright X$ is recurrent, then the G-action on X is extensively amenable.*

Proof. Assume that the action $G \curvearrowright X$ is recurrent. Let $x_0 \in X$ be a point. By Theorem 1.13 There exist functions $\theta_n \colon X \to \mathbb{R}_+$ with finite support, such that $\theta_n(x_0) = 1$ for all $n \in \mathbb{N}$ and we have

$$\|g\theta_n - \theta_n\|_2 \longrightarrow 0$$

for every $g \in G$. We can assume that $0 \leq \theta_n(x) \leq 1$ for all $x \in X$, $n \in \mathbb{N}$.

Let us define $\xi \colon \mathscr{P}_f(X) \to \mathbb{R}_+$ as follows. For $E \in \mathscr{P}_f(X)$ let

$$\xi_n(E) = \begin{cases} 1 & \text{if } E = \varnothing \\ \prod_{x\in E} \theta_n(x) & \text{otherwise.} \end{cases}$$

Since θ_n is finitely supported, we have $\xi_n \in \ell^2(\mathscr{P}_f(X))$. Let $f_n = \xi_n / \|\xi_n\|_2$. We would like to prove that the unit vectors $f_n \in \ell^2(\mathscr{P}_f(X))$ are approximately $\mathscr{P}_f(X) \rtimes G$-invariant.

First, consider the element $\{x_0\} \in \mathscr{P}_f(X)$. We have

$$\begin{aligned} \{x_0\} \cdot f_n(E) &= \frac{1}{\|\xi_n\|_2} \xi_n(\{x_0\} \triangle E) \\ &= \frac{1}{\|\xi_n\|_2} \prod_{x \in \{x_0\} \triangle E} \theta_n(x) \\ &= \frac{1}{\|\xi_n\|_2} \prod_{x\in E} \theta_n(x) \\ &= \frac{1}{\|\xi_n\|_2} \xi_n(E) = f_n(E) \end{aligned}$$

since $\theta_n(x_0) = 1$. Hence, for all $n \in \mathbb{N}$, the vector f_n is $\{x_0\}$-invariant.

It is sufficient to show that the vectors f_n are increasingly G-invariant, then the $\mathscr{P}_f(X) \rtimes G$-invariance will follow by the transitivity of the G-action on X. Take an element $g \in G$. Our goal is to prove that $\|g \cdot f_n - f_n\|_2 \to 0$. We have

$$\|g \cdot f_n - f_n\|_2^2 = \|f_n\|_2^2 + \|g \cdot f_n\|_2^2 - 2\langle g \cdot f_n, f_n\rangle = 2 - 2\langle g \cdot f_n, f_n\rangle,$$

so it is enough to show that $\langle g \cdot f_n, f_n\rangle \to 1$. We can compute

$$\begin{aligned} \|\xi_n\|_2^2 &= \langle \xi_n, \xi_n\rangle^2 \\ &= \sum_{E\in\mathscr{P}_f(X)} \left(\prod_{x\in E} \theta_n(x)\right)^2 \\ &= \prod_{x\in X} (1 + \theta_n(x)^2) \\ &= \prod_{x\in X} (1 + \theta_n(g^{-1}x)^2) \end{aligned}$$

and

$$\langle g \cdot \xi_n, \xi_n \rangle = \sum_{E \in \mathscr{P}_f(X)} \left(\prod_{x \in E} \theta_n(g^{-1}x)\theta_n(x) \right) \\ = \prod_{x \in X} (1 + \theta_n(g^{-1}x)\theta_n(x)).$$

Therefore, we have

$$\frac{1}{\langle g \cdot f_n, f_n \rangle^2} = \left(\frac{\langle \xi_n, \xi_n \rangle}{\langle g \cdot \xi_n, \xi_n \rangle} \right)^2 \\ = \prod_{x \in X} \frac{(1 + \theta_n(x)^2)(1 + \theta_n(g^{-1}x)^2}{(1 + \theta_n(g^{-1}x)\theta_n(x))^2}.$$

Using that $\log(t) \leq t - 1$ for all $t > 0$, we get

$$0 \leq 2 \log \frac{1}{\langle g \cdot f_n, f_n \rangle} \\ = \sum_{x \in X} \log \frac{(1 + \theta_n(x)^2)(1 + \theta_n(g^{-1}x)^2)}{(1 + \theta_n(g^{-1}x)\theta_n(x))} \\ \leq \sum_{x \in X} \left(\frac{(1 + \theta_n(x)^2)(1 + \theta_n(g^{-1}x)^2)}{(1 + \theta_n(g^{-1}x)\theta_n(x))} - 1 \right) \\ = \sum_{x \in X} \frac{\theta_n(x)^2 + \theta_n(g^{-1}x)^2 - 2\theta_n(g^{-1}x)\theta_n(x)}{(1 + \theta_n(g^{-1}x)\theta_n(x))^2} \\ \leq \sum_{x \in X} (\theta_n(x) - \theta_n(g^{-1}x))^2 \\ = \|g \cdot \theta_n - \theta_n\| \longrightarrow 0.$$

Hence, we have that $\langle g \cdot f_n, f_n \rangle \to 1$ for any $g \in G$, so the vectors f_n are approximately $\mathscr{P}_f(X) \rtimes G$-invariant. Therefore, the positive unit vectors $\{f_n^2\}$ are increasingly $\mathscr{P}_f(X) \rtimes G$-invariant, so by Reiter's condition the action $\mathscr{P}_f(X) \rtimes G \curvearrowright \mathscr{P}_f(X)$ is amenable. This proves the extensive amenability of the action $G \curvearrowright X$. □

Remark 2.18. Juschenko, Matte Bon, Monod and de la Salle presented an alternate proof for Theorem 2.17, using inverted orbits ([21]).

Remark 2.19. Garban defined a stronger property, *diffuse-extensive-amenability*, and proved that it is equivalent to recurrence for actions of wobbling groups ([13]). This led him to conjecture that the action of the wobling group $\mathrm{W}(\mathbb{Z}^d) \curvearrowright \mathbb{Z}^d$ is not extensively amenable for $d \geq 3$.

In the following proposition we give a sufficient condition for the action of the piecewise group to be extensively amenable. The same statement follows from the first part of Theorem 1.4 in [20] in the case of finitely generated groups.

Proposition 2.20. *Let G be a group acting on a set X. Assume that for all finitely generated subgroups $H \leq G$ and all H-orbits $Y \subseteq X$ the graph of the action $H \curvearrowright Y$ is recurrent. Then the action of the piecewise group* $\mathrm{PW}(G \curvearrowright X)$ *on X is extensively amenable.*

Remark 2.21. By Theorem 2.17 and Proposition 2.6, it immediately follows that the action of G on X is extensively amenable. Now we prove the extensive amenability of the action of the piecewise group by verifying the recurrence assumption for all finitely generated subgroups of the piecewise group.

Proof. According to Proposition 2.6, it is sufficient to show that for a finitely generated subgroup $F \leq \mathrm{PW}(G \curvearrowright X)$, the action of F on X is extensively amenable. So let F be any finitely generated subgroup of $\mathrm{PW}(G \curvearrowright X)$. For any $\varphi \in F$, there exists a finite set of group elements $S_\varphi = \{g_1, \ldots, g_k\} \subset G$ such that $\varphi(x) \in S_\varphi \cdot x$ for all $x \in X$. Hence there exists a finitely generated subgroup $H \leq G$ such that $F \leq \mathrm{PW}(H \curvearrowright X)$.

Let $p \in X$ be an arbitrary point and let $Y \subseteq X$ denote the H-orbit of p. By Proposition 2.6, it is enough to show that the F-action on Y is extensively amenable.

Let $\mathcal{G}_F$ be the graph of the action of F on Y, i.e., $\mathrm{V}(\mathcal{G}_F) = Y$ and $\mathrm{E}(\mathcal{G}_F) = \{(y, \varphi(y)) : y \in Y,\ \varphi \in T\}$ where $T = T^{-1} \subseteq F$ is a finite generating set of F. This graph might not be connected.

Now each $\varphi \in T$ is a piecewise H map on X, so we can find a finite set $S_\varphi \subset H$ such that $\varphi(x) \in S_\varphi \cdot x$ for all $x \in Y$. Let $\hat{T} = \bigcup_{\varphi \in T} S_\varphi \subset H$ and let S be a symmetric generating set of H such that $\hat{T} \subseteq S$. Let $\mathcal{G}_H$ denote the graph of the action of H on Y with generating set S.

The vertex set of $\mathcal{G}_F$ is equal to the vertex set of $\mathcal{G}_H$. Whenever $(y, \varphi(y))$ is an edge in $\mathcal{G}_F$, there exists $g \in \hat{T}$ such that $\varphi(y) = g \cdot y$. Since $\hat{T} \subseteq S$, this implies that $(y, \varphi(y))$ is also an edge of $\mathcal{G}_H$. Hence $\mathcal{G}_F$ is a subgraph of $\mathcal{G}_H$.

By the assumption in the statement the graph $\mathcal{G}_H$ is recurrent. By Proposition 1.15 all connected subgraphs of a recurrent graph are also recurrent. In particular, all connected components of $\mathcal{G}_F$ are recurrent. Hence by Theorem 2.17 and Proposition 2.6 the F-action on Y is extensively amenable. □

Corollary 2.22. *Let G be a group such that every finitely generated subgroup of G is either virtually cyclic or virtually $\mathbb{Z}^2$. Consider an action of G on a set X. Then the action of the piecewise group* $\mathrm{PW}(G \curvearrowright X)$ *on X is extensively amenable.*

Proof. We use Proposition 2.20 for G. Consider a finitely generated subgroup $H \leq G$, we know that H is virtually cyclic or virtually $\mathbb{Z}^2$. In both cases, the group H is recurrent by Theorem 1.19, so for every action of H, the graph of the action is also recurrent by Proposition 1.20. This verifies the assumption of Proposition 2.20, so the action of $\mathrm{PW}(G \curvearrowright X)$ on X is extensively amenable. □

Chapter 3

Amenability of full groups

This chapter presents some amenability results that use extensive amenability.

In the following two statements we consider functors $F\colon \mathbf{I} \to \mathbf{AG}$. Let e denote the unit element of $F(X)$.

Proposition 3.1 (Corollary 1.4 in [21])**.** *Let $F\colon \mathbf{I} \to \mathbf{AG}$ be extended from a functor $\mathbf{FI} \to \mathbf{AG}$ and let $G \curvearrowright X$ be an extensively amenable action. Consider a subgroup $H \leq F(X) \rtimes G$. If the intersection $H \cap (\{e\} \times G)$ is amenable, then the subgroup H is also amenable.*

Proof. Let Y be the the H-orbit of e in $F(X)$. Since $F(X) \rtimes G \curvearrowright F(X)$ is extensively amenable (by Theorem 2.13), the action $H \curvearrowright Y$ is also extensively amenable, hence amenable.

On the other hand, the stabilizer of e in H is precisely the subgroup $H \cap (\{e\} \times G)$. It is amenable by assumption. Therefore, by Lemma 1.25, H is amenable. □

The following is a special case of Proposition 3.1.

Corollary 3.2. *Let $F\colon \mathbf{I} \to \mathbf{AG}$ be extended from a functor $\mathbf{FI} \to \mathbf{AG}$ and let $G \curvearrowright X$ be an extensively amenable action. Assume that there exists an embedding $i\colon G \hookrightarrow F(X) \rtimes G$ of the form $g \mapsto (c_g, g)$, such that the set $\{g \in G : c_g = e\}$ is an amenable subgroup of G. Then G is amenable.*

Proof. Use Proposition 3.1 for $H = i(G)$. Then the subgroup $i(G) \cap (\{e\} \times G)$ is exactly $\{g \in G : c_g = e\}$. □

Remark 3.3. Consider a map $c\colon G \to F(X)$ and let $i\colon G \to F(X) \rtimes G$, $g \mapsto (c_g, g)$. Then for $g, h \in G$, we have

$$(c_{gh}, gh) = i(gh) = i(g)i(h) = (c_g, g)(c_h, h) = (c_g g(c_h), gh).$$

From these calculations we can see that i is a group homomorphism if and only if c satisfies the cocycle identity.

Also note that the set $\ker c = \{g \in G : c_g = e\}$ is a subgroup of G since e is a fixed point of the action of G on $F(X)$.

Definition 3.4. The map $c\colon G \to F(X)$, $g \mapsto c_g$ from Corollary 3.2 is called a *cocycle with amenable kernel*.

Thus, we can reformulate the statement of Corollary 3.2 as follows. Let $G \curvearrowright X$ be an extensively amenable action. If there exists a cocycle $c\colon G \to F(X)$ with amenable kernel for some $F\colon \mathbf{I} \to \mathbf{AG}$ that is extended from a functor $\mathbf{FI} \to \mathbf{AG}$, then G is amenable.

The following two sections both contain an application of Corollary 3.2.

3.1 Virtually cyclic groups

In this section we will consider a virtually $\mathbb{Z}$ group G acting minimally on a compact Hausdorff topological space C by homeomorphisms. Our goal is to prove the amenability of the topological full group $[[G \curvearrowright C]]$. In order to achieve this we will construct a cocycle on $[[G \curvearrowright C]]$ with an amenable kernel.

Remark 3.5. Note that minimality is a necessary requirement. It is possible to construct a $\mathbb{Z}$-action on a Cantor space such that the topological full group contains a non-amenable free group, so it cannot be amenable. We present such an action in Example 4.6.

We will always think of G as the set $\mathbb{Z} \times Q$ with the multiplication

$$(a, x)(b, y) = (f(x, y) + a + b^{\alpha_x}, xy) \text{ for } a, b \in \mathbb{Z}, x, y \in Q,$$

as described in Definition 1.27.

3.1.1 Action on an orbit

Take a point p in the space C. Since the action of G is minimal, the orbit of p is dense. This means in particular that the stabilizer G_p has infinite index in the virtually $\mathbb{Z}$ group G, so it is finite by Lemma 1.29. Let us denote the orbit of p by X, consider the action $G \curvearrowright X$ that is the restriction of the action on C. Let us define the map

$$\varepsilon_p\colon [[G \curvearrowright C]] \hookrightarrow \mathrm{PW}(G \curvearrowright X),$$
$$\varphi \longmapsto \varphi\big|_X.$$

Since X is dense in C, the action of φ on X determines φ on C, so this map is injective.

In many cases it will be more convenient to work with the piecewise group $\mathrm{PW}(G \curvearrowright X)$ instead of the topological full group $[[G \curvearrowright C]]$.

We will use the following technical lemma and remark in the construction of our cocycle.

Lemma 3.6. *If G is a virtually $\mathbb{Z}$ group acting on a compact space C minimally, then we can find a point $p \in C$ such that the normalizer $N_G(G_p)$ is infinite.*

Proof. Suppose for contradiction that for all points $q \in C$, the normalizer $N_G(G_q)$ is finite. The stabilizer of $g \cdot q$ is gG_qg^{-1}, so the finiteness of all normalizers implies that for every q, there are only finitely many points in the orbit of q that have the same stabilizer as q, and all the others have conjugate stabilizers.

Consider the map from $\mathrm{Sub}_f(G) = \{K \leq G : K \text{ is finite}\}$ to $\mathscr{C}(C) = \{C' \subseteq C : C' \text{ is closed}\}$ that gives us the set of points stabilized by a certain subgroup. For $K \leq G$, let

$$\xi(K) = \{q \in C : g \cdot q = q \text{ for every } g \in K\}.$$

This is indeed a closed set in C, since G acts by homeomorphisms. Note that $q \in \xi(K)$ implies $K \leq G_q$. We have

$$C = \bigcup_{\substack{K=G_q \text{ for} \\ \text{some } q\in C}} \xi(K).$$

Since $\mathrm{Sub}_f(G)$ is countable, we can have at most countably many subgroups as stabilizers. So C is the union of countably many closed sets. By the Baire category theorem, at least one of these sets has non-empty interior. Let $K_0 \leq G$ be a subgroup such that $\operatorname{int}(\xi(K_0)) \neq \varnothing$.

Choose a point $q \in C$ for which $K_0 = G_q$. As we saw in the beginning of the proof, there are only finitely many points in the orbit of q that are stabilized by K_0. Hence, this orbit intersects the non-empty open set $\operatorname{int}(\xi(K_0))$ at finitely many points, so it cannot be dense in C. This contradicts the minimality of the action. □

Remark 3.7. By Lemma 3.6, we can always find a point p such that $N_G(G_p)$ is infinite, so it has finite index in the virtually $\mathbb{Z}$ group G by Lemma 1.29. Hence, its intersection with $\mathbb{Z} \times \{e_Q\}$ is $k\mathbb{Z} \times \{e_Q\}$ for some $k \in \mathbf{N}$. We also know that G_p intersects $\mathbb{Z} \times \{e_Q\}$ trivially, since G_p is finite. Hence, G_p and $k\mathbb{Z} \times \{e_Q\}$ normalize each other and their intersection is trivial, so they commute. Therefore, we can assume (perhaps by passing to a finite index subgroup of $\mathbb{Z}$) that G_p commutes with the normal subgroup $\mathbb{Z} \times \{e_Q\}$ in G. We will use this assumption later.

Definition 3.8 (Generators)**.** As before, let G be a virtually $\mathbb{Z}$ group acting transitively on the space X, and let $p \in X$ be a point and $H = G_p$ its stabilizer. Using the notations of Definition 1.27, H intersects $(\mathbb{Z} \times \{e_Q\}) \leq G$ trivially. With π denoting the projection map $G \to Q$, the subgroup $\pi(H)$ is isomorphic to H. Let m be the index of $\pi(H)$ in Q, and let $\{x_1, \ldots, x_m\}$ be a system of left coset representatives of $\pi(H)$ in Q such that $x_1 = e_Q$. Let us introduce the notations

$$\begin{aligned} \tau &= (1, e_Q) \in G, \\ \sigma_i &= (0, x_i) \in G \text{ for } i \in [1, m]. \end{aligned}$$

Let $S = \{\tau, \sigma_2, \ldots, \sigma_m\}$.

The choice of the set $S = \{\tau, \sigma_2, \ldots, \sigma_m\}$ ensures that the graph of the action of G on X is connected with generating set S. (Since $\sigma_1 = e_G$, it is not necessary to have it among the generators, but this notation will be convenient in some calculations.)

Lemma 3.9. *We have $X = G \cdot p = \{\tau^k \sigma_i \cdot p = (k, x_i) \cdot p : k \in \mathbb{Z}, i \in [1, m]\}$. Furthermore $\tau^k \sigma_i \cdot p = \tau^\ell \sigma_j \cdot p$ if and only if $(k, i) = (\ell, j)$.*

Proof. Observe that

$$\begin{aligned} \tau\sigma_i &= (1, e_Q)(0, x_i) = (1, x_i), \\ \sigma_i\tau &= (0, x_i)(1, e_Q) = (1^{\alpha_{x_i}}, x_i) = \tau^{\pm 1}\sigma_i. \end{aligned}$$

Hence, $G \cdot p = \{\tau^k \sigma_i \cdot p = (k, x_i) \cdot p : k \in \mathbb{Z}, i \in [1, m]\}$. If $\tau^k \sigma_i \cdot p = \tau^\ell \sigma_j \cdot p$, then $\sigma_j^{-1} \tau^{k-\ell} \sigma_i = \tau^{\pm(k-\ell)} \sigma_j^{-1} \sigma_i \in H$. By taking the π-image we have

$$\pi(\tau^{\pm(k-\ell)} \sigma_j^{-1} \sigma_i) = x_j^{-1} x_i \in \pi(H).$$

Since these are coset representatives of $\pi(H)$, we have $x_i = x_j$ and $i = j$. Hence, $\sigma_i = \sigma_j$, and $\tau^{k-\ell} \in H$ implies $k = \ell$.

Therefore, $\tau^k \sigma_i \cdot p = \tau^\ell \sigma_j \cdot p$ if and only if $(k, i) = (\ell, j)$. □

3.1.2 Definition of the cocycle

Definition 3.10. Let G be a virtually $\mathbb{Z}$ group acting minimally on the compact space C, and let $p \in C$ with the assumption of Remark 3.7. Let $X = G \cdot p$ as before. Let us define $A \subseteq X$ as follows.

$$A = \{(k, x_i) \cdot p : k \in \mathbb{Z}, i \in [1, m], k^{\alpha_{x_i}} \geq 0\}$$

Observe that A is a collection of 'half lines'. Whenever α_{x_i} is the identity, A contains the positive half line $\{(k, x_i) \cdot p : k \in \mathbb{N}\}$, and when $\alpha_{x_i} = -1$, it contains the negative half line $\{(k, x_i) \cdot p : -k \in \mathbb{N}\}$.

Lemma 3.11. *For every $g \in G$, the set $gA \setminus A \subset X$ is finite.*

Proof. Let $g = (\ell, y)$ be an arbitrary element of G. We have

$$g \cdot ((k, x_i) \cdot p) = (\ell, y)(k, x_i) \cdot p = (f(y, x_i) + \ell + k^{\alpha_y}, yx_i) \cdot p.$$

There exists a unique $j \in [1, m]$, such that $yx_i\pi(H) = x_j\pi(H)$ and a unique $(t, z) = h \in H$, such that $yx_iz = x_j$. Using these notations, we have

$$\begin{aligned}(f(y, x_i) + \ell + k^{\alpha_y}, yx_i) \cdot p &= (f(y, x_i) + \ell + k^{\alpha_y}, yx_i)(t, z) \cdot p \\ &= (f(yx_i, z) + f(y, x_i) + \ell + k^{\alpha_y} + t^{\alpha_{yx_i}}, x_j) \cdot p.\end{aligned}$$

We are going to use the assumption that H commutes with $\mathbb{Z} \times \{e_Q\}$, mentioned in Remark 3.7. This implies that $\alpha(\pi(H)) = \{+1\}$, so whenever x and x' are in the same coset of $\pi(H)$ in Q, we have $\alpha_x = \alpha_{x'}$. Now we know that yx_i and x_j are in the same coset, so $\alpha_{yx_i} = \alpha_{x_j}$. Therefore,

$$(f(yx_i, z) + f(y, x_i) + \ell + k^{\alpha_y} + t^{\alpha_{yx_i}}, x_j) \cdot p \notin A$$

$$\Updownarrow$$

$$\begin{aligned}0 &> (f(yx_i, z) + f(y, x_i) + \ell + k^{\alpha_y} + t^{\alpha_{yx_i}})^{\alpha_{x_j}} \\ &= (f(yx_i, z) + f(y, x_i) + \ell + k^{\alpha_y} + t^{\alpha_{yx_i}})^{\alpha_{yx_i}} \\ &= f(yx_i, z)^{\alpha_{yx_i}} + f(y, x_i)^{\alpha_{yx_i}} + \ell^{\alpha_{yx_i}} + k^{\alpha_{x_i}} + t.\end{aligned}$$

For a fixed g and fixed $i \in [1, m]$, the number $f(yx_i, z)^{\alpha_{yx_i}} + f(y, x_i)^{\alpha_{yx_i}} + \ell^{\alpha_{yx_i}} + t$ is also fixed. For $i \in [1, m]$ let $A_i = \{(k, x_i) \cdot p : k \in \mathbb{Z}, k^{\alpha_{x_i}} \geq 0\}$. The previous calculations show that

$$|gA_i \setminus A| = \left|\{k \in \mathbb{Z} : k^{\alpha_{x_i}} \geq 0 \text{ and } f(yx_i, z)^{\alpha_{yx_i}} + f(y, x_i)^{\alpha_{yx_i}} + \ell^{\alpha_{yx_i}} + k^{\alpha_{x_i}} + t < 0\}\right|,$$

which is finite. Hence,

$$gA \setminus A = \bigcup_{i \in [1,m]} (gA_i \setminus A)$$

is also a finite set. □

Proposition 3.12. *For every piecewise map $\varphi \in \mathrm{PW}(G \curvearrowright X)$, the set $A \triangle \varphi(A)$ is finite.*

Proof. There exists a finite set $T = \{g_1, \ldots, g_t\} \subset G$, such that for every $h \in G$, $\varphi(h \cdot p) \in Th \cdot p$. We have the following inclusion.

$$\varphi(A) \setminus A \subseteq \left(\bigcup_{i=1}^{t} g_i A\right) \setminus A = \bigcup_{i=1}^{t} (g_i A \setminus A).$$

By Lemma 3.11, the sets on the right-hand side are all finite, so $\varphi(A) \setminus A$ is also finite. The same argument works for $\varphi^{-1}(A) \setminus A$, implying that $A \setminus \varphi(A)$ is also finite. Hence, $A \triangle \varphi(A)$ is finite. □

Proposition 3.12 allows us to define the following embedding.

Definition 3.13. For $\varphi \in \mathrm{PW}(G \curvearrowright X)$ let $c_\varphi = A \triangle \varphi(A) \in \mathscr{P}_f(X)$. The map

$$\begin{aligned}[[G \curvearrowright C]] \leq \mathrm{PW}(G \curvearrowright X) &\longrightarrow \mathscr{P}_f(X) \rtimes \mathrm{PW}(G \curvearrowright X), \\ \varphi &\longmapsto (c_\varphi, \varphi) = (A \triangle \varphi(A), \varphi)\end{aligned}$$

is an embedding.

3.1.3 Amenable kernel

This section is dedicated to proving that the kernel of the cocycle $[[G \curvearrowright C]] \to \mathscr{P}_f(X)$, $\varphi \mapsto c_\varphi = A \triangle \varphi(A)$ defined in the previous section is amenable. The kernel is the subgroup $\{\varphi \in [[G \curvearrowright C]] : \varphi(A) = A\}$, i.e., the stabilizer of the set $A \subseteq X$ in the full group $[[G \curvearrowright C]]$.

Definition 3.14. Let G be a group acting on the space C. Let $D \subset G$ be a finite set containing the identity element. For an element $\varphi \in \mathrm{PW}(G \curvearrowright C)$ and for two points $q_1, q_2 \in C$, we say that the *φ-action is the same on the D-neighborhood of q_1 and q_2*, if for every $d \in D$, φ acts by the same element of G on $d \cdot q_1$ and on $d \cdot q_2$ (i.e., there exists $g \in G$ such that $\varphi(d \cdot q_1) = gd \cdot q_1$ and $\varphi(d \cdot q_2) = gd \cdot q_2$).

Lemma 3.15. *Let G be a virtually $\mathbb{Z}$ group acting minimally on the compact space C, and let $p \in C$ be as before.*

For every finite subset $F \subset [[G \curvearrowright C]]$ and every $n \in \mathbb{N}$, there exists $\hat{n} = \hat{n}(n, F) \in \mathbb{N}$, such that for every interval $I \subset \mathbb{Z}$ of length $2\hat{n}$, there exists $t \in I$ such that $[t - n, t + n] \subseteq I$, and for every $\varphi \in F$, the φ-action is the same on the $[-n, n] \times Q$-neighborhood of p and $(t, e_Q) \cdot p$.

Proof. Let $F \subset [[G \curvearrowright C]]$ be a finite subset and $n \in \mathbb{N}$. We can find a finite partition $\mathcal{P}$ of C into clopen (closed and open) sets, such that every $\varphi \in F$ is just acting with an element of G when restricted to an element of $\mathcal{P}$. This implies that there exists an open neighborhood V of p, such that for all $j \in [-n, n]$ and all $x \in Q$, the set $(j, x) \cdot V$ is contained in some $P \in \mathcal{P}$. The union

$$\bigcup_{\ell \geq 1} \bigcup_{|r| \leq \ell} (r, e_Q) \cdot V$$

is nonempty, open and Z-invariant, where $Z = \mathbb{Z} \times \{e_Q\} \triangleleft G$. By Remark 1.10, there exists a minimal Z-invariant closed subset $C_0 \subseteq C$ with $p \in C_0$, a natural number $M \in \mathbb{N}$, and group elements $e_G = g_1, g_2, \dots, g_M \in G$, such that

$$C = \bigsqcup_{i=1}^{M} g_i \cdot C_0.$$

By Lemma 1.11, the nonempty, open and Z-invariant set $\bigcup_{\ell \geq 1} \bigcup_{|r| \leq \ell} (r, e_Q) \cdot V$ is the union of some $g_i \cdot C_0$'s. Since it contains p, it has a nonempty intersection with C_0, hence we have

$$C_0 \subseteq \bigcup_{\ell \geq 1} \bigcup_{|r| \leq \ell} (r, e_Q) \cdot V.$$

By the compactness of C_0, a finite union already contains it, so there is $\ell \in \mathbb{N}$ such that

$$C_0 \subseteq \bigcup_{|r|\leq \ell} (r, e_Q) \cdot V.$$

Let $\hat{n} = \hat{n}(n, F) = \ell + n$. Now if $I \subset \mathbb{Z}$ is an interval of length $2\hat{n}$, then $I = [s - \hat{n}, s + \hat{n}]$ for some $s \in \mathbb{Z}$. We have

$$C_0 = (-s, e_Q) \cdot C_0 \subseteq \bigcup_{|r|\leq \ell} (-s, e_Q)(r, e_Q) \cdot V = \bigcup_{|r|\leq \ell} (-s + r, e_Q) \cdot V.$$

Hence, there is an integer $-t \in [-s - \ell, -s + \ell]$, such that $p \in (-t, e_Q) \cdot V$. By the choice of $\hat{n}$, we have $[t - n, t + n] \subseteq [s - \hat{n}, s + \hat{n}]$. By multiplying with the inverse of $(-t, e_Q)$, we get

$$(-t, e_Q)^{-1} \cdot p = (t, e_Q) \cdot p \in V.$$

Thus for all $j \in [-n, n]$ and for all $x \in Q$, both $(j, x) \cdot p$ and $(j, x)(t, e_Q) \cdot p$ are in $(j, x) \cdot V$, so every element of F acts on them as the same element of G (by the choice of the partition and the open set V). In other words, the action of every element of F is the same on the $[-n, n] \times Q$-neighborhoods of p and $(t, e_Q) \cdot p$. $\square$

The picture we have in mind is the following. We think of $X = G \cdot p$ as a collection of lines, for each $i \in [1, m]$ the corresponding line is $(\mathbb{Z} \times \{x_i\}) \cdot p$, and the origin on this line is the point $(0, x_i) \cdot p$. Suppose that we only know the action of F on a neighborhood of the origin on each line (i.e., the $([-n, n] \times \{x_1, \ldots, x_m\})$-neighborhood of p). Lemma 3.15 states that on any chosen line in any long enough interval we can find a subinterval of length $2n$, where the action of F is determined by what we know.

Having these lines in mind we introduce a definition to get the coordinate of a given point on the corresponding line.

Definition 3.16. For $q \in X = G \cdot p$, there exists exactly one element of the form $(k, x_i) \in G$, such that $q = (k, x_i) \cdot p$. Let $|q| = |k|$.

The following lemma shows that there is a universal bound on the distance between the coordinate of a point in X and that of its image under the action of a piecewise G map.

Lemma 3.17. *Let* $\varphi \in \mathrm{PW}(G \curvearrowright X)$. *Then*

$$\sup\{\big||q| - |\varphi(q)|\big| : q \in X\}$$

is finite.

Proof. Consider the graph of the action $G \curvearrowright X$ with the generating set S introduced in Definition 3.8 (this assures that the graph is connected). Let d denote the distance function on this graph. The piecewise group $\mathrm{PW}(G \curvearrowright X)$ also acts on the graph, take $\varphi \in \mathrm{PW}(G \curvearrowright X)$. Since this is a piecewise G map, it can only move points to a limited distance. Let

$$|\varphi| = \max\{\mathrm{d}(q, \varphi(q)) : q \in X\}.$$

This means that we can get to $\varphi(q)$ from q by applying at most $|\varphi|$ many generators. When we multiply a point with the generator $\tau = (1, e)$, the coordinate changes by 1. Now take $\sigma_i = (0, x_i) \in S$. We would like to compute the difference of the coordinates of q and $\sigma_i \cdot q$. Let $q = (n, x_j) \cdot p$, then we have

$$\begin{aligned} \sigma_i \cdot q &= (0, x_i)(n, x_j) \cdot p \\ &= (f(x_i, x_j) + n^{\alpha_{x_i}}, x_i x_j) \cdot p. \end{aligned}$$

There exists a unique $h = (\ell, z) \in H = G_p$ such that $x_i x_j z = x_k$ (for some $k \in [1, m]$).

$$\begin{aligned} \sigma_i \cdot q &= (f(x_i, x_j) + n^{\alpha_{x_i}}, x_i x_j)(\ell, z) \cdot p \\ &= (f(x_i x_j, z) + f(x_i, x_j) + n^{\alpha_{x_i}} + \ell^{\alpha_{x_i x_j}}, x_k) \cdot p. \end{aligned}$$

The difference of the coordinates is

$$\begin{aligned} \big||q| - |\sigma_i(q)|\big| &= \big||f(x_i x_j, z) + f(x_i, x_j) + n^{\alpha_{x_i}} + \ell^{\alpha_{x_i x_j}}| - |n|\big| \\ &\leq |f[x_i x_j, z)| + |f(x_i, x_j)| + |\ell|. \end{aligned}$$

Recall that $f : Q \times Q \to \mathbb{Z}$ is a cocycle. Since Q is finite, there is an upper bound on the absolute values f can take, let us denote this number by $f_0 \in \mathbb{Z}$. We also know that H is finite, so there are only finitely many possible values for the first coordinate of an element of H, let us denote the maximum absolute value by $\ell_0 \in \mathbb{Z}$. We obtained that

$$\big||q| - |\sigma_i(q)|\big| \leq 2f_0 + \ell_0,$$

and this bound does not depend on the choice of the point q. So we have

$$\big||q| - |\varphi(q)|\big| \leq |\varphi|(2f_0 + \ell_0)$$

for every $q \in X$. This finishes the proof of the lemma. □

Notation 3.18. For a finite set $F \subset \mathrm{PW}(G \curvearrowright X)$, let us introduce the following notation.

$$d_F = \max\{\big||q| - |\varphi(q)|\big| : \varphi \in F,\ q \in X\}.$$

By Lemma 3.17, we know that this is finite.

Proposition 3.19. *The stabilizer $[[G \curvearrowright C]]_A$ is locally finite.*

Proof. Let $F \subset [[G \curvearrowright C]]_A$ be a finite subset of the stabilizer. We would like to prove that the subgroup generated by F in $[[G \curvearrowright X]]_A$ is finite.

By Lemma 3.15 for F and $n = d_F + 1$, we have $\hat{n} = \hat{n}(n, F)$. Let $I_0 = [-\hat{n}, \hat{n}]$, and decompose $\mathbb{Z}$ as the disjoint union of consecutive intervals $\{I_k\}_{k\in\mathbb{Z}}$ of length $2\hat{n}$. (So for example $I_1 = [\hat{n}+1, 3\hat{n}+1]$, $I_{-1} = [-3\hat{n}-1, -\hat{n}-1]$, etc.) According to the lemma, for each $k \in \mathbb{Z} \setminus \{0\}$, there exists $t_k \in I_k$ such that $[t_k - n, t_k + n] \subseteq I_k$, and for every $\varphi \in F$, the φ-action is the same on the $[-n, n] \times Q$-neighborhood of $(t_k, e_Q) \cdot p$ and of p. Let $t_0 = 0$.

Now for $k \in \mathbb{Z}$ and $i \in [1, m]$, let

$$\begin{aligned} B_{k,i} &= \{(\ell, x_i) \cdot p : \ell \in [t_k^{\alpha_{x_i}}, (t_{k+1} - 1)^{\alpha_{x_i}}]\} \\ &= \big([t_k^{\alpha_{x_i}}, (t_{k+1}-1)^{\alpha_{x_i}}] \times \{x_i\}\big) \cdot p \subset (\mathbb{Z} \times \{x_i\}) \cdot p. \end{aligned}$$

Note that for $\alpha_{x_i} = -1$ this interval becomes $[-t_{k+1} + 1, -t_k]$. For $k \in \mathbb{Z}$ we define

$$B_k = \bigcup_{i=1}^{m} B_{k,i} \subset X.$$

Claim 3.20. *For every $k \in \mathbb{Z}$, the finite set $B_k \subset X$ is invariant under the action of F.*

Proof. Fix $k \in \mathbb{Z}$. Take arbitrary elements $\varphi \in F$ and $(r, x_i) \cdot p \in B_k$, then there exists a group element $g = (\ell, z) \in G$, such that the action of φ on $(r, x_i) \cdot p$ is multiplication by g. So we have

$$\begin{aligned} \varphi((r, x_i) \cdot p) &= (\ell, z)(r, x_i) \cdot p \\ &= (f(z, x_i) + \ell + r^{\alpha_z}, z x_i) \cdot p. \end{aligned}$$

There is a unique $(L, y) \in H$, such that $z x_i y = x_j$ for some $j \in [1, m]$.

$$\begin{aligned} \varphi((r, x_i) \cdot p) &= (f(z, x_i) + \ell + r^{\alpha_z}, z x_i)(L, y) \cdot p \\ &= (f(z, x_i) + \ell + r^{\alpha_z} + L^{\alpha_{z x_i}} + f(z x_i, y), x_j) \cdot p. \end{aligned}$$

Let $R = f(z, x_i) + \ell + L^{\alpha_{zx_i}} + f(zx_i, y)$, by Lemma 3.17 we have $|R| \leq d_F = n - 1$, since this is the difference of the coordinates of $(r, x_i) \cdot p$ and its φ-image. There are three cases.

1. $r \in [(t_k + n)^{\alpha_{x_i}}, (t_{k+1} - n - 1)^{\alpha_{x_i}}]$,
2. $r \in [t_k^{\alpha_{x_i}}, (t_k + n - 1)^{\alpha_{x_i}}]$,
3. $r \in [(t_{k+1} - n)^{\alpha_{x_i}}, (t_{k+1} - 1)^{\alpha_{x_i}}]$.

In each case we conclude that the image of the point $(r, x_i) \cdot p$ stays in the set B_k.

1. Note that we have $\alpha_{x_j} = \alpha_{zx_i} = \alpha_z \alpha_{x_i}$, since they are in the same coset of $\pi(H)$ in Q. Therefore, in this case we have

   $$r^{\alpha_z} \in [(t_k + n)^{\alpha_{x_j}}, (t_{k+1} - n - 1)^{\alpha_{x_j}}].$$

   Moreover, we know that $|R| < n$, and hence by acting with φ we cannot leave the interval, i.e.,

   $$r^{\alpha_z} + R \in [(t_k)^{\alpha_{x_j}}, (t_{k+1} - 1]^{\alpha_{x_j}}].$$

   This means that $\varphi((r, x_i) \cdot p) \in B_k$.

2. In this case the point $(r, x_i) \cdot p$ is in the $[-n, n] \times Q$-neighborhood of $(t_k, e_Q) \cdot p$, so we can use that the φ-action on this neighborhood of $(t_k, e_Q) \cdot p$ is the same as on the $[-n, n] \times Q$-neighborhood of p. This, and the fact that φ is in the stabilizer of A, ensures that the image of $(r, x_i) \cdot p$ stays in B_k.

   Let $b = r - t_k^{\alpha_{x_i}}$, then $b^{\alpha_{x_i}} \in [0, n]$, so $(r, x_i) \cdot p = (b + t_k^{\alpha_{x_i}}, x_i) \cdot p = (b, x_i)(t_k, e_Q) \cdot p$. Now we can see the action of φ on $(r, x_i) \cdot p$ the following way.

   $$\begin{aligned}
   \varphi((r, x_i) \cdot p) &= \varphi((b, x_i)(t_k, e_Q) \cdot p) \\
   &= (\ell, z)(b, x_i)(t_k, e_Q) \cdot p \\
   &= (\ell, z)(b + t_k^{\alpha_{x_i}}, x_i) \cdot p \\
   &= (\ell, z)(t_k^{\alpha_{x_i}}, e_Q)(b, x_i) \cdot p \\
   &= (\ell + (t_k^{\alpha_{x_i}})^{\alpha_z}, z)(b, x_i) \cdot p \\
   &= (t_k^{\alpha_{x_i}\alpha_z}, e_Q)(\ell, z)(b, x_i) \cdot p \\
   &= (t_k^{\alpha_{x_i z}}, e_Q)\varphi((b, x_i) \cdot p).
   \end{aligned}$$

The last equality is due to the fact that $(b, x_i) \cdot p$ is in the $[-n, n] \times Q$-neighborhood of p, this is the corresponding point to $(r, x_i) \cdot p$, so the φ-action on this point is also multiplication by $g = (\ell, z)$. We have

$$\begin{aligned}\varphi((b, x_i) \cdot p) &= (t_k^{\alpha_{x_i z}}, e_Q)^{-1} \varphi((r, x_i) \cdot p) \\ &= (-t_k^{\alpha_{x_i z}}, e_Q)(r^{\alpha_z} + R, x_j) \cdot p \\ &= (r^{\alpha_z} + R - t_k^{\alpha_{x_i z}}, x_j) \cdot p.\end{aligned}$$

Since φ is in the stabilizer of the set $A \subset X$, the image of $(b, x_i) \cdot p$ is also in A, so we have

$$\begin{aligned}0 &\leq (r^{\alpha_z} + R - t_k^{\alpha_{x_i z}})^{\alpha_{x_j}} \\ &= \big((b + t_k^{\alpha_{x_i}})^{\alpha_z} + R - t_k^{\alpha_{x_i z}}\big)^{\alpha_{x_j}} \\ &= b^{\alpha_{x_i}} + R^{\alpha_{x_j}}.\end{aligned}$$

In the second equation we used that from $\alpha_{x_j} = \alpha_{x_i z} = \alpha_{z x_i} = \alpha_{x_i}\alpha_z$ we get $\alpha_z \alpha_{x_j} = \alpha_{x_i}$, since $\mathrm{im}(\alpha) \subseteq \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$. We also know that $|b| + |R| \leq 2n - 1$, so

$$0 \leq b^{\alpha_{x_i}} + R^{\alpha_{x_j}} \leq 2n - 1.$$

From the choice of the t_k's it is clear that $|t_{k+1} - t_k| \geq 2n$, so we have

$$\begin{aligned}[t_k, t_{k+1} - 1] \ni t_k + b^{\alpha_{x_i}} + R^{\alpha_{x_j}} \\ &= \big((t_k^{\alpha_{x_i}} + b)^{\alpha_z} + R\big)^{\alpha_{x_j}} \\ &= (r^{\alpha_z} + R)^{\alpha_{x_j}}.\end{aligned}$$

Hence, $r^{\alpha_z} + R \in [t_k, t_{k+1} - 1]^{\alpha_{x_j}}$. Therefore, $\varphi((r, x_i) \cdot p) \in B_k$ holds in the second case as well.

3. In this case we use that the complement of $A \subset X$ is invariant under the action of φ. Hence, exactly the same ideas and similar calculations as in the second case show that $\varphi((r, x_i) \cdot p) \in B_k$.

This proves that B_k is indeed invariant under the action of all elements of F. □

Since every B_k is invariant under the action of F, we can realize the group $\langle F \rangle$ as a subgroup of $\prod_{k \in \mathbb{Z}} \mathrm{Sym}(B_k)$. By the choice of $t_k \in I_k$, we have that $|t_{k+1} - t_k| \leq 4\hat{n}$, so $|B_{k,i}| \leq 4\hat{n}$ for every $i \in [1, m]$, and hence $|B_k| \leq 4\hat{n}m$ for all $k \in \mathbb{Z}$. This means that there is a uniform bound on the cardinality of the B_k's, so the direct product $\prod_{k\in\mathbb{Z}} \mathrm{Sym}(B_k)$ is locally finite. The group $\langle F\rangle$ is a finitely generated subgroup of this, consequently it is finite. □

Now we can prove Theorem A.

Theorem 3.21. *Let G be a virtually cyclic group. Then for any minimal action of G on a compact Hausdorff topological space C by homeomorphisms, the topological full group $[[G \curvearrowright C]]$ is amenable.*

Proof. The action $\mathrm{PW}(G \curvearrowright C) \curvearrowright C$ is extensively amenable by Corollary 2.22. As $[[G \curvearrowright C]]$ is a subgroup of the piecewise group, the action $[[G \curvearrowright C]] \curvearrowright C$ is also extensively amenable. The kernel of the cocycle $[[G \curvearrowright C]] \to \mathscr{P}_f(X)$, $\varphi \mapsto c_\varphi = A \triangle \varphi(A)$ is exactly the stabilizer $[[G \curvearrowright C]]_A$. By Proposition 3.19 this kernel is locally finite, hence amenable. Therefore, by Corollary 3.2, the topological full group $[[G \curvearrowright C]]$ is amenable. □

3.2 Interval exchange transformations

Definition 3.22. An *interval exchange transformation* is a right continuous bijection $g\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that the set of angles $\{gx - x : x \in \mathbb{R}/\mathbb{Z}\}$ is finite.

Since the composition of interval exchange transformations is also an interval exchange transformation, they form a group. We denote the group of all interval exchange transformations by IET. By construction, IET acts on the set $\mathbb{R}/\mathbb{Z}$.

In other words, if we cut the circle $\mathbb{R}/\mathbb{Z}$ into finitely many intervals and rearrange them, we get an interval exchange transformation. These transformations have been widely studied in dynamical systems, for example in connection with polygonal billiards with rational angles.

There are several unanswered questions concerning the subgroups of IET, for instance, Katok asked whether IET contains a non-abelian free group. Dahmani, Fujiwara and Guirardel [8] showed that free subgroups are *rare* in IET, in the sense that a subgroup generated by two generic elements is not free. A related open question asked in [10] is the following.

Question 1. *Is the group* IET *amenable?*

It was shown in [21] as an application of extensive amenability, that subgroups of low *rational rank* of IET are amenable. We present the proof of their result in this section.

Given a finitely generated group $G \leq \mathrm{IET}$ we define its *group of angles* $\Lambda(G) \leq \mathbb{R}/\mathbb{Z}$ as the subgroup of $\mathbb{R}/\mathbb{Z}$ generated by $\{gx - x : x \in \mathbb{R}/\mathbb{Z}, g \in G\}$. Since G is finitely generated, $\Lambda(G)$ is a finitely generated abelian group. Therefore, it is isomorphic to $\mathbb{Z}^d \times F$, where F is a finite abelian group.

Definition 3.23. For a finitely generated subgroup $G \leq \mathrm{IET}$ the *rational rank* of G is the supremum of all $d \in \mathbb{N}$ for which $\mathbb{Z}^d$ embeds in $\Lambda(G)$. The rational rank of G is denoted by $\mathrm{rk}_{\mathbb{Q}}(G)$.

As presented in [7], Chapter 5, it is possible to realize a finitely generated subgroup of IET as a subgroup of a topological full group of a Cantor minimal action of a finitely generated abelian group.

Let $G \leq \mathrm{IET}$ be a finitely generated subgroup and let us assume that G contains a rotation z by an irrational angle. Let $D \subset \mathbb{R}/\mathbb{Z}$ denote the set of all points of discontinuity of all elements of G. This set is countable since G is finitely generated, and it is dense in $\mathbb{R}/\mathbb{Z}$ since the z-orbit of a discontinuity point is also in D. Now let

$$\Sigma = (\mathbb{R}/\mathbb{Z} \setminus D) \sqcup \{x_-, x_+ : x \in D\}.$$

Let us define the factor map $\pi \colon \Sigma \to \mathbb{R}/\mathbb{Z}$ that collapses the two copies of each point of D.

$$\pi(y) = \begin{cases} y & \text{if } y \in (\mathbb{R}/\mathbb{Z} \setminus D), \\ x & \text{if } y = x_- \text{ or } y = x_+ \text{ with } x \in D. \end{cases}$$

We also define a topology on Σ as follows. For $x, y \in \mathbb{R}/\mathbb{Z}$ let $(x, y) \subset \mathbb{R}/\mathbb{Z}$ denote the open interval obtained by going from x to y counterclockwise. We declare for every $x, y \in D$ the set $[x, y] = \pi^{-1}((x, y)) \cup \{x_+, y_-\} \subset \Sigma$ to be open.

Notice that every element $g \in G \cup \Lambda(G)$ can be viewed as a permutation of Σ, for every point of discontinuity x, we extend g continuously to x_- and x_+. Therefore, G and $\Lambda(G)$ both act on Σ by homeomorphisms. For every $x, y \in D$, the set $[x, y]$ is clopen (since $[y, x]$ is the complement of $[x, y]$), so G is contained in the topological full group of the $\Lambda(G)$-action on Σ. Since $\Lambda(G)$ contains an irrational rotation, its action is minimal on $\mathbb{R}/\mathbb{Z}$, and also on Σ.

Proposition 3.24. *Let* $G \leq \mathrm{IET}$ *be a finitely generated subgroup such that* $\mathrm{rk}_{\mathbb{Q}}(G) = d \geq 1$. *Then* G *embeds in the topological full group of a minimal action of* $\Lambda(G)$ *on a Cantor space.*

Proof. Since the rational rank of G is at least 1, we can assume that G contains a rotation z by an irrational angle. Thus, the above construction yields a minimal $\Lambda(G)$-action on the space Σ, such that G is contained in the topological full group.

Note that Σ is a compact Hausdorff space that has no isolated points. The clopen sets of the form $[x, y]$ (with $x, y \in D$) form a countable base for the topology. Hence, Σ is a Cantor space. □

As we will see in the proof of the following theorem, there exists a cocycle with amenable kernel on the whole group IET. This reduces the question of amenability of a finitely generated subgroup $G \leq \mathrm{IET}$ to determining whether the action $G \curvearrowright \mathbb{R}/\mathbb{Z}$ is extensively amenable.

Theorem 3.25 (Proposition 5.3 in [21].)**.** *Let* $G \leq \mathrm{IET}$ *be a finitely generated subgroup. Then* G *is amenable if and only if the* G*-action on* $\mathbb{R}/\mathbb{Z}$ *is extensively amenable.*

Proof. If G is amenable, then any action of G is extensively amenable (by Proposition 2.7), proving the 'only if' direction of the statement.

For the other direction, our goal is to construct a $\mathrm{Sym}_f(\mathbb{R}/\mathbb{Z})$-cocycle $g \mapsto \sigma_g$ with amenable kernel. Then we can apply Corollary 3.2 for the Sym_f functor.

If we replace the convention that interval excange transformations are right continuous with left continuity, we get another group $\widetilde{\mathrm{IET}}$, also acting on $\mathbb{R}/\mathbb{Z}$. If $g \in \mathrm{IET}$, let $\tilde{g}$ denote the left continuous map that coincides with g except in the points of discontinuity of g. Then the map $\mathrm{IET} \to \widetilde{\mathrm{IET}}$, $g \mapsto \tilde{g}$ is a group isomorphism. Observe that

$$\sigma_g = \tilde{g}g^{-1}$$

is a permutation of $\mathbb{R}/\mathbb{Z}$ that is equal to the identity everywhere except for the points of discontinuity of g^{-1}. Hence, $\sigma_g \in \mathrm{Sym}_f(\mathbb{R}/\mathbb{Z})$. For $g, h \in \mathrm{IET}$ we have

$$\sigma_{gh} = \widetilde{gh}(gh)^{-1} = \tilde{g}\tilde{h}h^{-1}g^{-1} = \sigma_g g \sigma_h g^{-1}.$$

Let us define

$$\iota \colon \mathrm{IET} \longrightarrow \mathrm{Sym}_f(\mathbb{R}/\mathbb{Z}) \rtimes \mathrm{IET}$$
$$g \longmapsto (\sigma_g, g).$$

The map ι is an injective group homomorphism. Indeed, for all $g, h \in \mathrm{IET}$ we have

$$\begin{aligned}\iota(gh) &= (\sigma_{gh}, gh) = \\ &= (\sigma_g g \sigma_h g^{-1}, gh) = \\ &= (\sigma_g({}^g\sigma_h), gh) = \\ &= (\sigma_g, g)\cdot(\sigma_h, h) = \iota(g)\cdot\iota(h).\end{aligned}$$

Assume that $G \leq \mathrm{IET}$ is a subgroup and the action $G \curvearrowright \mathbb{R}/\mathbb{Z}$ is extensively amenable. The restriction $\iota\colon G \to \mathrm{Sym}_f(\mathbb{R}/\mathbb{Z}) \rtimes G$ is also an embedding. Note that $\sigma_g = \mathrm{id}$ if and only if g has no points of discontinuity, in other words, if g is a rotation of the circle. Therefore, $\iota(G) \cap (\{\mathrm{id}\} \times \mathrm{G})$ is a group of rotations and hence it is amenable. Thus, $g \mapsto \sigma_g$ is a cocycle with amenable kernel, so by Corollary 3.2, the group G is amenable. □

Corollary 3.26. *] Let $G \leq \mathrm{IET}$ be a subgroup with rational rank at most 2. Then G is amenable.*

Proof. If G has rational rank 0, then G is contained in a subgroup of rational rank 1. Since subgroups inherit amenability, we can assume that the rational rank of G is either 1 or 2. Then by Proposition 3.24 G embeds in the topological full group of an action of $\Lambda(G)$ which is either a virtually $\mathbb{Z}$ or a virtually $\mathbb{Z}^2$ group. In both cases Corollary 2.22 implies that the topological full group of $\Lambda(G)$ acts extensively amenably on Σ. Hence, the action of G is also extensively amenable on Σ.

Note that the map $\pi\colon \Sigma \to \mathbb{R}/\mathbb{Z}$ is a surjective G-map. We also observe that for $x \in D \subset \mathbb{R}/\mathbb{Z}$, any $g \in G_x$ fixes the points x_- and x_+ in Σ. Hence, the G_x-action on $\pi^{-1}(x)$ is the trivial action, in particular it is extensively amenable. By Proposition 2.9 the G-action on $\mathbb{R}/\mathbb{Z}$ is also extensively amenable. Therefore, by Theorem 3.25, G is amenable. □

Chapter 4

Free groups in full groups

In this chapter G will be an infinite, not virtually $\mathbb{Z}$ group. Our goal is to prove Theorem B.

4.1 Action on the space of colorings

Notation 4.1. We denote the identity element of G by e.

Let S be a symmetric generating set for the group G. If a and b are subsets or elements of G, then we will denote their distance in the Cayley graph $\mathrm{Cay}(G, S)$ by $\mathrm{d}(a, b)$. For $g \in G$, let $\mathrm{length}(g)$ denote $\mathrm{d}(e, g)$.

Let $k \in \mathbb{N}$. For $x \in G$ let $B_k(x)$ denote the k-ball around x in the Cayley graph. For a subset $X \subseteq G$, we will denote its k-neighborhood by $B_k(X)$, i.e., $B_k(X) = \bigcup_{x \in X} B_k(x)$.

We will work with edge colorings of graphs. We introduce the following notion.

Definition 4.2. Let $A, B, C, D_1, D_2, \dots D_k$ denote $k + 3$ different colors and let $X \in \{A, B, C, D_1, D_2, \dots, D_k\}$ be one of the colors. An edge coloring of a graph is *X-proper*, if there are no adjacent X-colored edges in the graph. We will call a coloring *3-proper* if it is A-proper, B-proper and C-proper.

Let G be a group with a symmetric generating set S. Let Σ_G denote the space of 3-proper edge colorings of the Cayley graph $\mathrm{Cay}(G, S)$ by the letters A, B, C, D, E, F.

Proposition 4.3. *If G is an infinite group, then Σ_G is a Cantor space with the topology of pointwise convergence.*

Proof. Let $E(\mathrm{Cay}(G,S))$ denote the set of edges of the Cayley graph. Then Σ_G is a non-empty subspace of the topological space $\{A,B,C,D,E,F\}^{E(\mathrm{Cay}(G,S))}$, where we consider the discrete topology on the set of colors. In the space of colorings, being 3-proper is a closed condition, i.e., 3-proper colorings converge to a 3-proper coloring. Hence, Σ_G is a closed subspace of a compact, metrizable and totally disconnected space, so it inherits these properties.

Let $\sigma \in \Sigma_G$ be any coloring. We are in one of the following two cases:

- There exists some $N \in \mathbb{N}$ such that outside of the ball $B_N(e)$ every edge has color E in σ: for every $n \in \mathbb{N}$, define σ_n to be equal to σ in the n-ball around the identity element, and the constant F coloring outside of it.
- There is no such $N \in \mathbb{N}$: for $n \in \mathbb{N}$ let σ_n be equal to σ in the ball $B_n(e)$, and the constant E coloring outside of it.

In both cases, we have $\sigma_n \in \Sigma_G$, and the sequence $(\sigma_n)_{n\in\mathbb{N}}$ converges to σ, but σ_n is never equal to σ.

Hence, Σ_G has no isolated points, so it is a Cantor space. □

Definition 4.4. Consider the natural (left) G-action on Σ_G defined by translations.

For each color $X \in \{A,B,C\}$, there exists a corresponding continuous involution on Σ_G, that we will denote by the same letter. On $\sigma \in \Sigma_G$ it is defined as follows: if the vertex $e \in G$ is adjacent to an edge labeled by X, then we translate the coloring towards the other endpoint of this edge (i.e., the origin is now at that other vertex), if e has no adjacent edges labeled by X then $X \cdot \sigma = \sigma$. (This is well-defined since the coloring is 3-proper, so we have at most one X-colored edge from every vertex.)

For $X \in \{A,B,C\}$, the involution X is contained in the topological full group $[[G \curvearrowright \Sigma_G]]$. Note that it preserves any G-invariant subset of Σ_G, so if M is a closed G-invariant subset, then A, B and C can also be viewed as elements of the topological full group $[[G \curvearrowright M]]$. This gives us a homomorphism from the free product $\Delta = \langle A\rangle * \langle B\rangle * \langle C\rangle$ to $[[G \curvearrowright M]]$. Since Δ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, it contains the free group on two generators. Our goal is to find a subspace M so that this homomorphism is injective, proving that $[[G \curvearrowright M]]$ contains a non-abelian free group.

Definition 4.5. We will think of elements of Δ as words using the letters A, B, C, that do not contain two consecutive instances of the same letter. There are no powers or inverses needed, since the generators are all involutions. For $w \in \Delta$, length(w) denotes its length as a word from the letters A, B, C.

Given a coloring $\sigma \in \Sigma_G$, we will say that a word $w \in \Delta$ is *written* along a path $p\colon [0,k] \to G$, $i \mapsto v_i$, if the length of w is k and the color of the edge (v_{i-1}, v_i) is the ith letter of w.

Example 4.6. Consider the usual Cayley graph of the integers, with the generating set $S = \{-1, 1\}$ and let us examine the $\mathbb{Z}$-action on the Cantor space $\Sigma_{\mathbb{Z}}$.

If $w \in \Delta = \langle A \rangle * \langle B \rangle * \langle C \rangle$ is any word, then there exists a coloring $\sigma \in \Sigma_{\mathbb{Z}}$ that is not fixed by w. Indeed, write w along a path ending at 0, and label the rest of the edges with the color E. Hence, the homomorphism $\Delta \to [[\mathbb{Z} \curvearrowright \Sigma_{\mathbb{Z}}]]$ is an embedding.

Therefore, $\mathbb{Z} \curvearrowright \Sigma_{\mathbb{Z}}$ is an action of $\mathbb{Z}$ on a Cantor space such that the topological full group contains a non-abelian free group. Note that this action is clearly not minimal.

Definition 4.7. Let w be a word from $\Delta = \langle A \rangle * \langle B \rangle * \langle C \rangle$. Assume that in a coloring σ the word w is written along a path $p\colon [0,k] \to G$, $i \mapsto v_i$. We will say that in this appearance of the word w, the point v_i is *marked* with the color X (with $X \in \{D, E, F\}$), if all edges going from v_i have color X except for (v_{i-1}, v_i) and (v_i, v_{i+1}). (In the case when $i = 0$ or $i = k$, then one of these does not exist, so there is only one exception.)

Let σ be a 3-proper coloring of Cay(G, S), i.e., $\sigma \in \Sigma_G$.

We will say that σ has *property* $(P1)$ if for any word $w \in \Delta$, there exists a number $R_\sigma(w) \in \mathbb{N}$, such that every ball of radius $R_\sigma(w)$ in Cay(G, S) contains the word w written along a path with the starting point marked with the color D and all other points marked with the color E.

We say that σ has *property* $(P2)$ if for every $g \in G \setminus \{e\}$, there exists a radius $N_\sigma(g) \in \mathbb{N}$, such that in every $N_\sigma(g)$-ball in Cay(G, S) we can find an edge (x, y), such that $g \cdot (x, y)$ is also in that $N_\sigma(g)$-ball and the color of (x, y) is different from the color of $g \cdot (x, y)$.

4.2 Construction

As we will see from the next proposition, property $(P1)$ ensures that the map $\Delta \to [[G \curvearrowright M]]$ is injective, while property $(P2)$ is responsible for the freeness of the action of G on M.

Proposition 4.8. *Let G be a finitely generated infinite group with a finite symmetric generating set S. Assume that there exists a coloring $\sigma \in \Sigma_G$ with property $(P1)$. Let M be a non-empty minimal closed G-invariant subset of the orbit closure $\overline{G \cdot \sigma}$. Then M is a Cantor space and the topological full group $[[G \curvearrowright M]]$ contains a non-abelian free group.*

If we assume furthermore that σ has property $(P2)$, then the action of G on M is free.

Proof. Since M is a closed subspace of Σ_G, it is also compact, metrizable and totally disconnected.

Suppose for contradiction that there exists an isolated point $\theta \in M$. Then all points in the orbit of θ are also isolated points. Due to the minimality of M, the orbit of θ is dense in M. Since it consists of isolated points, we have $M = \overline{G \cdot \theta} = G \cdot \theta$, so by compactness, M is finite.

Let us examine the property $(P1)$. The same property holds for every coloring in the orbit of σ, since G acts by translations. Let w be a word from $\langle A \rangle * \langle B \rangle * \langle C \rangle$. Take a convergent sequence $(\sigma_n)_{n \in \mathbf{N}}$ of colorings satisfying property $(P1)$, such that for all words w and every $i, j \in \mathbb{N}$ we have $R_{\sigma_i}(w) = R_{\sigma_j}(w)$. In the limit coloring we will see the word w written along a path with the starting point marked with D and other points marked with E in every ball of radius $R_{\sigma_1}(w)$, since this holds for every coloring in the sequence. Hence, property $(P1)$ (with the same $R(w)$'s) holds for any coloring in the orbit closure $\overline{G \cdot \sigma}$, including the colorings in M.

The finiteness of the orbit $M = G \cdot \theta$ means that the coloring of some finite neighborhood of the identity element determines the whole coloring θ. From property $(P1)$ it follows that every vertex in the Cayley graph can only be the starting point of one word. Hence, in the coloring θ we can only see finitely many words from $\langle A \rangle * \langle B \rangle * \langle C \rangle$ written along paths. This is a contradiction, so M cannot be finite. Therefore, there are no isolated points in M. Hence, M is a non-empty, compact, metrizable and totally disconnected topological space with no isolated points, so by definition it is a Cantor space.

Now consider the previously mentioned homomorphism $\Delta \to [[G \curvearrowright M]]$. Since Δ contains a non-abelian free group, it is enough to prove that this homomorphism is injective.

Take a word $w \in \Delta$. We would like to find an element of M that is not fixed by w. Take an arbitrary coloring $\lambda \in M$. Due to property $(P1)$ we can see w written along a marked path in the ball $B_{R_\lambda(w)}(e)$ in the coloring λ, say with endpoint $g \in G$. Consider the coloring $g \cdot \lambda \in M$. Here we can see w written along a path ending at the origin. Since the starting point is marked with the color D, and the endpoint with the color E, this path cannot be a cycle. Therefore, $w \cdot (g \cdot \lambda) \neq g \cdot \lambda$. Hence, w does not act as the identity on M, so the homomorphism $\Delta \to [[G \curvearrowright M]]$ is indeed injective.

Now assume that σ has property $(P2)$ and suppose for contradiction that there exists $\lambda \in M$ and $g \in G$, such that $g \cdot \lambda = \lambda$. Let $N = N_\sigma(g)$ be the radius for g given by property $(P2)$. Since λ is in the closure of $G \cdot \sigma$, we can find a coloring in the orbit of σ, say $h \cdot \sigma$, such that the coloring of the N-ball around the identity is the same in λ and $h \cdot \sigma$. In the coloring $h \cdot \sigma$, due to property $(P2)$, there is an edge (x, y) in the N-ball that has a different label than $g \cdot (x, y)$, and both edges are in the N-ball. On this ball the coloring coincides with λ, this contradicts the assumption that $g \cdot \lambda = \lambda$. So the action of G on M is free. □

Definition 4.9. Let $\mathcal{G} = (V, E)$ be a locally finite graph with distance function d. Recall that a path $p \colon I \to V$ is a *geodesic* if

- $I \subseteq \mathbb{Z}$ is a finite or infinite interval, i.e., if $a, b \in I$ and $a \leq k \leq b$, then $k \in I$,
- if $n, n+1 \in I$, then $(p(n), p(n+1)) \in E$,
- for every $a, b \in I$, we have $\mathrm{d}(p(a), p(b)) = |a - b|$.

We will call a subset $\ell \subseteq V$ a geodesic if there exists a geodesic $p \colon I \to V$ such that $\mathrm{im}(p) = \ell$.

For a vertex $v \in V$, we will say that ℓ is a *geodesic through* v if $v \in \ell$.

The main idea of the proof of the following proposition was communicated by Tamás Terpai.

Proposition 4.10. *Let G be a finitely generated infinite group that is not virtually $\mathbb{Z}$, with symmetric generating set S. Then there exists a coloring $\sigma \in \Sigma_G$ satisfying the properties $(P1)$ and $(P2)$.*

Lemma 4.11. *Let $G = \langle S \rangle$ be infinite but not virtually $\mathbb{Z}$. Then for any $n \in \mathbb{N}$, there exists an integer $K(n)$, such that whenever $g \in G$ and ℓ is a geodesic through g in $\mathrm{Cay}(G, S)$, we can find another vertex h with $\mathrm{d}(\ell, h) = n$ and $\mathrm{d}(g, h) \leq K(n)$.*

Proof. We can translate the vertex g to the identity element $e \in G$, so it is enough to prove the statement for $g = e$. Suppose for contradiction that there is $n \in \mathbb{N}$, such that for every $k \in \mathbb{N}$, there exists a geodesic ℓ_k through e with no suitable h. This implies that the n-neighborhood of ℓ_k covers the k-ball around e.

By local finiteness there exists a geodesic ℓ through e, such that the n-neighborhood of ℓ covers the whole Cayley graph.

We will estimate the growth of the Cayley graph. Let $k \in \mathbb{N}$, $k > n$, and look at $B_{k+1}(e) \setminus B_k(e)$. For $s \in \mathbb{N}$ let $\ell^s = \ell \cap B_s(e)$. By the triangle inequality we have the following inclusions:

$$B_n(\ell^{k-n}) \subseteq B_k(e) \subseteq B_{k+1}(e) \subseteq B_n(\ell^{k+n+1}).$$

Hence, we have

$$B_{k+1}(e) \setminus B_k(e) \subseteq B_n(\ell^{k+n+1}) \setminus B_n(\ell^{k-n}) \subseteq B_n(\ell^{k+n+1} \setminus \ell^{k-n}).$$

The set $\ell^{k+n+1} \setminus \ell^{k-n}$ has size at most $4n+2$, so the size of its n-neighborhood is bounded above by $(4n+2)|B_n(e)|$. This number does not depend on k, so the size of $B_{k+1}(e) \setminus B_k(e)$ is bounded by a constant. Therefore, the growth of the Cayley graph is at most linear, implying that the group is either finite or virtually $\mathbb{Z}$ (by [24]). This contradicts the assumption of the lemma, concluding the proof. □

Proof of Proposition 4.10. We enumerate the words $\Delta = \{w_1, w_2, w_3, \dots\}$ and the group elements $G \setminus \{e\} = \{g_1, g_2, g_3, \dots\}$ such that for every $i \geq 1$ we have $\text{length}(w_i) < \text{length}(g_i)$ (we allow repetitions). We will fix ranges $\{R_i\}_{i \geq 1}$, and for every k, we are going to construct a coloring $\sigma_k \in \Sigma_k$ that satisfies the requirements of $(P1)$ with respect to the first k words in Δ, with $R_{\sigma_k}(w_i) = R_i$ for each $i \leq k$, and the requirements of $(P2)$ for the first k elements of G, with $N_{\sigma_k}(g_i) = R_i$ for all $i \leq k$. Then we will take σ to be the limit point of a convergent subsequence of $\{\sigma_k\}_{k \geq 1}$. Thus, σ will have property $(P1)$ and $(P2)$ with $R_\sigma(w_i) = R_i$ and $N_\sigma(g_i) = R_i$ for every $i \geq 1$.

For each $i \geq 1$, let us define R_i and an auxiliary range R'_i as follows. Let $R'_1 = \text{length}(g_1) + 2$ and

$$R'_i = \max\{\text{length}(g_i) + 2, 2R'_{i-1} + K(2R'_{i-1} + 1)\} \quad \text{for } i \geq 2,$$
$$R_i = 6R'_i + K(2R'_i + 1) + \text{length}(g_i) + 1 \quad \text{for } i \geq 1,$$

where K is the function from Lemma 4.11.

For a fixed $k \in \mathbb{N}$, the construction of σ_k is as follows. The variable i will take the values $k, k-1, \ldots, 2, 1$ successively, and for every i we place copies of w_i, and also label additional edges that ensure property $(P2)$ for g_i, resulting in the partial coloring $\sigma_k^{(i)}$. In each step the following three conditions will hold.

(1) In every R_i-ball we can see w_i written along a path with the starting point marked with D, and other points marked with E.
(2) In every R_i-ball there is an edge (x, y) with label D, such that $g_i \cdot (x, y)$ has label E. We will call these the *labeled edges belonging to* g_i.
(3) If g is a vertex in a copy of the word w_i, then the R_i'-ball around g does not intersect any copy of a word w_j (other than the one containing g) for $i \leq j$ or labeled edges belonging to g_j for $i < j$.

In the nth step (when the value of i is $k+1-n$) we take a maximal set of points $T \subset G$, such that the $2R_i'$-balls around the points of T

- do not intersect each other,
- do not intersect a previously placed copy of a word w_j for all j such that $i < j$,
- do not contain previously labeled edges belonging to g_j for any j such that $i < j$.

Then for every point t in T, we take a geodesic path starting from t and ending at $g_i \cdot t$, write w_i along this path, and mark the starting point with the color D, all other points with E. (Here we use that $\text{length}(w_i) < \text{length}(g_i)$.) Now for each $t \in T$, pick an edge adjacent to t that is labeled by D, and label the g_i-translate of this edge by E, these will be the labeled edges belonging to g_i. This way condition (3) is immediately satisfied, since $R_i' \geq \text{length}(g_i) + 2$.

Now we will verify that conditions (1) and (2) hold for the partial coloring $\sigma_k^{(i)}$. Take an arbitrary vertex $g \in G$, we would like to find a copy of w_i and a labeled edge pair belonging to g_i in the ball $B_{R_i}(g)$. If $g \in T$, then the first condition clearly holds, so assume that $g \notin T$. The reason why g cannot be added to T is that the $2R_i'$-ball around g intersects the $2R_i'$-ball around a point of T, or it intersects a copy of another word w_j with $i \leq j$, or a labeled edge belonging to g_j with $i \leq j$.

- In the first case we have $p \in T$ such that $B_{2R_i'}(g) \cap B_{2R_i'}(p) \neq \varnothing$. This implies that $\text{d}(g, p) \leq 4R_i'$, so the R_i-ball around g contains the copy of w_i starting from p, a D-labeled edge from p and its g_i-translate labeled by E.

- In the second case there is a point $x \in B_{2R'_i}(g)$ belonging to a copy of the word w_j or an endpoint of a labeled edge belonging to g_j (so x is the g_j-translate of the starting point of a copy of w_j). In both cases we can consider the geodesic ℓ on which w_j is written, we have $x \in \ell$. By Lemma 4.11, there is a point $y \in G$, such that $\mathrm{d}(y, \ell) = 2R'_i + 1$ and $\mathrm{d}(y, x) \leq K(2R'_i + 1)$. This way the ball $B_{2R'_i}(y)$ does not intersect the geodesic ℓ and lies in the protective ball $B_{R'_j}(x)$ (since $i < j$), so it does not intersect any other previously placed word or labeled edge either. Hence, there exists a point $q \in T \cap B_{4R'_i}(y)$, otherwise y could be added to T to enlarge it (possibly $y = q$). We have that

$$\begin{aligned}\mathrm{d}(g, q) &\leq \mathrm{d}(g, x) + \mathrm{d}(x, y) + \mathrm{d}(y, q) \leq \\ &\leq 2R'_i + K(2R'_i + 1) + 4R'_i \leq \\ &\leq R_i - \mathrm{length}(g_i) - 1,\end{aligned}$$

  so the ball $B_{R_i}(g)$ contains a copy of w_i, and also an edge adjacent to q labeled by D and its g_i-translate labeled by E.

In both cases we can see that conditions (1) and (2) are satisfied, so we can continue the coloring. After k steps, we obtain the partial coloring $\sigma_k^{(1)}$. We label the remaining edges with the color F, concluding the construction of σ_k. □

Now we can prove Theorem B.

Theorem 4.12. *Let G be a finitely generated group that is not virtually cyclic. Then there exists a minimal free action of G on a Cantor space Σ by homeomorphisms, such that the topological full group $[[G \curvearrowright \Sigma]]$ contains a non-abelian free group.*

Proof of Theorem 4.12. Let S be a symmetric generating set for G. By Proposition 4.10 there exists a coloring in Σ_G with property $(P1)$ and $(P2)$. Hence, Proposition 4.8 implies the statement of the theorem. □

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