EXPONENTIAL INTEGRABILITY IN THE SPIRIT OF MOSER-TRUDINGER'S INEQUALITIES OF FUNCTIONS WITH FINITE NON-LOCAL, NON-CONVEX ENERGY

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ABSTRACT. Let $d \geq 1$, $p \geq d$, and let Ω be a smooth bounded open subset of $\mathbb{R}^d$. We prove some exponential integrability in the spirit of Moser-Trudinger's inequalities for measurable functions u defined in Ω such that

$$\underset{|u(x)-u(y)|>\delta}{\int_\Omega \int_\Omega} \frac{1}{|x-y|^{d+p}} \, dx \, dy < +\infty,$$

for some $\delta > 0$. This double integral appeared in characterizations of Sobolev spaces and involved in improvements of the Sobolev inequaliies, Poincaré inequalities, and Hardy inequalities.

1. Introduction

Let (ρ_n) be a sequence of non-negative radial functions satisfying

$$\lim_{n\to\infty} \int_\tau^\infty \rho_n(r) r^{N-1} \, dr = 0 \quad \forall \, \tau > 0, \quad \text{and} \quad \lim_{n\to\infty} \int_0^{+\infty} \rho_n(r) r^{N-1} \, dr = 1. \tag{1.1}$$

Set

$$K_{d,p} := \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p \, d\sigma, \tag{1.2}$$

for some $e \in \mathbb{S}^{d-1}$, the unit sphere in $\mathbb{R}^d$.

Jean Bourgain, Haim Brezis, and Petru Mironescu [10, Theorems 1 and 2] (see also [11] and [8]) proved the following BBM formula:

Proposition 1.1. *Let $d \geq 1$, $p > 1$ and let Ω be a smooth bounded open subset in $\mathbb{R}^d$ or $\Omega = \mathbb{R}^d$. Assume that $g \in L^p(\Omega)$ and let (ρ_n) satisfy (1.1). Then $g \in W^{1,p}(\mathbb{R}^d)$ if and only if*

$$\liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|) \, dx \, dy < +\infty.$$

Moreover, for $g \in W^{1,p}(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|) \, dx \, dy = K_{d,p} \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx.$$

A variant of Propposition 1.1 for $p = 1$ involving functions of bounded variations was obtained by Jean Bourgain, Haim Brezis, and Petru Mironescu [10] and Juan Davila [21]. Further studies related to this characterizations can be founded in [2, 6, 16, 17, 20, 25, 35, 38, 39, 42, 43, 44].

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We next discuss another characterization of Sobolev spaces in the spirit of the BBM formula. To this end, for $d \geq 1$, $p \geq 1$, and $\delta > 0$, for a measurable subset O of $\mathbb{R}^d$, and for a measurable function u defined in O, set

$$I_{\delta,p}(u, O) = \int_O \int_O_{|u(x)-u(y)|>\delta} \frac{\delta^p}{|x-y|^{d+p}} \, dx \, dy. \tag{1.3}$$

This quantity has its root in estimates for the topological degree in [13, 12, 29, 34, 45] which has the motivation from the study of the Ginzburg Landau equation [9].

It was shown [28, Theorems 2 and 5] and [7, Theorem 1] that

Proposition 1.2. *Let $d \geq 1$ and Ω be a smooth bounded open subset in $\mathbb{R}^d$ or $\Omega = \mathbb{R}^d$ and let $p > 1$ and $g \in L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$ if and only if*

$$\liminf_{\delta \to 0} I_{\delta,p}(u, \Omega) < +\infty.$$

Moreover, for $g \in W^{1,p}(\Omega)$,

$$\lim_{\delta \to 0} I_{\delta,p}(u, \Omega) = \frac{1}{p} K_{d,p} \int_\Omega |\nabla u(x)|^p \, dx, \tag{1.4}$$

where $K_{d,p}$ is defined by (1.2). We also have, for all $\delta > 0$,

$$I_{\delta,p}(u, \Omega) \leq C_{d,p} \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx \quad \forall \, u \in W^{1,p}(\Omega), \tag{1.5}$$

for some positive constant $C_{d,p}$ depending only on d and p.

The case $p = 1$ is more delicate. One has [28, Theorem 8] (see also [18, Proposition 2]), for $u \in W^{1,1}(\Omega)$,

$$\liminf_{\delta \to 0} I_{\delta,1}(u, \Omega) \geq K_{d,1} \int_\Omega |\nabla u| \, dx$$

and (see [28, Theorem 8] and [7, Theorem 1]) that $u \in BV(\Omega)$ provided that $u \in L^1(\Omega)$ and $\liminf_{\delta \to 0} I_{\delta,1}(u, \Omega) < +\infty$. Let B_r denote the ball centered at 0 and of radius r. An example due to Augusto Ponce presented in [28] showed that there exists $u \in W^{1,1}(B_1)$ such that $\lim_{\delta \to 0} I_{\delta,1}(u, B_1) = +\infty$. When $d = 1$, there exists $u \in W^{1,1}(0,1)$ [18, Pathology 2] such that

$$K_{1,1} \int_0^1 |\nabla u| \, dx = \liminf_{\delta \to 0} I_{\delta,1}(u, (0,1)) < \limsup_{\delta \to 0} I_{\delta,1}(u, (0,1)) = +\infty.$$

It turns out that the concept of Γ-convergence fits very well this setting. It was shown [30, 32] that the Γ-limit exists for $p \geq 1$. Surprisingly, the Γ-limit, which is positive, is strictly less than the pointwise limit [32, 30]. The quantity $I_{\delta,1}$ has a similar form with non-local filters using in denoising process [19], in particular with Yaroslavsky's ones [47, 48]. A discussion on a connection between nonlocal filters using $I_{\delta,1}$ and local ones involving the total variations via the Γ-convergence theory is given in [18, Section 5.2]. Further interesting investigations related to the Γ-limit of $I_{\delta,p}$ are given in [3, 4, 5, 18].

One can obtain new and improved variants of Poincaré's inequality, Sobolev's inequality and Rellich-Kondarachov's compactness criterion using the information of $I_{\delta,p}$ instead of the one of the gradient [33, Theorems 1, 2, and 3]. Concerning the Sobolev inequality, one has

Proposition 1.3. *Let $1 < p < d$ and set $q = dp/(d-p)$ and fix $\delta > 0$ arbitrary. We have, for $u \in L^p(\mathbb{R}^d)$,*

$$\Big(\int_{|u|>\lambda\delta} |u|^q\Big)^{1/q} \le C\Big(I_{\delta,p}(u,\mathbb{R}^d)\Big)^{1/p}, \tag{1.6}$$

for some positive constants λ and C independent of u.

Concerning the Poincaré inequality, one obtains

Proposition 1.4. *Let $d \ge 1$, $p \ge 1$, $\delta > 0$, let B be an open ball of $\mathbb{R}^d$, and let $u \in L^p(B)$. There exists a positive constant $C_{d,p}$ depending only on d and p such that*

$$\int_B\int_B |u(x)-u(y)|^p dxdy \le C_{d,p}\left(|B|^{\frac{d+p}{d}} I_{\delta,p}(u,B) + \delta^p |B|^2\right). \tag{1.7}$$

The proof of Sobolev's inequality (1.6) is based on the one of Poincaré's inequality (1.7) and uses the theory of sharp functions due to Charles Fefferman and Elias Stein [23] and the method of truncation due to Vladimir Mazya [26]. The proof of Poincaré's inequality (1.7) has its roots in [7] and uses John-Nirenberg's inequality [24].

Remark 1.1. For a measurable function defined in B, by applying (1.7) for u_k with $u_k = \min\big\{k, \max\{u,-k\}\big\}$ and letting $k \to +\infty$, one also obtains (1.7) for measurable functions.

With Marco Squassina, the second author also established new and improved variants of Hardy and Caffarelli, Kohn, Nirenberg's inequality [35] using the quantity $I_{\delta,p}$. The approach used in [35] does not involve the integration-by-parts arguments and can be extended for the fractional Sobolev spaces [36]. Other investigations related to $I_{\delta,p}$ can be found in [14, 18, 31, 33, 36, 37, 38].

Let Ω be a smooth bounded open subset of $\mathbb{R}^d$ and $p \ge d$. It follows from (1.7) that $u \in BMO(\Omega)$ provided that $u \in L^1(\Omega)$ and $I_{\delta,p}(u,\Omega) < +\infty$. More precisely, one has, for $p \ge d$,

$$\|u\|_{BMO(\Omega)} \le C_\Omega\Big(I_{\delta,p}(u,\Omega) + \delta^d\Big),$$

where

$$\|u\|_{BMO(\Omega)} := \sup_{\text{ball } B\subset\Omega} \frac{1}{|B|}\int_B |u-u_B|\,dx. \tag{1.8}$$

Here, for a given a measurable set O of $\mathbb{R}^d$ and a function $u \in L^1(O)$, one sets

$$|O| := \operatorname{meas}(O) \quad \text{and} \quad u_O = \fint_O u\,dx \text{ with } \fint_O u\,dx := \frac{1}{|O|}\int_O u\,dx. \tag{1.9}$$

One can then derive the exponential integrability of u from John-Nirenberg's inequality:

$$\fint_B e^{c|u-u_B|/\|u\|_{BMO(B)}} \le C, \tag{1.10}$$

for some positive constant c and C depending only on d and for any open ball B.

Using the Poincaré inequality, one can prove that $u \in W^{1,p}(\Omega)$ then $u \in BMO(\Omega)$, this yields the exponential integrability of u in (1.10). In fact, for $u \in W^{1,p}(\Omega)$ with $p \ge d$, one can improve (1.10). First, Morrey's inequality (see, e.g., [15]) states that $u \in C^\alpha(\Omega)$ with $\alpha = 1 - d/p$ if $u \in W^{1,p}_0(\Omega)$ for $p > d$. Second, Moser-Trudinger's inequality [27, 46, 40, 41] confirms that

$$\sup_{\|u\|_{W_0^{1,d}(\Omega)}\le 1} \int_\Omega e^{\alpha|u|^{d/(d-1)}} \le C,$$

for some positive constants α and C depending only on Ω.

The goal of this paper is to understand whether or not a better integrability property of u than (1.10) inequality holds when $u \in L^p(\Omega)$ and $I_{\delta,p}(u,\Omega) < +\infty$. It is worth noting that, for all $\delta > 0$, there exists $u \in L^\infty(\Omega) \setminus C(\bar{\Omega})$ such that $I_{\delta,p}(u,\Omega) = 0$ for all $p \geq 1$. A simple example is the function $u = \delta \mathbb{1}_B$ in Ω, for some ball $B \Subset \Omega$, where $\mathbb{1}_O$ denotes the characteristic function of a subset O of $\mathbb{R}^d$. One can also show that there exists a function u such that $I_\delta(u,\Omega) < +\infty$ and $u \notin L^\infty(\Omega)$. An example for this is the function $u(x) = (\ln \lambda)^{-1} \ln \ln |x|^{-1}$ for $x \in B_{1/e}$ and $\lambda > p/d$ (the verification is given in Section 3).

In this work, we address the gap between the exponential integrability (1.10) and the boundedness for functions u with $I_{\delta,p}(u,\Omega) < +\infty$ for some $\delta > 0$ and $p \geq d$. Our first result is

Theorem 1.1. *Let $p > d \geq 1$, $\delta > 0$, and let B be a an open ball of $\mathbb{R}^d$. We have,*

i) for $M > 0$ and $\alpha > 0$, there exists a constant $0 \leq \beta = \beta(\alpha, M) \leq 1$ depending only on M and α such that

$$\sup_{|B|^{\frac{p-d}{d}} \delta^{-p} I_{\delta,p}(u,B) \leq M} \fint_B e^{\alpha \left(\frac{p}{d}\right)^{\beta\delta^{-1}|u-u_B|}} dx \leq C, \tag{1.11}$$

ii) given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α, d, and p such that

$$\sup_{|B|^{\frac{p-d}{d}} \delta^{-p} I_{\delta,p}(u,B) \leq M_0} \fint_B e^{\alpha \left(\frac{p}{d}\right)^{\delta^{-1}|u-u_B|}} dx \leq C. \tag{1.12}$$

Here C denotes a positive constant depending only on d, p, and α.

As a consequence of Theorem 1.1, we obtain

Proposition 1.5. *Let $p > d \geq 1$, $\delta > 0$, and let Ω be a smooth bounded open subset of $\mathbb{R}^d$. We have,*

i) for $M > 0$ and $\alpha > 0$, there exists a constant $0 \leq \beta = \beta(\alpha, M) \leq 1$ depending only on α and M such that

$$\sup_{\delta^{-p} I_{\delta,p}(u,\Omega) \leq M} \int_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\beta\delta^{-1}|u|}} dx \leq C_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\beta\delta^{-1}\|u\|_{L^1(\Omega)}}},$$

ii) given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α, d, p, and Ω such that

$$\sup_{\delta^{-p} I_{\delta,p}(u,\Omega) \leq M_0} \int_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\delta^{-1}|u|}} dx \leq C_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\delta^{-1}\|u\|_{L^1(\Omega)}}}.$$

Here C_Ω denotes a positive constant depending only on d, p, α, and Ω.

Here is a variant of *ii)* of Theorem 1.1.

Theorem 1.2. *Let $p = d \geq 1$, $\delta > 0$, and let B be a an open ball of $\mathbb{R}^d$. Given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α and d such that*

$$\sup_{\delta^{-d} I_{\delta,p}(u,B) \leq M_0} \fint_B e^{\alpha\delta^{-1}|u-u_B|} dx \leq C, \tag{1.13}$$

for some positive constant C depending only on d and α.

Remark 1.2. Inequality (1.13) shares some similarities with John-Nirenberg's inequality but is different. In fact, fixing $\delta > 0$, as a consequence of (1.7), we have

$$\|u\|_{BMO(B)} \leq C(M + \delta),$$

if $I_{\delta,p}(u) \le M$. Hence $\|u\|_{BMO(B)}$ does not generally converge to 0 and (1.13) cannot be derived from (1.10).

As a consequence of Theorem 1.2, we have

Proposition 1.6. *Let $p = d \ge 1$, $\delta > 0$, and let Ω be a smooth bounded open subset of $\mathbb{R}^d$. Given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α, d, and Ω such that*

$$\sup_{\delta^{-p} I_{\delta,p}(u,\Omega) \le M_0} \int_\Omega e^{\alpha\delta^{-1}|u|} dx \le C_\Omega e^{\alpha\delta^{-1}\|u\|_{L^1(\Omega)}},$$

for some positive constant C_Ω depending only on d, α, and Ω.

The exponential growths in (1.12) and (1.13) are optimal. In fact, we have

Proposition 1.7. *Let $p \ge d \ge 1$, $\gamma > p/d$, and $\alpha > 0$, and let B be a an open ball of $\mathbb{R}^d$.*
i) If $p > d$ then for any $M > 0$ there exists $u \in L^1(B)$ such that

$$I_{\delta,p}(u, B) \le M \quad \text{and} \quad \int_B e^{\alpha\gamma^{\delta^{-1}|u-u_B|}} dx = +\infty. \tag{1.14}$$

i) If $p = d$ then there exists a bounded sequence $(u_n) \subset L^1(B)$ such that

$$\lim_{n\to+\infty} I_{\delta,p}(u_n, B) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_B e^{\alpha\left(\delta^{-1}|u_n-u_{nB}|\right)^\gamma} dx = +\infty. \tag{1.15}$$

2. Proofs of Theorems 1.1 and 1.2

This section contains the proof of the Theorems 1.1 and 1.2. We first establish two lemmas used in the proof of (1.11), (1.12), and (1.13) and then establish Theorems 1.1 and 1.2.

2.1. Two useful lemmas.

For $x \in \mathbb{R}^d$ and $\rho > 0$, let $B_\rho(x)$ denote the ball in $\mathbb{R}^d$ centered at x and of radius ρ. We have

Lemma 2.1. *Let $d \ge 1$, $\lambda > 0$, and let $E \subset F \subset \mathbb{R}^d$ be measurable subsets of $\mathbb{R}^d$ with $0 < |E| < |F| < \infty$. Let $\rho > 0$ be such that $|E| = |B_\rho|$ and let $x \in \mathbb{R}^d$ be such that $B_{2\rho}(x) \subset F$. Then*

$$\int_{F\setminus E} \frac{dy}{|x-y|^\lambda} \ge C_{d,\lambda} |E|^{1-\frac{\lambda}{d}}, \tag{2.1}$$

for some positive constant $C_{d,\lambda}$ depending only on d and λ. As a consequence, if $p \ge 1$, $|E| = |B_\rho|$ for some $\rho > 0$ and, for $p \ge 1$, D is measurable subset of F such that for almost every $x \in D$, the ball $B_{2\rho}(x) \subset F$, then

$$\int_D \int_{F\setminus E} \frac{dy\, dx}{|x-y|^{d+p}} \ge C_{d,p} |D|\, |E|^{-\frac{p}{d}}, \tag{2.2}$$

for some positive constant $C_{d,p}$ depending only on d and p.

Proof. For $y \in \mathbb{R}^d$, we have

$$B_\rho(y) = \big(B_\rho(y) \setminus E\big) \cup \big(B_\rho(y) \cap E\big) \text{ and } E = \big(E \setminus B_\rho(y)\big) \cup \big(E \cap B_\rho(y)\big).$$

Since $|E| = |B_\rho(y)|$, it follows that

$$|E \setminus B_\rho(y)| = |B_\rho(y) \setminus E| \text{ for } y \in \mathbb{R}^d. \tag{2.3}$$

Fix x such that $B_{2\rho}(x) \subset F$. We have

$$\int_{F\setminus E} \frac{dy}{|y-x|^\lambda} = \int_{(F\setminus E)\cap B_\rho(x)} \frac{dy}{|y-x|^\lambda} + \int_{(F\setminus E)\cap\left(F\setminus B_\rho(x)\right)} \frac{dy}{|y-x|^\lambda}$$

$$\geq \frac{1}{\rho^\lambda} |(F \setminus E) \cap B_\rho(x)| + \int_{(F\setminus E)\cap\big(F\setminus B_\rho(x)\big)} \frac{dy}{|y-x|^\lambda}. \tag{2.4}$$

Then

$$|(F \setminus E) \cap B_\rho(x)| \overset{B_\rho(x)\subset F}{=} |B_\rho(x) \setminus E| \overset{(2.3)}{=} |E \setminus B_\rho(x)| \overset{E\subseteq F}{=} |(F \setminus B_\rho(x)) \cap E|\,. \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\int_{F\setminus E} \frac{dy}{|y-x|^\lambda} \geq \frac{1}{\rho^\lambda} |(F \setminus B_\rho(x)) \cap E| + \int_{(F\setminus E)\cap\big(F\setminus B_\rho(x)\big)} \frac{dy}{|y-x|^\lambda}.$$

This yields

$$\begin{aligned}\int_{F\setminus E} \frac{dy}{|y-x|^\lambda} &\geq \int_{\big(F\setminus B_\rho(x)\big)\cap E} \frac{dy}{|y-x|^\lambda} + \int_{(F\setminus E)\cap\big(F\setminus B_\rho(x)\big)} \frac{dy}{|y-x|^\lambda} \\ &\overset{E\subseteq F}{\geq} \int_{F\setminus B_\rho} \frac{dy}{|y-x|^\lambda} \overset{B_{2\rho}(x)\subset F}{\geq} \int_{B_{2\rho}\setminus B_\rho} \frac{dy}{|y-x|^\lambda} \geq C_{d,p}|E|^{1-\lambda/d},\end{aligned}$$

which is (2.1).

Integrating (2.1) w.r.t. x in D, we obtain (2.2). $\square$

Remark 2.1. A similar version of inequality (2.1) has played crucial roles in deriving fractional versions of Sobolev [22] and Hardy [1] inequalities.

The following simple lemma is also used in the proof of Theorem 1.1.

Lemma 2.2. *Let $d \geq 1$, $p > 1$, $\delta > 0$, and let O be a ball in $\mathbb{R}^d$. Let $g \in L^1_{loc}(O)$. We have, $k \in \mathbb{N}_+$,*

$$\int_O \int_O_{|u(x)-u(y)|>2^k\delta} \frac{\delta^p}{|x-y|^{d+p}} \leq 2^{-k(p-1)} \int_O \int_O_{|u(x)-u(y)|>\delta} \frac{\delta^p}{|x-y|^{d+p}}. \tag{2.6}$$

Proof. By considering the function u/δ and by the recurrence, it suffices to consider the case $\delta = 1$ and $k = 1$. We have

$$\begin{aligned}\iint_{\substack{O\times O\\ |u(x)-u(y)|>2}} \frac{dx\,dy}{|x-y|^{d+p}} &= \iint_{\substack{O\times O\\ |u(x)-u(x/2+y/2)+u(x/2+y/2)-u(y)|>1}} \frac{dx\,dy}{|x-y|^{d+p}} \\ &\leq \iint_{\substack{O\times O\\ |u(x)-u(x/2+y/2)|>1}} \frac{dx\,dy}{|x-y|^{d+p}} + \iint_{\substack{O\times O\\ |u(x/2+y/2)-u(y)|>1}} \frac{dx\,dy}{|x-y|^{d+p}}.\end{aligned}$$

By a change of variables $z = x/2 + y/2$, we obtain

$$\iint_{\substack{O\times O\\ |u(x)-u(y)|>2}} \frac{dx\,dy}{|x-y|^{d+p}} \leq 2^{-(p-1)} \iint_{\substack{O\times O\\ |u(x)-u(y)|>1}} \frac{dx\,dy}{|x-y|^{d+p}},$$

which yields the conclusion for $\delta = 1$ and $k = 1$. $\square$

2.2. **Proof of part i) of Theorem 1.1.** In this proof, for notational ease, we denote $I_{\delta,p}$ by I_δ for $\delta > 0$. Without loss of generality we can assume $B = B_1$, $u_B = 0$, and $\delta = 1$. Define $\tilde{u}$ in $B_{3/2}$ by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in B_1 \\ u\left(\dfrac{(2-|x|)x}{|x|}\right) & \text{if } x \in B_{3/2} \setminus B_1. \end{cases}$$

We have, for all $\tau > 0$,

$$\Big|\Big\{x \in B_{3/2}; |\tilde{u}| \geq \tau\Big\}\Big| \leq C \Big|\Big\{x \in B_1; |u| \geq \tau\Big\}\Big| \tag{2.7}$$

and, see e.g., [18, Lemma 17],

$$I_1\left(\tilde{u}, B_{3/2}\right) \leq C I_1\left(u, B_1\right). \tag{2.8}$$

Using John-Nirenberg's inequality, we have

$$\Big|\Big\{x \in B_{3/2}; |\tilde{u}| \geq \ell\Big\}\Big| \leq 1/8^d, \tag{2.9}$$

if $\ell \geq c_1 M$ for some $c_1 > 0$.

We claim that, for $\ell \geq c_1 M$ and $\lambda > 2$,

$$\Big|\Big\{x \in B_1 : |u(x)| \geq \lambda \ell\Big\}\Big| \leq c_2 I_\ell\left(u, B_1\right) \Big|\Big\{x \in B_1 : |u(x)| \geq (\lambda - 1)\ell\Big\}\Big|^{\frac{p}{d}}. \tag{2.10}$$

In fact, fix an arbitrary $x \in B_{5/4}$ and let ρ be such that $|B_\rho(x)| = \big|\big\{x \in B_{3/2}; |u| \geq (\lambda-1)\ell\big\}\big|$. Since $\lambda > 2$, it follows from (2.9) that $\rho < 1/8$, which yields $B_{2\rho}(x) \subset B_{3/2}$. Applying Lemma 2.1 with $D = \{x \in B_{5/4}; |\tilde{u}| \geq \lambda\ell\} \cap O$, $E = \{x \in B_{3/2}; |\tilde{u}| \geq (\lambda-1)\ell\}$ and $F = B_{3/2}$, and using (2.7) and (2.8), we obtain (2.10).

Applying Lemma 2.2, we have, for $k \in \mathbb{N}$,

$$I_{2^k}(u, B_1) \leq 2^{-k(p-1)} I_1(u, B_1) \leq 2^{-k(p-1)} M.$$

Fix k_0 be such that for $k \geq k_0$, one has $c_2 2^{-k(p-1)} M \leq e^{-2\alpha}$, which yields

$$c_2 I_{2^k}(u, B_1) \leq e^{-\alpha (p/d)^3}. \tag{2.11}$$

Set

$$\ell_0 = \max\Big\{c_1 M, \Big(c_3 M e^{2\alpha}\Big)^{1/(p-1)}\Big\}. \tag{2.12}$$

Then, for some c_3 larger than c_2,

$$\ell_0 \geq \max\{c_1 M, 2^{k_0}\}.$$

Using (2.10), (2.11), and a standard iterative process, we have, for $\lambda \in \mathbb{N}$ and

$$\Big|\Big\{x \in B_1; |u| > \lambda \ell_0\Big\}\Big| \leq e^{-\alpha (p/d)^{\lambda+2}} \Big|\Big\{x \in B_1; |u| > \ell_0\Big\}\Big|. \tag{2.13}$$

This implies

$$\int_{B_1} e^{\alpha (p/d)^{|u|}} \, dx \leq \int_{\substack{B_1 \\ |u| \geq \ell_0}} e^{\alpha (p/d)^{|u|}} \, dx + \int_{\substack{B_1 \\ |u| \leq \ell_0}} e^{\alpha (p/d)^{\ell_0}} \, dx \leq C.$$

This implies the conclusion of part $i)$ with $\beta(\alpha, M) = \ell_0^{-1}$ where ℓ_0 is given by (2.12). □

2.3. **Proof of part** $ii)$ **of Theorem 1.1.** The proof of part $ii)$ is in the spirit of part $i)$. In fact, noting that if M_0 is small enough then (2.13) holds with $\ell_0 = 1$. The conclusion then follows. □

2.4. **Proof of** (1.13) **of Theorem 1.2.** The proof is similar to the one of part $ii)$ of Theorem 1.1 and is omitted. □

2.5. **Proof of Propositions 1.5 and 1.6.** Propositions 1.5 and 1.6 can be derived from Theorems 1.1 and 1.2 respectively after using local charts and appropriately extending u in a neighborhood of Ω (see, e.g., [18, Lemma 17]. The details are omitted. $\square$

3. Proof of Proposition 1.7

Without loss of generality, one might assume that $B = B_{1/e}$ and $\delta = 1$.

Proof of assertion (1.14). Fix $\gamma > \lambda > p/d > 1$, set, for $x \in B_{1/e}$,

$$u(x) = g(|x|) \quad \text{where} \quad g(r) = (\ln \lambda)^{-1} \ln \ln (1/r) \text{ for } r \in I := (0, 1/e).$$

It is clear that $g \in L^1(I)$. Using polar coordinates, we have

$$I_{1,p}(u, B_{1/e}) = \int_0^{1/e} \int_{\substack{0 \\ |g(r_1)-g(r_2)|>1}}^{1/e} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{r_1^{d-1} r_2^{d-1}}{|r_1\sigma_1 - r_2\sigma_2|^{p+d}} \, d\sigma_1 \, d\sigma_2 \, dr_1 \, dr_2.$$

$$\leq C_d \int_{\substack{0 \\ |g(r_1)-g(r_2)|>1}}^{1/e} \int_0^{1/e} \frac{r_1^{d-1} r_2^{d-1}}{|r_1 - r_2|^{p+d}} \, dr_1 \, dr_2. \tag{3.1}$$

We have, for $0 < r_1 < r_2 < e^{-1}$,

$$|g(r_1) - g(r_2)| > 1 \text{ if and only if } r_2 > r_1^{1/\lambda} \text{ and } 0 < r_1 < e^{-\lambda},$$

this yields

$$\frac{r_1 r_2}{(r_2 - r_1)^{1+p/d}} \leq \frac{C r_1}{r_2^{p/d}} \leq C,$$

for some positive constant C depending only on d, p, and λ. It follows that, for $0 < r_1 < r_2 < e^{-1}$ and $|g(r_1) - g(r_2)| > 1$,

$$\frac{r_1^{d-1} r_2^{d-1}}{|r_1 - r_2|^{p+d}} = \left(\frac{r_1 r_2}{(r_2 - r_1)^{\frac{p}{d}+1}} \right)^{d-1} \frac{1}{|r_1 - r_2|^{\frac{p}{d}+1}} \leq \frac{C}{|r_1 - r_2|^{\frac{p}{d}+1}}. \tag{3.2}$$

We derive from (3.1) and (3.2) that

$$I_{1,p}(u, B_{1/e}) \leq C I_{1,\frac{p}{d}}(g, I). \tag{3.3}$$

We have

$$I_{1,p/d}(g, I) = 2 \iint_{\substack{I \times I \\ |g(r_1)-g(r_2)|>1 \\ r_1 < r_2}} \frac{1}{|r_2 - r_1|^{1+\frac{p}{d}}} \, dr_1 \, dr_2 \tag{3.4}$$

$$\leq C \int_0^{e^{-\lambda}} \left(\frac{1}{\left(r_1^{1/\lambda} - r_1\right)^{\frac{p}{d}}} - \frac{1}{\left(e^{-1} - r_1\right)^{\frac{p}{d}}} \right) dr_1 < +\infty,$$

since $r_1^{1/\lambda} - r_1 \geq C r_1^{1/\lambda}$ and $e^{-1} - r_1 \geq C$ for $r_1 \in (0, e^{-\lambda})$.

On the other hand, for any $\tau \in I$, we have, with $\rho = \frac{\ln \gamma}{\ln \lambda} - 1$,

$$\int_{B_\tau} e^{\alpha \gamma^g} \, dx = \int_0^\tau e^{\alpha \left(\log r^{-1} \right)^{(1+\rho)}} r^{d-1} \, dr = +\infty, \tag{3.5}$$

since $\lim_{r \to 0_+} \left(\log r^{-1} \right)^{1+\rho} / \log r^{-1} = +\infty$.

Set, for $0 < \tau < e^{-1}$,

$$u_\tau(x) = u(\tau x) \text{ for } x \in B_{e^{-1}}.$$

Then

$$I_{1,p}(u_\tau, B_{e^{-1}}) = \tau^{p-d} I_{1,p}(u, B_{\tau e^{-1}}) \to 0 \text{ as } \tau \to 0. \tag{3.6}$$

Combining (3.5) and (3.6) yields the conclusion since for any $M > 0$ we can choose $\tau > 0$ small enough so that $I_{1,p}(u_\tau, B_{e^{-1}}) \le M$. □

Proof of assertion (1.15). Let $n \in \mathbb{N}$ large and fix $1 < q < \gamma$ and denote $q' = q/(q-1)$. Define

$$u_n(x) = g_n(|x|) \text{ where } g_n(r) = \begin{cases} \ln^{\frac{1}{q}} n & \text{if } 0 \le r \le \frac{1}{n}, \\ \dfrac{\ln\left(\frac{1}{r}\right)}{\ln^{\frac{1}{q'}} n} & \text{if } \frac{1}{n} \le r \le \frac{1}{e}. \end{cases}$$

As in (3.3), we have

$$I_{1,d}(u_n, B_{1/e}) \le C I_{1,1}(g_n, I), \tag{3.7}$$

where $I = (0, 1/e)$.

We now estimate $I_{1,1}(g_n, I)$. Denote $J_n = (0, 1/n)$, and $K_n = I \setminus J_n$. We have

$$I_{1,1}\left(g_n, I\right) = 2\mathcal{I}_1 + \mathcal{I}_2. \tag{3.8}$$

where

$$\mathcal{I}_1 = \iint_{\substack{I_n \times J_n \\ |g_n(r_1) - g_n(r_2)| > 1}} \frac{1}{|r_1 - r_2|^2} \, dr_1 \, dr_2 \quad \text{and} \quad \mathcal{I}_2 = \iint_{\substack{J_n \times J_n \\ |g_n(r_1) - g_n(r_2)| > 1}} \frac{1}{|r_1 - r_2|^2} \, dr_1 \, dr_2.$$

We next estimate $\mathcal{I}_1$ and $\mathcal{I}_2$. We begin with $\mathcal{I}_1$. For $(r_1, r_2) \in J_n \times K_n$, we have $|g_n(r_1) - g_n(r_2)| > 1$ if and only if

$$\left| \ln^{\frac{1}{q}} n - \frac{\ln\left(\frac{1}{r_2}\right)}{\ln^{\frac{1}{q'}} n} \right| > 1, \text{ this implies } \frac{1}{e} \ge r_2 > a_n := \frac{e^{(\log n)^{\frac{1}{q'}}}}{n}.$$

It follows that

$$\mathcal{I}_1 = \int_0^{\frac{1}{n}} \int_{a_n}^{\frac{1}{e}} \frac{dr_2 dr_1}{|r_1 - r_2|^2} \le \ln\left(\frac{a_n}{a_n - 1/n}\right) \to 0 \text{ as } n \to +\infty. \tag{3.9}$$

We next deal with $\mathcal{I}_2$. For $(r_1, r_2) \in K_n \times K_n$ with $r_2 \ge r_1$, we have $|g_n(r_1) - g_n(r_2)| > 1$ if and only if

$$\left| \frac{\ln\left(\frac{1}{r_1}\right) - \ln\left(\frac{1}{r_2}\right)}{\ln^{\frac{1}{q'}} n} \right| > 1, \text{ this implies } \frac{1}{e} \ge r_2 > r_1 b_n \text{ and } \frac{1}{n} \le r_1 < \frac{1}{e b_n} \text{ with } b_n = e^{\ln^{\frac{1}{q'}} n}.$$

We then have

$$\mathcal{I}_2 = 2 \int_{1/n}^{1/(eb_n)} \int_{r_1 b_n}^{\frac{1}{e}} \frac{dr_2}{(r_2 - r_1)^2} dr_1 \le 2 \int_{1/n}^{1/(eb_n)} \frac{1}{r_1(b_n - 1)} \, dr_1 \to 0 \text{ as } n \to +\infty. \tag{3.10}$$

Combining (3.8), (3.9), and (3.10) yields

$$\lim_{n \to \infty} I_{1,1}\left(g_n, I\right) = 0.$$

which yields, by (3.7),

$$\lim_{n \to \infty} I_{1,d}\left(u_n, B_{1/e}\right) = 0. \tag{3.11}$$

We have
$$\int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\ln\left(\frac{1}{r}\right)}{\ln^{\frac{1}{q'}} n} dr = \frac{1}{(\log n)^{\frac{1}{q'}}} \left(\frac{1}{2}\ln(2e) - \frac{1}{n}\ln(ne)\right) \to 0 \text{ as } n \to \infty.$$
This implies
$$0 \le \lim_{n\to+\infty} \int_{B_{1/e}} u_n \, dx \le \lim_{n\to+\infty} \int_I g_n \, dx = 0. \tag{3.12}$$
On the other hand, since $\gamma > q$, we have
$$\int_I r^{d-1} e^{\alpha g_n^r} \ge \int_0^{\frac{1}{n}} r^{d-1} e^{\alpha \ln^{\gamma/q} n} = \frac{1}{dn^d} e^{\alpha \ln^{\gamma/q} n} \to +\infty \text{ as } n \to +\infty. \tag{3.13}$$
The conclusion now follows from (3.11), (3.12), and (3.13). □

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