A gyrokinetic model for the tokamak periphery

R. Jorge¹,², B. Frei¹, A. Baillo³, P. Ricci¹, N. F. Loureiro³

Sherwood Fusion Theory Conference, Auburn AL - USA, April 23, 2018
The Tokamak Periphery = Edge + SOL

- Core boundary conditions
- Heat exhaust
- Plasma fueling and ashes removal
- Impurity control
Properties of Periphery Turbulence

- **Low-frequency** \( \omega \ll \Omega_i \)

- **Large Scale Fluctuations**
  \[ k_{\perp} \rho_i \ll 1 \quad \frac{e\phi}{T_e} \sim 1 \]

- **Small Scale Fluctuations**
  \[ k_{\perp} \rho_i \sim 1 \quad \frac{e\phi}{T_e} \ll 1 \]

- **Wide range of temperatures and densities**
  \[ T_e \sim 1 - 10^3 \text{ eV} \quad n \sim 10^{18} - 10^{20} \text{ m}^{-3} \]
Properties of Periphery Turbulence

- Low-frequency \( \omega \ll \Omega_i \)

- Large Scale Fluctuations
  \[ k_{\perp} \rho_i \ll 1 \quad \frac{e\phi}{T_e} \sim 1 \]

- Small Scale Fluctuations
  \[ k_{\perp} \rho_i \sim 1 \quad \frac{e\phi}{T_e} \ll 1 \]

- Wide range of temperatures and densities
  \[ T_e \sim 1 - 10^3 \text{ eV} \quad n \sim 10^{18} - 10^{20} \text{ m}^{-3} \]
# First Heroic Steps On The Way To An Edge Model

<table>
<thead>
<tr>
<th></th>
<th>Qin et al. 2006&lt;sup&gt;1&lt;/sup&gt;</th>
<th>Hahm et al. 2009&lt;sup&gt;2&lt;/sup&gt;</th>
<th>Dimits et al. 2012&lt;sup&gt;3&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Small Scale Fluctuations</strong></td>
<td>EM $O(\epsilon^2)$</td>
<td>EM $O(\epsilon^2)$</td>
<td>EM $O(\epsilon^2)$</td>
</tr>
<tr>
<td><strong>Large Scale Fluctuations</strong></td>
<td>EM $O(\epsilon)$</td>
<td>ES $O(\epsilon^2)$</td>
<td>ES $O(\epsilon^2)$</td>
</tr>
<tr>
<td><strong>Poisson’s Eq.</strong></td>
<td>$\int (...),d^3v$</td>
<td>Long-Wavelength Limit</td>
<td>$\int (...),d^3v$</td>
</tr>
<tr>
<td><strong>Collisions</strong></td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

<sup>1</sup> Qin et al., Contri. Plasma Phys. 46 (2006)

<sup>2</sup> Hahm et al., Physics of Plasmas 16 (2009)

<sup>3</sup> Dimits, Physics of Plasmas 19 (2012)
Our Model

Retain

- Both large scale (full-F) and small scale fluctuations
- Second order
- Full Coulomb collisions
- Numerical efficiency

How we proceed

Single Particle Dynamics \[\rightarrow\] Gyrokinetic Theory \[\rightarrow\] Moment Hierarchy
**Single Particle Dynamics - Ordering Assumptions**

**Magnetized Plasma**

\[
\frac{\omega}{\Omega_i} \sim \epsilon \quad k_\perp \rho_i \\
\rho_i \frac{1}{L_p} \sim \epsilon
\]

**Fluctuation Scales and Amplitude**

\[
k_\perp \rho_i \frac{e\phi}{T_e} = \epsilon \ll 1
\]

**Electromagnetic**

\[
v_\parallel A_\parallel \sim \phi
\]

**Arbitrary Collisionalities**

(still magnetized)

\[
\frac{\nu_{ei}}{\Omega_i} \lesssim \epsilon
\]
Single Particle Dynamics – Hamiltonian Perturbation Theory

Particle Lagrangian

\[ L(x, v) \]

2-Step Phase-Space Reduction

Lagrangian

\[ \Gamma(R, v_\parallel, \mu) \]

(cyclic coordinate \( \theta \), conserved \( \mu \))
Single Particle Dynamics – First Step

Large Scales \( \epsilon \sim k_{\perp} \rho_i \ll 1, \frac{e\phi}{T_e} \sim 1 \)

Start from

\[
L = qA \cdot \dot{x} - q\phi - \frac{mv^2}{2}
\]

Split between parallel and perpendicular velocity

Describe guiding center motion \( O(\epsilon^2) \)

\[
L = qA^* \cdot \dot{R} - q\phi^* - \frac{mv_\parallel^2}{2} + \mu \frac{m\dot{\theta}}{q}
\]
Single Particle Dynamics – Second Step

Small Scales \( \epsilon \sim \frac{e\phi}{T_e} \ll 1, \ k_{\perp} \rho_i \sim 1 \)

Start from

\[
L = q A^* \cdot \dot{R} - q \phi^* - \frac{mv^2_\parallel}{2} + \mu \frac{m\dot{\theta}}{q}
\]

Introduce small scale fluctuations

Describe gyro center motion \( O(\epsilon^2) \)

\[
L = q A^* \cdot \dot{R} - q \phi^* - \frac{mv^2_\parallel}{2} + \mu \frac{m\dot{\theta}}{q} - q \langle \phi - v_{\parallel} A_{\parallel} \rangle - q \frac{\partial \langle (\phi - v_{\parallel} A_{\parallel})^2 \rangle}{\partial \mu}
\]
Single Particle Dynamics – Equations of Motion

\[ \dot{R} = v_{\parallel} b + v_E \times B + \text{Other Drifts} \]

Including large scale \( \frac{\nabla \phi(R) \times B}{B^2} \) Curvature Drift

and small scale \( \frac{\nabla \langle \phi(x) \rangle \times B}{B^2} \) fluctuations Polarization Drift Non-Linear Drifts
Single Particle Dynamics – Equations of Motion

\[ \dot{\mathbf{R}} = v_{\parallel} \mathbf{b} + \mathbf{v}_{\mathbf{E} \times \mathbf{B}} + \text{Other Drifts} \]

\[ \dot{v}_{\parallel} = qE_{\parallel} + \mu \nabla_{\parallel} B + \text{Non-Linear Forces} \]

Including large scale \( \nabla_{\parallel} \phi(\mathbf{R}) \)

and small scale \( \nabla_{\parallel} \langle \phi(x) \rangle \) fluctuations
Single Particle Dynamics – Equations of Motion

\[ \dot{\mathbf{R}} = v_\parallel \mathbf{b} + \mathbf{v}_{E \times B} + \text{Other Drifts} \]

\[ \dot{v}_\parallel = qE_\parallel + \mu \nabla_\parallel B + \text{Non-Linear Forces} \]

\[ \dot{\mu} = 0 \]

Conserved Adiabatic Invariant
From Single Particle to Particle Distribution

Gyrokinetic Equation:

\[ \frac{\partial F}{\partial t} + \dot{\mathbf{R}} \cdot \nabla F + \dot{v}_\parallel \frac{\partial F}{\partial v_\parallel} = \langle C(F) \rangle \]

Challenges

- 5-D + time
- Full Coulomb Collisions
- Coupling to Maxwell’s equations (integro-differential system)

These challenges can be successfully approached by using a moment hierarchy
Our Goal – Turn Gyrokinetic Eq. Into a Hierarchy of Fluid-Like Eqs.

Expand GK Equation into a Set of 3D

Moment Hierarchy Equations

\[
\frac{dn}{dt} = \ldots
\]

\[
\frac{dv}{dt} = \ldots
\]

\[
\frac{dT}{dt} = \ldots
\]

\[
\ldots
\]

Retain necessary kinetic effects and no more
Advantages of a Moment Hierarchy Model

Set of fluid-like equations with reasonable computational cost

Tune the number of moments according to the desired level of accuracy (function of collisionality)

Reduce to the fluid model at high collisionality
3 Steps To Build a Moment Hierarchy Model

1. Choose an orthogonal polynomial basis for $F$

\[ F = F_M \sum_{p,j} N^{pj}(\mathbf{R}) H_p(v_\parallel) L_j(\mu) \]

2. Project kinetic equation onto basis

\[ \int (\text{GK Eq.}) \ H_p L_j \ dv_\parallel d\mu \]

3. Obtain evolution equation for the basis coefficients

\[ \frac{\partial N^{pj}}{\partial t} = \ldots \]
From Gyrokinetic Equation to Moment Hierarchy

\[ \int \text{(GK Eq.) } H_p L_j \, dv_{||} \, d\mu \]

\[ \frac{\partial N^{pj}}{\partial t} + \nabla \cdot \mathbf{R}^{pj} - \frac{\sqrt{2p}}{v_{th}} v^p_{||} - 1 \mathbf{F}^{pj} = C^{pj} \]

- **Spatial evolution of Moments + Fields**
- **Fluid Operator** (density, velocity, temperature)
- **Time Evolution**
- **Forces included at \( p > 0 \)**
- **Collisions**
Example – 1D Linear Gyrokinetic Moment Hierarchy

\[ \frac{\partial N_{pj}}{\partial t} + \frac{1}{2} \frac{\partial N_{p+1j}}{\partial z} + p \frac{\partial N_{p-1j}}{\partial z} = 2\delta_{p,1} K_j (k_{\perp} \rho) E_{||} + C_{pj} \]

- **Time Evolution**
- **Phase Mixing** (coupling with other moments)
- **Electric Field Drive**
- **Collisions**
Example – 1D Linear Gyrokinetic Moment Hierarchy

\[ \frac{\partial N^{pj}}{\partial t} + \frac{1}{2} \frac{\partial N^{p+1j}}{\partial z} + p \frac{\partial N^{p-1j}}{\partial z} = 2\delta_{p,1} K_j(k_{\perp \rho}) E_{||} + C^{pj} \]

Phase Mixing (coupling with other moments)

Time Evolution

Collisions

\[ K_j(x) = \frac{1}{j!} \left( \frac{x}{2} \right)^{2j} e^{-x^2/4} \]

Analytical Closed formula for the Gyroaverage operation

Full finite Larmor radius effects
Projection of the Full Coulomb Collision Operator

\[ C^{pj} = \int \langle C(F) \rangle H_p L_j dv \parallel d\mu \]

with \[ C(F) = \frac{\partial}{\partial v} [A(F)F] + \frac{\partial^2}{\partial v \partial v} : [D(F)F] \]

- Bilinear
- Tensorial Nature
- Gyroaveraging Operation
- No parallel/perpendicular velocity symmetries

Not immediate…
Collision Operator - Spherical Harmonic Decomposition

\[ C[F] \sim C[\mathbf{P}^{lk}(\mathbf{v})] \sim \mathbf{P}^{lk}(\mathbf{v}) \]

**Gyroaveraging Procedure**

**Large Scales** \( k_\perp \rho_i \ll 1 \)

\[
\langle \mathbf{P}^{lk}(\mathbf{v}) \rangle \quad \rightarrow \quad H_p(\nu_\parallel) L_j(\mu)
\]

**Small Scales** \( \frac{e \phi}{T_e} \ll 1 \)

\[
\langle \mathbf{P}^{lk}(\mathbf{v}) e^{in\theta} \rangle \quad \rightarrow \quad f(n) H_p(\nu_\parallel) K_j(k_\perp \rho) L_j(\mu)
\]
Moments of the Collisional Operator

\[ C^{pj} = \int \langle C(F) \rangle H_p L_j \, dv_\parallel \, d\mu \]

After integration

\[ C^{pj} = \text{Kernel} \times f(n) \times \text{moments of } F \]
Physics That We Are Able To Capture

Low Collisionality

Particle resonance effects retrieved

High-Collisionality
Semi-collisional closure

Drift-Reduced Braginskii equations retrieved

\[
\frac{\delta F}{F_M} \sim \frac{\lambda_{mfp}}{L||}
\]

\[
N^{30} \sim q|| = -\chi|| \nabla|| T
\]
Drift-Wave Instability Case Study

Peak Growth Rate

\[ \eta \sim 0.15 \]

\[ 10^{-2} \leq \nu \leq 10^{8} \]

Collison Frequency
Drift-Wave Instability Case Study

Peak Growth Rate

Collision Frequency

- Fluid
- Collisionless
- Moment-Hierarchy (2,0)
Drift-Wave Instability Case Study

Peak Growth Rate

Collision Frequency

- Fluid
- Collisionless
- Moment-Hierarchy (2,0)
- Moment-Hierarchy (3,1)
Drift-Wave Instability Case Study

Peak Growth Rate

Collision Frequency

- Fluid
- Collisionless
- Moment-Hierarchy (2,0)
- Moment-Hierarchy (3,1)
- Moment-Hierarchy (5,2)
Drift-Wave Instability Case Study

Peak Growth Rate

Collision Frequency
Drift-Wave Instability Case Study

Peak Growth Rate

Collision Frequency

- Fluid
- Collisionless
- Moment-Hierarchy (2,0)
- Moment-Hierarchy (3,1)
- Moment-Hierarchy (5,2)
- Moment-Hierarchy (8,4)
- Moment-Hierarchy (18,4)
<table>
<thead>
<tr>
<th></th>
<th>This Work</th>
<th>Qin et al. 2006</th>
<th>Hahm et al. 2009</th>
<th>Dimits et al. 2012</th>
<th>Madsen et al. 2013</th>
<th>Mandell et al. 2018</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Large Scales</strong></td>
<td>EM O(e^2)</td>
<td>EM O(e)</td>
<td>ES O(e^2)</td>
<td>ES O(e^2)</td>
<td>ES O(e)</td>
<td>ES O(e)</td>
</tr>
<tr>
<td><strong>Small Scales</strong></td>
<td>EM O(e^2)</td>
<td>EM O(e^2)</td>
<td>EM O(e^2)</td>
<td>EM O(e^2)</td>
<td>EM O(e)</td>
<td>ES O(e)</td>
</tr>
<tr>
<td><strong>Collisions</strong></td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td><strong>Poisson’s Eq.</strong></td>
<td>Moments</td>
<td>∫(...) d^3v</td>
<td>Long-Wavelength Limit</td>
<td>∫(...) d^3v</td>
<td>Long-Wavelength Limit, Padé</td>
<td>Moments</td>
</tr>
<tr>
<td>**B_</td>
<td></td>
<td>**</td>
<td>Exact</td>
<td>Exact</td>
<td>Exact</td>
<td>Exact</td>
</tr>
</tbody>
</table>

2 Hahm et al., Physics of Plasmas 16 (2009)
3 Dimits, Physics of Plasmas 19 (2012)
4 Madsen, Physics of Plasmas 20 (2013)
5 Mandell et al., J. Plasma Phys 84 (2018)
Acknowledgements

This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 under grant agreement No 633053.

RJ was partially funded by the Portuguese FCT - Fundação para a Ciência e Tecnologia, under grant PD/BD/105979/2014.

NFL was partially funded by US Department of Energy Grant no. DE-FG02-91ER54109.

The views and opinions expressed herein do not necessarily reflect those of the European Commission.