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for spatio-temporal processes

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MULTILEVEL ENSEMBLE KALMAN FILTERING FOR SPATIO-TEMPORAL PROCESSES

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ABSTRACT. This work concerns state-space models, in which the state-space is an infinite-dimensional spatial field, and the evolution is in continuous time, hence requiring approximation in space and time. The multilevel Monte Carlo (MLMC) sampling strategy is leveraged in the Monte Carlo step of the ensemble Kalman filter (EnKF), thereby yielding a multilevel ensemble Kalman filter (MLEnKF) for spatio-temporal models, which has provably superior asymptotic error/cost ratio. A practically relevant stochastic partial differential equation (SPDE) example is presented, and numerical experiments with this example support our theoretical findings.

Key words: Monte Carlo, multilevel, filtering, Kalman filter, ensemble Kalman filter, partial differential equations (PDE).

AMS subject classification: 65C30, 65Y20

1. INTRODUCTION

Filtering refers to the sequential estimation of the state v and/or parameters p of a system through sequential incorporation of online data y . The most complete estimation of the state v_n at time n is given by its probability distribution conditional on the observations up to the given time $\mathbb{P}(dv_n | y_1, \dots, y_n)$ [29, 1]. For linear Gaussian systems the analytical solution may be given in closed form, via update formulae for the mean and covariance known as the Kalman filter [33]. However, for more general cases there are no closed form solution. One must therefore resort to either algorithms which approximate the probabilistic solution by leveraging ideas from control theory in the data assimilation community [34, 29], or Monte Carlo methods to approximate the filtering distribution itself [1, 16, 12]. The ensemble Kalman filter (EnKF) [10, 18] combines elements of both approaches. In the linear Gaussian case it converges to the Kalman filter solution [43], and even in the nonlinear case, under suitable assumptions it converges [41, 40] to a limit which is optimal among those which incorporate the data linearly and uses a single update iteration [40, 42, 46]. In the case of spatially extended models approximated on a numerical grid, the state space itself may become very high-dimensional and even the linear solves may become intractable. Therefore, one may be inclined to use the EnKF filter even for linear Gaussian problems in which the solution is computationally intractable despite being given in closed form by the Kalman filter.

Herein the underlying problem will admit a hierarchy of approximations with cost inversely proportional to accuracy, and it will be necessary to approximate the target for a single prediction step. Very recently, a number of works have emerged which extend the multilevel Monte Carlo (MLMC) framework to the context of Monte Carlo algorithms designed for Bayesian inference. Examples include Markov chain Monte Carlo [15, 25], sequential Monte Carlo samplers [5, 28, 44], particle

filters [27, 22], and EnKF [26]. See also [24, 20] for seminal works on MLMC and the recent overview [21].

The filtering papers [27, 22, 26] thusfar consider only finite-dimensional SDE forward models, with the approximation error only arising from time discretization.

The present work considers multilevel EnKF (MLEnKF) [26] for spatio-temporal processes. It was mentioned above that the limiting EnKF distribution, the so-called mean-field EnKF (MFEEnKF), is in general not the Bayesian posterior filtering distribution and has a fixed bias. The error of the EnKF approximation may be decomposed into MC error and this Gaussian bias as shown in [40]. According to folklore, small sample sizes are suitable, and it may well be due to minimum error being limited by the bias. Nonetheless, the latter is difficult to quantify and deal with, while the MC error can be controlled and minimized. Unfortunately, scientists are often limited to small ensemble sizes anyway, due to an extremely high-dimensional underlying state space, which is approximating a spatial field. Within the MLEnKF framework developed here, a much smaller MC error can be obtained for the same fixed cost, which will lower the cost requirement for practitioners to ensure that the MC error is commensurate with the bias. Furthermore, it has been shown in [35, 49, 50, 36] that signal tracking stability of EnKF is based on a feedback control mechanism, which can be established for a single member ensemble in 3DVAR [9, 7, 39, 48, 45, 23, 19]. The greater accuracy of EnKF in comparison to 3DVAR [38] is afforded presumably by its use of the ensemble statistics, and the relation to the optimal linear update. Therefore, it is of interest to improve the MC approximation. Convergence of the square root EnKF in infinite-dimensions for linear systems was considered in [37]. As in that work, we will require that the limiting covariance is trace-class.

It is worth mentioning optimal filtering by particle filters here. Typically particle filters are not used for spatial processes, or even modestly high-dimensional processes, due to very bad scaling of importance sampling in high dimensions, which can even be exponential [6, 2]. There has been some work in the past few years which overcomes this issue either for particular examples [4] or by allowing for some bias [3, 52, 47, 53]. However, particle filters cannot yet be considered practically applicable for high-dimensional problems, let alone general spatial processes. MLMC has been applied recently to particle filters, in the context where the approximation arises due to time discretization of a finite-dimensional SDE [27, 22]. It is a very interesting open problem to design multilevel particle filters for spatial processes.

The rest of the paper will be organized as follows. In section 2 the notation and problem will be introduced, and the spatio-temporal multilevel EnKF (MLEnKF) will be introduced in sub-section 2.4. In section 3 it is proven that indeed the spatial MLEnKF inherits almost the same favorable asymptotic “cost-to-error” as the standard MLMC for a finite time horizon, and its mean-field limiting distribution is the filtering distribution in the linear and Gaussian case. In section 4 an example of a practically relevant SPDE and numerical method are given for the purposes of illustration, and the required assumptions are verified. In section 5 some numerical experiments are conducted on the example from section 4, and the results of these experiments corroborate the theory. Finally, conclusions and future directions are presented in section 6.

2. KALMAN FILTERING

2.1. General set-up. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where Ω is a set of events, \mathcal{F} is a sigma algebra of subsets of Ω and \mathbb{P} is the associated probability measure. Let \mathcal{V} be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and norm $\|\cdot\|_{\mathcal{V}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{V}}}$. Let further $V \subset \mathcal{V}$ denote a subspace which is closed in the topology induced by the stronger norm $\|\cdot\|_V = \sqrt{\langle \cdot, \cdot \rangle_V}$. For an arbitrary separable Banach space $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$, the associated L^p -Bochner space is denoted

$$L^p(\Omega, \mathcal{K}) = \{u : \Omega \rightarrow \mathcal{K} \mid u \text{ is measurable and } \mathbb{E}[\|u\|_{\mathcal{K}}^p] < \infty\}, \quad \text{for } p \in [1, \infty),$$

with norm $\|u\|_{L^p(\Omega, \mathcal{K})} = (\mathbb{E}[\|u\|_{\mathcal{K}}^p])^{1/p}$, or just $\|u\|_p$ when the meaning is clear. Furthermore, for a pair of arbitrary Banach spaces $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ into $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$, the complete and linear space of bounded linear operators from the former into the latter is denoted

$$L(\mathcal{J}, \mathcal{K}) := \{H : \mathcal{J} \rightarrow \mathcal{K} \mid H \text{ is linear and } \|H\|_{L(\mathcal{J}, \mathcal{K})} < \infty\},$$

with associated operator norm

$$(1) \quad \|H\|_{L(\mathcal{J}, \mathcal{K})} := \sup_{x \in \mathcal{J} \setminus \{0\}} \frac{\|Hx\|_{\mathcal{K}}}{\|x\|_{\mathcal{J}}}.$$

Consider the general stochastic evolution equation for the random variable $u_n \in L^p(\Omega, V)$, with dynamics

$$(2) \quad u_{n+1} = \Psi(u_n), \quad \text{for } n = 0, 1, \dots, N-1.$$

In particular, we will be concerned herein with the case in which $\Psi : L^p(\Omega, V) \times \Omega \rightarrow L^p(\Omega, V)$ is the finite-time evolution of an SPDE or, equivalently, a discrete random mapping (possibly nonlinear) of a spatially extended state given as a random $\cap_{p \geq 2} L^p$ -integrable element of the separable Hilbert space $V \subset \mathcal{V}$. Let $\{\phi_k\}_{k=1}^{\infty} \subset \mathcal{V}$ be a complete orthonormal basis of \mathcal{V} , so that any $u \in \mathcal{V}$ admits the representation $u = \sum_{k=1}^{\infty} \langle u, \phi_k \rangle_{\mathcal{V}} \phi_k$. For $m \in \mathbb{N}$, we use the notation $\mathcal{R}_m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle)$ to denote the m -dimensional Hilbert space with the Euclidean inner product and the induced norm $\|\cdot\|_{\mathcal{R}_m} := \langle \cdot, \cdot \rangle^{1/2}$.

Given the history of signal observations

$$(3) \quad y_n = Hu_n + \eta_n, \quad n = 1, 2, \dots, N,$$

for $H \in L(\mathcal{V}, \mathcal{R}_m)$, i.i.d. random variables $\{\eta_n\}$ with $\eta_n \sim N(0, \Gamma)$ and $\Gamma \in \mathbb{R}^{m \times m}$ symmetric positive definite, the objective is to track the signal u_n given the observations Y_n where $Y_n = (y_1, y_2, \dots, y_n)$. Notice that under the given assumptions we have a hidden Markov model. That is, the distribution of the random variable we seek to approximate admits the following sequential structure

$$(4) \quad \begin{aligned} \mathbb{P}(du_n | Y_n) &= \frac{1}{Z(Y_n)} \mathcal{L}(u_n; y_n) \mathbb{P}(du_n | Y_{n-1}), \\ \mathbb{P}(du_n | Y_{n-1}) &= \int_{u_{n-1} \in \mathcal{V}} \mathbb{P}(du_n | u_{n-1}) \mathbb{P}(du_{n-1} | Y_{n-1}), \\ \mathcal{L}(u_n; y_n) &= \exp \left\{ -\frac{1}{2} \|\Gamma^{-1/2}(y_n - Hu_n)\|_{\mathcal{R}_m}^2 \right\}, \\ Z(Y_n) &= \int_{u_n \in \mathcal{V}} \mathcal{L}(u_n; y_n) \mathbb{P}(du_n | Y_{n-1}). \end{aligned}$$

It will be assumed that $\Psi(\cdot)$ cannot be evaluated exactly, but that there exists a hierarchy of accuracy levels at which it can be approximated, each with its associated cost. The explicit dependence on ω will be suppressed where confusion is not possible. For notational simplicity, we will consider the particular case in which the

map $\Psi(\cdot)$ does not depend on n . Note that the results easily extend to the non-autonomous case, provided the assumptions on $\{\Psi_n\}_{n=1}^N$ are uniform with respect to n . The specialization is merely for notational convenience. In particular, we will denote by $\{\Psi^\ell : L^p(\Omega, \mathcal{V}) \times \Omega \rightarrow L^p(\Omega, \mathcal{V})\}_{\ell=0}^\infty$ a hierarchy of approximations to the solution $\Psi := \Psi^\infty$. First some assumptions must be made.

Assumption 1. *For every $p \geq 2$, it holds that $\psi : L^p(\Omega, V) \times \Omega \rightarrow L^p(\Omega, V)$, and for all $u, v \in L^p(\Omega, \mathcal{V})$, the solution operators $\{\Psi^\ell\}_{\ell=0}^\infty$ satisfy the following conditions: there exists a constant $0 < c_\Psi < \infty$ depending on p such that*

- (i) $\|\Psi^\ell(u) - \Psi^\ell(v)\|_{L^p(\Omega, \mathcal{V})} \leq c_\Psi \|u - v\|_{L^p(\Omega, \mathcal{V})}$,
- (ii) $\|\Psi^\ell(u)\|_{L^p(\Omega, \mathcal{V})} \leq c_\Psi (1 + \|u\|_{L^p(\Omega, \mathcal{V})})$.

2.2. Some details on Hilbert spaces, Hilbert-Schmidt operators, and Cameron-Martin spaces. Let \mathcal{K}_1 and \mathcal{K}_2 be two separable Hilbert spaces with respective inner products $\langle \cdot, \cdot \rangle_{\mathcal{K}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{K}_2}$, and the induced norms

$$(5) \quad \|u\|_{\mathcal{K}_1} := \langle u, u \rangle_{\mathcal{K}_1}^{1/2}, \quad \text{and} \quad \|u\|_{\mathcal{K}_2} := \langle u, u \rangle_{\mathcal{K}_2}^{1/2}.$$

The tensor product of \mathcal{K}_1 and \mathcal{K}_2 is a Hilbert space with the inner product defined by

$$(6) \quad \langle u \otimes v, u' \otimes v' \rangle_{\mathcal{K}_1 \otimes \mathcal{K}_2} = \langle u, u' \rangle_{\mathcal{K}_1} \langle v, v' \rangle_{\mathcal{K}_2} \quad \forall u, u' \in \mathcal{K}_1, \quad \forall v, v' \in \mathcal{K}_2,$$

for rank-1 tensors, and extended by linearity to arbitrary tensors of finite rank. The tensor product $\mathcal{K}_1 \otimes \mathcal{K}_2$ is the completion of this set with respect to the induced norm $\|\cdot\|_{\mathcal{K}_1 \otimes \mathcal{K}_2}$. It holds that

$$(7) \quad \|u \otimes v\|_{\mathcal{K}_1 \otimes \mathcal{K}_2} = \|u\|_{\mathcal{K}_1} \|v\|_{\mathcal{K}_2}.$$

Let $\{\phi_k\}$ and $\{e_k\}$ be orthonormal bases for \mathcal{K}_1 and \mathcal{K}_2 , respectively, and observe that finite sums of rank-1 tensors of the form $X := \sum_{i,j} \alpha_{ij} \phi_i \otimes e_j \in \mathcal{K}_1 \otimes \mathcal{K}_2$ can be identified with a bounded linear mapping

$$(8) \quad T_X : \mathcal{K}_2^* \rightarrow \mathcal{K}_1 \quad \text{with} \quad T_X(f) := \sum_{i,j} \alpha_{ij} f(e_j) \phi_i, \quad \text{for } f \in \mathcal{K}_2^*.$$

For two bounded linear operators $A, B : \mathcal{K}_2^* \rightarrow \mathcal{K}_1$ we recall the definition of the Hilbert-Schmidt inner product and norm

$$(9) \quad \langle A, B \rangle_{HS} = \sum_k \langle A e_k^*, B e_k^* \rangle_{\mathcal{K}_1}, \quad |A|_{HS} = \langle A, A \rangle_{HS}^{1/2},$$

where $\{e_k^*\}$ is the orthonormal basis of \mathcal{K}_2^* satisfying $e_k^*(e_j) = \delta_{jk}$ for all j, k in the considered index set. A bounded linear operator $A : \mathcal{K}_2^* \rightarrow \mathcal{K}_1$ is called a Hilbert-Schmidt operator if $|A|_{HS} < \infty$ and $HS(\mathcal{K}_2^*, \mathcal{K}_1)$ is the space of all such operators. In view of (8),

$$\begin{aligned} |T_X|_{HS}^2 &= \sum_k \left\langle \sum_{i,j} \alpha_{ij} e_k^*(e_j) \phi_i, \sum_{i',j'} \alpha_{i'j'} e_k^*(e_{j'}) \phi_{i'} \right\rangle_{\mathcal{K}_1} \\ &= \sum_{i,j} |\alpha_{ij}|^2 \\ &= \|X\|_{\mathcal{K}_1 \otimes \mathcal{K}_2}^2. \end{aligned}$$

By completion, the tensor product space $\mathcal{K}_1 \otimes \mathcal{K}_2$ is isometrically isomorphic to $HS(\mathcal{K}_2^*, \mathcal{K}_1)$ (and to $HS(\mathcal{K}_2, \mathcal{K}_1)$ by the Riesz representation theorem). For an element $A \in \mathcal{K}_1 \otimes \mathcal{K}_2$ we identify the norms

$$(10) \quad \|A\|_{\mathcal{K}_1 \otimes \mathcal{K}_2} = |A|_{HS},$$

and such elements will interchangeably be considered either as members of $\mathcal{K}_1 \otimes \mathcal{K}_2$ or of $HS(\mathcal{K}_2^*, \mathcal{K}_1)$. When viewed as $A \in HS(\mathcal{K}_2^*, \mathcal{K}_1)$, the mapping $A : \mathcal{K}_2^* \rightarrow \mathcal{K}_1$ is defined by

$$Af := \sum_{i,j} A_{ij} f(e_j) \phi_i, \quad \text{for } f \in \mathcal{K}_2^*,$$

where $A_{ij} := \langle \phi_i, Ae_j^* \rangle_{\mathcal{K}_1}$, and when viewed as $A \in \mathcal{K}_1 \otimes \mathcal{K}_2$, we use tensor-basis representation

$$A = \sum_{i,j} A_{ij} \phi_i \otimes e_j.$$

The covariance matrix for a pair of random variables $Z, X \in L^2(\Omega, V)$ will be denoted

$$\text{Cov}[Z, X] := \mathbb{E}[(Z - \mathbb{E}[Z]) \otimes (X - \mathbb{E}[X])] \in V \otimes V$$

with the shorthand $\text{Cov}[Z] := \text{Cov}[Z, Z]$.

Consider the Gaussian random variable $u \sim \mu_0 := N(0, C)$ with the trace-class covariance $C = \text{Cov}[u] \in V \otimes V$. The associated covariance operator $C : V \rightarrow V$ has an eigen-basis which is orthonormal with respect to V , in the sense that

$$C\hat{\phi}_k = \lambda_k \hat{\phi}_k, \quad \langle \hat{\phi}_j, \hat{\phi}_k \rangle_V = \delta_{jk}, \quad \text{and} \quad \sum_{k=0}^{\infty} \lambda_k < \infty,$$

for a monotonically decreasing non-negative sequence of eigenvalues $\{\lambda_k\}$. It is easy to see that $u \in V$ μ_0 -almost surely. The space $\mathcal{D}(C^{-1/2}) := \{v \in V; \|C^{-1/2}v\|_V < \infty\}$ is known as the Cameron-Martin space, and by Kolmogorov's three series theorem [11], it follows that $u \sim \mu_0 \Rightarrow u \notin \mathcal{D}(C^{-1/2})$ almost surely. In fact,

$$\mathcal{D}(C^{-1/2}) \subset V \subset (\mathcal{D}(C^{-1/2}))^*,$$

where $(\mathcal{D}(C^{-1/2}))^*$ denotes the dual of $\mathcal{D}(C^{-1/2})$ with respect to the inner product $\langle \cdot, \cdot \rangle_V$, and $C : (\mathcal{D}(C^{-1/2}))^* \rightarrow \mathcal{D}(C^{-1/2})$.

Proposition 1. *If $u \in L^2(\Omega, V)$, then $C := \mathbb{E}[(u - \mathbb{E}[u]) \otimes (u - \mathbb{E}[u])] \in V \otimes V$ and $C : V \rightarrow \mathcal{D}(C^{-1})$, where $\mathcal{D}(C^{-1}) := \{v \in V \mid \|C^{-1}v\|_V < \infty\} \subset \mathcal{D}(C^{-1/2})$.*

Proof. Observe first that if $u \in L^2(\Omega, V)$, then $\mathbb{E}[u] \in L^2(\Omega, V)$ since by Jensen's inequality, $\|\mathbb{E}[u]\|_V^2 \leq \mathbb{E}[\|u\|_V^2]$. Furthermore,

$$\begin{aligned} \|C\|_{V \otimes V} &= \|\mathbb{E}[(u - \mathbb{E}[u]) \otimes (u - \mathbb{E}[u])]\|_{V \otimes V} \\ &\leq \mathbb{E}[\|(u - \mathbb{E}[u]) \otimes (u - \mathbb{E}[u])\|_{V \otimes V}] \\ &= \|u - \mathbb{E}[u]\|_{L^2(\Omega, V)}^2 \\ &= \|u\|_{L^2(\Omega, V)}^2 - \|\mathbb{E}[u]\|_V^2 < \infty. \end{aligned}$$

Observe next that $C^{-1}(CV) \subset V$ implies that $CV \subset \mathcal{D}(C^{-1})$. And since the eigenvalue sequence $\{\lambda_k\}$ is monotonically decreasing and non-negative, it is clear that $\mathcal{D}(C^{-1}) \subset \mathcal{D}(C^{-1/2})$. \square

2.3. EnKF. EnKF uses an ensemble of particles to estimate means and covariance matrices appearing in the Kalman filter. However, the framework can be generalized to non-Gaussian models. Let $v_{n,i}, \hat{v}_{n,i}$ respectively denote the prediction and update of the i -th particle at simulation time n . One EnKF two-step transition consists of the propagation of an ensemble $\{\hat{v}_{n,i}\}_{i=1}^M \mapsto \{\hat{v}_{n+1,i}\}_{i=1}^M$. The two steps of the transition are the predict and the update. In the predict step, M particle paths are computed over one interval, i.e.,

$$(11) \quad v_{n+1,i} = \Psi(\hat{v}_{n,i}, \omega_i), \quad i = 1, 2, \dots, M,$$

where $\hat{v}_{n,i} \in \cap_{p \geq 2} L^p(\Omega, V)$ and $\Psi(\cdot, \omega_i)$ denotes the forward map using the driving noise realization $\omega_i \in \Omega$. For this presentation it suffices to assume a single infinite precision map, however there indeed may also be numerical approximation errors, i.e., Ψ^L may be used in place of Ψ for some satisfactory resolution L . The prediction step is completed by using the particle paths to compute sample mean and covariance operator:

$$\begin{aligned} m_{n+1}^{\text{MC}} &= E_M[v_{n+1}] , \\ C_{n+1}^{\text{MC}} &= \text{Cov}_M[v_{n+1}] , \end{aligned}$$

with the unbiased sample moments

$$(12) \quad E_M[v_n] := \frac{1}{M} \sum_{i=1}^M v_{n,i} ,$$

and

$$(13) \quad \text{Cov}_M[u_n, v_n] := \frac{M}{M-1} (E_M[u_n \otimes v_n] - E_M[u_n] \otimes E_M[v_n]) ,$$

and the shorthand $\text{Cov}_M[u_n] := \text{Cov}_M[u_n, u_n]$. The update step consists of computing *i.*) auxiliary operators

$$(14) \quad S_{n+1}^{\text{MC}} = H C_{n+1}^{\text{MC}} H^* + \Gamma \text{ and } K_{n+1}^{\text{MC}} = (C_{n+1}^{\text{MC}} H^*) (S_{n+1}^{\text{MC}})^{-1} ,$$

where $H^* \in L(\mathcal{R}_m, \mathcal{V}^*)$ is the adjoint of the operator H , defined by

$$(H^* a)(w) = \langle a, Hw \rangle_{\mathcal{R}_m} \text{ for all } a \in \mathcal{R}_m \text{ and } w \in \mathcal{V} ;$$

ii.) measurement corrected particle paths for $i = 1, 2, \dots, M$,

$$\begin{aligned} \tilde{y}_{n+1,i} &= y_{n+1} + \eta_{n+1,i} , \\ \hat{v}_{n+1,i} &= (I - K_{n+1}^{\text{MC}} H) v_{n+1,i} + K_{n+1}^{\text{MC}} \tilde{y}_{n+1,i} , \end{aligned}$$

where the sequence $\{\eta_{n+1,i}\}_{i=1}^M$ is i.i.d. with $\eta_{n+1,1} \sim N(0, \Gamma)$. This last procedure may appear somewhat ad-hoc. Indeed it was originally introduced in [10] to correct the statistical error induced in its absence in implementations following the original formulation of the ensemble Kalman filter in [17]. It has become known as the perturbed observation implementation. Due to the form of the update, all ensemble members are correlated to one another after the first update. So, even in the linear Gaussian case, the ensemble is no longer Gaussian after the first update. Nonetheless, it has been shown that the limiting ensemble converges to the correct Gaussian in the linear and finite-dimensional case [43, 41], with the rate $\mathcal{O}(N^{-1/2})$ in L^p for Lipschitz functionals with polynomial growth at infinity. Furthermore, it converges with the same rate in the nonlinear Lipschitzian case of Assumption 1 to a limiting distribution which will be discussed further in the subsection 2.7, cf. [41, 40]. The measurement corrected sample mean and covariance, would be given by:

$$\begin{aligned} \hat{m}_{n+1}^{\text{MC}} &= E_M[\hat{v}_{n+1}] , \\ \hat{C}_{n+1}^{\text{MC}} &= \text{Cov}_M[\hat{v}_{n+1}] . \end{aligned}$$

For later computing quantities of interest, we introduce the following notation for the empirical measure of the EnKF ensemble $\{\hat{v}_{n,i}\}_{i=1}^M$:

$$(15) \quad \hat{\mu}_n^{\text{MC}} = \frac{1}{M} \sum_{i=1}^M \delta_{\hat{v}_{n,i}} ,$$

and for any $\varphi : \mathcal{V} \rightarrow \mathbb{R}$, let

$$\hat{\mu}_n^{\text{MC}}(\varphi) := \int \varphi d\hat{\mu}_n^{\text{MC}} = \frac{1}{M} \sum_{i=1}^M \varphi(\hat{v}_{n,i}).$$

This section is concluded with a comment regarding the required computation of auxiliary operators (14). Introduce the orthonormal bases $\mathcal{R}_m = \text{span}\{e_i\}_{i=1}^m$, $\mathcal{V} = \text{span}\{\phi_j\}$ and $\mathcal{V}^* = \text{span}\{\phi_j^*\}$ with $\phi_j^*(\phi_i) = \delta_{ij}$ for all $1 \leq i \leq m$ and $j \in \mathbb{N}$. Since $H \in L(\mathcal{V}, \mathcal{R}_m)$, it can be represented by

$$H = \sum_{i=1}^m \sum_{j=1}^{\infty} H_{ij} e_i \otimes \phi_j^*$$

with $H_{ij} := \langle e_i, H\phi_j \rangle_{\mathcal{R}_m}$, and $\|H\|_{L(\mathcal{V}, \mathcal{R}_m)} < \infty$ implies the constraint

$$\sum_{j=1}^{\infty} H_{ij} \phi_j^* \in \mathcal{V}^*, \quad \text{for all } i \in \{1, 2, \dots, m\}.$$

For the covariance matrix, it holds almost surely that $C_{n+1}^{\text{MC}} \in V \otimes V \subset \mathcal{V} \otimes \mathcal{V}$, so it may be represented by

$$C_{n+1}^{\text{MC}} = \sum_{i,j=1}^{\infty} C_{n+1,ij}^{\text{MC}} \phi_i \otimes \phi_j, \quad \text{where } C_{n+1,ij}^{\text{MC}} := \langle \phi_i, C_{n+1}^{\text{MC}} \phi_j^* \rangle_{\mathcal{V}}.$$

For the auxiliary operator, it holds almost surely that

$$R_{n+1}^{\text{MC}} := C_{n+1}^{\text{MC}} H^* \in L(\mathcal{R}_m, V),$$

so it can be represented by

$$(16) \quad R_{n+1}^{\text{MC}} = \sum_{i=1}^m \sum_{j=1}^{\infty} R_{n+1,ij}^{\text{MC}} \phi_i \otimes e_j, \quad \text{where } R_{n+1,ij}^{\text{MC}} = \sum_{k=1}^{\infty} C_{n+1,ik}^{\text{MC}} H_{jk}.$$

Lastly, for the operators in (14) it holds that $(S^{\text{MC}})^{-1} \in L(\mathcal{R}_m, \mathcal{R}_m)$ and almost surely that $K^{\text{MC}} \in L(\mathcal{R}_m, V)$, so they can respectively be represented by

$$(17) \quad S_{n+1,ij}^{\text{MC}} = \left(\sum_{k=1}^{\infty} H_{ik} R_{n+1,kj}^{\text{MC}} \right) + \Gamma_{ij} \quad \text{and} \quad K_{n+1,ij}^{\text{MC}} = \sum_{k=1}^m R_{n+1,ik}^{\text{MC}} ((S_{n+1}^{\text{MC}})^{-1})_{kj}.$$

2.4. Multilevel EnKF. Herein a hierarchy of subspaces are introduced $\mathcal{V}_\ell = \text{span}\{\phi_k\}_{k=1}^{N_\ell}$, where $\{N_\ell\}$ is an exponentially increasing sequence of natural numbers further described in Assumption 2. By construction, $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}$. In correspondence with this hierarchy of subspaces, define a sequence of orthogonal projection operators $\{\mathcal{P}_\ell : \mathcal{V} \rightarrow \mathcal{V}_\ell\}$ by

$$(18) \quad \mathcal{P}_\ell v := \sum_{j=1}^{N_\ell} \langle \phi_j, v \rangle_{\mathcal{V}} \phi_j \in \mathcal{V}_\ell,$$

for $v \in \mathcal{V}$. It trivially follows that \mathcal{V}_ℓ is isometrically isomorphic to \mathcal{R}_{N_ℓ} , so that any element $v^\ell \in \mathcal{V}_\ell$ will, when convenient, be viewed as the unique corresponding element of \mathcal{R}_{N_ℓ} whose k -th component is given by $\langle \phi_k, v^\ell \rangle_{\mathcal{V}}$ for $k \in \{1, 2, \dots, N_\ell\}$.

In view of practical construction of numerical methods, it is worth introducing a second sequence of projection operators $\{\Pi_\ell : \mathcal{V} \rightarrow \mathcal{V}_\ell\}$ as, e.g., interpolant operators for which $\Pi_\ell \mathcal{V} = \mathcal{P}_\ell \mathcal{V} = \mathcal{V}_\ell$. Furthermore, $\{\Pi_\ell\}$ is assumed to be close to the corresponding orthogonal projectors in operator norm and the computational cost of applying Π_ℓ to any $v \in \mathcal{V}$ is $\mathcal{O}(N_\ell)$, cf. Assumption 2. This framework can accommodate spectral methods, for which typically $\Pi_\ell = \mathcal{P}_\ell$, as well as finite element type approximations, for which Π_ℓ more commonly will be taken as an

interpolant operator. In the latter case, the basis $\{\phi_j\}$ will be a hierarchical finite element basis, cf. [51, 8].

MLEnKF computes particle paths on a hierarchy of function spaces with accuracy levels determined by the solvers $\{\Psi^\ell : L^p(\Omega, \mathcal{V}) \times \Omega \rightarrow L^p(\Omega, \mathcal{V}_\ell)\}_\ell$. The case where the accuracy levels are given by refinement of the temporal discretization has already been covered for finite-dimensional state spaces in [26]. Let v_n^ℓ, \hat{v}_n^ℓ respectively denote the prediction and update of a particle on solution level ℓ and at simulation time n . A solution on level ℓ is computed by the numerical integrator $v_{n+1}^\ell = \Psi^\ell(\hat{v}_n^\ell)$. Furthermore, let the increment operator for level ℓ be given by

$$\Delta_\ell v_n := v_n^\ell - v_n^{\ell-1}, \quad \ell = 0, 1, \dots, L,$$

with the convention that $v_n^{-1} := 0$ for all $n \geq 0$. Then the transition from approximation of the conditional distribution $u_n|Y_n$ to the conditional distribution $u_{n+1}|Y_{n+1}$ in the MLEnKF framework consists of the predict/update step of generating *pairwise coupled* particle realizations on a set of levels $\ell = 0, 1, \dots, L$. However, it is important to note that here one has correlation between realization pairs and also between levels due to the update, unlike the standard MLMC in which all realization pairs are independent. We return to this point in the next section.

Similarly as for EnKF, the mapping $\{(\hat{v}_{n,i}^{\ell-1}, \hat{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L \mapsto \{(\hat{v}_{n+1,i}^{\ell-1}, \hat{v}_{n+1,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$ represents the transition of the MLEnKF ensemble over one predict-update step. In the predict step, particle paths are first computed one step on a hierarchy of levels. That is, the particle paths are computed one step forward by

$$(19) \quad \begin{aligned} v_{n+1,i}^{\ell-1} &= \Psi^{\ell-1}(\hat{v}_{n,i}^{\ell-1}, \omega_{\ell,i}), \\ v_{n+1,i}^\ell &= \Psi^\ell(\hat{v}_{n,i}^\ell, \omega_{\ell,i}), \end{aligned}$$

for the levels $\ell = 0, 1, \dots, L$ and particles $i = 1, 2, \dots, M_\ell$, with the convention that $\hat{v}_n^{-1} := v_n^{-1} = 0$ for all $n \geq 0$. Here the driving noise in the second argument of the $\Psi^{\ell-1}$ and Ψ^ℓ are correlated only within pairs, and are otherwise independent. Thereafter, sample mean and covariance matrices are computed as a sum of sample moments of increments over all levels:

$$(20) \quad \begin{aligned} m_{n+1}^{\text{ML}} &= \sum_{\ell=0}^L E_{M_\ell}[\Delta_\ell v_{n+1}], \\ C_{n+1}^{\text{ML}} &= \sum_{\ell=0}^L \text{Cov}_{M_\ell}[v_{n+1}^\ell] - \text{Cov}_{M_\ell}[v_{n+1}^{\ell-1}], \end{aligned}$$

where we recall the sample moment notation (12) and (13). Recalling the isometric isomorphism between \mathcal{V}_ℓ and \mathcal{R}_{N_ℓ} , and defining

$$(21) \quad X_{M_\ell} := \frac{1}{\sqrt{M_\ell - 1}} ([v_{n+1,1}^\ell, v_{n+1,2}^\ell, \dots, v_{n+1,M_\ell}^\ell] - E_{M_\ell}[v_{n+1}^\ell] \mathbf{1}),$$

where $\mathbf{1}$ is a row vector of M_ℓ ones, it holds almost surely that $X_{M_\ell} \in \mathbb{R}^{N_\ell \times M_\ell}$, and the sample covariance $\text{Cov}_{M_\ell}[v_{n+1}^\ell]$ can be represented by the matrix

$$(22) \quad \text{Cov}_{M_\ell}[v_{n+1}^\ell] = X_{M_\ell} X_{M_\ell}^\top.$$

Recalling (14) and (17), it is necessary for the stability of the algorithm that the matrix HR_n^{ML} appearing in the denominator of the gain (where $R_n^{\text{ML}} = C_{n+1}^{\text{ML}} H^*$ is the multilevel version of the operator defined in the Monte Carlo context in equation (16)) is positive semi-definite, a condition which is *not* guaranteed for multilevel estimators. This will therefore be *imposed* in the algorithm, similarly to

the strategy in the recent work [26]. Let

$$HR_n^{\text{ML}} = \sum_{i=1}^m \lambda_i q_i q_i^\top$$

denote the eigenvalue decomposition of HR_n^{ML} . Notice that the multilevel covariance does not ensure $\min_i(\lambda_i) \geq 0$. Define

$$(23) \quad (HR_n^{\text{ML}})^+ = \sum_{i=1; \lambda_i \geq 0}^m \lambda_i q_i q_i^\top.$$

In the update step the multilevel Kalman gain is defined as follows

$$(24) \quad K_{n+1}^{\text{ML}} = R_{n+1}^{\text{ML}} (S_{n+1}^{\text{ML}})^{-1}, \text{ where } S_{n+1}^{\text{ML}} := (HR_{n+1}^{\text{ML}})^+ + \Gamma.$$

Next, all particle paths are corrected according to measurements and perturbed observations are added:

$$(25) \quad \begin{aligned} \tilde{y}_{n+1,i}^\ell &= y_{n+1} + \eta_{n+1,i}^\ell \\ \hat{v}_{n+1,i}^{\ell-1} &= (I - \Pi_{\ell-1} K_{n+1}^{\text{ML}} H) v_{n+1,i}^{\ell-1} + \Pi_{\ell-1} K_{n+1}^{\text{ML}} \tilde{y}_{n+1,i}^\ell \\ \hat{v}_{n+1,i}^\ell &= (I - \Pi_\ell K_{n+1}^{\text{ML}} H) v_{n+1,i}^\ell + \Pi_\ell K_{n+1}^{\text{ML}} \tilde{y}_{n+1,i}^\ell, \end{aligned}$$

where the sequence $\{\eta_{n+1,i}^\ell\}_{i=1}^{M_\ell}$ is i.i.d. with $\eta_{n+1,1}^{\{0\}} \sim N(0, \Gamma)$. It is in this step that the particle pairs $\{(\hat{v}_{n+1,i}^{\ell-1}, \hat{v}_{n+1,i}^\ell)\}_{i=1}^{M_\ell}$ all become correlated and the situation becomes significantly more complicated than classical MLMC settings, where all increment pairs typically are independently distributed. This is the conclusion of the MLEnKF update step.

Proposition 2. *Assume that for any $\ell \geq 0$, the computational cost of applying Π_ℓ to any element of \mathcal{V} is proportional to N_ℓ . Then the cost arising from level ℓ in the construction of the M_ℓ sample updates (25) is proportional to $m \times N_\ell \times M_\ell$.*

Proof. Two separate operations are required at each level ℓ . The first arises in the construction of the multilevel gain K_{n+1}^{ML} in (24). Now shall become apparent the impetus for introducing the operator $R_{n+1}^{\text{ML}} = C_{n+1}^{\text{ML}} H^*$ in (16). Notice that at no point is the full C_{n+1}^{ML} required in order to build the MLEnKF Kalman gain, but rather only

$$(26) \quad R_{n+1}^{\text{ML}} = \sum_{\ell=0}^L \left(\text{Cov}_{M_\ell} [v_{n+1}^\ell, H v_{n+1}^\ell] - \text{Cov}_{M_\ell} [v_{n+1}^{\ell-1}, H v_{n+1}^{\ell-1}] \right).$$

The level ℓ contribution to this is dominated by the operation $X_{M_\ell} (H X_{M_\ell})^\top$, where X_{M_ℓ} is defined in (21). The cost of constructing $H X_{M_\ell} \in \mathbb{R}^{m \times M_\ell}$ is proportional to $m \times N_\ell \times M_\ell$, and so the cost of constructing $X_{M_\ell} (H X_{M_\ell})^\top$ is proportional to $2 \times m \times N_\ell \times M_\ell$. There is also an insignificant one time cost of constructing and inverting S_{n+1}^{ML} that can be bounded by $\mathcal{O}(m^2 N_L)$.

The second operation at level ℓ arises from actually computing the update (25) using $\Pi_\ell K_{n+1}^{\text{ML}}$. Since we have assumed that the cost of applying Π_ℓ is $\mathcal{O}(N_\ell)$, the cost of obtaining $\Pi_\ell K_{n+1}^{\text{ML}}$ from K_{n+1}^{ML} is negligible and it is clear that each sample incurs a cost $m \times N_\ell$. \square

2.5. MLEnKF algorithms. A subtlety with computing (26) efficiently is that the summands will be elements of different sized tensor spaces since for $\ell = 1, 2, \dots$ it almost surely holds that

$$\text{Cov}_{M_\ell} [v_n^{\ell-1}, H v_n^{\ell-1}] \in \mathbb{R}^{N_{\ell-1} \times m}$$

while

$$\text{Cov}_{M_\ell} [v_n^\ell, H v_n^\ell] \in \mathbb{R}^{N_\ell \times m}.$$

Algorithm 1 efficiently computes (26) through performing all arithmetic operations in the tensor space of lowest possible dimension. The computational cost of the algorithm is $\mathcal{O}(m \sum_{\ell=0}^L M_\ell N_\ell)$.

Algorithm 1 Computing the auxiliary variable R_n^{ML}

Input: Observation $H \in L(\mathcal{V}, \mathcal{R}_m)$ and prediction ensemble $\{(v_{n,i}^{\ell-1}, v_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$.

Output: R_n^{ML} .

Initialize $R_n^{\text{ML}} = 0 \in \mathbb{R}^{N_L \times m}$.

for $\ell = 0$ **to** $L - 1$ **do**

Update the submatrix $R_n^{\text{ML}}(1 : N_\ell, :) \in \mathbb{R}^{N_\ell \times m}$ consisting of the N_ℓ first rows and all columns of R_n^{ML} as follows:

$$R_n^{\text{ML}}(1 : N_\ell, :) = R_n^{\text{ML}}(1 : N_\ell, :) + \text{Cov}_{M_\ell}[v_n^\ell, H v_n^\ell] - \text{Cov}_{M_{\ell+1}}[v_n^\ell, H v_n^\ell].$$

end for

Lastly, add finest level sample covariance:

$$R_n^{\text{ML}} = R_n^{\text{ML}} + \text{Cov}_{M_L}[v_n^L, H v_n^L].$$

return R_n^{ML} .

In Algorithm 2, we summarize the main steps for one prediction-update cycle of MLEnKF.

Algorithm 2 MLEnKF prediction-update cycle

Input: Dynamics $\{\Pi_\ell : \mathcal{V} \rightarrow \mathcal{V}_\ell\}$, observation $H \in L(\mathcal{V}, \mathcal{R}_m)$, observation noise Γ , and multilevel update ensemble $\{(\hat{v}_{n,i}^{\ell-1}, \hat{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$.

Output: multilevel update ensemble $\{(\hat{v}_{n+1,i}^{\ell-1}, \hat{v}_{n+1,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$.

Prediction:

for $\ell = 0$ **to** L **do**

for $i = 0$ **to** M_ℓ **do**

Compute particle pair paths $(v_{n+1,i}^{\ell-1}, v_{n+1,i}^\ell)$ according to (19).

end for

end for

Update:

Compute R_{n+1}^{ML} by Algorithm 1, and S_{n+1}^{ML} and K_{n+1}^{ML} by (23) and (24).

for $\ell = 0$ **to** L **do**

for $i = 0$ **to** M_ℓ **do**

Generate the perturbed observation $\tilde{y}_{n+1,i}^\ell$ and update the particle pair $(\hat{v}_{n+1,i}^{\ell-1}, \hat{v}_{n+1,i}^\ell)$ by (25).

end for

end for

return $\{(\hat{v}_{n+1,i}^{\ell-1}, \hat{v}_{n+1,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$.

2.6. MLEnKF empirical measure. The following notation denotes the (signed) empirical measure of the multilevel ensemble $\{(\hat{v}_{n,i}^{\ell-1}, \hat{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$:

$$(27) \quad \hat{\mu}_n^{\text{ML}} = \frac{1}{M_0} \sum_{i=1}^{M_0} \delta_{\hat{v}_{n,i}^0} + \sum_{\ell=1}^L \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} (\delta_{\hat{v}_{n,i}^\ell} - \delta_{\hat{v}_{n,i}^{\ell-1}}),$$

and for any $\varphi : \mathcal{V} \rightarrow \mathbb{R}$, let

$$\hat{\mu}_n^{\text{ML}}(\varphi) := \int \varphi d\hat{\mu}_n^{\text{ML}} = \sum_{\ell=0}^L \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} (\varphi(\hat{v}_{n,i}^\ell) - \varphi(\hat{v}_{n,i}^{\ell-1})).$$

2.7. Nonlinear Kalman filtering. It will be useful to introduce the limiting process, in the case of (2) being a nonlinear non-Gaussian forward model. The following process defines the MFEnKF [40]:

$$(28) \quad \text{Prediction} \begin{cases} \bar{v}_{n+1} &= \Psi(\hat{v}_n), \\ \bar{m}_{n+1} &= \mathbb{E}[\bar{v}_{n+1}], \\ \bar{C}_{n+1} &= \mathbb{E}[(\bar{v}_{n+1} - \bar{m}_{n+1}) \otimes (\bar{v}_{n+1} - \bar{m}_{n+1})] \end{cases}$$

$$(29) \quad \text{Update} \begin{cases} \bar{S}_{n+1} &= (H\bar{C}_{n+1})H^* + \Gamma \\ \bar{K}_{n+1} &= (\bar{C}_{n+1}H^*)\bar{S}_{n+1}^{-1} \\ \tilde{y}_{n+1} &= y_{n+1} + \eta_{n+1} \\ \hat{v}_{n+1} &= (I - \bar{K}_{n+1}H)\bar{v}_{n+1} + \bar{K}_{n+1}\tilde{y}_{n+1}. \end{cases}$$

Here η_n are i.i.d. draws from $N(0, \Gamma)$. It is easy to see that in the linear Gaussian case, the mean and variance of the above process correspond to the mean and variance of the filtering distribution [38]. Moreover, it was shown in [43, 41] that for finite-dimensional state-space the single level EnKF converges to the Kalman filtering distribution with the standard rate $\mathcal{O}(M^{-1/2})$ in this case. It was further shown in [41] and [40] that for nonlinear Gaussian state-space models and fully non-Gaussian models (2), respectively, the EnKF converges to the above process with the same rate as long as the models satisfy a Lipschitz criterion as in Assumption 1. The work of [26] illustrated that the MLEnKF converges as well, and with an asymptotic cost-to-accuracy which is strictly smaller than its single level EnKF counterpart. The work of [37] extended convergence results to infinite-dimensional state-space for square root filters. In this work, the aim is to prove convergence of the MLEnKF for infinite-dimensional state-space, with the same favorable asymptotic cost-to-accuracy performance.

The following fact will be necessary in the subsequent section.

Proposition 3. *Consider the hidden Markov model defined by (2) and (3), and assume the initial data $u_0 \in \cap_{p \geq 2} L^p(\Omega, V)$. Then the MFEnKF process (28)–(29) satisfies $\bar{v}_n, \hat{v}_n \in \cap_{p \geq 2} L^p(\Omega, V)$ and $\|\bar{K}_n\|_{L(\mathcal{R}_m, V)} < \infty$ for all $n \in \mathbb{N}$.*

Proof. Since $\hat{v}_0 = u_0$, the property clearly holds for $n = 0$. Given $\hat{v}_n \in L^p(\Omega, V)$, Assumption 1 guarantees $\bar{v}_{n+1} \in L^p(\Omega, V)$. By Proposition 1, $\bar{C}_{n+1} \in V \otimes V$. Since $H\bar{C}_{n+1}H^* \geq 0$ and $\Gamma > 0$, it follows that $H^*\bar{S}_{n+1}^{-1} \in L(\mathcal{R}_m, \mathcal{V}^*)$ as

$$\begin{aligned} \|H^*\bar{S}_{n+1}^{-1}\|_{L(\mathcal{R}_m, \mathcal{V}^*)} &\leq \|H^*\|_{L(\mathcal{R}_m, \mathcal{V}^*)} \|\bar{S}_{n+1}^{-1}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} \\ &\leq \|H\|_{L(\mathcal{V}, \mathcal{R}_m)} \|\Gamma^{-1}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} \\ &< \infty. \end{aligned}$$

Furthermore, since $\mathcal{V}^* \subset V^*$ it also holds that $\|H^*\bar{S}_{n+1}^{-1}\|_{L(\mathcal{R}_m, V^*)} < \infty$ and

$$\begin{aligned} \|\bar{K}_{n+1}\|_{L(\mathcal{R}_m, V)} &\leq \|\bar{C}_{n+1}\|_{L(V^*, V)} \|H^*\bar{S}_{n+1}^{-1}\|_{L(\mathcal{R}_m, V^*)} \\ &\leq \|\bar{C}_{n+1}\|_{V \otimes V} \|H^*\bar{S}_{n+1}^{-1}\|_{L(\mathcal{R}_m, V^*)} \\ &< \infty. \end{aligned}$$

The result follows by recalling that $V \subset \mathcal{V}$ and the triangle inequality:

$$\begin{aligned} \|\hat{v}_{n+1}\|_{L^p(\Omega, V)} &\leq \|\bar{v}_{n+1}\|_{L^p(\Omega, V)} \\ &\quad + \|\bar{K}_{n+1}\|_{L(\mathcal{R}_m, V)} (\|Hv\|_{L^p(\Omega, \mathcal{R}_m)} + \|\tilde{y}_{n+1}\|_{L^p(\Omega, \mathcal{R}_m)}) \\ &\leq \|\bar{v}_{n+1}\|_{L^p(\Omega, V)} \\ &\quad + \|\bar{K}_{n+1}\|_{L(\mathcal{R}_m, V)} (\|H\|_{L(\mathcal{V}, \mathcal{R}_m)} \|\bar{v}_{n+1}\|_{L^p(\Omega, \mathcal{V})} + \|\tilde{y}_{n+1}\|_{L^p(\Omega, \mathcal{R}_m)}) \\ &< \infty. \end{aligned}$$

□

3. THEORETICAL RESULTS

In this section we derive theoretical results on the approximation error and computational cost of weakly approximating the MFEnKF filtering distribution by MLEnKF. Let us first state some assumptions used throughout this section.

Assumption 2. Consider the hidden Markov model defined by (2) and (3) and assume the initial data satisfies $u_0 \in \cap_{p \geq 2} L^p(\Omega, V)$. Assume the sequence of resolution dimensions $\{N_\ell\}$ fulfills the exponential growth constraint $N_\ell \approx \kappa^\ell$ for some $\kappa > 1$, where the notation $N_\ell \approx f(\ell)$ means there exist positive scalars $C > c > 0$ such that $c|f(\ell)| \leq N_\ell \leq C|f(\ell)|$ holds for all $\ell \in \mathbb{N} \cup \{0\}$. Assume further that Assumption 1 holds for the considered hierarchy of solution operators $\{\Psi^\ell\}$ which on level ℓ has spatial resolution parameter $h_\ell \approx N_\ell^{-1/d}$ and temporal resolution parameter $\Delta t_\ell = h_\ell^{\gamma_t}$, for some $\gamma_t > 0$. For a given set of constants $\beta, \gamma_x, \gamma_t > 0$, assume the following conditions are fulfilled for all $\ell \geq 0$:

- (i) $\|\Psi^\ell(u) - \Psi(u)\|_{L^p(\Omega, \mathcal{V})} \lesssim (1 + \|u\|_{L^p(\Omega, V)}) h_\ell^{\beta/2}$,
for all $p \geq 2$ and $u \in \cap_{p \geq 2} L^p(\Omega, V)$,
- (ii) for all $u \in V$,

$$\|(I - \mathcal{P}_\ell)u\|_{\mathcal{V}} \lesssim \|u\|_V h_\ell^{\beta/2} \quad \text{and} \quad \|(\Pi_\ell - \mathcal{P}_\ell)u\|_{\mathcal{V}} \lesssim \|u\|_V h_\ell^{\beta/2},$$

- (iii) the computational cost of applying Π_ℓ to any element of \mathcal{V} is $\mathcal{O}(N_\ell)$ and that of applying Ψ^ℓ to any element of \mathcal{V} is

$$\text{Cost}(\Psi^\ell) \approx h_\ell^{-(d\gamma_x + \gamma_t)},$$

where d denotes the dimension of the spatial domain of elements in \mathcal{V} , and $d\gamma_x + \gamma_t \geq d$.

Remark 1. The constraint $d\gamma_x + \gamma_t \geq d$ in Assumption 2(iii) ensures that the computational cost of the forward simulation, $\text{Cost}(\Psi^\ell) \approx h_\ell^{-(d\gamma_x + \gamma_t)}$, is at least linear in N_ℓ . Therefore, in view of Proposition 2, the share of the total cost of a single prediction and update step assigned to level ℓ is proportional to $M_\ell \times \text{Cost}(\Psi^\ell)$. However, it is important to observe that in settings with high dimensional observations, $m \geq N_0$, the MLEnKF algorithm presented in the preceding section will need to be modified to be efficient in the non-asymptotic regime, i.e., when the accuracy constraint ε , relatively speaking, is large.

The next corollary states direct consequences of the above assumption which will be useful for later reference.

Proposition 4. If Assumption 2 holds, then for all $\ell \geq 0$, $u, v \in \cap_{p \geq 2} L^p(\Omega, V)$, and globally Lipschitz continuous quantities of interest (QoI) $\varphi : \mathcal{V} \rightarrow \mathbb{R}$,

- (i) $\|\Psi^\ell(v) - \Psi^{\ell-1}(v)\|_{L^p(\Omega, \mathcal{V})} \lesssim (1 + \|v\|_{L^p(\Omega, V)}) h_\ell^{\beta/2}$, for all $p \geq 2$,
- (ii) $|\mathbb{E}[\varphi(\Psi^\ell(u)) - \varphi(\Psi(v))]| \lesssim \|u - v\|_{L^p(\Omega, \mathcal{V})} + (1 + \|u\|_{L^p(\Omega, V)}) h_\ell^{\beta/2}$, for all $p \geq 2$,
- (iii) $\|(I - \Pi_\ell)\bar{C}_n\|_{\mathcal{V} \otimes \mathcal{V}} \lesssim \|\Psi(\hat{v}_{n-1})\|_{L^2(\Omega, V)} h_\ell^{\beta/2}$.

Proof. Property (i) follows from Assumption 2(i) and the triangle inequality. Property (ii) follows from the Lipschitz continuity of φ followed by the triangle inequality, Assumption 1(i), and Assumption 2(i). For property (iii), Proposition 3, Jensen's inequality, definition (7), and Hölder's inequality implies that

$$\begin{aligned} \|(I - \Pi_\ell)\bar{C}_n\|_{\mathcal{V} \otimes \mathcal{V}} &= \|\mathbb{E}[(I - \Pi_\ell)(\bar{v}_n - \mathbb{E}[\bar{v}_n]) \otimes (\bar{v}_n - \mathbb{E}[\bar{v}_n])]\|_{\mathcal{V} \otimes \mathcal{V}} \\ &\leq \|(I - \Pi_\ell)(\bar{v}_n - \mathbb{E}[\bar{v}_n])\|_{L^2(\Omega, \mathcal{V})} \|\bar{v}_n - \mathbb{E}[\bar{v}_n]\|_{L^2(\Omega, \mathcal{V})} \\ &\leq \|(I - \Pi_\ell)\bar{v}_n\|_{L^2(\Omega, \mathcal{V})} \|\bar{v}_n\|_{L^2(\Omega, \mathcal{V})}. \end{aligned}$$

Since $(I - \mathcal{P}_\ell)\bar{v}_n = (I - \mathcal{P}_\ell)\Psi(\hat{v}_{n-1})$, Assumption 2(ii) implies that

$$\begin{aligned} \|(I - \Pi_\ell)\Psi(\hat{v}_{n-1})\|_{L^2(\Omega, \mathcal{V})} &\leq \|(I - \mathcal{P}_\ell)\Psi(\hat{v}_{n-1})\|_{L^2(\Omega, \mathcal{V})} + \|(\Pi_\ell - \mathcal{P}_\ell)\Psi(\hat{v}_{n-1})\|_{L^2(\Omega, \mathcal{V})} \\ &\leq 2\|\Psi(\hat{v}_{n-1})\|_{L^2(\Omega, \mathcal{V})}h_\ell^{\beta/2}. \end{aligned}$$

□

We now state the main theorem of this paper. It gives an upper bound for the computational cost of achieving a sought accuracy in $L^p(\Omega)$ -norm when using the MLEnKF method to approximate the expectation of a QoI. The theorem may be considered an extension to spatially extended models of the earlier work [26].

Theorem 1 (MLEnKF accuracy vs. cost). *Consider a Lipschitz continuous QoI $\varphi : \mathcal{V} \rightarrow \mathbb{R}$, and suppose Assumptions 1 and 2 hold. For a given $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined under the constraints $L = \lceil 2d \log_\kappa(\varepsilon^{-1})/\beta \rceil$ and*

$$(30) \quad M_\ell \approx \begin{cases} h_\ell^{(\beta+d\gamma_x+\gamma_t)/2} h_L^{-\beta}, & \text{if } \beta > d\gamma_x + \gamma_t, \\ h_\ell^{(\beta+d\gamma_x+\gamma_t)/2} L^2 h_L^{-\beta}, & \text{if } \beta = d\gamma_x + \gamma_t, \\ h_\ell^{(\beta+d\gamma_x+\gamma_t)/2} h_L^{-(\beta+d\gamma_x+\gamma_t)/2}, & \text{if } \beta < d\gamma_x + \gamma_t. \end{cases}$$

Then, for any $p \geq 2$ and $n \in \mathbb{N}$,

$$(31) \quad \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n(\varphi)\|_{L^p(\Omega)} \lesssim |\log(\varepsilon)|^n \varepsilon,$$

where $\hat{\mu}_n^{\text{ML}}$ denotes the multilevel empirical measure defined in (27) whose particle evolution is given by the multilevel predict (19) and update (25) formulae, approximating the time n mean-field EnKF distribution $\hat{\mu}_n$ (the filtering distribution $\hat{\mu}_n = N(\hat{m}_n, \hat{C}_n)$ in the linear Gaussian case).

The computational cost of the MLEnKF estimator over the time sequence becomes

$$(32) \quad \text{Cost (MLEnKF)} \approx \begin{cases} \varepsilon^{-2}, & \text{if } \beta > d\gamma_x + \gamma_t, \\ \varepsilon^{-2} |\log(\varepsilon)|^3, & \text{if } \beta = d\gamma_x + \gamma_t, \\ \varepsilon^{-2(d\gamma_x+\gamma_t)/\beta}, & \text{if } \beta < d\gamma_x + \gamma_t. \end{cases}$$

Following [26] and [41, 40, 43], we introduce the mean-field multilevel ensemble $\{(\bar{v}_{n,i}^{\ell-1}, \bar{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$, where every particle pair $(\bar{v}_{n,i}^{\ell-1}, \bar{v}_{n,i}^\ell)$ evolves by the respective forward mappings $\Psi^{\ell-1}$ and Ψ^ℓ using the same driving noise realization as the corresponding MLEnKF particle pair $(v_{n,i}^{\ell-1}, v_{n,i}^\ell)$, but in the update of the mean-field multilevel ensemble, the limiting covariance \bar{C}_n and the limiting Kalman gain \bar{K}_n are used. In other words, the evolution mean-field multilevel ensemble $\{(\bar{v}_{n,i}^{\ell-1}, \bar{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$ is described by the following equations running over all ensemble indices $\ell = 0, 1, \dots, L$ and $i = 1, 2, \dots, M_\ell$ and with $\bar{v}^{-1} = \hat{v}^{-1} := 0$:

$$(33) \text{Prediction} \quad \begin{cases} \bar{v}_{n+1,i}^{\ell-1} &= \Psi^{\ell-1}(\hat{v}_{n,i}^{\ell-1}, \omega_{\ell,i}), \\ \bar{v}_{n+1,i}^\ell &= \Psi^\ell(\hat{v}_{n,i}^\ell, \omega_{\ell,i}), \end{cases}$$

$$(34) \quad \text{Update} \quad \begin{cases} \tilde{y}_{n+1,i}^\ell &= y_{n+1} + \eta_{n+1,i}^\ell, \\ \hat{v}_{n+1,i}^{\ell-1} &= (I - \Pi_{\ell-1} \bar{K}_{n+1} H) \bar{v}_{n+1,i}^{\ell-1} + \Pi_{\ell-1} \bar{K}_{n+1} \tilde{y}_{n+1,i}^\ell, \\ \hat{v}_{n+1,i}^\ell &= (I - \Pi_\ell \bar{K}_{n+1} H) \bar{v}_{n+1,i}^\ell + \Pi_\ell \bar{K}_{n+1} \tilde{y}_{n+1,i}^\ell. \end{cases}$$

Note that by similar reasoning as in Proposition 3, it also holds that $\bar{v}_n^\ell, \hat{v}_n^\ell \in \cap_{p \geq 2} L^p(\Omega, V)$ for any $\ell \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N} \cup \{0\}$.

Before estimating error in between the multilevel and mean-field Kalman gains, let us recall that the multilevel Kalman gain is given by

$$K_n^{\text{ML}} = R_n^{\text{ML}}((HR_n^{\text{ML}})^+ + \Gamma)^{-1}$$

where

$$(35) \quad (HR_n^{\text{ML}})^+ := \sum_{i=1; \lambda_i \geq 0}^m \lambda_i q_i q_i^\top,$$

for eigenpairs $\{\lambda_i, q_i\}$ of HR_n^{ML} .

Lemma 1 (Multilevel covariance approximation error). *For the operator $(HR_n^{\text{ML}})^+ : \mathcal{R}_m \rightarrow \mathcal{R}_m$ defined by (35), it holds that*

$$(36) \quad \|(HR_n^{\text{ML}})^+ - HR_n^{\text{ML}}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} \leq \|H\|_{L(\mathcal{V}, \mathcal{R}_m)}^2 \|C_n^{\text{ML}} - \bar{C}_n\|_{\mathcal{V} \otimes \mathcal{V}}.$$

Proof. The representation (35) implies that $(HR_n^{\text{ML}})^+ - HR_n^{\text{ML}} : \mathcal{R}_m \rightarrow \mathcal{R}_m$ is self-adjoint and positive semi-definite and

$$\begin{aligned} \|(HR_n^{\text{ML}})^+ - HR_n^{\text{ML}}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} &= \max_{\|q\|_{\mathcal{R}_m}=1} q^* ((HR_n^{\text{ML}})^+ - HR_n^{\text{ML}})q \\ &= \max(-\min_{j; \lambda_j < 0} \lambda_j, 0). \end{aligned}$$

If $\{j \mid \lambda_j < 0\} = \emptyset$ then

$$\|(HR_n^{\text{ML}})^+ - HR_n^{\text{ML}}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} = 0,$$

and inequality (36) holds. It remains to verify the lemma for the setting $\{j \mid \lambda_j < 0\} \neq \emptyset$. Let the normalized eigenvector associated to the eigenvalue $\min_{\{j; \lambda_j < 0\}} \lambda_j$ be denoted q_{\max} . Then, since $(HR_n^{\text{ML}})^+ q_{\max} = 0$ and the mean-field covariance \bar{C}_n is self-adjoint and positive semi-definite,

$$\begin{aligned} \|(HR_n^{\text{ML}})^+ - HR_n^{\text{ML}}\|_{\text{op}} &= q_{\max}^* ((HR_n^{\text{ML}})^+ - HR_n^{\text{ML}}) q_{\max} \\ &\leq q_{\max}^* H \bar{C}_n H^* q_{\max} - q_{\max}^* H C_n^{\text{ML}} H^* q_{\max} \\ &\leq \|H(\bar{C}_n - C_n^{\text{ML}})H^*\|_{L(\mathcal{R}_m, \mathcal{R}_m)} \\ &\leq \|H\|_{L(\mathcal{V}, \mathcal{R}_m)}^2 \|\bar{C}_n - C_n^{\text{ML}}\|_{L(\mathcal{V}^*, \mathcal{V})} \\ &\leq \|H\|_{L(\mathcal{V}, \mathcal{R}_m)}^2 \|\bar{C}_n - C_n^{\text{ML}}\|_{\mathcal{V} \otimes \mathcal{V}}. \end{aligned}$$

□

The next step is to bound the Kalman gain error in terms of the covariance error.

Lemma 2 (Kalman gain error). *There exists a positive constant $\tilde{c}_n < \infty$, depending on $\|H\|_{L(\mathcal{V}, \mathcal{R}_m)}$, $\|\Gamma^{-1}\|_{L(\mathcal{R}_m, \mathcal{R}_m)}$, and $\|\bar{K}_n\|_{L(\mathcal{R}_m, \mathcal{V})}$, such that*

$$(37) \quad \|K_n^{\text{ML}} - \bar{K}_n\|_{L(\mathcal{R}_m, \mathcal{V})} \leq \tilde{c}_n \|C_n^{\text{ML}} - \bar{C}_n\|_{\mathcal{V} \otimes \mathcal{V}}.$$

Proof. Introducing the auxiliary operator $\bar{R}_n := \bar{C}_n H^*$ and observing that $H \bar{R}_n$ is positive semi-definite, we have

$$\begin{aligned} \bar{K}_n - K_n^{\text{ML}} &= \bar{R}_n (H \bar{R}_n + \Gamma)^{-1} - R_n^{\text{ML}} ((HR_n^{\text{ML}})^+ + \Gamma)^{-1} \\ &= \bar{R}_n ((H \bar{R}_n + \Gamma)^{-1} - ((HR_n^{\text{ML}})^+ + \Gamma)^{-1}) \\ &\quad + (\bar{C}_n - C_n^{\text{ML}}) H^* ((HR_n^{\text{ML}})^+ + \Gamma)^{-1}. \end{aligned}$$

Using the equality

$$(H \bar{R}_n + \Gamma)^{-1} - ((HR_n^{\text{ML}})^+ + \Gamma)^{-1} = (H \bar{R}_n + \Gamma)^{-1} ((HR_n^{\text{ML}})^+ - H \bar{R}_n) ((HR_n^{\text{ML}})^+ + \Gamma)^{-1},$$

we further obtain

$$\begin{aligned}\bar{K}_n - K_n^{\text{ML}} &= \bar{R}_n(H\bar{R}_n + \Gamma)^{-1}((HR_n^{\text{ML}})^+ - H\bar{R}_n)((HR_n^{\text{ML}})^+ + \Gamma)^{-1} \\ &\quad + (\bar{C}_n - C_n^{\text{ML}})H^*((HR_n^{\text{ML}})^+ + \Gamma)^{-1} \\ &= \bar{K}_n((HR_n^{\text{ML}})^+ - H\bar{R}_n)((HR_n^{\text{ML}})^+ + \Gamma)^{-1} \\ &\quad + (\bar{C}_n - C_n^{\text{ML}})H^*((HR_n^{\text{ML}})^+ + \Gamma)^{-1}.\end{aligned}$$

Next, since $(HR_n^{\text{ML}})^+$ and Γ respectively are positive semi-definite and positive definite,

$$\|((HR_n^{\text{ML}})^+ + \Gamma)^{-1}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} \leq \|\Gamma^{-1}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} < \infty.$$

And it follows by inequality (36), and

$$\begin{aligned}&\|(HR_n^{\text{ML}})^+ - H\bar{R}_n\|_{L(\mathcal{R}_m, \mathcal{R}_m)} \\ &\leq \|(HR_n^{\text{ML}})^+ - HR_n^{\text{ML}}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} + \|H(R_n^{\text{ML}} - \bar{R}_n)\|_{L(\mathcal{R}_m, \mathcal{R}_m)} \\ &= \|(HR_n^{\text{ML}})^+ - HR_n^{\text{ML}}\|_{L(\mathcal{R}_m, \mathcal{R}_m)} + \|H(C_n^{\text{ML}} - \bar{C}_n)H^*\|_{L(\mathcal{R}_m, \mathcal{R}_m)},\end{aligned}$$

that

$$(38) \quad \begin{aligned}\|\bar{K}_n - K_n^{\text{ML}}\|_{L(\mathcal{R}_m, \mathcal{V})} &\leq \left(1 + 2\|\bar{K}_n\|_{L(\mathcal{R}_m, \mathcal{V})}\|H\|_{L(\mathcal{V}, \mathcal{R}_m)}\right) \\ &\quad \times \|\Gamma^{-1}\|_{L(\mathcal{R}_m, \mathcal{R}_m)}\|H\|_{L(\mathcal{V}, \mathcal{R}_m)}\underbrace{\|\bar{C}_n - C_n^{\text{ML}}\|_{L(\mathcal{V}^*, \mathcal{V})}}_{\leq \|\bar{C}_n - C_n^{\text{ML}}\|_{\mathcal{V} \otimes \mathcal{V}}}.\end{aligned}$$

□

The next theorem bounds the distance between the MLEnKF and MFEnKF prediction covariance matrices. For that purpose, let us first recall the dynamics for the mean-field multilevel ensemble $\{(\bar{v}_{n,i}^{\ell-1}, \bar{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$ described in equations (33) and (34), and introduce

$$(39) \quad \bar{C}_n^{\text{ML}} := \sum_{\ell=0}^L \text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}].$$

Theorem 2. *Suppose Assumptions 1 and 2 hold and for any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 1. Then the following inequality holds for any $p \geq 2$ and $n \in \mathbb{N}$,*

$$(40) \quad \|C_n^{\text{ML}} - \bar{C}_n\|_{L^p(\Omega, \mathcal{V} \otimes \mathcal{V})} \lesssim \varepsilon + \|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_{L^p(\Omega, \mathcal{V} \otimes \mathcal{V})}.$$

Proof. Introducing the auxiliary covariance matrix

$$\bar{C}_n^L := \text{Cov}[\bar{v}_n^L]$$

and using the triangle inequality,

$$(41) \quad \|C_n^{\text{ML}} - \bar{C}_n\|_p \leq \|\bar{C}_n^L - \bar{C}_n\|_p + \|\bar{C}_n^{\text{ML}} - \bar{C}_n^L\|_p + \|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_p.$$

The proof of the theorem is concluded by Lemmas 3 and 4 below. □

Lemma 3. *Suppose Assumptions 1 and 2 hold and for any $\varepsilon > 0$, let L be defined as in Theorem 1. Then the following inequalities hold for any $n \in \mathbb{N}$ and $p \geq 2$,*

$$(42) \quad \max(\|\bar{v}_n^\ell - \bar{v}_n\|_{L^p(\Omega, \mathcal{V})}, \|\hat{v}_n^\ell - \hat{v}_n\|_{L^p(\Omega, \mathcal{V})}) \lesssim h_\ell^{\beta/2},$$

$$(43) \quad \max(\|\bar{v}_n^\ell - \bar{v}_n^{\ell-1}\|_{L^p(\Omega, \mathcal{V})}, \|\hat{v}_n^\ell - \hat{v}_n^{\ell-1}\|_{L^p(\Omega, \mathcal{V})}) \lesssim h_\ell^{\beta/2}, \quad \forall \ell \in \mathbb{N},$$

$$(44) \quad \|\bar{C}_n^L - \bar{C}_n\|_{\mathcal{V} \otimes \mathcal{V}} \lesssim \varepsilon.$$

Proof. The initial data of the limit mean-field methods is given by $\hat{v}_0 = u_0$ and on level ℓ by $\hat{v}_0^\ell := \Pi_\ell u_0$. Assumption 2(ii) implies that

$$\|\hat{v}_0 - \hat{v}_0^\ell\|_{L^p(\Omega, \mathcal{V})} \lesssim \|\hat{v}_0\|_{L^p(\Omega, \mathcal{V})} h_\ell^{\beta/2}$$

By Assumptions 1(i) and 2(i),

$$\|\bar{v}_n - \bar{v}_n^\ell\|_{L^p(\Omega, \mathcal{V})} \lesssim \|\hat{v}_{n-1} - \hat{v}_{n-1}^\ell\|_{L^p(\Omega, \mathcal{V})} + (1 + \|\hat{v}_{n-1}\|_{L^p(\Omega, \mathcal{V})}) h_\ell^{\beta/2},$$

and by Proposition 4(iii),

$$\begin{aligned} \|\hat{v}_n - \hat{v}_n^\ell\|_{L^p(\Omega, \mathcal{V})} &\leq \|I - \bar{K}_n H\|_{L(\mathcal{V}, \mathcal{V})} \|\bar{v}_n^\ell - \bar{v}_n\|_{L^p(\Omega, \mathcal{V})} + \|(I - \Pi_\ell) \bar{K}_n (H \bar{v}_n^\ell + \tilde{y}_n)\|_{L^p(\Omega, \mathcal{V})} \\ &\leq c (\|\bar{v}_n^\ell - \bar{v}_n\|_{L^p(\Omega, \mathcal{V})} + \|(I - \Pi_\ell) \bar{C}_n\|_{\mathcal{V} \times \mathcal{V}}) \\ &\lesssim \|\bar{v}_n^\ell - \bar{v}_n\|_{L^p(\Omega, \mathcal{V})} + \|\Psi(\hat{v}_{n-1})\|_{L^2(\Omega, \mathcal{V})} h_\ell^{\beta/2}. \end{aligned}$$

Inequality (42) consequently holds by induction, and thus also (43) by the triangle inequality. To prove inequality (44),

$$\begin{aligned} &\|\bar{C}_n^L - \bar{C}_n\|_{\mathcal{V} \otimes \mathcal{V}} \\ &= \|\mathbb{E}[(\bar{v}_n^L - \mathbb{E}[\bar{v}_n^L]) \otimes (\bar{v}_n^L - \mathbb{E}[\bar{v}_n^L]) - (\bar{v}_n - \mathbb{E}[\bar{v}_n]) \otimes (\bar{v}_n - \mathbb{E}[\bar{v}_n])]\|_{\mathcal{V} \otimes \mathcal{V}} \\ &= \|\mathbb{E}[(\bar{v}_n^L - \mathbb{E}[\bar{v}_n^L]) \otimes (\bar{v}_n^L - \mathbb{E}[\bar{v}_n^L]) - (\bar{v}_n - \mathbb{E}[\bar{v}_n]) \otimes (\bar{v}_n - \mathbb{E}[\bar{v}_n])]\|_{\mathcal{V} \otimes \mathcal{V}} \\ &\leq \|(\bar{v}_n^L - \mathbb{E}[\bar{v}_n^L]) \otimes (\bar{v}_n^L - \mathbb{E}[\bar{v}_n^L]) - (\bar{v}_n - \mathbb{E}[\bar{v}_n]) \otimes (\bar{v}_n - \mathbb{E}[\bar{v}_n])\|_{L^1(\Omega, \mathcal{V} \otimes \mathcal{V})} \\ &\leq (\|\bar{v}_n^L - \mathbb{E}[\bar{v}_n^L]\|_2 + \|(\bar{v}_n - \mathbb{E}[\bar{v}_n])\|_2) \|\bar{v}_n^L - \bar{v}_n\|_2 \\ &\lesssim \varepsilon. \end{aligned}$$

□

We next derive a bound for $\|\bar{C}_n^{\text{ML}} - \bar{C}_n^L\|_p$.

Lemma 4 (Multilevel i.i.d. sample covariance error). *Suppose Assumptions 1 and 2 hold and for any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 1. Then the following inequality holds for any $n \in \mathbb{N}$ and $p \geq 2$,*

$$(45) \quad \|\bar{C}_n^{\text{ML}} - \bar{C}_n^L\|_{L^p(\Omega, \mathcal{V} \otimes \mathcal{V})} \lesssim \varepsilon,$$

where we recall that $\bar{C}_n^L := \text{Cov}[\bar{v}_n^L]$.

Proof. Since the sample covariances in (39) are unbiased,

$$\mathbb{E}[\bar{C}_n^{\text{ML}}] = \text{Cov}[\bar{v}_n^L] = \sum_{\ell=0}^L \text{Cov}[\bar{v}_n^\ell] - \text{Cov}[\bar{v}_n^{\ell-1}],$$

and therefore

$$(46) \quad \|\bar{C}_n^{\text{ML}} - \bar{C}_n^L\|_p = \|\bar{C}_n^{\text{ML}} - \mathbb{E}[\bar{C}_n^{\text{ML}}]\|_p.$$

Next, introduce the linear centering operator $\Upsilon : L^1(\Omega, \mathcal{V} \otimes \mathcal{V}) \rightarrow L^1(\Omega, \mathcal{V} \otimes \mathcal{V})$, defined by $\Upsilon(Y) = Y - \mathbb{E}[Y]$. Then, by equation (39),

$$(47) \quad \begin{aligned} \|\bar{C}_n^{\text{ML}} - \mathbb{E}[\bar{C}_n^{\text{ML}}]\|_p &= \left\| \sum_{\ell=0}^L \Upsilon(\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}]) \right\|_p \\ &\leq \sum_{\ell=0}^L \|\Upsilon(\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}])\|_p \\ &\leq \sum_{\ell=0}^L \left(\|\Upsilon(\text{Cov}_{M_\ell}[\bar{v}_n^\ell, \Delta_\ell \bar{v}_n])\|_p + \|\Upsilon(\text{Cov}_{M_\ell}[\Delta_\ell \bar{v}_n, \bar{v}_n^{\ell-1}])\|_p \right), \end{aligned}$$

where we recall that $\Delta_\ell \bar{v}_n = \bar{v}_n^\ell - \bar{v}_n^{\ell-1}$, and that

$$\begin{aligned} \Upsilon(\text{Cov}_{M_\ell}[\bar{v}_n^\ell, \Delta_\ell \bar{v}_n]) &= \text{Cov}_{M_\ell}[\bar{v}_n^\ell, \Delta_\ell \bar{v}_n] - \text{Cov}[\bar{v}_n^\ell, \Delta_\ell \bar{v}_n], \\ \Upsilon(\text{Cov}_{M_\ell}[\Delta_\ell \bar{v}_n, \bar{v}_n^{\ell-1}]) &= \text{Cov}_{M_\ell}[\Delta_\ell \bar{v}_n, \bar{v}_n^{\ell-1}] - \text{Cov}[\Delta_\ell \bar{v}_n, \bar{v}_n^{\ell-1}]. \end{aligned}$$

By Lemmas 3 and 9,

$$\begin{aligned} \|\bar{C}_n^{\text{ML}} - \mathbb{E}[\bar{C}_n^{\text{ML}}]\|_p &\leq 2 \sum_{\ell=0}^L \frac{c}{\sqrt{M_\ell}} (\|\bar{v}_n^\ell\|_{2p} + \|\bar{v}_n^{\ell-1}\|_{2p}) \|\Delta_\ell \bar{v}_n\|_{2p} \\ &\lesssim \sum_{\ell=0}^L \frac{1}{\sqrt{M_\ell}} \|\Delta_\ell \bar{v}_n\|_{2p} \lesssim \sum_{\ell=0}^L M_\ell^{-1/2} h_\ell^{\beta/2} \lesssim \varepsilon. \end{aligned}$$

□

The previous two lemmas complete the proof of Theorem 2. We now turn to bounding the last term of the right-hand side of inequality (40).

Lemma 5. *Suppose Assumptions 1 and 2 hold and for any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 1. Then, for any $p \geq 2$ and $n \in \mathbb{N}$,*

(48)

$$\|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_{L^p(\Omega, \mathcal{V} \otimes \mathcal{V})} \leq 8 \sum_{l=0}^L \|v_n^\ell - \bar{v}_n^\ell\|_{L^{2p}(\Omega, \mathcal{V})} (\|v_n^\ell\|_{L^{2p}(\Omega, \mathcal{V})} + \|\bar{v}_n^\ell\|_{L^{2p}(\Omega, \mathcal{V})}).$$

Proof. From the definitions of the sample covariance (13) and multilevel sample covariance (20), one obtains the bounds

$$\begin{aligned} \|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_p &\leq \sum_{\ell=0}^L \left(\|\text{Cov}_{M_\ell}[v_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^\ell]\|_p \right. \\ &\quad \left. + \|\text{Cov}_{M_\ell}[v_n^{\ell-1}] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}]\|_p \right), \end{aligned}$$

and

$$\begin{aligned} \|\text{Cov}_{M_\ell}[v_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^\ell]\|_p &\leq \frac{M_\ell}{M_\ell - 1} \|E_{M_\ell}[v_n^\ell \otimes v_n^\ell] - E_{M_\ell}[\bar{v}_n^\ell \otimes \bar{v}_n^\ell]\|_p \\ &\quad + \frac{M_\ell}{M_\ell - 1} \|E_{M_\ell}[v_n^\ell] \otimes E_{M_\ell}[v_n^\ell] - E_{M_\ell}[\bar{v}_n^\ell] \otimes E_{M_\ell}[\bar{v}_n^\ell]\|_p \\ &=: I_1 + I_2. \end{aligned}$$

The bilinearity of the sample covariance yields that

$$(49) \quad I_1 \leq 2 \|E_{M_\ell}[(v_n^\ell - \bar{v}_n^\ell) \otimes v_n^\ell]\|_p + 2 \|E_{M_\ell}[\bar{v}_n^\ell \otimes (v_n^\ell - \bar{v}_n^\ell)]\|_p$$

and

$$I_2 \leq 2 \|E_{M_\ell}[(v_n^\ell - \bar{v}_n^\ell)] \otimes E_{M_\ell}[v_n^\ell]\|_p + 2 \|E_{M_\ell}[\bar{v}_n^\ell] \otimes E_{M_\ell}[(v_n^\ell - \bar{v}_n^\ell)]\|_p.$$

For bounding I_1 we use Jensen's and Hölder's inequalities:

$$\begin{aligned} \|E_{M_\ell}[(v_n^\ell - \bar{v}_n^\ell) \otimes v_n^\ell]\|_p^p &= \mathbb{E} \left[\|E_{M_\ell}[(v_n^\ell - \bar{v}_n^\ell) \otimes v_n^\ell]\|_{\mathcal{V} \otimes \mathcal{V}}^p \right] \\ &\leq \mathbb{E} \left[E_{M_\ell} \left[\|v_n^\ell - \bar{v}_n^\ell\|_{\mathcal{V}}^p \|v_n^\ell\|_{\mathcal{V}}^p \right] \right] \\ &= \mathbb{E} \left[\|v_n^\ell - \bar{v}_n^\ell\|_{\mathcal{V}}^p \|v_n^\ell\|_{\mathcal{V}}^p \right] \\ &\leq \|v_n^\ell - \bar{v}_n^\ell\|_{2p}^p \|v_n^\ell\|_{2p}^p. \end{aligned}$$

The second summand of inequality (49) is bounded similarly, and we obtain

$$I_1 \leq 2 \|v_n^\ell - \bar{v}_n^\ell\|_{2p} \left(\|v_n^\ell\|_{2p} + \|\bar{v}_n^\ell\|_{2p} \right).$$

The I_2 term can also be bounded with similar steps as in the preceding argument so that also

$$I_2 \leq 2 \|v_n^\ell - \bar{v}_n^\ell\|_{2p} \left(\|v_n^\ell\|_{2p} + \|\bar{v}_n^\ell\|_{2p} \right).$$

The proof is finished by summing the contributions of I_1 and I_2 over all levels. \square

The propagation of error in update steps of MLEnKF is governed by the magnitude $\|\bar{C}_n - C_n^{\text{ML}}\|$, i.e., the distance between the MFEnKF predict covariance and the MLEnKF predict covariance. The next lemma uses Lemma 5 in combination with an induction argument to control the distance between the mean-field multi-level ensemble $\{(\hat{v}_{n,i}^{\ell-1}, \hat{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$ and the MLEnKF ensemble $\{(\hat{v}_{n,i}^{\ell-1}, \hat{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=0}^L$:

Lemma 6 (Distance between ensembles.). *Suppose Assumptions 1 and 2 hold and for any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 1. Then the following inequality holds for any $n \in \mathbb{N}$ and $p \geq 2$,*

$$(50) \quad \sum_{\ell=0}^L \|\hat{v}_n^\ell - \bar{v}_n^\ell\|_{L^p(\Omega, \mathcal{V})} \lesssim |\log(\varepsilon)|^n \varepsilon.$$

Proof. We will use an induction argument to show that for arbitrary fixed $N \in \mathbb{N}$ and $p \geq 2$, it holds for all $n \leq N$ that

$$\sum_{\ell=0}^L \|\hat{v}_n^\ell - \bar{v}_n^\ell\|_{L^{p'}(\Omega, \mathcal{V})} \lesssim |\log(\varepsilon)|^n \varepsilon, \quad \forall p' \leq 4^{N-n} p.$$

The result then follows by the arbitrariness of N and p .

Notice first that by definition $\hat{v}_0^\ell = \bar{v}_0^\ell := \Pi_\ell u_0$, hence for any $p' \geq 2$,

$$\sum_{\ell=0}^L \|\hat{v}_0^\ell - \bar{v}_0^\ell\|_{p'} = 0.$$

Fix $p \geq 2$ and $N \in \mathbb{N}$, and assume that

$$(51) \quad \sum_{\ell=0}^L \|\hat{v}_{n-1}^\ell - \bar{v}_{n-1}^\ell\|_{p'} \lesssim |\log(\varepsilon)|^{n-1} \varepsilon, \quad \forall p' \leq 4^{N+1-n} p.$$

Then, by Assumption 1(i),

$$(52) \quad \sum_{\ell=0}^L \|v_n^\ell - \bar{v}_n^\ell\|_{p'} \leq \sum_{\ell=0}^L c_\Psi \|\hat{v}_{n-1}^\ell - \bar{v}_{n-1}^\ell\|_{p'} \lesssim |\log(\varepsilon)|^{n-1} \varepsilon, \quad \forall p' \leq 4^{N+1-n} p.$$

Furthermore, by Lemma 2,

$$(53) \quad \begin{aligned} \|\hat{v}_n^\ell - \bar{v}_n^\ell\|_{\mathcal{V}} &\leq \|I - \Pi_\ell \bar{K}_n H\|_{L(\mathcal{V}, \mathcal{V})} \|v_n^\ell - \bar{v}_n^\ell\|_{\mathcal{V}} \\ &\quad + \tilde{c}_n \|C_n^{\text{ML}} - \bar{C}_n\|_{\mathcal{V} \otimes \mathcal{V}} \|\tilde{y}_n^\ell - H v_n^\ell\|_{\mathcal{R}_m}, \end{aligned}$$

for all $\ell = 0, \dots, L$. Hölder's inequality then implies

$$(54) \quad \begin{aligned} \|\hat{v}_n^\ell - \bar{v}_n^\ell\|_{p'} &\leq \|I - \Pi_\ell \bar{K}_n H\|_{L(\mathcal{V}, \mathcal{V})} \|v_n^\ell - \bar{v}_n^\ell\|_{p'} \\ &\quad + \tilde{c}_n \|C_n^{\text{ML}} - \bar{C}_n\|_{L^{2p'}(\Omega, \mathcal{V} \otimes \mathcal{V})} (\|\tilde{y}_n^\ell\|_{2p'} + \|H\|_{L(\mathcal{V}, \mathcal{R}_m)} \|v_n^\ell\|_{2p'}). \end{aligned}$$

Plugging (52) into the right-hand side of the inequality (48) and using Theorem 2, we obtain that for all $p' \leq 4^{N-n} p$,

$$\begin{aligned} \|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_{2p'} &\lesssim \varepsilon + \sum_{\ell=0}^L \|v_n^\ell - \bar{v}_n^\ell\|_{4p'} (\|v_n^\ell\|_{4p'} + \|\bar{v}_n^\ell\|_{4p'}) \\ &\lesssim |\log(\varepsilon)|^{n-1} \varepsilon. \end{aligned}$$

Summing over the levels in (54), it holds for all $p' \leq 4^{N-n}p$ that

$$\begin{aligned} \sum_{\ell=0}^L \|\hat{v}_n^\ell - \hat{v}_n^\ell\|_{p'} &\lesssim \sum_{\ell=0}^L \left\{ \|v_n^\ell - \bar{v}_n^\ell\|_{p'} + |\log(\varepsilon)|^{n-1} \varepsilon (\|\tilde{y}_n^\ell\|_{2p'} + \|H\|_{L(\mathcal{V}, \mathcal{R}_m)} \|v_n^\ell\|_{2p'}) \right\} \\ &\lesssim |\log(\varepsilon)|^{n-1} \varepsilon \left(1 + \sum_{\ell=0}^L (\|\tilde{y}_n^\ell\|_{2p'} + \|H\|_{L(\mathcal{V}, \mathcal{R}_m)} \|v_n^\ell\|_{2p'}) \right) \\ &\lesssim |\log(\varepsilon)|^n \varepsilon. \end{aligned}$$

□

Induction is complete on the distance between the multilevel ensemble and its i.i.d. shadow in $L^p(\Omega, \mathcal{V})$, and we are finally ready to prove the main result.

Proof of Theorem 1. By the triangle inequality,

$$(55) \quad \begin{aligned} \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n(\varphi)\|_p &\leq \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n^{\text{ML}}(\varphi)\|_p + \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n^L(\varphi)\|_p \\ &\quad + \|\hat{\mu}_n^L(\varphi) - \hat{\mu}_n(\varphi)\|_p, \end{aligned}$$

where $\hat{\mu}_n^{\text{ML}}$ denotes the empirical measure associated to the mean-field multilevel ensemble $\{(\hat{v}_{n,i}^{\ell-1}, \hat{v}_{n,i}^\ell)\}_{i=1}^{M_\ell}\}_{\ell=0}^L$, and $\hat{\mu}_n^L$ denotes the probability measure associated to \hat{v}^L . Before treating each term separately, we note that the two first summands of the right-hand side of the inequality relate to the statistical error, whereas the last relates to the bias.

By the Lipschitz continuity of the QoI φ , the triangle inequality, and Lemma 6, the first term satisfies the following bound

$$(56) \quad \begin{aligned} \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n^{\text{ML}}(\varphi)\|_p &= \left\| \sum_{\ell=0}^L E_{M_\ell} [\varphi(\hat{v}_n^\ell) - \varphi(\hat{v}_n^{\ell-1}) - (\varphi(\hat{v}_n^\ell) - \varphi(\hat{v}_n^{\ell-1}))] \right\|_p \\ &\leq \sum_{\ell=0}^L \left(\|\varphi(\hat{v}_n^\ell) - \varphi(\hat{v}_n^{\ell-1})\|_p + \|\varphi(\hat{v}_n^{\ell-1}) - \varphi(\hat{v}_n^{\ell-1})\|_p \right) \\ &\leq c_\varphi \sum_{\ell=0}^L \left(\|\hat{v}_n^\ell - \hat{v}_n^{\ell-1}\|_p + \|\hat{v}_n^{\ell-1} - \hat{v}_n^{\ell-1}\|_p \right) \\ &\lesssim |\log(\varepsilon)|^n \varepsilon. \end{aligned}$$

For the second summand of (55), notice that we can write $\hat{\mu}_n^L = \sum_{\ell=0}^L \hat{\mu}_n^\ell - \hat{\mu}_n^{\ell-1}$, where $\hat{\mu}_n^\ell$ is the measure associated to the level ℓ limiting process \hat{v}^ℓ and $\hat{\mu}_n^{-1} := 0$. Then, by virtue of Lemmas 3 and 8 and the Lipschitz continuity of φ ,

$$(57) \quad \begin{aligned} \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n^L(\varphi)\|_p &\leq \sum_{\ell=0}^L \left\| E_{M_\ell} [\varphi(\hat{v}_n^\ell) - \varphi(\hat{v}_n^{\ell-1}) - \mathbb{E}[\varphi(\hat{v}_n^\ell) - \varphi(\hat{v}_n^{\ell-1})]] \right\|_p \\ &\leq c \sum_{\ell=0}^L M_\ell^{-1/2} \|\varphi(\hat{v}_n^\ell) - \varphi(\hat{v}_n^{\ell-1})\|_p \\ &\leq \tilde{c} \sum_{\ell=0}^L M_\ell^{-1/2} \|\hat{v}_n^\ell - \hat{v}_n^{\ell-1}\|_p \\ &\lesssim \sum_{\ell=0}^L M_\ell^{-1/2} h_\ell^{\beta/2} \lesssim \varepsilon. \end{aligned}$$

Finally, the bias term in (55) satisfies

$$(58) \quad \|\hat{\mu}_n^L(\varphi) - \hat{\mu}_n(\varphi)\|_p = |\hat{\mu}_n^L(\varphi) - \hat{\mu}_n(\varphi)| = |\mathbb{E}[\varphi(\hat{v}_n^L) - \varphi(\hat{v}_n)]| \lesssim \varepsilon,$$

where the last step follows from the Lipschitz continuity of the QoI and Lemma 3.

Inequalities (56), (57), and (58) together with inequality (55) complete the proof. \square

Remark 2. *Theorem 1 shows the cost-to-accuracy performance of MLEnKF. The geometrically growing logarithmic penalty in the error (31) is disconcerting. The same penalty appears in the work [26], yet the numerical results there indicate a time-uniform rate of convergence, and this may be an artifact of the rough bounds used in the proof. We suspect ergodicity of the MLEnKF process may allow us to obtain linear growth or even a uniform bound. There has been much recent work in this general direction. The interested reader is referred to the works [13, 14, 49].*

We conclude this section with a result on the cost-to-accuracy performance of EnKF. It shows that MLEnKF generally outperforms EnKF.

Theorem 3 (EnKF accuracy vs. cost). *Consider a Lipschitz continuous QoI $\varphi : \mathcal{V} \rightarrow \mathbb{R}$, and suppose Assumptions 1 and 2 hold. For a given $\varepsilon > 0$, let L and M be defined under the respective constraints $L = \lceil 2d \log_\kappa(\varepsilon^{-1})/\beta \rceil$ and $M \approx \varepsilon^{-2}$. Then, for any $n \in \mathbb{N}$ and $p \geq 2$,*

$$(59) \quad \|\hat{\mu}_n^{\text{MC}}(\varphi) - \hat{\mu}_n(\varphi)\|_{L^p(\Omega, \mathcal{V})} \lesssim \varepsilon,$$

where $\hat{\mu}_n^{\text{MC}}$ denotes the EnKF empirical measure, cf. equation (15), with particle evolution given by the EnKF predict and update formulae at resolution level L (i.e., using the numerical approximant Ψ^L in the prediction and the projection operator Π_L in the update).

The computational cost of the EnKF estimator over the time sequence becomes

$$(60) \quad \text{Cost}(\text{EnKF}) \approx \varepsilon^{-2(1+(d\gamma_x+\gamma_t)/\beta)}.$$

Sketch of proof. By the triangle inequality,

$$\begin{aligned} \|\hat{\mu}_n(\varphi) - \hat{\mu}_n^{\text{MC}}(\varphi)\|_{L^p(\Omega)} &\leq \|\hat{\mu}_n(\varphi) - \hat{\mu}_n^L(\varphi)\|_{L^p(\Omega)} + \|\hat{\mu}_n^L(\varphi) - \hat{\mu}_n^{\text{MC}}(\varphi)\|_{L^p(\Omega)} \\ &\quad + \|\hat{\mu}_n^{\text{MC}}(\varphi) - \hat{\mu}_n^{\text{MC}}(\varphi)\|_{L^p(\Omega)} =: I + II + III, \end{aligned}$$

where $\hat{\mu}_n^{\text{MC}}$ denotes the empirical measure associated to the EnKF ensemble $\{\hat{v}_{n,i}^L\}_{i=1}^M$ and $\hat{\mu}_n^L$ denotes the empirical measure associated to \hat{v}_n^L . It follows by inequality (58) that $I \lesssim \varepsilon$.

For the second term, the Lipschitz continuity of the QoI φ implies there exists a positive scalar c_φ such that $|\varphi(x)| \leq c_\varphi(1 + \|x\|_{\mathcal{V}})$. Since $\hat{v}_n^L \in L^p(\Omega, V)$ for any $n \in \mathbb{N}$ and $p \geq 2$, it follows by Lemma 8 (on the Hilbert space \mathcal{R}_1) that

$$II \leq \|E_M[\varphi(\hat{v}_n^L)] - \mathbb{E}[\varphi(\hat{v}_n^L)]\|_{L^p(\Omega)} \leq M^{-1/2} c_\varphi \|\hat{v}_n^L\|_{L^p(\Omega, \mathcal{V})} \lesssim \varepsilon.$$

For the last term, let us first assume that for any $p \geq 2$ and $n \in \mathbb{N}$,

$$(61) \quad \|\hat{v}_{n,1}^L - \hat{v}_{n,1}^L\|_{L^p(\Omega, \mathcal{V})} \lesssim \varepsilon,$$

for the single particle dynamics $\hat{v}_{n,1}^L$ and $\hat{v}_{n,1}^L$ respectively associated to the EnKF ensemble $\{\hat{v}_{n,i}^L\}_{i=1}^M$ and the mean-field EnKF ensemble $\{\hat{v}_{n,i}^L\}_{i=1}^M$. Then the Lipschitz continuity of φ , the fact that $\hat{v}_{n,1}^L, \hat{v}_{n,1}^L \in L^p(\Omega, V)$ for any $n \in \mathbb{N}$ and $p \geq 2$ holds (when assuming (61)), and the triangle inequality yield that

$$III = \|E_M[\varphi(\hat{v}_n^L)] - \varphi(\hat{v}_n^L)\|_{L^p(\Omega)} \leq c_\varphi \|\hat{v}_n^L - \hat{v}_n^L\|_{L^p(\Omega, \mathcal{V})} \lesssim \varepsilon.$$

All that remains is to verify (61), but we omit this as it can be done by similar steps as for the proof of inequality (50). \square

4. EXAMPLE AND NUMERICAL METHOD

4.1. The stochastic evolution equation. We consider the following one-dimensional stochastic partial differential equation (SPDE)

$$(62) \quad \begin{aligned} du &= (\Delta u + f(u))dt + BdW, & (t, x) \in (0, T] \times (0, 1), \\ u(0, \cdot) &= u_0, \\ u(t, 0) &= u(t, 1) = 0, & \forall t \in (0, T], \end{aligned}$$

where $T > 0$ and the mapping f , the cylindrical Wiener process W and the linear smoothing operator B will be introduced below. Our base-space is $\mathcal{K} = L^2(0, 1)$, we denote by $A : D(A) = H^2(0, 1) \cap H_0^1(0, 1) \rightarrow \mathcal{K}$ the Laplace operator Δ with zero-valued Dirichlet boundary conditions and $H^k(0, 1)$ denotes the Sobolev space of order $k \in \mathbb{N}$. A spectral decomposition of $-A$ yields the sequence of eigenpairs $\{(\lambda_j, \phi_j)\}_{j \in \mathbb{N}}$ where $-A\phi_j = \lambda_j\phi_j$ with $\phi_j := \sqrt{2}\sin(j\pi x)$ and $\lambda_j = \pi^2 j^2$. $\mathcal{K} = \overline{\text{span}\{\phi_j\}}$, it follows that

$$Av = \sum_{j \in \mathbb{N}} -\lambda_j \langle \phi_j, v \rangle_{\mathcal{K}} \phi_j, \quad \forall v \in D(A),$$

and eigenpairs of the spectral decomposition give rise to the following interpolation Hilbert space for any $r \in \mathbb{R}$

$$(63) \quad \begin{aligned} \mathcal{K}_r &:= D((-A)^r) = \left\{ v : [0, 1] \rightarrow \mathbb{R} \mid v \text{ is } \mathcal{B}([0, 1])/\mathcal{B}(\mathbb{R})\text{-measurable} \right. \\ &\quad \left. \text{and } \sum_{j \in \mathbb{N}} \lambda_j^{2r} |\langle \phi_j, v \rangle_{\mathcal{K}}|^2 < \infty \right\}, \end{aligned}$$

with norm $\|\cdot\|_{\mathcal{K}_r} := \|(-A)^r(\cdot)\|_{\mathcal{K}}$. Associated with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and normal filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, the $I_{\mathcal{K}}$ -cylindrical Wiener process is defined by

$$W(t, \cdot) = \sum_{j \in \mathbb{N}} W_j(t) \phi_j,$$

where $\{W_j : [0, T] \times \Omega \rightarrow \mathbb{R}\}_{j \in \mathbb{N}}$ is a sequence of independent $\{\mathcal{F}_t\}_{t \in [0, T]}/\mathcal{B}(\mathbb{R})$ -adapted standard Wiener processes. The smoothing operator is defined by

$$(64) \quad B := \sum_{j \in \mathbb{N}} \lambda_j^{-b} \phi_j \otimes \phi_j,$$

with $b \geq 0$. We assume the Hilbert spaces $V \subset \mathcal{V}$ considered in the filtering problems below are of the form $\mathcal{V} = \mathcal{K}_{r_1}$ and $V = \mathcal{K}_{r_2}$ with $r_1 < r_2 < b + 1/4$, that the initial data is $\{\mathcal{F}_0\}/\mathcal{B}(V)$ -measurable and satisfies $u_0 \in \cap_{p \geq 2} L^p(\Omega, V)$ and that $f \in \text{Lip}(\mathcal{K}_{r_2}, \mathcal{K}_{r_2}) \cap \text{Lip}(\mathcal{K}_{r_1}, \mathcal{K}_{r_1})$, where

$$\text{Lip}(\mathcal{K}_r, \mathcal{K}_r) := \left\{ f \in C(\mathcal{K}_r, \mathcal{K}_r) \mid \sup_{u, v \in \mathcal{K}_r, u \neq v} \frac{\|f(u) - f(v)\|_{\mathcal{K}_r}}{\|u - v\|_{\mathcal{K}_r}} < \infty \right\}.$$

Under these assumptions, there exists for any $r < b + 1/4$ an up to modifications unique $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0, T]})$ mild solution of (62) that is an $\{\mathcal{F}_t\}_{t \in [0, T]}/\mathcal{B}(\mathcal{K}_r)$ -adapted stochastic process satisfying

$$(65) \quad u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)}f(u(s))ds + \int_0^t e^{A(t-s)}dW_s,$$

almost surely for all $t \in [0, T]$. Moreover, for any $p \geq 2$ and $r < b + 1/4$,

$$(66) \quad \|u(T, \cdot)\|_{L^p(\Omega, \mathcal{K}_r)} \leq C(1 + \|u_0\|_{L^p(\Omega, \mathcal{K}_r)}),$$

where $C > 0$ depends on r and p , cf. [30].

Remark 3. *The Dirichlet zero-valued boundary conditions imposed in (62) only make pointwise sense provided $u(t, \cdot) \in \mathcal{K}_{1/2+\delta}$ for some $\delta > 0$ and all $t \in (0, T]$, \mathbb{P} -almost surely. In settings with lower regularity solutions the boundary condition should be interpreted in mild rather than pointwise form.*

4.2. The filtering problem. Let $\Psi(u_n) \in \cap_{p \geq 2} L^p(\Omega, V)$ denote the mild solution of (62) at time T with initial data $u_n \in \cap_{p \geq 2} L^p(\Omega, V)$, and recall the underlying dynamics

$$u_{n+1} = \Psi(u_n), \quad \text{for } n = 0, 1, \dots, N-1.$$

As $V \subset \mathcal{K}$, the mild solution may be written $u_{n+1} = \sum_{j \in \mathbb{N}} u_{n+1}^{(j)} \phi_j$, where $u_{n+1}^{(j)} := \langle u_{n+1}, \phi_j \rangle_{\mathcal{K}}$.

The finite-dimensional observation of the underlying dynamics at time n is of the form

$$(67) \quad y_n = H u_n + \eta_n, \quad \eta_n \sim N(0, \Gamma) \text{ i.i.d. } \perp u_n,$$

where $Hu = [H_1(u), \dots, H_m(u)]^\top$ with $H_i \in \mathcal{V}^*$ for $i = 1, 2, \dots, m$.

4.2.1. Spatial truncation. Before introducing a fully discrete approximation method for MLEnKF applications, let us first have a quick look at exact-in-time-truncated-in-space approximation method. It consists of the hierarchy of subspaces $\mathcal{V}_\ell = \mathcal{P}_\ell \mathcal{V} = \text{span}(\{\phi_k\}_{k=1}^{N_\ell})$, $\Pi_\ell = \mathcal{P}_\ell$ and

$$\Psi^\ell(u_n) := \mathcal{P}_\ell u_{n+1} = \sum_{j=1}^{N_\ell} u_{n+1}^{(j)} \phi_j, \quad \text{for any } u_n \in \mathcal{V}.$$

We recall that we assume that $r_1 < r_2 < b+1/4$, so that $\mathcal{V} = \mathcal{K}_{r_1}$ and $V = \mathcal{K}_{r_2}$, and note that since the approximation method integrates exactly in time, $\gamma_t = 0$, but the spatial discretization yields a spatial cost rate $\gamma_x = 1$ and $d = 1$, cf. Assumption 2.

To verify that the approximation method may be incorporated into the MLEnKF framework, it remains to verify that Assumptions 1 and 2 hold. By definition (65), that $f \in \text{Lip}(\mathcal{K}_{r_1}, \mathcal{K}_{r_1})$, that for any $r \in \mathbb{R}$ and $t \geq 0$,

$$(68) \quad \sup_{v \in \mathcal{K}_r \setminus \{0\}} \frac{\|e^{At}v\|_{\mathcal{K}_r}}{\|v\|_{\mathcal{K}_r}} \leq 1,$$

and Jensen's inequality, it follows that for any $p \geq 2$, there exists a $C > 0$ such that

$$\begin{aligned} & \|\Psi^\ell(u_0) - \Psi^\ell(v_0)\|_{L^p(\Omega, \mathcal{V})} \leq \|\Psi(u_0) - \Psi(v_0)\|_{L^p(\Omega, \mathcal{V})} \\ & \leq \|u_0 - v_0\|_{L^p(\Omega, \mathcal{V})} + \left\| \int_0^T e^{A(t-s)} (f(u(s)) - f(v(s))) ds \right\|_{L^p(\Omega, \mathcal{V})} \\ & \leq \|u_0 - v_0\|_{L^p(\Omega, \mathcal{V})} + \int_0^T \|f(u(s)) - f(v(s))\|_{L^p(\Omega, \mathcal{V})} ds \\ & = \|u_0 - v_0\|_{L^p(\Omega, \mathcal{V})} + C \int_0^T \|u(s) - v(s)\|_{L^p(\Omega, \mathcal{V})} ds. \end{aligned}$$

Hence by Gronwall's inequality,

$$\|\Psi^\ell(u_0) - \Psi^\ell(v_0)\|_{L^p(\Omega, \mathcal{V})} \leq C \|u_0 - v_0\|_{L^p(\Omega, \mathcal{V})},$$

which verifies Assumption 1(i). Assumption 1(ii) follows from (66). To verify that Assumption 2(i) holds with rate $\beta = 4(r_2 - r_1)$, observe that for any $p \geq 2$,

$$\begin{aligned}
 \|\Psi^\ell(u_0) - \Psi(u_0)\|_{L^p(\Omega, \mathcal{V})} &= \left\| \sum_{j > N_\ell} \lambda_j^{r_1} u_1^{(j)} \phi_j \right\|_{L^p(\Omega, \mathcal{K})} \\
 (69) \qquad \qquad \qquad &\leq \lambda_{N_\ell}^{-(r_2 - r_1)} \left\| \sum_{j > N_\ell} \lambda_j^{r_2} u_1^{(j)} \phi_j \right\|_{L^p(\Omega, \mathcal{K})} \\
 &\lesssim N_\ell^{-2(r_2 - r_1)} \|u_1\|_{L^p(\Omega, \mathcal{V})} \\
 &\lesssim (1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) h_\ell^{2(r_2 - r_1)},
 \end{aligned}$$

where the last inequality follows from (66) and $h_\ell \approx N_\ell^{-1}$. It follows straightforwardly that Assumptions 2(ii) and (iii) also hold.

4.3. A fully discrete approximation method. The fully discrete approximation method consists of Galerkin approximation of space in combination with numerical integration in time by the exponential Euler scheme, cf. [31, 32]. Given a timestep $\Delta t_\ell = T/J_\ell$, let $\{U_{\ell, k}\}_{k=0}^{J_\ell} \subset \mathcal{V}_\ell$ with $U_{\ell, 0} = \mathcal{P}_\ell u_0$ denote ℓ -th resolution numerical approximation of the SPDE (62) given by the scheme

$$\begin{aligned}
 U_{\ell, k+1} &= e^{A_\ell \Delta t_\ell} U_{\ell, k} + A_\ell^{-1} (e^{A_\ell \Delta t_\ell} - I) f_\ell(U_{\ell, k}) \\
 &\quad + \underbrace{\mathcal{P}_\ell \int_{k\Delta t_\ell}^{(k+1)\Delta t_\ell} e^{A_\ell((k+1)\Delta t_\ell - s)} B dW(s)}_{=: R_{\ell, k}},
 \end{aligned}$$

where $A_\ell := \mathcal{P}_\ell A$ and $f_\ell := \mathcal{P}_\ell f$. The j -th mode of the scheme $U_{\ell, k}^{(j)} := \langle U_{\ell, k}, \phi_j \rangle_{\mathcal{K}}$ for $j = 1, 2, \dots, N_\ell$, is given by

$$(70) \qquad U_{\ell, k+1}^{(j)} = e^{-\lambda_j \Delta t_\ell} U_{\ell, k}^{(j)} + \frac{1 - e^{-\lambda_j \Delta t_\ell}}{\lambda_j} (f_\ell(U_{\ell, k}))^{(j)} + R_{\ell, k}^{(j)},$$

for $k = 0, 1, \dots, J_\ell - 1$ with $\{R_{\ell, k}^{(j)}\}_{\ell, k, j}$ being the following i.i.d. sequence of Gaussians

$$(71) \qquad R_{\ell, k}^{(j)} \sim N\left(0, \frac{1 - e^{-2\lambda_j \Delta t_\ell}}{2\lambda_j^{1+2b}}\right),$$

for all $\ell = 0, 1, \dots, j \in \{1, 2, \dots, N_\ell\}$ and $k \in \{0, 1, \dots, J_\ell - 1\}$. In view of the mode-wise numerical solution, the ℓ -th level solution operator for the fully discrete approximation method is defined by

$$\tilde{\Psi}^\ell(u_0) := \sum_{j=1}^{N_\ell} U_{\ell, J_\ell}^{(j)} \phi_j.$$

Coupling of levels. In our implementation of MLEnKF we will use a hierarchy of temporal resolutions $\{\Delta t_\ell = T/J_\ell\}$ with $J_\ell = 2^\ell J_0$. Pairwise correlated particles $(\tilde{\Psi}^{\ell-1}(u_0), \tilde{\Psi}^\ell(u_0))$ are generated by first computing $\tilde{\Psi}^\ell(u_0)$ by (70) and thereafter conditioning on the fine level driving noise $\{R_{\ell, k}\}_k$ when computing the coarse level solution; $\tilde{\Psi}^{\ell-1}(u_0) | \{R_{\ell, k}\}_k$. Since $J_\ell/J_{\ell-1} = 2$, the scheme for $\tilde{\Psi}^{\ell-1}(u_0) | \{R_{\ell, k}\}_k$ becomes

$$\begin{aligned}
 (72) \qquad U_{\ell-1, k+1} &= e^{A_{\ell-1} \Delta t_{\ell-1}} U_{\ell-1, k} + A_{\ell-1}^{-1} (e^{A_{\ell-1} \Delta t_{\ell-1}} - I) f_{\ell-1}(U_{\ell-1, k}) \\
 &\quad + e^{A_{\ell-1} \Delta t_\ell} \mathcal{P}_{\ell-1} R_{\ell, 2k} + \mathcal{P}_{\ell-1} R_{\ell, 2k+1}.
 \end{aligned}$$

for $k = 0, 1, \dots, J_{\ell-1} - 1$, and with initial condition $U_{\ell-1,0} = \mathcal{P}_{\ell-1}u_0$. That the correct coupled coarse level scheme is given by (72) follows from

$$\begin{aligned} & \mathcal{P}_{\ell-1} \int_{k\Delta t_{\ell-1}}^{(k+1)\Delta t_{\ell-1}} e^{A((k+1)\Delta t_{\ell-1}-s)} B dW(s) \\ &= e^{A_{\ell-1}\Delta t_{\ell}} \mathcal{P}_{\ell-1} \int_{2k\Delta t_{\ell}}^{(2k+1)\Delta t_{\ell}} e^{A((2k+1)\Delta t_{\ell}-s)} B dW(s) \\ &+ \mathcal{P}_{\ell-1} \int_{(2k+1)\Delta t_{\ell}}^{2(k+1)\Delta t_{\ell}} e^{A((k+1)\Delta t_{\ell-1}-s)} B dW(s), \end{aligned}$$

which further implies that

$$R_{\ell-1,k} \begin{vmatrix} R_{\ell,2k} \\ R_{\ell,2k+1} \end{vmatrix} = e^{A_{\ell-1}\Delta t_{\ell}} \mathcal{P}_{\ell-1} R_{\ell,2k} + \mathcal{P}_{\ell-1} R_{\ell,2k+1}.$$

In conclusion, the scheme for the j -th mode of the coarse level solution is given by

$$(73) \quad \begin{aligned} U_{\ell-1,k+1}^{(j)} &= e^{-\lambda_j \Delta t_{\ell-1}} U_{\ell-1,k}^{(j)} + \frac{1 - e^{-\lambda_j \Delta t_{\ell-1}}}{\lambda_j} (f_{\ell-1}(U_{\ell-1,k}^{(j)}))^{(j)} \\ &+ e^{-\lambda_j \Delta t_{\ell}} R_{\ell,2k}^{(j)} + R_{\ell,2k+1}^{(j)}, \end{aligned}$$

for $j = 1, 2, \dots, N_{\ell-1}$ and $k = 0, 1, \dots, J_{\ell-1} - 1$, and the coarse level solution takes the form

$$\tilde{\Psi}^{\ell-1}(u_0) = \sum_{j=1}^{N_{\ell-1}} U_{\ell-1,J_{\ell-1}}^{(j)} \phi_j.$$

Assumptions and convergence rates. To show that the fully discrete numerical method is compatible with MLEnKF, it remains to verify that Assumptions 1 and 2 are fulfilled for the considered filtering problem described in sections 4.1 and 4.2.

Assumption 1(i): Let $U_{\ell,k}$ and $\bar{U}_{\ell,k}$ denote solutions at time $t = k\Delta t_{\ell}$ of the scheme (70) with respective initial data $U_{\ell,0} = \mathcal{P}_{\ell}u_0$ and $\bar{U}_{\ell,0} = \mathcal{P}_{\ell}v_0$, and $u_0, v_0 \in L^p(\Omega, \mathcal{V})$. Then, by (68), and the properties (a) for all $\ell \geq 0$ and $v \in \mathcal{V}_{\ell}$

$$(74) \quad \|A_{\ell}^{-1}(e^{A_{\ell}\Delta t_{\ell}} - I)v\|_{\mathcal{V}} = \left\| \int_0^{\Delta t_{\ell}} e^{A_{\ell}s} v ds \right\|_{\mathcal{V}} \leq \int_0^{\Delta t_{\ell}} \|e^{A_{\ell}s} v\|_{\mathcal{V}} ds \leq \|v\|_{\mathcal{V}} \Delta t_{\ell},$$

and (b) by $f \in \text{Lip}(\mathcal{V}, \mathcal{V})$, there exists a $C > 0$ such that

$$\begin{aligned} \|U_{\ell,J_{\ell}} - \bar{U}_{\ell,J_{\ell}}\|_{L^p(\Omega, \mathcal{V})} &\leq (1 + C\Delta t_{\ell}) \|U_{\ell,J_{\ell}-1} - \bar{U}_{\ell,J_{\ell}-1}\|_{L^p(\Omega, \mathcal{V})} \\ &\leq (1 + C\Delta t_{\ell})^{T/\Delta t_{\ell}} \|\mathcal{P}_{\ell}(u_0 - v_0)\|_{L^p(\Omega, \mathcal{V})} \\ &\leq e^{CT} \|u_0 - v_0\|_{L^p(\Omega, \mathcal{V})}. \end{aligned}$$

Consequently, for every $p \geq 2$, there exists a $c_{\Psi} > 0$ such that

$$\left\| \tilde{\Psi}^{\ell}(u_0) - \tilde{\Psi}^{\ell}(v_0) \right\|_p \leq c_{\Psi} \|u_0 - v_0\|_p$$

holds for all $\ell \geq 0$ and $u_0, v_0 \in L^p(\Omega, \mathcal{V})$.

Assumption 1(ii): Under the regularity constraints in section 4.1, it holds for all $\ell \in \mathbb{N}$ and $U_{\ell,k} := \sum_{j=1}^{N_{\ell}} U_{\ell,k}^{(j)} \phi_k$ that

$$(75) \quad \max_{k \in \{0,1,\dots,J_{\ell}\}} \|U_{\ell,k}\|_{L^p(\Omega, \mathcal{K}_r)} \leq C(1 + \|u_0\|_{L^p(\Omega, \mathcal{K}_r)}), \quad \forall p \geq 2 \text{ and } r < b + 1/4,$$

where $C > 0$ depends on r and p , but not on ℓ , cf. [30, Lemma 8.2.21].

Assumption 2(i): To verify this result we first introduce the ℓ -th level exact-in-time Galerkin approximation defined by

$$u^{\ell}(t) := e^{A_{\ell}t} u_0 + \int_0^t e^{A_{\ell}(t-s)} f_{\ell}(u^{\ell}(s)) ds + \int_0^t e^{A_{\ell}(t-s)} dW_s, \quad t \in [0, T],$$

and write $\widehat{\Psi}^\ell(u_0) := u^\ell(T)$. Under the assumptions in section 4.1 it holds for any $p \geq 2$ and $r \leq r_2$ that

$$\sup_{\ell \geq 0} \left\| \widehat{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{K}_r)} \lesssim (1 + \|u_0\|_{L^p(\Omega, \mathcal{K}_r)}),$$

cf. [30, Corollary 8.1.12]. By the triangle inequality and (69),

$$(76) \quad \left\| \Psi(u_0) - \widetilde{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{V})} \leq \left\| \Psi(u_0) - \widehat{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{V})} + \left\| \widehat{\Psi}^\ell(u_0) - \widetilde{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{V})}.$$

Under the assumptions in section 4.1, [30, Corollary 8.1.11-12] implies that for any $p \geq 2$ and all $\hat{r} < \min(r_1 + 1/2, r_2)$,

$$\left\| \Psi(u_0) - \widehat{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{V})} \lesssim (1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) N_\ell^{2(r_1 - \hat{r})},$$

and [30, Theorem 8.2.25] implies that for

$$\zeta := \max \left(2 \left(r_1 - b + \frac{1}{4} \right), 0 \right)$$

and any $p \geq 2$,

$$(77) \quad \left\| \widehat{\Psi}(u_0) - \widetilde{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{V})} \lesssim (1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) J_\ell^{-\bar{r}}, \quad \text{for all } \bar{r} < \frac{1 - \zeta}{2}.$$

This verifies Assumption 2(i) as it leads to the following bound: for any $p \geq 2$, all $\hat{r} < \min(r_1 + 1, r_2)$ and all $\bar{r} < \frac{1 - \zeta}{2}$,

$$\left\| \Psi(u_0) - \widetilde{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{V})} \lesssim (1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) (N_\ell^{2(r_1 - \hat{r})} + J_\ell^{-\bar{r}}).$$

By straightforward arguments one may verify that the two remaining conditions of Assumption 2 hold. This shows that our considered problem setting is compatible with MLEnKF.

4.4. Linear forcing. In what remains we will focus on the linear case $f(u) = u$ as it is more amenable to numerical studies than nonlinear settings. The exact solution of the j -th mode is in this setting given by

$$(78) \quad \begin{cases} \mathcal{E}_{n+1}^{(j)} = e^{(1 - \lambda_j)T} u_n^{(j)} + \xi_n^{(j)}, & \xi_n^{(j)} \sim N \left[0, \frac{\lambda_j^{-2b}}{2(\lambda_j - 1)} (1 - e^{2(1 - \lambda_j)T}) \right] \perp u_n^{(j)}. \end{cases}$$

Notice, however, that the model is still non-trivial as correlations between the modes $\{u_{n+1}^{(j)}\}_j$ will arise from the update (67) unless all the observation dimensions are of the form $H_i = \phi_j^*$ for some $j \in \mathbb{N}$.

Since the Galerkin and spatial approximation methods coincide in the linear setting, i.e., $\Psi^\ell = \widehat{\Psi}^\ell$, it holds by (69) that for any $p \geq 2$,

$$(79) \quad \left\| \Psi(u_0) - \widehat{\Psi}^\ell(u_0) \right\|_{L^p(\Omega, \mathcal{V})} \lesssim (1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) N_\ell^{2(r_1 - r_2)}.$$

Let us next show that the time discretization convergence rate (77) is improved from $(1 - \zeta)/2$ in the above nonlinear setting to 1 in the linear setting. We begin by studying the properties of the sequence $\{\mathcal{P}_\ell \widetilde{\Psi}^m(u_0)\}_{m=\ell}^\infty$ for a fixed $\ell \in \mathbb{N}$. The j -th mode projected difference of coupled solutions for $m > \ell$ is given by

$$\langle \mathcal{P}_\ell (\widetilde{\Psi}^m(u_0) - \widetilde{\Psi}^{m-1}(u_0)), \phi_j \rangle_{\mathcal{K}} = \begin{cases} U_{m, J_m}^{(j)} - U_{m-1, J_{m-1}}^{(j)}, & \text{if } j \leq N_\ell, \\ 0 & \text{otherwise,} \end{cases}$$

and the difference can be bounded as follows:

Lemma 7. Consider the SPDE (62) with $f(u) = u$, B given by (64) for some $b \geq 0$ and $\bar{u}_0 \in L^2(\Omega, \mathcal{V})$, where $\mathcal{V} = \mathcal{K}_{r_1}$ for some $r_1 < b + 1/4$. Then for any $m \in \mathbb{N}$, the sequence

$$I_{m,j} := \left\langle \mathcal{P}_{m-1} \left(\tilde{\Psi}^m(\bar{u}_0) - \tilde{\Psi}^{m-1}(\bar{u}_0) \right), \phi_j \right\rangle_{\mathcal{K}}, \quad j = 1, 2, \dots$$

can be split in three parts

$$I_{m,j} = I_{m,j,1} + I_{m,j,2} + I_{m,j,3},$$

where $I_{m,j,1}, I_{m,j,2}$, and $I_{m,j,3}$ for every $j = 1, 2, \dots$ is a triplet of mutually independent random variables and $I_{m,j,1} = I_{m,j,2} = I_{m,j,3} = 0$ for all $j > N_{m-1}$. Furthermore, there exists a constant $c > 0$ that depends on $T > 0$ and $\lambda_1 > 1$ such that for any $m \in \mathbb{N}$ and all $j \leq N_{m-1}$,

$$|I_{m,j,1}| \leq c |\bar{u}_0^{(j)}| \Delta t_m,$$

and $I_{m,j,2}$ and $I_{m,j,3}$ are mean zero-valued Gaussians with variance bounded by

$$(80) \quad \max \left(\mathbb{E} [I_{m,j,2}^2], \mathbb{E} [I_{m,j,3}^2] \right) \leq c \frac{\Delta t_m^2}{\lambda_j^{1+2b}}.$$

Proof. See Appendix B. □

By Lemma 7 and the assumptions in section 4.1, there exists a $C > 0$ depending on p, T, λ_1 and $b + 1/4 - r_1$ such that for any $m > \ell$,

$$(81) \quad \begin{aligned} & \left\| \mathcal{P}_\ell \left(\tilde{\Psi}^m(u_0) - \tilde{\Psi}^{m-1}(u_0) \right) \right\|_{L^p(\Omega, \mathcal{V})}^2 \leq \left\| \mathcal{P}_{m-1} \left(\tilde{\Psi}^m(u_0) - \tilde{\Psi}^{m-1}(u_0) \right) \right\|_{L^p(\Omega, \mathcal{V})}^2 \\ & \leq \left\| \sum_{j=1}^{\infty} (I_{m,j,1} + I_{m,j,2} + I_{m,j,3}) \phi_j \right\|_{L^p(\Omega, \mathcal{V})}^2 \\ & \leq 3 \left\| \sum_{j=1}^{\infty} I_{m,j,1} \phi_j \right\|_{L^p(\Omega, \mathcal{V})}^2 + 3 \left\| \sum_{j=1}^{\infty} I_{m,j,2} \phi_j \right\|_{L^p(\Omega, \mathcal{V})}^2 + 3 \left\| \sum_{j=1}^{\infty} I_{m,j,3} \phi_j \right\|_{L^p(\Omega, \mathcal{V})}^2 \\ & \leq 3c \Delta t_m^2 \|u_0\|_{L^p(\Omega, \mathcal{V})}^2 + 3 \left\| \sum_{j=1}^{\infty} I_{m,j,2}^2 \langle \phi_j, \phi_j \rangle_{\mathcal{V}} \right\|_{L^{p/2}(\Omega)} + 3 \left\| \sum_{j=1}^{\infty} I_{m,j,3}^2 \langle \phi_j, \phi_j \rangle_{\mathcal{V}} \right\|_{L^{p/2}(\Omega)} \\ & \leq 3c \Delta t_m^2 \|u_0\|_{L^p(\Omega, \mathcal{V})}^2 + 3 \sum_{j=1}^{\infty} \left(\|I_{m,j,2}\|_{L^{p/2}(\Omega)} + \|I_{m,j,3}\|_{L^{p/2}(\Omega)} \right) \lambda_j^{2r_1} \\ & \leq 3c \Delta t_m^2 \left(\|u_0\|_{L^p(\Omega, \mathcal{V})}^2 + 2 \sum_{j=1}^{\infty} \lambda_j^{2(r_1-b)-1} \right) \\ & \leq C \left(1 + \|u_0\|_{L^p(\Omega, \mathcal{V})} \right)^2 \Delta t_m^2. \end{aligned}$$

Here, the sixth inequality follows from $I_{m,j,2}$ and $I_{m,j,3}$ being mean zero-valued Gaussians with variance bounded by (80), which implies that for any $p \geq 2$, there exists a constant $C > 0$ depending on p such that

$$\max_{r \in \{2,3\}} \|I_{m,j,r}\|_{L^{p/2}(\Omega)} = \max_{r \in \{2,3\}} \|I_{m,j,r}\|_{L^p(\Omega)}^2 = C \frac{\Delta t_m^2}{\lambda_j^{1+2b}}$$

holds for all $j \in \mathbb{N}$. And the last inequality follows from the assumption $r_1 < b+1/4$, which implies that

$$\sum_{j=1}^{\infty} \lambda_j^{2(b-r_1)-1} < \infty.$$

From inequality (81) we deduce that $\{\mathcal{P}_\ell \tilde{\Psi}^m(u_0)\}_{m=\ell}^{\infty}$ is $L^p(\Omega, \mathcal{V})$ -Cauchy and that there exists a constant $C > 0$ depending on p, T, λ_1 and $b+1/4-r_1$ such that

$$\begin{aligned} \|\mathcal{P}_\ell \Psi(u_0) - \tilde{\Psi}^\ell(u_0)\|_{L^p(\Omega, \mathcal{V})} &\leq \sum_{m=\ell+1}^{\infty} \|\mathcal{P}_\ell(\tilde{\Psi}^m(u_0) - \tilde{\Psi}^{m-1}(u_0))\|_{L^p(\Omega, \mathcal{V})} \\ (82) \qquad &\leq C(1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) \sum_{m=\ell+1}^{\infty} \Delta t_m \\ &\leq C(1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) \Delta t_\ell \sum_{k=1}^{\infty} 2^{-k} \\ &= C(1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) J_\ell^{-1}. \end{aligned}$$

In view of the preceding inequality and (79) we obtain the following L^p -strong convergence rate for the fully discrete scheme:

Theorem 4. *Consider the SPDE (62) with $f(u) = u$ and other assumptions as stated in section 4.1. Then for all $p \geq 2$ and $\ell \in \mathbb{N} \cup \{0\}$, there exists a $C > 0$ such that*

$$(83) \quad \|\Psi(u_0) - \tilde{\Psi}^\ell(u_0)\|_{L^p(\Omega, \mathcal{V})} \leq C(1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) (N_\ell^{2(r_1-r_2)} + J_\ell^{-1}), \quad \forall \ell \geq 0,$$

where C depends on r_1, r_2 and p , but not on ℓ .

Remark 4. *To the best of our knowledge, the L^p -strong time discretization convergence rate we derived in (82) is an improvement of that in literature in two ways. First, for the $p = 2$ setting, it is slightly higher than $\mathcal{O}(\log(\Delta t^{-1})\Delta t)$, which is the best rate in literature, cf. [31]. And second, this is the first proof of order 1 L^p -strong time discretization convergence rate for any $p \geq 2$ (provided $u_0 \in L^p(\Omega, V)$).*

Error equilibration. To equilibrate the temporal and spatial discretization errors of (83), one needs to determine the sequence $\{N_\ell = N_0 \kappa^\ell\}$ so that $N_\ell^{2(r_1-r_2)} \approx J_\ell^{-1}$. This yields $\kappa = 2^{1/2(r_2-r_1)}$ and the following L^p -strong convergence rate

$$\|\Psi(u_0) - \tilde{\Psi}^\ell(u_0)\|_{L^p(\Omega, \mathcal{V})} \lesssim (1 + \|u_0\|_{L^p(\Omega, \mathcal{V})}) h_\ell^{2(r_2-r_1)}.$$

In view of Assumption 2, the fully discrete approximation method can be combined with MLEnKF, yielding the convergence rate $\beta = 4(r_2 - r_1)$ and the computational cost rates $\gamma_x = 1$ and $\gamma_t = 2(r_2 - r_1)$.

5. NUMERICAL EXAMPLES

The numerical examples here will consider only the linear version of (62), such that a high-resolution-in-space and exact-in-time approximation can be used as a benchmark ground truth for verification of the theoretical results.

If we neglect the logarithmic term appearing in (31), as motivated by Remark 2, then Theorems 1 and 3 respectively imply the following relations between mean squared error (MSE) and computational cost

$$\text{Cost (MLEnKF)}^{\min(1, \beta/(d\gamma_x + \gamma_t))} \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n(\varphi)\|_2^2 \lesssim \begin{cases} 1 & \text{if } \beta \neq d\gamma_x + \gamma_t, \\ L^3 & \text{if } \beta = d\gamma_x + \gamma_t, \end{cases}$$

and

$$\text{Cost (EnKF)}^{\beta/(\beta + d\gamma_x + \gamma_t)} \|\hat{\mu}_n^{\text{MC}}(\varphi) - \hat{\mu}_n(\varphi)\|_2^2 \lesssim 1.$$

In other words,

$$(84) \quad \|\hat{\mu}_n^{\text{ML}}(\varphi) - \hat{\mu}_n(\varphi)\|_2^2 \lesssim \begin{cases} \text{Cost}(\text{MLEnKF})^{-1} & \text{if } \beta > d\gamma_x + \gamma_t, \\ L^3 \text{Cost}(\text{MLEnKF})^{-1} & \text{if } \beta = d\gamma_x + \gamma_t, \\ \text{Cost}(\text{MLEnKF})^{-\beta/(d\gamma_x + \gamma_t)} & \text{if } \beta < d\gamma_x + \gamma_t, \end{cases}$$

and

$$(85) \quad \|\hat{\mu}_n^{\text{MC}}(\varphi) - \hat{\mu}_n(\varphi)\|_2^2 \lesssim \text{Cost}(\text{EnKF})^{-\beta/(\beta + d\gamma_x + \gamma_t)}.$$

In the examples below, we numerically verify that both the spatially discrete and the fully discrete approximation methods fulfill (84) and (85) for the filtering problem of sections 4.1 and 4.2. For all test problems, we set $T = 1/4$, $N = 40$ observation times, $N_\ell = 2^\ell$, when relevant $J_\ell = 2^\ell$ (i.e., for the fully discrete method). The approximation error, which we refer to as the MSE, is estimated by the sum of squared QoI error over the sequence of observation times averaged over 100 realizations of the respective filtering methods. That is,

$$\text{MSE}(\text{MLEnKF}) := \frac{1}{100} \sum_{i=1}^{100} \sum_{n=0}^N |\hat{\mu}_{n,i}^{\text{ML}}(\varphi) - \hat{\mu}_n(\varphi)|^2,$$

where $\{\hat{\mu}_{n,i}^{\text{ML}}(\varphi)\}_{i=1}^{100}$ is a sequence of i.i.d. QoI evaluations induced from i.i.d. realizations of the MLEnKF; and, similarly

$$\text{MSE}(\text{EnKF}) := \frac{1}{100} \sum_{i=1}^{100} \sum_{n=0}^N |\hat{\mu}_{n,i}^{\text{MC}}(\varphi) - \hat{\mu}_n(\varphi)|^2.$$

The reference sequence $\{\hat{\mu}_n(\varphi)\}_{n=0}^N$ is approximated by Kalman filtering computed on the 2^{13} dimensional subspace $\mathcal{V}_{13} \subset \mathcal{V}$. This yields an accurate approximation of the reference sequence, since when the underlying dynamics (62) is linear with Gaussian additive noise, the full space Kalman filter distribution equals the reference MFEnKF distribution $\hat{\mu}$. Furthermore, corresponding EnKF and MLEnKF solutions are computed at no higher spatial resolution than \mathcal{V}_{10} .

Example 1. Consider the filtering problem with parameters $b = 1/4 + v$, $v = 10^{-3}$, $r_1 = 0$, $r_2 = 1/2$, $\Gamma = 1/4$, the observation

$$H = \sum_{j \in \mathbb{N}} (-1)^{-j-1} (2j-1)^{-(1/2+v)} \phi_{2j-1}^* \in \mathcal{V}^*,$$

the QoI

$$\varphi = \sum_{j \in \mathbb{N}} j^{-(1/2+v)} \phi_j^* \in \mathcal{V}^*,$$

and the initial data

$$u_0 = \sum_{j \in \mathbb{N}} j^{-(3/2+v)} \phi_j \in V.$$

The convergence rates become $\beta = 2$, $d\gamma_x = 1$ and $\gamma_t = 1$ for the fully discrete method and $\gamma_t = 0$ for the spatially discrete method. Figure 1 and the left subplot of Figure 2 present the MSE versus computational cost, measured in runtime, for the spatially discrete and fully discrete method, respectively, both methods used in combination with EnKF and MLEnKF. The observed rates are consistent with (84) and (85). The right subplot of Figure 2 shows the graph of $(\text{Cost}(\text{MLEnKF}), \text{MSE}(\text{MLEnKF}) \times \text{Cost}(\text{MLEnKF}) / L^3)$, which is consistent with the constraint

$$\text{MSE}(\text{MLEnKF}) \times \text{Cost}(\text{MLEnKF}) / L^3 \lesssim 1,$$

that is a consequence of (84).

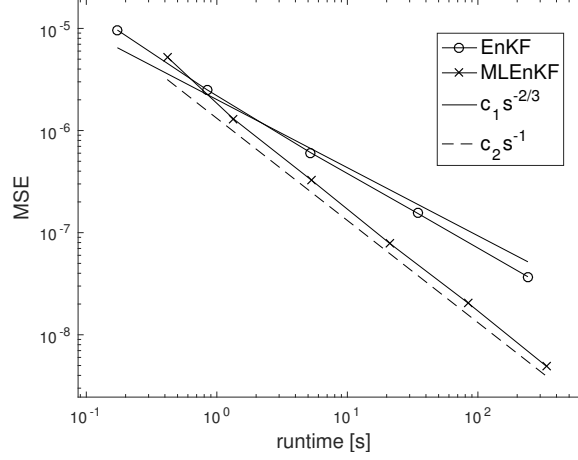


FIGURE 1. Comparison of MSE versus computational cost for Example 1 using the spatially discrete method.

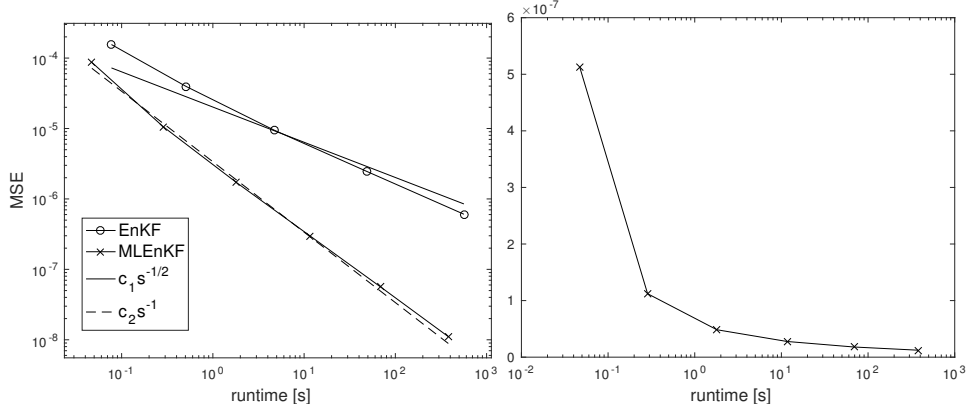


FIGURE 2. Left: MSE versus computational cost for Example 1 using the fully discrete method. Right: Graph of $(\text{Cost}(\text{MLEnKF}), \text{MSE}(\text{MLEnKF}) \times \text{Cost}(\text{MLEnKF}) / L^3)$ for Example 1 using the fully discrete method. $\text{Cost}(\text{MLEnKF})$ is measured in runtime.

Example 2. We consider problem with $b = 1/2 + v$, $v = 10^{-3}$, $r_1 = 1/4 + v/2$, $r_2 = 3/4 + v/2$, $\Gamma = 1/4$, the observation

$$H = \delta_{0.5} = \sqrt{2} \sum_{j=1}^{\infty} \sin(j\pi/2) \phi_j^* \in \mathcal{V}^*,$$

the QoI

$$\varphi = \sum_{j=1}^{\infty} \phi_j^* \in \mathcal{V}^*,$$

and the initial data

$$u_0 = \sum_{j \in \mathbb{N}} j^{-2+v} \phi_j \in V.$$

The approximation space \mathcal{V} considered here has higher regularity than the corresponding in Example 1 to allow for pointwise observations H . The approximation rates become $\beta \approx 2$ and $d_{\gamma_x} = 1$, $\gamma_t = 1$ for the fully discrete method and

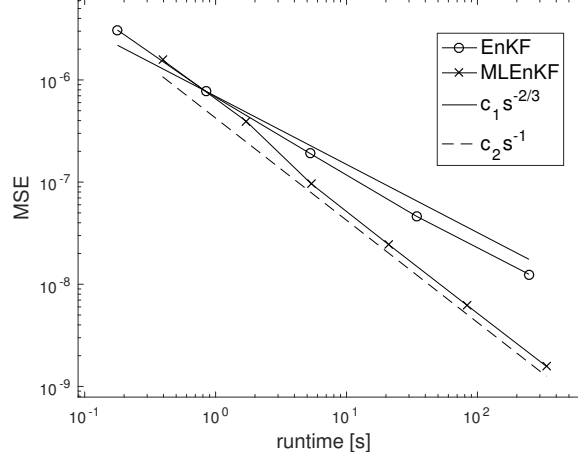


FIGURE 3. MSE versus computational cost for Example 2 using the spatially discrete method.

$\gamma_t = 0$ for the spatially discrete method. Figure 3 and the left subplot of Figure 4 shows the MSE versus computational cost, measured in runtime, for the spatially discrete and fully discrete method used in combination with EnKF and MLEnKF, respectively, and the right subplot of Figure 4 shows that the graph of $(\text{Cost}(\text{MLEnKF}), \text{MSE}(\text{MLEnKF}) \times \text{Cost}(\text{MLEnKF}) / L^3)$. The numerical observations are consistent with the theoretically predicted asymptotic behavior of (85) and (84).

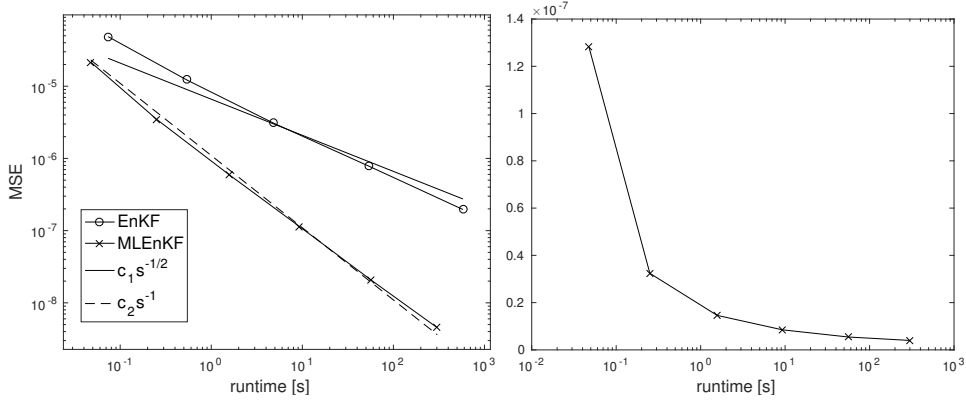


FIGURE 4. Left: MSE versus computational cost for Example 2 using the fully discrete method. Right: Graph of $(\text{Cost}(\text{MLEnKF}), \text{MSE}(\text{MLEnKF}) \times \text{Cost}(\text{MLEnKF}) / L^3)$ for Example 2 using the fully discrete method. $\text{Cost}(\text{MLEnKF})$ is measured in runtime.

Remark 5 (Multilevel particle filter). *To the best of our knowledge, there does not exist a multilevel particle filter for SPDE to this date. Provided the effective dimension on level ℓ is N_ℓ , the general requirement for particle filters is that the ensemble size on that level is bounded from below by ce^{N_ℓ} particles, for some constant $c > 0$. For MLEnKF, on the other hand, the level ℓ ensemble size is bounded from above by*

$\mathcal{O}(L^2 N_\ell^{\beta/d} N_\ell^{-(\beta+d\gamma_x+\gamma_t)/(2d)})$. Therefore, the problems accessible with this methodology are substantially larger than those accessible to existing consistent particle filter technology.

6. CONCLUSION

Multilevel EnKF for spatio-temporal processes are presented here, using a hierarchical decomposition based on the spatial resolution parameter. It is shown that an optimality rate similar to vanilla MLMC can extend to the case of sequential inference using EnKF for spatio-temporal models. The results were verified with several numerical experiments on a practically relevant SPDE model. One may therefore expect that value can be leveraged, for a fixed computational cost, by spreading work across a multilevel ensemble associated to models of multiple spatio-temporal resolutions rather than restricting to an ensemble associated only to the finest resolution model and using one very small ensemble. This has potential for broad impact across application areas in which there has been a recent explosion of interest in EnKF, for example weather prediction and subsurface exploration.

APPENDIX A. MARCINKIEWICZ–ZYGMUND INEQUALITIES FOR HILBERT SPACES

For closing the proof of Lemma 4 we make use a couple of lemmas extending the Marcinkiewicz–Zygmund inequality to separable Banach spaces.

Lemma 8. [37, Theorem 5.2] *Let $2 \leq p < \infty$ and $X_i \in L^p(\Omega, \mathcal{V})$ be i.i.d. samples of $X \in L^p(\Omega, \mathcal{V})$. Then*

$$(86) \quad \|E_M[X] - \mathbb{E}[X]\|_{L^p(\Omega, \mathcal{V})} \leq \frac{c_p}{\sqrt{M}} \|X - \mathbb{E}[X]\|_{L^p(\Omega, \mathcal{V})}$$

where c_p only depends on p .

Proof. Let r_1, r_2, \dots denote a sequence of real-valued i.i.d. random variables with $P(r_i = \pm 1) = 1/2$. A Banach space \mathcal{K} is said to be of R-type q if there exists a $c > 0$ such that for every $\bar{n} \in \mathbb{N}$ and for all (deterministic) $x_1, x_2, \dots, x_{\bar{n}} \in \mathcal{K}$,

$$\mathbb{E} \left[\left\| \sum_{i=1}^{\bar{n}} r_i x_i \right\|_{\mathcal{K}} \right] \leq c \left(\sum_{i=1}^{\bar{n}} \|x_i\|_{\mathcal{K}}^q \right)^{1/q}.$$

It is clear that all Hilbert spaces (and for our interest \mathcal{V} , in particular) are of R-type 2, since their norms are induced by an inner product. Following the proofs of [54, Proposition 2.1 and Corollary 2.1], let $\{X'_i\}$ denote an additional sequence of i.i.d. samples of $X \in L^p(\Omega, V)$ which fulfills that X'_i and X_j are mutually independent (and identically distributed) for any $i, j \in \mathbb{N}$. Introducing the symmetrization $\tilde{X}_i := (X_i - X'_i)$, and noting that almost surely,

$$\mathbb{E}[\tilde{X}_i | X_i] = X_i - \mathbb{E}[X],$$

we derive by the conditional Jensen's inequality that

$$\begin{aligned}
\mathbb{E} \left[\left\| \sum_{i=1}^{\bar{n}} X_i - \mathbb{E}[X] \right\|_{\mathcal{V}}^p \right] &\leq \mathbb{E} \left[\left\| \mathbb{E} \left[\sum_{i=1}^{\bar{n}} \tilde{X}_i \mid \{X_i\}_{i=1}^{\bar{n}} \right] \right\|_{\mathcal{V}}^p \right] \\
&\leq \mathbb{E} \left[\left\| \sum_{i=1}^{\bar{n}} \tilde{X}_i \right\|_{\mathcal{V}}^p \mid \{X_i\}_{i=1}^{\bar{n}} \right] \\
&= \mathbb{E} \left[\left\| \sum_{i=1}^{\bar{n}} \tilde{X}_i \right\|_{\mathcal{V}}^p \right] \\
(87) \quad &= \mathbb{E} \left[\left\| \sum_{i=1}^{\bar{n}} r_i \tilde{X}_i \right\|_{\mathcal{V}}^p \right] \\
&\leq c \mathbb{E} \left[\left(\sum_{i=1}^{\bar{n}} \|\tilde{X}_i\|_{\mathcal{V}}^2 \right)^{p/2} \right] \\
&\leq c 2^p \mathbb{E} \left[\left(\sum_{i=1}^{\bar{n}} \|X_i - \mathbb{E}[X]\|_{\mathcal{V}}^2 \right)^{p/2} \right].
\end{aligned}$$

And by another application of Hölder's inequality,

$$\begin{aligned}
\mathbb{E} \left[\left\| \sum_{i=1}^M \frac{X_i - \mathbb{E}[X]}{M} \right\|_{\mathcal{V}}^p \right] &\leq \hat{c} M^{-p} \mathbb{E} \left[\left(\sum_{i=1}^M \|X_i - \mathbb{E}[X]\|_{\mathcal{V}}^2 \right)^{p/2} \right] \\
&\leq \hat{c} M^{-p/2} \mathbb{E} [\|X - \mathbb{E}[X]\|_{\mathcal{V}}^p].
\end{aligned}$$

□

Lemma 9. *Suppose $X, Y \in L^p(\Omega, \mathcal{V})$, $p \geq 2$. Then, for $1 \leq r, s \leq \infty$ satisfying $1/r + 1/s = 1$, it holds that*

$$(88) \quad \|\text{Cov}_M[X, Y] - \text{Cov}[X, Y]\|_{L^p(\Omega, \mathcal{V} \otimes \mathcal{V})} \leq \frac{c}{\sqrt{M}} \|X\|_{L^{pr}(\Omega, \mathcal{V})} \|Y\|_{L^{ps}(\Omega, \mathcal{V})}$$

where the upper bound for the constant $c = \frac{M}{M-1} \left(2c_p + \frac{c_{pr}c_{ps} + 1}{\sqrt{M}} \right)$ only depends on r, s and p .

Proof. Since $\text{Cov}[X, Y] = \text{Cov}[X - \mathbb{E}[X], Y - \mathbb{E}[Y]]$ and $\text{Cov}_M[X, Y] = \text{Cov}_M[X - \mathbb{E}[X], Y - \mathbb{E}[Y]]$, cf. (13), we may without loss of generality assume that $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Using the triangle inequality,

$$\begin{aligned}
(89) \quad &\frac{M-1}{M} \|\text{Cov}_M[X, Y] - \text{Cov}[X, Y]\|_p \\
&\leq \|E_M[X \otimes Y] - \mathbb{E}[X \otimes Y]\|_p + \|E_M[X] \otimes E_M[Y]\|_p + \frac{1}{M} \|\mathbb{E}[X \otimes Y]\|_{\mathcal{V} \otimes \mathcal{V}}.
\end{aligned}$$

Estimate (86) and Hölder's inequality yield

$$\begin{aligned}
\|E_M[X \otimes Y] - \mathbb{E}[X \otimes Y]\|_p &\leq \frac{c_p}{\sqrt{M}} \|X \otimes Y - \mathbb{E}[X \otimes Y]\|_p \\
&\leq \frac{2c_p}{\sqrt{M}} \|X \otimes Y\|_p \leq \frac{2c_p}{\sqrt{M}} \|X\|_{pr} \|Y\|_{ps}.
\end{aligned}$$

Similarly, since $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ by assumption, we obtain by (86) and Hölder's inequality

$$(90) \quad \|E_M[X] \otimes E_M[Y]\|_p \leq \|E_M[X]\|_{pr} \|E_M[Y]\|_{ps} \leq \frac{c_{pr} c_{ps}}{M} \|X\|_{pr} \|Y\|_{ps}.$$

And, finally, for the last term

$$\frac{1}{M} \|\mathbb{E}[X \otimes Y]\|_{\mathcal{V} \otimes \mathcal{V}} \leq \frac{1}{M} \|X \otimes Y\|_{L^1(\Omega, \mathcal{V} \otimes \mathcal{V})} \leq \frac{1}{M} \|X\|_{L^{pr}(\Omega, \mathcal{V})} \|Y\|_{L^{ps}(\Omega, \mathcal{V})}.$$

□

APPENDIX B. PROOF OF LEMMA 7

Proof. Introducing the function $g : (1, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ defined by

$$g(\lambda, s) = e^{-\lambda s} + \frac{1 - e^{-\lambda s}}{\lambda},$$

consecutive iterations of the scheme (70) for $j \leq N_m$ yield

$$(91) \quad \begin{aligned} U_{m, J_m}^{(j)} &= g(\lambda_j, \Delta t_m) U_{m, J_m-1}^{(j)} + R_{m, J_m-1} \\ &= (g(\lambda_j, \Delta t_m))^2 U_{m, J_m-2}^{(j)} + g(\lambda_j, \Delta t_m) R_{m, J_m-2}^{(j)} + R_{m, J_m-1}^{(j)} \\ &= \dots \\ &= (g(\lambda_j, \Delta t_m))^{J_m} U_{m, 0}^{(j)} + \sum_{k=0}^{J_m-1} (g(\lambda_j, \Delta t_m))^{J_m-(k+1)} R_{m, k}^{(j)}, \end{aligned}$$

where we recall that the initial data is given by $U_{m, 0} = \mathcal{P}_m \bar{u}_0$ with $\bar{u}_0 \in L^2(\Omega, \mathcal{V})$. And since $J_m = 2J_{m-1}$, consecutive iterations of the coupled coarse scheme (73) for $j \leq N_{m-1}$ yield

$$(92) \quad \begin{aligned} U_{m-1, J_{m-1}}^{(j)} &= g(\lambda_j, \Delta t_{m-1}) U_{m-1, J_{m-1}-1}^{(j)} + e^{-\lambda_j \Delta t_m} R_{m, J_m-2}^{(j)} + R_{m, J_m-1}^{(j)} \\ &= (g(\lambda_j, \Delta t_{m-1}))^2 U_{m-1, J_{m-1}-2}^{(j)} \\ &\quad + g(\lambda_j, \Delta t_{m-1}) \left(e^{-\lambda_j \Delta t_m} R_{m, J_m-4}^{(j)} + R_{m, J_m-3}^{(j)} \right) \\ &\quad + e^{-\lambda_j \Delta t_m} R_{m, J_m-2}^{(j)} + R_{m, J_m-1}^{(j)} \\ &= \dots \\ &= (g(\lambda_j, \Delta t_{m-1}))^{J_{m-1}} U_{m-1, 0}^{(j)} \\ &\quad + \sum_{k=0}^{J_{m-1}-1} (g(\lambda_j, \Delta t_{m-1}))^{J_{m-1}-(k+1)} \left(e^{-\lambda_j \Delta t_m} R_{m, 2k}^{(j)} + R_{m, 2k+1}^{(j)} \right). \end{aligned}$$

The j -th mode final time difference of the coupled solutions for $j \leq N_{m-1}$ thus becomes

$$(93) \quad \begin{aligned} U_{m, J_m}^{(j)} - U_{m-1, J_{m-1}}^{(j)} &= \left((g(\lambda_j, \Delta t_m))^{2J_{m-1}} - (g(\lambda_j, \Delta t_{m-1}))^{J_{m-1}} \right) U_{\ell, 0}^{(j)} \\ &\quad + \sum_{k=0}^{J_{m-1}-1} \left((g(\lambda_j, \Delta t_m))^{2k} - (g(\lambda_j, \Delta t_{m-1}))^k \right) R_{m, J_m-2k+1}^{(j)} \\ &\quad + \sum_{k=0}^{J_{m-1}-1} \left((g(\lambda_j, \Delta t_m))^{2k} g(\lambda_j, \Delta t_m) - (g(\lambda_j, \Delta t_{m-1}))^k e^{-\lambda_j \Delta t_m} \right) R_{m, J_m-2(k+1)}^{(j)} \\ &=: I_{m, j, 1} + I_{m, j, 2} + I_{m, j, 3}. \end{aligned}$$

For bounding these three terms, we need to estimate the difference between powers of $(g(\lambda_j, \Delta t_m))^2$ and $g(\lambda_j, \Delta t_{m-1})$. Note first that

$$(94) \quad \begin{aligned} (g(\lambda_j, \Delta t_m))^2 &= e^{-2\lambda_j \Delta t_m} + 2e^{-\lambda_j \Delta t_m} \frac{1 - e^{-\lambda_j \Delta t_m}}{\lambda_j} + \left(\frac{1 - e^{-\lambda_j \Delta t_m}}{\lambda_j} \right)^2 \\ &= e^{-\lambda_j \Delta t_{m-1}} + \frac{1 - e^{-\lambda_j \Delta t_{m-1}}}{\lambda_j} + (1 - \lambda_j) \left(\frac{1 - e^{-\lambda_j \Delta t_m}}{\lambda_j} \right)^2. \end{aligned}$$

Since $\inf_{j \in \mathbb{N}} \lambda_j = \lambda_1 > 1$, it holds for all $j \in \mathbb{N}$ that $(g(\lambda_j, \Delta t_m))^2 < g(\lambda_j, \Delta t_{m-1})$ and

$$\left| (g(\lambda_j, \Delta t_m))^2 - g(\lambda_j, \Delta t_{m-1}) \right| < (1 - e^{-\lambda_j \Delta t_m}) \frac{1 - e^{-\lambda_j \Delta t_m}}{\lambda_j} \leq (1 - e^{-\lambda_j \Delta t_m}) \Delta t_m,$$

where the last inequality follows from (74). By the mean-value theorem, it holds for any $j, k \geq 1$ that

$$(95) \quad \left| (g(\lambda_j, \Delta t_m))^{2k} - (g(\lambda_j, \Delta t_{m-1}))^k \right| \leq (g(\lambda_j, \Delta t_{m-1}))^{k-1} k (1 - e^{-\lambda_j \Delta t_m}) \Delta t_m.$$

Let us further note that since

$$\frac{\partial}{\partial \lambda} g(\lambda, s) = \frac{\partial}{\partial \lambda} \frac{1 + (\lambda - 1)e^{-\lambda s}}{\lambda} = \frac{(-\lambda^2 s + \lambda s + 1)e^{-\lambda s} - 1}{\lambda^2},$$

it follows that

$$\frac{\partial}{\partial \lambda} g(\lambda, s) \leq 0, \quad \forall s > 0 \text{ and } \lambda \geq 1.$$

Consequently, for any $s > 0$,

$$(96) \quad \sup_{\lambda \geq \lambda_1} g(\lambda, s) \leq g(\lambda_1, s) < 1.$$

And we also have

$$(97) \quad \sup_{\lambda \geq \lambda_1} \lambda e^{-\lambda s} \leq \frac{e^{-1}}{s}, \quad \text{for any } s > 0.$$

By (95), (96), (97), the mean-value theorem and recalling that $\Delta t_{m-1} = 2\Delta t_m$, it holds for any $1 < k \leq J_{m-1}$ and $j \geq 1$ and some $\theta_{jk} \in [0, 1]$ that

$$(98) \quad \begin{aligned} &\left| (g(\lambda_j, \Delta t_m))^{2k} - (g(\lambda_j, \Delta t_{m-1}))^k \right| \leq (g(\lambda_j, \Delta t_{m-1}))^{k-1} k \lambda_j \left(\frac{1 - e^{-\lambda_j \Delta t_m}}{\lambda_j} \right) \Delta t_m \\ &\leq \left(e^{-\lambda_j \Delta t_{m-1}} + \frac{1 - e^{-\lambda_j \Delta t_{m-1}}}{\lambda_j} \right)^{k-1} k \lambda_j \Delta t_m^2 \\ &\leq e^{-\lambda_j (k-1) \Delta t_{m-1}} k \lambda_j \Delta t_m^2 \\ &+ \left(e^{-\lambda_j \Delta t_{m-1}} + \theta_{jk} \frac{1 - e^{-\lambda_j \Delta t_{m-1}}}{\lambda_j} \right)^{k-2} (k-1) k \Delta t_{m-1} \Delta t_m^2 \\ &\leq \frac{e^{-1} k}{(k-1) \Delta t_{m-1}} \Delta t_m^2 + \frac{T^2}{2} \Delta t_m \\ &\leq \frac{1 + T^2}{2} \Delta t_m. \end{aligned}$$

From (98), we conclude that for $j \leq N_{m-1}$,

$$(99) \quad |I_{m,j,1}| \leq \frac{1 + T^2}{2} |U_{\ell,0}^{(j)}| \Delta t_m = \frac{1 + T^2}{2} |\bar{u}_0^{(j)}| \Delta t_m.$$

For bounding the terms $I_{m,j,2}$ and $I_{m,j,3}$, note by (93) that both terms are linear combinations of i.i.d. Gaussians from the sequence

$$R_{m,k}^{(j)} \sim N\left(0, \frac{1 - e^{-\lambda_j \Delta t_m}}{2\lambda_j^{1+2b}}\right), \quad k = 0, 1, \dots, J_m - 1,$$

cf. (71), and hence, both terms mean zero-valued Gaussians. Furthermore, $I_{m,j,2}$ and $I_{m,j,3}$ are mutually independent as any summand of the former term is independent of any summand from the latter. Consequently, $I_{m,j,2} + I_{m,j,3}$ is a mean zero-valued Gaussian with variance

$$\mathbb{E}[(I_{m,j,2} + I_{m,j,3})^2] = \mathbb{E}[I_{m,j,2}^2] + \mathbb{E}[I_{m,j,3}^2].$$

By the mutual independence of all terms in $I_{m,j,2}$, it holds for $j \leq N_{m-1}$ that

(100)

$$\begin{aligned} \mathbb{E}[I_{m,j,2}^2] &= \sum_{k=0}^{J_m-1} \left((g(\lambda_j, \Delta t_m))^{2k} - (g(\lambda_j, \Delta t_{m-1}))^k \right)^2 \mathbb{E} \left[\left(R_{m, J_m-2k+1}^{(j)} \right)^2 \right] \\ &\leq \frac{1 - e^{-\lambda_j \Delta t_m}}{2\lambda_j^{1+2b}} \sum_{k=0}^{\infty} \left((g(\lambda_j, \Delta t_m))^{2k} - (g(\lambda_j, \Delta t_{m-1}))^k \right)^2 \\ &= \frac{1 - e^{-\lambda_j \Delta t_m}}{2\lambda_j^{1+2b}} \sum_{k=0}^{\infty} \left[(g(\lambda_j, \Delta t_m))^{4k} + (g(\lambda_j, \Delta t_{m-1}))^{2k} \right. \\ &\quad \left. - 2 \left((g(\lambda_j, \Delta t_m))^2 g(\lambda_j, \Delta t_{m-1}) \right)^k \right]. \end{aligned}$$

By the strict inequality (96), we are dealing with three sums of geometric series, an observation which in combination with (94) yields that

$$\sum_{k=0}^{\infty} \left((g(\lambda_j, \Delta t_m))^2 g(\lambda_j, \Delta t_{m-1}) \right)^k = \frac{1}{1 - \left(g(\lambda_j, \Delta t_{m-1}) (g(\lambda_j, \Delta t_m))^2 \right)},$$

$$\begin{aligned} \sum_{k=0}^{\infty} (g(\lambda_j, \Delta t_m))^{4k} &= \frac{1}{1 - (g(\lambda_j, \Delta t_m))^4} \\ &= \frac{1}{1 - (g(\lambda_j, \Delta t_m))^2 \left(g(\lambda_j, \Delta t_{m-1}) + (1 - \lambda_j) \left(\frac{1 - e^{-\lambda_j \Delta t_m}}{\lambda_j} \right)^2 \right)}, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} (g(\lambda_j, \Delta t_{m-1}))^{2k} &= \frac{1}{1 - (g(\lambda_j, \Delta t_{m-1}))^2} \\ &= \frac{1}{1 - g(\lambda_j, \Delta t_{m-1}) \left((g(\lambda_j, \Delta t_m))^2 - (1 - \lambda_j) \left(\frac{1 - e^{-\lambda_j \Delta t_m}}{\lambda_j} \right)^2 \right)}. \end{aligned}$$

The mean value theorem applied to functions of the form $(c+x)^{-1}$ up to second order, equation (96) and $g(\lambda_j, \Delta t_m) < g(\lambda_j, \Delta t_{m-1}) < 1$ yields

$$\begin{aligned} &\sum_{k=0}^{\infty} (g(\lambda_j, \Delta t_m))^{4k} + (g(\lambda_j, \Delta t_{m-1}))^{2k} \\ &\leq \frac{2}{1 - \left(g(\lambda_j, \Delta t_{m-1}) (g(\lambda_j, \Delta t_m))^2 \right)} + \frac{3}{(1 - g(\lambda_j, \Delta t_{m-1}))^3} \frac{(1 - e^{-\lambda_j \Delta t_m})^4}{\lambda_j^2}. \end{aligned}$$

It follows from (100) that for all $j \leq N_{m-1}$,

$$\begin{aligned}
\mathbb{E}[I_{m,j,2}^2] &\leq \frac{3}{(1-g(\lambda_j, \Delta t_{m-1}))^3} \frac{(1-e^{-\lambda_j \Delta t_m})^5}{\lambda_j^{3+2b}} \\
(101) \quad &\leq 3 \frac{\lambda_j^3}{(\lambda_j-1)^3 (1-e^{-\lambda_j \Delta t_m})^3} \frac{(1-e^{-\lambda_j \Delta t_m})^3}{\lambda_j^{1+2b}} \Delta t_m^2 \\
&\leq 3 \left(\frac{\lambda_1}{\lambda_1-1} \right)^3 \frac{\Delta t_m^2}{\lambda_j^{1+2b}}.
\end{aligned}$$

The last term is bounded by a similar argument: For all $j \leq N_{m-1}$,

$$\begin{aligned}
\mathbb{E}[I_{m,j,3}^2] &= \sum_{k=0}^{J_{m-1}-1} \left((g(\lambda_j, \Delta t_m))^{2k} \left(e^{-\lambda_j \Delta t_m} + \frac{1-e^{-\lambda_j \Delta t_m}}{\lambda_j} \right) \right. \\
&\quad \left. - (g(\lambda_j, \Delta t_{m-1}))^k e^{-\lambda_j \Delta t_m} \right)^2 \mathbb{E} \left[\left(R_{m, J_{m-2}(k+1)}^{(j)} \right)^2 \right] \\
(102) \quad &\leq 2 \frac{1-e^{\lambda_j \Delta t_m}}{\lambda_j^{1+2b}} \sum_{k=0}^{\infty} \left[\left((g(\lambda_j, \Delta t_m))^{2k} - (g(\lambda_j, \Delta t_{m-1}))^k \right)^2 \right. \\
&\quad \left. + (g(\lambda_j, \Delta t_m))^{2k} \left(\frac{1-e^{\lambda_j \Delta t_m}}{\lambda_j} \right)^2 \right] \\
&\leq 8 \left(\frac{\lambda_1}{\lambda_1-1} \right)^3 \frac{\Delta t_m^2}{\lambda_j^{1+2b}}.
\end{aligned}$$

Here, the last inequality follows by observing that as for $\mathbb{E}[I_{m,j,2}^2]$,

$$\frac{1-e^{\lambda_j \Delta t_m}}{\lambda_j^{1+2b}} \sum_{k=0}^{\infty} \left((g(\lambda_j, \Delta t_m))^{2k} - (g(\lambda_j, \Delta t_{m-1}))^k \right)^2 \leq 3 \left(\frac{\lambda_1}{\lambda_1-1} \right)^3 \frac{\Delta t_m^2}{\lambda_j^{1+2b}},$$

and

$$\begin{aligned}
\frac{(1-e^{\lambda_j \Delta t_m})^3}{\lambda_j^{3+2b}} \sum_{k=0}^{\infty} (g(\lambda_j, \Delta t_m))^{2k} &\leq \frac{(1-e^{\lambda_j \Delta t_m})}{\lambda_j^{1+2b}} \frac{1}{1-g(\lambda_j, \Delta t_m)} \Delta t_m^2 \\
&\leq \frac{(1-e^{\lambda_j \Delta t_m})}{\lambda_j^{1+2b}} \frac{\lambda_j}{(\lambda_j-1)(1-e^{\lambda_j \Delta t_m})} \Delta t_m^2 \\
&\leq \frac{\lambda_1}{\lambda_1-1} \frac{\Delta t_m^2}{\lambda_j^{1+2b}} \\
&\leq \left(\frac{\lambda_1}{\lambda_1-1} \right)^3 \frac{\Delta t_m^2}{\lambda_j^{1+2b}}.
\end{aligned}$$

□

Remark 6. Equation (94) shows that to leading order the additive noise from two consecutive iterations of the fine level scheme equals the additive noise from the corresponding single iteration of the coupled coarse scheme. This cancellation, and thus the correct coupling of the coarse and fine schemes, is crucial for achieving the order 1 a priori time discretization convergence rate.

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