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Symplectic Dynamical Low Rank approximation of wave equations with random parameters

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Abstract

In this paper we propose a dynamical low-rank strategy for the approximation of second order wave equations with random parameters. The governing equation is rewritten in Hamiltonian form and the approximate solution is expanded over a set of $2S$ dynamical symplectic-orthogonal deterministic basis functions with time-dependent stochastic coefficients. The reduced (low rank) dynamics is obtained by a symplectic projection of the governing Hamiltonian system onto the tangent space to the approximation manifold along the approximate trajectory. The proposed formulation is equivalent to recasting the governing Hamiltonian system in complex setting and looking for a dynamical low rank approximation in the low dimensional manifold of all complex-valued random fields with rank equal to S. Thanks to this equivalence, we are able to properly define the approximation manifold in the real setting, endow it with a differential structure and obtain a proper parametrization of its tangent space, in terms of orthogonal constraints on the dynamics of the deterministic modes. Finally, we recover the Symplectic Dynamically Orthogonal reduced order system for the evolution of both the stochastic coefficients and the deterministic basis of the approximate solution. This consists of a system of S deterministic PDEs coupled to a reduced Hamiltonian system of dimension $2S$. As a result, the approximate solution preserves the mean Hamiltonian energy over the flow.

Introduction

The last decades have witnessed a growing demand of mathematical modelling and uncertainty quantification in applications across science and technology. The goal is to perform reliable numerical simulations which accurately and effectively take into account the presence of variability and/or lack of precise characterization of the input data. In this paper, we focus on second order wave equations with random parameters, such as acoustic or elastic waves with uncertain/random speed and/or source terms. Applications are found for instance in seismology, where the propagation of the seismic waves strongly depends on some epistemic uncertainties as the location of the epicenter or the density and elastic modulus of the medium. In this context the quantification of the uncertainty in the solution is particularly challenging for large-scale problems which require long time integration. On the one hand sampling techniques, such as Monte Carlo, typically require a lot of problem solves, leading to a very high and sometimes unaffordable computational cost. On the other hand, spectral approximations of polynomial chaos type are negatively affected by the long-time integration as the structure of the random solution might considerably change over time. In particular, the parameter-to-solution map may become more and more complex in time, thus demanding a higher and higher number of terms in the Polynomial Chaos expansion, as time evolves, to maintain an acceptable accuracy level (see

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e.g. [34]). A promising approach for uncertainty propagation in time dependent problems, which has been so far applied to parabolic equations, is provided by the Dynamical Low Rank (DLR) method. [24, 25, 31, 32]. It can be seen as a reduced order method, thus solvable at a relatively low computational cost, in which the solution is a low rank function, i.e. it can be expanded at each time instant as a linear combination of few spatial deterministic modes with linearly independent random coefficients. Its peculiarity is that both the spatial and the stochastic bases are computed on the fly and are free to evolve, thus adjusting at each time to the current structure of the random solution. From a variational point of view, DLR can be seen as a Galerkin projection of the governing equation onto the tangent space to the manifold of functions of fixed rank. Our goal is to extend the DLR method to second order wave equations. This approach is indeed very attractive in this context since the solution manifold may strongly change during the propagation. The first question that we address is how to adapt the DLR approach, originally designed for parabolic equations, to approximate second order wave equations. Moreover, given the hyperbolic nature of the problem, a special attention has to be paid to preserve the stability of the reduced system.

It has been shown in literature [28] that reduced order models obtained by projecting a hyperbolic system on a fixed subspace constructed by Principal Orthogonal Decomposition (POD) may become unstable even if the original system was not. To overcome this issue, in [14] the authors derive reduced order systems which preserve the Lagrangian structure of the full order system, whereas in [5] the same strategy is combined with the Gappy POD method to further reduce the computational cost. In the context of parametric Hamiltonian systems, recent works [18,27] have proposed a reduced order method with symplectic basis, designed in analogy to the POD technique, in which the standard Galerkin projection is replaced by a symplectic projection and the solution is approximated in a low dimensional symplectic space. As a result, the reduced order system consists of a Hamiltonian system of small size which preserves the symplectic structure of the full order system, is energy conservative and preserve stability.

In this work we combine the ideas in [27] with the DLR approach and propose a Symplectic Dynamical Low Rank (SDLR) method for wave equations rewritten in Hamiltonian form, which preserves the underlying geometrical structure of the full order system. The SDLR method shares with the symplectic order reduction the use of a symplectic deterministic basis, and, as the "classic" DLR approximation, allows both the stochastic and the deterministic modes to evolve in time. As a result, the dynamical low rank approximate solution preserves the expected value of the Hamiltonian which is a crucial property for stability preservation [27].

The main challenge in constructing the SDLR method is to find a correct characterization of the manifold of approximate solutions and a parametrization of its tangent space at each point. To do so we restrict our attention to a deterministic reduced basis which is at the same time symplectic and orthonormal. This enables us to show that the approximation manifold of all functions that can be expanded over a set of $2S$ symplectic-orthonormal deterministic modes with stochastic coefficients that are subject to a specific rank condition on their second moment matrix, has the structure of a differential manifold and we are able to obtain a proper parameterization of the tangent space, in terms of a orthogonal constraint on the variations of the deterministic modes, from which the reduced system can be derived, which consists of a set of S equations for the constrained dynamics of the deterministic modes, coupled with a reduced order Hamiltonian system of dimension $2S$ for the evolution of the stochastic coefficients. Consequently, the approximation problem can be defined as the symplectic projection of the governing Hamiltonian system into the tangent space to the approximation manifold along the approximate trajectory. The characterization of the manifold and its tangent space has been obtained by exploiting the isomorphism with the manifold of complex-valued random fields of rank S. More precisely, we showed that, when recast in the complex setting, the SDLR variational principle coincides with the dynamical low-rank approximation of the governing complex-valued Hamiltonian system, into the low dimensional manifold of the complex-valued random fields with rank S.

The paper is organized as follows: in Section 1 we introduce the problem setting and the notation used throughout; in Section 3 we review some standard results concerning symplectic manifolds which will be used in Section 4 to redefine the problem in the Hamiltonian framework; in Section 2 and Section 5 we recall respectively the DLR approximation (in DO formulation) for a general parabolic problem and the symplectic order reduction for Hamiltonian systems, and then in Section 6 we present the SDLR approximation, both in real and complex setting, and we derive the reduced system. We conclude in Section 7 with some numerical test cases.

1 Notation and problem setting

Let $\mathbb{F}$ stand for $\mathbb{R}$ or $\mathbb{C}$, and D be an open bounded subset of $\mathbb{R}^d$, $1 \leq d \leq 3$, with a smooth boundary ∂D. We denote by $L^2(D,\mathbb{F})$ (respectively $H^1(D,\mathbb{F})$) the Hilbert space of square integrable functions (respectively with square integrable partial derivatives) on D with values in $\mathbb{F}$. When $\mathbb{F}$ is omitted we always refer to $\mathbb{R}$. As usual $H_0^1(D,\mathbb{F})$ denotes the subspace of all functions in $H^1(D,\mathbb{F})$ which vanish on the boundary. We denote by $\langle \cdot,\cdot \rangle$ the real inner product in $L^2(D,\mathbb{R})$, and with $\langle \cdot,\cdot \rangle_h$ the Hermitian inner product in $L^2(D,\mathbb{C})$, which is defined as:

$$\langle \hat{u},\hat{v} \rangle_h := \langle u^q, v^q \rangle + \langle u^p, v^p \rangle + i(\langle u^p, v^q \rangle - \langle u^q, v^p \rangle) \qquad \forall \hat{u} = u^q + iu^p, \hat{v} = v^q + iv^p \in L^2(D,\mathbb{C}), \quad (1)$$

Hereafter, complex valued functions are denoted with the overhat symbol ($\hat{u}$), with real and imaginary components labeled with the apex q and p respectively ($\hat{u} = u^q + iu^p$).
We define the Stiefel manifold $St(S, H^1(D,\mathbb{F}))$, as the set of L^2-orthonormal frames of S functions in $H^1(D,\mathbb{F})$, i.e.:

$$St(S,H^1(D,\mathbb{F})) = \left\{ \mathbf{V} = (V_1,...,V_S) : \, V_i \in H^1(D,\mathbb{F}) \text{ and } \langle V_i, V_j \rangle_* = \delta_{ij} \; \forall i,j = 1,...,S \right\} \quad (2)$$

where $\langle V_i, V_j \rangle_*$ is the real L^2 product if $\mathbb{F} = \mathbb{R}$ and the hermitian product if $\mathbb{F} = \mathbb{C}$. We denote by $\mathcal{G}(S, H^1(D,\mathbb{F}))$ the Grassmann manifold of dimension S that consists of all the $S-$dimensional linear subspaces of $H^1(D,\mathbb{F})$. The definition of Stiefel and Grassmann manifold can be generalized to vector-valued functions in $[H^1(D,\mathbb{F})]^d$.
Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space, where Ω is the set of outcomes, $\mathcal{A}$ a σ-algebra and $P : \mathcal{A} \to [0,1]$ a probability measure. Let $y : \Omega \to \mathbb{F}$ be an integrable random variable; we define the mean of y as:

$$\bar{y} = \mathbb{E}[y] = \int_\Omega y(\omega) d\mathcal{P}(\omega).$$

$L^2(\Omega,\mathbb{F})$ (respectively $L_0^2(\Omega,\mathbb{F})$) denotes the Hilbert space of square integrable random variables (respectively with zero mean), that is:

$$\begin{aligned} L^2(\Omega,\mathbb{F}) \quad &:= \left\{ y : \Omega \to \mathbb{F} \, : \, \mathbb{E}[y^2] = \textstyle\int_\Omega (y(\omega))^2 d\mathcal{P}(\omega) < \infty \right\}, \\ L_0^2(\Omega,\mathbb{F}) \quad &:= \left\{ y : \Omega \to \mathbb{F} \, : \, \mathbb{E}[y] = 0, \, \mathbb{E}[y^2] = \textstyle\int_\Omega (y(\omega))^2 d\mathcal{P}(\omega) < \infty \right\}. \end{aligned}$$

We also recall that $L^2(D \times \Omega, \mathbb{F})$ denotes the space of all square integrable random fields, i.e.:

$$L^2(D \times \Omega, \mathbb{F}) := \left\{ u : D \times \Omega \to \mathbb{F} \text{ s.t. } \mathbb{E}\big[\|u\|^2_{L^2(D,\mathbb{F})} \big] < \infty \right\}.$$

Observe that $L^2(D \times \Omega, \mathbb{F})$ is isometrically isomorphic to the tensor product space $L^2(D,\mathbb{F}) \otimes L^2(\Omega,\mathbb{F})$. We denote by $\mathrm{B}(S, L^2(\Omega,\mathbb{F}))$ the set of all S frames of linearly independent random variables in $L^2(\Omega,\mathbb{F})$, i.e.:

$$\mathrm{B}(S, L^2(\Omega,\mathbb{F})) = \{ \mathbf{Y} = (Y_1,...,Y_S)^T \in [L^2(\Omega,\mathbb{F})]^S \text{ s.t. } \mathrm{rank}(\mathbb{E}[\mathbf{Y}\mathbf{Y}^*]) = S \}, \quad (3)$$

where $\mathbf{Y}^*$ is the complex conjugate of $\mathbf{Y}$, which simply coincides with the transpose if $\mathbf{Y}$ is a real valued random vector. Similarly, we denote by $\mathcal{G}(S, L^2(\Omega, \mathbb{F}))$ the Grassmann manifold of dimension S associated to $L^2(\Omega, \mathbb{F})$.

1.1 Wave equation with random parameters

We consider the following initial boundary value problem: find a random function $u : \bar{D} \times [0,T] \times \Omega \to \mathbb{R}$, such that P-almost everywhere in Ω (almost surely) the following holds:

$$\begin{cases} \ddot{u}(\mathbf{x},t,\omega) = \nabla \cdot \Big(c(\mathbf{x},\omega) \nabla u(\mathbf{x},t,\omega) \Big) + f(u(\mathbf{x},t,\omega),\omega) & \mathbf{x} \in D,\ t \in (0,T],\ \omega \in \Omega, \\ u(\mathbf{x},0,\omega) = p_0(\mathbf{x},\omega) & \mathbf{x} \in D,\ \omega \in \Omega, \\ \dot{u}(\mathbf{x},0,\omega) = q_0(\mathbf{x},\omega) & \mathbf{x} \in D,\ \omega \in \Omega, \\ u(\boldsymbol{\sigma},t,\omega) = 0 & \boldsymbol{\sigma} \in \partial D,\ t \in (0,T],\ \omega \in \Omega, \end{cases} \tag{4}$$

For convenience we restrict in this work to homogeneous Dirichlet boundary conditions, although the development hereafter generalizes easily to other types of boundary conditions, either homogeneous or non-homogeneous with deterministic forcing terms. The case of non-homogeneous stochastic boundary conditions can be treated as in [24] but will not be detailed in this work. For the well-posedness of problem (4), we assume that the random wave speed c is bounded and uniformly coercive [23, 33]:

$$0 < c_{min} \leq c(\mathbf{x},\omega) \leq c_{max} < \infty \qquad \forall \mathbf{x} \in D,\ a.s.,$$

and the initial data satisfy: $q_0 \in L^2(\Omega, H_0^1(D))$, $p_0 \in L^2(\Omega, L^2(D))$. Here the randomness may affect the wave speed c as well as the initial conditions p_0, q_0 and the (possibly non linear) source term f. Our goal is to find a dynamical low rank approximation of the solution of problem (4).

2 DLR approximation

We recall that the Dynamical Low Rank approximation (DLR) [12, 25, 31] is a reduced basis technique used for the approximation of parabolic equations with random parameters. Consider the following general real valued problem:

$$\begin{cases} \dot{u}(\mathbf{x},t,\omega) = \mathcal{L}(u(\mathbf{x},t,\omega),t,\omega), & \mathbf{x} \in D,\ t \in (0,T],\ \omega \in \Omega, \\ u(\mathbf{x},0,\omega) = u_0(\mathbf{x},\omega) & \mathbf{x} \in D,\ \omega \in \Omega, \\ \mathcal{B}(u(\boldsymbol{\sigma},t,\omega),\omega) = g(\boldsymbol{\sigma},t) & \boldsymbol{\sigma} \in D,\ t \in (0,T],\ \omega \in \Omega, \end{cases} \tag{5}$$

where $\mathcal{L}$ is a linear or non-linear differential operator, $x \in D$ is the spatial coordinate, t is the time variable in $[0,T]$ and $\mathcal{B}$ a suitable boundary operator. Here $\omega \in \Omega$ represents a random elementary event which may affect the operator $\mathcal{L}$ (as e.g. a coefficient or a forcing term) or the initial conditions. Let us assume that the solution $u(\cdot,t,\omega)$ to problem (5) is in a certain real Hilbert space $\mathcal{H} \subset L^2(D)$ for (almost) all $t \in [0,T]$ and $\omega \in \Omega$ and that $\mathcal{L}(u,t,\omega) \in \mathcal{H}'$ for all $u \in \mathcal{H}$ and almost everywhere in $[0,T]$ and Ω. Hereafter, whenever no confusion arises, we may write simply $\mathcal{L}(u)$ instead of $\mathcal{L}(u,t,\omega)$. The approximation manifold consists of the collection of all S rank random fields, i.e functions that can be exactly expressed as linear combination of S linearly independent deterministic modes combined with S linearly independent stochastic modes.

Definition 2.1. *We define $\mathcal{M}_S \subset \mathcal{H} \otimes L^2(\Omega)$ the manifold of all S rank random fields, i.e.:*

$$\mathcal{M}_S = \Big\{ u_S \in \mathcal{H} \otimes L^2(\Omega) : u_S = \textstyle\sum_{i=1}^S U_i Y_i \;|\quad \begin{array}{l} \textit{span}(U_1, ..., U_S) \in \mathcal{G}(S, \mathcal{H}), \\ \textit{span}(Y_1, ..., Y_S) \in \mathcal{G}(S, L^2(\Omega)) \end{array} \Big\} \tag{6}$$

The DLR approximate solution is sought in $\mathcal{M}_S$ and satisfies the following variational principle:

DLR Variational Principle. *At each $t \in [0, T]$, find $u_S(t) \in \mathcal{M}_S$ such that: $u_S(0) = u_{0,S}$ and*

$$\mathbb{E}\left[\langle \dot{u}_S(\cdot, t, \cdot) \,-\, \mathcal{L}(u_S(\cdot, t, \cdot)),\, v\rangle\right] = 0, \quad \forall v \in \mathcal{T}_{u_S(t)}\mathcal{M}_S,\, t \in (0, T] \tag{7}$$

where $\mathcal{T}_{u_S(t)}\mathcal{M}_S$ is the tangent space to $\mathcal{M}_S$ at $u_S(t)$.

The variational principle (7) enforces the approximate solution u_S to satisfy the governing equation projected onto the tangent space to the approximation manifold along the solution trajectory. The initial datum $u_{0,S}$ is a suitable S rank approximation of u_0 by e.g. a truncated Karhunen-Loève expansion (best S rank approximation in the $L^2(D) \otimes L^2(\Omega)$ norm). In quantum mechanic this is known as Dirac-Frenkel time-dependent variational principle (see e.g. [17]) and leads to the MCTDH method [4,7,13] for the approximation of deterministic time-dependent Schrödinger equations.
There exist several possible parameterizations of a S rank random field. One option, which leads to the so-called Dynamically Orthogonal (DO) method [31, 32], consists in expanding the approximate solution over a set of S $L^2(D)$-orthonormal deterministic modes:

$$u_S(\mathbf{x}, \omega) = \sum_{i=1}^{S} Y_i(\omega) U_i(x) = \mathbf{U}\mathbf{Y} \tag{8}$$

where:

- $\mathbf{U} \in St(S, \mathcal{H})$ is a row vector of L^2-orthonormal deterministic functions,
- $\mathbf{Y} \in \mathrm{B}(S, L^2(\Omega))$ is a column vector of S random variables with full rank second moment matrix $\mathbf{C} = \mathbb{E}[\mathbf{Y}\mathbf{Y}^T]$.

One easily sees that the representation (8) is not unique. For any orthogonal matrix $\mathbf{O} \in \mathcal{O}(S) \subset \mathbb{R}^{S \times S}$ one can always find a new couple of bases $\mathbf{W} = \mathbf{U}\mathbf{O} \in St(S, \mathcal{H})$ and $\mathbf{Z} = \mathbf{O}^T\mathbf{Y} \in \mathrm{B}(S, L^2(\Omega))$ which represents the same S rank random field: $u_S = \mathbf{U}\mathbf{Y} = \mathbf{W}\mathbf{Z}$. The uniqueness of the decomposition (8), in terms of $\mathbf{U} \in St(S, \mathcal{H})$ and $\mathbf{Y} \in \mathrm{B}(S, L^2(\Omega))$, is recovered by imposing the following constraint on the dynamics of $\mathbf{U}$ [12]:

$$\langle \dot{U}_i(t), U_j(t)\rangle = 0 \qquad i, j = 1, ..., S \tag{9}$$

This condition represents a quotientation of $St(S, \mathcal{H})$ with respect to the group of rotations $\mathcal{O}(S)$ and leads to the diffeomorphic identification of $\mathcal{M}_S$ with $\big(St(S, \mathcal{H})/\mathcal{O}(S)\big) \times \mathrm{B}(S, L^2(\Omega))$. In particular (9) implies that the tangent bundle to $\big(St(S, \mathcal{H})/\mathcal{O}(S)\big)$ is parametrized in terms of the tangent vectors of $St(S, \mathcal{H})$ which are orthogonal to the equivalent classes of the quotientification. This procedure is based on classical results of fiber bundle theory. We refer interested readers to [1,11,15] for further details.
By means of (9), the tangent space to $\mathcal{M}_S$ at $u_S = \mathbf{U}\mathbf{Y}$ is parametrized as:

$$\mathcal{T}_{u_S}\mathcal{M}_{\mathcal{S}} \quad = \Big\{ \delta u = \quad \sum_{i=1}^{S} \big(\delta Y_i U_i + Y_i \delta U_i\big) \in \mathcal{H} \otimes L^2(\Omega) : \,\langle U_i, \delta U_j\rangle = 0,\, \forall i, j = 1, ..., S \Big\} \tag{10}$$

Then (7) leads to the DO reduced system [31, 32]: find $u_S(t) = \mathbf{U}(t)\mathbf{Y}(t)$, $t \in (0, T]$ such that

$$\begin{cases} \sum_{i=1}^{S} \dot{U}_i \mathbf{C}_{ij} = \mathcal{P}_{\mathbf{U}}^{\perp}\Big(\mathbb{E}[\mathcal{L}(u_S)Y_j]\Big) & \forall j = 1, ..., S \\ \dot{Y}_j = \langle \mathcal{L}(u_S), U_j \rangle & \forall j = 1, ..., S \end{cases} \tag{11}$$

where $\mathbf{C} = \mathbb{E}[\mathbf{Y}\mathbf{Y}^T] \in \mathbb{R}^{S\times S}$ and $\mathcal{P}_{\mathbf{U}}^{\perp}$ is the projection operator from the space $L^2(D)$ to the orthogonal complement of the S dimensional subspace $\mathcal{U} = \text{span}\{U_1, \dots, U_S\}$, i.e. $\mathcal{P}_{\mathbf{U}}^{\perp}(v) = v - \mathcal{P}_{\mathbf{U}}(v) = v - \sum_{i=1}^{S} \langle v, U_i \rangle U_i$, $\forall v \in L^2(D)$. The projector $\mathcal{P}_{\mathbf{U}}^{\perp}(\cdot)$ can actually be defined on the larger space $\mathcal{H}'$ by interpreting $\langle \cdot, \cdot \rangle$ as a duality pairing. The initial condition is given by the truncated Karhunen-Loève expansion (the best S rank approximation in L^2-norm) and the DO approximate solution is determined by solving (11). The peculiarity of the DO method is that both the spatial and stochastic bases are computed on the fly and are free to evolve in time, thus adjusting at each time to the current structure of the random solution.

3 Symplectic Manifolds

Symplectic manifolds are the natural setting for Hamiltonians systems, due to the intrinsic symplectic structure of the canonical phase-space coordinates. We review in this section the main definitions and results concerning symplectic manifolds. For a comprehensive treatment see e.g. [20, 22].

Definition 3.1. *A symplectic manifold is a pair $(\mathcal{V}, \vartheta)$ consisting of a differential manifold $\mathcal{V}$ and a 2-form:*

$$\begin{aligned} \vartheta_u : \quad \mathcal{T}_u\mathcal{V} \times \mathcal{T}_u\mathcal{V} &\to \mathbb{R} \\ (y, z) &\to \vartheta_u(y, z) \end{aligned}$$

for any $u \in \mathcal{V}$, which is:

- *closed, i.e $\mathbf{d}\vartheta = 0$ where $\mathbf{d}$ is the exterior derivative.*
- *non-degenerate, i.e. for any $u \in \mathcal{V}$ and $y \in \mathcal{T}_u\mathcal{V}$, $\vartheta_u(y, z) = 0$ for all $z \in \mathcal{T}_u\mathcal{V}$ if and only if $y = 0$.*

The form ϑ is called symplectic form.

If $\mathcal{V}$ is a vector space, the requirement $\mathbf{d}\vartheta = 0$ is automatically satisfied since the ϑ_u is constant in u and Definition 3.1 is simplified as follows:

Definition 3.2. *Let $\mathcal{V}$ be a vector space and ϑ a bilinear map: $\vartheta : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that:*

- *ϑ is not degenerate, i.e. $\vartheta(y, z) = 0$ for all $z \in \mathcal{V}$ if and only if $y = 0$,*
- *ϑ is antisymmetric, i.e. $\vartheta(y, z) = -\vartheta(z, y)$ for any $y, z \in \mathcal{V}$.*

The pair $(\mathcal{V}, \vartheta)$ is called symplectic vector space.

Definition 3.3. *Let $(\mathcal{V}, \vartheta)$ be a symplectic vector space. A smooth submanifold $\mathcal{W} \subset \mathcal{V}$ is said symplectic if the restriction of ϑ to $\mathcal{W}$ is not-degenerate.*

Definition 3.4. *Let $(\mathcal{V}, \vartheta)$ be a symplectic vector space and $\mathcal{U}$ a subspace of $\mathcal{V}$. The symplectic complement of $\mathcal{U}$ is defined as:*

$$\mathcal{U}^{\perp,sym} = \big\{z \in \mathcal{V} \text{ such that } \vartheta(z, y) = 0, \ \forall y \in \mathcal{U}\big\} \tag{12}$$

Unlike orthogonal complements, $\mathcal{U}^{\perp,sym} \cap \mathcal{U}$ is not necessary trivial. We start by recalling some properties of finite dimensional symplectic manifolds [16]. Afterwards, we look at the infinite dimensional case [20, 35].

Proposition 3.1. *All finite dimension symplectic vector spaces are even dimensional.*

This can be verified by observing that real skew-symplectic matrices of odd dimension must have a non trivial kernel. Since a symplectic form makes the tangent spaces into symplectic vector spaces, Proposition 3.1 actually applies to all finite dimension symplectic manifolds. Without further specification, in the following $\mathcal{V}_{2N}$ will always denote a finite dimensional manifold of dimension $2N$.

Theorem 3.1 (Darboux' theorem). *Let $(\mathcal{V}_{2N}, \vartheta)$ be a symplectic manifold. For any $\mathbf{u} \in \mathcal{V}_{2N}$ there exists a neighborhood $\mathcal{B}_{\mathbf{u}} \subseteq \mathcal{V}_{2N}$ of $\mathbf{u}$ and a local coordinate chart in which ϑ is constant.*

Definition 3.5. *Let $(\mathcal{V}_{2N}, \vartheta)$ be a symplectic vector space. A basis $(\mathbf{e}_1, ..., \mathbf{e}_N, \mathbf{f}_1, ..., \mathbf{f}_N)$ of $\mathcal{V}_{2N}$ is said symplectic if:*

- $\vartheta(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij} = -\vartheta(\mathbf{f}_j, \mathbf{e}_i)$, $\forall i, j = 1, ..., N$,
- $\vartheta(\mathbf{e}_i, \mathbf{e}_j) = 0 = \vartheta(\mathbf{f}_i, \mathbf{f}_j)$, $\forall i, j = 1, ..., N$.

Darboux' theorem implies that, for any $\mathbf{u}$ in the symplectic manifold $(\mathcal{V}_{2N}, \vartheta)$ there is a neighborhood $\mathcal{B}_{\mathbf{u}} \subseteq \mathcal{V}_{2N}$ and a symplectic basis with respect to which the symplectic form is written as $\vartheta_{\mathbf{u}}(\mathbf{w}, \mathbf{v}) = \mathbf{w}^T \mathbf{J}_{2N} \mathbf{v}$ for all $\mathbf{w}, \mathbf{v} \in \mathcal{B}_{\mathbf{u}}$ (column vectors), where $\mathbf{J}_{2N} \in \mathbb{R}^{2N \times 2N}$ is the Poisson matrix, i.e.

$$\mathbf{J}_{2N} = \begin{pmatrix} 0 & \mathbb{I}_N \\ -\mathbb{I}_N & 0 \end{pmatrix}$$

and $\mathbb{I}_N$ is the identity matrix in $\mathbb{R}^{N \times N}$. It is easy to verify that $\mathbf{J}_{2N}\mathbf{J}_{2N}^T = \mathbf{J}_{2N}^T\mathbf{J}_{2N} = \mathbb{I}_{2N}$ and $\mathbf{J}_{2N}\mathbf{J}_{2N} = \mathbf{J}_{2N}^T\mathbf{J}_{2N}^T = -\mathbb{I}_{2N}$. When $\mathcal{V}_{2N}$ is a vector space, $\vartheta_{\mathbf{u}}$ is constant in $\mathbf{u}$ and $\mathcal{B}_{\mathbf{u}}$ corresponds to the whole space, namely $\vartheta(\mathbf{w}, \mathbf{v}) = \mathbf{w}^T \mathbf{J}_{2N} \mathbf{v}$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{V}_{2N}$. If this symplectic basis coincides with the canonical basis of $\mathbb{R}^{2N}$ we call ϑ canonical symplectic form and we denote by $(\mathcal{V}_{2N}, \mathbf{J}_{2N})$ the corresponding symplectic manifold. A prototypical example of symplectic vector space arises from the identification of the complex space $\mathbb{C}^N$ with the real space $\mathbb{R}^{2N}$. Let us write elements of $\mathbb{C}^N$ as N-tuples of complex numbers $\hat{\mathbf{u}} = (\hat{u}_1, ..., \hat{u}_N)$, for each term $\hat{u}_i = u_i^q + i u_i^p$, the apex q and p denoting respectively the real and the complex components. Let $\mathbb{C}^N$ be equipped with the usual Hermitian inner product:

$$\langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle_h = \sum_{i=1}^{S} \hat{u}_i \hat{v}_i^* = \sum_{i=1}^{S} (u_i^q v_i^q + u_i^p v_i^p) + i \sum_{i=1}^{S} (v_i^q u_i^p - u_i^q v_i^p),$$

for any $\hat{\mathbf{u}}, \hat{\mathbf{v}} \in \mathbb{C}^N$. The realification, namely the identification of $\mathbb{C}^N$ with $\mathbb{R}^{2N}$, consists in associating to any $\hat{\mathbf{u}} \in \mathbb{C}^N$ the elements $\mathbf{u} = (\mathbf{u}^q, \mathbf{u}^p) \in \mathbb{R}^{2N}$, where $\mathbf{u}^q = (u_1^q, ..., u_N^q)$ and $\mathbf{u}^p = (u_1^p, ..., u_N^p)$. In the following we always use the overhat to distinguish complex elements ($\hat{\mathbf{u}}$) and corresponding real representations ($\mathbf{u}$). One can easily see that the canonical symplectic form of $\mathbb{R}^{2N}$ coincides with the imaginary part of the Hermitian product, with changed sign: $\mathbf{u}^T \mathbf{J}_{2N} \mathbf{v} = -\operatorname{Im}\langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle_h$, for all $\hat{\mathbf{u}}, \hat{\mathbf{v}} \in \mathbb{C}^N$.
We call symplectic matrix any $\mathbf{A} \in \mathbb{R}^{2N \times 2N}$ such that $\mathbf{A}^T \mathbf{J}_{2N} \mathbf{A} = \mathbf{J}_{2N}$. The collection of all symplectic matrices of $\mathbb{R}^{2N \times 2N}$ forms a group, called symplectic group.

Definition 3.6. *The symplectic group, denoted by $Sp(2N, \mathbb{R}^{2N})$, is the subset of $\mathbb{R}^{2N \times 2N}$ defined as:*

$$Sp(2N, \mathbb{R}^{2N}) := \{\mathbf{A} \in \mathbb{R}^{2N \times 2N} : \mathbf{A}^T \mathbf{J}_{2N} \mathbf{A} = \mathbf{J}_{2N}\}.$$

The unitary group is the subgroup of $Sp(2N,\mathbb{R}^{2N})$ of all unitary matrices

Definition 3.7. *The unitary group, denoted by $U(N,\mathbb{R}^{2N})$, is the subset of $\mathbb{R}^{2N\times 2N}$ defined as:*

$$U(N,\mathbb{R}^{2N}) := \{\mathbf{A} \in Sp(2N,\mathbb{R}^{2N}) : \mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbb{I}_{2N}\}.$$

In other words, $U(N,\mathbb{R}^{2N}) = Sp(2N,\mathbb{R}^{2N}) \cap \mathcal{O}(2N,\mathbb{R}^{2N})$, where $\mathcal{O}(2N,\mathbb{R}^{2N})$ denotes the group of orthogonal matrices in $\mathbb{R}^{2N\times 2N}$. Definitions 3.6 and 3.7 can be generalized to rectangular matrices $\mathbf{A} \in \mathbb{R}^{2N\times 2S}$ for any $0 < S < N$:

Definition 3.8. *We denote by $Sp(2S,\mathbb{R}^{2N})$ the sub-manifold of $\mathbb{R}^{2N\times 2S}$ defined as:*

$$Sp(2S,\mathbb{R}^{2N}) := \{\mathbf{A} \in \mathbb{R}^{2N\times 2S} : \mathbf{A}^T\mathbf{J}_{2N}\mathbf{A} = \mathbf{J}_{2S}\}$$

and by $U(S,\mathbb{R}^{2N})$, the submanifold of $\mathbb{R}^{2N\times 2S}$ defined as:

$$U(S,\mathbb{R}^{2N}) := \{\mathbf{A} \in Sp(2S,\mathbb{R}^{2N}) : \mathbf{A}^T\mathbf{A} = \mathbb{I}_{2S}\}.$$

We call symplectic (respectively unitary) matrix any $\mathbf{A} \in Sp(2S,\mathbb{R}^{2N})$ (respectively $\mathbf{A} \in U(S,\mathbb{R}^{2N})$).

Definition 3.9. *A linear map $\phi : \mathbb{R}^{2S} \to \mathbb{R}^{2N}$ defined as:*

$$\begin{aligned} \phi : \quad & \mathbb{R}^{2S} && \to \mathbb{R}^{2N} \\ & \mathbf{x} && \mapsto \phi(\mathbf{x}) := \mathbf{A}\mathbf{x} \end{aligned} \tag{13}$$

is said symplectic if it preserves the canonical form, i.e

$$\mathbf{x}^T\mathbf{J}_{2S}\mathbf{x} = \phi(\mathbf{x})^T\mathbf{J}_{2N}\phi(\mathbf{x}) = (\mathbf{A}\mathbf{x})^T\mathbf{J}_{2N}\mathbf{A}\mathbf{x} \qquad \forall \mathbf{x} \in \mathbb{R}^{2S}$$

.

Observe that ϕ in (13) is symplectic if and only if $\mathbf{A} \in Sp(2S,\mathbb{R}^{2N})$.
In the same way as $\mathbb{R}^{2N}$ admits a (canonical) symplectic structure associated to the Euclidean product, all inner product vector spaces can be equipped with the symplectic form associated to their inner product, called again canonical form. In particular, consider the (possibly infinite dimensional) Hilbert space $\mathcal{H}$ and the product space $\boldsymbol{\mathcal{H}} = [\mathcal{H}]^2$, for which we use the notation $\mathbf{u} = (u^q, u^p)$ to denote the first and second component of any $\mathbf{u} \in \boldsymbol{\mathcal{H}}$. Let $\boldsymbol{\mathcal{H}}$ be equipped with the usual inner product: $\langle \mathbf{u},\mathbf{v}\rangle_{\boldsymbol{\mathcal{H}}} := \langle u^q, v^q\rangle_{\mathcal{H}} + \langle u^p, v^p\rangle_{\mathcal{H}}$, for any $\mathbf{u},\mathbf{v}$ in $\boldsymbol{\mathcal{H}}$; we denote by $\mathcal{J}_2 : \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ the following linear operator:

$$\begin{aligned} \mathcal{J}_2 : \quad & \boldsymbol{\mathcal{H}} && \to \boldsymbol{\mathcal{H}} \\ & \mathbf{u} && \mapsto \mathcal{J}_2(\mathbf{u}) := \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \begin{bmatrix} u^q \\ u^p \end{bmatrix} = \begin{bmatrix} u^p \\ -u^q \end{bmatrix} \end{aligned}$$

where I_d is the identity operator in $\mathcal{H}$. Then, the canonical form of $\boldsymbol{\mathcal{H}}$ is defined as

$$\begin{aligned} \vartheta : \quad & \boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{H}} && \to \quad \mathbb{R} \\ & (\mathbf{u},\mathbf{v}) && \mapsto \quad \langle \mathbf{u}, \mathcal{J}_2(\mathbf{v})\rangle_{\boldsymbol{\mathcal{H}}} = \langle u^q, v^p\rangle_{\mathcal{H}} - \langle u^p, v^q\rangle_{\mathcal{H}} \end{aligned}$$

The form ϑ is antisymmetric, being $\mathcal{J}_2 \circ \mathcal{J}_2(\mathbf{u}) = -\mathbf{u}$, for all $\mathbf{u} \in \boldsymbol{\mathcal{H}}$, and non degenerate since $\vartheta(\mathcal{J}_2(\mathbf{u}),\mathbf{u}) = \langle \mathcal{J}_2(\mathbf{u}), \mathcal{J}_2(\mathbf{u})\rangle_{\boldsymbol{\mathcal{H}}} = \|\mathbf{u}\|^2_{\boldsymbol{\mathcal{H}}}$ which is non zero for any $0 \neq \mathbf{u} \in \boldsymbol{\mathcal{H}}$. Hence $(\boldsymbol{\mathcal{H}}, \vartheta)$ is a symplectic vector space. We generally write $(\mathcal{V}, \mathcal{J}_2)$ to refer to a symplectic manifold $\mathcal{V} \subset \boldsymbol{\mathcal{H}}$, when equipped with the canonical form of $\boldsymbol{\mathcal{H}}$.

Proposition 3.2. *Let $\mathcal{H}^{\mathbb{C}}$ be a complex Hilbert space and $\mathcal{H} \times \mathcal{H}$ its realification. The Hermitian product of $\mathcal{H}^{\mathbb{C}}$, defined as:*

$$\langle \hat{u}, \hat{v} \rangle_{\mathcal{H}^{\mathbb{C}}} := \langle u^q, v^q \rangle_{\mathcal{H}} + \langle u^p, v^p \rangle_{\mathcal{H}} + i(\langle u^p, v^q \rangle_{\mathcal{H}} - \langle v^p, u^q \rangle_{\mathcal{H}}) \qquad \forall \hat{u} = u^q + iu^p, \hat{v} = v^q + iv^p \in \mathcal{H}^{\mathbb{C}},$$

satisfies:

$$\langle (u^q, u^p), \mathcal{J}_2(v^q, v^p) \rangle_{\mathcal{H} \times \mathcal{H}} = -\operatorname{Im}(\langle \hat{u}, \hat{v} \rangle_{\mathcal{H}^{\mathbb{C}}})$$

for any $\mathbf{u} = (u^q, u^p), \mathbf{v} = (v^q, v^p) \in \mathcal{H} \times \mathcal{H}$ and $\hat{u} = u^q + iu^p, \hat{v} = v^q + iv^p \in \mathcal{H}^{\mathbb{C}}$.

This construction applies straightforwardly to $\boldsymbol{\mathcal{H}} = [L^2(D)]^2$ equipped with the $[L^2(D)]^2$ inner product and to $\boldsymbol{\mathcal{H}} = [H^1(D)]^2$ equipped either with the $L^2(D) \times L^2(D)$ or the $[H^1(D)]^2$ inner product. The identification in complex setting leads to $\mathcal{H}^{\mathbb{C}} = L^2(D, \mathbb{C})$ and $\mathcal{H}^{\mathbb{C}} = H^1(D, \mathbb{C})$ for $\boldsymbol{\mathcal{H}}$ respectively equal to $\boldsymbol{\mathcal{H}} = [L^2(D)]^2$ or $\boldsymbol{\mathcal{H}} = [H^1(D)]^2$.

In view of the Symplectic Dynamical Low Rank approximation of wave equations we need to recast problem (4) into a Hamiltonian system, in terms of the phase-space coordinates $(u, \dot{u}) \in H^1(D) \times L^2(D)$. For this aim, we are interested to equip $\boldsymbol{\mathcal{H}} = H^1(D) \times L^2(D)$ with the symplectic form associated to the $L^2(D) \times L^2(D)$ inner product and verify that what we obtain is still a symplectic space. The issue is due to the fact that now $\boldsymbol{\mathcal{H}}$ is a product of two different Hilbert spaces. With a little abuse of notation, we use the same symbol $\mathcal{J}_2$ to denote the restriction of $\mathcal{J}_2$ to $H^1(D) \times L^2(D)$ (respectively $L^2(D) \times H^1(D)$), i.e. the linear operator:

$$\begin{aligned} \mathcal{J}_2 : \quad H^1(D) \times L^2(D) \quad &\to \quad L^2(D) \times H^1(D) \\ \mathbf{u} = (u^q, u^p)^T \quad &\mapsto \quad \mathcal{J}_2(\mathbf{u}) := \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \begin{bmatrix} u^q \\ u^p \end{bmatrix} = \begin{bmatrix} u^p \\ -u^q \end{bmatrix} \end{aligned} \tag{14}$$

where I_d is the identity operator defined in $L^2(D)$ or restricted to $H^1(D)$. Then the bilinear form associated to $\mathcal{J}_2$ is clearly antisymmetric and non degenerate in $H^1(D) \times L^2(D)$, thanks to the fact that $H^1(D)$ is dense in $L^2(D)$. This allows us to conclude that $H^1(D) \times L^2(D)$ is a symplectic (pre-Hilbert) vector space when endowed with the canonical form associated to the $[L^2(D)]^2$ inner product. On the other hand, in this case we loose the identification in complex setting, namely Proposition 3.2 does not apply to $H^1(D) \times L^2(D)$ since we are dealing with the cartesian product of two different spaces. We denote by ϑ_D the symplectic form of $H^1(D) \times L^2(D)$ associated to the $L^2(D) \times L^2(D)$ inner product, i.e:

$$\vartheta_D(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathcal{J}_2 \mathbf{v} \rangle_{[L^2(D)]^2}, \qquad \mathbf{u}, \mathbf{v} \in H^1(D) \times L^2(D). \tag{15}$$

Hereafter, when confusion does not arise, we omit the subscript and we write $< \cdot, \cdot >$ to indicate the $L^2(D) \times L^2(D)$-product in $H^1(D) \times L^2(D)$ (or any other Sobolev space $\boldsymbol{\mathcal{H}} \subset [L^2(D)]^2$). The same considerations apply to $\big(H^1(D) \times L^2(D)\big) \otimes L^2(\Omega)$ and $[H^1(D)]^S \times [L^2(D)]^S$, for any $S > 0$.

In analogy with (2), one can define the Stiefel manifold $Sp(2S, H^1(D) \times L^2(D))$ of all possible $2S$ dimensional symplectic bases in $H^1(D) \times L^2(D)$ with respect to the symplectic form ϑ_D.

Definition 3.10. *We denote with $Sp(2S, H^1(D) \times L^2(D))$ the Stiefel manifold of all S dimensional symplectic bases of $(H^1(D) \times L^2(D), \vartheta_D)$, i.e.:*

$$\begin{aligned} Sp(2S, H^1(D) \times L^2(D)) := \big\{ \mathbf{U} \ &= (\mathbf{U}_1, ..., \mathbf{U}_{2S}) \in [H^1(D) \times L^2(D)]^{2S}, \text{ such that} \\ &\vartheta_D(\mathbf{U}_i, \mathbf{U}_j) = (\mathbf{J}_{2S})_{ij}, \, \forall i, j = 1, ..., 2S \big\}. \end{aligned} \tag{16}$$

We denote by $\mathcal{U}^{sym} \subset H^1(D) \times L^2(D)$ the subspace spanned by $\mathbf{U}$, for any $\mathbf{U} \in Sp(2S, H^1(D) \times L^2(D))$, and we call $\mathbf{U}$ a symplectic basis of $\mathcal{U}^{sym}$. Note that the symplectic form ϑ_D, when restricted to $\mathcal{U}^{sym}$, can be identified with the canonical form of $\mathbb{R}^S$, that is for any $\mathbf{R}$, $\mathbf{G} \in \mathbb{R}^S$ and $\mathbf{u} = \mathbf{U}\mathbf{R}$, $\mathbf{v} = \mathbf{U}\mathbf{G} \in \mathcal{U}^{sym}$:

$\vartheta_D(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{2S} \mathbf{R}_i \langle \mathbf{U}_i, \mathcal{J}_2 \mathbf{U}_j \rangle \mathbf{G}_j = \mathbf{R}^T \mathbf{J}_{2S} \mathbf{G}$. This implies that ϑ_D is non degenerate in $\mathcal{U}^{sym}$ and $\mathcal{U}^{sym}$, is a symplectic submanifold of $H^1(D) \times L^2(D)$. We define in $Sp(2S, H^1(D) \times L^2(D))$ the following equivalence relation:

$$\mathbf{W} \sim \mathbf{U} \iff \mathcal{W}^{sym} = \mathcal{U}^{sym}$$

meaning that two equivalent elements span the same symplectic subspace.

Lemma 3.1. *Two symplectic bases $\mathbf{W}, \mathbf{U} \in Sp(2S, H^1(D) \times L^2(D))$ are equivalent if and only if there exists a symplectic matrix $\mathbf{B} \in Sp(2S, \mathbb{R}^{2S})$ such that $\mathbf{W} = \mathbf{U}\mathbf{B}$.*

Proof. The sufficient condition is obvious: $\mathbf{B}^{\mathrm{T}} \mathbf{J}_{2S} \mathbf{B} = \mathbf{J}_{2S}$ implies $\langle (\mathbf{W})_i, (\mathcal{J}_2 \mathbf{W})_j \rangle_{L^2(D)} = (\mathbf{J}_{2S})_{ij}$. On the other hand, if $\mathbf{U} \in Sp(2S, H^1(D) \times L^2(D))$, then $\mathbf{U}_1, ..., \mathbf{U}_{2S}$ are linearly independent. Hence, if $\mathbf{W}, \mathbf{U} \in Sp(2S, H^1(D) \times L^2(D))$ span the same subspace, there necessarily exists a (unique) full rank matrix $\mathbf{B} \in \mathbb{R}^{2S \times 2S}$ such that $\mathbf{W} = \mathbf{U}\mathbf{B}$. Then $\langle (\mathbf{W})_i, (\mathcal{J}_2 \mathbf{W})_j \rangle_{L^2(D)} = (\mathbf{J}_{2S})_{ij}$ implies that $\mathbf{B}$ belongs to $Sp(2S, \mathbb{R}^{2S})$. ■

4 Hamiltonian formulation of wave equations with random parameters

From a physics point of view, a Hamiltonian, denoted in the following by H, is a smooth function which expresses the total energy of a dynamical system in terms of the position and the momentum of its particles. In more abstract setting we can state the following [19]:

Definition 4.1. *Let $(\mathcal{V}, \vartheta)$ be a symplectic manifold. A vector field X_H on $\mathcal{V}$ is called Hamiltonian if there is a function $H : \mathcal{V} \to \mathbb{R}$ such that:*

$$\vartheta_{\mathbf{u}}(X_H(\mathbf{u}), \mathbf{v}) = \mathbf{d}H(\mathbf{u}) \cdot \mathbf{v}$$

where $\mathbf{d}H(\mathbf{u}) \cdot \mathbf{v}$ is the directional derivative of H along $\mathbf{v}$. Hamilton's equations are the evolution equations:

$$\dot{\mathbf{u}} = X_H(\mathbf{u}) \tag{17}$$

If $(\mathcal{V}_{2N}, \vartheta)$ is a symplectic vector space and $(\mathbf{q}, \mathbf{p}) = (q_1, ..., q_N, p_1, ..., p_N)$ denote the canonical coordinates with respect to which ϑ has matrix $\mathbf{J}_{2N}$, the Hamiltonian equations become:

$$\dot{\mathbf{u}} = \mathbf{J}_{2N} \nabla H(\mathbf{u}).$$

Let ϕ_t denote the flow of the Hamiltonian X_H, that is $\phi_t(\mathbf{u}_0)$ is the solution to (17) with initial condition $\mathbf{u}_0 \in \mathcal{V}$, we have that ϕ_t conserves the energy of H.

Proposition 4.1. *Let ϕ_t be the flow of X_H on the symplectic manifold $(\mathcal{V}, \vartheta)$. Then $H \circ \phi_t = H$, where defined.*

Proof.

$$\begin{aligned} \tfrac{d}{dt}\big(H \circ \phi_t(\mathbf{u})\big) &= \mathbf{d}H(\phi_t(\mathbf{u})) \cdot X_H(\phi_t(\mathbf{u})) \\ &= \vartheta_{\phi_t(\mathbf{u})}(X_H(\phi_t(\mathbf{u})), X_H(\phi_t(\mathbf{u}))) = 0 \end{aligned}$$

■

The flow ϕ_t of a Hamiltonian vector field consists of symplectic transformations, namely ϕ_t (whenever it is defined) preserves the symplectic form ϑ. Formally, for all $\mathbf{u} \in \mathcal{V}$ and $\mathbf{v}, \mathbf{z} \in \mathcal{T}_{\mathbf{u}}\mathcal{V}$, we have:

$$\vartheta_{\mathbf{u}}(\mathbf{v}, \mathbf{z}) = \vartheta_{\phi_t(\mathbf{u})}(D_{\mathbf{u}}[\phi_t](\mathbf{v}), D_{\mathbf{u}}[\phi_t](\mathbf{z})) \tag{18}$$

where $D_{\mathbf{u}}[\phi_t]$ is the differential of ϕ_t at $\mathbf{u}$. It follows from Poincaré lemma [10, 20] that the flow ϕ_t of a vector field X is symplectic if and only if it is locally Hamiltonian, that is there locally exists a Hamiltonian function H such that $\vartheta_{\mathbf{u}}(X(\mathbf{u}), \mathbf{v}) = \mathbf{d}H(\mathbf{u}) \cdot \mathbf{v}$. The link between symplecticity and energy preservation has been widely studied and exploited to derive numerical time discretization schemes that share the same symplectic structure of the original system, in order to preserve the geometric properties. The same idea can be used to formulate reduced order methods which preserve the symplectic structure underlying the original full order Hamiltonian system, thus being energy conservative and preserving stability.

We start by looking for a suitable Hamiltonian formulation for wave equations with random parameters. As shown in literature [23, 33], problem (4) admits a unique solution $u \in L^\infty([0,T], H_0^1(D) \otimes L^2(\Omega))$ with time derivative $\dot{u} \in L^\infty([0,T], L^2(D) \otimes L^2(\Omega))$, provided that the random wave speed c is bounded and uniformly coercive and the initial data (q_0, p_0) belong to $\big(H_0^1(D) \otimes L^2(\Omega)\big) \times \big(L^2(D) \otimes L^2(\Omega)\big)$. Let us introduce the phase space variables $(p, q) = (u, \dot{u})$, then problem (4) can be rewritten into a first order system in $H^1(D) \times L^2(D)$ for almost all $\omega \in \Omega$:

$$\begin{cases} \dot{q}(\mathbf{x},t,\omega) = p(\mathbf{x},t,\omega) & \mathbf{x} \in D,\ t \in (0,T],\ \omega \in \Omega, \\ \dot{p}(\mathbf{x},t,\omega) = \nabla \cdot \big(c(\mathbf{x},\omega)\nabla q(\mathbf{x},t,\omega)\big) - f(q(\mathbf{x},t,\omega),\omega) & \mathbf{x} \in D,\ t \in (0,T],\ \omega \in \Omega, \\ q(\mathbf{x},0,\omega) = q_0(\mathbf{x},\omega) & \mathbf{x} \in D,\ \omega \in \Omega, \\ p(\mathbf{x},0,\omega) = p_0(\mathbf{x},\omega) & \mathbf{x} \in D,\ \omega \in \Omega, \\ q(\mathbf{x},t,\omega) = 0 & \mathbf{x} \in \partial D,\ t \in (0,T],\ \omega \in \Omega, \end{cases} \tag{19}$$

analogously written in matrix form as:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \mathcal{J}_2 \begin{pmatrix} -\nabla \cdot \big(c\nabla \cdot q\big) + f(q) \\ p \end{pmatrix}$$

Problem (19) can be interpreted as a Hamiltonian system in the symplectic space $(H_0^1(D) \times L^2(D), \vartheta_D)$ with symplectic form ϑ_D defined in (15). In this case, the Hamiltonian energy associated to (19) is defined pointwise in ω as:

$$H_\omega(q,p) = \frac{1}{2}\int_D \Big(|p|^2 + c(\omega)|\nabla q|^2 + F(q)\Big), \qquad F'(q) = f(q).$$

Thus, by denoting with $\nabla_q H_\omega$, $\nabla_p H_\omega$ the functional derivatives of H_ω with respect to q and p respectively, i.e.:

$$\begin{aligned} \langle \nabla_q H_\omega, \delta q \rangle &= \textstyle\int_D c\nabla q \nabla \delta q + \int_D f(q)\delta q \qquad \text{and} \qquad \langle \nabla_p H_\omega, \delta p \rangle = \int_D p\delta p. \\ &= \textstyle\int_D \big(-\nabla \cdot (c\nabla \cdot q) + f(q)\big)\delta q, \end{aligned}$$

for any $\delta q \in H_0^1(D)$, $\delta p \in L^2(D)$, where the term $\int_D -\nabla \cdot (c\nabla \cdot q)\delta q$ should be interpreted in distributional sense, equation (4) is recast into the following canonical Hamiltonian system, written with respect to $\mathbf{u} = (q, p)$:

$$\begin{cases} \dot{\mathbf{u}}(\mathbf{x},t,\omega) = \mathcal{J}_2 \nabla H_\omega\big(\mathbf{u}(\mathbf{x},t,\omega),\omega\big), \\ \mathbf{u}(\mathbf{x},0,\omega) = (q_0(\mathbf{x},\omega), p_0(\mathbf{x},\omega))^T \end{cases} \tag{20}$$

for almost every $\mathbf{x} \in D$ and $\omega \in \Omega$. Observe that both the flow of the solutions and the Hamiltonian depend on the random input, and that the conservation of energy applies point-wise in the parameter space, which means that, for any realization ω, the flow ϕ_t of (20) with initial conditions evaluated in ω, conserves the Hamiltonian evaluated in ω. This immediately implies that the expected value, and generally any finite moment of H_ω, are constant along the flow of the solutions.

Alternatively, in a setting more suited to our context, the conservation of energy can be derived directly in

$\big(H^1(D)\otimes L^2(\Omega)\big)\times\big(L^2(D)\otimes L^2(\Omega)\big) = \big(H^1(D)\times L^2(D)\big)\otimes L^2(\Omega)$ equipped with the following symplectic form:

$$\begin{aligned}\vartheta(\mathbf{u},\mathbf{v}) \quad &= \mathbb{E}[\langle \mathbf{u}, \mathcal{J}_2\mathbf{v}\rangle_{[L^2(D)]^2}],\\ &= \mathbb{E}[\langle u^q, v^p\rangle_{L^2(D)}] - \mathbb{E}[\langle u^p, v^q\rangle_{L^2(D)}]\end{aligned} \tag{21}$$

for any $\mathbf{u} = (u^q, u^p)$, $\mathbf{v} = (v^q, v^p) \in \big(H^1(D)\otimes L^2(\Omega)\big)\times\big(L^2(D)\otimes L^2(\Omega)\big)$, with $u^q, v^q \in H^1(D)\otimes L^2(\Omega)$, $u^p, v^p \in L^2(D)\otimes L^2(\Omega)$. The pair $(\big(H^1(D)\otimes L^2(\Omega)\big)\times\big(L^2(D)\otimes L^2(\Omega)\big))$ is the symplectic space that will be used in Section 6 to derive the Symplectic Dynamical Low Rank method. In this setting the Hamiltonian energy associated to (19) is defined as:

$$H(q,p) = \frac{1}{2}\mathbb{E}\Big[\int_D \Big(|p|^2 + c(\omega)|\nabla q|^2 + F(q)\Big)\Big], \qquad F'(q) = f(q).$$

In particular, if $X_{H(\omega)}$ denotes the Hamiltonian vector field associated to (20), for $\mathbf{u}$ sufficiently smooth, system (20) can be rewritten as $\dot{\mathbf{u}} = X_H(\mathbf{u})$ and the conservation of mean energy along the flow of the solutions can be rederived in terms of the symplectic form (21) as:

$$\begin{aligned}\tfrac{d}{dt}H(\mathbf{u}(t)) \quad &= \langle \nabla H(\mathbf{u}(t)), \dot{\mathbf{u}}(t)\rangle\\ &= \langle \nabla H(\mathbf{u}(t)), X_{H(\omega)}(\mathbf{u}(t))\rangle\\ &= -\vartheta\big(X_{H(\omega)}(\mathbf{u}(t)), X_{H(\omega)}(\mathbf{u}(t))\big) = 0\end{aligned} \tag{22}$$

thanks to the antisymmetry of ϑ.

5 Symplectic Order Reduction

We recall here the symplectic order reduction for parametric Hamiltonian systems proposed in [27]. This method is designed in analogy to the proper orthogonal decomposition where the standard inner product is replaced by the symplectic form and leads to approximate solutions which belong to a low dimensional symplectic space. This method has the desirable property of preserving the symplectic structure of the full order system, which allows one to derive conservative schemes.

Definition 5.1. *Let $\mathbf{U} \in Sp(2S, H^1(D)\times L^2(D))$, the symplectic inverse of $\mathbf{U}$, denoted by $\mathbf{U}^+$, is the $2S$ vector function written as:*

$$\mathbf{U}^+ := \mathcal{J}_2^T\mathbf{U}\mathbf{J}_{2S} \in Sp(2S, L^2(D)\times H^1(D)). \tag{23}$$

If we write $\mathbf{U}$ component-wise, with $\mathbf{U}_i = (U_i^q, U_i^p)^T \in H^1(D)\times L^2(D)$:

$$\begin{aligned}\mathbf{U} \quad &= \begin{bmatrix} U_1^q & \dots & U_S^q & U_{S+1}^q & \dots & U_{2S}^q\\ U_1^p & \dots & U_S^p & U_{S+1}^p & \dots & U_{2S}^p\end{bmatrix} \qquad && \mathbf{Q}_{I,S} = U_1^q,\dots,U_S^q, \qquad && \mathbf{Q}_{II,S} = U_{S+1}^q,\dots,U_{2S}^q,\\ &= \begin{bmatrix}\mathbf{Q}_{I,S} & \mathbf{Q}_{II,S}\\ \mathbf{P}_{I,S} & \mathbf{P}_{II,S}\end{bmatrix}, && \mathbf{P}_{I,S} = U_1^p,\dots,U_S^p, && \mathbf{P}_{II,S} = U_{S+1}^p,\dots,U_{2S}^p,\end{aligned}$$

then $\mathbf{U}^+$ is explicitly given by:

$$\mathbf{U}^+ = \begin{bmatrix}\mathbf{P}_{II,S} & -\mathbf{P}_{I,S}\\ -\mathbf{Q}_{II,S} & \mathbf{Q}_{I,S}\end{bmatrix}$$

It is straightforward to verify that $\langle \mathbf{U}_i, \mathbf{U}_j^+\rangle = \delta_{ij}$, $\forall i,j = 1,\dots,2S$. The notion of symplectic inverse is used to define the symplectic Galerkin projection. Precisely:

Definition 5.2. *Let $\mathbf{v} = (v^q, v^p)^{\mathrm{T}}$ be a square integrable random field in $\big(H^1(D)\times L^2(D)\big)\otimes L^2(\Omega)$ and $\mathbf{U} \in Sp(2S, H^1(D)\times L^2(D))$ a symplectic basis, spanning $\mathcal{U}^{sym}$. The symplectic projection of $\mathbf{v}$ into $\mathcal{U}^{sym}\otimes L^2(\Omega)$*

is defined as:

$$\mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}] := \sum_{i=1}^{2S} \langle \mathbf{v}, \mathbf{U}_i^+ \rangle \mathbf{U}_i, \tag{24}$$

where $\mathbf{U}^+ \in Sp(2S, L^2(D) \times H^1(D))$ *is the symplectic inverse of* $\mathbf{U}$*, defined in (23). Namely,*

$$\mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}] = \mathbf{U}(\mathbf{x})\mathbf{Y}(\omega) \tag{25}$$

where $\mathbf{Y} = Y_1, ..., Y_{2S}$ *is a vector of* $2S$ *square integrable random variables defined by* $Y_i = \langle \mathbf{v}, \mathbf{U}_i^+ \rangle$ *for any* $i = 1, ..., S$*. Moreover we say that* $\mathbf{v}$ *is in the subspace spanned by* $\mathbf{U}$ *if* $\mathbf{v} = \mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}]$*, or namely if there exists a vector of (square integrable) random variables* $\mathbf{Y}$*, such that* $\mathbf{v} = \mathbf{U}\mathbf{Y}$*. Observe that* $\mathbf{Y}$ *is uniquely determined by* $\mathbf{U}$ *by means of symplectic projection as* $Y_i = \langle \mathbf{v}, \mathbf{U}_i^+ \rangle$*. On the contrary, we say that* $\mathbf{v} \in \big(H^1(D) \times L^2(D)\big) \otimes L^2(\Omega)$ *is in the symplectic orthogonal complement of* $\mathbf{U}$ *if* $\mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}] = 0$*. We denote by* $\mathcal{P}^{sym,\perp}_{\mathbf{U}}[\cdot] = \mathbb{I} - \mathcal{P}^{sym}_{\mathbf{U}}[\cdot]$ *the projection onto the symplectic orthogonal complement of* $\mathbf{U}$*.*

The Symplectic Order Reduction method consists of two steps:

- an off-line stage for computing the basis functions $\mathbf{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_{2S})$. They can be extracted by means of Principal Symplectic Decomposition (PSD) procedures from snapshots $\mathbf{u}(\cdot, t_j, \omega_k)$ collected at different times and for different values of the parameters [27], or following a greedy-PSD approach as described in [18].
- an on-line stage which consists in low-cost reduced-order simulations for computing the coefficients $\mathbf{Y} = (Y_1, \ldots, Y_{2S})$ at each time and for different values of the parameters. The reduced order system is obtained by preforming a symplectic Galerkin projection of the governing Hamiltonian equations in the subspace spanned by $\mathbf{U}$.

The use of the symplectic Galerkin projection aims at preserving the symplectic structure of the original problem, in order to ensure the stability of the reduced order system [14]. More precisely, the approximate solution $\mathbf{u}_S = (q_S, p_S)^T$ to problem (20), which is written as:

$$\mathbf{u}_S(\mathbf{x}, t, \omega) = \mathbf{U}(\mathbf{x})\mathbf{Y}(\omega, t),$$

satisfies the following variational principle at each time and for any $\omega \in \Omega$:

$$\big\langle \dot{\mathbf{u}}_S - \mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S, \omega), \mathcal{J}_2^T \mathbf{v} \big\rangle = 0, \quad \forall \mathbf{v} \in \mathcal{U}^{sym}, \tag{26}$$

where $\mathcal{U}^{sym}$ is the subspace spanned by $\mathbf{U} \in Sp(2S, H^1(D) \times L^2(D))$. This can be written formally as a symplectic projection of the governing equation (20) into $\mathcal{U}^{sym}$:

$$\dot{\mathbf{u}}_S(t) = \mathcal{P}^{sym}_{\mathbf{U}}[\mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S(t), \omega)], \qquad \forall t,\, \omega \in (0, T] \times \Omega$$

where the definition of $\mathcal{P}^{sym}_{\mathbf{U}}[\cdot]$ is properly extended to all $\mathbf{v} \in \big(L^2(D) \times H^{-1}(D)\big) \otimes L^2(\Omega)$ as $\mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}] = \sum_{i=1}^{2S} \mathbf{U}_i \langle \mathbf{v}, \mathbf{U}_i^+ \rangle$ and $\langle \cdot, \cdot \rangle$ denoting the H_0^1-H^{-1} duality pair. Moreover, let us define the following composite function:

$$\begin{aligned} \tilde{H}_\omega := H_\omega \circ \phi_{\mathbf{U}} : \quad [L^2(\Omega)]^{2S} &\to L^2(\Omega) \\ \mathbf{Y} &\to H_\omega\Big(\sum_{i=1}^{2S} \mathbf{U}_i Y_i, \omega\Big). \end{aligned}$$

Then, if we write the solution component-wise, the position and momentum are respectively approximated as:

$$q(\mathbf{x},t,\omega) \approx q_S(\mathbf{x},t,\omega) = \sum_{i=1}^{2S} U_i^q(\mathbf{x}) Y_i(\omega,t), \qquad p(\mathbf{x},t,\omega) \approx p_S(\mathbf{x},t,\omega) = \sum_{i=1}^{2S} U_i^p(\mathbf{x}) Y_i(\omega,t),$$

where the stochastic coefficients $\mathbf{Y} = (Y_1, ..., Y_{2S})$, which belong to $[L^2(\Omega)]^{2S}$, satisfy the following system of ordinary differential equations (ODEs):

$$\begin{aligned} \dot{\mathbf{Y}}(\omega) &= \langle \mathcal{P}_{\mathbf{U}}^{sym}[\mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S,\omega)], \mathbf{U}^+ \rangle \\ &= \langle \mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S,\omega), \mathbf{U}^+ \rangle \\ &= \langle \nabla H_\omega(\mathbf{u}_S,\omega), \mathbf{U}\mathbf{J}_{2S}^{\mathrm{T}} \rangle = \mathbf{J}_{2S} \nabla_{\mathbf{Y}} \tilde{H}_\omega(\mathbf{Y},\omega) \end{aligned} \tag{27}$$

with initial conditions $Y_i(0) = \langle (q_0,p_0)^T, \mathbf{U}_i^+ \rangle$ for all $i = 1, ..., 2S$, obtained by performing a symplectic projection of the initial datum (19) on $\mathbf{U}$.

Remark 1. *The reduced system (27) consists of Hamiltonian equations in the symplectic Hilbert space $[L^2(\Omega)]^{2S}$ equipped with the canonical form: $\mathbb{E}[\mathbf{Y}^T \mathbf{J}_{2S} \mathbf{Z}]$, $\forall \mathbf{Y}, \mathbf{Z} \in [L^2(\Omega)]^{2S}$.*

Lemma 5.1 (from [27])**.** *Let $\mathbf{U}$ belong to $Sp(2S, H^1(D) \times L^2(D))$ and $\phi_{\mathbf{U}}$ be the linear map associated to $\mathbf{U}$, defined as:*

$$\begin{aligned} \phi_{\mathbf{U}} \quad &: [L^2(\Omega)]^{2S} &\to \quad & [H^1(D) \times L^2(D)] \otimes L^2(\Omega) \\ &\quad \mathbf{Y} &\mapsto \quad & \phi_{\mathbf{U}}(\mathbf{Y}) := \mathbf{U}\mathbf{Y}. \end{aligned}$$

Then $\phi_{\mathbf{U}}$ is a symplectic linear map between $\big([L^2(\Omega)]^{2S}, \mathbf{J}_{2S}\big)$ and $\big([H^1(D) \times L^2(D)] \otimes L^2(\Omega), \mathcal{J}_2\big)$, i.e. $\phi_{\mathbf{U}}$ preserves the symplectic form:

$$\mathbb{E}[\mathbf{Y}^T \mathbf{J}_{2S} \mathbf{Z}] = \mathbb{E}[\langle \mathbf{U}\mathbf{Z}, \mathcal{J}_2 \mathbf{U}\mathbf{Y} \rangle]$$

for any $\mathbf{Y}, \mathbf{Z} \in [L^2(\Omega)]^{2S}$. Moreover, the function $\tilde{H}_\omega$, defined in (27), is a first integral, pointwise in ω, of $\mathbf{Y}(t)$. This means that the flow of (27) preserves the energy of $\tilde{H}_\omega$ at each time and for each ω.

In conclusion, the original problem (19), set in $(H^1(D) \times L^2(D)) \otimes L^2(\Omega)$, is reduced to a Hamiltonian ODE system of dimension $2S$, set in $[L^2(\Omega)]^{2S}$, describing the evolution of the random coefficients $Y_1, ..., Y_{2S}$. To verify that $\tilde{H}_\omega$ is conserved by the solution of (27), note that $\frac{d}{dt}\tilde{H}_\omega(\mathbf{Y}(t)) = \sum_{i=1}^{2S} \nabla_{Y_i(t)} \tilde{H}_\omega(\mathbf{Y}(t)) \cdot \dot{Y}_i(t) = (\nabla_{\mathbf{Y}} \tilde{H}_\omega(\mathbf{Y}(t)))^T \mathbf{J}_{2S} \nabla_{\mathbf{Y}} \tilde{H}_\omega(\mathbf{Y}(t)) = 0$ a.s. in Ω. The energy of the approximate solution, that is $\tilde{H}_\omega(\mathbf{Y}(t)) = H_\omega(\mathbf{U}\mathbf{Y}(t)) = H_\omega(\mathbf{u}_S(t))$, is not necessary equal to the exact one, namely the energy of the exact solution $H_\omega(\mathbf{u}(t))$, but the discrepancy between the exact and the approximate energy remains constant in time and can be evaluated at initial time. The drawback of the Symplectic Reduced Order approach with a fixed basis $\mathbf{U}$, is that if the solution manifolds $\mathcal{M}(t) = \{u(\cdot,t,\omega),\ \omega \in \Omega\}$ significantly change during the time evolution, as it typically happens in wave propagation phenomena, the *fixed* reduced basis $\mathbf{U} = (\mathbf{U}_1, \dots, \mathbf{U}_{2S})$ has to be sufficiently rich to be able to approximate such manifolds for all $t \in [0,T]$. This leads to a fairly large reduced model thus compromising its efficiency.

6 Symplectic Dynamical Low Rank approximation

In this paper we propose the Symplectic Dynamical Low Rank (Symplectic DO) approximation for wave equations with random parameters which combines the Dynamically Orthogonal approach described in Section 2 with the Symplectic Order Reduction strategy summarized in Section 5. This method shares with the symplectic order reduction the use of a symplectic deterministic basis, and, as the “classic” DO approximation, allows both the stochastic and the deterministic modes to evolve in time. This aims to both preserve the Hamiltonian structure of the original problem and guarantee more flexibility to the approximation. The approximate solution, indeed, preserves the (approximated) mean Hamiltonian energy and continuously adapts in time to the structure of the solution. The reduced dynamical system consists of a set of equations for the constrained dynamics of the deterministic modes in a submanifold of $Sp(2S, H^1(D) \times L^2(\Omega))$, coupled with a reduced order Hamiltonian system for the evolution of the stochastic coefficients.

Definition 6.1. *We denote $U(S, [H^1(D)]^2)$ the submanifold of $Sp(2S, H^1(D) \times L^2(D))$ consisting of all L^2-orthonormal symplectic bases in $[H^1(D)]^2$, i.e.:*

$$U(S, [H^1(D)]^2) := \big\{\mathbf{U} = \quad (\mathbf{U}_1, ..., \mathbf{U}_{2S}) \in [H^1(D) \times H^1(D)]^{2S} \text{ such that } \vartheta_D(\mathbf{U}_i, \mathbf{U}_j) = (\mathbf{J}_{2S})_{ij} \text{ and } \langle \mathbf{U}_j, \mathbf{U}_i \rangle_{L^2(D)} = \delta_{ij},\ \forall i, j = 1, ..., 2S\big\},$$

with ϑ_D defined in (15).

The advantage in restricting $Sp(2S, H^1(D) \times L^2(D))$ to $U(S, [H^1(D)]^2)$ is the possibility to identify the latter with the Stiefel manifold $St(S, H^1(D, \mathbb{C}))$ of all S-dimensional orthonormal complex bases in $H^1(D, \mathbb{C})$ (while the same clearly does not applies to $Sp(2S, H^1(D) \times L^2(D))$). We postpone this discussion to Section 6.1, and we go forward here with the construction of the approximation manifold.

Proposition 6.1. *The following properties hold for any $\mathbf{U} \in U(S, [H^1(D)]^2)$:*

a) let $\mathbf{U} \in Sp(2S, H^1(D) \times L^2(D))$, then $\mathbf{U} \in U(S, [H^1(D)]^2)$ if and only if:

$$\mathbf{U}^+ = \mathcal{J}_2^T \mathbf{U} \mathbf{J}_{2S} = \mathcal{J}_2 \mathbf{U} \mathbf{J}_{2S}^T = \mathbf{U}; \tag{28}$$

b) $\mathbf{U} \in U(S, [H^1(D)]^2)$ if and only if:

$$\mathbf{U} = \begin{pmatrix} \mathbf{Q} & -\mathbf{P} \\ \mathbf{P} & \mathbf{Q} \end{pmatrix} \tag{29}$$

with $\mathbf{Q}, \mathbf{P} \in [H^1(D)]^S$ row vector functions such that:

$$\langle P_i, Q_j \rangle = \langle Q_i, P_j \rangle \quad \text{and} \quad \langle Q_i, Q_j \rangle + \langle P_i, P_j \rangle = \delta_{ji}, \tag{30}$$

for all $i, j = 1, ..., S$.

Proof. Here we use the notation $\ll \mathbf{U}, \mathbf{V} \gg$ to denote the $2S \times 2S$ matrix with entries $\ll \mathbf{U}, \mathbf{V} \gg_{ij} = \langle \mathbf{U}_j, \mathbf{V}_i \rangle$, for all $\mathbf{U}, \mathbf{V} \in [H^1(D) \times H^1(D)]^{2S}$ (Analogous definition for $\mathbf{U}, \mathbf{V} \in [H^1(D)]^S$).

a) If (28) holds then $\ll \mathbf{U}, \mathbf{U} \gg = \ll \mathcal{J}_2^T \mathbf{U} \mathbf{J}_{2S}, \mathbf{U} \gg = \ll \mathbf{U}^+, \mathbf{U} \gg = \mathbb{I}_{2S}$ implies $\mathbf{U} \in U(S, [H^1(D)]^2)$. Conversely, if $\mathbf{U} \in U(S, [H^1(D)]^2)$, then $\mathbf{U}$ is an orthonormal basis and $\mathcal{P}_{\mathbf{U}}[\mathbf{U}^+] = \mathbf{U}$. This implies that $\mathbf{U}^+$ can be written as $\mathbf{U}^+ = \mathbf{U} + \mathbf{B}$ with $\langle \mathbf{B}_i, \mathbf{U}_j \rangle = 0$ for all $i, j = 1, ..., 2S$. By observing that $\langle \mathbf{U}_j^+, \mathbf{U}_i^+ \rangle = \langle (\mathbf{U}\mathbf{J}_{2S})_j, (\mathbf{U}\mathbf{J}_{2S})_i \rangle = \delta_{ij}$ for all $i, j = 1, ..., 2S$, we necessarily have that $\mathbf{B} = \mathbf{0}$ which implies $\mathbf{U}^+ = \mathbf{U}$.

b) If $\mathbf{U} \in U(S, [H^1(D)]^2)$ then $\ll \mathbf{U}, \mathcal{J}_2 \mathbf{U} \gg = \mathbf{J}_{2S}$. Block-wise, this is written as:

$$\mathbf{U} = \begin{bmatrix} \mathbf{Q}_{I,S} & \mathbf{Q}_{II,S} \\ \mathbf{P}_{I,S} & \mathbf{P}_{II,S} \end{bmatrix} \tag{31}$$

with $\mathbf{Q}_{I,S}, \mathbf{Q}_{II,S}, \mathbf{P}_{I,S}, \mathbf{P}_{II,S} \in [H^1(D)]^S$ such that:

$$\begin{aligned} &\ll \mathbf{P}_{II,S}, \mathbf{Q}_{I,S} \gg - \ll \mathbf{Q}_{II,S}, \mathbf{P}_{I,S} \gg = \mathbb{I}_S, \\ \ll \mathbf{P}_{I,S}, \mathbf{Q}_{I,S} \gg = \ll \mathbf{Q}_{I,S}, \mathbf{P}_{I,S} \gg \quad &\text{and} \quad \ll \mathbf{P}_{II,S}, \mathbf{Q}_{II,S} \gg = \ll \mathbf{Q}_{II,S}, \mathbf{P}_{II,S} \gg \end{aligned} \tag{32}$$

From a) we have that $\mathbf{U}^+ = \mathbf{U}$, i.e.:

$$\mathbf{U}^+ = \begin{bmatrix} \mathbf{P}_{II,S} & -\mathbf{P}_{I,S} \\ -\mathbf{Q}_{II,S} & \mathbf{Q}_{I,S} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{I,S} & \mathbf{Q}_{II,S} \\ \mathbf{P}_{I,S} & \mathbf{P}_{II,S} \end{bmatrix} = \mathbf{U}$$

which implies $\mathbf{Q} := \mathbf{Q}_{I,S} = \mathbf{P}_{II,S}$ and $\mathbf{P} := \mathbf{P}_{I,S} = -\mathbf{Q}_{II,S}$. The proof is concluded by combining the last relation with (32). The other implication is obvious.

■

From Proposition 6.1 it follows that the symplectic projection coincides with the standard projection:

Proposition 6.2. *For any* $\mathbf{v} = (v^q, v^p)^{\mathrm{T}} \in \left(H^1(D) \times L^2(D)\right) \otimes L^2(\Omega)$ *and* $\mathbf{U} \in U(S, [H^1(D)]^2)$ *it holds that:*

$$\mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}] = \mathcal{P}_{\mathbf{U}}[\mathbf{v}] \tag{33}$$

Additionally the following properties hold:

Proposition 6.3. *Let* $\mathbf{v}$ *be a square integrable random field* $\mathbf{v} = (v^q, v^p)^{\mathrm{T}} \in (H^1(D) \times L^2(D)) \otimes L^2(\Omega)$*. For any* $\mathbf{U} \in U(S, [H^1(D)]^2)$ *we have that:*

$$\mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}] = \mathcal{P}_{\mathbf{U}}[\mathbf{v}] = \mathcal{P}_{\mathcal{J}_2\mathbf{U}}[\mathbf{v}] = \mathcal{P}^{sym}_{\mathcal{J}_2\mathbf{U}}[\mathbf{v}]; \tag{34}$$

where $\mathcal{P}_{\mathbf{U}}$, $\mathcal{P}_{\mathcal{J}_2\mathbf{U}}$ (respectively $\mathcal{P}^{sym}_{\mathbf{U}}$, $\mathcal{P}^{sym}_{\mathcal{J}_2\mathbf{U}}$) are the standard (respectively symplectic) projections in the subspace spanned by $\mathbf{U}$ and $\mathcal{J}_\mathbf{2}\mathbf{U}$ respectively (which coincide in this case).

Proposition 6.4. *Let* $\mathbf{v}$ *be a square integrable random field* $\mathbf{v} = (v^q, v^p)^{\mathrm{T}} \in (H^1(D) \times L^2(D)) \otimes L^2(\Omega)$*. For any* $\mathbf{U} \in U(S, [H^1(D)]^2)$ *we have that:*

$$\mathcal{J}_2\mathcal{P}^{sym}_{\mathbf{U}}[\mathbf{v}] = \mathcal{J}_2\mathcal{P}_{\mathbf{U}}[\mathbf{v}] = \mathcal{P}_{\mathbf{U}}[\mathcal{J}_2\mathbf{v}] = \mathcal{P}^{sym}_{\mathbf{U}}[\mathcal{J}_2\mathbf{v}]. \tag{35}$$

The same property is satisfied by the projector into the symplectic -orthogonal complement of $\mathbf{U}$*:*

$$\mathcal{J}_2\mathcal{P}^{sym,\perp}_{\mathbf{U}}[\mathbf{v}] = \mathcal{J}_2\mathcal{P}^{\perp}_{\mathbf{U}}[\mathbf{v}] = \mathcal{P}^{\perp}_{\mathbf{U}}[\mathcal{J}_2\mathbf{v}] = \mathcal{P}^{sym,\perp}_{\mathbf{U}}[\mathcal{J}_2\mathbf{v}].$$

The Symplectic DO approximate solution of problem (19) is sought in the approximation manifold defined as follows:

Definition 6.2. *We call symplectic manifold of rank* S*, denoted by* $\mathcal{M}^{sym}_S$*, the collection of all random fields* $\mathbf{u}_S = (q_S, p_S)^T \in [H^1(D)]^2 \otimes L^2(\Omega)$ *that can be written as:* $\mathbf{u}_S = \mathbf{U}\mathbf{Y}$ *where*

- $\mathbf{U} \in U(S, [H^1(D)]^2)$*,*
- $\mathbf{Y} = Y_1, ..., Y_{2S}$ *is a* $2S$ *dimensional vector of square integrable random variables* $Y_i \in L^2(\Omega)$*, such that rank*$(\mathbb{E}[\mathbf{Y}\mathbf{Y}^T] + \mathbf{J}^T_{2S}\mathbb{E}[\mathbf{Y}\mathbf{Y}^T]\mathbf{J}_{2S}) = 2S$*.*

We call symplectic S rank random field any function $\mathbf{u}_S \in \mathcal{M}_S^{sym}$. This can be written component-wise as follows:

$$q_S(\mathbf{x},\omega) = \sum_{i=1}^{S} Q_i(\mathbf{x})Y_i(\omega) - \sum_{i=1}^{S} P_i(\mathbf{x})Y_{S+i}(\omega), \quad p_S(\mathbf{x},\omega) = \sum_{i=1}^{S} P_i(\mathbf{x})Y_i(\omega) + \sum_{i=1}^{S} Q_i(\mathbf{x})Y_{S+i}(\omega). \tag{36}$$

In the following we denote by $\mathrm{B}^{sym}(2S, L^2(\Omega)) \subset [L^2(\Omega)]^{2S}$ the set of all $2S$-vectors $\mathbf{Z} = (Z_1, ..., Z_{2S}) \in [L^2(\Omega)]^{2S}$, that satisfy the full rank condition on $\mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}] + \mathbf{J}_{2S}^T\mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}]\mathbf{J}_{2S}$. Observe that (29) implies that the first S components of $\mathbf{U} \in U(S, [H^1(D)]^2)$ characterize the whole vectors $\mathbf{U}$ and $\mathbf{U}^+$, for all $\mathbf{U} \in U(S, [H^1(D)]^2)$ (which motivates the name of symplectic S rank random field instead of $2S$). Hence, for (29) to be verified, the same regularity has to be assumed for both the position and momentum components. This means that, when we look for an approximate solution of problem (19) in $\mathcal{M}_S^{sym}$, we necessary have to assume some extra-regularity on the approximate momentum. In other words, the orthonormality combined to the symplectic condition forces to set the approximation problem in $[H^1(D)]^2 \otimes L^2(\Omega)$ while the more natural setting would be $(H^1(D) \times L^2(D)) \otimes L^2(\Omega)$.

Remark 2. *The representation of $\mathbf{u}_S \in \mathcal{M}_S^{sym}$ in terms of $\mathbf{U} \in U(S, [H^1(D)]^2)$ and $\mathbf{Y} \in \mathrm{B}^{sym}(2S, L^2(\Omega))$ (decomposition (36)) is not unique. Let $\mathbf{u}_S = \mathbf{U}\mathbf{Y} \in \mathcal{M}_S^{sym}$ with $\mathbf{U} \in U(S, [H^1(D)]^2)$, $\mathbf{Y} \in \mathrm{B}^{sym}(2S, L^2(\Omega))$, then for any $\mathbf{B} \in U(S, \mathbb{R}^{2S})$ we have that $\mathbf{W} = \mathbf{U}\mathbf{B} \in U(S, [H^1(D)]^2)$, $\mathbf{Z} = (\mathbf{B}^+)^T\mathbf{Y} = \mathbf{B}^{\mathrm{T}}\mathbf{Y} \in \mathrm{B}^{sym}(2S, L^2(\Omega))$ and $\mathbf{W}\mathbf{Z} = \mathbf{u}_S$. Indeed:*

- *showing that $\mathbf{W} \in U(S, [H^1(D)]^2)$:*

$$\begin{aligned} \langle \mathbf{W}_j, (\mathcal{J}_2\mathbf{W})_i \rangle &= \langle (\mathbf{U}\mathbf{B})_j, (\mathcal{J}_2\mathbf{U}\mathbf{B})_i \rangle \\ &= \mathbf{B}_{kj}\langle \mathbf{U}_k, (\mathcal{J}_2\mathbf{U})_l \rangle \mathbf{B}_{li} \\ &= \mathbf{B}_{jk}^{\mathrm{T}}(\mathbf{J}_{2S})_{kl}\mathbf{B}_{li} = (\mathbf{J}_{2S})_{ij} \end{aligned}$$

$$\begin{aligned} \langle \mathbf{W}_j, \mathbf{W}_i \rangle = &\ \langle \mathbf{U}_l\mathbf{B}_{li}, \mathbf{U}_s\mathbf{B}_{sj} \rangle \\ &= \mathbf{B}_{il}^{\mathrm{T}}\langle \mathbf{U}_l, \mathbf{U}_s \rangle \mathbf{B}_{sj} \ ; \\ &= \mathbf{B}_{il}^{\mathrm{T}}\delta_{ls}\mathbf{B}_{sj} = \delta_{ij} \end{aligned}$$

  Here the Einstein notation is used.

- *showing that $\mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}] + \mathbf{J}_{2S}\mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}]\mathbf{J}_{2S}$ is full rank:*

$$\begin{aligned} \mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}] + \mathbf{J}_{2S}\mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}]\mathbf{J}_{2S} &= \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B} + \mathbf{J}_{2S}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\mathbf{J}_{2S} \\ &= \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{J}_{2S}^{\mathrm{T}}\mathbf{C}\mathbf{J}_{2S}\mathbf{B} \\ &= \mathbf{B}^{\mathrm{T}}\big(\mathbf{C} + \mathbf{J}_{2S}^{\mathrm{T}}\mathbf{C}\mathbf{J}_{2S}\big)\mathbf{B}. \end{aligned}$$

  and $\mathbf{B}$ and $\big(\mathbf{C} + \mathbf{J}_{2S}^{\mathrm{T}}\mathbf{C}\mathbf{J}_{2S}\big)$ are the both full rank.

A necessary condition for $\mathbf{Z}$ to belong to $\mathrm{B}^{sym}(2S, L^2(\Omega))$ is that the second moments matrix $\mathbb{E}[\mathbf{Z}\mathbf{Z}^T]$ has rank at least equal to S. Indeed, since $\mathbf{J}_{2S}$ is a full rank matrix, the rank of $\mathbf{J}_{2S}^T\mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}]\mathbf{J}_{2S}$ is equal to the rank of $\mathbb{E}[\mathbf{Z}\mathbf{Z}^{\mathrm{T}}]$. Then the conclusion is drawn by recalling that the sum of ranks is greater or equal to the rank of the sum (i.e. rank($\mathbf{A}$) + rank($\mathbf{B}$) $\geq$ rank($\mathbf{A} + \mathbf{B}$), $\forall \mathbf{A}, \mathbf{B}$).

Remark 3. *We recall that in the standard DO approximation of parabolic equations (Section 2) with rank $2S$, one assumes that the second moments matrix $\mathbf{C} = \mathbb{E}[\mathbf{Y}\mathbf{Y}^T]$ is full rank (rank($\mathbf{C}$) $= 2S$). Here we need the weaker assumption rank($\mathbf{C} + \mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}$) $= 2S$. The motivation is related to the fact that we work in the phase-space coordinates: we need to uniquely determine the couple (q_S, p_S) and not the position and* *the momentum separately. Namely asking $Y_1, ..., Y_{2S}$ linearly independent is a too strong assumption in our model, as emphasized by the following example:*

$$\begin{cases} \ddot{q}(x,t,\omega) = \Delta q(x,t,\omega) & x \in (0,2\pi),\ t \in (0,T],\ \omega \in \Omega \\ q(0,t,\omega) = q(2\pi,t,\omega) = 0 & t \in (0,T],\ \omega \in \Omega \\ q(x,0,\omega) = q_0(x,\omega) = Z_1(\omega)\frac{1}{\sqrt{\pi}}\sin(x) & x \in [0,2\pi],\ \omega \in \Omega \\ \dot{q}(x,0,\omega) = p_0(x,\omega) = 0 & x \in [0,2\pi],\ \omega \in \Omega \end{cases} \tag{37}$$

Here Z_1 is a square integrable random variable. We start looking for a symplectic decomposition of the initial data (q_0, p_0) in $\mathcal{M}_S^{sym}$. Problem (37) is linear with only one random variable that multiplies the initial datum, which suggests to set $S = 1$. Hence we look for $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2) \in U(1, [H^1(D)]^2)$ and $\mathbf{Y} = (Y_1, Y_2)^T \in \mathrm{B}^{sym}(2, L^2(\Omega))$ such that $\sum_{i=1}^2 \mathbf{U}_i Y_i = (Z_1(\omega)\frac{1}{\sqrt{\pi}}\sin(x), 0)^T$ (observe that, by working with symplectic bases, we can not decrease further the number of modes). The solution can be obtained by setting $Y_1 = Z_1$ and $Y_2 = 0$. Then the deterministic basis $\mathbf{U} \in U(1, [H^1(D)]^2)$ is uniquely determined by:

$$\mathbf{U} = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \sin(x) & 0 \\ 0 & \sin(x) \end{pmatrix}. \tag{38}$$

Conversely, we can not find a symplectic basis $\mathbf{U} \in U(1, [H^1(D)]^2)$, or more generally $\mathbf{U} \in Sp(2, H^1(D) \times L^2(D))$, if we assume that $\mathbb{E}[\mathbf{Y}\mathbf{Y}^T]$ is full rank. Let us write $\mathbf{U} = \begin{bmatrix} Q_{I,1} & Q_{II,1} \\ P_{I,1} & P_{II,1} \end{bmatrix}$; if Y_1, Y_2 are linearly independent, then $P_{I,1}(x)Y_1(\omega) + P_{II,1}(x)Y_2(\omega) = p_0(x,\omega) = 0$ implies $P_{I,1} = P_{II,1} = 0$ and hence $\mathbf{U}^T \mathcal{J}_2 \mathbf{U} \neq \mathbf{J}_2$. On the other hand we have seen that the symplectic decomposition of (q_0, p_0) in $\mathcal{M}_1^{sym}$ is well defined when the assumption of linear independence of Y_1, Y_2 is relaxed to $\mathbf{C} + \mathbf{J}_2^T \mathbf{C} \mathbf{J}_2$ full rank. This consideration generally applies to the solution of (37) at any time. The solution, given by $\mathbf{u}_1(t) = (q(t), \dot{q}(t)) = (Z_1(\omega)\cos(t)\frac{1}{\sqrt{\pi}}\sin(x), -Z_1(\omega)\sin(t)\frac{1}{\sqrt{\pi}}\sin(x)) \in \mathcal{M}_1^{sym}$, is characterized by a covariance matrix $\mathbf{C}(t) = \mathbb{E}[\mathbf{Y}(t)\mathbf{Y}^{\mathrm{T}}(t)]$ with $Y_1(t) = Z_1\cos(t)$ and $Y_2(t) = -Z_1\sin(t)$ of defective rank while $\mathbf{C}(t) + \mathbf{J}_{2S}^T \mathbf{C}(t)\mathbf{J}_{2S}$ is full rank at any time. (We will see in the following that for this particular case the symplectic DO approximation degenerates to the Symplectic Proper Decomposition described in Section 5: the deterministic basis does not evolve, the coefficients evolve according to (27) and the approximation (with $S = 1$) is exact).

We emphasize that the full rank condition for $\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}$ guarantees the uniqueness of the representation on $\mathbf{U}$ once $\mathbf{Y}$ is fixed. Namely, let $\mathbf{u}_S$ be in $\mathcal{M}_S^{sym}$; if $\mathbf{u}_S = \mathbf{U}\mathbf{Y} = \mathbf{W}\mathbf{Y}$ with $\mathbf{U}, \mathbf{W} \in U(S, [H^1(D)]^2)$ and $\mathbf{Y} \in \mathrm{B}^{sym}(2S, L^2(\Omega))$, then necessarily $\mathbf{U} = \mathbf{W}$. Indeed:

$$\begin{aligned} \mathbf{0} &= (\mathbf{U} - \mathbf{W})\mathbf{Y} && \Rightarrow (\mathbf{U} - \mathbf{W})\mathbf{C} = \mathbf{0} \\ &= \mathcal{J}_2(\mathbf{U} - \mathbf{W})\mathbf{J}_{2S}^T \mathbf{Y} && \Rightarrow (\mathbf{U} - \mathbf{W})\mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S} = \mathbf{0} \end{aligned} \tag{39}$$

By summing the two equations on the right, we get $(\mathbf{U} - \mathbf{W})(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}) = \mathbf{0}$, which implies $\mathbf{U} = \mathbf{W}$ thanks to the full rank condition on $\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}$. The same result does not apply if we extend the submanifold $U(S, [H^1(D)]^2)$ to the whole $Sp(2S, H^1(D) \times L^2(D))$. Consider for instance the random field $\mathbf{u}_1 = (q, p) = (Z(\omega)\frac{1}{2\sqrt{\pi}}\sin(x), Z(\omega)\frac{1}{\sqrt{\pi}}\sin(2x))$ with $x \in [0, 2\pi]$ and $Z \in L^2(\Omega)$. This can be represented, for instance, as:

$$\mathbf{U} = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{2}\sin(x) & -\sin(2x) \\ \sin(2x) & 0 \end{pmatrix} \qquad \mathbf{Y} = \begin{pmatrix} Z \\ 0 \end{pmatrix} \tag{40}$$

or equivalently as:

$$\mathbf{W} = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{2}\sin(x) & 0 \\ \sin(2x) & 2\sin(x) \end{pmatrix} \qquad \mathbf{Y} = \begin{pmatrix} Z \\ 0 \end{pmatrix} \tag{41}$$

where $\mathbf{U}, \mathbf{W} \in Sp(2, H^1(D) \times L^2(D))$ and $\mathbf{Y} \in \mathrm{B}^{sym}(2, L^2(\Omega))$. This implies that, if we replace $U(1, [H^1(D)]^2)$ with $Sp(2, H^1(D) \times L^2(D))$ in Definition 6.2, what we get is not a manifold anymore. Indeed, to get a manifold we need that the decomposition of $\mathbf{u}_S \in \mathcal{M}_S$, even though it is not unique in terms of $\mathbf{U}, \mathbf{Y}$, is uniquely characterized when one of the two bases is fixed. This implies that a stronger condition on $\mathbf{C}$ should be required when $U(1, [H^1(D)]^2)$ is extended to $Sp(2, H^1(D) \times L^2(D))$.

6.1 Parametrization of the tangent space by means of complex representation

In this section we discus how to equip $\mathcal{M}_S^{sym}$ with a differential manifold structure and parametrize the tangent space. This is achieved by identifying $\mathcal{M}_S^{sym}$, i.e. the manifold of all real valued symplectic random fields of rank S, with the manifold of all complex valued functions of rank S. To do so, let us introduce the complex variable $\hat{v} = q + ip$ and its complex conjugate $\hat{v}^* = q - ip$. The Hamiltonian system (20), written in terms of the new variables $(\hat{v}, \hat{v}^*)$, becomes:

$$\begin{aligned} i\dot{\hat{v}} &= 2\partial_{\hat{v}^*} H(\hat{v}, \hat{v}^*, \omega) \\ i\dot{\hat{v}}^* &= -2\partial_{\hat{v}} H(\hat{v}, \hat{v}^*, \omega) \end{aligned} \tag{42}$$

Observe that the second equation can be obtained from the first one by complex conjugation, thus it is redundant. The Hamiltonian function in (42) is now expressed with respect to the new complex variables $\hat{v}$ and $\hat{v}^*$ and satisfies the reality condition:

$$H(\hat{v}, \hat{v}^*, \omega) = \big(H(\hat{v}, \hat{v}^* \omega)\big)^* =: H^*(\hat{v}^*, \hat{v}, \omega)$$

where with the symbol * we always denote the complex conjugate. We emphasize that the solution of (42), which is completely characterized by solving only one of the two equations in (42), is a complex valued function $\hat{v} : \bar{D} \times [0, T] \times \Omega \to \mathbb{C}$, whose real and imaginary parts correspond respectively to the position and momentum in system (20). In what follows, complex functions will be written as $\hat{v} = v^q + iv^p$, according to which the apex q and p will denote respectively the real and the imaginary part.

Definition 6.3. *We call complex S rank random field any function $\hat{u}_S \in H^1(D, \mathbb{C}) \otimes L^2(\Omega, \mathbb{C})$ which can be exactly expressed as:*

$$\hat{u}_S(\mathbf{x}, \omega) = \sum_{i=1}^{S} \hat{Y}_i(\omega) \hat{U}_i(\mathbf{x}) = \sum_{i=1}^{S} \big(Y_i^q(\omega) + iY_i^p(\omega)\big)(U_i^q(\mathbf{x}) + iU_i^p(\mathbf{x})) \tag{43}$$

with:

- $\hat{\mathbf{U}} = (\hat{U}_1, ..., \hat{U}_S) \in St(S, H^1(D, \mathbb{C}))$,
- $\hat{\mathbf{Y}} = \hat{Y}_1, ..., \hat{Y}_S \in \mathrm{B}(S, L^2(\Omega, \mathbb{C}))$.

Definition 6.4. *We define complex manifold of dimension S the collection of all complex S rank random fields:*

$$\begin{aligned} \mathcal{M}_S^{\mathbb{C}} &= \big\{\hat{u}_S = \sum_{i=1}^{S} \hat{U}_i \hat{Y}_i \,|\, \textit{span}(\hat{U}_1, ..., \hat{U}_S) \in \mathcal{G}(S, H^1(D, \mathbb{C})),\ \textit{span}(\hat{Y}_1, ..., \hat{Y}_S) \in \mathcal{G}(S, L^2(\Omega, \mathbb{C}))\big\} \\ &= \big\{\hat{u}_S = \hat{\mathbf{U}}\hat{\mathbf{Y}},\ \hat{\mathbf{U}} \in St(S, H^1(D, \mathbb{C})),\ \hat{\mathbf{Y}} = (\hat{Y}_1, .., \hat{Y}_S) \in \mathrm{B}(S, L^2(\Omega, \mathbb{C}))\big\} \end{aligned}$$

Observe that $\mathcal{M}_S^{\mathbb{C}}$ is the complex version of the manifold $\mathcal{M}_S$, introduced in Section 2 to describe the DO approximation of real parabolic equations. Hence, $\mathcal{M}_S^{\mathbb{C}}$, as well as $\mathcal{M}_S$, can be equipped with a differential manifold structure by means of the same standard tools of differential geometry, recalled in Section 2. Complex manifolds of fixed rank have been already used in literature e.g. for the approximation of deterministic Schrödinger equations, see [7,13]. Let us define the following map:

$$\begin{aligned} \pi : (St(S, H^1(D,\mathbb{C})), \mathrm{B}(S, L^2(\Omega,\mathbb{C})) &\to \mathcal{M}_S^{\mathbb{C}} \\ (\hat{\mathbf{U}}, \hat{\mathbf{Y}}) &\to \sum_{i=1}^{S} \hat{U}_i \hat{Y}_i = \hat{u}_S \end{aligned}$$

This map is surjective, i.e. $\mathcal{M}_S^{\mathbb{C}}$ is the image of $(St(S, H^1(D,\mathbb{C})), \mathrm{B}(S, L^2(\Omega,\mathbb{C}))$ by π, but clearly non injective. The triple $\big(St(S, H^1(D,\mathbb{C})) \times \mathrm{B}(S, L^2(\Omega,\mathbb{C})), \mathcal{M}_S^{\mathbb{C}}, \pi\big)$ defines a fiber bundle with fibers given by the group of the unitary matrices $U(S,\mathbb{C}^S) = \{\hat{\mathbf{W}} \in \mathbb{C}^{S\times S} : \hat{\mathbf{W}}^*\hat{\mathbf{W}} = \hat{\mathbf{W}}\hat{\mathbf{W}}^* = \mathbb{I}\}$ and $\mathcal{M}_S^{\mathbb{C}}$ is isomorphic to the quotient space $(St(S, H^1(D,\mathbb{C})/U(S,\mathbb{C}^S)) \times \mathrm{B}(S, L^2(\Omega,\mathbb{C}))$. The uniqueness of the representation of $\hat{u}_S \in \mathcal{M}_S^{\mathbb{C}}$ in terms of bases $(\hat{\mathbf{U}}, \hat{\mathbf{Y}}) \in (St(S, H^1(D,\mathbb{C})), \mathrm{B}(S, L^2(\Omega,\mathbb{C}))$ is recovered in terms of unique decomposition in the tangent space, by imposing the following Gauge constraints [8,21]:

$$\langle \delta\hat{U}_i, \hat{U}_j\rangle_h = \langle \delta U_i^q, U_j^q\rangle + \langle \delta U_i^p, U_j^p\rangle + i(\langle \delta U_i^p, U_j^q\rangle - \langle \delta U_i^q, U_j^p\rangle) = 0, \quad \forall i,j = 1,...,S \tag{44}$$

for any $\delta\hat{\mathbf{U}} = (\delta\hat{U}_1, ..., \delta\hat{U}_S) \in \mathcal{T}_{\hat{u}_S}\mathcal{M}_S^{\mathbb{C}}$ and $\hat{\mathbf{U}} = (\hat{U}_1, ..., \hat{U}_S) \in \mathcal{M}_S^{\mathbb{C}}$. This leads to the following parametrization of the tangent space to $\mathcal{M}_S^{\mathbb{C}}$ at $\hat{u}_S = \sum_{i=1}^{S} \hat{U}_i\hat{Y}_i$:

$$\begin{aligned} \mathcal{T}_{\hat{u}_S}\mathcal{M}_S^{\mathbb{C}} \quad = \Big\{ \hat{\delta v} = \sum_{i=1}^{S} \big(\delta\hat{U}_i\hat{Y}_i + \hat{U}_i\delta\hat{Y}_i\big) \quad &\text{with } \delta\hat{Y}_i \in L^2(\Omega,\mathbb{C}) \text{ and } \delta\hat{U}_i \in H^1(D,\mathbb{C}), \\ &\text{s.t. } \langle \delta\hat{U}_i, \hat{U}_j\rangle_h = 0,\ \forall i,j = 1,...,S \Big\} \end{aligned} \tag{45}$$

Remark 4. *$\mathcal{T}_{\hat{u}_S}\mathcal{M}_S^{\mathbb{C}}$ is a complex linear space, hence $\delta\hat{v}$ belongs to $\mathcal{T}_{\hat{u}_S}\mathcal{M}_S^{\mathbb{C}}$ if and only if $i\delta\hat{v}$ belongs to $\mathcal{T}_{\hat{u}_S}\mathcal{M}_S^{\mathbb{C}}$.*

The complex Hilbert space $H^1(D,\mathbb{C}) \otimes L^2(\Omega,\mathbb{C})$, equipped with the usual hermitian L^2 product, can be identified with the real space $[H^1(D,\mathbb{R}) \otimes L^2(\Omega,\mathbb{R})]^2$, equipped with the complex structure associated to $\mathcal{J}_2$. Namely the following map is bijective

$$\begin{aligned} H^1(D,\mathbb{C}) \otimes L^2(\Omega,\mathbb{C}) &\to [H^1(D,\mathbb{R}) \otimes L^2(\Omega,\mathbb{R})]^2 \\ \hat{u} = u^q + iu^p &\to (u^q, u^p)^T =: \mathbf{u} \end{aligned}$$

and for all $\hat{u}, \hat{v} \in H^1(D,\mathbb{C}) \otimes L^2(\Omega,\mathbb{C})$ we have:

$$\mathbb{E}[\langle \hat{u}, \hat{v}\rangle_h] = \mathbb{E}[\langle \mathbf{u}, \mathbf{v}\rangle] - i\mathbb{E}[\langle \mathbf{u}, \mathcal{J}_2\mathbf{v}\rangle], \tag{46}$$

where $\langle\cdot,\cdot\rangle$ is the standard L^2 product in the real space. Observe that the imaginary part of the Hermitian product (46) coincides with the canonical symplectic form of $[H^1(D,\mathbb{R}) \otimes L^2(\Omega,\mathbb{R})]^2$ defined in (21) with changed sign:

$$\mathrm{Im}(\mathbb{E}[\langle \hat{u}, \hat{v}\rangle_h]) = -\vartheta(\mathbf{u}, \mathbf{v}) = -\mathbb{E}[\langle \mathbf{u}, \mathcal{J}_2\mathbf{v}\rangle] \tag{47}$$

Similarly $[L^2(\Omega,\mathbb{C})]^S$ can be identified with $[L^2(\Omega,\mathbb{R})]^{2S}$, i.e:

$$\begin{aligned} [L^2(\Omega,\mathbb{C})]^S &\to [L^2(\Omega,\mathbb{R})]^{2S} \\ \hat{\mathbf{Z}} = (Z_1^q + iZ_1^p, ..., Z_S^q + iZ_S^p) &\to (Z_1^q, ..., Z_S^q, Z_1^p, ..., Z_S^p)^T =: (\mathbf{Z}^q, \mathbf{Z}^p)^T = \mathbf{Z} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[\hat{\mathbf{Z}}^*\hat{\mathbf{Y}}] &= \mathbb{E}[\mathbf{Y}^T\mathbf{Z}] - i\mathbb{E}[\mathbf{Y}^T\mathbf{J}_{2S}\mathbf{Z}] \\ &= \mathbb{E}[\mathbf{Y^q}^T\mathbf{Z^q}] + \mathbb{E}[\mathbf{Y^p}^T\mathbf{Z^p}] + i(\mathbb{E}[\mathbf{Y^p}^T\mathbf{Z^q}] - \mathbb{E}[\mathbf{Y^q}^T\mathbf{Z^p}]). \end{aligned}$$

Let $GL\big([L^2(\Omega,\mathbb{C})]^S, H^1(D,\mathbb{C})\otimes L^2(\Omega,\mathbb{C})\big)$ be the set of all bounded linear maps from $[L^2(\Omega,\mathbb{C})]^S$ to $H^1(D,\mathbb{C})\otimes L^2(\Omega,\mathbb{C})$. We denote by $\hat{\phi}_{\hat{\mathbf{A}}}$ the map of $GL\big([L^2(\Omega,\mathbb{C})]^S, H^1(D,\mathbb{C})\otimes L^2(\Omega,\mathbb{C})\big)$ which can be represented as:

$$\begin{aligned} \hat{\phi}_{\hat{\mathbf{A}}}: \quad & [L^2(\Omega,\mathbb{C})]^S && \to H^1(D,\mathbb{C})\otimes L^2(\Omega,\mathbb{C}) \\ & \hat{\mathbf{Z}} && \mapsto \hat{\phi}_{\hat{\mathbf{A}}}(\hat{\mathbf{Z}}) = \hat{\mathbf{A}}\hat{\mathbf{Z}} \end{aligned} \tag{48}$$

for any (row) vector of deterministic complex functions $\hat{\mathbf{A}} \in [H^1(D,\mathbb{C})]^S$. Let $[L^2(\Omega,\mathbb{C})]^S$ and $H^1(D,\mathbb{C})\otimes L^2(\Omega,\mathbb{C})$ be identified with $[L^2(\Omega,\mathbb{R})]^{2S}$ and $[H^1(D,\mathbb{R})\otimes L^2(\Omega,\mathbb{R})]^2$ respectively, and let ϕ be the function $\hat{\phi}_{\hat{\mathbf{A}}}$ in real setting. Then, ϕ must satisfies:

$$\begin{aligned} \phi: \quad & [L^2(\Omega,\mathbb{R})]^{2S} && \to [H^1(D,\mathbb{R})\otimes L^2(\Omega,\mathbb{R})]^2 \\ & \mathbf{Z} && \mapsto \phi(\mathbf{Z}) = \mathbf{u} \iff \hat{u} = \hat{\mathbf{A}}\hat{\mathbf{Z}} \end{aligned} \tag{49}$$

where $\mathbf{Z}$ and $\mathbf{u}$ are the realification of $\hat{\mathbf{Z}}$ and $\hat{\phi}(\hat{\mathbf{Z}})$ respectively. The map ϕ is linear and can be written in terms of a matrix of functions $\mathbf{A} \in [H^1(D,\mathbb{R})\times H^1(D,\mathbb{R})]^{2S}$ such that $\phi(\mathbf{Z}) = \mathbf{A}\mathbf{Z} \iff \hat{u} = \hat{\mathbf{A}}\hat{\mathbf{Z}}$. Precisely for any $\hat{\mathbf{A}} = (\mathbf{A}^q + i\mathbf{A}^p) \in [H^1(D,\mathbb{C})]^S$, the map $\hat{\phi}_{\hat{\mathbf{A}}}$ is identified in real setting with $\phi_{\mathbf{A}} : [L^2(\Omega,\mathbb{R})]^{2S} \to [H^1(D,\mathbb{R})\otimes L^2(\Omega,\mathbb{R})]^2$ where $\mathbf{A}$ is given:

$$\mathbf{A} := \begin{pmatrix} \mathbf{A}^q & -\mathbf{A}^p \\ \mathbf{A}^p & \mathbf{A}^q \end{pmatrix} \tag{50}$$

The proof is an exercise of linear algebra [29, 30]. We say that $\mathbf{A}$ is the real matrix representation of $\hat{\mathbf{A}}$ and write $\hat{\mathbf{A}} \sim \mathbf{A}$. This motivates the real identification of row-vector complex functions which will be used in the following. Observe that in this setting the complex conjugate simply corresponds to the transpose: $\hat{\mathbf{A}}^* \sim \mathbf{A}^T$ and the hermitian product $\langle A_i^q, B_j^q\rangle + \langle A_i^p, B_j^p\rangle + i(\langle A_i^p, B_j^q\rangle - \langle A_i^q, B_j^p\rangle)$ can be computed by matrix multiplication as:

$$\begin{aligned} \langle \hat{A}_i, \hat{B}_j\rangle_h &\sim \langle \begin{pmatrix} A_i^q & -A_i^p \\ A_i^p & A_i^q \end{pmatrix} \begin{pmatrix} B_i^q & B_i^p \\ -B_i^p & B_i^q \end{pmatrix} \rangle \\ &= \begin{pmatrix} \langle A_i^q, B_j^q\rangle + \langle A_i^p, B_j^p\rangle & \langle A_i^q, B_j^p\rangle - \langle A_i^p, B_j^q\rangle \\ \langle A_i^p, B_j^q\rangle - \langle A_i^q, B_j^p\rangle & \langle A_i^q, B_j^q\rangle + \langle A_i^p, B_j^p\rangle \end{pmatrix} \end{aligned}$$

where the last matrix is indeed the real matrix representation of $\langle \hat{A}_i, \hat{B}_j\rangle_h$. Moreover the real multiplication by $\mathcal{J}_2$ corresponds to the complex multiplication with the imaginary unit i. Namely if $\mathbf{A}$ is the real matrix representation of $\hat{\mathbf{A}}$, then $\mathcal{J}_2\mathbf{A}$ is the real matrix representation of $i\hat{\mathbf{A}}$. The same procedure in finite dimension leads to representing a complex matrix by a real matrix of double dimension, i.e. $\hat{\mathbf{A}} = \mathbf{A}^q + i\mathbf{A}^p \in \mathbb{C}^{S\times S}$ is represented by $\mathbf{A} \in \mathbb{R}^{2S\times 2S}$, written as in (50), with $\mathbf{A}^q$ and $\mathbf{A}^p$ real matrices in $\mathbb{R}^{S\times S}$.

Lemma 6.1. *The manifold $\mathcal{M}_S^{\mathbb{C}}$ of all S rank complex random fields is isomorphic to the manifold $\mathcal{M}_S^{sym}$ in Definition 6.2.*

Proof. The proof is based on the real representation of complex valued functions introduced before. Let $\hat{\mathbf{U}} = (\hat{U}_1, ..., \hat{U}_S) \in St(S, H^1(D,\mathbb{C}))$ and U_i^q, U_i^p denote respectively the real and imaginary part of $\hat{U}_i$ for any $i = 1, ..., S$. The orthonormality condition $\langle \hat{U}_i, \hat{U}_j\rangle_h = \delta_{ij}$ is written component-wise as:

$$\langle U_i^q, U_j^q\rangle + \langle U_i^p, U_j^p\rangle = \delta_{ij}, \text{ and } \langle U_i^p, U_j^q\rangle - \langle U_i^q, U_j^p\rangle = 0, \qquad \forall i,j = 1,...,S. \tag{51}$$

Let $\mathbf{U}$ be the real matrix representation of $\hat{\mathbf{U}}$ as defined in (50). This is written as:

$$\hat{\mathbf{U}} \sim \mathbf{U} = \begin{pmatrix} \mathbf{Q} & -\mathbf{P} \\ \mathbf{P} & \mathbf{Q} \end{pmatrix} \quad \text{with} \quad \begin{array}{ll} \mathbf{Q} = (Q_1, ..., Q_S) & : Q_i = \hat{U}_i^q \in H^1(D, \mathbb{R}), \quad \forall i = 1, ..., S, \\ \mathbf{P} = (P_1, ..., P_S) & : P_i = \hat{U}_i^p \in H^1(D, \mathbb{R}), \quad \forall i = 1, ..., S. \end{array}$$

Observe that condition (51) coincides with condition (30). Thus, from Proposition 6.1 (point b) we have that $\hat{\mathbf{U}} \in St(S, H^1(D, \mathbb{C}))$ if and only if $\mathbf{U} \in U(S, [H^1(D, \mathbb{R})]^2)$. It follows that any element $\hat{\mathbf{U}} \in St(S, H^1(D, \mathbb{C}))$ can be uniquely identified with an element $\mathbf{U} \in U(S, [H^1(D, \mathbb{R})]^2)$.
Consider now $\hat{\mathbf{Z}} \in [L^2(\Omega, \mathbb{C})]^S$ and its realification $\mathbf{Z} = (\mathbf{Z}^q, \mathbf{Z}^p)^T \in [L^2(\Omega, \mathbb{R})]^{2S}$. The components $(\hat{Z}_1, ..., \hat{Z}_S)$ of $\hat{\mathbf{Z}}$ are linearly independent if and only if the following matrix

$$\mathbb{E}[\hat{\mathbf{Z}}\hat{\mathbf{Z}}^*] = \mathbb{E}[\mathbf{Z}^{\mathbf{q}}\mathbf{Z}^{\mathbf{q}T}] + \mathbb{E}[\mathbf{Z}^{\mathbf{p}}\mathbf{Z}^{\mathbf{p}T}] + i(\mathbb{E}[\mathbf{Z}^{\mathbf{p}}\mathbf{Z}^{\mathbf{q}T}] - \mathbb{E}[\mathbf{Z}^{\mathbf{q}}\mathbf{Z}^{\mathbf{p}T}]) \in \mathbb{C}^{S\times S} \tag{52}$$

has full rank. Observe that the real matrix representation of $\mathbb{E}[\hat{\mathbf{Z}}\hat{\mathbf{Z}}^*]$ is given by:

$$\begin{pmatrix} \mathbb{E}[\mathbf{Z}^{\mathbf{q}}\mathbf{Z}^{\mathbf{q}T}] + \mathbb{E}[\mathbf{Z}^{\mathbf{p}}\mathbf{Z}^{\mathbf{p}T}] & \mathbb{E}[\mathbf{Z}^{\mathbf{q}}\mathbf{Z}^{\mathbf{p}T}] - \mathbb{E}[\mathbf{Z}^{\mathbf{p}}\mathbf{Z}^{\mathbf{q}T}] \\ \mathbb{E}[\mathbf{Z}^{\mathbf{p}}\mathbf{Z}^{\mathbf{q}T}] - \mathbb{E}[\mathbf{Z}^{\mathbf{q}}\mathbf{Z}^{\mathbf{p}T}] & \mathbb{E}[\mathbf{Z}^{\mathbf{q}}\mathbf{Z}^{\mathbf{q}T}] + \mathbb{E}[\mathbf{Z}^{\mathbf{p}}\mathbf{Z}^{\mathbf{p}T}] \end{pmatrix} = \left(\mathbb{E}[\mathbf{Z}\mathbf{Z}^T] + \mathbf{J}_{2S}^T\mathbb{E}[\mathbf{Z}\mathbf{Z}^T]\mathbf{J}_{2S}\right) \in \mathbb{R}^{2S\times 2S} \tag{53}$$

This implies that $(\hat{Z}_1, ..., \hat{Z}_S)$ are linearly independent if and only if $\mathbb{E}[\mathbf{Z}\mathbf{Z}^T] + \mathbb{E}[\mathbf{J}_{2S}\mathbf{Z}\mathbf{Z}^T\mathbf{J}_{2S}^T]$ is full rank. Observe also that $\mathbb{E}[\mathbf{Z}\mathbf{Z}^T] + \mathbb{E}[\mathbf{J}_{2S}\mathbf{Z}\mathbf{Z}^T\mathbf{J}_{2S}^T]$ is the real matrix representation of $\mathbb{E}[\hat{\mathbf{Z}}\hat{\mathbf{Z}}^*]$, hence the two identifications are consistent. It follows that $\mathrm{B}(S, L^2(\Omega, \mathbb{C}))$ can be uniquely identified with $\mathrm{B}^{sym}(2S, L^2(\Omega, \mathbb{R}))$. Finally any $\hat{u}_S = \hat{\mathbf{U}}\hat{\mathbf{Y}} \in \mathcal{M}_S^{\mathbb{C}}$, with $\hat{\mathbf{U}} \in St(S, H^1(D, \mathbb{C}))$ and $\hat{\mathbf{Y}} \in \mathrm{B}(S, L^2(\Omega, \mathbb{C}))$, can be uniquely represented in real setting as $\mathbf{u}_S = \mathbf{U}\mathbf{Y} \in \mathcal{M}_S^{sym}$ where $\mathbf{U} \in U(S, [H^1(D, \mathbb{R})]^2)$ and $\mathbf{Y} \in \mathrm{B}^{sym}(2S, L^2(\Omega, \mathbb{R}))$ are the real representations of $\hat{\mathbf{U}}$ and $\hat{\mathbf{Y}}$ respectively. ■

We now rewrite Lemma 6.1 in real setting to recover a unique representation of S-rank random fields $\mathbf{u}_S \in \mathcal{M}_S^{sym}$ in terms of the bases in $(\mathbf{U}, \mathbf{Y}) \in (U(S, [H^1(D)]^2), \mathrm{B}(S, L^2(\Omega, \mathbb{C})))$.

Proposition 6.5. *In real setting, the orthogonal condition* (44) *is reinterpreted as:*

$$\langle \delta\mathbf{U}_i, \mathbf{U}_j^+ \rangle = \langle \delta\mathbf{U}_i, \mathbf{U}_j \rangle = 0 \qquad \forall i, j = 1, ..., 2S \tag{54}$$

We mention that condition (54) can be directly derived, without making use of the isomorphism with $\mathcal{M}_S^{\mathbb{C}}$, by quotienting $U(S, [H^1(D)]^2)$ by $U(S, \mathbb{R}^{2S})$. This is perfectly consistent with the construction discussed before, being $U(S, \mathbb{R}^{2S})$ isomorphic to $\mathcal{O}(S, \mathbb{C})$. Condition (54) can be seen as a symplectic orthogonality condition: we ask that $\delta\mathbf{U}$ belongs to the symplectic orthogonal complement to $\mathbf{U}$ at each time:

$$\mathcal{P}_{\mathbf{U}}^{sym}[\delta\mathbf{U}] = \mathcal{P}_{\mathbf{U}}[\delta\mathbf{U}] = \mathbf{0}$$

Observe that condition (54) preserves the orthogonal-symplectic structure of the basis, namely if $\mathbf{U}(t)$ is an integrable curve passing through $\mathbf{U}(0) \in U(S, [H^1(D)]^2)$, of a vector field which satisfies (54), then $\mathbf{U}(t) \in U(S, [H^1(D)]^2)$ at any time, as the following proposition shows.

Proposition 6.6. *Let $\mathbf{U}(t)$ be a smooth curve in $[H^1(D)]^{2S}$ such that:*

1. $\mathbf{U}(0) \in U(S, [H^1(D)]^2)$,
2. $\vartheta_D(\dot{\mathbf{U}}_i(t), \mathbf{U}_j(t)) = 0$, $\forall i, j = 1, ..., 2S$ *and* $\forall t \in [0, T]$,
3. $\dot{\mathbf{U}}(t) = \mathcal{J}_2\dot{\mathbf{U}}(t)\mathbf{J}_{2S}$, $\forall t \in [0, T]$,

then $\mathbf{U}(t) \in U(S, [H^1(D)]^2)$ for all t.

Proof. We start by showing that the orthogonality is preserved. First of all we combine condition 2 and condition 3 and we get:

$$0 = \vartheta_D(\dot{\mathbf{U}}_i(t), \mathbf{U}_j(t)) = \langle \dot{\mathbf{U}}_i(t), \mathcal{J}_2 \mathbf{U}_j(t) \rangle = \langle \mathcal{J}_2^T \dot{\mathbf{U}}_i(t), \mathbf{U}_j(t) \rangle = \langle (\dot{\mathbf{U}}(t)\mathbf{J}_{2S}^T)_i, \mathbf{U}_j(t) \rangle$$

which implies $\langle \dot{\mathbf{U}}_j(t), \mathbf{U}_i(t) \rangle = 0$ for all $i, j = 1, ..., 2S$ and $t \in [0, T]$. Then

$$\frac{d}{dt} \langle \mathbf{U}_j(t), \mathbf{U}_i(t) \rangle = \langle \dot{\mathbf{U}}_j(t), \mathbf{U}_i(t) \rangle + \langle \mathbf{U}_j(t), \dot{\mathbf{U}}_i(t) \rangle = 0$$

implies that $\langle \mathbf{U}_j(t), \mathbf{U}_i(t) \rangle = \langle \mathbf{U}_j(0), \mathbf{U}_i(0) \rangle = \delta_{ij}$, $\forall i, j = 1, ..., 2S$ and $\forall t \in [0, T]$.
Similarly for the symplecticity:

$$\begin{aligned} \tfrac{d}{dt} \langle \mathcal{J}_2^T \mathbf{U}_i(t), \mathbf{U}_j(t) \rangle &= \langle \mathcal{J}_2^T \dot{\mathbf{U}}_i(t), \mathbf{U}_j(t) \rangle + \langle \mathcal{J}_2^T \mathbf{U}_i(t), \dot{\mathbf{U}}_j(t) \rangle \\ &= \langle (\dot{\mathbf{U}}(t)\mathbf{J}_{2S})_i, \mathbf{U}_j(t) \rangle + \langle \mathbf{U}_i(t), \mathcal{J}_2 \dot{\mathbf{U}}_j(t) \rangle \\ &= \langle (\dot{\mathbf{U}}(t)\mathbf{J}_{2S})_i, \mathbf{U}_j(t) \rangle + \vartheta_D(\mathbf{U}_i(t), \dot{\mathbf{U}}_j(t)) = 0 \end{aligned}$$

which implies $\langle \mathbf{U}_i(t), \mathcal{J}_2 \mathbf{U}_j(t) \rangle = \langle \mathbf{U}_i(0), \mathcal{J}_2 \mathbf{U}_j(0) \rangle = (\mathbf{J}_{2S})_{ij}$, $\forall i, j = 1, ..., 2S$ and $\forall t \in [0, T]$. ■

The dynamic condition (54) induces a bijection between $(U(S, [H^1(D)]^2)/U(S, \mathbb{R}^{2S})) \times \mathrm{B}^{sym}(2S, L^2(\Omega, \mathbb{R})))$ and $\mathcal{M}_S^{sym}$ which allows to equip $\mathcal{M}_S^{sym}$ with a differential manifold structure. In particular, for any $\mathbf{u}_S \in \mathcal{M}_S^{sym}$, the tangent space to $\mathcal{M}_S^{sym}$ at $\mathbf{u}_S = \mathbf{U}\mathbf{Y}$ is parametrized as follows:

Lemma 6.2. *For any $\mathbf{u}_S = \mathbf{U}\mathbf{Y} \in \mathcal{M}_S^{sym}$, the tangent space to $\mathcal{M}_S^{sym}$ at $\mathbf{u}_S$ is the subspace of $[H^1(D)]^2 \otimes L^2(\Omega)$ given by:*

$$\begin{aligned} \mathcal{T}_{\mathbf{u}_S} \mathcal{M}_S^{sym} &= \Big\{ \delta \mathbf{u}_S = (\delta \mathbf{U})\mathbf{Y} + \mathbf{U}\delta \mathbf{Y} \in [H^1(D)]^2 \otimes L^2(\Omega) : \quad \delta \mathbf{Y} \in [L^2(\Omega, \mathbb{R})]^{2S}, \\ &\qquad\qquad \delta \mathbf{U} \in \mathcal{U}^{sym\perp} : \mathcal{J}_2^T (\delta \mathbf{U}) \mathbf{J}_{2S} = \delta \mathbf{U} \Big\} \\ &= \Big\{ \delta \mathbf{u}_S = \sum_{i=1}^{2S} (\delta \mathbf{U}_i Y_i + \mathbf{U}_i \delta Y_i) : \qquad \delta Y_i \in L^2(\Omega, \mathbb{R}) \textit{ and } \delta \mathbf{U}_i \in [H^1(D)]^2, \\ &\qquad\qquad \textit{s.t. } \mathcal{J}_2^T \delta \mathbf{U} \mathbf{J}_{2S} = \delta \mathbf{U},\ \langle \delta \mathbf{U}_i, \mathbf{U}_j \rangle = 0,\ \forall i, j = 1, ..., 2S \Big\} \end{aligned} \tag{55}$$

The following property holds for any $\mathbf{u}_S = \mathbf{U}\mathbf{Y} \in \mathcal{M}_S^{sym}$:

Proposition 6.7. $\mathbf{v} \in \mathcal{T}_{\mathbf{u}_S} \mathcal{M}_S^{sym}$ *if and only if* $\mathcal{J}_2 \mathbf{v} \in \mathcal{T}_{\mathbf{u}_S} \mathcal{M}_S^{sym}$

Proposition 6.7 follows directly from the diffeomorphism between $\mathcal{M}_S^{\mathbb{C}}$ and $\mathcal{M}_S^{sym}$, see Remark 4 for the same result in complex setting. We emphasize that this property does not apply to arbitrary symplectic manifolds, and in particular, does not hold when the space of symplectic deterministic bases is not restricted to $U(S, [H^1(D)]^2)$. Observe that Proposition 6.7 implies that the symplectic form defined in (21) is not degenerate in $\mathcal{M}_S^{sym}$. Indeed for any $\mathbf{v} \in \mathcal{T}_{\mathbf{u}_S} \mathcal{M}_S^{sym}$ such that $\mathbf{v} \neq \mathbf{0}$, $\vartheta(\mathcal{J}_2 \mathbf{v}, \mathbf{v}) = \mathbb{E}[\langle \mathcal{J}_2 \mathbf{u}, \mathcal{J}_2 \mathbf{v} \rangle] = \|\mathbf{v}\|^2_{[L^2(D)]^2 \otimes L^2(\Omega)} > 0$.

Lemma 6.3. *Let $\mathbf{u}_S \in \mathcal{M}_S^{sym}$ be written as $\mathbf{u}_S = \mathbf{U}\mathbf{Y}$. For any $\mathbf{v} = (\delta \mathbf{U})\mathbf{Y} + \mathbf{U}\delta \mathbf{Y} \in \mathcal{T}_{\mathbf{u}_S} \mathcal{M}_S^{sym}$, $\delta \mathbf{U}$ and $\delta \mathbf{Y}$ are uniquely characterized as:*

$$\begin{aligned} \delta \mathbf{Y} &= \langle \mathbf{v}, \mathbf{U} \rangle \\ \delta \mathbf{U} &= \mathcal{P}_{\mathbf{U}}^{\perp, sym} \Big[\mathbb{E}[\mathbf{v}\mathbf{Y}^T] + \mathcal{J}_2 \mathbb{E}[\mathbf{v}\mathbf{Y}^T \mathbf{J}_{2S}^T] \Big] (\mathbf{C} + \mathbf{J}_{2S} \mathbf{C} \mathbf{J}_{2S}^T)^{-1} \end{aligned} \tag{56}$$

Proof. Let $\tilde{\mathbf{v}} \in [H^1(D)]^2 \otimes L^2(\Omega)$ and $\mathbf{v} = (\delta \mathbf{U})\mathbf{Y} + \mathbf{U}\delta \mathbf{Y}$ be the projection of $\tilde{\mathbf{v}}$ in the tangent space

$\mathcal{T}_{\mathbf{u}_S}\mathcal{M}_S^{sym}$, that is:

$$\mathbb{E}[\langle \tilde{\mathbf{v}}, \mathbf{w} \rangle] = \mathbb{E}[\langle \mathbf{v}, \mathbf{w} \rangle] \qquad \forall \mathbf{w} \in \mathcal{T}_{\mathbf{u}_S}\mathcal{M}_S^{sym} \tag{57}$$

According to (55) this can be written as:

$$\mathbb{E}[\langle \tilde{\mathbf{v}}, \mathbf{W}\mathbf{Y} + \mathbf{U}\mathbf{Z} \rangle] = \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{Y} + \mathbf{U}\delta\mathbf{Y}, \mathbf{W}\mathbf{Y} + \mathbf{U}\mathbf{Z} \rangle] \tag{58}$$

for any $\mathbf{Z} \in [L^2(\Omega, \mathbb{R})]^{2S}$ and $\mathbf{W} \in \mathcal{U}^{sym\perp}$ which satisfies $\mathcal{J}_2^T \mathbf{W}\mathbf{J}_{2S} = \mathbf{W}$. We need to verify that $\delta\mathbf{U}$ and $\delta\mathbf{Y}$ are uniquely characterized only in terms of $\tilde{\mathbf{v}}$, $\mathbf{U}$ and $\mathbf{Y}$.

- By testing against $\mathbf{U}\mathbf{Z}$ (i.e. setting $\mathbf{W} = 0$), we easily recover the characterization of $\delta\mathbf{Y}$:

$$\begin{aligned} & \mathbb{E}[\langle \tilde{\mathbf{v}}, \mathbf{U}\mathbf{Z} \rangle] && = \mathbb{E}[\langle \mathbf{U}\delta\mathbf{Y}, \mathbf{U}\mathbf{Z} \rangle], \\ \Rightarrow \quad & \mathbb{E}[\langle \tilde{\mathbf{v}}, \mathbf{U} \rangle \mathbf{Z}] && = \mathbb{E}[\mathbf{Z}^T \delta\mathbf{Y}] \qquad \forall \mathbf{Z} \in [L^2(\Omega, \mathbb{R})]^{2S}, \end{aligned} \tag{59}$$

  which leads to:

$$\delta\mathbf{Y} = \langle \mathcal{P}_{\mathbf{U}}[\tilde{\mathbf{v}}], \mathbf{U} \rangle.$$

- We now want to test against $\mathbf{W}\mathbf{Y}$ for $\mathbf{W}$ arbitrary in $\mathcal{U}^{sym\perp}$ and satisfying $\mathcal{J}_2^T \mathbf{W}\mathbf{J}_{2S} = \mathbf{W}$. The last condition can be replaced by setting $\mathbf{W} = \frac{1}{2}\big(\mathcal{J}_2^T \mathbf{V}\mathbf{J}_{2S} + \mathbf{V}\big)$ with arbitrary $\mathbf{V} \in \mathcal{U}^{sym\perp}$. Observe indeed that, for any $\mathbf{V} \in \mathcal{U}^{sym\perp}$, we have that $\mathcal{P}_{\mathbf{U}}^{sym}(\mathcal{J}_2^T \mathbf{V}\mathbf{J}_{2S}) = \mathcal{J}_2^T \mathcal{P}_{\mathbf{U}}^{sym}(\mathbf{V}\mathbf{J}_{2S}) = 0$ which implies $\mathbf{W} \in \mathcal{U}^{sym\perp}$. Thus we have:

$$\mathbb{E}[\langle \tilde{\mathbf{v}}, \mathcal{J}_2^T \mathbf{V}\mathbf{J}_{2S}\mathbf{Y} + \mathbf{V}\mathbf{Y} \rangle] = \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{Y}, \mathcal{J}_2^T \mathbf{V}\mathbf{J}_{2S}\mathbf{Y} + \mathbf{V}\mathbf{Y} \rangle] \qquad \forall \mathbf{V} \in \mathcal{U}^{sym\perp} \tag{60}$$

  The left hand side can be rewritten as

$$\mathbb{E}[\langle \tilde{\mathbf{v}}, \mathcal{J}_2^T \mathbf{V}\mathbf{J}_{2S}\mathbf{Y} + \mathbf{V}\mathbf{Y} \rangle] = \langle \mathbb{E}[\mathcal{J}_2 \tilde{\mathbf{v}}\mathbf{Y}^T \mathbf{J}_{2S}^T], \mathbf{V} \rangle + \langle \mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T], \mathbf{V} \rangle,$$

  while for the right hand side we have:

$$\begin{aligned} \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{Y}, \mathcal{J}_2^T \mathbf{V}\mathbf{J}_{2S}\mathbf{Y} + \mathbf{V}\mathbf{Y} \rangle] &= \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{Y}, \mathcal{J}_2^T \mathbf{V}\mathbf{J}_{2S}\mathbf{Y} \rangle] + \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{Y}, \mathbf{V}\mathbf{Y} \rangle] \\ &= \mathbb{E}[\langle \mathcal{J}_2(\delta\mathbf{U})\mathbf{Y}, \mathbf{V}\mathbf{J}_{2S}\mathbf{Y} \rangle] + \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{Y}, \mathbf{V}\mathbf{Y} \rangle] \\ &= \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{J}_{2S}\mathbf{Y}, \mathbf{V}\mathbf{J}_{2S}\mathbf{Y} \rangle] + \mathbb{E}[\langle (\delta\mathbf{U})\mathbf{Y}, \mathbf{V}\mathbf{Y} \rangle] \\ &= \langle \delta\mathbf{U}, \mathbf{V} \rangle \mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T + \langle \delta\mathbf{U}, \mathbf{V} \rangle \mathbf{C} \end{aligned} \tag{61}$$

  where we used the fact that $\mathcal{J}_2^T \delta\mathbf{U}\mathbf{J}_{2S} = \delta\mathbf{U}$. By combining the two parts we get:

$$\langle \mathbb{E}[\mathcal{J}_2 \tilde{\mathbf{v}}\mathbf{Y}^T \mathbf{J}_{2S}^T], \mathbf{V} \rangle + \langle \mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T], \mathbf{V} \rangle = \langle \delta\mathbf{U}, \mathbf{V} \rangle \mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T + \langle \delta\mathbf{U}, \mathbf{V} \rangle \mathbf{C} \tag{62}$$

  for any $\mathbf{V} \in \mathcal{U}^{sym\perp}$. By using Preposition 6.4 we finally obtain:

$$\delta\mathbf{U}(\mathbf{C} + \mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T) = \mathcal{P}_{\mathbf{U}}^{\perp, sym}\Big[\mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T] + \mathcal{J}_2 \mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T \mathbf{J}_{2S}^T]\Big]. \tag{63}$$

  Observe that $\delta\mathbf{U}$ is completely characterized, thanks to the full rank assumption on $\mathbf{C} + \mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T$.

It is worth checking that the term $\delta\mathbf{U}$ thus obtained does indeed satisfy the condition $\mathcal{J}_2(\delta\mathbf{U})\mathbf{J}_{2S}^T = \delta\mathbf{U}$. We observe that $\mathcal{J}_2(\delta\mathbf{U})\mathbf{J}_{2S}^T = \delta\mathbf{U}$ applies if and only if

$$\delta\mathbf{U}(\mathbf{C} + \mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T)\mathbf{J}_{2S}^T = \mathcal{J}_2(\delta\mathbf{U})\mathbf{J}_{2S}^T(\mathbf{C} + \mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T)\mathbf{J}_{2S}^T = -\mathcal{J}_2(\delta\mathbf{U})(\mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T + \mathbf{C}).$$

Then, from (63) and Proposition 6.4 follows that:

$$\begin{aligned}\delta\mathbf{U}(\mathbf{C}+\mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T)\mathbf{J}_{2S}^T &= \mathcal{P}_{\mathbf{U}}^{\perp,sym}\Big[\mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T]+\mathcal{J}_2\mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T\mathbf{J}_{2S}^T]\Big]\mathbf{J}_{2S}^T\\ &= \mathcal{P}_{\mathbf{U}}^{\perp,sym}\Big[\mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T\mathbf{J}_{2S}^T]+\mathcal{J}_2^T\mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T]\Big]\\ &= \mathcal{J}_2^T\mathcal{P}_{\mathbf{U}}^{\perp,sym}\Big[\mathcal{J}_2\mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T\mathbf{J}_{2S}^T]+\mathbb{E}[\tilde{\mathbf{v}}\mathbf{Y}^T]\Big]\\ &= \mathcal{J}_2^T\delta\mathbf{U}(\mathbf{C}+\mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T)\\ &= -\mathcal{J}_2\delta\mathbf{U}(\mathbf{C}+\mathbf{J}_{2S}\mathbf{C}\mathbf{J}_{2S}^T)\end{aligned}$$

which concludes the proof. ■

6.2 DLR Variational Principle in complex and real setting

Our goal is to find a dynamical low rank approximation $\mathbf{u}_S \in \mathcal{M}_S^{sym}$ of problem (4), which is written as:

$$\mathbf{u}_S(\mathbf{x},t,\omega)=\sum_{i=1}^{2S}\mathbf{U}_i(\mathbf{x},t)Y_i(t,\omega) \tag{64}$$

To do so we exploit the diffeomorphism between $\mathcal{M}_S^{sym}$ and $\mathcal{M}_S^{\mathbb{C}}$.
We start by considering problem (42). In complex setting, since this is a first order PDE we can apply the DO approximation described in Section 2. The DO variational principle for problem (42) reads as follows:

Complex DLR Variational Principle. *At each $t \in (0,T]$, find $\hat{u}_S(t) \in \mathcal{M}_S^{\mathbb{C}}$ such that:*

$$\mathbb{E}\big[\big\langle i\dot{\hat{u}}_S - 2\partial_{\hat{u}_S^*}H_\omega(\hat{u}_S,\hat{u}_S^*,\cdot),\hat{v}\big\rangle_h\big]=0, \qquad \forall\hat{v}\in\mathcal{T}_{\hat{u}_S(t)}\mathcal{M}_S^{\mathbb{C}}. \tag{65}$$

with initial condition $\hat{u}_{0,S}$ given by a suitable S rank approximation of $\hat{u}_0$ by e.g. a truncated Karhunen-Loève expansion.

Since $\mathcal{T}_{\hat{u}_S(t)}\mathcal{M}_S^{\mathbb{C}}$ is a complex linear space (which means that $\hat{v}\in\mathcal{T}_{\hat{u}_S(t)}\mathcal{M}_S^{\mathbb{C}}$ if and only if $i\hat{v}\in\mathcal{T}_{\hat{u}_S(t)}\mathcal{M}_S^{\mathbb{C}}$), we get the same conditions if we take only the real part or the imaginary part of (65). Following the discussion of Section 6.1, and in particular by means of (47) and Lemma 6.1, we can recast problem (65) in the real setting as follows:

Symplectic DLR Variational Principle. *At each $t \in (0,T]$, find $\mathbf{u}_S(t) \in \mathcal{M}_S^{sym}$ such that:*

$$\mathbb{E}\left[\big\langle\mathcal{J}_2\dot{\mathbf{u}}_S+\nabla H_\omega(\mathbf{u}_S,\cdot),\mathbf{v}\big\rangle\right]=0, \qquad \forall\mathbf{v}\in\mathcal{T}_{\mathbf{u}_S(t)}\mathcal{M}_S^{sym}, \tag{66}$$

with initial conditions given by the symplectic projection of the initial data onto $\mathcal{M}_S^{sym}$.

The term $\mathbb{E}\left[\big\langle\nabla H_\omega(\mathbf{u}_S,\cdot),\mathbf{v}\big\rangle\right]$ in (66) is interpreted as $\frac{d}{dt}_{|t=0}\mathbb{E}[H_\omega(\boldsymbol{\gamma}_S(t))]$, i.e. the directional derivative along a curve $\boldsymbol{\gamma}_S(t)\in\mathcal{M}_S^{sym}$ with $\boldsymbol{\gamma}_S(0)=\mathbf{u}_S$ and $\dot{\boldsymbol{\gamma}}_S(0)=\mathbf{v}$.
Observe that the variational principle (66) corresponds to a symplectic projection of the governing equation onto the (time-dependent) tangent space to the manifold along the trajectory of the approximate solution. We call symplectic dynamical low rank (or symplectic DO) approximation of problem (19) the solution to (66). This belongs to $\mathcal{M}_S^{sym}$ at any t and is written as:

$$\mathbf{u}_S(\mathbf{x},t,\omega)=\begin{pmatrix}q_S(\mathbf{x},t,\omega)\\ p_S(\mathbf{x},t,\omega)\end{pmatrix}=\sum_{i=1}^{2S}\mathbf{U}_i(\mathbf{x},t)Y_i(\omega,t)=\begin{pmatrix}\sum_{i=1}^{S}Q_iY_i-\sum_{i=1}^{S}P_iY_{S+i}\\ \sum_{i=1}^{S}P_iY_i+\sum_{i=1}^{S}Q_iY_{S+i}\end{pmatrix}, \tag{67}$$

with $\mathbf{U}(t) \in U(S, [H^1(D)]^2)$, $\mathbf{Y}(t) \in \mathrm{B}^{sym}(2S, L^2(\Omega))$. The peculiarity of the symplectic dynamical low rank approximation is the conservation of energy:

Lemma 6.4. *Assuming that problem (66) admits a smooth solution $\mathbf{u}_S$, for all $t \in [0, T]$, the expected value of the Hamiltonian is conserved along the approximate solution.*

Proof. Equation (66) can be rewritten as

$$\vartheta(\dot{\mathbf{u}}_S, \mathbf{v}) = -\mathbb{E}\left[\left\langle \nabla H_\omega(\mathbf{u}_S, \cdot), \mathbf{v} \right\rangle\right].$$

where ϑ is the symplectic form defined in (21). Then, taking $\mathbf{v} = \dot{\mathbf{u}}_S$ we get:

$$0 = \vartheta(\dot{\mathbf{u}}_S, \dot{\mathbf{u}}_S) = -\mathbb{E}\left[\left\langle \nabla H_\omega(\mathbf{u}_S, \cdot), \dot{\mathbf{u}}_S \right\rangle\right] = -\frac{d}{dt}_{|t=0} \mathbb{E}[H_\omega(\boldsymbol{u}_S(t))],$$

which implies $\mathbb{E}[H_\omega(\boldsymbol{u}_S(t))] = \mathbb{E}[H_\omega(\boldsymbol{u}_S(0))]$ for all $t \in [0, T]$. ■

Similarly to the Symplectic Order reduction, the energy that is conserved by the approximate solution, i.e. $H(\boldsymbol{u}_S(t)) = \mathbb{E}[H_\omega(\boldsymbol{u}_S(t))]$, is not necessary equal to the energy of the exact solution $H(\boldsymbol{u}(t)) = \mathbb{E}[H_\omega(\boldsymbol{u}(t))]$. However, such discrepancy is constant in time and depends only the approximation of the initial data:

$$|\mathbb{E}[H_\omega(\boldsymbol{u}(t)) - H_\omega(\boldsymbol{u}_S(t))]| = |\mathbb{E}[H_\omega(\boldsymbol{u}(0))] - H_\omega(\boldsymbol{u}_S(0))]|.$$

Moreover, thanks to the analogy with the complex DO, the Symplectic Dynamical Low-rank approximation has the same approximation properties as the standard DO approach. In particular, if the differential operator is linear and deterministic, i.e. $\nabla H = L$ with L deterministic, linear and self-adjoint, the following holds:

Proposition 6.8. *The symplectic dynamical low-rank approximation of linear deterministic Hamiltonian systems with random initial condition coincides with the exact solution, provided that the initial condition belongs to $\mathcal{M}_S^{sym}$.*

More generally, the symplectic dynamical low-rank approximation of linear deterministic Hamiltonian systems is optimal in L^2-sense provided that there is no crossing between the omitted and not omitted singular values of the exact solution. As discussed in [25], this condition is an intrinsic limitation of dynamical low rank methods, and generally can not be avoided without data-driven adaptivity strategies or closure models: when such crossings occur, the neglected modes, which become dominant in the exact solution, can not be tracked by the reduced system, which evolves only the modes that were dominant at initial time.
By using the parametrization of the tangent space in (55) we finally derive the symplectic DO reduced system. The variational problem (66) is rewritten in terms of dynamic equations for $(\mathbf{Y}, \mathbf{U})$ as follows:

Proposition 6.9. *Let $(\mathbf{U}(t), \mathbf{Y}(t)) \in U(S, [H^1(D)]^2) \times \mathrm{B}^{sym}(2S, L^2(\Omega))$ be a solution of the following system:*

$$\begin{cases} \dot{\mathbf{Y}} = \langle \mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S), \mathbf{U}^+ \rangle = \mathbf{J}_{2S} \nabla_{\mathbf{Y}} \tilde{H}_\omega(\mathbf{Y}) & \text{(68a)} \\ \dot{\mathbf{U}}(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}) = \mathcal{P}_{\mathbf{U}}^{\perp}\left[\nabla H(\mathbf{u}_S) \mathbf{Y}^T \mathbf{J}_{2S} + \mathcal{J}_2 \nabla H(\mathbf{u}_S) \mathbf{Y}^T\right] & \\ \qquad = \mathcal{P}_{\mathbf{U}}^{\perp}\left[\mathbb{E}[\nabla H_\omega(\mathbf{u}_S) \mathbf{Y}^T \mathbf{J}_{2S}] + \mathbb{E}[\mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S) \mathbf{Y}^T]\right] & \text{(68b)} \end{cases}$$

with initial conditions given by the complex SVD. Then $\mathbf{u}_S(t) = \mathbf{U}(t)\mathbf{Y}(t) \in \mathcal{M}_S^{sym}$ satisfies the DO variational principle (66) at any $t \in [0, T]$.

Proof. The equations in (68a)-(68b) can be simply obtained by replacing $\tilde{\mathbf{v}}$ with $\mathcal{J}_2 \nabla H(\mathbf{u}_S)$ in the proof of Lemma 6.3. ■

Observe that system (68a)-(68b) consists of $2S$ random ODEs coupled to $2S$ deterministic PDEs. However, exploiting the unitary structure of $\mathbf{U}$ (29), we actually need to solve only S PDEs to completely characterize the deterministic basis at each time. Indeed, the dynamic condition (54) preserves at continuous level this unitary structure (29), provided that $\mathbf{U} \in U(S, [H^1(D)]^2)$ at initial time. This can be directly verified by looking at the set of equations for $\dot{\mathbf{U}}$ (68b). First of all, let $\mathbf{A} = \mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}$ or $\mathbf{A} = \left(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}\right)^{-1}$, in both cases, it holds: $\mathbf{J}_{2S} \mathbf{A} \mathbf{J}_{2S}^T = \mathbf{J}_{2S}^T \mathbf{A} \mathbf{J}_{2S} = \mathbf{A}$. The analogous property is satisfied by the term on the right hand side of (68b):

$$
\begin{aligned}
&\mathcal{J}_2\Big(\mathcal{P}_{\mathbf{U}}^{\perp}\big[\mathbb{E}[\nabla H_\omega(\mathbf{u}_S)\mathbf{Y}^T\mathbf{J}_{2S}] + \mathbb{E}[\mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S)\mathbf{Y}^T]\big]\Big)\mathbf{J}_{2S}^T \\
&= \Big(\mathcal{P}_{\mathbf{U}}^{\perp}\big[\mathbb{E}[\mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S)\mathbf{Y}^T] - \mathbb{E}[\nabla H_\omega(\mathbf{u}_S)\mathbf{Y}^T\mathbf{J}_{2S}^T]\big]\Big)
\end{aligned}
$$

where we use Proposition 6.4 and the properties of the Poisson matrix. This implies that the same property is necessarily satisfied by $\dot{\mathbf{U}}$, i.e. $\mathcal{J}_2^T \dot{\mathbf{U}} \mathbf{J}_{2S} = \mathcal{J}_2 \dot{\mathbf{U}} \mathbf{J}_{2S}^T = \dot{\mathbf{U}}$ and the structure (29) is preserved by the dynamic system. On the other hand at discrete level the time discretization scheme has to be carefully chosen to preserve the unitary structure of $\mathbf{U}$.

6.3 Isolating the mean

In our context of partial differential equations with random parameters, since we are usually interested in computing the statistics of the solution, it may be worth approximating separately the mean of the solution, as proposed by [31] and adopted in [32], [25], [6] for the DO approximation of parabolic equations. For this aim we re-define S rank random field as follows.

Definition 6.5. *We call S rank random field (in the isolated mean format) any function that can be exactly expressed as $\mathbf{u}_S = \bar{\mathbf{u}}_S + \mathbf{U}\mathbf{Y}$, where:*

- $\bar{\mathbf{u}}_S = \mathbb{E}[\mathbf{u}_S] \in [H^1(D)]^2 \otimes L^2(D)$.
- $\mathbf{U} \in U(S, [H^1(D)]^2)$,
- $\mathbf{Y} = (Y_1, ..., Y_{2S}) \in \mathrm{B}^{sym}(2S, L^2(\Omega))$ *such that* $\mathbb{E}[Y_i] = 0$ *for any* $i = 1, ..., S$.

We define $\mathring{\mathcal{M}}_S^{sym} \subset (H^1(D) \times L^2(D)) \otimes L_0^2(\Omega)$ the manifold of all symplectic S rank random fields with zero mean.

In this setting, the symplectic Low Rank approximation of problem (4) is sought in $([H^1(D)]^2 \otimes L^2(D)) \times \mathring{\mathcal{M}}_S^{sym}$ and satisfies:

$$
\begin{cases}
\dot{\mathbf{Y}} = \mathbf{J}_{2S} \nabla_{\mathbf{Y}} \tilde{H}_\omega^\circ(\mathbf{Y}) \\
\dot{\bar{\mathbf{u}}}_S = \mathbb{E}[\mathcal{J}_2 \nabla H_\omega(\mathbf{u}_S)] \\
\dot{\mathbf{U}}(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}) = \mathcal{P}_{\mathbf{U}}^{\perp}\big[\mathbb{E}[\nabla H_\omega^\circ(\mathbf{u}_S)\mathbf{Y}^T\mathbf{J}_{2S}] + \mathbb{E}[\mathcal{J}_2 \nabla H_\omega^\circ(\mathbf{u}_S)\mathbf{Y}^T]\big]
\end{cases}
\tag{69}
$$

where $H_\omega^\circ(\cdot) = H_\omega(\cdot) - \mathbb{E}[H_\omega]$ and $\tilde{H}_\omega^\circ = \tilde{H}_\omega \circ \mathbf{U}$.

7 Numerical tests

7.1 Linear Deterministic Hamiltonian: validation test 1

For the validation of the Symplectic DO method we consider the following straightforward problem in the one dimensional domain $D = (0, 2\pi)$:

$$\begin{cases} \ddot{q}(x,t,\omega) = \Delta q(x,t,\omega) & x \in (0,2\pi),\, \omega \in \Omega,\, t \in (0,T] \\ q(0,t,\omega) = q(2\pi,t,\omega) = 0 & \omega \in \Omega,\, t \in (0,T] \\ q(x,0,\omega) = Z(\omega)\frac{1}{\sqrt{\pi}}\sin(x) & x \in (0,2\pi),\, \omega \in \Omega \\ \dot{q}(x,0,\omega) = 0 & x \in (0,2\pi),\, \omega \in \Omega \end{cases} \tag{70}$$

where Z is a uniformly distributed random variable in $[-1, 1]$. The analytical solution, given by:

$$q(x,t,\omega) = Z(\omega)\cos(t)\frac{1}{\sqrt{\pi}}\sin(x), \qquad p(x,t,\omega) = -Z(\omega)\sin(t)\frac{1}{\sqrt{\pi}}\sin(x)$$

is clearly a $1-$rank symplectic function, namely $\mathbf{u} = (q, p)$ belongs to $\mathcal{M}_1^{sym}$ and can be written as $\mathbf{u} = \mathbf{U}\mathbf{Y}$ with:

$$\mathbf{U} = \frac{1}{\sqrt{\pi}}\begin{bmatrix} \sin(x) & 0 \\ 0 & \sin(x) \end{bmatrix} \in U(1, [H_0^1(D)]^2), \quad \mathbf{Y} = \begin{bmatrix} Z(\omega)\cos(t) \\ -Z(\omega)\sin(t) \end{bmatrix} \in \mathrm{B}^{sym}(2, L^2(\Omega)).$$

In particular, this means that the rank of the exact solution, which is equal to 1 at $t = 0$, remains constant in time. The same generally applies to any solution of linear deterministic Hamiltonian systems with finite rank initial condition. We start by rewriting problem (70) in Hamiltonian form:

$$\begin{cases} \dot{\mathbf{u}}(x,t,\omega) = \mathcal{J}_2\mathbf{L}\mathbf{u}(x,t,\omega) \\ \mathbf{u}(x,0,\omega) = (Z(\omega)\frac{1}{\sqrt{\pi}}\sin(x), 0) \\ u_1(0,t,\omega) = u_1(2\pi,t,\omega) = 0 \end{cases} \quad \text{with} \quad \mathbf{L} = \begin{bmatrix} -\Delta & 0 \\ 0 & \mathbb{I} \end{bmatrix}. \tag{71}$$

Then by following (68a)-(68b), one can easily derive the reduced Symplectic DO system, which is given by:

$$\begin{cases} \dot{\mathbf{Y}}(t,\omega) = < \mathcal{J}_2\mathbf{L}\mathbf{U}(\cdot,t), \mathbf{U}(\cdot,t) > \mathbf{Y}(t,\omega) & \omega \in \Omega,\, t \in (0,T] \\ \dot{\mathbf{U}}(t)(\mathbf{C}(t) + \mathbf{J}_2^T\mathbf{C}(t)\mathbf{J}_2) = \mathcal{P}^{\perp}_{\mathbf{U}(t)}\big[\mathcal{J}_2\mathbf{L}\mathbf{U}(t)\mathbf{C}(t) + \mathbf{L}\mathbf{U}(t)\mathbf{C}(t)\mathbf{J}_2\big] & x \in (0,2\pi),\, t \in (0,T] \end{cases} \tag{72}$$

with initial conditions:

$$\mathbf{U}(0) = \begin{bmatrix} Q(0) & -P(0) \\ P(0) & Q(0) \end{bmatrix} = \frac{1}{\sqrt{\pi}}\begin{bmatrix} \sin(x) & 0 \\ 0 & \sin(x) \end{bmatrix}, \quad \mathbf{Y}(0) = \begin{bmatrix} Z(\omega) \\ 0 \end{bmatrix} \tag{73}$$

and completed with homogeneous Dirichlet boundary conditions: $Q(0,t) = Q(2\pi,t) = P(0,t) = P(2\pi,t) = 0$ for all $t \in [0,T]$. After observing that $\mathbf{U}(0)$ is an eigenfunction of $\mathbf{L}$ with eigenvalue equal to 1, i.e. $\mathbf{L}\mathbf{U}(0) = \mathbf{U}(0)$, we claim that the Symplectic DO system (72) recovers the exact solution of problem (70) and keeps the deterministic modes constant in time. Namely we want to show that the exact solution, written as $\mathbf{u}(t) = \mathbf{U}(t)\mathbf{Y}(t)$ with $\mathbf{U}(t) = \mathbf{U}(0)$ and $\mathbf{Y}(t) = (Z(\omega)\cos(t), -Z(\omega)\sin(t))^T$, satisfies (72). To verify this, we start by assuming that $\mathbf{C} + \mathbf{J}_2^T\mathbf{C}\mathbf{J}_2$ has full rank, with $\mathbf{C}$ denoting the moments matrix, i.e. $\mathbb{E}[\mathbf{Y}\mathbf{Y}^T]$. Under this assumption, one can easily see that equations (72) are automatically satisfied by $\mathbf{U}(t) = \mathbf{U}(0)$,

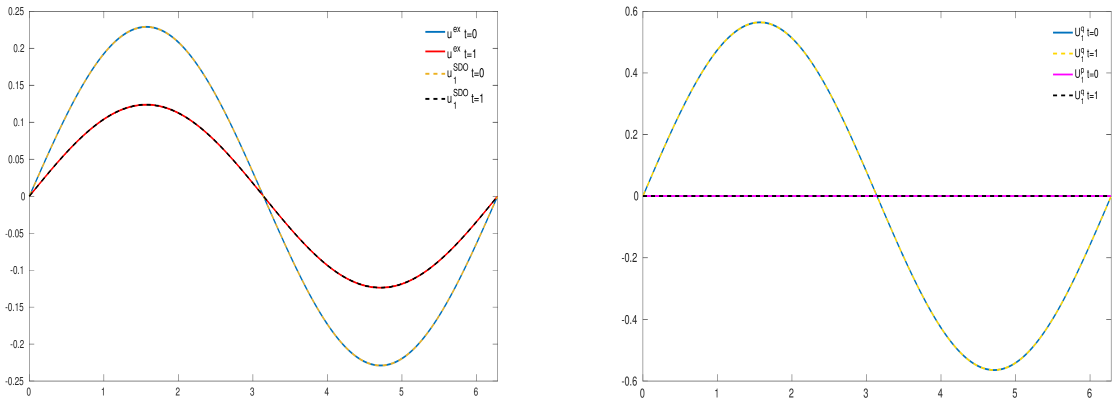

Figure 1: Left: the exact solution (solid line) and the symplectic DO approximate solution with $S = 1$ (dotted line) for $Z = 0.4058$ at $t = 0$ and $t = 1$: the two solutions coincide. Right: the deterministic modes of the symplectic DO approximate solution with $S = 1$ at $t = 0$ and $t = 1$: the modes are constant in time. Discretization parameters: number of Gauss-Legendre collocation points $N_y = 7$, spatial discretization $h = 0.01$, time-step $\Delta t = 0.01$.

by observing that

$$\begin{aligned} 0 = \dot{\mathbf{U}}(\mathbf{C} + \mathbf{J}_2^T \mathbf{C} \mathbf{J}_2) &= \mathcal{P}_{\mathbf{U}}^{\perp}\big[\mathcal{J}_2 \mathbf{U} \mathbf{C} - \mathbf{U} \mathbf{C} \mathbf{J}_2^T\big] \\ &= \mathcal{P}_{\mathbf{U}}^{\perp}\big[\mathbf{U}\big](\mathbf{J}_2 \mathbf{C} - \mathbf{C} \mathbf{J}_2^T) \\ &= 0 \end{aligned}$$

since $\mathcal{P}_{\mathbf{U}}^{\perp}\big[\mathbf{U}\big]$ is clearly equal to zero. Thus, the Symplectic DO system, which is reduced to the Hamiltonian system for the evolution of the coefficients $\mathbf{Y}$, degenerates to the proper symplectic decomposition proposed in [27]. Specifically we have $\dot{\mathbf{Y}} = \mathbf{J}_2 \mathbf{Y}$ with initial condition $\mathbf{Y}(0)$, which admits a unique solution given by $\mathbf{Y}(t) = (Z(\omega)\cos(t), -Z(\omega)\sin(t))^T$. We finally verify that the assumption on the rank of $\mathbf{C} + \mathbf{J}_2^T \mathbf{C} \mathbf{J}_2$ is actually fulfilled, by observing that $\mathbf{C} + \mathbf{J}_2^T \mathbf{C} \mathbf{J}_2 = \begin{pmatrix} \mathbb{E}[Z^2] & 0 \\ 0 & \mathbb{E}[Z^2] \end{pmatrix}$ at any time. This allows us to conclude that the Symplectic DO method recovers the exact solution by keeping the deterministic basis constant in time. The numerical results perfectly agree with the previous analysis, with the only care in choosing a symplectic time discretization scheme, see Figure 1.

7.2 Linear Deterministic Hamiltonian: validation test 2

Next, we consider again a linear wave equation but with a more general initial condition:

$$\begin{cases} \ddot{q}(x,t,\omega) = c^2 \Delta q(x,t,\omega) & x \in (0,1),\, \omega \in \Omega,\, t \in (0,T] \\ q(0,t,\omega) = q(1,t,\omega) = 0 & \omega \in \Omega,\, t \in (0,T] \\ q(x,0,\omega) = Z(\omega) h(10 \times |x - 0.5|) & x \in (0,1),\, \omega \in \Omega \\ \dot{q}(x,0,\omega) = 0 & x \in (0,1),\, \omega \in \Omega \end{cases} \tag{74}$$

with $c^2 = 0.1$ and:

$$h(s) = \begin{cases} 1 - 1.5 s^2 + 0.75 s^3 & 0 \geq s \leq 1 \\ 0.25(2 - s)^3 & 1 < s \leq 2 \\ 0 & s > 2 \end{cases} \tag{75}$$

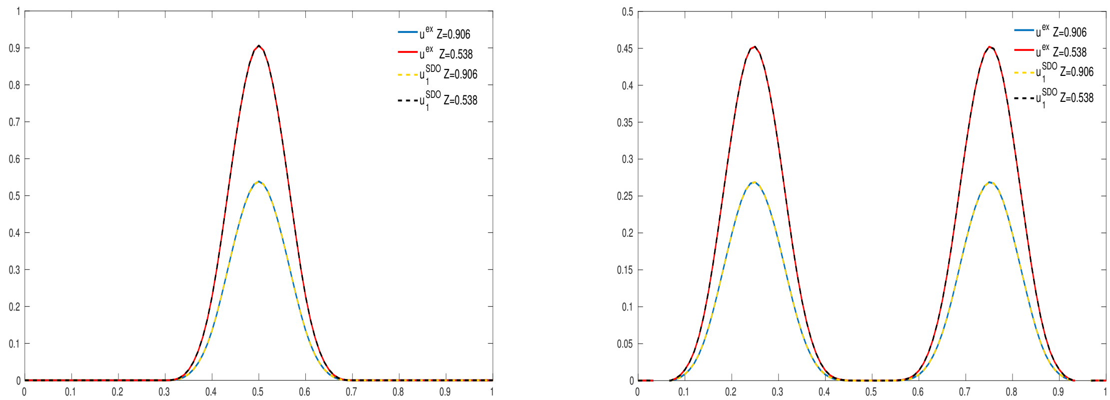

Figure 2: The solution for two different realizations of Z,i.e. $Z = 0.906$ and $Z = 0.538$, at $t = 0$ on the left and $t = 0.8$ on the right. The symplectic DO solution coincides with the reference solution computed with the Stochastic collocation method. Discretization parameters: number of Gauss-Legendre collocation points $N_y = 5$, spatial discretization $h = 0.01$, time-step $\Delta t = 0.01$.

Since the Hamiltonian is linear and deterministic, the exact solution, which at time $t = 0$ is a symplectic 1-rank function, has rank which is constant in time and can be written as $\mathbf{u}(x,t,\omega) = \mathbf{Z}(\omega)(q(x,t), p(x,t))$. By observing that $\mathcal{J}_2\mathbf{Lu}$ and $\mathbf{Lu}$ belong to the tangent space $\mathcal{T}_{\mathbf{u}(t)}\mathcal{M}_1^{sym}$ at any time, we claim that the Symplectic DO method recovers again the exact solution. In particular, the Symplectic DO approximate solution, which is initialized as:

$$\mathbf{U}_0 = \begin{bmatrix} \frac{h(10\times|x-0.5|)}{\|h(10\times|x-0.5|)\|} & 0 \\ 0 & \frac{h(10\times|x-0.5|)}{\|h(10\times|x-0.5|)\|} \end{bmatrix} \quad \mathbf{Y}_0 = \begin{bmatrix} \|h(10\times|x-0.5|)\|Z \\ 0 \end{bmatrix},$$

is expected to evolve as $(\mathbf{U}(t), Y(t) = Z\mathbf{X}(t))$, where $\mathbf{X}(t)$ is a rescaling factor, and satisfies $\mathbf{U}(t)\mathbf{Y}(t) = \mathbf{u}$ at any time. The numerical results validate the exactness of the Symplectic DO method for the problem under consideration, up to the numerical discretization error in time and space. The validation is done by comparing the Symplectic DO approximate solution to the reference solution computed with the Stochastic Collocation method with Gauss-Legendre points ([3]). Figure 2 shows the solution for two different realizations of Z and at two different times $t = 0$ and $t = 0.6$: we see that the DO solution and the reference solution coincide. Contrary to the previous example (in which the deterministic basis remains fixed in time), Figure 3 shows that in this case, the deterministic modes evolve in time by following the wave propagation. In particular, we observe that the mode $\mathbf{P}_1$, initialized to zero, will be automatically activated by the method, which means that the approximation will not be restricted to the diagonal structure of $\mathbf{U}_0$, used for the initialization. This shows the potential of the Symplectic DO method with respect to a reduced order method with fixed (in time) bases.

7.3 Wave equation with random wave speed

We now consider a linear wave equation with random speed and random initial data, in the 2-dimensional physical domain $D = (0,1)^2$, with boundary $\partial D = \bar{\Gamma}_N \cup \bar{\Gamma}_D$, $\bar{\Gamma}_N = \{(x,y) \in \mathbb{R}^2, x \in (0,1), y = 1\}$,

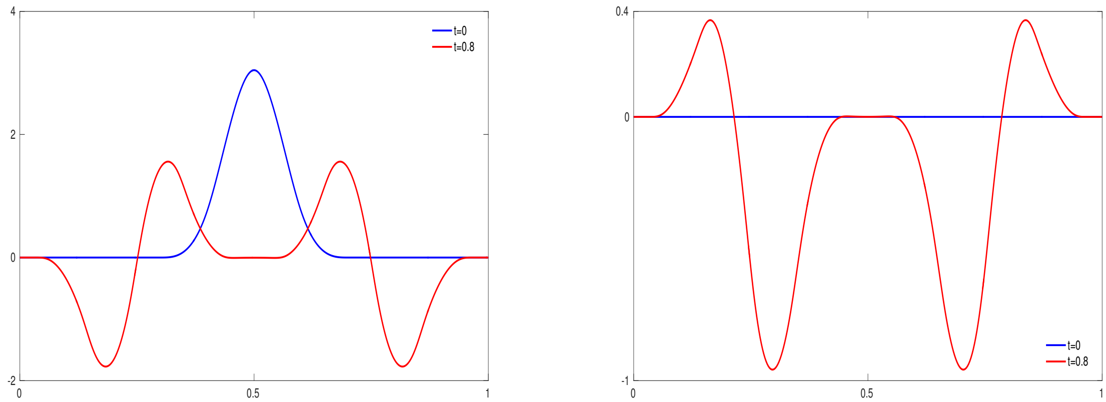

Figure 3: The deterministic modes Q (left) and P (right) of the symplectic DO approximate solution at time $t = 0$ and $t = 0.8$. We observe that both the modes evolve in time by following the variability spread of the solutions. Discretization parameters: number of Gauss-Legendre collocation points $N_y = 5$, spatial discretization $h = 0.01$, timestep $\Delta t = 0.01$.

$\Gamma_D = \partial D \backslash \Gamma_N$. The problem reads as follows:

$$\begin{cases} \ddot{q}(\mathbf{x}, t, \omega) = c^2(\omega)\Delta q(\mathbf{x}, t, \omega) & \mathbf{x} \in (0,1)^2,\, t \in (0,T],\, \omega \in \Omega \\ q(\mathbf{x}, t, \omega) = 0 & \mathbf{x} \in \Gamma_D,\, t \in (0,T],\, \omega \in \Omega \\ \partial_n q(\mathbf{x}, t, \omega) = 0 & \mathbf{x} \in \Gamma_N,\, t \in (0,T],\, \omega \in \Omega \\ q(\mathbf{x}, 0, \omega) = \alpha(\omega) q_0(\mathbf{x}) & \mathbf{x} \in (0,1)^2,\, \omega \in \Omega \\ \dot{q}(\mathbf{x}, 0, \omega) = 0 & \mathbf{x} \in (0,1)^2,\, \omega \in \Omega \end{cases} \tag{76}$$

with:

$$q_0(\mathbf{x}) = \begin{cases} e^{\frac{-\|\mathbf{x}-0.5\|^2}{2(0.1)^2}} & \|\mathbf{x}-0.5\|^2 < 0.8 \\ 0 & \|\mathbf{x}-0.5\|^2 \geq 0.8 \end{cases} \tag{77}$$

Here the randomness arises form both the diffusion coefficient and the initial data. We assume that the uncertainty in the initial condition is independent from the randomness of the wave speed. The stochastic space is parametrized in terms of 2 independent random variables Z_1, Z_2, affecting respectively the initial condition $q(0)$ and the diffusion coefficient. Specifically we assume $\alpha = (Z_1+0.1)^2$ and $c^2 = 0.1+0.05Z_2$ with Z_1, Z_2 linearly independent and uniformly distributed in $[-1, 1]$. The goal here is to test the symplectic DO method on a problem in which the probability distribution (and consequently the rank) of the exact solution changes over time. We start by rewriting problem (76) in Hamiltonian form:

$$\begin{cases} \dot{\mathbf{u}}(\mathbf{x}, t, \omega) = \mathcal{J}_2 \mathbf{L}(\omega)\mathbf{u}(\mathbf{x}, t, \omega) \\ \mathbf{u}(\mathbf{x}, 0, \omega) = \big((Z_1(\omega)+0.1)^2 q_0(\mathbf{x}), 0\big) \end{cases} \quad \text{with} \quad \mathbf{L} = \begin{bmatrix} -c^2(\omega)\Delta & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \tag{78}$$

Observe that the Hamiltonian explicitly depends on the random variable c^2:

$$H_\omega(q, p, \omega) = \frac{1}{2}\int_D (|p|^2 - c(\omega)^2 |\nabla q|^2).$$

We look for an approximate solution $\mathbf{u}_S \in \mathcal{M}_S^{sym}$ written as:

$$\mathbf{u}_S(\mathbf{x},t,\omega) = \begin{pmatrix} q_S(\mathbf{x},t,\omega) \\ p_S(\mathbf{x},t,\omega) \end{pmatrix} = \sum_{i=1}^{2S} \mathbf{U}_i(\mathbf{x},t)Y_i(\omega,t) = \begin{pmatrix} \sum_{i=1}^{2S} Q_i(\mathbf{x},t)Y_i(t,\omega) - \sum_{i=1}^{2S} P_i(\mathbf{x},t)Y_i(t,\omega) \\ \sum_{i=1}^{2S} P_i(\mathbf{x},t)Y_i(t,\omega) + \sum_{i=1}^{2S} Q_i(\mathbf{x},t)Y_i(t,\omega) \end{pmatrix}, \tag{79}$$

which satisfies

$$\begin{cases} \dot{\mathbf{Y}} = < \mathcal{J}_2 \mathbf{L}\mathbf{U}, \mathbf{U} > \mathbf{Y} & \text{(80a)} \\ \dot{\mathbf{U}}(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S}) = \mathcal{P}_{\mathbf{U}}^{\perp} \left[\mathcal{J}_2 \mathbb{E}[\mathbf{L}\mathbf{U}\mathbf{Y}\mathbf{Y}^T] - \mathbb{E}[\mathbf{L}\mathbf{U}\mathbf{Y}\mathbf{Y}^T]\mathbf{J}_{2S}^T \right] & \text{(80b)} \end{cases}$$

at any time and for some $S \geq 1$. Despite the initial condition is a 1-rank function, the rank of the exact solution is expected to increase in time. Indeed, even if the governing equation is linear, the parameters-to-solution maps is non-linear, due the randomness which affects the differential operator. Thus it is reasonable to expect that the Symplectic DO approximation needs $S > 1$ modes to achieve good levels of accuracy.
We look for a Symplectic DO approximate solution in $\mathcal{M}_S^{sym}$ for $S > 1$, and for the initialization of the modes we adopt the same strategy used in [24,25]; namely the deterministic modes are initialized randomly and the redundant stochastic coefficients are set to zero. Precisely, after setting $\tilde{Q}_1 = q_0$ and $\tilde{Y}_1 = (Z_1 + 0.1)^2$, we initialize $\tilde{Q}_2, ..., \tilde{Q}_S$ randomly with associated stochastic coefficients $\tilde{Y}_2, ..., \tilde{Y}_S$ equal to zero. Then, in order to get a symplectic orthogonal basis, we factorize $\tilde{\mathbf{Q}} = (\tilde{Q}_1, ..., \tilde{Q}_S)$, by using the (real) QR factorization, in $\tilde{\mathbf{Q}} = \mathbf{Q}\mathbf{R}$ and we initialize:

$$\mathbf{U} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix}, \quad \mathbf{Y}_i = \sum_{j=1}^{S} \mathbf{R}_{ij} \tilde{Y}_j \text{ and } \tilde{Y}_{S+i} = 0, \, \forall i = 1, ..., S.$$

Roughly speaking, we use a number of modes larger then what needed to approximate the initial data (although the approximate solution thus constructed has deficient rank), but we evolve in time only the "active" modes (possibly after a suitable rotation of the basis), i.e. those corresponding to non vanishing singular values. The problem of dealing with approximate solutions with deficient rank is however an issue which generally affects the dynamically low rank approximation with fixed rank, at initial and successive times. To deal with it, we implemented two alternative strategies: the first one simply consists in multiplying both sides of (80b) by the pseudo inverse of $(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S})$; the second is based on the complex diagonalization of $(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S})$. Detailing more the second strategy, let $\tilde{\mathbf{C}}$ denotes the sum $(\mathbf{C} + \mathbf{J}_{2S}^T \mathbf{C} \mathbf{J}_{2S})$. Observe that $\tilde{\mathbf{C}}$ satisfies $\tilde{\mathbf{C}} = \mathbf{J}_{2S}^T \tilde{\mathbf{C}} \mathbf{J}_{2S}$, so it can be written as: $\tilde{\mathbf{C}} = \begin{bmatrix} \tilde{\mathbf{C}}_1 & -\tilde{\mathbf{C}}_2 \\ \tilde{\mathbf{C}}_2 & \tilde{\mathbf{C}}_1 \end{bmatrix}$, with $\tilde{\mathbf{C}}_1$, $\tilde{\mathbf{C}}_2 \in \mathbb{R}^{S \times S}$. This means that $\tilde{\mathbf{C}}$ can be identified by the complex hermitian matrix $\hat{\mathbf{C}} = \tilde{\mathbf{C}}_1 + i\tilde{\mathbf{C}}_2 \in \mathbb{C}^{S \times S}$. Let $\hat{\mathbf{D}}$, $\hat{\mathbf{V}}$ be respectively the (complex) eigenvalues and eigenvectors of $\hat{\mathbf{C}}$, and $\mathbf{V}$ the real matrix representation of $\hat{\mathbf{V}}$, i.e. $\mathbf{V} = \begin{bmatrix} \mathrm{Re}(\hat{\mathbf{V}}) & -\mathrm{Im}(\hat{\mathbf{V}}) \\ \mathrm{Im}(\hat{\mathbf{V}}) & \mathrm{Re}(\hat{\mathbf{V}}) \end{bmatrix}$.
We define $\tilde{\mathbf{U}}_i = (\tilde{U}_i^Q, \tilde{U}_i^P)^T = \sum_{j=1}^{2S} \mathbf{U}_j \mathbf{V}_{ji}$ and we rewrite equations (80b) with respect to the rotated basis $\tilde{\mathbf{U}}$ (by neglecting the time variation on $\mathbf{V}$). Observe that the complex diagonalization guarantees that the rotated basis $\tilde{\mathbf{U}}$ belongs to $U(S, [H^1(D)]^2)$, since the product of symplectic orthogonal matrices is as well symplectic orthogonal. Then we actually solve only the equations corresponding to not vanish eigenvalues, i.e. the equations in $\dot{\tilde{\mathbf{U}}}_i$ for which $\mathbf{D}_{ii}$ (which is real) is larger then a prescribed tolerance, for any $i = 1, ..., S$. Denoting by r the rank of $\mathbf{D}$, the remaining modes $\tilde{\mathbf{U}}_{r+1}, ..., \tilde{\mathbf{U}}_S$ are kept constant to the previous time iteration. Finally, by exploiting the unitary structure in (29), we reconstruct the complete basis as:

$$\tilde{\mathbf{U}} = \begin{bmatrix} \tilde{\mathbf{U}}^Q & -\tilde{\mathbf{U}}^P \\ \tilde{\mathbf{U}}^P & \tilde{\mathbf{U}}^Q \end{bmatrix} \tag{81}$$

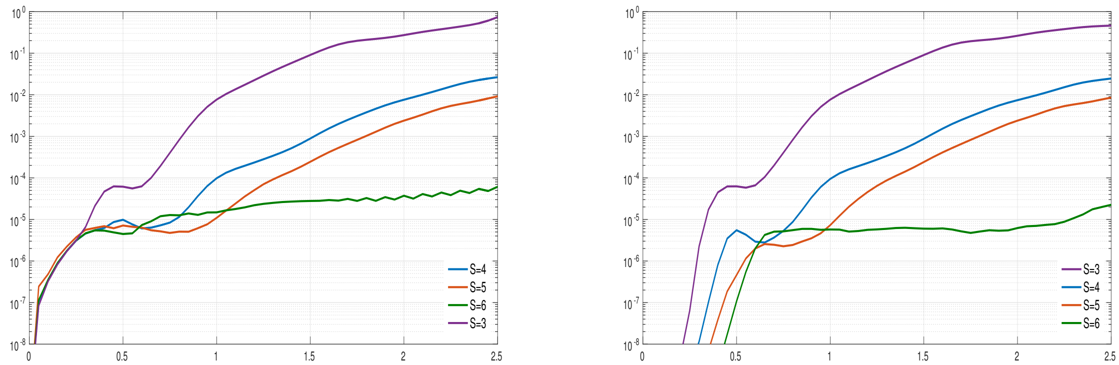

Figure 4: Evolution in time of the approximation error of the Symplectic DO method with different number of modes (S=3,4,5,6). The error is computed in norm $H^1(D) \otimes L^2(\Omega)$ with respect to a reference solution computed with the Stochastic Collocation method. On the left the approximation error with the Strang splitting combined with the symplectic Euler scheme. On the right the Lie-Trotter splitting combined with the implicit midpoint scheme (right). Discretization parameters: stochastic tensor grid with Gauss-Legendre collocation points, number of points: $N_y = 49$, spatial discretization: triangular mesh with edge $h = 0.04$, uniform time-step $\Delta t = 0.001$.

and we get the updated modes in the original coordinates by multiplying by $\mathbf{V}^T$. Despite the two strategies lead to comparable numerical results, the technique based on the complex diagonalization, has the computational advantage of solving the minimum number of equations required. In practice, in the results reported here, the rank r is computed with respect to a threshold ϵ that is weighted by the largest eigenvalue of $\mathbf{D}$ at each time, specifically we set the threshold equal to $\epsilon = 10^{-15} \underset{i=1,\ldots,S}{max} D^n_{ii}$ at any $t^n = n\Delta t$.

7.4 Numerical Discretization

The implementation of all numerical tests in this Section has been developed within the open source Finite Element library FEniCs [2]. The Finite Element method is used for the discretization in the physical space, namely for solving (80b) and for computing the $L^2(D)$-projection in (80a). Specifically we use $P1$ finite elements on a uniform triangular grid of equal edges $h = 0.04$. For what concerns the discretization of the random modes, we parametrize the stochastic space in terms of a uniformly distributed random vector $\boldsymbol{\eta}$, in accordance with the distribution of the input random data. Thus the stochastic space $(\Omega, \mathcal{A}, \mathcal{P})$ is replaced by $(\boldsymbol{\Lambda}, \mathcal{B}(\boldsymbol{\Lambda}), f(\boldsymbol{\eta})d\boldsymbol{\eta})$ where here $\boldsymbol{\Lambda} = [-1, 1]^2$, $\mathcal{B}(\boldsymbol{\Lambda})$ and $f = \frac{1}{4}$ denote respectively the domain, the Borel σ-algebra and the density function of $\boldsymbol{\eta}$. Then, equations (80a) are solved with the Stochastic Collocation method on Gauss-Legendre collocation points with tensorized Gaussian grid [3]. The corresponding quadrature formula is used to compute the covariance matrix and any expected value in (80b). However, the use of *sparse* stochastic collocation grids is recommended for problems in higher dimensional stochastic spaces. For details see e.g. [3, 26, 36].
The time discretization scheme has to be carefully chosen in order to preserve the symplectic structure of the problem. For a complete review of symplectic schemes we refer to [10] and references therein. Moreover, since numerical symplectic schemes do not necessarily preserve the orthogonal structure (29) at the discrete level, especially for approximate solutions with deficient rank, special attention has been paid to preserve both the orthogonal and symplectic structure of the deterministic modes. We propose two possible time discretization strategies, described hereafter, both finalized to preserve the symplecticity of the flow and guarantee the orthogonality of the deterministic basis. Based on the linear reversibility of wave equations, which states that the time reversed solution of a wave equation is also solution to the same wave equation, we

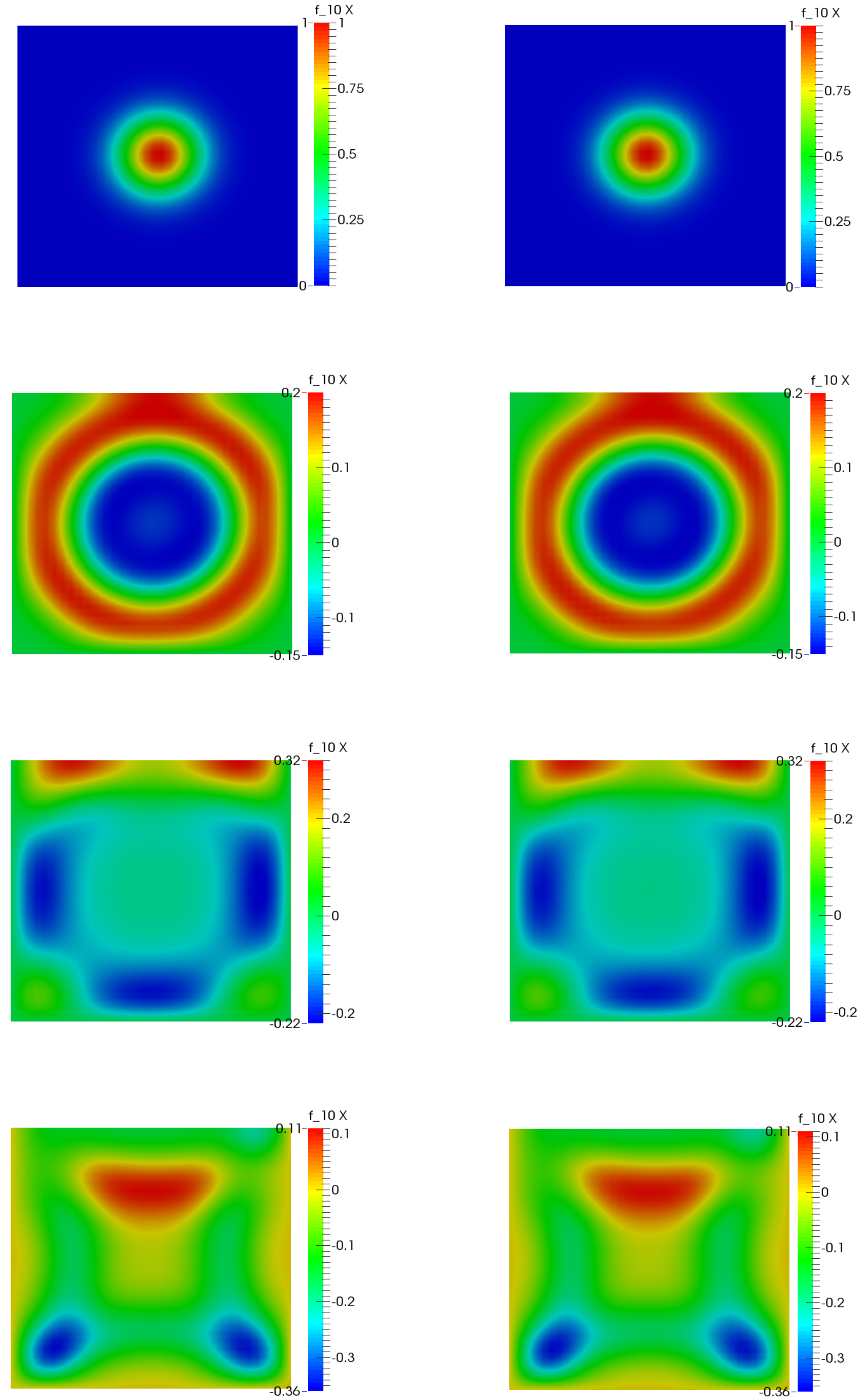

Figure 5: Reference solution (left) and Symplectic DO approximate solution with $S = 5$ (right) for $\alpha = 1$ and $c^2 = 0.121$ at $t = 0$, $t = 1$, $t = 1.5$, and $t = 2$. Discretization parameters: stochastic tensor grid with Gauss-Legendre collocation points, number of points: $N_y = 49$, spatial discretization: $P1$ finite elements over a triangular mesh with edge $h = 0.04$, uniform time-step $\Delta t = 0.001$ (with implicit midpoint scheme).

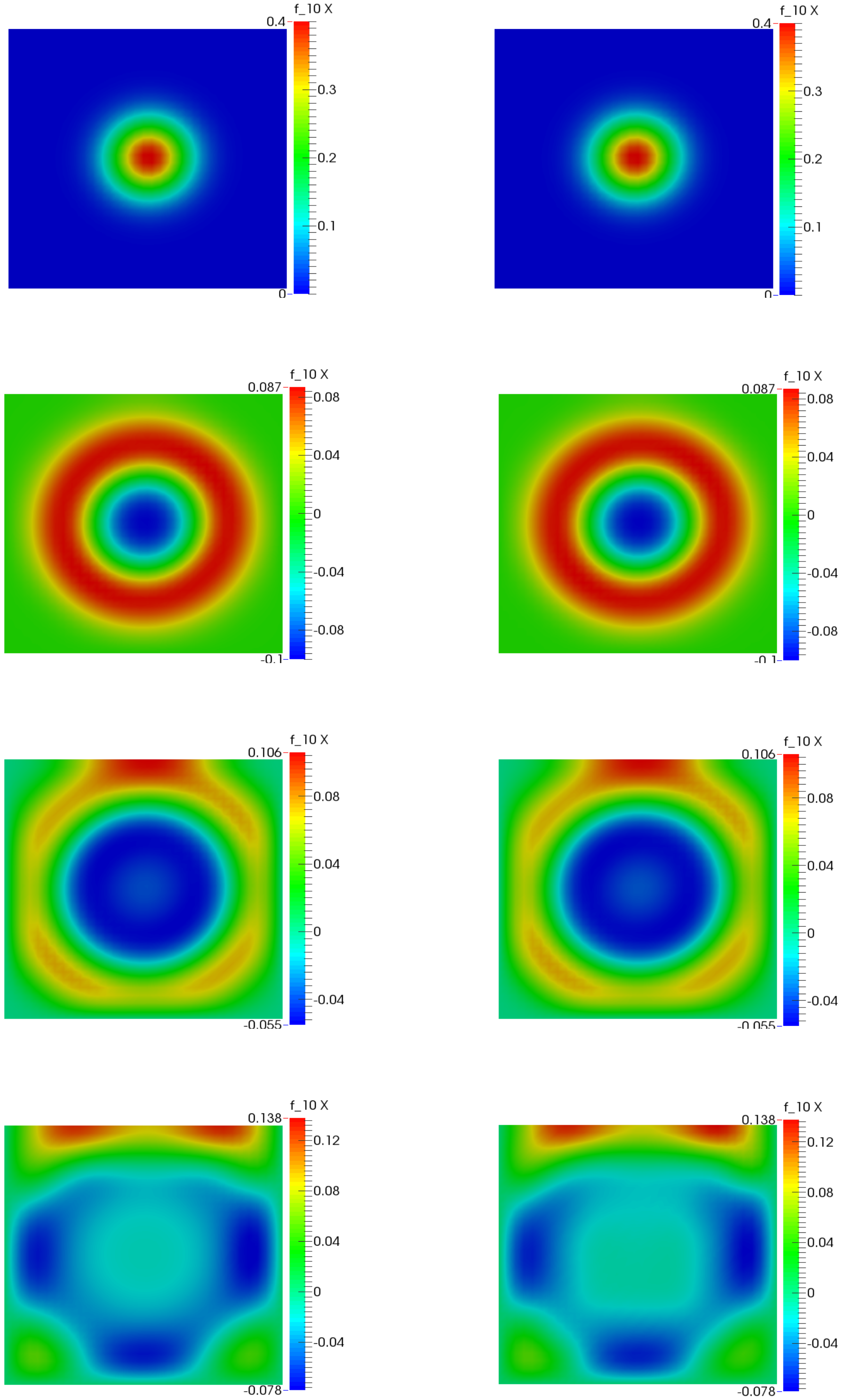

Figure 6: Reference solution (left) and Symplectic DO approximate solution with $S = 5$ (right) for $\alpha = 0.4$ and $c^2 = 0.063$ at $t = 0$, $t = 1$, $t = 1.5$, and $t = 2$. Discretization parameters: stochastic tensor grid with Gauss-Legendre collocation points, number of points: $N_y = 49$, spatial discretization: triangular mesh with edge $h = 0.04$, uniform time-step $\Delta t = 0.001$ (with implicit midpoint scheme).

look for a numerical scheme which, when applied to a reversible differential equation, produces a reversible numerical flow, in order to get a consistent long-time behavior. Based on the link between reversibility and symmetric schemes [10], we propose two possible symplectic time discretization methods based respectively on a symmetric splitting and on the implicit midpoint rule (which is a symmetric scheme). The two procedures can be summarized as follows:

- Strang splitting in $\mathbf{U}, \mathbf{Y}$ combined with the symplectic Euler scheme. Starting from $\mathbf{u}_S^n = \mathbf{U}^n \mathbf{Y}^n$ at $t = t^n$:
  - we compute $\mathbf{Y}^{n+1/2} \approx \mathbf{Y}(t^n + \frac{\Delta t}{2})$ by solving system (80a) discretized in time with the Symplectic Euler scheme for half time step;
  - we compute $\mathbf{U}^{n+1} \approx \mathbf{U}(t^n + \Delta t)$ by solving system (80b) with the Symplectic Euler scheme and the updated coefficients $\mathbf{Y}^{n+1/2}$;
  - we re-orthogonalize $\mathbf{U}^{n+1}$ by using the complex QR factorization;
  - we compute $\mathbf{Y}^{n+1} \approx \mathbf{Y}(t^n + \Delta t)$ by solving system (80a) for half time step, with initial values $\mathbf{Y}^{n+1/2}$ and the updated deterministic basis. The equations are discretized by the adjoint Symplectic Euler scheme with respect to the one used in the first half-step.
- Standard Lie-Trotter splitting in $\mathbf{U}, \mathbf{Y}$ combined with the implicit midpoint scheme for the time discretization of both system (80a) and system (80b). We apply the complex diagonalization strategy to (80b) and we denote by $\mathbf{u}_S^n = \mathbf{U}^n \mathbf{Y}^n = \tilde{\mathbf{U}}^n \tilde{\mathbf{Y}}^n$ the approximate solution at time $t^n = n\Delta t$ in standard and rotated bases respectively. Equations (80a)-(80b) are discretized in time as follows:

$$
\begin{aligned}
&\tfrac{1}{\Delta t}\mathbf{Y}^{n+1} - \tfrac{1}{2} < \mathcal{J}_2 \mathbf{L}\mathbf{U}^n, \mathbf{U}^n > \mathbf{Y}^{n+1} = \tfrac{1}{\Delta t}\mathbf{Y}^n + \tfrac{1}{2} < \mathcal{J}_2 \mathbf{L}\mathbf{U}^n, \mathbf{U}^n > \mathbf{Y}^n \\
&\tfrac{1}{\Delta t}\tilde{\mathbf{U}}_i^{n+1}\hat{\mathbf{D}}_{ii}^{n+1} - \tfrac{1}{2}\mathcal{P}^{\perp}_{\mathbf{U}^n}\left[\mathcal{J}_2\mathbb{E}[\mathbf{L}\tilde{\mathbf{U}}_i^{n+1}\tilde{\mathbf{Y}}_i^{n+1}\tilde{\mathbf{Y}}_i^{n+1}] + \mathbb{E}[\mathbf{L}\tilde{\mathbf{U}}_i^{n+1}\tilde{\mathbf{Y}}_i^{n+1}\tilde{\mathbf{Y}}_{S+i}^{n+1}]\right] = \tfrac{1}{\Delta t}\tilde{\mathbf{U}}_i^n\hat{\mathbf{D}}_{ii}^{n+1} \\
&\qquad + \tfrac{1}{2}\mathcal{P}^{\perp}_{\mathbf{U}^n}\left[\mathcal{J}_2\mathbb{E}[\mathbf{L}\tilde{\mathbf{U}}_i^n\tilde{\mathbf{Y}}_i^{n+1}\tilde{\mathbf{Y}}_i^{n+1}] + \mathbb{E}[\mathbf{L}\tilde{\mathbf{U}}_i^n\tilde{\mathbf{Y}}_i^{n+1}\tilde{\mathbf{Y}}_{S+i}^{n+1}]\right] \\
&\qquad + \mathcal{P}^{\perp}_{\mathbf{U}^n}\left[\mathcal{J}_2\mathbb{E}[\sum_{\substack{j=1\\ j\neq i}}^{2S}\mathbf{L}\tilde{\mathbf{U}}_j^n\tilde{\mathbf{Y}}_j^{n+1}\tilde{\mathbf{Y}}_i^{n+1}] + \mathbb{E}[\sum_{\substack{k=1\\ k\neq S+i}}^{2S}\sum_{j=1}^{S}\mathbf{L}\tilde{\mathbf{U}}_j^n\tilde{\mathbf{Y}}_j^{n+1}\tilde{\mathbf{Y}}_k^{n+1}(\mathbf{J}_{2S})_{ki}]\right]
\end{aligned} \tag{82}
$$

  Observe that the time discretization in (82) allows decoupling the equations in $\mathbf{U}_1, ..., \mathbf{U}_S$ which are then solved separately. Concerning the re-orthogonalization of the deterministic modes in the second strategy, we recall that the midpoint rule has the convenient property of conserving quadratic invariants and in particular the implicit midpoint scheme is a unitary integrator [9]. We numerically observe that the implicit midpoint scheme helps in preserving the symplectic orthogonal structure of the deterministic basis, thus reducing the number of the (computationally expensive) re-orthogonalizations. However, we emphasize that the midpoint scheme proposed here does not preserve exactly the unitary structure of $\mathbf{U}$ and reorthogonalization is still needed for approximate solutions with deficient rank. In particular, the unitary structure is slightly compromised by the explicit treatment of the coupling terms in (82). However for the problem under consideration, this scheme allows us to apply a complex QR re-orthogonalization only in the very first time steps, when the solution has deficient rank, and then around every 100 iterations (one possibility is to apply the complex QR decomposition only when the error in the orthonormalization of $\mathbf{U}$ is larger then a prescribed tolerance). Figure 4 shows the time evolution of the approximation error of the Symplectic DO approximate solution, implemented with the Strang splitting combined with Symplectic Euler method on the left, and with the Lie-Trotter splitting combined with the midpoint scheme on the right, with different numbers of modes. The error is computed in norm $H^1(D) \otimes L^2(\Omega)$ with respect to a reference solution computed with the Stochastic Collocation method on Gauss-Legendre points using 7 points in each direction and the same discretization parameters in time and space, i.e. a triangular mesh with edge $h = 0.04$ and uniform time-step $\Delta t = 0.001$. We observe that a good level of accuracy can be reached with only a few modes and in particular, for $S = 6$ the magnitude of error tends to stay constant in time and lower than 10^{-4}. Despite the two strategies lead to comparable numerical results, we point out that the second one is generally computationally more efficient since a smaller number of re-orthogonalizations is required. We conclude by reporting here same qualitative results to show the effectiveness of the Symplectic DO method in reproducing the exact flow of the solutions. In Figure 5 and Figure 6 we compare the exact and the approximate solution with $S = 5$, evaluated in $\alpha = 1$, $c^2 = 0.121$ and $\alpha = 0.4$, $c^2 = 0.063$ respectively, at different times. One can see that, even if the two realizations (i.e. for $\alpha = 1$, $c^2 = 0.121$ and $\alpha = 0.4$, $c^2 = 0.063$) are quite different, the Symplectic DO is able to effectively reproduce both of them at the same time.

7.5 Conservation of energy

We consider again a linear wave equation with random speed and random initial data as in Section 7.3, but in the 1-dimensional domain $[0, 1]$. The goal here is to validate that the expected value of the Hamiltonian is conserved along the DO symplectic approximate solution, as theoretically derived in Lemma 6.4, and verify how the numerical scheme may affect this result. The problem reads as in (74), where here the wave speed is random, i.e. $c^2(\omega) = 0.06 + 0.02Z_2(\omega)$ with Z_2 uniformly distributed in $[-1, 1]$. The initial condition is given by $(q(x, 0, \omega), \dot{q}(x, 0, \omega)) = ((Z_1(\omega) + 2)^2 h(10 \times |x - 0.5|), 0)$ with h defined in (75) and Z_1, Z_2 independent and identically uniform distributed. We derive the symplectic DO reduced system (68a)-(68b). Observe that the equations for the deterministic modes are all coupled. To verify the conservation of energy we first discretize the symplectic DO reduced system by applying the Lie-Trotter splitting in $\mathbf{U}$, $\mathbf{Y}$ combined with the implicit midpoint scheme for the discretization in time but, differently to what done in the previous examples, we solve the system of S deterministic PDEs without decoupling the equations in $\mathbf{U}$. Figure 7 (left) shows that the expected value of the Hamiltonian is exactly conserved, up to machine precision. Observe that, since there is no truncation error in the initial datum, the conservation of energy does not depend on the number of modes. Secondly, we apply the discretization scheme described in (82), which allows to decouple the equations for the deterministic modes and considerably reduce the computational cost. Figure 7 (left) shows that the conservation of the expected value of the Hamiltanian is slightly affected by the discretization and it also depends on the number of modes. In Figure 7 (right) we compare the approximation errors of the two discretization schemes with number of modes $S = 4$ and $S = 6$. As expected, better level of accuracy can be reached without decoupling the system. However for large scale problems this leads to a very large computational cost. On the other hand, the discretization scheme (82) is substantially cheaper and still able to reach a good level of accuracy. Moreover the conservation of energy may be considered as an indicator in the quality of the approximation and in the number of modes to use.

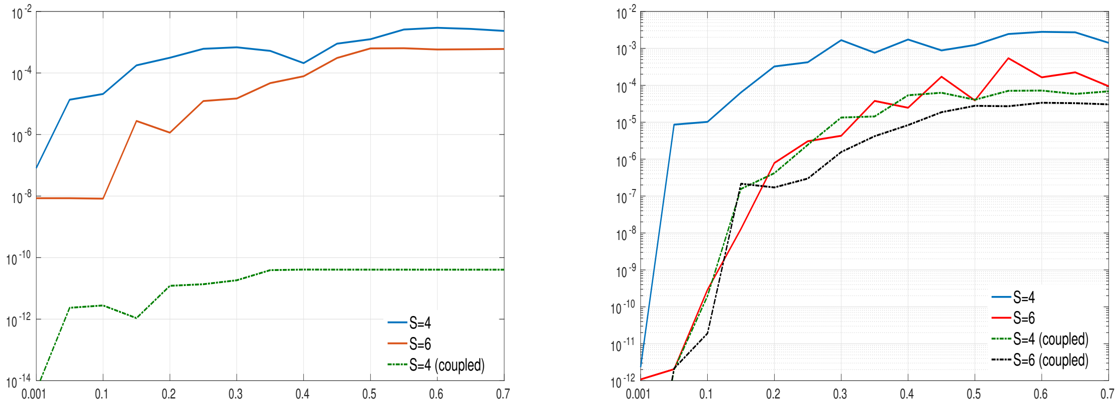

Figure 7: Left: Evolution in time of error in the expected value of the Hamiltonian. The dashed (solid) line corresponds to approximation error obtained by solving the coupled (decoupled) system for the evolution of the deterministic modes. Right: Evolution in time of the approximation error of the Symplectic DO method with number of modes $S = 4$ and $S = 6$. The dashed (solid) lines correspond to approximation error obtained by solving the coupled (decoupled) system of the evolution of the deterministic modes. The error is computed in norm $H^1(D) \otimes L^2(\Omega)$ with respect to a reference solution computed with the Stochastic Collocation method. Discretization parameters: stochastic anisotropic tensor grid with Gauss-Legendre collocation points, 7 in the first and 13 in the second dimension, spatial discretization: triangular mesh with edge $h = 0.02$, uniform time-step $\Delta t = 0.001$.

8 Conclusion

In this paper, we developed a dynamical low-rank technique for the approximation of wave equations with random parameters, which combines the DLR approach with the use of symplectic deterministic (dynamic) bases. The governing equation is rewritten in the Hamiltonian form in a suitable symplectic space, and the approximate solution is sought in the set of all random fields which can be expanded, in separable form, over a symplectic-orthogonal deterministic basis of dimension $2S$. After deriving the proper conditions on the stochastic coefficients to equip this set, denoted by $\mathcal{M}_S^{sym}$, with a manifold structure, we formulated the Symplectic DLR variational principle as the symplectic projection of the Hamiltonian system onto the tangent space to $\mathcal{M}_S^{sym}$ along the approximate trajectory. We showed that this coincides with rewriting the governing Hamiltonian system in complex variables and looking for a DLR approximation in the manifold $\mathcal{M}_S^{\mathbb{C}}$ of all the complex-valued random fields with rank S. We used the analogy between the complex manifold $\mathcal{M}_S^{\mathbb{C}}$ and its real representation $\mathcal{M}_S^{sym}$ to determine a suitable parametrization of the tangent space to $\mathcal{M}_S^{sym}$ (in real form). After deriving the associated orthogonal constraints on the dynamics of the deterministic modes, we recovered the reduced dynamical system which, in the real framework, consists of a set of equations for the constrained dynamics of the deterministic modes, coupled with a reduced order Hamiltonian system for the evolution of the stochastic coefficients. The Symplectic DO shares with the symplectic order reduction the use of symplectic deterministic bases, and, as the “classic” DO approximation, allows both the stochastic and the deterministic modes to evolve in time. As a result, the approximate solution preserves the (approximated) mean Hamiltonian energy and continuously adapts in time to the structure of the solution. Open question remains instead the well-posedness of the Symplectic DLR problem. To prove the existence and uniqueness of the approximate solution a possible strategy to be investigated, may consist in exploiting the conservation of energy. Envisaged future investigations concern also the generalization of the Symplectic DLR approach to dynamical low-rank approximations with arbitrary (not necessarily orthonormal) symplectic bases.

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