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F. Bonizzoni, F. Nobile

http://mathicse.epfl.ch

Phone: +41 21 69 37648
Address:
EPFL - SB - MATHICSE (Bâtiment MA)
Station 8 - CH-1015 - Lausanne - Switzerland
Fax: +41 21 69 32545

Perturbation analysis for the Darcy problem with log-normal permeability *

Francesca Bonizzoni [♯,†], Fabio Nobile [♯,†]

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[♯] MOX– Modellistica e Calcolo Scientifico
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano
via Bonardi 9, 20133 Milano, Italy.
[†] CSQI – Calcul Scientifique et Quantification de l'Incertitude
MATHICSE, École Polytechnique Fédérale de Lausanne
Bâtiment des Mathématiques, Station 8, CH-1015 Lausanne, Switzerland.
`francesca.bonizzoni@polimi.it, fabio.nobile@epfl.ch`

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Abstract

We study the single-phase flow in a saturated, bounded heterogeneous porous medium. We model the permeability as a log-normal random field. We perform a perturbation analysis, expanding the solution in Taylor series. The series is directly computable in the case of a random field parametrized by a finite number of random variables. On the other hand, in the case of an infinite dimensional random field, suitable equations satisfied by the derivatives of the stochastic solution can be derived. We give a theoretical upper bound for the norm of the residual of the Taylor expansion which predicts the divergence of the series as the polynomial degree goes to infinity. We provide a formula to compute the optimal degree for the Taylor polynomial and the maximum achievable accuracy of the perturbation approach. Our theoretical findings are confirmed by numerical experiments in the simple case where the permeability field is described using one random variable.

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1 Introduction

In many mathematical models, the input parameters are affected by uncertainty, which may be due to the incomplete knowledge or the intrinsic variability of certain phenomena. Some illustrative examples are flows in porous media, combustion problems, earthquake engineering, biomedical engineering, finance, etc.

Starting from a suitable Partial Differential Equation (PDE) model, we describe the uncertain parameters as random variables or random fields. The aim of the *Uncertainty Quantification* is to infer on the solution of the Stochastic PDE (SPDE) by computing the statistics of the solution or of functionals of it.

The situation we are interested in is the study of single-phase flow of a fluid in a bounded heterogeneous saturated porous medium. Randomness typically arises in the forcing terms, as for instance pressure gradients, (see e.g. [15, 38, 39, 44, 11]), as well as in the permeability tensor, due to the impossibility of a full characterization of conductivity properties of subsurface media (see e.g. [46, 40, 26, 27, 19, 8]). In this work, we focus on the following linear elliptic SPDE posed in the bounded domain $D \subset \mathbb{R}^d$

$$-\operatorname{div}(a(\omega, x)\nabla u(\omega, x)) = f(x), \quad x \in D \tag{1}$$

where $u(\omega, x)$ represents the hydraulic head, the forcing term $f(x)$ is deterministic and the permeability tensor $a(\omega, x)$ is modeled as a *log-normal random field*, i.e. $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$ with $Y(\omega, x)$ a Gaussian random field. Here ω represents a random elementary event. Note that in (1) the differential operators refer to the spatial variable $x \in D$. The log-normal model is widely used in geophysical applications: see e.g. [8, 19, 26, 27, 40]. In recent years, it has appeared also in the mathematical literature [12, 13, 21, 24]. Given complete statistical information on the permeability field $a(\omega, x)$, the aim of this work is to infer on the statistical moments of the stochastic solution $u(\omega, x)$.

The Monte Carlo sampling method is the easiest way to compute the statistics of $u(\omega, x)$. It features a rate of convergence independent of the dimension of the probability space. On the other hand, it does not exploit any regularity of the solution and the rate is of the order of $M^{-1/2}$, M being the number of independent realizations, so that a large number of realizations has to be considered to reach a satisfactory accuracy. In recent years, a number of improvements have been proposed and applied to SPDEs. Between them, we recall the Multilevel Monte Carlo method [7, 16, 41] and the Quasi Monte Carlo method [25, 29].

The generalized Polynomial Chaos Expansion of the stochastic solution gives rise to a second family of methods. It can be coupled with a projection strategy (Stochastic Galerkin method [6, 20, 23, 24, 33, 38, 43]) or an interpolation strategy (Stochastic Collocation method [5, 22, 34, 35, 45]). These approaches strongly exploit the regularity of the solution in the random variables, but can not handle high dimensional probability spaces.

In this work, we address the case of small randomness and consider a *perturbation approach*, alternative to Monte Carlo sampling or Polynomial Chaos

Expansion, based on the Taylor expansion of the solution u with respect to the Gaussian random field Y. Perturbation approaches have been widely used in the literature. In the context of perturbations with respect to random fields (infinite-dimensional parameter space), we mention for instance the work [28], which considers equation (1) in a random domain, the contributions [26, 27, 40] from the hydrology literature, where log-normal random models for the permeability field are considered, and [17] which addresses problem (1) with a permeability field described as a linear combination of countably many uniform random variables.

In the literature, whenever an infinite-dimensional random field is considered, the majority of the authors compute only a second order correction to the mean and variance of the stochastic solution. The aim of the present work, concerning the log-normal model, is to understand if it is reasonable to compute higher order corrections and to investigate the approximation properties of the Taylor polynomial.

The main achievements of the work are the following: we derive an upper bound on the norm of the Taylor residual; we predict the divergence of the Taylor series and the existence of an optimal degree K_{opt} of the Taylor polynomial such that, adding new terms to the Taylor polynomial will deteriorate the accuracy instead of improving it; we provide an explicit formula for the optimal degree K_{opt} for the computation of the expected value of the solution. Our theoretical results are confirmed by some numerical tests performed in a one dimensional setting with a permeability field parametrized by a single random variable.

We underline that the divergence of the Taylor series strongly depends on the chosen log-normal model. Indeed, in [4] (see also [42] the authors show that, if the permeability field is described as a finite linear combination of bounded random variables, then the Taylor series converges.

The outline of the paper is the following. Section 2 introduces the problem at hand and states some results on the statistical moments of the L^∞-norm of a sufficiently smooth Gaussian random field extending the results in [12]. In Section 3 we expand the stochastic solution of the SPDE in Taylor series, provide bounds on the norm of both the Taylor polynomial and the Taylor residual, and predict the divergence of the Taylor series. In Section 4 we state the existence and provide a formula to compute the optimal degree of the Taylor polynomial such that, adding new terms to the Taylor polynomial will deteriorate the accuracy instead of improving it. Finally, Section 5 is focused on some numerical tests in a one-dimensional case which confirm the divergence of the Taylor series predicted in Section 3.

2 Problem setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where Ω is the set of outcomes, $\mathcal{F}$ the σ-algebra of events and $\mathbb{P} : \Omega \to [0, 1]$ a probability measure. Let D be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with locally Lipschitz boundary. We are interested in the Darcy boundary value problem with stochastic permeability: given $f \in L^2(D)$ and $g \in H^{1/2}(\Gamma_D)$, find $u \in L^p\left(\Omega; H^1(D)\right)$ s.t. $u|_{\Gamma_D} = g$, and

$$\int_D a(\omega, x)\nabla u(\omega, x) \cdot \nabla v(x)\, dx = \int_D f(x)v(x)\, dx \quad \forall v \in H^1_{\Gamma_D}(D), \text{ a.s. in } \Omega \quad (2)$$

where $\{\Gamma_D,\ \Gamma_N\}$ is a partition of the boundary of the domain ∂D, and homogeneous Neumann boundary conditions are imposed on Γ_N. We denoted with $H^1_{\Gamma_D}(D)$ the subspace of $H^1(D)$ of functions whose trace vanishes on Γ_D, and with $L^p\left(\Omega; H^1(D)\right)$ the Bochner space of functions $v(\omega, x)$ such that $\|v\|_{L^p(\Omega;H^1)} := \left(\int_\Omega \|v(\omega)\|^p_{H^1}\, \mathrm{d}\mathbb{P}(\omega)\right)^{1/p} < \infty$.

We describe the permeability field as a log-normal random field $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, where $Y : \Omega \times \bar{D} \to \mathbb{R}$ is a Gaussian random field. The log-normal model is frequently used in geophysical applications: see for example [46, 19, 8, 26, 27, 40]. Let us define the mean-free Gaussian random field $Y'(\omega, x) = Y(\omega, x) - \mathbb{E}[Y](x)$, and assume that its covariance kernel $Cov_{Y'} : \overline{D \times D} \to \mathbb{R}$ is Hölder continuous with exponent t for some $0 < t \leq 1$. In [10] the following proposition is proved, which extends the result in [12] obtained only for centered second order *stationary* random fields Y with covariance function:

$$Cov_Y(x_1, x_2) = \nu(\|x_1 - x_2\|)$$

for some $\nu \in C^{0,1}(\mathbb{R}^+)$.

Proposition 2.1 *Let $Y : \Omega \times \bar{D} \to \mathbb{R}$ be a Gaussian random field with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$ for some $0 < t \leq 1$. Suppose $\mathbb{E}[Y] \in \mathcal{C}^{0,t/2}(\bar{D})$. Then there exists a version of Y whose trajectories belong to $\mathcal{C}^{0,\alpha}(\bar{D})$ a.s. for $0 < \alpha < t/2$.*

In what follows, we identify the Hölder regular version of the field with $Y(\omega, x)$, so that $\|Y(\omega)\|_{L^\infty(D)}$, $a_{min}(\omega) := \min_{x \in \bar{D}} a(\omega, x)$ and $a_{max}(\omega) := \max_{x \in \bar{D}} a(\omega, x)$ are well-defined random variables. Using the Fernique's theorem (see e.g. [18]), in [12] the author shows that

$$\frac{1}{a_{min}(\omega)} \in L^p(\Omega, \mathbb{P}), \quad a_{max}(\omega) \in L^p(\Omega, \mathbb{P}), \quad \forall\ 0 < p < +\infty.$$

The well-posedness of problem (2) follows from the Lax Milgram Lemma applied for almost all $\omega \in \Omega$ and the L^p integrability of $\frac{1}{a_{min}(\omega)}$. See [21, 24, 12].

Remark 2.1 *From the point of view of applications it is very interesting to study also the case of a random field conditioned to available observations. Take for example the fluid flow in a heterogeneous porous medium: the permeability varies randomly, and can be measured only in a certain number of spatial points. Assuming that N_{oss} point-wise measurements of the permeability have been collected* *(e.g. by exploratory wells), one can construct a conditioned random field Y whose covariance function is non-stationary, but still Hölder continuous with the same exponent, so that Proposition 2.1 holds.*

2.1 Upper bounds for the statistical moments of $\|Y'\|_{L^\infty(D)}$

Let us denote by $\sigma^2 := \dfrac{1}{|D|}\displaystyle\int_D \mathbb{V}ar\,[Y(\cdot,x)]\;dx$. If $Y(\omega,x)$ is a stationary field, then its variance is independent of $x \in D$ and coincides with σ^2. By a little abuse of notation, in what follows we will refer to σ^2 as the variance of Y also in the case of a non-stationary random field.

Let us start from the Karhunen-Loève expansion of the Gaussian random field $Y(\omega,x)$ (see e.g. [31, 32]):

$$Y(\omega,x) = \mathbb{E}\,[Y]\,(x) + \sigma \sum_{j=1}^{+\infty} \sqrt{\lambda_j}\;\phi_j(x)\;\xi_j(\omega), \quad (\omega,x) \in \Omega \times D, \tag{3}$$

where $\{\lambda_j\}_{j\geq 1}$ is the decreasing sequence of non-negative eigenvalues of the operator $L^2(D) \ni v \mapsto \displaystyle\int_D Cov_Y(x_1,x_2)v(x_2)\;dx_2 \in L^2(D)$, $\{\phi_j(x)\}_{j\geq 1}$ are the corresponding eigenfunctions, which form an orthonormal basis for $L^2(D)$, and $\{\xi_j(\omega)\}_{j\geq 1}$ are the centered independent Gaussian random variables with unit variance defined as $\xi_j(\omega) = \dfrac{1}{\sqrt{\lambda_j}}\displaystyle\int_D \left(Y(\omega,x) - \mathbb{E}\,[Y]\,(x)\right)\phi_j(x)\;dx$. Under the assumption $R_\gamma := \displaystyle\sum_{j=1}^{+\infty} \lambda_j\;\|\phi_j\|^2_{\mathcal{C}^{0,\gamma}(\bar{D})} < +\infty$, in [13] the author shows that

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq C_{Y'}\;R_\gamma^{k/2}\sigma^k\;(k-1)!!, \quad \forall\; k > 0 \text{ integer}, \tag{4}$$

where $C_{Y'}$ is a positive constant independent of σ.

An estimate of the type (4) can also be obtained with the *Euler characteristic heuristic* method proposed in [1] and further analyzed in [14], which, however, is valid only for smooth fields:

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq \widetilde{C}_{Y'}\;\sigma^{k-2}\;k\;(k-1)!!, \quad \forall\; k > 0 \text{ integer}, \tag{5}$$

where $\widetilde{C}_{Y'}$ is a positive constant independent of k and σ. We refer to [10] for the proof of (5) in the case of a field defined on a d-dimensional rectangle $D = [0,T]^d$.

The bound (5) is weaker than (4) as it predicts a scaling σ^{k-2} instead of σ^k for the k-th moment of the random variable $\|Y'\|_{L^\infty(D)}$. On the other hand, the bound (4) involves the exponential term $R_\gamma^{k/2}$ where R_γ depends on the covariance function of the random field.

To lighten the notations, in the rest of the paper we assume the Gaussian random field $Y(\omega,x)$ to be centered.

3 Perturbation analysis in the infinite dimensional case

Thanks to the Doob-Dynkin Lemma [36], the solution u of problem (2) is a function of Y: $u = u(Y, x)$. In this section, under the assumption of small standard deviation of Y, we perform a perturbation analysis based on the Taylor expansion of the solution u in a neighborhood of the zero-mean of Y and we study the approximation properties of the Taylor polynomial of u. We exhibit upper bounds for the norm of the K-th degree Taylor polynomial $T^K u$ and residual $R^K = u - T^K u$. The divergence of the Taylor series *for any* $\sigma > 0$ is predicted.

3.1 Taylor expansion

Let $0 < \sigma < 1$ be the standard deviation of the centered Gaussian random field $Y(\omega, x)$. Given a function $u(Y) : L^\infty(D) \to H^1(D)$ which is $(K+1)$-times Gateaux differentiable, we denote its k-th $(0 \leq k \leq K+1)$ Gateaux derivative in $\bar{Y} \in L^\infty(D)$ evaluated in the vector $\underbrace{(Y, \dots, Y)}_{k \text{ times}}$ as $D^k u(\bar{Y})[Y]^k$. The K-th order Taylor polynomial of u centered in 0 is:

$$T^K u(Y, x) := \sum_{k=0}^{K} \frac{D^k u(0)[Y]^k}{k!}, \quad K \geq 1 \tag{6}$$

where $D^0 u(0)[Y]^0 := u^0(x)$ is independent of the random field Y. The K-th order residual of the Taylor expansion $R^K u(Y, x) := u(Y, x) - T^K u(Y, x)$ can be expressed as

$$R^K u(Y, x) := \frac{1}{K!} \int_0^1 (1-t)^K D^{K+1} u(tY)[Y]^{K+1} dt. \tag{7}$$

See for example [3, 2].

In the case where Y is a finite dimensional random vector $Y = (Y_1, \dots, Y_N)$ the Taylor polynomial can be explicitly computed, so that we can approximate u with $T^K u$ and the statistics of u with the statistics of $T^K u$. This situation can be achieved for example by truncating a Karhunen-Loève or Fourier expansion of Y. On the other hand, in the infinite-dimensional setting, it is possible to derive deterministic recursive equations solved by the increasing order corrections of the statistical moments of $T^K u$. See [10]. For example, for the computation of the expected value of u, one can write deterministic recursive problems for the k-th order term $\mathbb{E}\left[D^k u(0)[Y]^k\right]$ and approximate $\mathbb{E}[u]$ as

$$\mathbb{E}[u] \approx \mathbb{E}\left[T^K u\right] = \sum_{k=0}^{K} \frac{1}{k!} \mathbb{E}\left[D^k u(0)[Y]^k\right],$$

This approach is known in literature as *moment equations* (see e.g. [39, 44, 4, 28, 37, 40, 26]). We do not detail here the derivation and algorithmic implementation of the moment equations, which can be found in [10]. Rather, we investigate the accuracy of the Taylor expansion for the problem at hand.

3.2 Upper bound on the norm of the Taylor polynomial

The problem solved by u^0 is the deterministic Laplacian problem: given $f \in L^2(D)$ and $g \in H^{1/2}(\Gamma_D)$, find $u^0 \in H^1(D)$ such that $u|_{\Gamma_D} = g$ and

$$\int_D \nabla u^0(x) \cdot \nabla v(x)\, dx = \int_D f(x) v(x)\, dx \quad \forall\, v \in H^1_{\Gamma_D}(D). \tag{8}$$

The problem solved by the k-th Gateaux derivative of u, $D^k u(0)[Y]^k$ ($k \geq 1$) is (see e.g. [5, 26, 40])

k-th derivative problem - log-normal random field

Given $u^0 \in H^1(D)$ and all lower order derivatives

$$D^l u(0)[Y]^l \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right),\ l < k,$$

$$\textit{find } D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right) \ \textit{s.t.}$$

$$\int_D \nabla D^k u(0)[Y]^k \cdot \nabla v\, dx = -\sum_{l=1}^{k} \binom{k}{l} \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla v\, dx$$

$$\forall\, v \in H^1_{\Gamma_D}(D), \quad \text{a.s. in } \Omega. \tag{9}$$

By the Lax Milgram lemma, the boundness of $\|Y\|_{L^\infty(D)}$ and a recursion argument, we can state the following result.

Theorem 3.1 *Problem* (9) *is well-posed, that is it admits a unique solution $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ for any $0 < p < +\infty$, that depends continuously on the data. Moreover, it holds*

$$\|D^k u(0)[Y]^k\|_{H^1(D)} \leq C \left(\frac{\|Y\|_{L^\infty(D)}}{\log 2}\right)^k k! < +\infty, \quad \forall k \geq 1 \quad \textit{a.s. in } \Omega \tag{10}$$

where $C = C\left(C_P, \left\|u^0\right\|_{H^1(D)}\right)$, C_P being the Poincaré constant.

Proof. For every fixed $\omega \in \Omega$, problem (9) is of the form: find $w \in H^1_{\Gamma_D}(D)$ such that

$$\mathscr{A}(w, v) = \mathscr{L}(v) \quad \forall v \in H^1_{\Gamma_D}(D),$$

where the bilinear form $\mathscr{A}$ and the linear form $\mathscr{L}$ are respectively defined as

$$\mathscr{A} : H^1_{\Gamma_D}(D) \times H^1_{\Gamma_D}(D) \to \mathbb{R}, \quad \mathscr{A}(w, v) = \int_D \nabla w(x) \cdot \nabla v(x)\, dx$$

$$\mathscr{L}: H^1_{\Gamma_D}(D) \to \mathbb{R}, \quad \mathscr{L}(v) = -\sum_{l=1}^{k} \binom{k}{l} \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla v \, dx.$$

It is easy to verify that $\mathscr{A}$ is continuous and coercive. Moreover, $\mathscr{L}$ is continuous:

$$\begin{aligned} |\mathscr{L}(v)| &\leq \sum_{l=1}^{k} \binom{k}{l} \left| \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla v \, dx \right| \\ &\leq \sum_{l=1}^{k} \binom{k}{l} \|Y\|^l_{L^\infty} \left\| D^{k-l} u(0)[Y]^{k-l} \right\|_{H^1} \|v\|_{H^1} . \end{aligned}$$

Thanks to the Lax Milgram Lemma we conclude the well-posedness of problem (9) a.s. in Ω. To prove (10), let us take $v = D^k u(0)[Y]^k$ in (9). By the Cauchy-Schwarz inequality

$$\begin{aligned} \int_D \left|\nabla D^k u(0)[Y]^k\right|^2 \mathrm{dx} &\leq \sum_{l=1}^{k} \binom{k}{l} \left| \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla D^k u(0)[Y]^k \mathrm{dx} \right| \\ &\leq \sum_{l=1}^{k} \binom{k}{l} \|Y\|^l_{L^\infty} \left\| \nabla D^{k-l} u(0)[Y]^{k-l} \right\|_{L^2} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2} \end{aligned}$$

By defining $S_k := \frac{1}{k!} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2}$, we have:

$$S_k \leq \sum_{l=1}^{k} \frac{\|Y\|^l_{L^\infty}}{l!} S_{k-l}. \tag{11}$$

We prove by induction that

$$S_k \leq C_k \|Y\|^k_{L^\infty} S_0, \tag{12}$$

where $\{C_k\}_{k\geq 1}$ are defined by recursion as

$$\begin{cases} C_0 = 1 \\ C_k = \sum_{l=1}^{k} \frac{1}{l!} C_{k-l}. \end{cases} \tag{13}$$

If $k = 1$, (12) easily follows from (11). Now, let us suppose that (12) is verified for every S_j with $j = 1, \ldots, k-1$. Then, using (11), the inductive hypothesis and the definition of the coefficients C_k in (13),

$$\begin{aligned} S_k &\leq \sum_{l=1}^{k} \frac{\|Y\|^l_{L^\infty}}{l!} S_{k-l} = \sum_{l=1}^{k-1} \frac{\|Y\|^l_{L^\infty}}{l!} S_{k-l} + \frac{\|Y\|^k_{L^\infty}}{k!} S_0 \\ &\leq \sum_{l=1}^{k-1} \frac{\|Y\|^l_{L^\infty}}{l!} C_{k-l} \|Y\|^{k-l}_{L^\infty} S_0 + \frac{\|Y\|^k_{L^\infty}}{k!} S_0 \\ &= \|Y\|^k_{L^\infty} \left(\sum_{l=1}^{k-1} \frac{C_{k-l}}{l!} + \frac{1}{k!} \right) S_0 = \|Y\|^k_{L^\infty} C_k \, S_0, \end{aligned}$$

so that (12) is verified. In [9], the authors show by induction that $C_k \leq \left(\frac{1}{\log 2}\right)^k$ $\forall\ k \geq 0$. Hence,

$$S_k \leq \left(\frac{\|Y\|_{L^\infty}}{\log 2}\right)^k S_0.$$

In conclusion,

$$\begin{aligned}\left\|D^k u(0)[Y]^k\right\|_{H^1} &\leq \sqrt{C_P^2+1}\left\|\nabla D^k u(0)[Y]^k\right\|_{L^2}\\ &\leq \sqrt{C_P^2+1}\ S_0\left(\frac{\|Y\|_{L^\infty}}{\log 2}\right)^k\ k!\\ &\leq \left(\sqrt{C_P^2+1}\left\|u^0\right\|_{H^1}\right)\left(\frac{\|Y\|_{L^\infty}}{\log 2}\right)^k\ k!,\end{aligned}$$

so that (10) is proved with $C = \sqrt{C_P^2+1}\left\|u^0\right\|_{H^1}$. Moreover, since $\|Y\|_{L^\infty} \in L^q(\Omega, \mathbb{P})$ for any $0 < q < +\infty$, we conclude that $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}\right)$ for any $0 < p < +\infty$. □

Combining (10) and (5) we give an estimate for the $L^p\left(\Omega; H^1(D)\right)$-norm of the Taylor polynomial $T^K u$.

Theorem 3.2 *Under the assumptions such that the upper bound* (5) *holds, then for every $p \geq 1$ integer,*

$$\left\|T^K u\right\|_{L^p(\Omega;H^1(D))} \leq \left\|u^0\right\|_{H^1(D)} + C\sum_{k=1}^{K}\left(\frac{\sigma}{\log 2}\right)^k\left(\sigma^{-2}\ kp\ (kp-1)!!\right)^{1/p} \quad (14)$$

where $C = C\left(C_P, \left\|u^0\right\|_{H^1(D)}, \widetilde{C}_{Y'}\right)$.

Proof. Applying the L^p-norm in probability to both sides of (10) and using (5), we have

$$\begin{aligned}\left\|T^K u\right\|_{L^p(\Omega;H^1)} &\leq \sum_{k=0}^{K}\frac{1}{k!}\left\|D^k u(0)[Y]^k\right\|_{L^p(\Omega;H^1)}\\ &\leq \left\|u^0\right\|_{H^1(D)} + C\sum_{k=1}^{K}\frac{1}{k!}\left(\frac{1}{\log 2}\right)^k\ k!\ \left(\mathbb{E}\left[\|Y\|_{L^\infty}^{pk}\right]\right)^{1/p}\\ &\leq \left\|u^0\right\|_{H^1(D)} + C\sum_{k=1}^{K}\left(\frac{\sigma}{\log 2}\right)^k\left(\sigma^{-2}\ kp\ (kp-1)!!\right)^{1/p},\end{aligned}$$

where $C = \left(\widetilde{C}_{Y'}\right)^{1/p}\sqrt{C_P^2+1}\left\|u^0\right\|_{H^1(D)}$. □

The behavior of the upper bound (14), given by the product of an exponential term and a bifactorial term, is depicted in Figure 1 for three value of the standard deviation $\sigma = 0.05,\ 0.1,\ 0.15$ and $p = 1$ (left), $p = 2$ (right).

Exploiting the upper bound (4) instead of (5), we obtain similar results as in Theorem 3.2, where a behavior σ^k is predicted, but the constant R_γ depending on the covariance function of Y is involved.

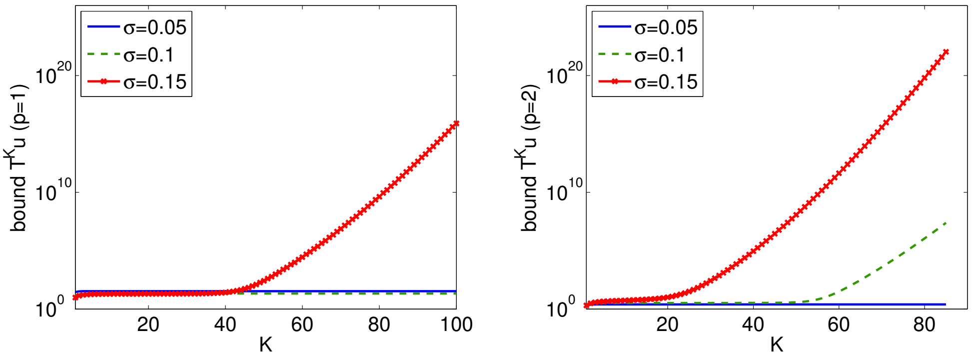

Figure 1: Upper bound (14) for three values of σ: $\sigma = 0.05,\ 0.1,\ 0.15$. Left $p = 1$, right $p = 2$.

3.3 Upper bound on the norm of the Taylor residual

The problem solved by $D^K u(tY)[Y]^K$, $t \in (0,1)$, is: given $u^0 \in H^1(D)$ and all lower order derivatives $D^l u(tY)[Y]^l \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$, $l < K$, find $D^K u(tY)[Y]^K \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ s.t.

$$\int_D e^{tY} \nabla D^K u(tY)[Y]^K \cdot \nabla v \; dx = \\ -\sum_{l=1}^{K} \left(\begin{array}{c} K \\ l \end{array} \right) \int_D Y^l e^{tY} \nabla D^{K-l} u(tY)[Y]^{K-l} \cdot \nabla v \; dx \tag{15}$$

$\forall\ v \in H^1_{\Gamma_D}(D)$, a.s. in Ω. Following an analogous reasoning as in Theorem 3.2, we find that problem (15) is well-posed and

$$\|D^K u(tY)[Y]^K\|_{H^1(D)} \leq C\ e^{t\|Y\|_{L^\infty(D)}} \left(\frac{\|Y\|_{L^\infty(D)}}{\log 2} \right)^K K! < +\infty, \tag{16}$$

$\forall\ K \geq 1$ a.s. in Ω, where $C = \sqrt{C_P^2 + 1}\, \|u^0\|_{H^1(D)}$.

Theorem 3.3 *Under the assumptions such that the upper bound* (5) *holds, then, for every* $p \geq 1$ *integer,*

$$\|R^K u\|_{L^p(\Omega; H^1(D))} \leq C\ (K+1)! \left(\frac{1}{\log 2} \right)^{K+1} \left\| \sum_{j=K+1}^{+\infty} \frac{\|Y\|^j_{L^\infty}}{j!} \right\|_{L^p(\Omega; H^1(D))} < +\infty, \tag{17}$$

where $C = C\left(C_P, \left\|u^0\right\|_{H^1(D)}, \widetilde{C}_{Y'}\right)$. *In particular, for* $p = 1$,

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\sigma^{j-2}}{(j-2)!!}. \tag{18}$$

Proof. Using (16), we find

$$\begin{aligned}\left\|R^K u\right\|_{H^1} &\leq \frac{1}{K!} \int_0^1 (1-t)^K \left\|D^{K+1}u(tY)[Y]^{K+1}\right\|_{H^1} dt \\ &\leq C\ (K+1) \left(\frac{\|Y\|_{L^\infty}}{\log 2}\right)^{K+1} \int_0^1 (1-t)^K \mathrm{e}^{t\|Y\|_{L^\infty}} dt,\end{aligned}$$

where $C = \sqrt{C_P^2 + 1}\left\|u^0\right\|_{H^1(D)}$. Let

$$I_K := \int_0^1 (1-t)^K \mathrm{e}^{t\|Y\|_{L^\infty}} dt. \tag{19}$$

By induction, we show that

$$I_K = \frac{K!}{\|Y\|_{L^\infty}^{K+1}} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}. \tag{20}$$

Indeed, for $K = 0$, using the integration by parts formula we find:

$$I_0 = \int_0^1 \mathrm{e}^{t\|Y\|_{L^\infty}} dt = \frac{\left(\mathrm{e}^{\|Y\|_{L^\infty}} - 1\right)}{\|Y\|_{L^\infty}} = \frac{1}{\|Y\|_{L^\infty}} \sum_{j=1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}.$$

Suppose now that relation (20) holds for $K-1$. Then, integrating by parts,

$$\begin{aligned} I_K &= \left[(1-t)^K \frac{\mathrm{e}^{t\|Y\|_{L^\infty}}}{\|Y\|_{L^\infty}}\right]_0^1 + \frac{K}{\|Y\|_{L^\infty}} \int_0^1 (1-t)^{K-1} \mathrm{e}^{t\|Y\|_{L^\infty}} dt \\ &= -\frac{1}{\|Y\|_{L^\infty}} + \frac{K}{\|Y\|_{L^\infty}} I_{K-1} \\ &= -\frac{1}{\|Y\|_{L^\infty}} + \frac{K}{\|Y\|_{L^\infty}} \frac{(K-1)!}{\|Y\|_{L^\infty}^K} \sum_{j=K}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!} \\ &= \frac{K!}{\|Y\|_{L^\infty}^{K+1}} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}. \end{aligned}$$

Hence,

$$\left\|R^K u(Y,x)\right\|_{H^1} \leq C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}.$$

Observe that, since $\displaystyle\sum_{j=K+1}^{N} \frac{\|Y\|_{L^\infty}^j}{j!} \leq \mathrm{e}^{\|Y\|_{L^\infty}}\ \forall\, N$ and $\mathrm{e}^{\|Y\|_{L^\infty}}$ is $L^p(\Omega, \mathbb{P})$-integrable for each $1 \leq p < \infty$, then the dominated convergence theorem states that $\displaystyle\sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}$ is

$L^p(\Omega,\mathbb{P})$-integrable for each $1 \le p < \infty$ and relation (17) follows. Moreover, if $p=1$,

$$\mathbb{E}\left[\sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}\right] = \sum_{j=K+1}^{+\infty} \frac{\mathbb{E}\left[\|Y\|_{L^\infty}^j\right]}{(j!)}.$$

Using (5), we conclude

$$\left\|R^K u(Y,x)\right\|_{L^1(\Omega;H^1)} \le C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\sigma^{j-2}}{(j-2)!!},$$

with $C = \widetilde{C}_{Y'}\sqrt{C_P^2+1}\,\|u^0\|_{H^1(D)}$. □

Using the upper bound (4) instead of (5), we predict that the $L^p\left(\Omega; H^1(D)\right)$-norm of $R^K u$ behaves as σ^{K+1} as a function of σ.

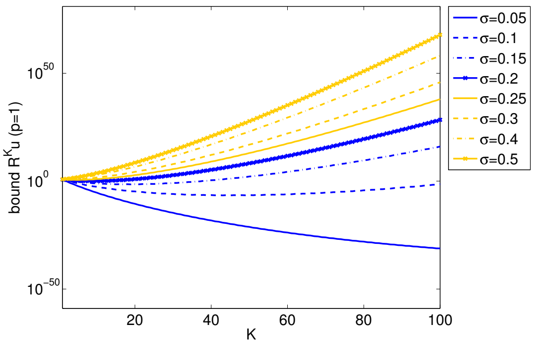

Figure 2: Semilogarithmic plot of the estimate (18) as a function of K, for different values of the standard deviation σ.

In Figure 2 we plot in semilogarithmic scale the estimate (18) as a function of the order of the residual K. We deduce the *divergence of the Taylor series for any* $\sigma > 0$ and the existence of an optimal degree K_{opt}^{σ} depending on σ and p such that adding new terms to the Taylor polynomial will deteriorate the accuracy instead of improving it. We highlight that we simply predicted and did not actually prove the divergence of the Taylor series. To do that, it is necessary to show the divergence of a lower bound for the norm of the residual $R^K u$. Nevertheless, in Section 5 we focus on the simple case of a single random variable and we perform numerical tests which confirm the divergence of the Taylor series.

4 Optimal order of approximation and minimal error

In the previous section we have predicted the divergence of the Taylor series $\forall\, \sigma > 0$ and the existence of an optimal degree K_{opt}^{σ} of the Taylor polynomial, which can be estimated as the argmin of the right-hand side in (17). Let $b(\sigma,K) = C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \left\|\sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}}{j!}\right\|_{L^p(\Omega;H^1(D))}$. The estimate (17) states that, for every $\sigma > 0$ fixed, the minimal error err^{σ}_{min} we can commit using a perturbation approach is bounded by

$$err^{\sigma}_{min} \leq \text{argmin}_K b(\sigma, K) = b(\sigma, K^{\sigma}_{opt}). \tag{21}$$

Here, we provide an approximation for both K^{σ}_{opt} and err^{σ}_{min} in the case $p = 1$ (estimate (18)).

Proposition 4.1 *Let* $0 < \sigma \leq \dfrac{\log 2}{\sqrt{5}}$. *Then, the optimal degree of the Taylor expansion can be estimated as*

$$\bar{K}^{\sigma} := \left\lfloor \frac{\log^2 2}{\sigma^2} \right\rfloor - 4. \tag{22}$$

Proof. The first step of the proof consists in showing that

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq C \frac{1}{(\log 2)^2(1-\sigma)} v(K), \tag{23}$$

where $v(K) = \left(\dfrac{\sigma}{\log 2}\right)^{K-1} (K+2)!!$ and C independent of K. Starting from (18) and using that

$$\sum_{j=K+1}^{+\infty} \frac{\sigma^{j-2}}{(j-2)!!} \leq \frac{1}{1-\sigma} \frac{\sigma^{K-1}}{(K-1)!!}$$

we find:

$$\begin{aligned}
\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} &\leq C \frac{1}{1-\sigma} \left(\frac{1}{\log 2}\right)^{K+1} (K+1)! \, \frac{\sigma^{K-1}}{(K-1)!!} \\
&= C \frac{1}{1-\sigma} \left(\frac{1}{\log 2}\right)^{K+1} \sigma^{K-1} (K+1) K (K-2)!! \\
&\leq C \frac{1}{1-\sigma} \left(\frac{1}{\log 2}\right)^{K+1} \sigma^{K-1} (K+2)!!,
\end{aligned}$$

so that (23) is proved. To find the argmin of $v(K)$, we consider $\log(v(K))$:

$$\log(v(K)) = \begin{cases} (2n-3)\log\alpha + \log(2n)!!, & \text{if } K = 2n-2, \\ (2n-4)\log\alpha + \log(2n-1)!!, & \text{if } K = 2n-3. \end{cases}$$

where $\alpha = \dfrac{\sigma}{\log 2}$. We analyze the two cases K odd or even separately, using that $(2n)!! = 2^n n!$, $(2n-1)!! = \dfrac{(2n)!}{2^n n!}$ and $\mathrm{e}\left(\dfrac{n}{\mathrm{e}}\right)^n \leq n! \leq \mathrm{e}\, n \left(\dfrac{n}{\mathrm{e}}\right)^n$. We conclude that

$$\log(v(n)) \leq w(n) + \bar{C}$$

where

$$w(n) := 2n \log\alpha + n \log 2 + (n+1)\log(n+1) - n \tag{24}$$

Table 1: This Table contains the optimal $K^{\sigma}_{opt} = \text{argmin}_K b(\sigma, K)$ $(p = 1)$ and its estimate $\bar{K}^{\sigma}$ in (22).

σ	K^{σ}_{opt}	$\bar{K}^{\sigma}$
0.10	45	44
0.15	19	17
0.18	11	10
0.20	9	8

and $\bar{C}$ is the positive constant

$$\bar{C} = \begin{cases} -3\log\alpha + 1, & \text{if } K = 2n-2, \\ -4\log\alpha + \log 2, & \text{if } K = 2n-3. \end{cases} \tag{25}$$

Note that we have bounded $(n+1)\log n$ with $(n+1)\log(n+1)$ in view of having a simpler derivative $\frac{d}{dn}w(n)$. We look for the $\text{argmin}(w(n))$ by imposing $\frac{d}{dn}w(n) = 0$, that is

$$2\log\alpha + \log 2 + \log(n+1) = 0,$$

which implies $n = \left\lfloor \frac{1}{2\alpha^2} \right\rfloor - 1$, so that we can choose $\bar{K}^{\sigma} = \left\lfloor \frac{1}{\alpha^2} \right\rfloor - 4$. $\square$

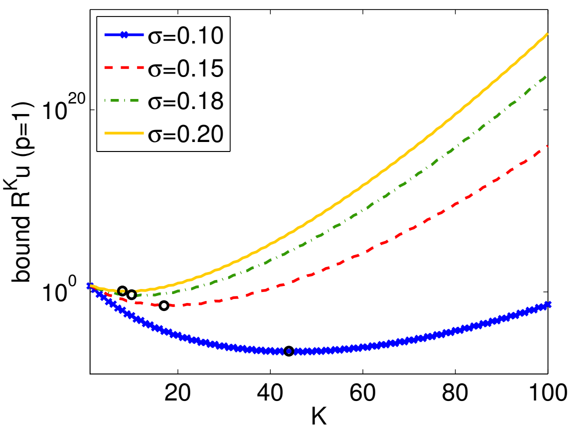

Figure 3: Semilogarithmic plot of $b(\sigma, K)$, the right-hand side of (18) (continuous line), and of the points $\left(\bar{K}^{\sigma}, b(\sigma, \bar{K}^{\sigma})\right)$ (black dot) for different values of σ.

In Table 1 we report the optimal $K^{\sigma}_{opt} = \text{argmin}_K b(\sigma, K)$ and its estimate $\bar{K}^{\sigma}$ (22) for different values of σ. Figure 3 represents the upper bound $b(\sigma, K)$ of the error (see (21)) and the points $\left(\bar{K}^{\sigma}, b(\sigma, \bar{K}^{\sigma})\right)$ (black dot) for different values of σ. We take the values $b(\sigma, \bar{K}^{\sigma})$ as an estimate of the minimal error we can commit (maximum accuracy achievable) by performing a perturbation approach as in the previous section.
As Table 1 and Figure 3 suggest, the estimate (22) of the optimal K is quite

sharp. Moreover, the smaller is σ, the bigger is K^{σ}_{opt} and the smaller is the minimal error we can commit.

Remark 4.1 *Suppose that the permeability field is modeled using a finite number of independent standard Gaussian random variables:*

$$Y(\omega, x) = \sigma \sum_{n=1}^{N} \sqrt{\lambda_n} Y_n(\omega)\phi_n(x),$$

and define $\mathbf{Y}(\omega) = (Y_1(\omega), \ldots, Y_N(\omega))$. *This situation can be achieved for example by approximating the Gaussian field* $Y(\omega, x)$ *by a* N*-terms Karhunen-Loève expansion (see e.g. [23, 31, 32, 30]). The stochastic solution* $u(\mathbf{Y}(\omega), x)$ *of the Darcy problem belongs to* $L^p_\rho\left(\mathbb{R}^N; H^1(D)\right)$, *the Banach space of functions* $v : \mathbb{R}^N \times D \to \mathbb{R}$ *such that* $\|v\|_{L^p_\rho\left(\mathbb{R}^N;H^1(D)\right)} := \left(\int_{\mathbb{R}^N} \|v(\mathbf{Y}, \cdot)\|^p_{H^1} \rho(\mathbf{Y}) d\mathbf{Y}\right)^{1/p} < \infty$, *where* $\rho(\mathbf{Y}) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{\|\mathbf{Y}\|^2}{2}}$ *is the joint probability density of the vector* $\mathbf{Y}(\omega)$. *In this setting the Gateaux derivative* $D^K u(0)[Y]^K$ *simplifies:* $D^K u(0)[Y]^K = \sum_{|\mathbf{k}|=K} \partial^{\mathbf{k}}_{\mathbf{Y}} u(\mathbf{0}, x)\mathbf{Y}^{\mathbf{k}}$, *and the Taylor polynomial is explicitly computable.*

The theoretical estimates on the norm of the Taylor polynomial (Section 3.2) and Taylor residual (Section 3.3) still hold with $R_{\gamma,N} := \sum_{n=1}^{N} \lambda_n \|\phi_n\|^2_{C^{0,\gamma}}$ *instead of* R_γ.

Remark 4.2 *In [4] (see also [17]) the authors study the Darcy problem* (2) *where the permeability is a linear combination of independent* bounded *random variables:* $a(\omega, x) = \mathbb{E}[a](x) + \sum_{n=1}^{N} \phi_n(x)Y_n(\omega)$, *with* $Y_n \sim \mathcal{U}([-\gamma_n, \gamma_n])$, $0 < \gamma_n < +\infty \ \forall \ n$, *and* $\phi_n \in L^\infty(D) \ \forall \ n$. *In this case, under the assumption of small variability of the field, the Taylor series is proved to be convergent. Hence, the divergence of the Taylor series predicted in Theorem 3.3 is strongly related to the log-normal permeability model.*

5 Single random variable - Numerical results

In the previous sections, we predicted the divergence of the Taylor series of the stochastic solution u in the case where the permeability field $a(\omega, x)$ is described as a log-normal random field. Recall that the Taylor polynomial is directly computable only in the finite-dimensional setting. Here we consider a simple case, where $a(\omega, x) = \mathrm{e}^{\phi(x)Y(\omega)}$, with $Y \simeq \mathcal{N}(0, \sigma^2)$. We compute the Taylor polynomial of u and perform some numerical tests, which *confirm the divergence of the Taylor series for every* $\sigma > 0$.

Suppose $Y \sim \mathcal{N}(0, \sigma^2)$, with $0 < \sigma < 1$ and $\phi \in L^\infty(D)$. Theorem 3.1 states that the boundary value problem solved by the k-th derivative of u, $\partial^k_Y u(0, x)$, is well-posed, and

$$\left\|\partial^k_Y u(0, x)\right\|_{H^1(D)} \leq C \left(\frac{\|\phi\|_{L^\infty}}{\log 2}\right)^k k!, \tag{26}$$

where $C = C\left(C_P, \|u^0\|_{H^1(D)}\right)$. In the same way, (16) implies

$$\left\|\partial_Y^k u(tY,x)\right\|_{H^1(D)} \leq C\ \mathrm{e}^{t|Y|\|\phi\|_{L^\infty}} \left(\frac{\|\phi\|_{L^\infty}}{\log 2}\right)^k k!. \tag{27}$$

Using the upper bound (26) and the value of the statistical moments of $|Y|$

$$\mathbb{E}\left[|Y|^p\right] = C\ \sigma^p (p-1)!!, \quad C = \begin{cases} 1 & \text{if } p \text{ is even} \\ \sqrt{\frac{2}{\pi}} & \text{if } p \text{ is odd} \end{cases} \tag{28}$$

we deduce

$$\left\|T^K u\right\|_{L^p_\rho(\mathbb{R};H^1(D))} \leq \left\|u^0\right\|_{H^1(D)} + C \sum_{k=1}^{K} \left(\frac{\|\phi\|_{L^\infty}\,\sigma}{\log 2}\right)^k ((pk-1)!!)^{1/p} \tag{29}$$

where $T^K u(Y,x) := \displaystyle\sum_{k=0}^{K} \frac{\partial_Y^k u(0,x)}{k!} Y^k$ is the K-th order Taylor polynomial and $C = C\left(C_P, \|u^0\|_{H^1(D)}\right)$. Similarly, using (27), we derive the following estimate for the K-th order integral residual $R^K u(Y,x) := \dfrac{1}{K!}\displaystyle\int_0^1 (1-t)^K \partial_Y^{K+1} u(tY,x) Y^{K+1} dt$:

$$\left\|R^K u\right\|_{L^p_\rho(\mathbb{R};H^1(D))} \leq C(K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \left\| \sum_{j=K+1}^{+\infty} \frac{(|Y|\,\|\phi\|_{L^\infty})^j}{j!} \right\|_{L^p_\rho(\mathbb{R};H^1(D))},$$

which can be particularized if $p=1$ as follows:

$$\left\|R^K u\right\|_{L^1_\rho(\mathbb{R};H^1(D))} \leq C(K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{(\sigma\,\|\phi\|_{L^\infty})^j}{j!!}, \tag{30}$$

where $C = C\left(C_P, \|u^0\|_{H^1(D)}\right)$.

We develop some numerical computations in a 1D case, with $D = [0,1]$, homogeneous Dirichlet boundary conditions imposed on $\Gamma_D = \{0,1\}$, $f(x) = x$ and $\phi(x) = \cos(\pi x)$. The problems solved by $u^0(x)$ and $\partial_Y^k u(0,x)$ respectively are:

$$\int_0^1 (u^0(x))' v'(x) dx = \int_0^1 f(x) v(x) dx, \quad u^0(0) = u^0(1) = 0 \tag{31}$$

$\forall\ v \in H_0^1([0,1])$, and

$$\int_0^1 (\partial_Y^k u(0,x))' v'(x) dx = -\sum_{l=1}^{k} \binom{k}{l} \int_0^1 \phi(x)^l (\partial_Y^{k-l} u(0,x))' v'(x) dx, \tag{32}$$

$\partial_Y^k u(0,0) = \partial_Y^k u(0,1) = 0$, $\forall\ v \in H_0^1([0,1])$, $\forall k \geq 1$. Note that the apex in problems (31) and (32) means the derivative with respect to x. Let $\{\varphi_i\}_{i=1}^{N_h-1}$ be the piecewise linear finite element basis associated with a uniform partition of $[0,1]$ in N_h subintervals of length $h = 1/N_h$. Applying the finite element method (FEM) to problem (31), we end up with the following system:

$$AU^0 = F^0, \tag{33}$$

where the stiffness matrix is tridiagonal, symmetric and its generic element is given by $A_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx$, the right-hand side is a vector whose j-th element is $F_j^0 = \int_0^1 f(x)\varphi_j(x)dx$, and U^0 is the unknown vector. Similarly, applying the linear FEM to the k-th problem (32), we end up with the following system:

$$AU^k = -\sum_{l=1}^{k} \begin{pmatrix} k \\ l \end{pmatrix} F^l U^{k-l}, \tag{34}$$

where the stiffness matrix is the same as in (33), and the right-hand side contains the solutions $U^0, \dots, U^{k-1}$ of the l-th problem for $l = 0, \dots, k-1$ and the matrices $F_{ij}^l = \int_0^1 (\phi(x))^l \varphi_j'(x)\varphi_i'(x)dx$, for $l = 1, \dots, k$.

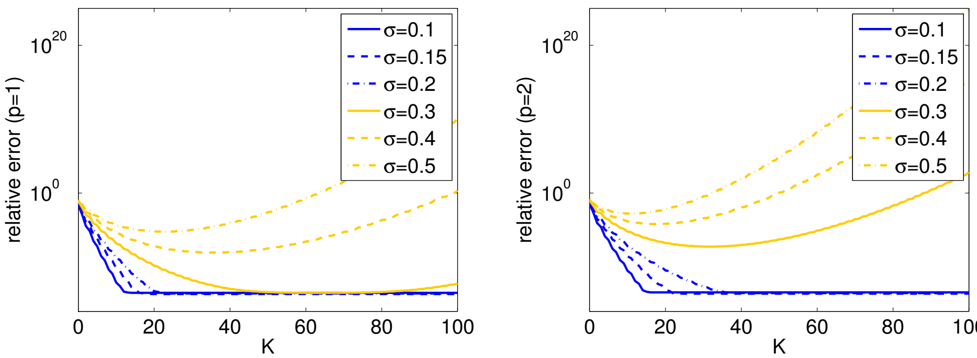

Figure 4: Relative error $\frac{\|u_h - T^K u_h\|_{L_\rho^p(\mathbb{R};L^2(D))}}{\|u_h\|_{L_\rho^p(\mathbb{R};L^2(D))}}$ computed by linear FEM in space and a high order Hermite quadrature formula in probability, for $p = 1$ (left) and $p = 2$ (right).

Let Y be fixed and let us denote with $u_h(Y,x)$ the linear FEM solution of the Darcy problem collocated in Y, so that

$$T^K u_h(Y,x) = \sum_{k=0}^{K} \sum_{i=1}^{N_h-1} \frac{U_i^k}{k!} \varphi_i(x) Y^k.$$

In Figure 4 we plot in semilogarithmic scale the relative error $\frac{\left\|u_h - T^K u_h\right\|_{L^p_\rho(\mathbb{R};L^2(D))}}{\|u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}}$ ($p = 1, 2$) computed by linear FEM in space and a high order Hermite quadrature formula in the Y variable, for different values of the standard deviation $0 < \sigma < 1$. Note that we have chosen the same spatial discretization both for u_h and $T^K u_h$, so that we observe only the truncation error of the Taylor series. These figures give numerical evidence of both the divergence of the Taylor series $\forall\ \sigma$, and the existence of an optimal degree of the Taylor polynomial K^σ_{opt} depending on σ (see Section 4). Moreover, the higher is p, the worse is the behavior of the norm of the residual, since it starts diverging for a smaller K.

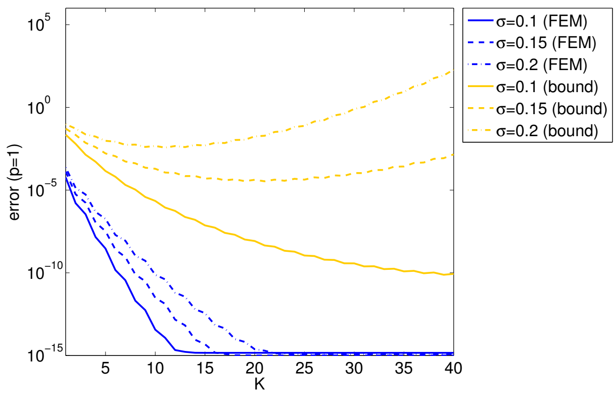

Figure 5: Comparison between the computed error $\left\|u_h - T^K u_h\right\|_{L^1_\rho(\mathbb{R};L^2(D))}$ and the theoretical estimate (30).

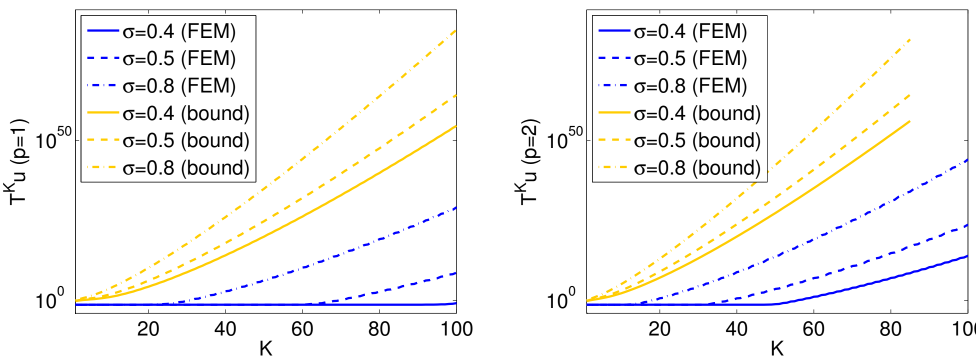

Figure 6: Comparison between the computed norm $\left\|T^K u_h\right\|_{L^p_\rho(\mathbb{R};L^2(D))}$ and its theoretical estimate (29), for $p = 1$ (left) and $p = 2$ (right).

Figure 5 compares the computed absolute error $\left\|u_h - T^K u_h\right\|_{L^1_\rho(\mathbb{R};L^2(D))}$ with the theoretical estimate (30). Figure 6 compares the theoretical upper bound for the $L^1_\rho\left(\mathbb{R}; H^1(D)\right)$ and $L^2_\rho\left(\mathbb{R}; H^1(D)\right)$ norms of the Taylor polynomial (see (29)) with the same quantities computed by linear FEM in space and a high order Hermite quadrature formula in the Y variable.

Both the estimates for the Taylor polynomial (29) and the Taylor residual (30) are quite pessimistic. This is a consequence of the estimate on $\left\|\partial^k_Y u(0,x)\right\|_{H^1(D)}$, which is itself very pessimistic.
With the aim of improving the theoretical bounds on the norm of the Taylor polynomial and residual, we assume that the growth of the derivatives follows the ansatz:

$$\left\|\partial^k_Y u(0,x)\right\|_{L^2(D)} \sim \left(\frac{\gamma \left\|\phi\right\|_{L^\infty}}{\log 2}\right)^k k! \tag{35}$$

for a suitable value of γ. Then we try to fit the value of γ starting from the numerical results obtained. In this specific example, the fitting procedure gives $\gamma = \dfrac{1}{3.5}$. Nevertheless, we highlight that the choice of γ strongly depends on $\phi(x)$, whereas it seems rather insensitive to other quantities such as the loading term $f(x)$, the boundary conditions or the number of intervals in the mesh N_h.
In Figure 7 we plot in semilogarithmic scale the quantity $\left\|\partial^k_Y u(0,x)\right\|_{L^2(D)}$ computed by linear FEM, compared with the theoretical estimate (26) and the fitted one (35) with $\gamma = \dfrac{1}{3.5}$. The agreement of the computed norm $\left\|\partial^k_Y u(0,x)\right\|_{L^2(D)}$ with the fitted estimate (35) is remarkable, which strongly indicates that the ansatz (35) is appropriate.

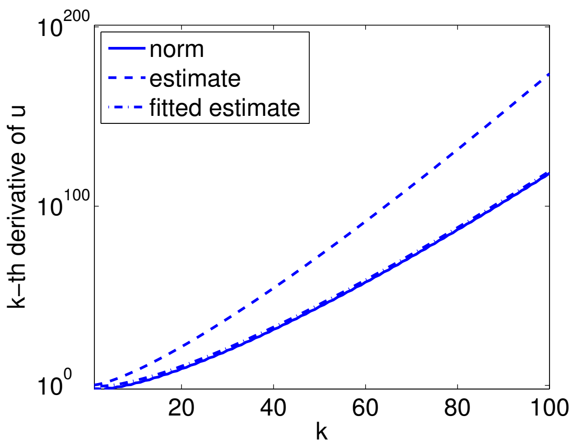

Figure 7: Comparison between the quantity $\left\|\partial^k_Y u(0,x)\right\|_{L^2(D)}$ computed by linear FEM, its theoretical estimate (26) and the fitted one (35) with $\gamma = \dfrac{1}{3.5}$.

We then use the fitted value $\gamma = \dfrac{1}{3.5}$ in the estimate (29) of the norm of the

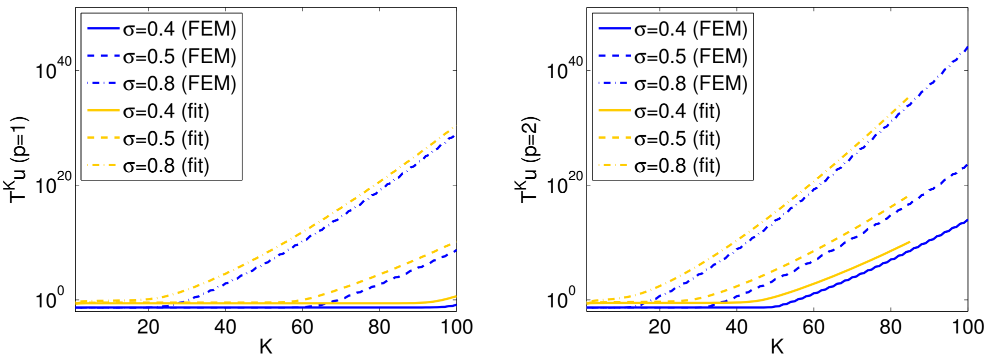

Figure 8: Comparison between the computed quantity $\|T^K u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}$ and its theoretical estimate (36) with the fitted value $\gamma = \dfrac{1}{3.5}$, for $p = 1$ (left) and $p = 2$ (right).

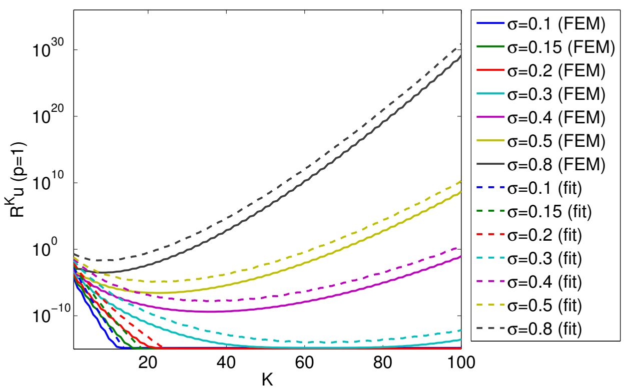

Figure 9: Comparison between the computed quantity $\|R^K u_h\|_{L^1(\Omega;L^2(D))}$ and its theoretical estimate (37) with the fitted value $\gamma = \dfrac{1}{3.5}$.

Taylor polynomial

$$\|T^K u\|_{L^p_\rho(\mathbb{R};H^1(D))} \leq \|u^0\|_{H^1(D)} + C \sum_{k=1}^{K} \left(\frac{\gamma\ \sigma\ \ \|\phi\|_{L^\infty}}{\log 2} \right)^k ((pk-1)!!)^{1/p} \quad (36)$$

as well as on the norm of the residual (30)

$$\|R^K u\|_{L^1_\rho(\mathbb{R};H^1(D))} \leq C\ (K+1)! \left(\frac{\gamma}{\log 2} \right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{(\|\phi\|_{L^\infty}\ \sigma)^j}{j!!}. \quad (37)$$

Figures 8 and 9 compare the computed quantities $\left(\|T^K u_h\|_{L^p_\rho(\mathbb{R};H^1(D))}\right.$ for $p = 1,2$ and $\left.\|R^K u_h\|_{L^1_\rho(\mathbb{R};H^1(D))}\right.$ respectively$\left.\right)$ with the fitted bounds (36) and (37) respectively. We underline that, with the ansatz (35) on the growth of the derivatives we are able to sharply predict the optimal degree of the Taylor polynomial K^σ_{opt}.

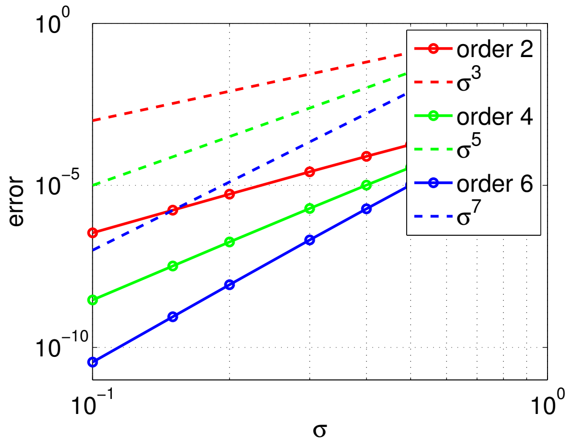

Figure 10: Error $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ as a function of σ.

Finally, we analyze the behavior of the error $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ as a function of σ. Figure 10 shows this error in logarithmic scale. Observe that the exponential behavior σ^{K+1} predicted in (30), is confirmed.

6 Conclusions

The present work addresses the Darcy problem describing the single-phase flow in a bounded heterogeneous porous medium occupying the domain $D \subset \mathbb{R}^d$, $d = 2,3$, where the permeability tensor is modeled as a log-normal random field: $a(\omega, x) = e^{Y(\omega,x)}$. Under the assumption of small variability of the field Y, we perform a perturbation analysis and study the approximation properties of the

Taylor polynomial of order K. We predict the divergence of the Taylor series, and we confirm it by numerical examples with just one random variable. We state the existence of an optimal degree K_{opt}^{σ} of the Taylor polynomial, and provide a formula to compute it in the case where the $L^1\left(\Omega; H^1(D)\right)$-norm is considered.

The results obtained in this work are very important in view of deriving an approximation of the statical moments of u. For example, if we look for an approximation of the expected value $\mathbb{E}[u]$, the underlying idea consists in deriving and numerically solving the recursive deterministic problem for the expected value of the k-th order derivative $D^k u(0)[Y]^k$, $k = 0, \ldots, K_{opt}^{\sigma}$, and then linearly combine them: $\mathbb{E}[u] \approx \sum_{k=0}^{K_{opt}^{\sigma}} \frac{1}{k!} \mathbb{E}\left[D^k u(0)[Y]^k\right]$. The k-th order derivative equation requires in turn the study of the problems solved by the correlations between $D^k u(0)[Y]^k$ and Y. These quantities belongs to tensor product spaces and, when discretized, are represented by high dimensional tensors, so that suitable numerical technique have to be adopted. This discussion can be found in [10] and is a topic of a forthcoming paper.

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