A Gauss-Bonnet Theorem for Asymptotically Conical Manifolds and Manifolds with Conical Singularities.

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Abstract

The purpose of this thesis is to provide an intrinsic proof of a Gauss-Bonnet-Chern formula for complete singular Riemannian manifolds with finitely many conical singularities and asymptotically conical ends. A geometric invariant is associated to the link of both the conical singularities and the asymptotically conical ends and is used to quantify the Gauss-Bonnet defect of such manifolds. This invariant is constructed by contracting powers of a tensor involving the curvature tensor of the link. Moreover this invariant can be written in terms of the total Lipschitz-Killing curvatures of the link. A detailed study of the Lipschitz-Killing curvatures of Riemannian manifolds is presented as well as a complete modern intrinsic proof of the Gauss-Bonnet-Chern Theorem for compact manifolds with boundary.

Résumé

Le résultat principal de cette thèse est un théorème de Gauss-Bonnet-Chern pour des variétés riemanniennes singulières, complètes ayant un nombre fini de singularités coniques et de bouts asymptotiquement coniques. On associe un invariant géométrique au link de chaque singularité conique et de chaque bout asymptotiquement conique qui permet de quantifier le défaut de Gauss-Bonnet de telles variétés. Cet invariant est construit en contractant des puissances d'un tensor qui dépend du tenseur de courbure du link. On montre que cet invariant peut être écrit comme une combinaison linéaire des courbures de Lipschitz-Killing totales du link. Une étude détaillée de ces courbures de Lipschitz-Killing ainsi qu'une preuve intrinsèque moderne du théorème de Gauss-Bonnet-Chern pour des variétés compactes à bord sont présentées.

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Introduction

The classical Gauss-Bonnet theorem, which goes back to the nineteenth century, can be stated as follows: let (S, g) be a closed surface without boundary, let K denote its Gauss curvature. Then

$$\frac{1}{2\pi}\int_S K dA = \chi(S), \tag{1}$$

where $\chi(S)$ denotes the Euler characteristic of S and dA is the area measure of S. This remarkable result establishes that although the curvature depends on the metric g, when it is summed up over the whole surface, this dependence disappears and the total amount of curvature becomes a topological invariant. If the surface is compact but admits a boundary, then the total geodesic curvature of ∂S has to be taken into account. In the 1930s, Cohn-Vossen extended formula (1) to non compact surfaces with finite total curvature in the form of an inequality: let (S, g) be a complete Riemannian surface with finite total curvature. Then

$$\frac{1}{2\pi}\int_S K dA \leq \chi(S). \tag{2}$$

The strict inequality is achieved for instance by taking $S = \mathbb{R}^2$ endowed with its standard flat metric. In this case the total curvature is zero whereas the Euler characteristic of $\mathbb{R}^2$ is equal to one.

In this search of generalization an important milestone was achieved in the 1940s by W. Fenchel [Fen40], C. Allendoerfer and A. Weil [AW43] on one hand, and S.-S. Chern [Che44], [Che45] on the other hand. They proved using radically different approaches that the total curvature of a compact Riemannian manifold *of arbitrary dimension* is a topological invariant. The method of Fenchel, Allendoerfer and Weil requires to compute the volume of tubes around submanifolds and is referred to as *extrinsic* because they assume that the manifold is embedded in some Euclidean space (at least locally). By contrast, Chern developped a completely *intrinsic* method in his proof by introducing differential forms on the manifold and on its unit tangent bundle. Chern's theorem can be stated as follows: let (M, g) be a closed n-dimensional Riemannian manifold. Let $\mathrm{Pf}(\Omega)$ be the Pfaffian of the curvature form of (M, g) then

$$(-1)^n\chi(M) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_M \mathrm{Pf}(\Omega). \tag{3}$$

The Pfaffian of the curvature form, which will be defined later, is an n-form on M depending only on the curvature tensor of M, which generalizes the Gauss curvature in higher dimensions. Note that if the dimension is odd, then this equation simplifies to $\chi(M) = 0$ since the Pfaffian vanishes. Once again, if the manifold M admits a boundary, the total curvature of ∂M appears in the formula.
Just a few years before those results, H. Weyl published his famous article "On the Volume of Tubes" [Wey39] in which he proved that for $r \leq \varepsilon$ sufficiently small the volume of a tube of radius

$$M_r = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, M) < r\},$$

around a compact submanifold $M^n \subset \mathbb{R}^N$ is given by a polynomial in the radius r of the tube, whose coefficients $\mathcal{K}_{2k}(M)$ depend only on the curvature tensor of the submanifold itself:

$$\mathrm{Vol}_N(M_r) = \frac{\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\mathcal{K}_{2k}(M) r^{q+2k}}{q(q+2)(q+4)\cdot \ldots \cdot (q+2k)}, \tag{4}$$

where $q = N - n$ is the codimension of M in $\mathbb{R}^N$ and Γ is the Gamma function. These coefficients $\mathcal{K}_{2k}(M)$ are called the *total Lipschitz-Killing curvatures* of the submanifold and will be of particular importance throughout this thesis. These curvatures can be defined as in [Gra04], without referring to any embedding by taking contractions of powers of the curvature endomorphism of M. Although these curvatures are complex objects, it appears that some of them are well-known quantities e.g. the first Lipschitz-Killing curvature is the volume of the manifold $\mathcal{K}_0(M) = \mathrm{Vol}(M)$, the second is the integral of the scalar curvature up to a constant $\mathcal{K}_2(M) = \frac{1}{2}\int_M S_g \mathrm{dvol}_M$ and if the dimension of the manifold is even, the last one is its the Euler-characteristic $\mathcal{K}_n(M) = (2\pi)^{\frac{n}{2}}\chi(M)$ up to a constant.
Searching for a generalization of this theorem, it is natural to ask, as in Cohn-Vossen's inequality (2), whether the compactness assumption can be replaced by some weaker condition enabling the integral of the curvature to converge. In [KZ01], R. Kellerhals and T. Zehrt show that in the case of an even-dimensional non compact complete hyperbolic manifold having *finite volume* a Gauss-Bonnet formula holds. The assumption of finite volume in the context of non compact hyperbolic manifolds is a strong geometric assumption that allows the total curvature to be well-defined.
Since the total curvature has to be finite, it is natural to look at complete non compact Riemannian manifolds which are of *finite topological type* that is manifolds which are diffeomorphic to the interior of a compact manifold with boundary. This condition ensures for example that the "unbounded parts" of the manifold are in finite number. For such manifolds, S. Rosenberg gives in [Ros85] several classes of complete metrics for which a Gauss-Bonnet formula holds.
Allowing a weaker control on the geometry of the non compact parts, one can ask if it is possible to obtain a quantification of the *Gauss-Bonnet defect*, that is the

difference between the Euler-characteristic and the total curvature. In dimension two it was shown by Finn [Fin65] and T. Shiohama [Shi85] that if S has finite total curvature, then

$$\chi(S) - \frac{1}{2\pi}\int_S K dA = \lim_{t\to\infty}\frac{L(t)}{t} = \lim_{t\to\infty}\frac{L^2(t)}{2A(t)},$$

where $L(t)$ is the length of $S(t) = \{x \in S \mid d(p,x) = t\}$ for any $p \in S$ and $A(t)$ is the area of $B(t) = \{x \in S \mid d(p,x) \leq t\}$. Therefore the Gauss-Bonnet defect is controlled by the geometry "at infinity".

In higher dimensions, the total curvature is finite if the non compact parts are sufficiently "flat" as for example if they are isometric to cones. In [DK05], F. Dillen and W. Kühnel study n-dimensional submanifolds of $\mathbb{R}^N$ which have conical ends in the sense that these submanifolds consist of a compact core and finitely many non compact parts that are isometric to subsets of $\mathbb{R}^N$ of the form

$$C(N) = \{p + tx \mid x \in N, \quad t \in [0,\infty)\} \subset \mathbb{R}^N,$$

with N an $(n-1)$-dimensional submanifold of the unit sphere $\mathbb{S}^{N-1}$ called the *link* and $p \in \mathbb{R}^N$ called the *apex* of the cone. They prove that if M^n is a complete submanifold of $\mathbb{R}^N$ with finitely many conical ends with links $N_1, \dots N_r$, then the Gauss-Bonnet defect can be expressed as a sum of curvature quantities of the links of the cones:

$$\chi(M) - \frac{1}{\alpha_{N-1}}\int_{S^\perp M}\det(A_\xi)d\xi = \sum_{j=1}^{r}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{\alpha_{N-1-n+2k}\alpha_{n-1-2k}}\Theta_{2k}(N_j), \qquad (5)$$

where $\alpha_j = \mathrm{Vol}(\mathbb{S}^j)$, the integrand $\det(A_\xi)$ is the determinant of the shape operator in the normal direction $\xi \in S^\perp M = \{(p,v) \in T^\perp M \mid \|v\| = 1\}$ and on the right-hand side, the $\Theta_{2k}(N_j)$ are defined by

$$\Theta_{2k}(N_j) = \int_{S^\perp N_j}\sigma_{2k}(\xi)d\xi,$$

where $\sigma_{2k}(\xi)$ is the $2k$-th elementary symmetric polynomial of the shape operator in direction ξ of the embeddind $N_j \hookrightarrow \mathbb{S}^{N-1}$.

Of course this theorem is strongly extrinsic as it requires the embedding of M in $\mathbb{R}^N$ and of N_j in $\mathbb{S}^{N-1}$. Moreover the various constants depend on the dimension of the ambient space $\mathbb{R}^N$. It is noteworthy to mention that the condition of being conical can be relaxed as it is the asymptotic behaviour of the end that matters in the quantification. Therefore Dillen and Kühnel introduce the notion of a *cone-like end* and show that the same statement holds for manifolds admitting cone-like ends.

Another way of generalizing the Gauss-Bonnet-Chern theorem is to switch from the class of smooth Riemannian manifolds to a larger class in which the metric is allowed to lack some smoothness at a finite number of points. However the metric is asked to take a certain form in the neighbourhood of these points. This was investigated in dimension two in [Tro91],[HT92],[Tro93], where the following Gauss-Bonnet formula for compact surfaces with *simple singularities* is shown: let (S,g) be a compact surface with n simple singularities p_i of order β_i, then

$$\frac{1}{2\pi}\int_S K dA = \chi(S) + \sum_{i=1}^{n} \beta_i, \tag{6}$$

where β_i is a local invariant. This formula can be thought as a quantification of the Gauss-Bonnet defect in terms of the curvature concentrated at each singular point. Note that simple singularities include conical singularities, conical ends as well as cusps, and cylindrical and parabolic ends. For surfaces with conical ends, Formula (6) goes back to R. Finn [Fin65].
The main purpose of this thesis is to give an intrinsic proof of a Gauss-Bonnet theorem for Riemannian manifolds with finitely many conical singularities and asymptotically conical ends, answering along the way a question raised in [DK05] of whether it is possible to prove (5) intrinsically.
Asymptotically conical manifolds have been studied recently in [CEV17], [Con11] and they are defined to be topological cones over compact manifolds endowed with a Riemannian metric which converges (as well as its derivatives up to order r) towards the standard cone metric. As in the case of (asymptotically) conical ends, a conical singularity comes together with a link (N, g_N) which is actually the key object to study when we want to work without referring to any ambient space. To each link, we will associate an geometric invariant $\tau(N)$ which depends only on the curvature of N as a Riemannian manifold. More precisely, $\tau(N)$ is (up to some dimensional constants) the sum of the total Lipschitz-Killing curvatures of N:

$$\tau(N) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \lambda_{n,k} \mathcal{K}_{2k}(N),$$

where $\lambda_{n,k}$ are explicit constants depending on n and k (see Theorem 4.9). Our main theorem can be stated

Main Theorem. Let $(\hat{M}^n, g)$ be a complete even dimensional singular Riemannian manifold with finitely many conical singularities $\{p_1,\ldots,p_r\}$ and finitely many asymptotically conical ends $\{E_{r+1},\ldots,E_s\}$. Then the total curvature of $\hat{M}$ is finite and we have

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}}\int_M \mathrm{Pf}(\Omega) = -\sum_{i=1}^{r} \tau(N_i) + \sum_{i=r+1}^{s} \tau(N_i), \tag{7}$$

where $M = \hat{M} \setminus \{p_1,\ldots,p_r\}$ and the N_i are the links of the conical singularities and of the asymptotically conical ends.

The precise definitions of conical singularities and asymptotically conical ends will be given in Chapter 4 as well as the asymptotic behaviour of the metric near those singularities. It will be shown that, as in the Gauss-Bonnet-Chern Theorem, the formula (7) for odd dimensional manifolds contains no geometric information as both the left-hand side and the right-hand side simplifies to $\chi(N)$.
In the case where $\hat{M}$ is two dimensional, then (7) reduces to formula (6) as explained in Remark 4.29.

The proof of the Main Theorem 7 is articulated in three parts. First we deal with the case where M is assumed to have only conical ends, that is each end of M is a standard cone. Applying the Gauss-Bonnet-Chern theorem to an exhaustion $\{M_t\}_{t>1}$ of M by compact manifolds with boundary, we obtain a quantification of the Gauss-Bonnet defect of M form the asymptotic behaviour of the boundary term given by the Gauss-Bonnet-Chern theorem. Using the Gauss equation this boundary term can be expressed using only quantities defined on the link of each cone, namely the Lipschitz-Killing curvatures of each link. Cartan's formalism of moving frames [Car01],[Spi99] is particularly well-suited to this problem. In particular, it is the approach used by Chern in his proof of the Gauss-Bonnet-Chern theorem in [Che44] and [Che45].
The second part is devoted to manifold whose ends are asymptotically conical. The strategy of proof is once again to apply the Gauss-Bonnet-Chern theorem to an exhaustion and then to find estimates of the additional terms that appear in this case.
Finally, in the third part, we adapt the method used for asymptotically conical ends to the case of conical singularities.

We conclude this introduction by mentioning the related work of some other authors dealing with conical singularties or conical ends. The paper [CEV17] by O. Chodosh, M. Eichmair and A. Volkmann studies isoperimetric inequalities in asymptotically conical manifolds, their definition is similar to our Definition 4.11, but they use a slightly weaker asymptotic condition than ours.
Note that our conical singularties are *point singularities*, meaning the singular locus is a zero-dimensional set. However one may also consider *higher dimensional conical singularities*. A local model for a standard k-dimensional conical singularity in a (singular) Riemannian manifold (M, g) is a Riemannian product of a smooth submanifold W with a cone over some $(n-k-1)$ Riemannian manifold N (this is the *link* of the conical singularity). Manifolds with codimension 2 conical singularities play a major role in the work of G. Tian and S. Donaldson and his collaborators to prove the existence of Kähler-Einstein metrics on Fano manifolds. We refer to P. Eyssidieux's talk at Seminaire Bourbaki [Eys16] for a survey of this very rich subject, as well as the PhD thesis of G. De Borbon [deB15].
One obtains the more general class of cone manifolds in a stratified sense if one allows the link N to itself be a stratified cone manifold (the definition being inductive starting with the zero dimensional strata being a finite set).

For such a type of stratified cone Manifolds, C. McMullen recently obtained a Gauss-Bonnet formula in [McM17]. However, this paper assumes the links to be spherical cone manifolds, as a result McMullen's Gauss-Bonnet formula does not contain our Main Theorem.
We finally mention the very recent work of R. Buzano and H.T. Nguyen [BN17, BN18]. In these papers the authors obtain a Gauss-Bonnet formula for manifolds with finitely many tame ends and isolated singular points. Their results have some similarities with our Main Theorem, but with some important differences. Their results assume some topological restrictions, conformal flatness and some non-negativity condition on the scalar curvature near the ends and the point singularities.

Organization of the Thesis

The text is organized as follows. The first chapter is a review of some definitions and results in Riemannian geometry that are to be used throughout the thesis. In particular, we develop in details the moving frame formalism as it is of utmost importance to define the Lipschitz-Killing curvatures and to understand both the proof of the Gauss-Bonnet-Chern theorem and our main theorem.
The second chapter is dedicated to the Lipschitz-Killing curvatures of a Riemannian manifold. After introducing the algebra of double-forms, we define and establish several properties of the Lipschitz-Killing curvatures. In particular we compute them in the case where the manifold is of constant sectional curvature as well as in the case where the manifold is a cone over a compact manifold. The Weyl formula is also presented since it makes the Lipschitz-Killing curvatures appear naturally in the expression of the volume of the tube around a submanifold of $\mathbb{R}^N$. In the last part of this chapter we make a detour in the world of principal bundles in order to present a proof of the Gauss-Bonnet-Chern theorem using this modern language. This yields a definition of the Lipschitz-Killing curvatures as global differential forms on the $SO(n)$ principal bundle of orthonormal positively oriented moving frames.
In the third chapter we present Chern's intrinsic proof of his Gauss-Bonnet-Chern Theorem using the language of principal bundles. The Hopf-Poincaré Theorem on the indices of a vector field is recalled and illustrated as it is crucial for the proof, especially when the manifold is assumed to have a boundary.

The main result of the thesis is proved in the fourth chapter. As we already mentioned, the proof is divided in three distinct steps: manifolds with conical ends, manifolds with asymptotically conical ends and manifolds with conical singularities. This chapter also includes a discussion about the invariant $\tau(N)$ in some special cases.

Chapter 1

Background on Riemannian Geometry

In this chapter we present the necessary background to understand the rest this thesis. The common thread consists of the formalism of moving frames, which is particularly well developed in Spivak's books [Spi99]. This approach to Riemannian geometry goes back to Élie Cartan [Car01] and can be seen as an alternative to working with coordinates. The problem of defining a connection usually requires either to choose local coordinates, which induce coordinate vector fields, or to introduce the global operator ∇. The method of moving frames provides a third way of defining a connection by considering *any* n-tuple of linearly independent vector field on an open subset of the manifold and not only vector fields induced by some coordinates. Note that throughout all the thesis we will use Einstein's convention on the summation of repeated indices.

1.1 Tensor fields

Definition 1.1. Let V be an n-dimensional vector space over $\mathbb{R}$ with dual V^* and consider the two following vector spaces

$$\mathrm{Tens}_l^k(V) = \{T : \underbrace{V^* \times \ldots \times V^*}_{l} \times \underbrace{V \times \ldots \times V}_{k} \to \mathbb{R} \mid T \text{ is multilinear}\},$$
$$\mathcal{L}_l^k(V) = \{L : \underbrace{V^* \times \ldots \times V^*}_{l} \times \underbrace{V \times \ldots \times V}_{k} \to V \mid L \text{ is multilinear}\}.$$

Elements of $\mathrm{Tens}_l^k(V)$ are called *tensors of type* (k, l), the index k is called the degree of *covariance* and the index l the degree of *contravariance*. It is clear that $\mathrm{Tens}_l^k(V)$ is a vector space of dimension n^{k+l}.

These two spaces are closely related one to each other.

Lemma 1.2. There is a canonical isomorphism

$$\mathcal{L}_l^k(V) \simeq \mathrm{Tens}_{l+1}^k(V)$$

Proof. To an element $L \in \mathcal{L}^k_l(V)$ we associate $T_L \in \mathrm{Tens}^k_{l+1}(V)$ the tensor defined by

$$T_L(\eta^1, \dots, \eta^l, \eta, v_1, \dots, v_k) = \eta \left(L(\eta^1, \dots, \eta^l, v_1, \dots, v_k) \right).$$

Conversely, given a tensor $T \in \mathrm{Tens}^k_{l+1}(V)$, we associate the following linear map

$$L_T(\eta^1, \dots, \eta^l, v_1, \dots, v_k) = T(\eta^1, \dots, \eta^l, -, v_1, \dots, v_k) : V^* \to \mathbb{R}$$

which is associated to a vector via the usual identification of V to the bidual V^{**}. Those applications are clearly inverse one to each other and $\mathbb{R}$-linear. □

Now let M be an n-dimensional differentiable manifold. We construct the vector bundles associated to $\mathrm{Tens}^k_l(T_pM)$ and $\mathcal{L}^k_l(T_pM)$ for each $p \in M$, i.e.

$$\mathrm{Tens}^k_l(TM) = \bigsqcup_{p \in M} \mathrm{Tens}^k_l(T_pM) \quad \mathcal{L}^k_l(TM) = \bigsqcup_{p \in M} \mathcal{L}^k_l(T_pM)$$

The smooth sections of $\mathrm{Tens}^k_l(TM)$ are called *tensor fields* and the space of all such sections is usually denoted by $\Gamma(\mathrm{Tens}^k_l(TM))$ but whenever there is no ambiguity we will make the following abuse of notations:

$$\mathrm{Tens}^k_l(M) = \Gamma(\mathrm{Tens}^k_l(TM))$$
$$\mathcal{L}^k_l(M) = \Gamma(\mathcal{L}^k_l(TM))$$

The familiar examples of tensor fields and elements of $\mathcal{L}^k_l(M)$ include any vector field $X \in \mathrm{Tens}^0_1(M)$, any one-form $\omega \in \mathrm{Tens}^1_0(M)$, any Riemannian metric g is a tensor field of type $(2,0)$, i.e. $g \in \mathrm{Tens}^2_0(M)$ or the curvature endomorphism $R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ is an element of $\mathcal{L}^3_0(M)$ and, through the isomorphism in lemma (1.2) extended to the bundles, also an element of $\mathrm{Tens}^3_1(M)$.

1.1.1 The musical isomorphisms

Suppose now that M is endowed with a Riemannian metric g. This allows us to define the *flat isomorphism* between vector fields and one-forms:

$$\flat : \mathrm{Tens}^0_1(M) \longrightarrow \mathrm{Tens}^1_0(M), \quad X^\flat(Y) = g(X,Y).$$

If we are given a coordinate system $(x^1, \dots, x^n)$ on an open subset $U \subset M$ and if ∂_i denotes the i-th coordinate vector field $\frac{\partial}{\partial x^i}$, then on U the one-form $X^\flat$ writes as

$$X^\flat = g(a^i \partial_i, -) = g_{ij} a^i dx^j$$

where $X = a^i \partial_i$. We usually set $a_j = g_{ij} a^i$ to write $X^\flat = a_j dx^j$ and we say that we have *lowered an index*.

If g^{ij} denote the components of the inverse $(g_{ij})^{-1}$, then the inverse of $\flat$ is given by

$$\sharp : \mathrm{Tens}^1_0(M) \longrightarrow \mathrm{Tens}^0_1(M), \quad \omega^\sharp = g^{ij} b_i \partial_j =: b^j \partial_j,$$

where $\omega = b_i dx^i$ and we say that we have *raised an index.*
Observe that we have

$$g(\omega^\sharp, X) = \omega(X).$$

These isomorphisms can be extended to arbitrary tensor fields as follows:

$$\begin{aligned} \flat : \mathrm{Tens}^k_l(M) &\longrightarrow \mathrm{Tens}^{k+1}_{l-1}(M) \quad \text{if } l \geq 1, \\ \sharp : \mathrm{Tens}^k_l(M) &\longrightarrow \mathrm{Tens}^{k-1}_{l+1}(M) \quad \text{if } k \geq 1. \end{aligned}$$

but we have to specify which index is lowered or raised. If $T \in \mathrm{Tens}^k_l(M)$ we lower the i-th index by setting

$$T^\flat(\omega^1, \ldots, \omega^{l-1}, X_1, \ldots, X_{k+1}) = T(X_i^\flat, \omega^1, \ldots, \omega^{l-1}, X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}),$$

where the $\hat{X}_i$ means that we have omitted X_i. Similarly, we raise the j-th index by setting

$$T^\sharp(\omega^1, \ldots, \omega^{l+1}, X_1, \ldots, X_{k-1}) = T(\omega^1, \ldots, \hat{\omega}^j, \ldots, \omega^{l+1}, (\omega^j)^\sharp, X_1, \ldots X_{k-1})$$

In coordinates, we have:

Lemma 1.3. If $(x^1, \ldots, x^n)$ is a coordinate system on an open subset $U \subset M$, any tensor $T \in \mathrm{Tens}^k_l(U)$ can be written as

$$T = T^{j_1 \ldots j_l}_{i_1 \ldots i_k} dx^{i_1} \otimes \ldots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \ldots \otimes \partial_{j_l}, \quad T^{j_1 \ldots j_l}_{i_1 \ldots i_k} \in C^\infty(U).$$

Then the components of $T^\flat$ (resp. $T^\sharp$) when we lower (resp. raise) the r-th (resp. s-th) index, are given by

$$\begin{aligned} \left(T^\flat\right)^{\nu_1 \ldots \nu_{l-1}}_{\mu_1 \ldots \mu_{k+1}} &= g_{\mu_r \alpha} T^{\alpha \nu_1 \ldots \nu_{l-1}}_{\mu_1 \ldots \hat{\mu}_r \ldots \mu_{k+1}} \\ \left(T^\sharp\right)^{\nu_1 \ldots \nu_{l+1}}_{\mu_1 \ldots \mu_{k-1}} &= g^{\nu_s \alpha} T^{\nu_1 \ldots \hat{\nu}_s \ldots \nu_{l+1}}_{\alpha \mu_1 \ldots \mu_{k-1}} \end{aligned}$$

Proof. Since $(\partial_i)^\flat = g_{ij} dx^j$, we have

$$\begin{aligned} \left(T^\flat\right)^{\nu_1 \ldots \nu_{l-1}}_{\mu_1 \ldots \mu_{k+1}} &= T^\flat(\partial_{\mu_1}, \ldots, \partial_{\mu_{k+1}}, dx^{\nu_1}, \ldots, dx^{\nu_{l-1}}) \\ &= T\left(\partial_{\mu_1}, \ldots, \hat{\partial}_{\mu_r}, \ldots, \partial_{\mu_{k+1}}, (\partial_{\mu_r})^\flat, dx^{\nu_1}, \ldots, dx^{\nu_{l-1}}\right) \\ &= g_{\mu_r \alpha} T\left(\partial_{\mu_1}, \ldots, \hat{\partial}_{\mu_r}, \ldots, \partial_{\mu_{k+1}}, dx^\alpha, dx^{\nu_1}, \ldots, dx^{\nu_{l-1}}\right) \\ &= g_{\mu_r \alpha} T^{\alpha \nu_1 \ldots \nu_{l-1}}_{\mu_1 \ldots \hat{\mu}_r \ldots \mu_{k+1}}. \end{aligned}$$

Similarly, since $(dx^i)^\sharp = g^{ij}\partial_j$, we have

$$\begin{aligned}
(T^\sharp)^{\nu_1\dots\nu_{l+1}}_{\mu_1\dots\mu_{k-1}} &= T^\sharp(\partial_{\mu_1},\dots,\partial_{\mu_{k-1}},dx^{\nu_1},\dots,dx^{\nu_{l+1}}) \\
&= T\left((dx^{\nu_s})^\sharp,\partial_{\mu_1},\dots,\partial_{\mu_{k-1}},dx^{\nu_1},\dots,\hat{dx}^{\nu_s},\dots,dx^{\nu_{l+1}}\right) \\
&= g^{\nu_s\alpha}T\left(\partial_\alpha,\partial_{\mu_1},\dots,\partial_{\mu_{k-1}},dx^{\nu_1},\dots,\hat{dx}^{\nu_s},\dots,dx^{\nu_{l+1}}\right) \\
&= g^{\nu_s\alpha}T^{\nu_1\dots\hat{\nu}_s\dots\nu_{l+1}}_{\alpha\mu_1\dots\mu_{k-1}}.
\end{aligned}$$

□

1.1.2 Contractions of tensors

The notion of trace of an endomorphism is easily generalized to arbitrary mixed tensors by means of contractions . Recall that if V is a finite dimensional real vector space, then $\mathrm{Tens}^1_1(V) = \mathrm{End}(V)$ and therefore the trace of $h \in \mathrm{Tens}^1_1(V)$ is well-defined as a map

$$\mathrm{Tr} : \mathrm{Tens}^1_1(V) \longrightarrow \mathbb{R} = \mathrm{Tens}^0_0(V)$$

Therefore, if $k, l \geq 1$ we can define the *contraction of the μ-th and ν-th indices* to be the map

$$C^\mu_\nu : \mathrm{Tens}^k_l(V) \longrightarrow \mathrm{Tens}^{k-1}_{l-1}(V),$$

where its action on a basis of $\mathrm{Tens}^k_l(V)$ is given by

$$C^\mu_\nu(e_{i_1}\otimes\dots\otimes e_{i_l}\otimes\epsilon^{j_1}\otimes\dots\otimes\epsilon^{j_k}) = \epsilon^{j_\nu}(e_{i_\mu})e_{i_1}\otimes\dots\otimes\hat{e}_{i_\mu}\otimes\dots\otimes e_{i_l}\otimes\epsilon^{j_1}\otimes\dots\otimes\hat{\epsilon}^{j_\nu}\otimes\dots\otimes\epsilon^{j_k},$$

with $(e_1,\dots,e_n)$ any basis of V and $(\epsilon^1,\dots,\epsilon^n)$ its dual basis. It is easy to show that this definition does not depend on the choice of the basis.
The contraction can obviously be extended to the bundle $\mathrm{Tens}^k_l(M)$ by taking $V = T_pM$. In particular, if M is of dimension n we have:

Lemma 1.4. If $(x^1,\dots,x^n)$ is a coordinate system on an open set $U \subset M$, then if $T \in \mathrm{Tens}^k_l(U)$, the coordinates of $C^\mu_\nu(T)$ are given by

$$(C^\mu_\nu(T))^{j_1\dots j_{l-1}}_{i_1\dots i_{k-1}} = T^{j_1\dots j_{\mu-1}\alpha j_{\mu+1}\dots j_{l-1}}_{i_1\dots i_{\nu-1}\alpha i_{\nu+1}\dots i_{k-1}},$$

where we sum over α.

Proof. By definition

$$\begin{aligned}
(C^\mu_\nu(T))^{j_1\dots j_{l-1}}_{i_1\dots i_{k-1}} &= C^\mu_\nu(T)(dx^{j_1},\dots,dx^{j_{l-1}},\partial_{i_1},\dots,\partial_{i_{k-1}}) \\
&= T^{b_1\dots b_l}_{a_1\dots a_k}C^\mu_\nu(\partial_{b_1}\otimes\dots\otimes\partial_{b_l}\otimes dx^{a_1}\otimes\dots\otimes dx^{a_k})(dx^{j_1},\dots,dx^{j_{l-1}},\partial_{i_1},\dots,\partial_{i_{k-1}}) \\
&= T^{b_1\dots b_l}_{a_1\dots a_k}dx^{a_\nu}(\partial_{b_\mu})\partial_{b_1}(dx^{j_1})\cdots\partial_{b_{\mu-1}}(dx^{j_{\mu-1}})\partial_{b_{\mu+1}}(dx^{j_\mu})\cdots\partial_{b_l}(dx^{j_{l-1}}) \\
&\qquad\cdot dx^{a_1}(\partial_{i_1})\cdots dx^{a_{\nu-1}}(\partial_{i_{\nu-1}})dx^{a_{\nu+1}}(\partial_{i_\nu})\cdots dx^{a_k}(\partial_{i_{k-1}}) \\
&= T^{b_1\dots b_l}_{a_1\dots a_k}\delta^{a_\nu}_{b_\mu}\delta^{j_1}_{b_1}\cdots\delta^{j_{\mu-1}}_{b_{\mu-1}}\delta^{j_\mu}_{b_{\mu-1}}\cdots\delta^{j_{l-1}}_{b_l}\delta^{a_1}_{i_1}\cdots\delta^{a_{\nu-1}}_{i_{\nu-1}}\delta^{a_{\nu+1}}_{i_\nu}\cdots\delta^{a_k}_{i_{k-1}} \\
&= T^{j_1\dots j_{\mu-1}\alpha j_{\mu+1}\dots j_{l-1}}_{i_1\dots i_{\nu-1}\alpha i_{\nu+1}\dots i_{k-1}}.
\end{aligned}$$

□

Observe that the contractions are not defined for tensors that are purely covariant or purely contravariant since both the degree of covariance and contravariance must be at least one. But if we are given a Riemannian metric g on the manifold M, then one can extend the contractions to $\mathrm{Tens}^0_l(M)$ and $\mathrm{Tens}^k_0(M)$ provided $k, l \geq 2$ by setting

$$C^1_\nu(T) := C^1_\nu(T^\flat), \quad \text{if } T \in \mathrm{Tens}^0_l(M),$$
$$C^\mu_1(T) := C^\mu_1(T^\sharp), \quad \text{if } T \in \mathrm{Tens}^k_0(M).$$

We use the metric to raise or lower an index in order to be able to contract. Obviously, this is not defined on $\mathrm{Tens}^0_1(M)$ or $\mathrm{Tens}^1_0(M)$.
As an example, we compute the contraction of the Riemannian metric $g \in \mathrm{Tens}^2_0(M)$. In coordinates we have

$$g^\sharp = g^{ik} g_{kj} \partial_i \otimes dx^j = \delta^i_j \partial_i \otimes dx^j \Rightarrow C^1_1(g) = C^1_1(g^\sharp) = \delta^i_i = n.$$

1.2 The method of moving frames

In Riemannian geometry, we are used to see the objects either in coordinates or with the global language, but there exists a third approach developed by Elie Cartan, called the *method of moving frames*. The basic tools of this formalism are linearly independant vector fields (i.e. moving frames) and their dual covector fields. Obviously, such frames do not exist globally on every manifold, therefore it is *local*, but the gain is that on some open subset of the manifold, one can consider *orthonormal* vector fields, which are simpler than coordinate vector fields. Throughout this section, (M, g) will be an n-dimensional Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor.

Definition 1.5. Let $U \subset M$ be an open subset. A *moving frame* on U is an n-tuple of vector fields $(X_1, \ldots, X_n)$ such that for every $p \in U$, the list $(X_1(p), \ldots, X_n(p))$ is a basis of T_pM. The *moving coframe* associated to $(X_1, \ldots, X_n)$ is the n-tuple of differential 1-forms $(\theta^1, \ldots, \theta^n)$ that are dual to the X_i's, that is they satisfy $\theta^i(X_j) = \delta^i_j$.

Obviously at each point $p \in U$ the list $(\theta^1(p), \ldots, \theta^n(p))$ forms a basis of the cotangent space T^*_pM.

Remark 1.6. Let us make a few remarks about this definition.

(a) Given a moving frame $(X_1, \ldots, X_n)$ on an open subset $U \subset M$, the X_i's are not necessarily coordinate vector fields. Therefore their Lie brackets $[X_i, X_j]$ do not vanish in general.

(b) The domain $U \subset M$ on which a moving frame is defined has no relation with an eventual coordinate chart. For example, it is possible to define a global moving frame on the whole torus $\mathbb{T}^2$ although it does not admit a global chart.

(c) If the manifold is endowed with a Riemannian metric g, the coframe is given by $\theta^i = g(X_i, \cdot)$. Moreover by applying the Gram-Schmidt algorithm, one can always assume that a given moving frame $(X_1, \dots, X_n)$ is *orthonormal* and therefore the components of the metric with respect to the moving frame are simply $g_{ij} = g(X_i, X_j) = \delta_{ij}$.

1.3 The connection forms

Let $(X_1, \dots, X_n)$ be a moving frame on $U \subset M$ with associated coframe $(\theta^1, \dots, \theta^n)$. We define n^3 functions $\Gamma_{ij}^k \in C^\infty(U)$ by

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k. \tag{1.1}$$

It is important to note that the Γ_{ij}^k are *not* the Christoffel symbols associated to some coordinate system. Hence they do not satisfy the usual symmetry relations. Nonetheless we have the following lemma:

Lemma 1.7. The Γ_{ij}^k's satisfy the relations:

(i) $\Gamma_{ij}^k - \Gamma_{ji}^k = \theta^k([X_i, X_j])$.

(ii) In addition if the moving frame is assumed to be orthonormal then

$$\Gamma_{ij}^k = -\Gamma_{ik}^j.$$

Proof. These two relations come from the properties of the Levi-Civita connection.

(i) Since the ∇ is torsion-free we have for all $X, Y \in \Gamma(M)$ that $[X, Y] = \nabla_X Y - \nabla_Y X$, hence

$$\begin{aligned}\theta^k([X_i, X_j]) &= \theta^k(\nabla_{X_i} X_j - \nabla_{X_j} X_i) \\ &= \theta^k(\Gamma_{ij}^l X_l - \Gamma_{ji}^l X_l) \\ &= \Gamma_{ij}^l \delta_l^k - \Gamma_{ji}^l \delta_l^k \\ &= \Gamma_{ij}^k - \Gamma_{ji}^k.\end{aligned}$$

(ii) Since ∇ is compatible with the metric we have

$$\begin{aligned}0 = \nabla_{X_i} \delta_{jk} = \nabla_{X_i} g(X_j, X_k) &= g(\nabla_{X_i} X_j, X_k) + g(X_j, \nabla_{X_i} X_k) \\ &= g(\Gamma_{ij}^l X_l, X_k) + g(X_j, \Gamma_{ik}^l X_l) \\ &= \Gamma_{ij}^l \delta_{lk} + \Gamma_{ik}^l \delta_{jl} \\ &= \Gamma_{ij}^k + \Gamma_{ik}^j\end{aligned}$$

□

We now investigate how an element of the coframe varies under covariant differentiation.

Lemma 1.8. We have

$$\nabla\theta^k = -\Gamma^k_{ij}\theta^j \otimes \theta^i, \tag{1.2}$$

or equivalently

$$\nabla_{X_i}\theta^k = -\Gamma^k_{ij}\theta^j. \tag{1.3}$$

Proof. Recall that the covariant derivative of a 1-form ω is given by

$$\nabla\omega(X,Y) = (\nabla_Y\omega)(X) = Y(\omega(X)) - \omega(\nabla_Y X),$$

for all $X, Y \in \Gamma(M)$. Applying this to θ^k we get

$$\begin{aligned}\nabla\theta^k(X_i, X_j) = (\nabla_{X_j}\theta^k)(X_i) &= X_j(\theta^k(X_i)) - \theta^k(\nabla_{X_j}X_i)\\ &= X_j(\delta^k_i) - \theta^k(\Gamma^l_{ji}X_l)\\ &= -\Gamma^l_{ji}\delta^k_l\\ &= -\Gamma^k_{ji}\end{aligned}$$

□

The n^3 functions Γ^k_{ij} completely determine the connection and this motivates the following definition:

Definition 1.9. The n^2 differential 1-forms defined by

$$\omega^i_j := \Gamma^i_{kj}\theta^k \tag{1.4}$$

are called the *connection* 1*-forms* of M. Observe that we have

$$\omega^i_j(X) = g(\nabla_X X_j, X_i)$$

The connection 1-forms are anti-symmetric and satisfy the *first structure equation*:

Lemma 1.10. The connection forms ω^i_j satisfy

$$\begin{cases} \omega^i_j = -\omega^j_i, \\ d\theta^i = \theta^j \wedge \omega^i_j. \end{cases}$$

Proof. The anti-symmetry is a direct consequence of lemma 1.7 (ii). The relation between the exterior derivative of a 1-form ω and its covariant derivative is the following

$$\begin{aligned}d\omega(X,Y) &= X\omega(Y) - Y\omega(X) - \omega([X,Y])\\ &= X\omega(Y) - Y\omega(X) - \omega(\nabla_X Y - \nabla_Y X)\\ &= X\omega(Y) - \omega(\nabla_X Y) - (Y\omega(X) - \omega\nabla_Y X)\\ &= \nabla\omega(Y,X) - \nabla\omega(X,Y).\end{aligned}$$

Therefore, applying this to θ^i we get by lemma (1.8)

$$\begin{aligned} d\theta^i(X_j, X_k) &= \nabla\theta^i(X_k, X_j) - \nabla\theta^i(X_j, X_k) \\ &= -\Gamma^i_{jk} + \Gamma^i_{kj}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \theta^l \wedge \omega^i_l(X_j, X_k) &= \delta^l_j \omega^i_l(X_k) - \delta^l_k \omega^i_l(X_j) \\ &= \omega^i_j(X_k) - \omega^i_k(X_j) \\ &= \Gamma^i_{kj} - \Gamma^i_{jk}. \end{aligned}$$

□

With respect to the basis $(\theta^j \wedge \theta^k)_{1\leq j,k\leq n}$ of $\Lambda^2(M)$ the exterior derivative of θ^i can be written as

$$d\theta^i = \frac{1}{2}\lambda^i_{jk}\theta^j \wedge \theta^k, \quad \lambda^i_{jk} = -\lambda^i_{kj},$$

where $\lambda^i_{jk} \in C^\infty(M)$. The first structure equation provides the following relations between the Γ^i_{kj} and the λ^i_{jk}:

Lemma 1.11. We have

$$\begin{cases} \lambda^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}, \\ \Gamma^i_{kj} = \frac{1}{2}\left(\lambda^i_{jk} + \lambda^j_{ki} - \lambda^k_{ij}\right). \end{cases}$$

Proof. By the first structure equation we have

$$\frac{1}{2}\lambda^i_{jk}\theta^j \wedge \theta^k = d\theta^i = \theta^j \wedge \omega^i_j = \Gamma^i_{kj}\theta^j \wedge \theta^k$$

and since $\lambda^i_{jk} = -\lambda^i_{kj}$ we have

$$\lambda^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}.$$

Moreover, from $\Gamma^i_{kj} = -\Gamma^j_{ki}$ we have

$$\begin{aligned} \lambda^i_{jk} + \lambda^j_{ki} - \lambda^k_{ij} &= \left(\Gamma^i_{kj} - \Gamma^i_{jk}\right) + \left(\Gamma^j_{ik} - \Gamma^j_{ki}\right) - \left(\Gamma^k_{ji} - \Gamma^k_{ij}\right) \\ &= \Gamma^i_{kj} - \Gamma^i_{jk} + \Gamma^j_{ik} + \Gamma^i_{kj} + \Gamma^i_{jk} - \Gamma^j_{ik} \\ &= 2\Gamma^i_{kj} \end{aligned}$$

□

1.3.1 The second fundamental form of a hypersurface

It will be useful later on to have an explicit expression for the second fundamental form of a hypersurface in terms of a moving coframe. So let $N \subset (M^n, g)$ be a Riemannian submanifold of dimension $n-1$ isometrically embedded in M, and let $(X_1, \ldots, X_{n-1})$ be an orthonormal moving frame on $U \subset N$. We extend this moving frame on N to an orthonormal moving frame on M by taking $X_n \perp TN$ of norm 1. We denote by $\nabla^\top$ the tangential part of the Levi-Civita connection of M. It coincides with the Levi-Civita connection of N and *the second fundamental form* b of N in M is given for all $X, Y \in \Gamma(N)$ by

$$\nabla_X Y = \nabla_X^\top Y + b(X,Y)X_n.$$

The second fundamental form is a $(2,0)$-tensor on N which satisfies the *Weingarten equation*: for all $X, Y \in \Gamma(N)$ we have

$$b(X,Y) = g(\nabla_X Y, X_n) = -g(\nabla_X X_n, Y).$$

Denoting with a and b indices varying between 1 and $n-1$, we have by the Weingarten equation and lemma 1.8:

$$b(X_a, X_b) = -g(\nabla_{X_a} X_n, X_b) = -\Gamma_{an}^b = \Gamma_{ab}^n.$$

Therefore the second fundamental form can be written as

$$b = \Gamma_{ab}^n \theta^a \otimes \theta^b \qquad (1.5)$$

The last equation provides a simple proof of the symmetry of b.

Lemma 1.12. For all $X, Y \in \Gamma(N)$ we have

$$b(X,Y) = b(Y,X).$$

Proof. Since the X_a's are tangent to N, it follows that $[X_a, X_b] \in \Gamma(N)$, i.e. $[X_a, X_b]$ is also tangent to N, for all $1 \leq a, b \leq n-1$. Therefore

$$\Gamma_{ab}^n - \Gamma_{ba}^n = \theta^n([X_a, X_b]) = \theta^n(\mu_{ab}^c X_c) = 0$$

with $\mu_{ab}^c \in C^\infty(U)$ and $1 \leq c \leq n-1$. So it follows that

$$b(X_a, X_b) = \Gamma_{ab}^n = \Gamma_{ba}^n = b(X_b, X_a).$$

□

1.3.2 The second fundamental form of a submanifold of arbitrary codimension

Suppose now that the submanifold N is of arbitrary dimension $1 \leq r \leq n-1$. As before let $U \subset N$ be an open subset on which an orthonormal moving frame $(X_1, \ldots, X_r)$ is defined. In the present case the normal space is of dimension $n-r$ at each point, therefore we choose $n-r$ unit vector fields $X_{r+1}, \ldots, X_n \in \Gamma(S^\perp N)$ that are normal to N. Here

$$S^\perp N = \{(p, v) \in SM \mid p \in N, v \perp T_pN\},$$

is the *unit normal bundle* of N in M. Again denoting by $\nabla^\top$ the tangent part of the Levi-Civita connection of M we define the *second fundamental form* B of N in M to be the normal part of the connection $\nabla_X Y$

$$\nabla_X Y = \nabla_X^\top Y + B(X, Y),$$

where X, Y arbitrary extensions to M of vector fields on N. The second fundamental form is a $(2,1)$-tensor on N and with respect to the frame $(X_1, \ldots, X_n)$ we have

$$B(X, Y) = B^\alpha(X, Y)X_\alpha, \quad n-r \leq \alpha \leq n.$$

Given a normal direction $\xi \in S^\perp N$, we define the second fundamental form in direction ξ by

$$B^\xi(X, Y) = g\left(B(X, Y), \xi\right).$$

If $\xi = X_\alpha$ and $X = X_a, Y = X_b$ we set

$$B^\alpha_{ab} = g\left(B(X_a, X_b), X_\alpha\right),$$

so that

$$B^\alpha(X_a, X_b) = B^\alpha_{ab} X_\alpha.$$

As before we have

Lemma 1.13. (Weingarten Equation) Let $X, Y \in \Gamma(N)$ and $\xi \in S^\perp N$ and extend these fields arbitrarily to M. Then

$$B^\xi(X, Y) = g(B(X, Y), \xi) = -g(\nabla_X \xi, Y). \tag{1.6}$$

In terms of connection forms we have using Equation 1.6

$$B^\alpha_{ab} = -g(\nabla_{X_a} X_\alpha, X_b) = -g(\Gamma^A_{a\alpha} X_A, X_b) = -\Gamma^b_{a\alpha} = \Gamma^\alpha_{ab},$$

therefore the second fundamental form in direction $\xi = \lambda^\alpha X_\alpha \in S^\perp N$ can be written as

$$B^\xi = \sum_{\alpha=n-r}^{n} \lambda^\alpha \Gamma^\alpha_{ab} \theta^a \otimes \theta^b.$$

As in the preceding section the symmetry of B comes directly from the latter expression for the second fundamental form.

1.4 The curvature forms

After having studied the connection in light of the moving frame technique, we will now express the curvature tensor in terms of the connection forms and their derivatives. Recall that the curvature tensor of M is the $(3,1)$ tensor defined for $X, Y, Z \in \Gamma(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The following lemma gives the components of R with respect to an orthonormal moving frame $(X_1, \dots, X_n)$. We write

$$R(X_k, X_l)X_j = R^i_{jkl} X_i.$$

Then

Lemma 1.14. We have

$$R^i_{jkl} = X_k(\Gamma^i_{lj}) - X_l(\Gamma^i_{kj}) + \Gamma^\mu_{lj}\Gamma^i_{k\mu} - \Gamma^\mu_{kj}\Gamma^i_{l\mu} + (\Gamma^\mu_{lk} - \Gamma^\mu_{kl})\,\Gamma^i_{\mu j}. \tag{1.7}$$

Proof. We have $R^i_{jkl} = \theta^i(R(X_k, X_l)X_j)$, and

$$\begin{aligned} R(X_k, X_l)X_j &= \nabla_{X_k}\nabla_{X_l}X_j - \nabla_{X_l}\nabla_{X_k}X_j - \nabla_{[X_k,X_l]}X_j \\ &= \nabla_{X_k}(\Gamma^\mu_{lj}X_\mu) - \nabla_{X_l}(\Gamma^\mu_{kj}X_\mu) - \nabla_{(\nabla_{X_k}X_l - \nabla_{X_l}X_k)}X_j \\ &= X_k(\Gamma^\mu_{lj})X_\mu + \Gamma^\mu_{lj}\Gamma^\nu_{k\mu}X_\nu - X_l(\Gamma^\mu_{kj})X_\mu - \Gamma^\mu_{kj}\Gamma^\nu_{l\mu}X_\nu - \Gamma^\mu_{kl}\Gamma^\nu_{\mu j}X_\nu + \Gamma^\mu_{lk}\Gamma^\nu_{\mu j}X_\nu \end{aligned}$$

which implies that

$$R^i_{jkl} = X_k(\Gamma^i_{lj}) - X_l(\Gamma^i_{kj}) + \Gamma^\mu_{lj}\Gamma^i_{k\mu} - \Gamma^\mu_{kj}\Gamma^i_{l\mu} + (\Gamma^\mu_{lk} - \Gamma^\mu_{kl})\,\Gamma^i_{\mu j}$$

□

The terms in the expression of R^i_{jkl} can be expressed in terms of the connection forms and their derivatives. Indeed, observe that

$$\begin{aligned} &\nabla\omega^i_j(X_l, X_k) = X_k(\omega^i_j(X_l)) - \omega^i_j(\nabla_{X_k}X_l) = X_k(\Gamma^i_{lj}) - \Gamma^\mu_{kl}\Gamma^i_{\mu j}, \\ &\nabla\omega^i_j(X_k, X_l) = X_l(\omega^i_j(X_k)) - \omega^i_j(\nabla_{X_l}X_k) = X_l(\Gamma^i_{kj}) - \Gamma^\mu_{lk}\Gamma^i_{\mu j}, \\ &\omega^i_\mu \wedge \omega^\mu_j(X_k, X_l) = \Gamma^\mu_{lj}\Gamma^i_{k\mu} - \Gamma^\mu_{kj}\Gamma^i_{l\mu}. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \theta^i(R(X_k, X_l)X_j) &= \nabla\omega^i_j(X_l, X_k) - \nabla\omega^i_j(X_k, X_l) + \omega^i_\mu \wedge \omega^\mu_j(X_k, X_l) \\ &= d\omega^i_j(X_k, X_l) + \omega^i_\mu \wedge \omega^\mu_j(X_k, X_l). \end{aligned}$$

This motivates the following definition:

Definition 1.15. The n^2 differential 2-forms defined by

$$\Omega^i_j := d\omega^i_j + \omega^i_k \wedge \omega^k_j \tag{1.8}$$

are called the *curvature forms*. They obviously satisfy $\Omega^i_j(X_k, X_l) = R^i_{jkl}$, so that we can write

$$\Omega^i_j = R^i_{jkl}\theta^k \otimes \theta^l,$$

or using the symmetry in the last two indices $R^i_{jkl} = R^i_{jlk}$:

$$\Omega^i_j = \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l.$$

Equation (1.8) is called the *second structure equation*.

As well as the connection forms, the curvature forms are anti-symmetric:

Lemma 1.16. We have

$$\Omega^i_j = -\Omega^j_i.$$

Proof. It a direct consequence of the anti-symmetry of the connection forms and the wedge product:

$$\begin{aligned}\Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j \\ &= -d\omega^j_i + \omega^k_i \wedge \omega^j_k \\ &= -d\omega^j_i - \omega^j_k \wedge \omega^k_i \\ &= -\Omega^j_i.\end{aligned}$$

□

In the rest of this thesis, we will often use the 2-forms Ω_{ij} obtained from the curvature forms by lowering an index. As the components of the metric with respect to the orthonormal moving $(X_1, \ldots, X_n)$ are simply $g_{ij} = \delta_{ij}$ the components of Ω_{ij} are the same as those of Ω^i_j, i.e.

$$\Omega_{ij} = g_{ik}\Omega^k_j = \delta_{ik}\Omega^k_j = \Omega^i_j.$$

1.5 First and Second Bianchi Identities

In the case of a connection with vanishing torsion, we have the following identities.

Proposition 1.17. We have

(a) the first Bianchi identity

$$\Omega^i_j \wedge \theta^j = 0, \tag{1.9}$$

(b) the second Bianchi identity

$$d\Omega^i_j = \Omega^i_k \wedge \omega^k_j - \omega^i_k \wedge \Omega^k_j. \tag{1.10}$$

Proof. (a) We have by the first and the second structure equations

$$\begin{aligned} 0 &= d(d\theta^j) \\ &= d(\theta^i \wedge \omega^j_i) \\ &= d\theta^i \wedge \omega^j_i - \theta^i \wedge d\omega^j_i \\ &= (\theta^k \wedge \omega^i_k) \wedge \omega^j_i - \theta^i \wedge (\Omega^j_i - \omega^j_k \wedge \omega^k_i) \\ &= \Omega^j_i \wedge \theta^i. \end{aligned}$$

(b) Similarly

$$\begin{aligned} 0 &= d(d\omega^i_j) \\ &= d(\Omega^i_j - \omega^i_k \wedge \omega^k_j) \\ &= d\Omega^i_j - d\omega^i_k \wedge \omega^k_j + \omega^i_k \wedge d\omega^k_j \\ &= d\Omega^i_j - (\Omega^i_k + \omega^i_l \wedge \omega^l_k) \wedge \omega^k_j + \omega^i_k \wedge (\Omega^k_j + \omega^k_l \wedge \omega^l_j) \\ &= -\Omega^i_k \wedge \omega^k_j + \omega^i_k \wedge \Omega^k_j. \end{aligned}$$

□

1.6 Transformation law for the connection and curvature forms

When working with charts on a manifold, it is crucial to know how quantities defined locally behave under a change of coordinates. The same question arises in the formalism of moving frames, but instead of change of coordinates we consider change of moving frames. We reproduce here a standard argument that may be found for example in [Spi99], Vol.2, Chapter 7. It is particularly useful in this paragraph to introduce the following notations. Let $X = (X_1, \ldots, X_n)$ and $X' = (X'_1, \ldots, X'_n)$ be two moving frame on an open subset $U \subset M$. Then there exists a smooth map $A : U \to \mathrm{GL}_n(\mathbb{R})$ such that $X' = X \cdot A$, meaning that each vector field of X' is given at $p \in U$ by

$$X'_i(p) = X_j(p) A^j_i(p),$$

where $A^j_i(p)$ are the coefficients of the invertible matrix $A(p)$. Let ∇ be a connection (not necessarily Levi-Civita).The equation

$$\nabla_{X_i} X_j = \omega^k_j(X_i) X_k,$$

can be written in the new notation as $\nabla X = X \cdot \omega$, where ω is the matrix formed by the connection forms ω^i_j. Finally denote by ω' the matrix of the connection forms of X'. Then ω and ω' are related by the following formula:

Lemma 1.18. We have

$$\omega' = A^{-1}dA + A^{-1}\omega A.$$

Proof. From the relation $X' = X \cdot A$ we have

$$\begin{aligned} X' \cdot \omega' &= \nabla X' \\ &= \nabla(X \cdot A) \\ &= (\nabla X) \cdot A + X \cdot \nabla A \\ &= (X \cdot \omega) \cdot A + X \cdot dA \\ &= X \cdot (\omega \cdot A + dA) \\ &= X' \cdot A^{-1} \cdot (\omega \cdot A + dA)\,, \end{aligned}$$

whence $\omega' = A^{-1} \cdot dA + A^{-1} \cdot \omega \cdot A$ □

Assuming now that the manifold is endowed with a Riemannian metric and that the moving frames X and X' are orthonormal, we establish a similar transformation law for the curvature forms. Denote by Ω and Ω' the matrices formed by the Ω^i_j and $(\Omega')^i_j$. Then

Lemma 1.19. We have

$$\Omega' = A^{-1} \cdot \Omega \cdot A.$$

Equivalently denoting then inverse of A by $A^{-1} = (b^k_l)$ we have for all $1 \leq i, j \leq n$

$$(\Omega')^i_j = b^i_k \Omega^k_l a^l_j.$$

Proof. The second structure equation in the matrix notation reads

$$\Omega = d\omega + \omega \wedge \omega,$$

Hence by Lemma 1.18 we get

$$\begin{aligned} \Omega' &= d\omega' + \omega' \wedge \omega' \\ &= d\left(A^{-1} \cdot dA + A^{-1} \cdot \omega \cdot A\right) + \left(A^{-1} \cdot dA + A^{-1} \cdot \omega \cdot A\right) \wedge \left(A^{-1} \cdot dA + A^{-1} \cdot \omega \cdot A\right) \end{aligned}$$

Since $d(A^{-1}) = -A^{-1} \cdot dA \cdot A^{-1}$ and $d(dA) = 0$ we get

$$\begin{aligned} \Omega' = &- A^{-1} \cdot dA \cdot A^{-1} \wedge dA + d(A^{-1} \cdot \omega \cdot A) + (A^{-1} \cdot dA) \wedge (A^{-1} \cdot dA) \\ &+ (A^{-1} \cdot dA) \wedge (A^{-1} \cdot \omega \cdot A) + (A^{-1} \cdot \omega \cdot A) \wedge (A^{-1} \cdot dA) + (A^{-1} \cdot \omega \cdot A) \wedge (A^{-1} \cdot \omega \cdot A) \\ = &\, dA^{-1} \wedge (\omega \cdot A) + A^{-1} \cdot d\omega \cdot A - A^{-1} \cdot \omega \wedge dA + (A^{-1} \cdot dA \cdot A^{-1}) \wedge (\omega \cdot A) \\ &+ (A^{-1} \cdot \omega) \wedge dA + (A^{-1} \cdot \omega) \wedge (\omega \cdot A) \\ = &\, (-A^{-1} \cdot dA \cdot A^{-1}) \wedge (\omega \cdot A) + A^{-1} \cdot d\omega \cdot A + (A^{-1} \cdot dA \cdot A^{-1}) \wedge (\omega \cdot A) \\ &+ (A^{-1} \cdot \omega) \wedge (\omega \cdot A) \\ = &\, A^{-1} \cdot d\omega \cdot A + (A^{-1} \cdot \omega) \wedge (\omega \cdot A) \\ = &\, A^{-1}\,(d\omega + \omega \wedge \omega) \\ = &\, A^{-1} \Omega A. \end{aligned}$$

□

1.7 The Gauss equation

One of the fundamental equations of the theory of Riemannian submanifolds is the *Gauss equation*, relating the curvature of the ambient manifold with the curvature of a submanifold. Let (M, g) be an m-dimensional Riemannian manifold and $(N, \overline{g})$ an n-dimensional Riemannian submanifold of (M, g). Then, on an open subset $U \subset M$ we consider an orthonormal moving frame $(e_1, \ldots, e_m)$ such that the first n vector fields form an orthonormal moving frame on $N \cap U$. Let $(\theta^1, \ldots, \theta^m)$ be the dual 1-forms and let ω^A_B be the connection forms of M. Observe that since $\theta^\alpha(X) = 0$ for all $X \in \Gamma(N)$, the restriction to N of the first structure equation for M gives

$$d\theta^i = \theta^A \wedge \omega^i_A = \theta^j \wedge \omega^i_j \quad \text{and} \quad \omega^i_j = -\omega^j_i,$$

therefore the ω^i_j are the connection forms of the Levi-Civita connection on N. Now, denote by Ω^A_B the curvature forms of M and by $\overline{\Omega}^i_j$ the curvature forms of N. The Gauss equation is the object of the next proposition.

Proposition 1.20. (Gauss equation)
On $N \cap U$ we have

$$\Omega^i_j - \overline{\Omega}^i_j = \omega^i_\alpha \wedge \omega^\alpha_j.$$

Proof. The second structure equation for M but restricted to N is

$$\begin{aligned}\Omega^A_B &= d\omega^A_B + \omega^A_C \wedge \omega^C_B \\ &= d\omega^i_j + \omega^i_C \wedge \omega^C_j \\ &= d\omega^i_j + \omega^i_k \wedge \omega^k_j + \omega^i_\alpha \wedge \omega^\alpha_j.\end{aligned}$$

On the other hand, the second structure equation for N is

$$\overline{\Omega}^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j,$$

therefore

$$\Omega^i_j - \overline{\Omega}^i_j = \omega^i_\alpha \wedge \omega^\alpha_j.$$

□

The Gauss equation can be written in terms of the components Γ^A_{CB}:

$$\begin{aligned}\Omega^i_j - \overline{\Omega}^i_j &= \omega^i_\alpha \wedge \omega^\alpha_j \\ &= \sum_{\alpha=n+1}^{m} -\Gamma^\alpha_{ki}\theta^k \wedge \Gamma^\alpha_{lj}\theta^l \\ &= -\sum_{\alpha=n+1}^{m} \Gamma^\alpha_{ik}\Gamma^\alpha_{jl}\theta^k \wedge \theta^l,\end{aligned}$$

since $\Gamma^\alpha_{kl} = \Gamma^\alpha_{lk}$ by lemma 1.12. Also observe that evaluating the Gauss equation on two vectors fields e_k and e_l of the moving frame gives

$$R_{ijkl} - \overline{R}_{ijkl} = -\sum_{\alpha=n+1}^{m} \Gamma^\alpha_{ip}\Gamma^\alpha_{jq}(\delta^p_k\delta^q_l - \delta^p_l\delta^q_k) = -\sum_{\alpha=n+1}^{m} \left(\Gamma^\alpha_{ik}\Gamma^\alpha_{jl} - \Gamma^\alpha_{il}\Gamma^\alpha_{jk}\right)$$

1.8 Conformal change of the metric

We shall now investigate how the connection and curvature forms behave when one performs a conformal change of the metric, i.e. when the metric is multiplied by a positive function. Consider an n-dimensional Riemannian manifold (M, g) as well as a positive function $f : M \to \mathbb{R}^*_+$. Define a new Riemannian metric $\tilde{g}$ on M by setting $\tilde{g} = f^2 g$ and let $(e_1, \dots, e_n)$ be an orthonormal moving frame with respect to the metric g on a subset $U \subset M$, with dual coframe $(\theta^1, \dots, \theta^n)$. Then an orthonormal moving frame with respect to $\tilde{g}$ is given by $\tilde{e}_i = \frac{1}{f} e_i$ and its dual coframe is simply $\tilde{\theta}^i = f\theta^i$. In the basis (θ^i), write the differential of f as $df = a_j\theta^j$ with $a_j \in C^\infty(M)$. The relation between the connection forms of $(M, \tilde{g})$ and (M, g) is given in the following lemma:

Lemma 1.21. We have

$$\tilde{\omega}^i_j = \omega^i_j + \frac{1}{f}(a_j\theta^i - a_i\theta^j)$$

Proof. The first structure equation for both $(M, \tilde{g})$ and (M, g) give

$$\begin{aligned} \tilde{\theta}^j \wedge \tilde{\omega}^i_j &= \tilde{\theta}^i \\ &= d(f\theta^i) \\ &= df \wedge \theta^i + f d\theta^i \\ &= df \wedge \theta^i + f(\theta^j \wedge \omega^i_j) \\ &= \theta^j \wedge \left(a_j\theta^i + f\omega^i_j\right), \end{aligned}$$

but the left-hand side can also be written as $\theta^j \wedge f\tilde{\omega}^i_j$, therefore we get

$$\theta^j \wedge \left(a_j\theta^i + f\omega^i_j - f\tilde{\omega}^i_j\right) = 0.$$

Since $a_i\theta^j \wedge \theta^j = 0$, the last equation can be written

$$\left(a_j\theta^i - a_i\theta^j + f\omega^i_j - f\tilde{\omega}^i_j\right) \wedge \theta^j = 0.$$

Let $\eta^i_j = a_j\theta^i - a_i\theta^j + f\omega^i_j - f\tilde{\omega}^i_j$ which we write in the basis (θ^k) as $\eta^i_j = b^i_{jk}\theta^k$. Then observe that $\eta^i_j = -\eta^j_i$ and that $\eta^i_j \wedge \theta^j = 0$, so that we have the (anti-)symmetries

$$b^i_{jk} = -b^j_{ik} \quad \text{and} \quad b^i_{jk} = b^i_{kj}.$$

But this implies that $b^i_{jk} = 0$, indeed

$$b^i_{jk} = -b^j_{ik} = -b^j_{ki} = b^k_{ji} = b^k_{ij} = -b^i_{kj} = -b^i_{jk},$$

therefore $\eta^i_j = 0$, i.e.

$$a_j\theta^i - a_i\theta^j + f\omega^i_j - f\tilde{\omega}^i_j = 0 \Longleftrightarrow \tilde{\omega}^i_j = \omega^i_j + \frac{1}{f}(a_j\theta^i - a_i\theta^j)$$

□

This relation between 1-forms yields an expression relating the components $\tilde{\Gamma}^i_{kj}$ and Γ^i_{kj}. Indeed from the last lemma we get the equation

$$f\tilde{\Gamma}^i_{lj}\theta^l = \Gamma^i_{lj}\theta^l + \frac{1}{f}\left(a_j\theta^i - a_i\theta^j\right),$$

since $\tilde{\omega}^i_j = \tilde{\Gamma}^i_{lj}\tilde{\theta}^l = f\tilde{\Gamma}^i_{lj}\theta^l$. Evaluating both sides of the equation over e_k gives

$$\tilde{\Gamma}^i_{kj} = \frac{1}{f}\Gamma^i_{kj} + \frac{1}{f^2}\left(a_j\delta^i_k - a_i\delta^j_k\right) \tag{1.11}$$

The change of the curvature tensor in the case of a conformal change of the metric is rather complicated. Although the general form of $\tilde{\Omega}^i_j$ is quite messy, we will be interested in a case where the conformal factor f is sufficiently simple for the next formula to be manageable.

Proposition 1.22. The curvature forms of $(M, \tilde{g})$ are given by

$$\tilde{\Omega}^i_j = \Omega^i_j + \theta^i \wedge \epsilon_j - \theta^j \wedge \epsilon_i - a_k a^k \theta^i \wedge \theta^j,$$

where

$$\epsilon_i = \frac{2a_i}{f^2}df + \frac{1}{f}(a_k\omega^k_i - da_i)$$

Proof. The second structure equation for $(M, \tilde{g})$ reads

$$\tilde{\Omega}^i_j = d\tilde{\omega}^i_j + \tilde{\omega}^i_k \wedge \tilde{\omega}^k_j.$$

Using Lemma 1.21 this equation can be rewritten

$$\begin{aligned}\tilde{\Omega}^i_j &= d\left(\omega^i_j + \frac{1}{f}(a_j\theta^i - a_i\theta^j)\right) + \left(\omega^i_k + \frac{1}{f}(a_k\theta^i - a_i\theta^k)\right) \wedge \left(\omega^k_j + \frac{1}{f}(a_j\theta^k - a_k\theta^j)\right)\\ &= d\omega^i_j + d\left(\frac{1}{f}\right)\left(a_j\theta^i - a_i\theta^j\right) + \frac{1}{f}d\left(a_j\theta^i - a_i\theta^j\right) + \omega^i_k \wedge \omega^k_j + \omega^i_k \wedge \frac{1}{f}\left(a_j\theta^k - a_k\theta^j\right)\\ &\quad \frac{1}{f}\left(a_k\theta^i - a_i\theta^k\right) \wedge \omega^k_j + \frac{1}{f^2}\left(a_k\theta^i - a_i\theta^k\right) \wedge \left(a_j\theta^k - a_k\theta^j\right).\end{aligned}$$

But we know the following things:

(a) the terms $d\omega^i_j + \omega^i_k \wedge \omega^k_j$ are equal to Ω^i_j by the second structure equation for (M, g);

(b) the differential of the fraction $\frac{1}{f}$ is given by

$$d\left(\frac{1}{f}\right) = -\frac{1}{f^2}df = -\frac{1}{f^2}a_k\theta^k;$$

(c) the terms $d\theta^i$ are equal via the first structure equation for (M, g) to

$$d\theta^i = \theta^j \wedge \omega^i_j;$$

therefore the above expression for $\tilde{\Omega}^i_j$ can be written as

$$\begin{aligned}\tilde{\Omega}^i_j = \Omega^i_j &- \frac{1}{f^2}\left(a_k a_j \theta^k \wedge \theta^i - a_k a_i \theta^k \wedge \theta^j\right) \\ &+ \frac{1}{f}\left(da_j \wedge \theta^i - da_i \wedge \theta^j + a_j\theta^k \wedge \omega^i_k - a_i\theta^k \wedge \omega^j_k\right) \\ &+ \frac{1}{f}\left(a_k\theta^j \wedge \omega^i_k\right) - a_j\theta^k \wedge \omega^i_k) + \frac{1}{f}\left(a_k\theta^i \wedge \omega^k_j - a_i\theta^k \wedge \omega^k_j\right) \\ &+ \frac{1}{f^2}\left(a_k a_j \theta^i_\wedge \theta^k - a_k a^k \theta^i \wedge \theta^j + a_i a_k \theta^k \wedge \theta^j\right)\end{aligned}$$

and by simplifying and reorganizing the terms we get

$$\begin{aligned}\tilde{\Omega}^i_j = \Omega^i_j &+ \theta^i \wedge \left(\frac{2a_j}{f^2}a_k\theta^k + \frac{1}{f}(a_k\omega^k_j - da_j)\right) \\ &- \theta^j \wedge \left(\frac{2a_i}{f^2}a_k\theta^k + \frac{1}{f}(a_k\omega^k_i - da_i)\right) - a_k a^k \theta^i \wedge \theta^j,\end{aligned}$$

which gives the desired result by remembering that $a_k\theta^k = df$ and by setting

$$\epsilon_i = \frac{2a_i}{f^2}df + \frac{1}{f}(a_k\omega^k_i - da_i).$$

□

Remark 1.23. Proposition 1.22 is the moving frame version of the well-known formula giving the $(0, 4)$ curvature tensor after a conformal change of the metric $\tilde{g} = e^{2\varphi}g$:

$$\tilde{R} = e^{2\varphi}\left(R - g \owedge \left(\nabla d\varphi - d\varphi \circ d\varphi + \frac{1}{2}|d\varphi|^2 g\right)\right),$$

where $\owedge$ is the Kulkarni-Nomizu product which is defined on p. 30. This formula can be found in [Bes08].

Remark 1.24. A case, which will be of particular interest in the sequel, is when the conformal factor is simply given by a constant, i.e. the new metric writes $\tilde{g} = t^2 g$, with $t \in \mathbb{R}_+^*$. In this case, writing $f : M \to \mathbb{R}_+^*$ the constant function $f(p) = t$ for all $p \in M$, we have that $df = 0$, i.e. with the previous notations $a_k = 0$ for all $1 \leq k \leq n$. By Lemma 1.21 and Proposition 1.22 we obtain that

$$\tilde{\omega}_j^i = \omega_j^i \quad \text{and} \quad \tilde{\Omega}_j^i = \Omega_j^i.$$

1.9 The Pfaffian

In this section, we introduce the *Pfaffian* of an even-dimensional squared matrix. It is a combinatorial expression depending on the coefficients of the considered matrix and it appears in a crucial way in the Gauss-Bonnet-Chern theorem. We begin with the case where the matrix has coefficients in $\mathbb{R}$, but what will be important is the case where the considered matrix is a matrix of 2-forms, more precisely with the curvature forms Ω_j^i as coefficients.

Definition 1.25. Let $n = 2k$ be an even integer and let $A \in M_n(\mathbb{R})$ be a squared matrix. The *Pfaffian* of A, is the following scalar

$$\mathrm{Pf}(A) = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \epsilon_\sigma A_{\sigma_1 \sigma_2} \cdots A_{\sigma_{2k-1} \sigma_{2k}} \tag{1.12}$$

One of the main features of the Pfaffian is that for a skew-symmetric matrix A we have

$$\mathrm{Pf}(A)^2 = \det(A).$$

See [Spi99, Vol. 5, p. 289] for additional details. For the expression (1.12) to make sense, it is not necessary that the coefficients of the matrix belong to $\mathbb{R}$ or $\mathbb{C}$. Actually, it is sufficient for them to be in a commutative ring. Following Spivak, we consider at each point $p \in M$ the ring (for the wedge product) of even-dimensional differential forms on T_pM

$$\Lambda^e(T_pM) = \mathbb{R} \oplus \Lambda^2(T_pM) \oplus \ldots \oplus \Lambda^{2k}(T_pM),$$

which is commutative. Then at each point $p \in M$, we consider the anti-symmetric matrix $\Omega(p) = (\Omega_{ij}(p)) \in M_{2k}(\Lambda^e(T_pM))$ with coefficients the curvature forms of M at p. Choosing an orthonormal moving frame $(e_1, \ldots, e_n)$ on an open subset $U \subset M$, it makes therefore sense to look at the Pfaffian of Ω, which is a $2k$-form on U:

$$\mathrm{Pf}(\Omega) = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \epsilon_\sigma \Omega_{\sigma_1 \sigma_2} \wedge \ldots \wedge \Omega_{\sigma_{2k-1} \sigma_{2k}}. \tag{1.13}$$

Although the Pfaffian is defined on the domain of the chosen moving frame it appears that if M is oriented, then it is a differential form on the whole of the manifold M.

Proposition 1.26. Let (M, g) be an oriented even dimensional Riemannian manifold and let $E = (e_1, \ldots, e_n)$ and $\overline{E} = (\overline{e}_1, \ldots, \overline{e}_n)$ be two orthonormal moving frames on a open subset $U \subset M$ with associated matrices of curvature forms Ω and Ω'. Then

$$\mathrm{Pf}(\Omega') = \det(A)\,\mathrm{Pf}(\Omega),$$

where $\overline{E} = E \cdot A$.

Proof. On U the two moving frames are related by

$$\overline{E} = E \cdot A$$

with $A : U \to \mathrm{O}(n)$ a smooth map. By Lemma 1.19 we know that the curvature forms are related by the equation

$$\overline{\Omega}^i_j = \sum_{r,s=1}^{n} a^r_i \Omega^r_s a^s_j,$$

since the coefficients of $A = (a^i_j)$ are given by a^j_i. Lowering an index we get

$$\overline{\Omega}_{ij} = \sum_{r,s=1}^{n} a_{ri} a_{sj} \Omega_{rs}$$

therefore we have

$$\begin{aligned}
\mathrm{Pf}(\overline{\Omega}) &= \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \varepsilon_\sigma \overline{\Omega}_{\sigma_1\sigma_2} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k-1}\sigma_{2k}} \\
&= \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \sum_{i_1,\ldots,i_{2k}=1}^{n} \varepsilon_\sigma a_{i_1\sigma_1} a_{i_2\sigma_2} \cdots a_{i_{2k-1}\sigma_{2k-1}} a_{i_{2k}\sigma_{2k}} \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2k-1}i_{2k}} \\
&= \frac{1}{2^k k!} \sum_{i_1,\ldots,i_{2p}=1}^{n} \left(\sum_{\sigma \in \mathfrak{S}_{2k}} \varepsilon_\sigma a_{i_1\sigma_1} \cdots a_{i_{2k}\sigma_{2k}} \right) \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2k-1}i_{2k}} \\
&= \frac{1}{2^k k!} \sum_{i_1,\ldots,i_{2k}=1}^{n} \varepsilon_{i_1\ldots i_{2k}} \det(A) \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2k-1}i_{2k}} \\
&= \det(A) \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \varepsilon_\sigma \Omega_{\sigma_1\sigma_2} \wedge \ldots \wedge \Omega_{\sigma_{2k-1}\sigma_{2k}} \\
&= \det(A)\,\mathrm{Pf}(\Omega)
\end{aligned}$$

□

As a corollary we get

Corollary 1.27. The Pfaffian of Ω of an *oriented* Riemannian manifold (M, g) is a globally defined n-differential form on M, i.e. $\mathrm{Pf}(\Omega) \in \Omega^n(M)$.

Proof. Since the manifold is oriented, we can choose the two orthonormal moving frames to be positively oriented. Therefore the map A of the last proof take values in $\mathrm{SO}(n)$ rather than in $\mathrm{O}(n)$. Whence

$$\mathrm{Pf}(\overline{\Omega}) = \det(A)\,\mathrm{Pf}(\Omega) = \mathrm{Pf}(\Omega).$$

□

Example 1.28. Recall that in dimension 2, there is only one non zero curvature form (up to sign), namely $\Omega^1_2 = -\Omega^2_1$. Therefore, given a Riemannian surface (S, g), the Pfaffian of its curvature form matrix is simply

$$\mathrm{Pf}(\Omega) = \frac{1}{2}(\Omega_{12} - \Omega_{21}) = \Omega_{12} = \frac{1}{2}R_{12ij}\theta^i \wedge \theta^j = R_{1212}\mathrm{dvol}_S = K\mathrm{dvol}_S,$$

where K is the Gauss curvature of the surface S. So if S is closed, we can write the Gauss-Bonnet theorem for S in terms of the Pfaffian of its curvature form:

$$\int_S \mathrm{Pf}(\Omega) = 2\pi\chi(S).$$

1.10 Conical Warped-Product Manifolds

Let (N, g_N) be an $(n-1)$-dimensional compact Riemannian manifold and consider the manifold $M = N \times (0, \infty)$ endowed with the following warped-product metric

$$g = f^2 g_N + dt^2,$$

where $f : M \to (0, \infty)$ is defined by $f(x, t) = t$ for $(x, t) \in N \times (0, \infty)$. Such a warped-product metric will be called a *conical warped-product metric*. Let $(e_1, \dots, e_{n-1})$ be an orthonormal moving frame on a subset $U \subset N$ and let $e_n = \frac{\partial}{\partial t}$, with t the arclentgh parameter on $(0, \infty)$. Then, at each point $(x, t) \in M$ we have a splitting of the tangent space as

$$T_{(x,t)}M = T_x N \oplus \mathbb{R},$$

and we can extend the vector fields e_A to M. In order to get an orthonormal moving frame on $U \times (0, \infty)$ we need to scale those extensions by setting

$$\overline{e}_i = \frac{1}{f}e_i \quad \text{and} \quad \overline{e}_n = e_n.$$

Denote by $\overline{\theta}^A$ and θ^A the dual forms to $\overline{e}_A$ and e_A so that we have the relations $\overline{\theta}^i = f\theta^i$ and $\overline{\theta}^n = \theta^n = dt$.

The warped-product structure on M yields an expression for the curvature forms of M in terms of those of the submanifold N. Indeed, via the Gauss equation, one can relate the curvature forms of M with those of a hypersurface of the form

$(N_t, g_t) = (N, t^2 g_N)$ for some $t \in (0, \infty)$. Since (N_t, g_t) is conformally equivalent to (N, g) with a constant conformal factor, we know by Proposition 1.22 that

$$ {}^t\omega^i_j = \omega^i_j \quad \text{and} \quad {}^t\Omega^i_j = \Omega^i_j, $$

where ${}^t\omega^i_j$ and ${}^t\Omega^i_j$ denote the connection and curvature forms of (N_t, g_t)[1]. On the other hand, the Gauss equation for the submanifold (N_t, g_t) of (M, g) gives

$$ \overline{\Omega}^i_j - {}^t\Omega^i_j = \overline{\omega}^i_n \wedge \overline{\omega}^n_j, $$

therefore the curvature forms of M can be written as

$$ \overline{\Omega}^i_j = \Omega^i_j - \overline{\omega}^i_n \wedge \overline{\omega}^j_n, $$

so it remains to compute the connection forms $\overline{\omega}^i_n$.

Proposition 1.29. We have

$$ \begin{cases} \overline{\omega}^i_n &= \theta^i, \\ \overline{\Omega}^i_n &= 0. \end{cases} $$

Proof. On one hand, the first structure equation for M gives

$$ \begin{aligned} d\overline{\theta}^i &= \overline{\theta}^A \wedge \overline{\omega}^i_A \\ &= \overline{\theta}^n \wedge \overline{\omega}^i_n + \overline{\theta}^j \wedge \overline{\omega}^i_j \\ &= \overline{\theta}^n \wedge \overline{\Gamma}^i_{An} \overline{\theta}^A + \overline{\theta}^j \wedge \overline{\Gamma}^i_{kj} \overline{\theta}^k \\ &= \overline{\theta}^n \wedge \overline{\Gamma}^i_{jn} \overline{\theta}^j + \overline{\theta}^j \wedge \overline{\Gamma}^i_{kj} \overline{\theta}^k \\ &= \overline{\Gamma}^i_{jn} \overline{\theta}^n \wedge \overline{\theta}^j + \overline{\Gamma}^i_{kj} \overline{\theta}^j \wedge \overline{\theta}^k. \end{aligned} $$

On the other hand, using the fact that $\overline{\theta}^i = f\theta^i$ and the first structure equation for N we get

$$ \begin{aligned} d\overline{\theta}^i &= d(f\theta^i) \\ &= df \wedge \theta^i + f d\theta^i \\ &= \frac{\partial f}{\partial t} dt \wedge \theta^i + f(\theta^j \wedge \omega^i_j) \\ &= \overline{\theta}^n \wedge \theta^i + \overline{\theta}^j \wedge \Gamma^i_{kj} \theta^k \\ &= \frac{1}{f} \overline{\theta}^n \wedge \overline{\theta}^i + \frac{1}{f} \Gamma^i_{kj} \overline{\theta}^j \wedge \overline{\theta}^k, \end{aligned} $$

[1]Here the superscript t refers to the arclength parameter and *not* to the transpose of the matrix

therefore, by comparing the coefficients of $\overline{\theta}^n \wedge \overline{\theta}^j$ we get

$$\overline{\Gamma}^i_{jn} = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{f} & \text{if } i = j. \end{cases} \iff \overline{\Gamma}^n_{ji} = \begin{cases} 0 & \text{if } i \neq j, \\ -\frac{1}{f} & \text{if } i = j. \end{cases}$$

Moreover, since $\overline{\theta}^n = dt$ we get that

$$0 = ddt = d\overline{\theta}^n = \overline{\theta}^A \wedge \overline{\omega}^n_A = \overline{\theta}^j \wedge \overline{\omega}^n_j = \overline{\Gamma}^n_{Aj} \overline{\theta}^j \wedge \overline{\theta}^A,$$

so that we deduce $\overline{\Gamma}^n_{Aj} = 0$ for all $A \neq j$ and in particular $\overline{\Gamma}^n_{nj} = 0$ for all $j = 1, \ldots, n-1$. Hence

$$\overline{\omega}^n_i = \overline{\Gamma}^n_{Ai} \overline{\theta}^A = \overline{\Gamma}^n_{ii} \overline{\theta}^i = -\frac{1}{f} \overline{\theta}^i = -\theta^i,$$

i.e. $\overline{\omega}^i_n = \theta^i$ by anti-symmetry.
As a consequence we immediately get the other equation:

$$\begin{aligned} \overline{\Omega}^n_i &= d\overline{\omega}^n_i + \overline{\omega}^n_A \wedge \overline{\omega}^A_i \\ &= -d\theta^i + \overline{\omega}^n_j \wedge \overline{\omega}^j_i \\ &= -d\theta^i + \theta^j \wedge \overline{\omega}^i_j \\ &= -d\theta^i + d\theta^i \\ &= 0. \end{aligned}$$

□

As a corollary of this proposition we obtain that the Pfaffian of the curvature forms of the conical warped-product (M, g) vanishes at each point.

Corollary 1.30. We have
$$\mathrm{Pf}(\overline{\Omega}) = 0$$

Proof. If the dimension n of M is odd, then the Pfaffian is zero by definition. If $n = 2p$, then

$$\mathrm{Pf}(\overline{\Omega}) = \frac{1}{2^p p!} \sum_{\sigma \in \mathfrak{S}_{2p}} \varepsilon_\sigma \overline{\Omega}_{\sigma_1 \sigma_2} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2p-1} \sigma_{2p}},$$

so each summand contains a curvature form of the type $\overline{\Omega}^i_n$ or $\overline{\Omega}^n_i$ which is zero by the last proposition. Whence $\mathrm{Pf}(\Omega) = 0$. □

So we have shown that the curvature forms of M are given by

$$\overline{\Omega}^i_j = \Omega^i_j - \theta^i \wedge \theta^j. \tag{1.14}$$

It appears that the form $\theta^i \wedge \theta^j$ can be written using the Kulkarni-Nomizu product of the metric with itself:

$$\theta^i \wedge \theta^j = \frac{1}{2}\left(g_N \owedge g_N\right)(e_i, e_j, \cdot, \cdot) : \Gamma(U) \times \Gamma(U) \longrightarrow C^\infty(U), \tag{1.15}$$

where $\owedge$ denotes the following product on the algebra of symmetric $(2,0)$-tensors: given α and β two symmetric $(2,0)$-tensors, their *Kulkarni-Nomizu product* is the $(4,0)$-tensor given by

$$\begin{aligned}\alpha \owedge \beta(X,Y,Z,W) = {} & \alpha(X,Z)\beta(Y,W) + \alpha(Y,W)\beta(X,Z) \\ & - \alpha(X,W)\beta(Y,Z) - \alpha(Y,Z)\beta(X,W).\end{aligned}$$

In particular, since g_N is a symmetric $(2,0)$-tensor, we have

$$(g_N \owedge g_N)(X,Y,Z,W) = 2\left(g_N(X,Z)g_N(Y,W) - g_N(X,W)g_N(Y,Z)\right).$$

The 2-forms given in (1.15) does actually have a deep geometrical meaning; indeed, the curvature tensor of a constant sectional curvature κ Riemannian manifold (V, g_V) can be written

$$R_\kappa = \frac{\kappa}{2} g_V \owedge g_V,$$

and we recall that the curvature tensor can also be written (as a double-form) in terms of the curvature forms as

$$R = \frac{1}{2}\Omega_{ij} \otimes \theta^i \wedge \theta^j.$$

Hence it follows that if N happens to be the unit sphere, the 2-forms given by $\theta^i \wedge \theta^j$ are actually its curvature forms. In this case, the curvature forms $\overline{\Omega}^i_j$ vanish identically, which is easily explained since the product manifold $\mathbb{S}^{n-1} \times (0,\infty)$ endowed with the warped-product metric $g = f^2 g_{\mathbb{S}^{n-1}} + dt^2$ is isometric to $\mathbb{R}^n \setminus \{0\}$ with the euclidean metric and is therefore flat.
Let us summarize the situation: the connection and curvature forms of M are given in terms of those of N by

$$\begin{cases} \overline{\omega}^i_j = \omega^i_j & \text{and} \quad \overline{\omega}^i_n = \theta^i, \\ \overline{\Omega}^i_j = \Omega^i_j - \theta^i \wedge \theta^j & \text{and} \quad \overline{\Omega}^i_n = 0. \end{cases}$$

Proposition 1.31. The curvature tensor $\overline{R}$ of M is given by

$$\overline{R} = t^2\left(R - D\right), \tag{1.16}$$

where we have set $D = \frac{1}{2} g_N \owedge g_N$ and R is the curvature tensor of N.

Remark 1.32. There is a slight abuse of notation in Equation (1.16) since $\overline{R}$ is defined on M while $R - D$ is defined on N. What it really means is that the tensor $\overline{R}$ is given by $t^2(R - D)$ in the directions that are tangent to N and vanishes in the normal direction $\overline{e}_n$. At the level of components with respect to the basis $(\overline{e}_1, \dots, \overline{e}_n)$ this gives $\overline{R}_{ABCD} = 0$ if any of the indices is equal to n and if $1 \leq i, j, k, l \leq n - 1$

$$\overline{R}_{ijkl} = \frac{1}{t^2}(R_{ijkl} - D_{ijkl}),$$

indeed

$$\begin{aligned}\overline{R}_{ijkl} &= \overline{R}(\overline{e}_i, \overline{e}_j)(\overline{e}_k, \overline{e}_l) \\ &= \frac{1}{t^4} R(e_i, e_j)(e_k, e_l) \\ &= \frac{1}{t^2}(R - D)(e_i, e_j)(e_k, e_l) \\ &= \frac{1}{t^2}(R_{ijkl} - D_{ijkl}).\end{aligned}$$

Proof. Indeed on $U \times (0, \infty)$ we have since $\overline{\Omega}^i_n = 0$:

$$\begin{aligned}\overline{R} &= \frac{1}{2}\overline{\Omega}_{AB} \otimes \left(\overline{\theta}^A \wedge \overline{\theta}^B\right) \\ &= \frac{1}{2}\overline{\Omega}_{ij} \otimes \left(\overline{\theta}^i \wedge \overline{\theta}^j\right) \\ &= \frac{t^2}{2}\left(\Omega_{ij} - \theta^i \wedge \theta^j\right) \otimes \left(\theta^i \wedge \theta^j\right) \\ &= t^2 \left(\frac{1}{2}\Omega_{ij} \otimes \left(\theta^i \wedge \theta^j\right) - \frac{1}{2}\left(\theta^i \wedge \theta^j\right) \otimes \left(\theta^i \wedge \theta^j\right)\right) \\ &= t^2 \left(R - D\right).\end{aligned}$$

□

To conclude this study of conical warped-product, let us compute the Ricci, scalar and sectional curvature of (M, g).

Proposition 1.33. The Ricci tensor, the scalar curvature and the sectional curvature of (M, g) are given by

$$\begin{aligned}\mathrm{Ric}_g &= \mathrm{Ric}_{g_N} - C^1_2(D), \\ \mathrm{Scal}_g &= \frac{1}{t^2}\left(\mathrm{Scal}_{g_N} - (n-1)(n-2)\right), \\ K_g &= \frac{1}{t^2}\left(K_{g_N} - 1\right)\end{aligned}$$

Where C^1_2 denotes the contraction of the first upper index and the second lower index of the tensor D as defined in Section 1.1.2.

It is actually worth noting that if $N = \mathbb{S}^{n-1}$, then $C_2^1(D)$ is the Ricci tensor of the sphere since D is the curvature tensor of $\mathbb{S}^{n-1}$. Moreover, the constant $(n-1)(n-2)$ appearing in the formula for the scalar curvature is in fact the scalar curvature of the unit sphere.

Proof. Let us consider the orthonormal moving frame $(\overline{e}_1, \dots, \overline{e}_n)$ described above. Then denoting by $\overline{R}_{ij}$ the components of Ric_g with respect to this moving frame as well as by R_{ij} the components of Ric_{g_N} with respect to $(e_1, \dots, e_{n-1})$ we have for all $1 \leq i, j, k \leq n-1$:

$$\begin{aligned}
\overline{R}_{ij} &= \overline{R}_{ikj}^k \\
&= \sum_{k=1}^{n} \overline{R}_{kikj} \\
&= \sum_{k=1}^{n-1} g(\overline{R}(\overline{e}_k, \overline{e}_j)\overline{e}_i, \overline{e}_k) \\
&= t^{-4} \sum_{k=1}^{n-1} g\left(t^2 (R - D)(e_k, e_j)e_i, e_k\right) \\
&= t^{-2} \sum_{k=1}^{n-1} \left[g_N\left(R(e_k, e_j)e_i, e_k\right) - g_N\left(D(e_k, e_j)e_i, e_k\right)\right] \\
&= \frac{1}{t^2} R_{ij} - \frac{1}{t^2} \sum_{k=1}^{n-1} D_{kikj} \\
&= \frac{R_{ij} - (C_2^1(D))_{ij}}{t^2},
\end{aligned}$$

where $(C_2^1(D))_{ij}$ is the component (ij) of $C_2^1(D)$. Therefore we have shown that

$$\mathrm{Ric}_g = \mathrm{Ric}_{g_N} - C_2^1(D).$$

Let us now compute the scalar curvature. Using the fact that

$$D_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk},$$

we get

$$\begin{aligned}
\sum_{k=1}^{n-1} D_{kikj} &= \sum_{k=1}^{n-1} (\delta_{kk}\delta_{ij} - \delta_{kj}\delta_{ki}) \\
&= \sum_{k=1}^{n-1} (\delta_{ij} - \delta_{kj}\delta_{ki}) \\
&= (n-1)\delta_{ij} - \sum_{k=1}^{n-1} \delta_{kj}\delta_{ki} \\
&= \begin{cases} 0 & \text{if } i \neq j, \\ n-2 & \text{if } i = j. \end{cases}
\end{aligned}$$

Hence by definition of the scalar curvature we have

$$\begin{aligned}
\mathrm{Scal}_g &= \sum_{i=1}^{n} \overline{R}_{ii} \\
&= \frac{1}{t^2} \sum_{i=1}^{n-1} \left(R_{ii} - \sum_{k=1}^{n-1} D_{kiki} \right) \\
&= \frac{1}{t^2} \left(\mathrm{Scal}_{g_N} - \sum_{k=1}^{n-1} (n-2) \right) \\
&= \frac{1}{t^2} \left(\mathrm{Scal}_{g_N} - (n-1)(n-2) \right) \\
&= \frac{1}{t^2} \left(\mathrm{Scal}_{g_N} - \mathrm{Scal}_{\mathbb{S}^{n-1}} \right).
\end{aligned}$$

Finally for the sectional curvature we have for $X, Y \in \Gamma(N)$ extended by 0 in the normal direction $\overline{e}_n$

$$\begin{aligned}
K_g(X,Y) &= \frac{\overline{R}(X,Y)(Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} \\
&= \frac{t^2(R-D)(X,Y)(Y,X)}{t^2 g_N(X,X) t^2 g_N(Y,Y) - t^4 g_N(X,Y)^2} \\
&= \frac{1}{t^2} \frac{R(X,Y)(Y,X)}{g_N(X,X)g_N(Y,Y) - g_N(X,Y)^2} - \frac{1}{t^2} \frac{D(X,Y)(Y,X)}{g_N(X,X)g_N(Y,Y) - g_N(X,Y)^2} \\
&= \frac{1}{t^2} \left(K_{g_N}(X,Y) - 1 \right)
\end{aligned}$$

□

Chapter 2

Lipschitz-Killing Curvatures

When working with Riemannian manifolds of arbitrary dimension, one encounters a large variety of quantities constructed from the curvature tensor R such as the Ricci curvature or the scalar curvature. In this section, we define certain curvatures, called the *Lipschitz-Killing curvatures* of the manifold, by contracting some powers of the Riemann tensor. Those curvatures generalize in some sense the mean and Gauss curvatures that appear in dimension two. In order to be able to define them, we need to introduce the *algebra of double forms* which is constructed from the standard algebra of differential forms. This algebraic detour will yield a natural interpretation of the curvature tensor as a double-form, which will be convenient in view of taking exterior powers of this tensor.

2.1 The Algebra of double forms

Let V be an n-dimensional real vector space. Denote by

$$\Lambda(V) = \bigoplus_{p=0}^{n} \Lambda^p(V)$$

the usual algebra of differential forms (also called the Grassmann algebra).

Definition 2.1. For $1 \leq p, q \leq n$, the vector space of *double-forms of type* (p, q) is

$$\mathcal{D}^{p,q}(V) = \Lambda^p(V) \otimes_{\mathbb{R}} \Lambda^q(V).$$

Therefore an element $\omega \in \mathcal{D}^{p,q}(V)$ can be written as

$$\omega = \sum_{i=1}^{n} \alpha_i \otimes \beta_i,$$

where α_i are p-forms and β_i are q-forms.

Note that if $q = 0$ then $\mathcal{D}^{p,0}(V) = \Lambda^p(V)$. Using the exterior product on $\Lambda^p(V)$, one can define an exterior product on $\mathcal{D}^{p,q}(V)$ by setting for $\omega_1 = \alpha_1 \otimes \beta_1 \in \mathcal{D}^{p,q}$ and $\omega_2 = \alpha_2 \otimes \beta_2 \in \mathcal{D}^{r,s}(V)$:

$$\omega_1 \wedge \omega_2 = (\alpha_1 \wedge \alpha_2) \otimes (\beta_1 \wedge \beta_2) \in \mathcal{D}^{p+r,q+s}(V).$$

By abuse of notation, the same symbol is used for the exterior product on $\Lambda^p(V)$ and $\Lambda^{p,q}(V)$. Hence $\omega_1 \wedge \omega_2$ is a double-form of type $(p+r, q+s)$ and its action on vectors $v_1, \dots, v_{p+r}, w_1, \dots, w_{q+s} \in V$ is given by

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{p+r})(w_1, \dots, w_{q+s}) = \frac{1}{p!r!q!s!} \sum_{\substack{\sigma \in \mathfrak{S}_{p+r} \\ \tau \in \mathfrak{S}_{q+s}}} \epsilon_\sigma \epsilon_\tau \alpha_1(v_{\sigma_1}, \dots, v_{\sigma_p}) \beta_1(w_{\tau_1}, \dots, w_{\tau_q}) \cdot \alpha_2(v_{\sigma_{p+1}}, \dots, v_{\sigma_{p+r}}) \beta_2(w_{\tau_{q+1}}, \dots, w_{\tau_{q+s}})$$

This product is associative and it anti-commutes in the sense that

$$\omega_1 \wedge \omega_2 = (-1)^{pr+qs} \omega_2 \wedge \omega_1. \tag{2.1}$$

Definition 2.2. Let

$$\mathcal{D}(V) := \bigoplus_{p,q=0}^{n} \mathcal{D}^{p,q}(V).$$

Then $(\mathcal{D}(V), +, \wedge)$ is an algebra called the *algebra of double-forms*.

Given ω a double-form of type (p, q), we denote by ω^k the k-fold wedge product of ω with itself, i.e.

$$\omega^k = \underbrace{\omega \wedge \ldots \wedge \omega}_{k \text{ times}},$$

which is a double-form of type (kp, kq) and by convention we set $\omega^0 = 1 \in \mathcal{D}^{0,0}(V) = \mathbb{R}$. Now we define operators called *contractions* acting on $\mathcal{D}(V)$ which allows us to lower the degree of a double-form. If $(e_1, \dots, e_n)$ is any orthonormal basis of V

$$C^0(\omega)(v_1, \dots, v_p)(w_1, \dots, w_q) = \omega(v_1, \dots, v_p)(w_1, \dots, w_q)$$

$$C^k(\omega)(v_1, \dots, v_{p-k})(w_1, \dots, w_{q-k}) = \sum_{i=1}^{n} C^{k-1}(v_1, \dots, v_{p-k}, e_i)(w_1, \dots, w_{q-k}, e_i)$$

It is not difficult to see that this definition does not depend on the chosen orthonormal basis.
Note that if ω is of type (p, p), then the kp-th contraction of ω^k is a scalar given by

$$C^{kp}(\omega^k) = \sum_{i_1, \dots, i_{kp}=1}^{n} \omega^k(e_{i_1}, \dots, e_{i_{kp}})(e_{i_1}, \dots, e_{i_{kp}}) \tag{2.2}$$

Now, given an n-dimensional manifold we consider the bundles associated to $\mathcal{D}^{p,q}(V)$ and $\mathcal{D}(V)$.

Definition 2.3. Let M be a manifold of dimension n, the *bundle of double-forms of type* (p,q), denoted $\mathcal{D}^{p,q}(M)$ is the tensor product bundle

$$\mathcal{D}^{p,q}(M) = \Lambda^p(M) \otimes_{C^\infty(M)} \Lambda^q(M).$$

Then the *bundle of double-forms* is

$$\mathcal{D}(M) = \bigoplus_{p,q=0}^{n} \mathcal{D}^{p,q}(M).$$

Finally, the sets of all smooth sections of $\mathcal{D}^{p,q}(M)$ and $\mathcal{D}(M)$ are denoted by

$$\Gamma(\mathcal{D}^{p,q}(M)) \quad \text{and} \quad \Gamma(\mathcal{D}(M)).$$

We will be particularly interested in those double-forms that are *symmetric*, i.e. that are of type (p,p) and satisfy

$$\omega(X_1,\dots,X_p)(Y_1,\dots,Y_p) = \omega(Y_1,\dots,Y_p)(X_1,\dots,X_p),$$

for all $X_1,\dots,X_p,Y_1,\dots,Y_p \in \Gamma(M)$.

Example 2.4. (a) Riemannian metrics are fields of symmetric double-forms of type $(1,1)$.

(b) An important example of a field of symmetric double-forms of type $(2,2)$ is the Riemannian curvature tensor R of a Riemannian manifold (M,g). Indeed, if $(e_1,\dots,e_n)$ is an orthonormal moving frame on $U \subset M$ with associated coframe $(\theta^1,\dots,\theta^n)$, then one can write the curvature tensor as the following tensor product

$$R = \frac{1}{2}\Omega_{ij} \otimes \theta^i \wedge \theta^j, \tag{2.3}$$

where $\Omega_{ij} = \delta_{ik}\Omega^k_j = \Omega^i_j$ are the curvature forms. This expression is indeed symmetric since

$$\begin{aligned}
R(e_i,e_j)(e_k,e_l) &= \frac{1}{2}\Omega_{\mu\nu} \otimes \theta^\mu \wedge \theta^\nu(e_i,e_j)(e_k,e_l)\\
&= \frac{1}{2}R_{\mu\nu ij}(\delta^\mu_k\delta^\nu_l - \delta^\mu_l\delta^\nu_k)\\
&= \frac{1}{2}(R_{klij} - R_{lkij})\\
&= R_{klij}\\
&= R_{ijkl}\\
&= R(e_k,e_l)(e_i,e_j)
\end{aligned}$$

2.1.1 Double-forms as endomorphisms

Through the identification of a finite dimensional vector space V endowed with a scalar product g with its dual V^*, we have

$$\mathrm{End}(V) = V^* \otimes V \cong V \otimes V.$$

An element $v \otimes w \in V \otimes V$ is sent onto the endomorphism defined by $g(v, \cdot)w : V \to V$. The trace of an endomorphism of V can be defined without using any basis of V by using its identification with an element of $V^* \otimes V$.

Definition 2.5. For $T \in \mathrm{End}(V)$, let $\eta^i \otimes v_i \in V^* \otimes V$ its image under the above isomorphism and set

$$\mathrm{Tr}(T) = \eta^i(v_i).$$

This definition of the trace coincides with the usual one:

Lemma 2.6. Let $(e_1, \ldots, e_n)$ be a basis of V and let (a_i^j) be the matrix of T with respect to this basis. Then

$$\mathrm{Tr}(T) = a_i^i$$

Proof. Let $(\theta^1, \ldots, \theta^n)$ be the dual basis to $(e_1, \ldots, e_n)$. As an element of $V^* \otimes V$, the endomorphism T is

$$T = \theta^i \otimes T(e_i).$$

Indeed

$$T(v) = T(v^i e_i) = v^i T(e_i) = \theta^i(v) T(e_i).$$

Therefore,

$$\mathrm{Tr}(T) = \theta^i(T(e_i)) = \theta^i(a_i^j e_j) = a_i^i$$

□

Now, by taking $\Lambda^p(V)$ as vector space, we can naturally identify the space of double-forms of type (p, p) with $\mathrm{End}(\Lambda^p(V))$, indeed

$$\mathcal{D}^{p,p}(V) = \Lambda^p(V) \otimes \Lambda^p(V) \cong \Lambda_p(V) \otimes \Lambda^p(V) = \mathrm{End}(\Lambda^p(V)),$$

where $\Lambda_p(V)$ is the space of p-vectors, which is the dual of $\Lambda^p(V)$. So we have

$$\underbrace{\alpha \otimes \beta}_{\in \mathcal{D}^{p,p}(V)} \longmapsto \underbrace{g(\alpha, \cdot) \otimes \beta}_{\in \Lambda_p(V) \otimes \Lambda^p(V)} \longmapsto \underbrace{g(\alpha, \cdot)\beta}_{\in \mathrm{End(V)}} =: T(\alpha \otimes \beta)$$

Observe that according to definition 2.5 the trace of $T(\alpha \otimes \beta)$ is given by

$$\mathrm{Tr}(T(\alpha \otimes \beta)) = g(\alpha, \beta).$$

The trace of $T(\alpha \otimes \beta)$ corresponds to the contraction of $\alpha \otimes \beta$:

Lemma 2.7. Let $A \in \mathcal{D}^{p,p}(V)$ and let $T(A) \in \mathrm{End}(\Lambda^p V)$ be the image of A under the above isomorphism. Then

$$C^p(A) = \mathrm{Tr}(T(A)) \tag{2.4}$$

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V with dual basis $(\theta^1, \ldots, \theta^n)$. Then the set

$$\{\theta^{i_1} \wedge \ldots \wedge \theta^{i_p} \mid 1 \leq i_1 < \ldots < i_p \leq n\}$$

is an orthonormal basis of $\Lambda^p(V)$. It is enough to show the result for A of the form $A = \alpha \otimes \beta$. Hence, writing

$$\alpha = \alpha_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p} \quad \text{and} \quad \beta = \beta_{j_1 \ldots j_p} \theta^{j_1} \wedge \ldots \wedge \theta^{j_p},$$

we have on one hand by the definition of the contraction operator (2.2)

$$\begin{aligned} C^p(\alpha \otimes \beta) &= \sum_{\mu_1, \ldots, \mu_p = 1}^{n} \alpha(e_{\mu_1}, \ldots, e_{\mu_p}) \beta(e_{\mu_1}, \ldots, e_{\mu_p}) \\ &= \sum_{\mu_1, \ldots, \mu_p = 1}^{n} \alpha_{\mu_1 \ldots \mu_p} \beta_{\mu_1 \ldots \mu_p}. \end{aligned}$$

On the other hand

$$\begin{aligned} \mathrm{Tr}(T(\alpha \otimes \beta)) &= g(\alpha, \beta) \\ &= \alpha_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p} g\left(\theta^{i_1} \wedge \ldots \wedge \theta^{i_p}, \theta^{j_1} \wedge \ldots \wedge \theta^{j_p}\right) \\ &= \sum_{i_1, \ldots, i_p = 1}^{n} \alpha_{i_1 \ldots i_p} \beta_{i_1 \ldots i_p}. \end{aligned}$$

□

Example 2.8. In case where the considered double-form is the curvature tensor R (see example 2.4), the correspondant endomorphism is the *curvature endomorphism*

$$R : \Lambda^2(M) \longrightarrow \Lambda^2(M), \quad R(\theta^i \wedge \theta^j) = g(R(e_i, e_j)-, -),$$

where $(\theta^1, \ldots, \theta^n)$ is the dual coframe to some moving frame $(e_1, \ldots, e_n)$ on M.

Remark 2.9. Note that despite $\mathrm{End}(\Lambda(V))$ being an algebra for the composition of endomorphisms, the above isomorphism between $\mathcal{D}^{p,p}(V)$ and $\mathrm{End}(\Lambda^p(V))$ does not extend to an algebra isomorphism between $\bigoplus_{p \geq 0} \mathcal{D}^{p,p}(V)$ and $\mathrm{End}(\Lambda(V))$ since the composition of endomorphisms is not commutative (in the sense described above). In particular, if $\omega \in \mathcal{D}^{p,p}(V)$, then, denoting by $T(\omega)$ its corresponding endomorphism, we have in general that

$$\omega \wedge \ldots \wedge \omega = \omega^k \neq T(\omega)^k = T(\omega) \circ \ldots \circ T(\omega),$$

since $\omega^k \in \mathcal{D}^{kp,kp}(V)$ and $T(\omega)^k \in \mathrm{End}(\Lambda^p(V))$.

2.2 The Lipschitz-Killing curvatures

It is now possible to define the Lipschitz-Killing curvatures of a Riemannian manifold (M, g). The definition relies on the interpretation of the curvature tensor as a double-form of type $(2, 2)$.

Definition 2.10. For $0 \leq j \leq n$ the *j-th Lipschitz-Killing curvature* of M is the following n-form

$$\kappa_j(M) = \begin{cases} \frac{1}{j!(j/2)!} C^j(R^{\frac{j}{2}}) \mathrm{dvol}_M & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases} \tag{2.5}$$

The constant factor $\frac{1}{j!(j/2)!}$ finds its justification both in Weyl's tube formula and in the Gauss-Bonnet-Chern Theorem. With this precise normalization the last Lipschitz-Killing curvature is precisely the Pfaffian of the curvature forms (see Lemma 2.13).
It is useful to work out an expression of the scalar $C^j(R^{j/2})$ with respect to some orthonormal moving frame.

Proposition 2.11. Let $(e_1, \ldots, e_n)$ be an orthonormal moving frame on $U \subset M$ and let $j = 2m$ be even. Then

$$C^{2m}(R^m) = \frac{(2m)!}{2^{2m}(n-2m)!} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\tau \in \mathfrak{S}_{2m}} \varepsilon_\tau R_{\sigma_1 \sigma_2 \sigma_{\tau_1} \sigma_{\tau_2}} \cdots R_{\sigma_{2m-1} \sigma_{2m} \sigma_{\tau_{2m-1}} \sigma_{\tau_{2m}}}$$

Proof. From equation (2.3) we have that

$$\begin{aligned} R^m &= \frac{1}{2^m} \left(\Omega_{ij} \otimes \theta^i \wedge \theta^j \right)^m \\ &= \frac{1}{2^m} \sum_{i_1, \ldots, i_{2m} = 1}^{n} \left(\Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2m-1} i_{2m}} \right) \otimes \left(\theta^{i_1} \wedge \ldots \wedge \theta^{i_{2m}} \right) \\ &= \frac{1}{2^m (n-2m)!} \sum_{\sigma \in \mathfrak{S}_n} \left(\Omega_{\sigma_1 \sigma_2} \wedge \ldots \wedge \Omega_{\sigma_{2m-1} \sigma_{2m}} \right) \otimes \left(\theta^{\sigma_1} \wedge \ldots \wedge \theta^{\sigma_{2m}} \right), \end{aligned}$$

since there are exactly $(n-2m)!$ permutations $\sigma \in \mathfrak{S}_n$ that send $\{1, \ldots, 2m\}$ to the set $\{i_1, \ldots, i_{2m}\}$. Now, by definition of the wedge product, we have for some fixed $\sigma, \tau \in \mathfrak{S}_n$

$$\begin{aligned} \Omega_{\sigma_1 \sigma_2} \wedge \ldots \wedge \Omega_{\sigma_{2m-1} \sigma_{2m}} (e_{\tau_1}, \ldots, e_{\tau_{2m}}) &= \frac{1}{2^m} \sum_{\rho \in \mathfrak{S}_{2m}} \epsilon_\rho \Omega_{\sigma_1 \sigma_2} (e_{\tau_{\rho_1}}, e_{\tau_{\rho_2}}) \cdots \Omega_{\sigma_{2m-1} \sigma_{2m}} (e_{\tau_{\rho_{2m-1}}}, e_{\tau_{\rho_{2m}}}) \\ &= \frac{1}{2^m} \sum_{\rho \in \mathfrak{S}_{2m}} \epsilon_\rho R_{\tau_{\rho_1} \tau_{\rho_2} \sigma_1 \sigma_2} \cdots R_{\tau_{\rho_{2m-1}} \tau_{\rho_{2m}} \sigma_{2m-1} \sigma_{2m}} \\ &= \frac{1}{2^m} \sum_{\rho \in \mathfrak{S}_{2m}} \epsilon_\rho R_{\sigma_1 \sigma_2 \tau_{\rho_1} \tau_{\rho_2}} \cdots R_{\sigma_{2m-1} \sigma_{2m} \tau_{\rho_{2m-1}} \tau_{\rho_{2m}}}, \end{aligned}$$

and since $\theta^{\sigma_1} \wedge \ldots \wedge \theta^{\sigma_{2m}}(e_{\tau_1}, \ldots, e_{\tau_{2m}}) = \delta^{\sigma_1 \ldots \sigma_{2m}}_{\tau_1 \ldots \tau_{2m}}$ is the generalized Kronecker symbol, we have finally that

$$\begin{aligned} C^{2m}(R^m) &= \sum_{i_1,\ldots,i_{2m}=1}^{n} R^m(e_{i_1}, \ldots, e_{i_{2m}})(e_{i_1}, \ldots, e_{i_{2m}}) \\ &= \frac{1}{(n-2m)!} \sum_{\tau \in \mathfrak{S}_n} R^m(e_{\tau_1}, \ldots, e_{\tau_{2m}})(e_{\tau_1}, \ldots, e_{\tau_{2m}}) \\ &= \frac{1}{2^{2m}((n-2m)!)^2} \sum_{\sigma,\tau \in \mathfrak{S}_n} \sum_{\rho \in \mathfrak{S}_{2m}} \epsilon_\rho \delta^{\sigma_1 \ldots \sigma_{2m}}_{\tau_1 \ldots \tau_{2m}} R_{\sigma_1 \sigma_2 \tau_{\rho_1} \tau_{\rho_2}} \cdots R_{\sigma_{2m-1} \sigma_{2m} \tau_{\rho_{2m-1}} \tau_{\rho_{2m}}}. \end{aligned}$$

In order for $\delta^{\sigma_1 \ldots \sigma_{2m}}_{\tau_1 \ldots \tau_{2m}}$ not to vanish, the set $(\tau_1, \ldots, \tau_{2m})$ must be a permutation of the set $(\sigma_1, \ldots, \sigma_{2m})$, and there are $(2m)!$ such permutations. Moreover, once the elements $(\tau_1, \ldots, \tau_{2m})$ are fixed, one can permute the $n - 2m$ remaining terms $(\tau_{2m+1}, \ldots, \tau_n)$ and there are $(n - 2m)!$ such permutations. In other words if $\sigma \in \mathfrak{S}_n$ is fixed then:

$$\mathrm{Card}\left(\{\tau \in \mathfrak{S}_n \mid \delta^{\sigma_1 \ldots \sigma_{2m}}_{\tau_1 \ldots \tau_{2m}} \neq 0\}\right) = (2m)!(n-2m)!.$$

Observe also that if $\delta^{\sigma_1 \ldots \sigma_{2m}}_{\tau_1 \ldots \tau_{2m}} \neq 0$, then there exists $\pi \in \mathfrak{S}_{2m}$ such that

$$(\tau_1, \ldots, \tau_{2m}) = (\sigma_{\pi_1}, \ldots, \sigma_{\pi_{2m}}) \quad \text{and} \quad \delta^{\sigma_1 \ldots \sigma_{2m}}_{\tau_1 \ldots \tau_{2m}} = \epsilon_\pi.$$

Therefore, we can write

$$\begin{aligned} &\sum_{\tau \in \mathfrak{S}_n} \sum_{\rho \in \mathfrak{S}_{2m}} \epsilon_\rho \delta^{\sigma_1 \ldots \sigma_{2m}}_{\tau_1 \ldots \tau_{2m}} R_{\sigma_1 \sigma_2 \tau_{\rho_1} \tau_{\rho_2}} \cdots R_{\sigma_{2m-1} \sigma_{2m} \tau_{\rho_{2m-1}} \tau_{\rho_{2m}}} \\ &\qquad = (n-2m)! \sum_{\rho,\pi \in \mathfrak{S}_{2m}} \epsilon_\rho \epsilon_\pi R_{\sigma_1 \sigma_2 \sigma_{\pi_{\rho_1}} \sigma_{\pi_{\rho_2}}} \cdots R_{\sigma_{2m-1} \sigma_{2m} \sigma_{\pi_{\rho_{2m-1}}} \sigma_{\pi_{\rho_{2m}}}} \\ &\qquad \overset{\tau = \pi \circ \rho}{=} (n-2m)!(2m)! \sum_{\tau \in \mathfrak{S}_{2m}} \epsilon_\tau R_{\sigma_1 \sigma_2 \sigma_{\tau_1} \sigma_{\tau_2}} \cdots R_{\sigma_{2m-1} \sigma_{2m} \sigma_{\tau_{2m-1}} \sigma_{\tau_{2m}}}, \end{aligned}$$

so that finally

$$C^{2m}(R^m) = \frac{(2m)!}{2^{2m}(n-2m)!} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\tau \in \mathfrak{S}_{2m}} \epsilon_\tau R_{\sigma_1 \sigma_2 \sigma_{\tau_1} \sigma_{\tau_2}} \cdots R_{\sigma_{2m-1} \sigma_{2m} \sigma_{\tau_{2m-1}} \sigma_{\tau_{2m}}}$$

□

We can express every Lipschitz-Killing curvature in terms of the curvature forms of the manifold.

Proposition 2.12. For all $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ we have

$$\kappa_{2k}(M) = \frac{1}{2^k k!(n-2k)!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma \Omega_{\sigma_1 \sigma_2} \wedge \ldots \Omega_{\sigma_{2k-1} \sigma_{2k}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_n}$$

Proof. For $\sigma \in \mathfrak{S}_n$ fixed, set

$$\eta_{2k}^{\sigma} = \varepsilon_\sigma \Omega_{\sigma_1\sigma_2} \wedge \ldots \Omega_{\sigma_{2k-1}\sigma_{2k}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_n}$$

then, since $\Omega_{ij} = \frac{1}{2} R_{ijkl}\theta^k \wedge \theta^l$ we have

$$\begin{aligned}
\eta_{2k}^{\sigma} &= \frac{1}{2^k}\varepsilon_\sigma R_{\sigma_1\sigma_2 i_1 i_2} \cdots R_{\sigma_{2k-1}\sigma_{2k} i_{2k-1} i_2 k}\theta^{i_1} \wedge \ldots \wedge \theta^{i_{2k}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_n} \\
&= \frac{1}{2^k}\sum_{\tau \in \mathfrak{S}_{2k}} \varepsilon_\sigma R_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}} \cdots R_{\sigma_{2k-1}\sigma_{2k}\sigma_{\tau_{2k-1}}\sigma_{\tau_{2k}}}\theta^{\sigma_{\tau_1}} \wedge \ldots \wedge \theta^{\sigma_{\tau_{2k}}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_n} \\
&= \frac{1}{2^k}\sum_{\tau \in \mathfrak{S}_{2k}} \varepsilon_\sigma \varepsilon_\tau R_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}} \cdots R_{\sigma_{2k-1}\sigma_{2k}\sigma_{\tau_{2k-1}}\sigma_{\tau_{2k}}}\theta^{\sigma_1} \wedge \ldots \wedge \theta^{\sigma_n} \\
&= \frac{1}{2^k}\sum_{\tau \in \mathfrak{S}_{2k}} \varepsilon_\tau R_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}} \cdots R_{\sigma_{2k-1}\sigma_{2k}\sigma_{\tau_{2k-1}}\sigma_{\tau_{2k}}}\mathrm{dvol}_M.
\end{aligned}$$

Then comparing the expression

$$\sum_{\sigma \in \mathfrak{S}_n} \eta_{2k}^{\sigma} = \frac{1}{2^k}\sum_{\tau \in \mathfrak{S}_{2k}} \varepsilon_\tau R_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}} \cdots R_{\sigma_{2k-1}\sigma_{2k}\sigma_{\tau_{2k-1}}\sigma_{\tau_{2k}}}\mathrm{dvol}_M,$$

with the one for $\kappa_{2k}(M)$ in Proposition 2.11, we get that

$$\kappa_{2k}(M) = \frac{1}{2^k k!(n-2k)!}\sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma \Omega_{\sigma_1\sigma_2} \wedge \ldots \Omega_{\sigma_{2k-1}\sigma_{2k}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_n}.$$

□

Observe that some of those $\kappa_j(M)$ are already well-known. Indeed we have the following lemma:

Lemma 2.13. The 0-th and 2-th Lipschitz-Killing curvatures are respectively

$$\begin{aligned}
\kappa_0(M) &= \mathrm{dvol}_M, \\
\kappa_2(M) &= \frac{1}{2}S_g \mathrm{dvol}_M,
\end{aligned}$$

where S_g is the scalar curvature. In addition, if M is of even dimension $n = 2p$, then

$$\kappa_n(M) = \mathrm{Pf}(\Omega),$$

where $\mathrm{Pf}(\Omega)$ is the Pfaffian of the curvature forms of M.

Proof. For $j = 0$ it's clear since $R^0 = 1$. For $j = 2$, we have if $(e_1, \ldots, e_n)$ is an orthonormal moving frame on an open subset $U \subset M$:

$$C^2(R^1) = \sum_{i,j=1}^{n} R(e_i, e_j)(e_i, e_j) = \sum_{i,j=1}^{n} R_{ijij} = S_g,$$

therefore

$$\kappa_2(M) = \frac{1}{2}S_g \mathrm{dvol}_M.$$

Now if $n = 2p$, recall that the Pfaffian of the matrix of the curvature forms $\Omega = (\Omega_{ij})$ is the $2p$-form given by

$$\mathrm{Pf}(\Omega) = \frac{1}{2^p p!}\sum_{\sigma\in\mathfrak{S}_{2p}} \epsilon_\sigma \Omega_{\sigma_1\sigma_2}\wedge\ldots\wedge\Omega_{\sigma_{2p-1}\sigma_{2p}}.$$

Its action on the orthonormal frame $(e_1,\ldots,e_{2p})$ is given by

$$\mathrm{Pf}(\Omega)(e_1,\ldots,e_{2p}) = \frac{1}{2^{2p}p!}\sum_{\sigma,\tau\in\mathfrak{S}_{2p}} \varepsilon_\sigma\varepsilon_\tau R_{\sigma_1\sigma_2\tau_1\tau_2}\cdots R_{\sigma_{2p-1}\sigma_{2p}\tau_{2p-1}\tau_{2p}}.$$

On the other hand, since R^p is a double-form of type $(2p, 2p)$ we have

$$\begin{aligned} C^{2p}(R^p) &= \sum_{i_1,\ldots,i_{2p}=1}^{2p} R^p(e_{i_1},\ldots,e_{i_{2p}})(e_{i_1},\ldots,e_{i_{2p}}) \\ &= (2p)!R^p(e_1,\ldots,e_{2p})(e_1,\ldots,e_{2p}) \\ &= \frac{(2p)!}{2^{2p}}\sum_{\sigma,\tau\in\mathfrak{S}_{2p}} \varepsilon_\sigma\varepsilon_\tau R_{\sigma_1\sigma_2\tau_1\tau_2}\cdots R_{\sigma_{2p-1}\sigma_{2p}\tau_{2p-1}\tau_{2p}} \end{aligned}$$

therefore

$$\kappa_{2p}(M) = \frac{1}{(2p)!p!}C^{2p}(R^p)\mathrm{dvol}_M = \mathrm{Pf}(\Omega)$$

□

As we shall see, the Pfaffian $\mathrm{Pf}(\Omega)$ is exactly the integrand in the Gauss-Bonnet-Chern theorem for compact manifolds.

Example 2.14. As a first example, let us compute the Lipschitz-Killing curvatures of a Riemannian surface (S, g). The 0-th Lipschitz-Killing curvature is, by lemma 2.13, the volume form of S, i.e. $\kappa_0(S) = \mathrm{dvol}_S$. The other non-trivial Lipschitz-Killing curvature is the second one. Using again lemma 2.13 we have that $\kappa_2(S) = \frac{S_g}{2}\mathrm{dvol}_S$. But in dimension 2, the scalar curvature is twice the Gauss curvature K, i.e. $S_g = 2K$. Note that the Gauss curvature K coincides with the component R_{1212} of the curvature tensor of S. Therefore we have

$$\begin{cases} \kappa_0(S) = \mathrm{dvol}_S, \\ \kappa_1(S) = 0, \\ \kappa_2(S) = K\mathrm{dvol}_S. \end{cases}$$

We will now establish several properties of the Lipschitz-Killing curvatures. The first one is their behaviour under a change of scale of the metric.

Proposition 2.15. Let (M,g) a Riemannian manifold and let $\lambda \in \mathbb{R}$. Consider the metric $\tilde{g} = \lambda^2 g$. Then if $\tilde{\kappa}_j(M)$ is the j-th Lipschitz-Killing curvature associated to $\tilde{g}$ we have

$$\tilde{\kappa}_j(M) = \lambda^{n-j}\kappa_j(M)$$

Proof. Under the change $g \mapsto \tilde{g}$, the $(0,4)$ curvature tensor R is also modified by a factor λ^2, i.e. $\tilde{R} = \lambda^2 R$. Now, if $(e_1, \dots, e_n)$ is an orthonormal moving frame with respect to g, then

$$\left(\tilde{e}_1 := \frac{1}{\lambda}e_1, \dots, \tilde{e}_n := \frac{1}{\lambda}e_n\right)$$

is an orthonormal moving frame with respect to $\tilde{g}$. The contraction operator is modified as follows

$$\begin{aligned} C^{2m}(\tilde{R}^m) &= \sum_{i_1,\dots,i_{2m}=1}^{n} \tilde{R}^m(\tilde{e}_{i_1}, \dots, \tilde{e}_{i_{2m}})(\tilde{e}_{i_1}, \dots, \tilde{e}_{i_{2m}}) \\ &= \lambda^{2m}\lambda^{-4m} \sum_{i_1,\dots,i_{2m}=1}^{n} R^m(e_{i_1}, \dots, e_{i_{2m}})(e_{i_1}, \dots, e_{i_{2m}}) \\ &= \lambda^{-2m}C^{2m}(R^m). \end{aligned}$$

The volume forms of (M,g) and $(M,\tilde{g})$ are related by

$$\mathrm{dvol}_{\tilde{g}} = \lambda^n \mathrm{dvol}_g,$$

therefore

$$\tilde{\kappa}_j(M) = \lambda^{n-j}\kappa_j(M).$$

□

Remark 2.16. Observe that this homogeneity gives for $j = n$:

$$\tilde{\kappa}_n(M) = \kappa_n(M),$$

which is consistent with the fact that $\kappa_n(M) = \mathrm{Pf}(\Omega)$ whose integral is a topological invariant by Gauss-Bonnet-Chern's Theorem.

2.2.1 Lipschitz-Killing curvatures of space forms

In order to give some concreteness to the definition of the Lipschitz-Killing curvatures, we now compute them for the three usual model spaces: the Euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n$ et the hyperbolic space $\mathbb{H}^n$ endowed with their standard Riemannian metrics. The result is the following

Theorem 2.17. Let (M_λ, g_λ) be a Riemannian manifold of constant curvature λ. Then for all $1 \leq j \leq \frac{n}{2}$

$$\kappa_{2j}(M_\lambda) = \begin{cases} 0 & \text{if } \lambda = 0, \\ \dfrac{n! r^{n-2j}}{2^j j!(n-2j)!} \mathrm{dvol}_{M_1} & \text{if } \lambda = \frac{1}{r^2}, \\ (-1)^j \dfrac{n! r^{n-2j}}{2^j j!(n-2j)!} \mathrm{dvol}_{M_{-1}} & \text{if } \lambda = -\frac{1}{r^2}, \end{cases} \tag{2.6}$$

and $\kappa_0(M_\lambda) = \mathrm{dvol}_{M_\lambda}$ in each case.

The proof of this theorem is separated in three cases according to the sign of λ. The result will follow from Corollaries 2.19 and 2.21.

Curvature zero

This case is the easiest one since the curvature tensor vanishes identically, i.e. $R \equiv 0$. Therefore, there is only one non zero Lipschitz-Killing curvature:

$$\kappa_j(M_0) = \begin{cases} \mathrm{dvol}_{M_0} & \text{if } j = 0, \\ 0 & \text{else.} \end{cases}$$

Positive constant curvature

The components of the curvature tensor of a Riemannian manifold M_1 of constant sectional curvature equal to one (with respect to some orthonormal moving frame $(e_1, \ldots, e_n)$) are given by

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Then using Proposition 2.11, we can work out the Lipschitz-Killing curvatures of M_1.

Proposition 2.18. The Lipschitz-Killing curvatures of M_1 are given by

$$\kappa_j(M_1) = \begin{cases} \dfrac{n!}{2^{j/2}(j/2)!(n-j)!} \mathrm{dvol}_{M_1} & \text{if } j \text{ is even}, \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

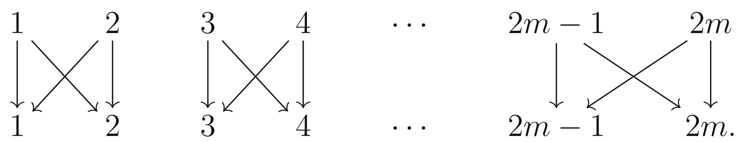
Proof. Let us denote the curvature tensor of (M_1, g_1) by R. We use the explicit formula given in proposition 2.11.

$$
\begin{aligned}
C^{2m}(R^m) &= \frac{(2m)!}{2^{2m}(n-2m)!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\tau\in\mathfrak{S}_{2m}} \varepsilon_\tau R_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}}\cdots R_{\sigma_{2m-1}\sigma_{2m}\sigma_{\tau_{2m-1}}\sigma_{\tau_{2m}}} \\
&= \frac{(2m)!}{2^{2m}(n-2m)!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\tau\in\mathfrak{S}_{2m}} \varepsilon_\tau(\delta_{\sigma_1\sigma_{\tau_1}}\delta_{\sigma_2\sigma_{\tau_2}}-\delta_{\sigma_1\sigma_{\tau_2}}\delta_{\sigma_2\sigma_{\tau_1}})\cdot \\
&\qquad \cdots(\delta_{\sigma_{2m-1}\sigma_{\tau_{2m-1}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m}}}-\delta_{\sigma_{2m-1}\sigma_{\tau_{2m}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m-1}}})
\end{aligned}
$$

Although this expression looks complicated, it appears that a lot of the summands vanish. Indeed, the product

$$
(\delta_{\sigma_1\sigma_{\tau_1}}\delta_{\sigma_2\sigma_{\tau_2}}-\delta_{\sigma_1\sigma_{\tau_2}}\delta_{\sigma_2\sigma_{\tau_1}})\cdots(\delta_{\sigma_{2m-1}\sigma_{\tau_{2m-1}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m}}}-\delta_{\sigma_{2m-1}\sigma_{\tau_{2m}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m-1}}}) \quad (2.7)
$$

is non-zero only if the permutation τ is a product of transpositions of the type $(2i-1\ 2i)$ or the identity, for $i=1,\dots,m$ i.e. transpositions of the form



Indeed, since

$$
\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk} = \begin{cases} +1 & \text{if } (i,j)=(k,l), \\ -1 & \text{if } (i,j)=(l,k), \\ 0 & \text{else.} \end{cases}
$$

the expression (2.7) is non zero if and only if in every term of the product we have

$$
(\sigma_i,\sigma_{i+1}) = (\sigma_{\tau_i},\sigma_{\tau_{i+1}}) \quad \text{or} \quad (\sigma_i,\sigma_{i+1}) = (\sigma_{\tau_{i+1}},\sigma_{\tau_i}),
$$

for $i=1,3,5,\dots,2m-1$, which is the same as

$$
(\tau_i,\tau_{i+1}) = (i,i+1) \quad \text{or} \quad (\tau_i,\tau_{i+1}) = (i+1,i).
$$

It is easy to see from the above diagram that there are exactly 2^m permutations $\tau\in\mathfrak{S}_{2m}$ that make the expression (2.7) non zero. Now, let us study the term

$$
\epsilon_\tau(\delta_{\sigma_1\sigma_{\tau_1}}\delta_{\sigma_2\sigma_{\tau_2}}-\delta_{\sigma_1\sigma_{\tau_2}}\delta_{\sigma_2\sigma_{\tau_1}})\cdots(\delta_{\sigma_{2m-1}\sigma_{\tau_{2m-1}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m}}}-\delta_{\sigma_{2m-1}\sigma_{\tau_{2m}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m-1}}}) \quad (2.8)
$$

for such a permutation τ.

i. If the permutation τ is even, i.e. if $\varepsilon_\tau=+1$, then the decomposition of τ in transpositions consists simply of an even number of transpositions corresponding to the couples $(\tau_i,\tau_{i+1})=(i+1,i)$ since all the other couples are of the form $(\tau_i,\tau_{i+1})=(i,i+1)$ and therefore do not appear in the decomposition in transpositions. Hence there is an even number of terms that are equal to -1 in the product so that (2.8) is equal to $+1$.

ii. If the permutation τ is odd, then by the same argument, there is an odd number of terms that are equal to -1, but in this case we also have $\varepsilon_\tau = -1$ so that (2.8) is also equal to $+1$.

From this, we conclude that

$$\begin{aligned} C^{2m}(R^m) &= \frac{(2m)!}{2^{2m}(n-2m)!} \sum_{\sigma\in\mathfrak{S}_n} \sum_{\tau\in\mathfrak{S}_{2m}} \varepsilon_\tau (\delta_{\sigma_1\sigma_{\tau_1}}\delta_{\sigma_2\sigma_{\tau_2}} - \delta_{\sigma_1\sigma_{\tau_2}}\delta_{\sigma_2\sigma_{\tau_1}})\cdot \\ &\qquad \cdots (\delta_{\sigma_{2m-1}\sigma_{\tau_{2m-1}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m}}} - \delta_{\sigma_{2m-1}\sigma_{\tau_{2m}}}\delta_{\sigma_{2m}\sigma_{\tau_{2m-1}}}) \\ &= \frac{(2m)!}{2^{2m}(n-2m)!} \sum_{\sigma\in\mathfrak{S}_n} 2^m \\ &= \frac{(2m)!n!}{2^m(n-2m)!} \end{aligned}$$

and thus

$$\kappa_{2m}(M_1) = \frac{1}{(2m)!m!} C^{2m}(R^m)\mathrm{dvol}_{M_1} = \frac{n!}{2^m m!(n-2m)!}\mathrm{dvol}_{M_1}$$

□

Observe that the contractions $C^{2m}(R^m)$ of the curvature tensor of M_1 can be written using the binomial coefficient.

$$C^{2m}(R^m) = \frac{((2m)!)^2}{2^m}\binom{n}{2m}. \tag{2.9}$$

Corollary 2.19. For $\lambda = \frac{1}{r^2} > 0$ the Lipschitz-Killing curvatures of M_λ are given by

$$\kappa_j(M_\lambda) = \begin{cases} \dfrac{n!r^{n-j}}{2^{j/2}(j/2)!(n-j)!}\mathrm{dvol}_{M_1} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

Proof. Let us denote by g the standard metric of M_1. Then, (M_λ, g_λ) is isometric to $(M_1, \frac{1}{\lambda}g)$ so that by proposition 2.15 we have

$$\kappa_j(M_\lambda) = r^{n-j}\kappa_j(M_1).$$

□

Negative constant curvature

The curvature tensor of a manifold M_{-1} of constant sectional curvature equal to $\lambda = -1$ is given by $\overline{R} = -R$, where R is the curvature tensor of (M_1, g_1).

Proposition 2.20. The Lipschitz-Killing curvatures of (M_{-1}, g_{-1}) are given by

$$\kappa_j(M_{-1}) = \begin{cases} \dfrac{n!(-1)^{j/2}}{2^{j/2}(j/2)!(n-j)!}\mathrm{dvol}_{M_{-1}} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

Proof. The proof is essentially the same as the one for the unit sphere. Let $(e_1, \dots, e_n)$ be an orthonormal moving frame on $U \subset M_{-1}$. Then the components of $\overline{R}$ with respect to this moving frame are given by

$$\overline{R}_{ijkl} = -R_{ijkl} = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}.$$

Hence by Propositions 2.11 and 2.18 we have

$$\begin{aligned} C^{2m}(\overline{R}^m) &= \frac{(2m)!}{2^{2m}(n-2m)!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\tau\in\mathfrak{S}_{2m}} \varepsilon_\tau \overline{R}_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}}\cdots\overline{R}_{\sigma_{2m-1}\sigma_{2m}\sigma_{\tau_{2m-1}}\sigma_{\tau_{2m}}} \\ &= \frac{(2m)!(-1)^m}{2^{2m}(n-2m)!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\tau\in\mathfrak{S}_{2m}} \varepsilon_\tau R_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}}\cdots R_{\sigma_{2m-1}\sigma_{2m}\sigma_{\tau_{2m-1}}\sigma_{\tau_{2m}}} \\ &= \frac{(2m)!n!(-1)^m}{2^m(n-2m)!} \end{aligned}$$

and finally

$$\kappa_{2m}(M_{-1}) = \frac{n!(-1)^m}{2^m m!(n-2m)!}\mathrm{dvol}_{M_{-1}}$$

□

As in the case of positive curvature, we obtain as a corollary the Lipschitz-Killing curvatures of a space of negative constant curvature:

Corollary 2.21. Let $\lambda = -\frac{1}{r^2} < 0$. The Lipschitz-Killing curvatures of (M_λ, g_λ) are given by

$$\kappa_j(M_\lambda) = \begin{cases} \dfrac{n!(-1)^{j/2}r^{n-j}}{2^{j/2}(j/2)!(n-j)!}\mathrm{dvol}_{M_{-1}} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

Remark 2.22. Using the fact that the second contraction of the curvature tensor is precisely the scalar curvature i.e. using the fact that $C^2(R) = S_g$, we recover the well-known formula for the scalar curvature of space forms:

$$S_g = \lambda n(n-1).$$

Indeed, writing $\lambda = \pm\frac{1}{r^2}$ we have from Theorem 2.17 that

$$S_g = C^2(R) = \frac{n!\mathrm{sgn}(\lambda)}{(n-2)!r^2} = \lambda n(n-1),$$

with $\mathrm{sgn}(\lambda)$ the sign of λ.

2.2.2 Lipschitz-Killing Curvatures of Conical Warped-Product

As in section 1.10, we consider now a manifold $M = N \times (0, \infty)$, where N is compact, endowed with the Riemannian metric $g = t^2 g_N + dt^2$, the metric g_N being a Riemannian metric on N. Let $(e_1, \dots, e_{n-1})$ be an orthonormal directly oriented moving frame on $U \subset N$ with dual coframe $(\theta^1, \dots, \theta^{n-1})$, and set $e_n = \frac{\partial}{\partial t}$, where t is the arclength parameter on $(0, \infty)$. Then on $U \times (0, \infty) \subset M$ we have the following orthonormal positively oriented moving frame:

$$\overline{e}_i = \frac{1}{f} e_i \quad \text{and} \quad \overline{e}_n = e_n = dt,$$

where $f : U \times (0, \infty) \to (0, \infty)$ is the map defined by $f(p, t) = t$. Recall that we established the following relations between the connection and curvature forms $\overline{\omega}_B^A$ and $\overline{\Omega}_B^A$ of M and the connection and curvature forms ω_j^i and Ω_j^i of N:

$$\begin{cases} \overline{\omega}_j^i = \omega_j^i, \\ \overline{\omega}_n^i = \overline{\theta}^i = f\theta^i, \\ \overline{\Omega}_j^i = \Omega_j^i - \theta^i \wedge \theta^j, \\ \overline{\Omega}_n^i = 0. \end{cases} \tag{2.10}$$

Moreover we showed in Proposition 1.31 that this gives rise to the following expression for the curvature tensor $\overline{R}$ of M:

$$\overline{R} = t^2 \left(R - D \right),$$

where $D = \frac{1}{2} g_N \owedge g_N$.
Imitating the argument made in the proof of Proposition 2.11, it is easy to show that

Proposition 2.23. The $2k$-th contraction of $\overline{R}^k$ is given by

$$C^{2k}(\overline{R}^k) = \frac{(2k)!}{2^{2k}(n-1-2k)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sum_{\tau \in \mathfrak{S}_{2k}} \varepsilon_\tau \overline{R}_{\sigma_1 \sigma_2 \sigma_{\tau_1} \sigma_{\tau_2}} \cdots \overline{R}_{\sigma_{2k-1} \sigma_{2k} \sigma_{\tau_{2k-1}} \sigma_{\tau_{2k}}}$$

Proof. It is the same argument as in the proof of Proposition 2.11 but with the additional assumption that $\overline{\Omega}_n^i = 0$ for all $1 \leq i \leq n-1$. □

There is a relation between the contractions of $\overline{R}^k$ as a tensor on M and the contractions of $R - D$ as a tensor on N. To avoid any confusion we denote in this paragraph the contraction operator on M by C_M and the one on N by C_N. Then

Lemma 2.24. We have for all $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$

$$C_M^{2k}(\overline{R}^k) = \frac{1}{t^{2k}} C_N^{2k}(R - D). \tag{2.11}$$

Proof. With respect to the orthonormal moving frames on $U \subset N$ and $U \times (0, \infty) \subset M$ described above we have

$$\begin{aligned} C_M^{2k}(\overline{R}^k) &= \sum_{i_1,\dots,i_{2k}=1}^{n} \overline{R}^k(\overline{e}_{i_1},\dots,\overline{e}_{i_{2k}})(\overline{e}_{i_1},\dots,\overline{e}_{i_{2k}}) \\ &= \sum_{i_1,\dots,i_{2k}=1}^{n} t^{2k}(R-D)^k(\overline{e}_{i_1},\dots,\overline{e}_{i_{2k}})(\overline{e}_{i_1},\dots,\overline{e}_{i_{2k}}) \\ &= \frac{1}{t^{2k}} \sum_{i_1,\dots,i_{2k}=1}^{n-1} (R-D)^k(e_{i_1},\dots,e_{i_{2k}})(e_{i_1},\dots,e_{i_{2k}}) \\ &= \frac{1}{t^{2k}} C_N^{2k}((R-D)^k) \end{aligned}$$

since R and D vanish if one of the argument is e_n. □

2.2.3 Total Lipschitz-Killing Curvatures

Since the $\kappa_j(M)$ defined above are differential n-forms on an n-dimensional manifold M, one can naturally integrate them. Obviously, these integrals are not necessary defined since the manifold can be non-compact.

Definition 2.25. Let (M, g) be an n-dimensional Riemannian manifold. The *total Lipschitz-Killing curvatures* are defined by

$$\mathcal{K}_j(M) = \begin{cases} \int_M \kappa_j(M) & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

The first obvious observation is that since $\kappa_0(M) = \mathrm{dvol}_M$, then if M is compact, the 0-th total Lipschitz-Killing curvature of M is simply its volume.

$$\mathcal{K}_0(M) = \mathrm{Vol}(M).$$

The second observation that can easily be made is the following: if (M, g) is a closed Riemannian manifold of even dimension $n = 2p$, then the Gauss-Bonnet-Chern Theorem 3.7 can be written using the last total Lipschitz-Killing curvature:

$$\frac{1}{(2\pi)^p} \mathcal{K}_{2p}(M) = \chi(M). \tag{2.12}$$

Indeed by Lemma 2.13 we know that since M is of even dimension we have

$$\kappa_{2p}(M) = \mathrm{Pf}(\Omega),$$

therefore integrating over M we get

$$\frac{1}{(2\pi)^p} \int_M \mathrm{Pf}(\Omega) = \chi(M).$$

2.3 Weyl's tube formula

Historically the Lipschitz-Killing curvatures appeared in the famous article [Wey39] by H. Weyl in 1939 in which it is shown that for $r > 0$ sufficiently small, the volume of the r-neighbourhood of a compact submanifold $M \subset \mathbb{R}^N$ is a polynomial in the variable r whose coefficients are, modulo some universal constants, the Lipschitz-Killing curvatures of M.

Theorem 2.26. (H. Weyl, 1939**)**
Let $M^n \subset \mathbb{R}^N$ be a compact Riemannian manifold embedded in $\mathbb{R}^N$. Then for all $r > 0$ sufficiently small we have

$$\mathrm{Vol}_N(M_r) = \frac{\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\mathcal{K}_{2m}(M) r^{q+2m}}{q(q+2)(q+4)\cdot \ldots \cdot (q+2m)}, \tag{2.13}$$

where $q = N - n$ is the codimension.

Recall that for $r > 0$, the r-neighbourhood of a subset $A \subset \mathbb{R}^N$ is the following set

$$A_r = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, M) < r\}.$$

The theorem of Weyl is remarkable from at least two viewpoints. First it shows that the volume of M_r is *polynomial* in r. Secondly and most importantly, it shows that this volume *does not depend* on the particular embedding of M in $\mathbb{R}^N$. Indeed, since the Lipschitz-Killing curvatures of M are intrinsic quantities, it follows that $\mathrm{Vol}_N(M_\epsilon)$ is intrinsic as well.
The proof of this theorem can be found in the original paper [Wey39] in which Weyl refers to its own *theory of invariants* but there are self-contained references that do not make use of this theory such as [Gra04].

Example 2.27. As an application of Weyl's tube formula and second proof of Proposition 2.18 we use Theorem 2.26 to compute the Lipschitz-Killing curvatures of the unit sphere. So consider the usual (isometric) embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Given $r > 0$ sufficiently small it is easy to compute the volume of $\mathbb{S}^n_r$ since it is the difference of the volumes of the $(n+1)$-balls of radii $r_1 = 1 + r$ and $r_2 = 1 - r$:

$$\begin{aligned}
\mathrm{Vol}_{n+1}(\mathbb{S}^n_r) &= \mathrm{Vol}_{n+1}(\mathbb{B}^{n+1}(r_1)) - \mathrm{Vol}_{n+1}(\mathbb{B}^{n+1}(r_2)) \\
&= \beta_{n+1}(1+r)^{n+1} - \beta_{n+1}(1-r)^{n+1} \\
&= \beta_{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} (1 - (-1)^k) r^k \\
&= \beta_{n+1} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} r^{2m+1},
\end{aligned}$$

since $(1-(-1)^k)$ vanishes whenever k is even. On the other hand, Theorem 2.26 gives

$$\begin{aligned}\mathrm{Vol}_{n+1}(\mathbb{S}^n_r) &= \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\frac{\mathcal{K}_{2m}(\mathbb{S}^n)}{1\cdot 3\cdot\ldots\cdot(2m+1)}r^{2m+1}\\ &= \sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\frac{\mathcal{K}_{2m}(\mathbb{S}^n)}{(2m+1)!!}r^{2m+1},\end{aligned}$$

where the double factorial is the operation defined by

$$(2m+1)!! = (2m+1)(2m-1)\cdot\ldots\cdot 5\cdot 3\cdot 1,$$

and we have used the fact that $\Gamma\left(\frac{1}{2}\right)=\pi^{\frac{1}{2}}$. Therefore by comparing the coefficients of the powers of r we get that

$$\mathcal{K}_{2m}(\mathbb{S}^n) = \beta_{n+1}\binom{n+1}{2m+1}(2m+1)!!.$$

Now, using that

$$\beta_{n+1} = \frac{\alpha_n}{n+1} \quad \text{and} \quad (2k-1)!! = \frac{(2k)!}{2^k k!},$$

we can rewrite the total Lipschitz-Killing curvatures of $\mathbb{S}^n$ as

$$\begin{aligned}\mathcal{K}_{2m}(\mathbb{S}^n) &= \frac{\alpha_n}{n+1}\frac{(n+1)!}{(2m+1)!(n-2m)!}\frac{(2m+2)!}{2^{m+1}(m+1)!}\\ &= \frac{\alpha_n n!}{2^m m!(n-2m)!}\\ &= \int_{\mathbb{S}^n}\kappa_{2m}(\mathbb{S}^n),\end{aligned}$$

where the $\kappa_{2m}(\mathbb{S}^n)$ are the one given by Proposition 2.18.

2.4 Connections in Principal Bundles

The whole approach of Cartan by moving frames can be reformulated in the general context of principal bundles. We will be using this framework in the proof of the Gauss-Bonnet-Chern Theorem in Chapter 3.
Instead of constructing the connection and curvature forms locally in an oriented manifold M, it is possible to define them globally on a larger manifold which turns out to be the total space of a bundle over M and then pull-back those global forms using sections of this freshly constructed bundle. The natural choice for the total space, which will be denoted by $SO(M)$, is to take as the fibre over a point $p \in M$, all orthonormal positively oriented bases of T_pM and therefore any section

$s : U \to SO(M)$ represent what we used to call a moving frame on the open subset $U \subset M$. There is a natural action of the group $\mathrm{SO}(n)$ on each fibre since every positively oriented orthonormal basis can be sent onto any other positively oriented orthonormal basis by an element of $\mathrm{SO}(n)$. This turns the bundle $(SO(M), M, \pi)$ into a principal bundle with structural group $G = \mathrm{SO}(n)$.
This construction requires to define an analogue of the connection forms on the total space of a principal bundle, which will be called an *Ehresmann connection*. By differentiation, we will obtain a notion of *curvature forms* which will satisfy a structural equation. Finally, by choosing a section of the bundle of orthonormal positively oriented moving frames, i.e. by choosing a moving frame on an open subset of the base manifold, we will recover Cartan's formalism. Although the definitions of a connection form and of a curvature form are valid for an arbitrary principal bundle, we will restrict our attention to the bundle of orthonormal positively oriented moving frames.
The content of this section comes mainly from the second volume of [Spi99] and the first volume of [KN63] and we refer to those books for additional background about connections in principal bundles.

2.4.1 The Connection Form

Let $F(M) \to M$ be the frame bundle, i.e. the principal bundle with structural group $\mathrm{GL}_n(\mathbb{R})$ and fibre

$$\{(p, X_1, \ldots, X_n) \mid (X_1, \ldots, X_n) \text{ is a basis of } T_pM\}.$$

Sections of this bundle are what we previously called moving frames. Since M is oriented and endowed with a Riemannian metric, we can restrict our attention to the following bundle: let $\pi : SO(M) \to M$ be the bundle of orthonormal positively oriented frames on M with structure group $\mathrm{SO}(n)$. The structure group acts on the right on $SO(M)$:

$$\begin{array}{ccc} SO(M) \times \mathrm{SO}(n) & \longrightarrow & SO(M) \\ (u, A) & \mapsto & u \cdot A. \end{array}$$

More precisely, if $u = (X_1, \ldots, X_n)$ and $A \in \mathrm{SO}(n)$, then the i-th element of $u \cdot A$ is

$$(u \cdot A)_i = A_i^j X_j,$$

since

$$u \cdot A = \begin{pmatrix} X_1^1 & \cdots & X_n^1 \\ \vdots & \ddots & \vdots \\ X_1^n & \cdots & X_n^n \end{pmatrix} \begin{pmatrix} A_1^1 & \cdots & A_n^1 \\ \vdots & \ddots & \vdots \\ A_1^n & \cdots & A_n^n \end{pmatrix} = \begin{pmatrix} X_i^1 A_1^i & \cdots & X_i^1 A_n^i \\ \vdots & \ddots & \vdots \\ X_i^n A_1^i & \cdots & X_i^n A_n^i \end{pmatrix}$$

Definition 2.28. An *Ehresmann connection* on the principal bundle $SO(M)$ is a smooth $\mathfrak{o}(n)$-valued 1-form ω on $SO(M)$ satisfying the two following algebraic conditions:

(a) for all $X \in \mathfrak{o}(n)$ we have $\omega(\sigma(X)) = X$;

(b) for all $A \in \mathrm{SO}(n)$ and all $Y \in \Gamma(SO(M))$ we have

$$\omega(dR_A(Y)) = A^{-1}\omega(Y)A;$$

where $R_A(u) = u \cdot A$ for all $u \in SO(M)$ is the right action of $\mathrm{SO}(n)$ on $SO(M)$ and $\sigma : \mathfrak{o}(n) \to \Gamma(SO(M))$ is the map defined by

$$\sigma(X)(u) = d\sigma_u(X),$$

with $\sigma_u : \mathrm{SO}(n) \to SO(M)$ given by $\sigma_u(A) = u \cdot A$, the dot standing for the action on the right of $\mathrm{SO}(n)$ on the manifold $SO(M)$.

Remark 2.29. An Ehresmann connection induces a distribution on $SO(M)$ as follows: for all $u \in SO(M)$ the map

$$\omega_u : T_uSO(M) \longrightarrow \mathfrak{o}(n)$$

is surjective and therefore the space $H_u := \ker(\omega_u) \subset T_uSO(M)$, called the *horizontal subspace at* u has the same dimension as M. Vectors in H_u are called *horizontal vectors*. Together with this subspace comes $V_u \subset T_uSO(M)$ the *vertical subspace at* u defined by $V_u = \ker(d\pi_u)$ and satisfying $T_uSO(M) = H_u \oplus V_u$, so that every tangent vector Y at u can be decomposed as

$$Y = h(Y) + v(Y),$$

with $h(Y)$ the horizontal component and $v(Y)$ the vertical component. Since for all $A \in \mathrm{SO}(n)$ the map $R_A : SO(M) \to SO(M)$ sends a fibre to itself, it follows that

$$\{\sigma(X)(u) \in T_uSO(M) \mid X \in \mathfrak{o}(n)\} = V_u.$$

The above definition of an Ehresmann connection is actually invariant in the sense that is does not depend on a particular moving frame.
However one can prove that it is equivalent to requiring that for every section $s : U \to SO(M)$ and every smooth map $A : U \to \mathrm{SO}(n)$ we have

$$(s \cdot A)^*\omega = A^{-1}dA + A^{-1}s^*\omega A, \tag{2.14}$$

where $s \cdot A : U \to SO(M)$ is defined by

$$(s \cdot A)(p) = s(p) \cdot A(p) = R_{A(p)}(s(p)).$$

This is consistent with the transformation law for the connection forms stated in Lemma 1.19. The proof of equation (2.14) is based on the following proposition on the differential of the map $s \cdot A : U \to SO(M)$, whose proof can be found in [Spi99], Vol.2, p.312.

Proposition 2.30. Let $s : U \to SO(M)$ be a section of $SO(M)$ and let $A : U \to$ SO(n) be a smooth map. Then for all $X_p \in T_pM)$ we have

$$d(s \cdot A)_p(X_p) = dR_{A(p)}(ds_p(X_p)) + \sigma(A^{-1}(p)X_p(A))(s(p) \cdot A(p)). \tag{2.15}$$

The argument of σ has to be an element of $\mathfrak{o}(n)$, but $X_p(A)$ the derivative of the map A in the direction X, and therefore an element of $\mathfrak{o}(n)$; it follows that $A^{-1}(p)X_p(A)$ is also an element of $\mathfrak{o}(n)$ and that $\sigma(A^{-1}(p)X_p(A))(s(p) \cdot A(p))$ is well-defined. Observe that the right-hand side of equation (2.15) is the decomposition of the left-hand side in the horizontal and vertical directions i.e.

$$\begin{aligned} h(d(s \cdot A)_p(X_p)) &= dR_{A(p)}(ds_p(X_p)) \in H_{s(p)A(p)}, \\ v(d(s \cdot A)_p(X_p)) &= \sigma(A^{-1}(p)X_p(A))(s(p) \cdot A(p)) \in V_{s(p)A(p)}. \end{aligned}$$

Another observation is that if A is constant on U, then the vertical part vanishes and one has for all $p \in U$:

$$d(s \cdot A)_p(X_p) = dR_A(ds_p(X_p))$$

The $\mathfrak{o}(n)$-valued 1-form ω can be seen as a matrix of usual 1-forms on $SO(M)$. Given a basis (E^i_j) of the Lie algebra $\mathfrak{o}(n)$ (which is nothing but the skew-symmetric matrices), one can write

$$\omega = \omega^i_j \cdot E^j_i,$$

with $\omega^i_j \in \Omega^1(SO(M))$. Now, the link between the usual notion of a (Koszul) connection on a a manifold and an Ehresmann connection ω is given by the following consideration: let $s : U \to SO(M)$ be the section given by

$$s(p) = (X_1(p), \ldots, X_n(p))$$

and define an operation $\nabla : \Gamma(U) \times \Gamma(U) \longrightarrow \Gamma(U)$ by

$$\nabla_{X_i} X_j := (s^* \omega^k_j)(X_i) X_k.$$

Then one can easily show that ∇ is a Koszul connection on M.

2.4.2 The Curvature and Torsion Forms

As in the case of the Cartan formalism (see Section 1.2) an Ehresmann connection ω comes together with a curvature form and a torsion form. The *curvature form of* ω is the $\mathfrak{o}(n)$-valued 2-form defined by $\Omega = D\omega$, i.e.

$$D\omega(Y, Z) = (d\omega)(h(Y), h(Z)),$$

with d the ordinary differential and $h(Y)$ and $h(Z)$ the horizontal components of Y and Z.
Now, as in the case of the connection form ω, we want to study the behaviour of the pull-back of the curvature form Ω by a section $s : U \to SO(M)$.

Proposition 2.31. Let $s : U \to SO(M)$ be a section of $SO(M)$ and let $A : U \to \mathrm{SO}(n)$ be a smooth map. Then

$$(s \cdot A)^*\Omega = A^{-1}s^*\Omega A.$$

Proof. If $X_p, Y_p \in T_pM$, then by Proposition 2.30 and by the definition of an Ehresmann connection we have

$$\begin{aligned}
(s \cdot A)^*\Omega(X, Y) &= \Omega\left(d(s \cdot A)(X), d(s \cdot A)(Y)\right) \\
&= d\omega\left(h(d(s \cdot A)(X)), h(d(s \cdot A)(Y))\right) \\
&= d\omega\left(dR_A(ds(X)), dR_A(ds(Y))\right) \\
&= R_A^* d\omega\left(ds(X), ds(Y)\right) \\
&= d\left(R_A^*\omega\right)\left(ds(X), ds(Y)\right) \\
&= d(A^{-1}\omega A)\left(ds(X), ds(Y)\right).
\end{aligned}$$

But $ds(X)$ and $ds(Y)$ are horizontal i.e $\omega(ds(X)) = \omega(ds(Y)) = 0$. Therefore, since

$$d(A^{-1}\omega A) = dA^{-1} \wedge \omega A + A^{-1}d\omega A + A^{-1}\omega \wedge dA,$$

it follows that $d(A^{-1}\omega A)(ds(X), ds(Y)) = A^{-1}d\omega(ds(X), ds(Y))A$. Hence

$$(s \cdot A)^*\Omega(X, Y) = A^{-1}d\omega(ds(X), ds(Y))A = A^{-1}s^*\Omega(X, Y)A.$$

□

2.4.3 The Structure Equations

Although an Ehresmann connection and its associated curvature form can be defined on any arbitrary principal bundle, it is not the case for the torsion form which is defined only on the principal bundle of frames (or in our case, the bundle of orthonormal positively oriented frames). First, we define the analogue of a moving coframe but in the setting of the principal bundle $SO(M)$. The *canonical form* of the principal bundle $SO(M)$ is the $\mathbb{R}^n$-valued 1-form defined at $u \in SO(M)$ by

$$\theta_u(Y_u) = u^{-1}(d\pi_u(Y_u)) \quad \text{for all } Y_u \in T_uSO(M),$$

where u^{-1} is the inverse of the isomorphism induced by any element $u \in SO(M)$ and defined by $u : \mathbb{R}^n \to T_{\pi(u)}M$ which sends the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ to the orthonormal basis $(u_1, \ldots, u_n)$ of $T_{\pi(u)}M$. The fact that θ can be compared to a coframe associated to a frame comes from the following observation: if we are given a section $s : U \to SO(M)$ with $s = (X_1, \ldots, X_n)$, then the pullback of θ by this section is given by

$$s^*\theta(Y_p) = \theta_{s(p)}(ds_p(Y_p)) = s(p)^{-1}(Y_p),$$

so that the i-th component of $s^*\theta(Y_p)$ is the i-th component of Y_p with respect to the basis $(X_1(p), \ldots, X_n(p))$, i.e. the $s^*\theta^i$ are the dual 1-forms to the moving frame $(X_1, \ldots, X_n)$.
Using the canonical form θ, we define the torsion form Θ of the connection ω by

$$\Theta = D\theta,$$

which is an $\mathbb{R}^n$-valued 2-form.
The next theorem, whose proof can be found in the [Spi99, Vol. 2, p. 327], consists of the structure equations. We state the theorem in the case where the considered principal bundle is $SO(M)$ although the second structure equation is true for an arbitrary principal bundle.

Theorem 2.32. Let ω be an Ehresmann connection on the principal bundle $\pi : SO(M) \to M$, with canonical 1-form θ, torsion 1-form Θ and curvature form Ω. Then we have

(a) the *first structure equation*

$$d\theta = -\omega \wedge \theta + \Theta,$$

(b) the *second structure equation*

$$d\omega = -\omega \wedge \omega + \Omega.$$

As well as for the connection form, it is useful to express the canonical form, the torsion form and the curvature form in terms of ordinary differential forms. If $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$, then there exist 1-forms $\theta^1, \ldots, \theta^n, \Theta^1, \ldots, \Theta^n \in \Omega^1(SO(M))$ such that

$$\theta = \theta^i \cdot e_i \quad \text{and} \quad \Theta = \Theta^i \cdot e_i.$$

For the curvature form, if (E^i_j) is the standard basis of $\mathfrak{o}(n)$, then there exist 2-forms $\Omega^i_j \in \Omega^2(SO(M))$ such that

$$\Omega = \Omega^i_j \cdot E^j_i.$$

Using these notations, the structural equations can be rewritten in the following (familiar if the torsion form vanishes) form:

$$\begin{aligned} d\theta^i &= -\omega^i_j \wedge \theta^j + \Theta^i \\ d\omega^i_j &= -\omega^i_k \wedge \omega^k_j + \Omega^i_j. \end{aligned}$$

It is important to note here that these two equations are defined on the manifold $SO(M)$ and not on the base space M. However, the construction is made so that if $s : U \to SO(M)$ is any section, then the structure equations pulled back on M are

$$\begin{aligned} d(s^*\theta^i) &= -s^*\omega^i_j \wedge s^*\theta^j + s^*\Theta^i \\ d(s^*\omega^i_j) &= -s^*\omega^i_k \wedge s^*\omega^k_j + s^*\Omega^i_j. \end{aligned}$$

After all these general considerations on connections, we will restrict our attention to the Levi-Civita connection on M. One can prove (e.g. in [KN63], p.158) that there is a *unique* connection ω on $SO(M)$ with vanishing torsion, i.e. there exist a unique connection ω such that the structure equations write

$$\begin{aligned} d\theta^i &= -\omega^i_j \wedge \theta^j, \\ d\omega^i_j &= -\omega^i_k \wedge \omega^k_j + \Omega^i_j. \end{aligned}$$

Therefore, in this case, the pulled back forms $s^*\theta^i$, $s^*\omega^i_j$ and $s^*\Omega^i_j$ are precisely the dual, connection and curvature forms associated to the moving frame s as presented in Chapter 1.

Remark 2.33. The Lipschitz-Killing Curvatures can be easily transposed to this new framework of principal bundles. On the manifold $SO(M)$ the following (global) differential form can be defined

$$\eta_{2k} := \frac{1}{2^k k!(n-2k)!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma \Omega_{\sigma_1 \sigma_2} \wedge \ldots \Omega_{\sigma_{2k-1}\sigma_{2k}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_n} \in \Omega^n(SO(M)).$$

Then if $s : U \to SO(M)$ is any section we recover the Lipschitz-Killing Curvatures of Proposition 2.12 by taking the pull-back by s of η_{2k}:

$$\kappa_{2k}(M) = s^*(\eta_{2k}) \in \Omega^n(U).$$

Chapter 3

The Gauss-Bonnet-Chern Theorem

In this chapter, we present the celebrated Gauss-Bonnet-Chern Theorem which will be a key ingredient in the proof of our result. This theorem has been proved almost simultaneously by Fenchel, Allendoerfer and Weil, on one hand and by Chern on the other hand in the early 40's. Although they established the same result, their methods to prove it are quite different. Fenchel considered in [Fen40] only submanifolds of $\mathbb{R}^N$ whereas Allendoerfer-Weil used in [AW43] a local embedding theorem due to Cartan to generalize Fenchel's result. Since, by Nash's embedding theorem, every Riemannian manifold can be isometrically embedded in some Euclidean space this is not actually a restriction, but this theorem was proved only a decade later. In 1944 Chern's article *A Simple Intrinsic Proof of the Gauss-Bonnet Formula for Closed Riemannian Manifolds* [Che44] was published, in which he proves the same result but in a completely intrinsic way, that is without assuming that the manifold is embedded in some Euclidean space. His method relies on a phenomenon called *transgression*, which is the property of a closed differential form to be exact provided it is pulled-back to a fiber bundle. In this context, the considered fiber bundle is the bundle of unit vectors in the tangent space of the manifold. In 1945 Chern's improved his proof in [Che45], therefore this article will be our main reference throughout this chapter. However we refer to [Li11] for a detailed analysis of the proof in [Che44]. Apart from the extrinsic and intrinsic proofs of the Gauss-Bonnet-Chern Theorem, there are other classical proofs using characteristic classes ([Li11],[MS74],[MT97]) or via the heat flow. Our choice of working with the original proof of Chern is motivated by the fact that we want to obtain an explicit for the boundary term which can be seen as the Gauss-Bonnet defect in the case of a compact manifold with boundary.

3.1 The Poincaré-Hopf Theorem

The Gauss-Bonnet-Chern theorem partially relies on an earlier result in differential topology, that establishes a relation between the differential structure of a compact manifold and its topology. More precisely, this theorem, due to Poincaré in dimension two and Hopf in higher dimensions, states that on a closed manifold, the sum of the indices of a vector field having isolated singularities is equal to the Euler characteristic of the manifold. The generalization to compact manifolds with boundary is probably due to Lefschetz. This theorem is now to be explained in some details but we refer to [Mil97] for a complete proof. Let us begin this section by recalling some topological notions.

Definition 3.1. A manifold M is of finite topological type if there exists a compact subset $K \subset M$ such that $M \setminus K$ is homeomorphic to $\partial K \times (1, \infty)$. For the sake of simplicity we will often assume that the boundary of K is connected as in Figure 3.1 although in the general case it consists of the union of finitely many connected compact components, i.e. there exist $N_1, \ldots, N_r \subset \partial K$ connected compact submanifolds of the boundary of K such that

$$\partial K = N_1 \sqcup \ldots \sqcup N_r.$$

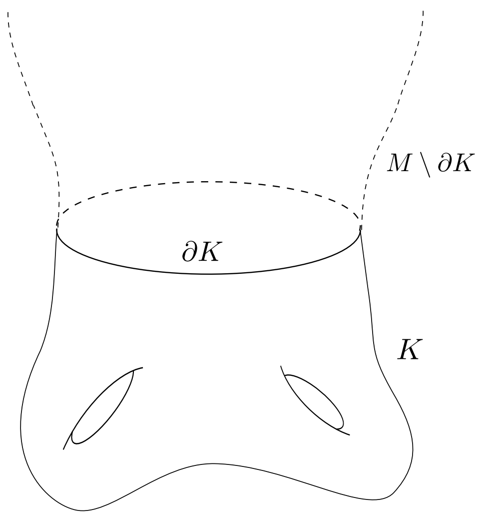


Figure 3.1: Manifold of finite topological type

The submanifolds $E_i := N_i \times (1, \infty) \subset M$ are called the *ends* of M. In other words, this hypothesis means that all the topology of M is contained in the compact K.

Note that in particular any compact manifold M is of finite topological and has no ends by taking $K = M$.

Definition 3.2. Let M be a manifold of dimension n which is of finite topological type. The *Euler characteristic* $\chi(M)$ of M is the following integer number

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{R}}(H_k(M)),$$

where $H_k(M)$ is the k-th homology group (on $\mathbb{R}$) of M.

It is well-known that χ is a homotopy invariant and that it has the following properties. Let M and N be two n-dimensional compact differentiable manifolds such that their intersection $M \cap N$ is a submanifold. Then

(i) we have an *inclusion-exlusion principle*

$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N),$$

(ii) if M is closed (compact without boundary) and the dimension n is odd

$$\chi(M) = 0.$$

(iii) for any triangulation of M with exactly n_k simplices of dimension $k = 0, \dots, n$, then

$$\chi(M) = \sum_{i=0}^{n} (-1)^i n_i.$$

It is worth noting that if M has a boundary, then since the interior $\mathring{M}$ and $M = \mathring{M} \cup \partial M$ have the same homotopy type we have $\chi(\mathring{M}) = \chi(M)$
Given a vector field X on a manifold M, one can associate a number to each isolated zero of X called the *index* of the vector field at this point. It is a measure of the behaviour of the vector field around the zero and represents in some sense the winding of X around the singularity.

Definition 3.3. Let $p \in M$ be an isolated zero of the vector field X and consider a smooth chart $\psi : U \to \mathbb{R}^n$ around p. Since p is isolated, there exist a small ball $B(x, \varepsilon) \subset \mathbb{R}^n$ around $x = \psi(p)$ such that x is the only zero of the vector field $\tilde{X} = X \circ \psi^{-1} \in \Gamma(\mathbb{R}^n)$ in $B(x, \varepsilon)$. Therefore, the following vector field is well-defined on $B(x, \varepsilon) \setminus \{x\}$:

$$\nu = \frac{\tilde{X}}{\|\tilde{X}\|}.$$

In particular, if $\partial B(x, \varepsilon) = S(x, \varepsilon)$ then $\nu|_{S(x,\varepsilon)} : S(x, \varepsilon) \to \mathbb{S}^{n-1}$ is well-defined. The *index of* X *at* p is the integer given by

$$\operatorname{Ind}(X, p) = \deg\left(\nu|_{S(x,\varepsilon)}\right).$$

It is now possible to state the Poincaré-Hopf theorem.

Theorem 3.4. (Poincaré-Hopf theorem) Let M be a compact manifold possibly with boundary and let X be a smooth vector field with isolated zeroes $p_1, \dots, p_k \in M$. If M has a boundary then assume that X is transverse to the boundary and points outwards. Then

$$\sum_{i=1}^{k} \operatorname{Ind}(X, p_i) = \chi(M).$$

In the case where the manifold has a boundary, then if instead of pointing outwards, the vector field is pointing inwards, the theorem is modified as follows:

$$\sum_{i=1}^{k} \mathrm{Ind}(X, p_i) = \chi(M) - \chi(\partial M).$$

We refer to [Mil97] for the proof in the case where M has no boundary and to [BSS09] if $\partial M \neq \emptyset$.

Example 3.5. As an illustration of the Poincaré-Hopf theorem, consider $M = \overline{\mathbb{B}}^n$ to be the closed unit ball. Then $\partial M = \mathbb{S}^{n-1}$ and let $X \in \Gamma(M)$ be the radial vector field . The only zero of X in M is at the origin. If X is taken to point outwards, then the origin is a source and $\mathrm{Ind}(X, 0) = 1$, which corresponds to the Euler characteristic of M since the unit ball has the homotopy type of a point. Now, if X points inwards, then the origin is a sink and therefore $\mathrm{Ind}(X, 0) = (-1)^n$. On the other hand, since

$$\chi(\partial M) = \chi(\mathbb{S}^{n-1}) = 1 + (-1)^{n-1} = \begin{cases} 2 & \text{if } n-1 \text{ is even,} \\ 0 & \text{if } n-1 \text{ is odd,} \end{cases}$$

we get that

$$\chi(M) - \chi(\partial M) = 1 - 1 - (-1)^{n-1} = (-1)^n = \mathrm{Ind}(X, 0).$$

3.2 Statement of the Gauss-Bonnet-Chern Theorem

We now present the essence of Chern's second article on the Gauss-Bonnet-Chern Theorem [Che45]. Let (M, g) be an n-dimensional oriented compact Riemannian manifold (possibly with boundary). The first step is the construction of what is the generalization of the Gauss curvature in higher dimension. This quantity is often called the *Pfaffian of the curvature form* or the *Gauss-Bonnet integrand* and is defined using the Cartan formalism. Let $(e_1, \ldots, e_n)$ be an orthonormal moving frame on an open subset $U \subset M$ with dual coframe $(\theta^1, \ldots, \theta^n)$ and let ω_B^A and Ω_B^A be the connection and curvature forms associated to the Levi-Civita connection of M. The Gauss-Bonnet integrand is the following n-form:

$$\mathrm{Pf}(\Omega) = \begin{cases} \dfrac{1}{2^p p!} \displaystyle\sum_{\sigma \in \mathfrak{S}_{2p}} \varepsilon_\sigma \Omega_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge \Omega_{\sigma_{2p}}^{\sigma_{2p-1}} & \text{if } n = 2p \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.1}$$

Although $\mathrm{Pf}(\Omega)$ is defined locally, one can easily show that it is in fact global by using the transformation law for the curvature forms. In his paper, Chern used a slightly modified n-form which simplifies the constants but which is also a little bit misleading in regard of the modern notations. More precisely, Chern defines

$$\Omega = \frac{1}{(2\pi)^p}\,\mathrm{Pf}(\Omega).$$

This notation is slightly ambiguous as Ω can denote both the Gauss-Bonnet integrand and the matrix of the curvature forms of M. However we shall carefully precise, whenever it is necessary, to which definition Ω refers.

Let $SM = \{(p,v) \in TM \mid v \in T_pM, \|v\| = 1\} \subset TM$ be the unit tangent bundle and denote by $\pi : SM \to M$ the canonical projection. The n-form Ω is not exact in general, however its pull-back by π on SM is exact:

Lemma 3.6. (Transgression Lemma) There exists $\tilde{\Pi} \in \Omega^{n-1}(SM)$ such that

$$\pi^*\Omega = -d\tilde{\Pi}.$$

With this lemma, the Gauss-Bonnet-Chern theorem can be stated:

Theorem 3.7. (Gauss-Bonnet-Chern Theorem) Let (M,g) be an n-dimensional oriented compact Riemannian manifold and let ν be the inward-pointing unit normal to the boundary. Then we have

$$(-1)^n\left(\chi(M) - \chi(\partial M)\right) - \frac{1}{(2\pi)^{\frac{n}{2}}}\int_M \mathrm{Pf}(\Omega) = \int_{\nu(\partial M)} \tilde{\Pi} \tag{3.2}$$

Let us make a few remark about this iconic theorem.

Remark 3.8. (1) Observe that for even dimensions $\chi(\partial M) = 0$ since ∂M is a closed odd-dimensional manifold and therefore Theorem 3.7 reduces to

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}}\int_M \mathrm{Pf}(\Omega) = \int_{\nu(\partial M)} \tilde{\Pi}. \tag{3.3}$$

(2) If the manifold has no boundary, then the Theorem reads

$$(-1)^n\chi(M) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_M \mathrm{Pf}(\Omega),$$

and if n is odd then both the Euler-characteristic and the Pfaffian vanish so that the theorem holds but gives no information.

(3) By a standard topological argument we know that the Euler-characteristic of a compact odd-dimensional manifold with boundary is equal to half of the Euler-characteristic of its boundary i.e.

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

Therefore in odd dimensions, the Theorem reads

$$\frac{1}{2}\chi(\partial M) = \int_{\nu(\partial M)} \tilde{\Pi}.$$

(4) If $(M_\lambda^{2p}, g_\lambda)$ is a compact even-dimensional Riemannian manifold without boundary of constant sectional curvature λ, we obtain as a consequence of the Theorem that the volume of M_λ is proportional to its Euler-characteristic. Indeed recall that the Gauss-Bonnet-Chern Theorem can be stated using the last Lipschitz-Killing curvature as

$$\chi(M_\lambda) = \frac{1}{(2\pi)^p}\mathcal{K}_{2p}(M_\lambda).$$

Moreover the Lipschitz-Killing curvatures of manifolds of constant curvature were given in Theorem 2.17, so that we know that

$$\mathcal{K}_{2p}(M_\lambda) = \lambda^p \frac{(2p)!}{2^p p!}\mathrm{Vol}(M_\lambda),$$

therefore

$$\frac{\lambda^p (2p)!}{2^{2p}\pi^p p!}\mathrm{Vol}(M_\lambda) = \chi(M_\lambda).$$

Observe that the sign of the Euler-characteristic depends on the dimension. In particular if $\lambda = -1$ then

$$(-1)^p \chi(M_{-1}) > 0.$$

Which is a particular case of the so called Hopf conjecture.

(5) In the case where the manifold has a boundary, observe that a choice of unit normal vector ν is made. In particular its orientation (inward/outward pointing) is important and modifies the statement of the Theorem as we shall see hereafter.

3.3 Chern's Proof

3.3.1 First Step: The Transgression Lemma

Chern's main idea in the proof of his theorem is to switch from the manifold M to the manifold formed by all unit tangent vectors, which we denote by SM and to show that the pullback of the Pfaffian form $\mathrm{Pf}(\Omega)$ (here Ω denotes the matrix of the curvature forms on M and not on $SO(M)$) on this manifold by the standard projection is exact. The following proposition, whose proof can be found in [KN63], p.57, will help us to establish a natural link between the connnection and curvature forms on M and SM.

Proposition 3.9. Let $\pi : P \to M$ be a principal G-bundle and let $H \subset G$ be a closed subgroup of G. Then $N := P/H$ is a differentiable manifold and we have a principal H-bundle given by $\pi_1 : P \to N$. Moreover there is a canonical associated fibre bundle $\pi_2 : N \to M$ with fiber G/H such that the following diagram commutes

$$\begin{array}{ccc} P & & \\ {\scriptstyle \pi_1}\downarrow & \searrow^{\pi} & \\ N & \xrightarrow[\pi_2]{} & M. \end{array}$$

Let us apply this proposition to the bundle of positively oriented orthonormal moving frames $P = SO(M)$ with structure group $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1)$. There is no canonical way for $\mathrm{SO}(n-1)$ to be a subgroup of $\mathrm{SO}(n)$ since the orientation preserving isometries of $\mathbb{R}^{n-1}$ are not canonically orientation preserving isometries of $\mathbb{R}^n$. The choice of a direction $v \in \mathbb{S}^{n-1}$ in $\mathbb{R}^n$ must be made in order to split $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \oplus \mathbb{R}v$ and make $SO(n-1)$ act on $\mathbb{R}^{n-1}$, leaving the direction v invariant. For the rest of this section, we make the following convention: if $(v_1, \ldots, v_n)$ is an orthonormal directly oriented basis of $\mathbb{R}^n$, then the chosen direction in making $\mathrm{SO}(n-1)$ a subgroup of $\mathrm{SO}(n)$ is the last one $v = v_n$. This means that the action of $\mathrm{SO}(n-1)$ on $SO(M)$ is defined as follows:

$$\begin{array}{ccc} SO(M) \times \mathrm{SO}(n-1) & \longrightarrow & SO(M) \\ (u, A) & \mapsto & u \cdot \bar{A}, \end{array}$$

where

$$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(n).$$

Therefore, if $u = (p, X_1, \ldots, X_n)$ then $(u \cdot A) = (p, A_1^j X_j, \ldots, A_{n-1}^j X_j, X_n)$ where we sums over repeated indices run from 1 to $n-1$.

The quotient manifold $N = SO(M)/\mathrm{SO}(n-1)$ is therefore diffeomorphic to the unit-tangent bundle SM since the above action of $\mathrm{SO}(n-1)$ is transitive on the orthonormal directly oriented bases of $\mathbb{R}^{n-1}$.

Moreover, we know that the quotient $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ is diffeomorphic to the oriented sphere $\mathbb{S}^{n-1}$, therefore Proposition 3.9 gives rise to the following bundles:

(a) A principal $\mathrm{SO}(n)$-bundle $\pi : SO(M) \to M$. We denote this principal bundle by ξ i.e. $\xi = (SO(M), M, \mathrm{SO}(n), \pi)$.

(b) A principal $\mathrm{SO}(n-1)$-bundle $\pi_1 : SO(M) \to SM$. We denote this principal bundle by ξ_1 i.e. $\xi_1 = (SO(M), SM, \mathrm{SO}(n-1), \pi_1)$.

(c) A fibre bundle $\pi_2 : SM \to M$ with fibre $\mathbb{S}^{n-1}$, which is nothing different but the usual unit tangent bundle of M. We denote this fibre bundle by ξ_2 i.e. $\xi_2 = (SM, M, \pi_2)$.

In terms of a diagram:

$$\begin{array}{ccc} SO(M) & & \\ {\scriptstyle \pi_1}\downarrow & \searrow^{\pi} & \\ SM & \xrightarrow[\pi_2]{} & M \end{array}$$

Chern's construction of the form $\tilde{\Pi}$ in Lemma 3.6 can now be explained using the modern language of principal bundles. The particularity of his construction is that although the form $\tilde{\Pi}$ is built up from the connection and curvature forms, and is therefore local in essence, it appears that it does not in fact depend on the chosen section of the bundle $\pi_1 : SO(M) \to SM$.

As before, write $\omega = \omega^i_j \cdot E^j_i$ and $\Omega = \Omega^i_j \cdot E^j_i$ with (E^i_j) the standard basis of $\mathfrak{o}(n)$, and $\omega^i_j \in \Omega^1(SO(M))$ and $\Omega^i_j \in \Omega^2(SO(M))$. Moreover, if $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$, the canonical form θ can be written as $\theta = \theta^i \cdot e_i$ with $\theta^i \in \Omega^1(SO(M))$.
On the total space $SO(M)$ consider the following (global) forms:

$$\Phi_k = \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \dots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \dots \wedge \omega^{\sigma_{n-1}}_n,$$
$$\Psi_k = 2(k+1) \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \dots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \Omega^{\sigma_{2k+1}}_n \wedge \omega^{\sigma_{2k+2}}_n \wedge \dots \wedge \omega^{\sigma_{n-1}}_n.$$

Remark 3.10. The cautious reader may note that our definition of Φ_k and Ψ_k slightly differs from the one given by Chern in [Che45]. If we denote his forms by Φ_k^{Chern} and Ψ_k^{Chern} the following relations hold:

$$\Phi_k = (-1)^{n-1} \Phi_k^{\text{Chern}}$$
$$\Psi_k = (-1)^n \Psi_k^{\text{Chern}}.$$

The same is true for the n-form Ω:

$$\Omega = (-1)^{\frac{n}{2}} \Omega^{\text{Chern}}.$$

This is actually due to the convention on the indices in the connection and curvature forms ω^i_j and Ω^i_j. The relation between his convention and ours is the following

$$\omega_{ij}^{\text{Chern}} = \omega^j_i = -\omega^i_j \quad \text{and} \quad \Omega_{ij}^{\text{Chern}} = \Omega^j_i = -\Omega^i_j.$$

The apparent complexity of the two differential forms Φ_k and Ψ_k hides a powerful property of invariance when they are pulled back on the unit tangent bundle SM and therefore that they are globally defined on SM. Indeed, it appears that they do not depend on a particular section. More precisely we have

Lemma 3.11. *Let $v : U \to SO(M)$ be a section of ξ_2 and let $s_i : \tilde{U}_i \to SO(M)$ be two sections of ξ_1 such that $v(U) \subset \tilde{U}_i$. Then*

$$(s_1 \circ v)^* \Phi_k = (s_2 \circ v)^* \Phi_k \in \Omega^{n-1}(U),$$
$$(s_1 \circ v)^* \Psi_k = (s_2 \circ v)^* \Psi_k \in \Omega^n(U).$$

Proof. For $i = 1, 2$, let $r_i := s_i \circ v : U \to SO(M)$ and observe that they are sections of ξ. On U, the sections r_1 and r_2 are moving frames with the same last vector, so there exists a smooth map $A : U \to \mathrm{SO}(n-1)$ such that

$$r_2 = r_1 \cdot \bar{A},$$

where $\bar{A} : U \to \mathrm{SO}(n)$ is defined by $\bar{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. By equation (2.14) and Proposition 2.31 we know that

$$\begin{aligned} r_2^*\omega &= (r_1 \cdot A)^*\omega = \bar{A}^{-1} d\bar{A} + \bar{A}^{-1} r_1^* \omega \bar{A}, \\ r_2^*\Omega &= (r_1 \cdot A)^*\Omega = \bar{A}^{-1} r_1^* \Omega \bar{A}. \end{aligned}$$

Moreover

$$\bar{A}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad d\bar{A} = \begin{pmatrix} dA & 0 \\ 0 & 0 \end{pmatrix}.$$

Only some particular connection and curvature forms are involved in the expressions of Φ_k and Ψ_k, namely the ω^i_n and the Ω^i_j for $1 \leq i, j \leq n-1$. Using the transformation laws we obtain directly

$$\begin{aligned} r_2^*\omega^i_n &= (A^{-1})^i_j r_1^* \omega^j_n = \sum_{j=1}^{n-1} A^j_i r_1^* \omega^j_n = A_{ji} r_1^* \omega^j_n, \\ r_2^*\Omega^i_j &= (A^{-1})^i_k r_1^* \Omega^k_l A^l_j = \sum_{k,l=1}^{n-1} A^k_i A^l_j r_1^* \Omega^k_l, \end{aligned}$$

since $A^{-1} = A^T$. Therefore,

$$\begin{aligned}
r_2^*\Phi_k &= \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma r_2^*\Omega_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge r_2^*\Omega_{\sigma_{2k}}^{\sigma_{2k-1}} \wedge r_2^*\omega_n^{\sigma_{2k+1}} \wedge \ldots \wedge r_2^*\omega_n^{\sigma_{n-1}} \\
&= \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \left(\sum_{\mu_1,\mu_2=1}^{n-1} A_{\sigma_1}^{\mu_1} A_{\sigma_2}^{\mu_2} r_1^*\Omega_{\mu_2}^{\mu_1} \right) \wedge \ldots \wedge \left(\sum_{\mu_{2k-1},\mu_{2k}=1}^{n-1} A_{\sigma_{2k-1}}^{\mu_{2k-1}} A_{\sigma_{2k}}^{\mu_{2k}} r_1^*\Omega_{\mu_{2k}}^{\mu_{2k-1}} \right) \\
&\qquad \wedge \left(\sum_{\mu_{2k+1}=1}^{n-1} A_{\sigma_{2k+1}}^{\mu_{2k+1}} r_1^*\omega_n^{\mu_{2k+1}} \right) \wedge \ldots \wedge \left(\sum_{\mu_{n-1}=1}^{n-1} A_{\sigma_{n-1}}^{\mu_{n-1}} r_1^*\omega_n^{\mu_{n-1}} \right) \\
&= \sum_{\sigma\in\mathfrak{S}_{n-1}} \sum_{\mu_1,\ldots,\mu_{n-1}=1}^{n-1} \varepsilon_\sigma A_{\sigma_1}^{\mu_1} \cdots A_{\sigma_{n-1}}^{\mu_{n-1}} r_1^*\Omega_{\mu_2}^{\mu_1} \wedge \ldots \wedge r_1^*\Omega_{\mu_{2k}}^{\mu_{2k-1}} \wedge r_1^*\omega_n^{\mu_{2k+1}} \wedge \ldots \wedge r_1^*\omega_n^{\mu_{n-1}} \\
&= \sum_{\mu_1,\ldots,\mu_{n-1}=1}^{n-1} \left(\sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma A_{\sigma_1}^{\mu_1} \cdots A_{\sigma_{n-1}}^{\mu_{n-1}} \right) r_1^*\Omega_{\mu_2}^{\mu_1} \wedge \ldots \wedge r_1^*\Omega_{\mu_{2k}}^{\mu_{2k-1}} \wedge r_1^*\omega_n^{\mu_{2k+1}} \wedge \ldots \wedge r_1^*\omega_n^{\mu_{n-1}} \\
&= \sum_{\tau\in\mathfrak{S}_{n-1}} \underbrace{\left(\sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \varepsilon_\tau A_{\sigma_1}^{\tau_1} \cdots A_{\sigma_{n-1}}^{\tau_{n-1}} \right)}_{\varepsilon_\tau \det(A)} r_1^*\Omega_{\tau_2}^{\tau_1} \wedge \ldots \wedge r_1^*\Omega_{\tau_{2k}}^{\tau_{2k-1}} \wedge r_1^*\omega_n^{\tau_{2k+1}} \wedge \ldots \wedge r_1^*\omega_n^{\tau_{n-1}} \\
&= \sum_{\tau\in\mathfrak{S}_{n-1}} \varepsilon_\tau r_1^*\Omega_{\tau_2}^{\tau_1} \wedge \ldots \wedge r_1^*\Omega_{\tau_{2k}}^{\tau_{2k-1}} \wedge r_1^*\omega_n^{\tau_{2k+1}} \wedge \ldots \wedge r_1^*\omega_n^{\tau_{n-1}} \\
&= r_1^*\Phi_k.
\end{aligned}$$

The same proof can be carried out to show that $r_1^*\Psi_k = r_2^*\Psi_k$. □

It is important to note here that the sections r_i are not arbitrary sections of the bundle ξ. Indeed, they share the same last vector. Therefore, at each point $p \in U$ we only need the subgroup $\mathrm{SO}(n-1)$, and not the whole group $\mathrm{SO}(n)$, to act on the fibre to send $r_1(p)$ onto $r_2(p)$. This would not be true if r_1 and r_2 were *arbitrary* sections, since they would be related at each point by an element of $\mathrm{SO}(n)$ and not only $\mathrm{SO}(n-1)$. It is then natural to take a closer look at the situation pulled-back on the unit tangent bundle since we know that $SM \cong SO(M)/\mathrm{SO}(n-1)$. Therefore, if we consider the sections of ξ_1 defined by

$$\tilde{s}_i := r_i \circ \pi_2 : \pi_2^{-1}(U) \to SO(M),$$

we obviously have $\tilde{s}_1^*\Phi_k = \tilde{s}_2^*\Phi_k$ and $\tilde{s}_1^*\Psi_k = \tilde{s}_2^*\Psi_k$ on $\pi_2^{-1}(U)$, but actually these forms are *global* i.e. they are defined on the whole unit tangent bundle SM.

Proposition 3.12. Let $(p_0, v_0) \in SM$ and let $v : U \to SM$ be a vector field such that $v(p_0) = (p_0, v_0)$. Let also $s : U \to SO(M)$ be a section of ξ with v as last vector and set $\tilde{\Phi}_k = \pi_2^* s^*\Phi_k \in \Omega^{n-1}\left(\pi_2^{-1}(U)\right)$. Then $\tilde{\Phi}_k\Big|_{(p_0,v_0)}$ is independent of v.

Proof. The differential $d\pi_2 : TSM \to TM$ vanishes on vectors that are tangent to the fibre $\pi_2^{-1}(p)$. □

The consequence of this proposition is that at each point $(p_0, v_0) \in SM$ we can define the form $\tilde{\Phi}_k\Big|_{(p_0,v_0)}$ as above i.e. using a vector field $v : U \to SM$, where U is an open neighbourhood of p_0, and a section $s : U \to SO(M)$ having v as last vector. By Lemma 3.11 the form $\tilde{\Phi}_k$ does not depend on s and by Proposition 3.12 it does not depend on v either. Therefore we have

Corollary 3.13. The form $\tilde{\Phi}_k$ is defined on the whole manifold SM i.e. $\tilde{\Phi}_k \in \Omega^{n-1}(SM)$.

The last two results hold also for the form $\tilde{\Psi}_k = \pi_2^* s^* \Psi_k$. The next step in proving the transgression lemma is to establish a relation between the $\tilde{\Phi}_k$ and the $\tilde{\Psi}_k$. This is summed up in the following lemma:

Lemma 3.14. Setting $\tilde{\Psi}_{-1} = 0$ for convenience, we have for all $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$

$$d\tilde{\Phi}_k = \tilde{\Psi}_{k-1} + \frac{n-2k-1}{2(k+1)} \tilde{\Psi}_k$$

Proof. Let us compute the exterior derivative of $\tilde{\Phi}_k$, which is an n-form on SM.

$$\begin{aligned}
d\Phi_k = \sum_{\sigma \in \mathfrak{S}_{n-1}} & \varepsilon_\sigma \big(d\Omega^{\sigma_1}_{\sigma_2} \wedge \Omega^{\sigma_3}_{\sigma_4} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\
& + \Omega^{\sigma_1}_{\sigma_2} \wedge d\Omega^{\sigma_3}_{\sigma_4} \wedge \Omega^{\sigma_5}_{\sigma_6} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\
& + \ldots \\
& + \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-3}}_{\sigma_{2k-2}} \wedge d\Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\
& + \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge d\omega^{\sigma_{2k+1}}_n \wedge \omega^{\sigma_{2k+2}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\
& + \ldots \\
& + \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}} \wedge \ldots \wedge \omega^{\sigma_{n-2}}_n \wedge d\omega^{\sigma_{n-1}}_n \big) \\
= k \sum_{\sigma \in \mathfrak{S}_{n-1}} & \varepsilon_\sigma d\Omega^{\sigma_1}_{\sigma_2} \wedge \Omega^{\sigma_3}_{\sigma_4} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\
& + (n-1-2k) \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge d\omega^{\sigma_{2k+1}}_n \wedge \omega^{\sigma_{2k+2}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n
\end{aligned}$$

since the Ω^i_j are two forms and therefore commute without change of the sign with any other k-forms. The second structure equation (1.8) and the second Bianchi identity (1.10) are given in matrix notation by

$$\begin{aligned}
d\omega &= \Omega - \omega \wedge \omega, \\
d\Omega &= \Omega \wedge \omega - \omega \wedge \Omega,
\end{aligned}$$

or with the indices

$$\begin{aligned} d\omega^A_B &= \Omega^A_B - \omega^A_C \wedge \omega^C_B, \\ d\Omega^A_B &= \Omega^A_C \wedge \omega^C_B - \omega^A_C \wedge \Omega^C_B. \end{aligned}$$

Replacing $d\Omega^{\sigma_1}_{\sigma_2}$ and $d\omega^{\sigma_{2k+1}}_n$ using the last two equations and separating the terms who do contain some ω^i_j (for $1 \leq i, j \leq n-1$) and those who don't, we get

$$\begin{aligned} d\Phi_k = k &\sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_3}_{\sigma_4} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \Omega^{\sigma_1}_n \wedge \omega^n_{\sigma_2} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\ &- k \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_3}_{\sigma_4} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \Omega^n_{\sigma_2} \wedge \omega^{\sigma_1}_n \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\ &+ (n-1-2k) \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \Omega^{\sigma_{2k+1}}_n \wedge \omega^{\sigma_{2k+2}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\ &\underbrace{+k \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_3}_{\sigma_4} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \Omega^{\sigma_1}_i \wedge \omega^i_{\sigma_2} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n}_{A_k} \\ &\underbrace{-k \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_3}_{\sigma_4} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \Omega^i_{\sigma_2} \wedge \omega^{\sigma_1}_i \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n}_{B_k} \\ &\underbrace{+(n-1-2k) \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_i \wedge \omega^i_n \wedge \omega^{\sigma_{2k+2}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n}_{C_k}. \end{aligned}$$

In this huge expression for $d\Phi_k$ there are actually several known and simple terms. Indeed, the third line is exactly $\frac{n-1-2k}{2(k+1)}\Psi_k$. Moreover the first two lines are the same up to the transposition that exchanges σ_1 and σ_2. Therefore we can write

$$\begin{aligned} d\Phi_k = \frac{n-1-2k}{2(k+1)}\Psi_k &+ 2k \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_3}_{\sigma_4} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \Omega^{\sigma_2}_n \wedge \omega^{\sigma_1}_n \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\ &+ A_k + B_k + C_k \end{aligned}$$

The second term of the right-hand side is (up to a relabeling of the permutations)

$$2k \sum_{\tau\in\mathfrak{S}_{n-1}} \varepsilon_\tau \Omega^{\tau_1}_{\tau_2} \wedge \ldots \wedge \Omega^{\tau_{2k-3}}_{\tau_{2k-2}} \wedge \Omega^{\tau_{2k-1}}_n \wedge \omega^{\tau_{2k}}_n \wedge \omega^{\tau_{2k+1}}_n \wedge \ldots \wedge \omega^{\tau_{n-1}}_n = \Psi_{k-1},$$

so that

$$d\Phi_k = \frac{n-1-2k}{2(k+1)}\Psi_k + \Psi_{k-1} + A_k + B_k + C_k. \tag{3.4}$$

Finally, all the terms containing some ω^i_j must simplify since the pullback of $d\Phi_k$ on SM is a global differential form and therefore cannot contain connection forms that are not of the type ω^i_n. This can be seen by taking a moving frame satisfying $\omega^i_j(p) = 0$ for some point $p \in U$ (this corresponds to choosing normal coordinates in the neighbourhood of the point p). Since $d\tilde{\Phi}_k$ is a global form on SM then for every choice of a moving frame in the neighbourhood of p the terms in $d\tilde{\Phi}_k$ containing connection forms of the type ω^i_j will simplify at the point p and since the point is arbitrary. It follows that $A_k + B_k + C_k = 0$ and pulling back Equation (3.4) on the unit tangent bundle SM we get

$$d\tilde{\Phi}_k = \frac{n-1-2k}{2(k+1)}\tilde{\Psi}_k + \tilde{\Psi}_{k-1}.$$

□

The last lemma yields the following formula for $\tilde{\Psi}_k$ in terms of $d\tilde{\Phi}_0, \ldots, d\tilde{\Phi}_k$:

$$\tilde{\Psi}_k = \sum_{i=0}^{k} (-1)^{k-i} \frac{(2k+2)\cdots(2i+2)}{(n-1-2i)\cdots(n-1-2k)} d\tilde{\Phi}_i, \tag{3.5}$$

for every $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. Therefore we have

(a) if $\dim(M) = n = 2m$:

$$\begin{aligned} \tilde{\Psi}_{m-1} &= 2m \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \tilde{\Omega}^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \tilde{\Omega}^{\sigma_{2m-3}}_{\sigma_{2m-2}} \wedge \tilde{\Omega}^{\sigma_{2m-1}}_{2m} \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \tilde{\Omega}^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \tilde{\Omega}^{\sigma_{2m-3}}_{\sigma_{2m-2}} \wedge \tilde{\Omega}^{\sigma_{2m-1}}_{\sigma_{2m}} \\ &= 2^m m! \pi_2^* \operatorname{Pf}(\Omega), \end{aligned}$$

(b) and if $\dim(M) = n = 2m+1$:

$$\tilde{\Psi}_{m-1} = 2m \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \tilde{\Omega}^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \tilde{\Omega}^{\sigma_{2m-3}}_{\sigma_{2m-2}} \wedge \tilde{\Omega}^{\sigma_{2m-1}}_{2m} \wedge \tilde{\omega}^{\sigma_{2m}}_{2m} = d\tilde{\Phi}_m,$$

since $\tilde{\Psi}_m = 0$.

So finally defining on SM the following *global* form $\tilde{\Pi}$ of degree $n-1$

$$\tilde{\Pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^{k+1}}{2^n \pi^{\frac{n-1}{2}} k! \Gamma\left(\frac{n-2k+1}{2}\right)} \tilde{\Phi}_k \in \Omega^{n-1}(SM), \tag{3.6}$$

we can show the transgression Lemma 3.6 using Lemma 3.14:

Proof. **(Lemma 3.6)**
Using the formula (3.5) one can easily show both for $n = 2m$ even and $n = 2m+1$ odd that the constants simplify so that

$$-d\tilde{\Pi} = \pi_2^* \Omega,$$

where Ω is the Gauss-Bonnet integrand (3.1) and is not to be confused with the curvature form of $SO(M)$, or equivalently

$$-d\tilde{\Pi} = \frac{1}{(2\pi)^{\frac{n}{2}}}\pi_2^* \operatorname{Pf}(\Omega)$$

□

3.3.2 Step Two: Application of The Hopf-Poincaré and The Stokes Theorems

Throughout this section, let (M, g) be a closed oriented Riemannian manifold of dimension n. The case where M has a boundary will be dealt with in a second time.

The next and last step in Chern's proof is to establish the link between curvature and topology. This is done by choosing a unit vector field ν with (possibly) isolated singularities. Denote by $I = \{x_1, \dots, x_r\} \subset M$ the set of singularities of ν, so that we can write $\nu \in \Gamma(M \setminus I, SM)$. Without any loss of generality we will assume that there is only one singularity x. Around this singularity, one can consider a small ball $B(x, \varepsilon)$ such that ν is well-defined on

$$M_\varepsilon = M \setminus B(x, \varepsilon).$$

Let N_ε denote the image by ν of M_ε i.e. $N_\varepsilon = \nu(M_\varepsilon)$. In order to be able to use the transgression lemma, we switch from M to SM via the pull-back of the projection $\pi_2|_{N_\varepsilon} : N_\varepsilon \to M_\varepsilon$. Restricted to M_ε and N_ε, ν and π_2 are inverse to each other. Therefore applying the transgression lemma 3.6 as well as Stoke's theorem we get

$$\begin{aligned}\int_M \Omega &= \int_{M_\varepsilon} \Omega + \int_{\overline{B}(x,\varepsilon)} \Omega \\ &= \int_{N_\varepsilon} \pi_2|_{N_\varepsilon}^* \Omega + \int_{\overline{B}(x_i,\varepsilon_i)} \Omega \\ &= \int_{N_\varepsilon} -d\tilde{\Pi} + \int_{\overline{B}(x,\varepsilon)} \Omega \\ &= -\int_{\partial N_\varepsilon} \tilde{\Pi} + \int_{\overline{B}(x,\varepsilon)} \Omega.\end{aligned}$$

Now, letting $\varepsilon \to 0$, we obtain on one hand

$$\lim_{\varepsilon\to 0} \int_{\overline{B}(x,\varepsilon)} \Omega = 0.$$

On the other hand, denoting by Σ_ε the boundary of $\overline{B}(x, \varepsilon)$ i.e. the sphere centered at x of radius ε, we have

$$\lim_{\varepsilon\to 0} \int_{\partial N_\varepsilon} \tilde{\Pi} = -\lim_{\varepsilon\to 0} \int_{\nu(\Sigma_\varepsilon)} \tilde{\Pi}$$

where the minus sign comes from the orientations of ∂N_ε as the boundary of a manifold and of $\nu(\Sigma_\varepsilon)$ as the image of the sphere.

3.3.3 Conclusion of the proof

We reproduce here the argument made by Chern to complete the proof. First, using normal coordinates he notices that the integral over Σ_ε of $\nu^*\Phi_k$ for $k \geq 1$ is bounded by a constant times ε i.e. there exists $C \in \mathbb{R}$ such that for all $k \geq 1$ we have

$$\left| \int_{\nu(\Sigma_\varepsilon)} \Phi_k \right| < C \cdot \varepsilon,$$

therefore

$$\lim_{\varepsilon \to 0} \int_{\nu(\Sigma_\varepsilon)} \tilde{\Pi} = -\frac{1}{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)} \lim_{\varepsilon \to 0} \int_{\nu(\Sigma_\varepsilon)} \tilde{\Phi}_0.$$

Since

$$\begin{aligned} \tilde{\Phi}_0 &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \omega_n^{\sigma_1} \wedge \ldots \wedge \omega_n^{\sigma_{n-1}} \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \omega_n^1 \wedge \ldots \wedge \omega_n^{n-1} \\ &= (n-1)! \omega_n^1 \wedge \ldots \wedge \omega_n^{n-1}, \end{aligned}$$

we get

$$\lim_{\varepsilon \to 0} \int_{\nu(\Sigma_\varepsilon)} \tilde{\Pi} = -\frac{(n-1)!}{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)} \lim_{\varepsilon \to 0} \int_{\nu(\Sigma_\varepsilon)} \omega_n^1 \wedge \ldots \wedge \omega_n^{n-1}.$$

Finally, by Lemma 3.15 we have

$$\lim_{\varepsilon \to 0} \frac{1}{\alpha_{n-1}} \int_{\nu(\Sigma_\varepsilon)} \omega_n^1 \wedge \ldots \wedge \omega_n^{n-1} = \frac{(-1)^{n-1} \deg(f)}{\alpha_{n-1}} \int_{\mathbb{S}^{n-1}} \mathrm{dvol}_{\mathbb{S}^{n-1}} = (-1)^{n-1} \mathrm{Ind}(\nu, x)$$

where $f : \Sigma_\varepsilon \longrightarrow S^{n-1}$ is the map defined by $f(p) = \nu_p$ for $p \in \Sigma_\varepsilon$ and $\alpha_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the volume of $\mathbb{S}^{n-1}$. Thus, since we assumed that x was the only singularity of ν on M, we also have by Poincaré-Hopf theorem that $\mathrm{Ind}(\nu, x) = \chi(M)$ and therefore

$$\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} \nu^* \tilde{\Pi} = \frac{(-1)^n (n-1)!}{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \chi(M).$$

Using the identity $\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ (often called the *duplication formula*) we obtain

$$\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} \nu^* \tilde{\Pi} = (-1)^n \chi(M).$$

Recalling that $\Omega = (2\pi)^{-\frac{n}{2}}\,\mathrm{Pf}(\Omega)$, this completes the proof of the Gauss-Bonnet-Chern Theorem in the case where the manifold has no boundary:

$$(-1)^n\chi(M) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_M \mathrm{Pf}(\Omega). \tag{3.7}$$

Observe that if n is odd the theorem reduces to $\chi(M) = 0$.
By introducing normal coordinates, one can show that the pullback of the volume form of $\mathbb{S}^{n-1}$ on Σ_ε by ν can be written by means of the second fundamental form of ∂M in M:

Lemma 3.15. Let $f : \Sigma_\varepsilon \longrightarrow \mathbb{S}^{n-1}$ be the map defined by $f(p) = \nu_p$ for $p \in \Sigma_\varepsilon$. Then

$$f^*\mathrm{dvol}_{\mathbb{S}^{n-1}} = (-1)^{n-1}\omega_n^1 \wedge \ldots \wedge \omega_n^{n-1} + o(\varepsilon).$$

Proof. Let us make general consideration about the pullback of volume forms. Let $F : M_1^n \to M_2^m$ be a differentiable map between two manifolds and let g be a Riemannian metric on M_2 as well as $p \in M_1$ and $q = F(p)$. Let $U \subset M_1$ be a an open neighbourhood of p and let $(e_1, \ldots, e_n)$ be a directly oriented orthonormal moving frame on $F(U) \subset M_2$. Then on U we have

$$dF = \sum_{i=1}^m \lambda^i e_i = \sum_{i=1}^m e_i \otimes \lambda^i \in TM_2 \otimes T^*M_1,$$

where $\lambda^i \in \Omega^1(U)$ is the 1-form defined on U by

$$\lambda^i = F^*\theta^i, \quad \text{with} \quad \theta^i(e_j) = \delta^i_j.$$

This implies that $F^*\mathrm{dvol}_{M_2} = \lambda^1 \wedge \ldots \wedge \lambda^m$.
Now, recall that on an open neighbourhood U of a Riemannian manifold (M, g) we have

$$\nabla e_i = \omega_i^j e_j = e_j \otimes \omega_i^j \in TM \otimes T^*M \tag{3.8}$$

Going back our particular situation of a map $f : \Sigma_\varepsilon \longrightarrow \mathbb{S}^{n-1}$, consider a directly oriented orthonormal moving frame $(e_1, \ldots, e_{n-1}, e_n = \nu)$, we have by Equation (3.8)

$$\nabla\nu = \sum_{j=1}^{n-1}\omega_n^j,$$

since $\omega_n^n = 0$. Now, by taking a system of normal coordinates at p we identify the vector field ν and the map f we get

$$df = \nabla\nu + o(\varepsilon) = \sum_{i=1}^{n-1}\omega_n^j e_j + o(\varepsilon).$$

Hence we get by the general consideration made above

$$f^*\mathrm{dvol}_{\mathbb{S}^{n-1}} = \omega_n^1 \wedge \ldots \wedge \omega_n^{n-1}.$$

□

3.3.4 The Case of Manifolds with Boundary

Suppose now that (M, g) is an n-dimensional compact oriented Riemannian manifold *with boundary.* In this case, the behaviour (i.e. inward/outward pointing) of ν at boundary points is important in view of the Poicaré-Hopf theorem. Following Chern, let ν be inward pointing at each boundary point. The main difference is that now the boundary of N_ε consists of two distinct components, the image of the boundary of M under ν i.e. $\nu(\partial M)$ and the image of Σ_ε under ν. Therefore, the same argument as before gives

$$\begin{aligned}\int_M \Omega &= \int_{M_\varepsilon} \Omega + \int_{\overline{B}(x,\varepsilon)} \Omega \\ &= \int_{N_\varepsilon} \pi_2|_{N_\varepsilon}^* \Omega + \int_{\overline{B}(x,\varepsilon)} \Omega \\ &= \int_{N_\varepsilon} -d\tilde{\Pi} + \int_{\overline{B}(x,\varepsilon)} \Omega \\ &= -\int_{\nu(\partial M)} \tilde{\Pi} + \int_{\nu(\Sigma_\varepsilon)} \tilde{\Pi} + + \int_{\overline{B}(x,\varepsilon)} \Omega.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain by Theorem 3.4 the Gauss-Bonnet-Chern theorem for manifold with boundary:

$$(-1)^n \left(\chi(M) - \chi(\partial M)\right) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = \int_{\nu(\partial M)} \tilde{\Pi} \tag{3.9}$$

Equations (3.7) and (3.9) together prove the Gauss-Bonnet-Chern Theorem 3.7.

Remark 3.16. Observe that when n is even, then ∂M is a closed manifold of odd dimension, therefore

$$(-1)^n \left(\chi(M) - \chi(\partial M)\right) = \begin{cases} \chi(M) & \text{if } n \text{ is even,} \\ \chi(\partial M) - \chi(M) & \text{if } n \text{ is odd.} \end{cases}$$

This quantity is what Chern, Allendoerfer, Weil and other people call the *inner Euler-Poincaré characteristic of* M and that they usually denote by $\chi'(M)$.

Example 3.17. If $M = S$ is a compact orientable surface with boundary, then one recover the classical Gauss-Bonnet theorem. Let $e_2 := \nu$ be an inward-pointing vector field defined on a neighbourhood U of the boundary. In dimension 2 there is a unique vector field e_1 such that (e_1, e_2) is an orthonormal directly oriented moving frame on U. Then by Example 1.28 we already know that

$$\mathrm{Pf}(\Omega) = K \mathrm{dvol}_S,$$

where K is the Gaussian curvature of S. Moreover

$$\nu^* \tilde{\Pi} = \frac{1}{4\sqrt{\pi}} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \nu^* \tilde{\Phi}_0 = \frac{1}{2\pi} \nu^* \tilde{\omega}_2^1 = \frac{1}{2\pi} \omega_2^1 = \frac{1}{2\pi} k_g \mathrm{dvol}_{\partial S},$$

where k_g is the geodesic curvature of the curve ∂S. Therefore the Gauss-Bonnet-Chern Theorem in dimension 2 reduces to

$$\chi(S) - \frac{1}{2\pi}\int_S K\text{dvol}_S = \frac{1}{2\pi}\int_{\partial S} k_g \text{dvol}_{\partial S}.$$

3.3.5 A Remark About the Orientation in the Case of Manifolds with Boundary

Recall that the changing the orientation at the boundary (inward/outward pointing) of the normal unit vector ν in Theorem 3.4 changes the statement. This remark is obviously also true concerning the Gauss-Bonnet-Chern Theorem since Poincaré-Hopf's Theorem is used. However, the change in the statement of the Theorem is this time quite subtle. Let (M, g) be a compact oriented n-dimensional Riemannian manifold with boundary. Denoting as before the inward-pointing unit-normal vector by ν consider the *outward-pointing* unit normal vector field $\overline{\nu} = -\nu$. Let $(e_1, \ldots, e_n)$ be an oriented orthonormal moving frame such that $e_n = \nu$ and let $(\overline{e}_1, \ldots, \overline{e}_n)$ be the oriented moving frame defined by

$$\overline{e}_i = e_i \quad \text{and} \quad \overline{e}_n = -e_n = \overline{\nu},$$

for all $1 \leq i \leq n-1$. These two frames define an opposite orientation on M (and therefore on ∂M) and the connection and curvature forms are modified as follows:

Lemma 3.18. Let $\omega^A_B, \Omega^A_B, \overline{\omega}^A_B$ and $\overline{\Omega}^A_B$ be the connection and curvature forms associated to the two above moving frames. Then

$$\begin{cases} \overline{\omega}^i_j = \omega^i_j, \\ \overline{\omega}^i_n = -\omega^i_n, \\ \overline{\Omega}^i_j = \Omega^i_j, \\ \overline{\Omega}^i_n = \Omega^i_n. \end{cases}$$

Proof. Using the first structure equation (Lemma 1.10) we have on one hand

$$\begin{aligned} d\overline{\theta}^i &= \overline{\theta}^A \wedge \overline{\omega}^i_A \\ &= \overline{\theta}^n \wedge \overline{\omega}^i_n + \overline{\theta}^j \wedge \overline{\omega}^i_j \\ &= -\theta^n \wedge \overline{\omega}^i_n + \theta^j \wedge \overline{\omega}^i_j, \end{aligned}$$

and on the other hand

$$d\overline{\theta}^i = d\theta^i = \theta^n \wedge \omega^i_n + \theta^j \wedge \omega^i_j.$$

Hence by comparing the terms we get $\omega^i_j = \overline{\omega}^i_j$ and $\omega^i_n = -\overline{\omega}^i_n$. Now the second structure equation (1.8) gives

$$\begin{aligned}\overline{\Omega}^i_j &= d\overline{\omega}^i_j + \overline{\omega}^i_A \wedge \overline{\omega}^A_j \\ &= d\omega^i_j + \omega^i_k \wedge \omega^k_j + (-\omega^i_n) \wedge (-\omega^n_j) \\ &= d\omega^i_j + \omega^i_A \wedge \omega^A_j \\ &= \Omega^i_j,\end{aligned}$$

and since $\overline{\omega}^n_n = \omega^n_n = 0$:

$$\begin{aligned}\overline{\Omega}^i_n &= d\overline{\omega}^i_n + \overline{\omega}^i_A \wedge \overline{\omega}^A_n \\ &= -d\omega^i_n - \omega^i_k \wedge \omega^k_n \\ &= -\Omega^i_n\end{aligned}$$

□

This lemma can be interpreted by recalling that the second fundamental forms b^ν and $b^{\overline{\nu}}$ of ∂M in M with respect to the two unit normals ν and $\overline{\nu}$ can be written as

$$\begin{aligned}b^\nu &= \Gamma^n_{ij}\theta^i \otimes \theta^j, \\ b^{\overline{\nu}} &= \overline{\Gamma}^n_{ij}\overline{\theta}^i \otimes \overline{\theta}^j.\end{aligned}$$

Hence the change of sign $\overline{\omega}^i_n = -\omega^i_n$ simply comes from the fact that the sign of the second fundamental form depends on the orientation of the chosen unit normal. By Proposition 1.26 we know that the Pfaffian changes as

$$\operatorname{Pf}(\overline{\Omega}) = -\operatorname{Pf}(\Omega),$$

but since the orientation of M is also reversed the integral of the Pfaffian remains unchanged:

$$\int_{(M,\overline{\nu})} \operatorname{Pf}(\overline{\Omega}) = -\int_{(M,\nu)} -\operatorname{Pf}(\Omega) = \int_{(M,\nu)} \operatorname{Pf}(\Omega),$$

where we have denoted by $(M,\overline{\nu})$ and (M,ν) the manifolds with the orientations corresponding to $\overline{\nu}$ and ν. The form $\tilde{\Pi}$ is modified as follows:

$$\overline{\Pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^{k+1}}{2^n \pi^{\frac{n-1}{2}} k! \Gamma\left(\frac{n-2k+1}{2}\right)} \overline{\Phi}_k,$$

where

$$\begin{aligned}\overline{\Phi}_k &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \overline{\Omega}^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \overline{\omega}^{\sigma_{2k+1}}_n \wedge \ldots \wedge \overline{\omega}^{\sigma_{n-1}}_n \\ &= (-1)^{n-1-k} \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \\ &= (-1)^{n-1-k}\tilde{\Phi}_k\end{aligned}$$

whence

$$\overline{\Pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^n}{2^n \pi^{\frac{n-1}{2}} k! \Gamma\left(\frac{n-2k+1}{2}\right)} \tilde{\Phi}_k. \tag{3.10}$$

Therefore going through the same proof as the one for manifolds with boundary and taking into account the Poincaré-Hopf Theorem, the Gauss-Bonnet-Chern Theorem 3.9 can be rewritten in terms of an outward-pointing unit normal vector field ν as

$$(-1)^n \chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = \int_{\nu(\partial M)} \overline{\Pi}. \tag{3.11}$$

It will actually be important thereafter to know both the statements for an inward and an outward point normal unit vector.

Chapter 4

Asymptotically Conical Ends and Conical Singularities

The Main Theorem 7 stated in the introduction shall now be demonstrated. The choice of separating the proof in three different step has been made in order to emphasize the particularity of each case. First we assume that the manifold admits *conical ends* in the sense that each end is isometric to a Riemannian cone without perturbation of the metric. This assumption implies in particular that the Pfaffian of the curvature form Ω vanishes on every end and therefore the total curvature is well-defined. In this case, the idea of the proof is to consider an exhaustion of M by compact manifolds with boundary and apply the Gauss-Bonnet-Chern theorem to each element of the exhaustion. This provides a quantification of the Gauss-Bonnet defect in terms of an integral on the boundary, and a careful analysis of the asymptotic behaviour of this boundary term yield the formula.
In a second time, the metric on each end is assumed to be *asymptotically conical*, meaning that the metric can be written as the standard conical metric plus a perturbation term which vanishes asymptotically (as well as his first two covariant derivatives). The technical details about the convergence of the metric are tackled in Appendix A. In this case, the integral of the Pfaffian of Ω (i.e. the total curvature) is not necessary equal to zero but it does vanish asymptotically so that the total curvature is well-defined. We then have to ensure that the boundary term in the Gauss-Bonnet-Chern theorem converge to the one obtained in the previous (strictly conical) case.
Finally we deal with the case of conical singularities.

4.1 Manifolds with Conical Ends

All the preparation that has been done so far will now serve to show a version of the Gauss-Bonnet-Chern theorem for complete manifolds that are not necessarily compact. Assumptions have to be made both on the geometry and the topology for the integral of the Pfaffian of the curvature forms to converge and for the Euler

characteristic to be an integer.
On the topological side, the natural hypothesis is to assume that the manifold M is of finite topological type (see Definition 3.1), i.e. that there exists a compact submanifold of M in which all the topology of M is contained.
The geometric assumptions have to ensure the convergence of the integral

$$\int_M |\mathrm{Pf}(\Omega)|\,,$$

therefore we will first consider the case where all the ends of M are *conical*.

Remark 4.1. The expression

$$\int |\mathrm{Pf}(\Omega)|$$

is to be understood in the sense of densities. More precisely, we say that an n-form $\beta \in \Omega^n(M)$ is *integrable* if

$$\int_M |\beta| < +\infty,$$

where β is the natural density associated to β (See [Lee13], pp. 427-434).

This means that the restriction of the ambient metric g to each end E_i takes the form

$$g = t^2 g_{N_i} + dt^2,$$

with g_{N_i} a metric on N_i. Recall that by Corollary 1.30 the Pfaffian vanishes identically on a conical warped-product.
This geometrical hypothesis will be relaxed hereafter as the Pfaffian does not need to vanish for its integral over M to converge, but only to decrease sufficiently fast.
Let us first suppose that M has no boundary. We will work out an expression for the Gauss-Bonnet defect of M in terms of the Lipschitz-Killing curvatures of the N_i's. More precisely, we have the following theorem:

Theorem 4.2. Let (M, g) be an even n-dimensional complete oriented Riemannian of finite topological type. Assume that M has one end $E = N \times (1, \infty)$ with $N \subset M$ compact and moreover that E is conical. Then the Gauss-Bonnet defect of M is given by

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = \tau(N), \tag{4.1}$$

where

$$\tau(N) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} A(n,k) \int_N C^{2k}((R - D)^k)\mathrm{dvol}_N \tag{4.2}$$

with R the curvature tensor of N and $D = \frac{1}{2} g_N \owedge g_N$, and the constants $A(n,k)$ are given by

$$A(n,k) = \frac{(n-1-2k)!}{2^{n-k}\pi^{\frac{n-1}{2}} k!(2k)!\Gamma\left(\frac{n-2k+1}{2}\right)} > 0. \tag{4.3}$$

Remark 4.3. The requirement for the manifold to have only one end is not a restriction. The choice of this assumption is purely motivated by the readability of the formula. If M has r ends $N_1, \ldots, N_r$, then the statement is the same except that one has to sum over each end separately.

4.2 On the invariant $\tau(N)$

The geometric invariant $\tau(N)$ can be easily computed in low dimensions. Table 4.1 gives some values of the constant $A(n,k)$ for some small n and $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$:

k\n	2	3	4	5	6
0	$\frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{2\pi^2}$	$\frac{3}{8\pi^2}$	$\frac{1}{\pi^3}$
1	-	$\frac{1}{8\pi}$	$\frac{1}{8\pi^2}$	$\frac{1}{16\pi^2}$	$\frac{1}{8\pi^3}$
2	-	-	-	$\frac{1}{384\pi^2}$	$\frac{1}{384\pi^2}$

Figure 4.1: Some values of $A(n,k)$.

Example 4.4. (Dimension 1**)** Let γ be a closed curve. Denoting by ds the length element of γ, the invariant $\tau(\gamma)$ is given by

$$\tau(\gamma) = A(2,0)\int_\gamma ds = \frac{1}{2\pi}\text{length}(\gamma).$$

Example 4.5. (Dimension 3**)** Let (N, g_N) be a compact 3-dimensional Riemannian manifold. The invariant $\tau(N)$ is given by

$$\tau(N) = A(4,0)\int_N \text{dvol}_N + A(4,1)\int_N C^2(R-D)\text{dvol}_N.$$

But given an orthonormal moving frame (e_1, e_2, e_3) on a open subset $U \subset N$ we have by definition of the contraction:

$$\begin{aligned} C^2(R-D) &= \sum_{i,j=1}^{3} (R-D)(e_i,e_j)(e_i,e_j) \\ &= \sum_{i,j=1}^{3} R(e_i,e_j)(e_i,e_j) - \sum_{i,j=1}^{3} D(e_i,e_j)(e_i,e_j) \\ &= \text{Scal}_{g_N} - 6 \end{aligned}$$

since $D(e_i,e_j)(e_k,e_l) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. Therefore

$$\begin{aligned} \tau(N) &= \frac{1}{2\pi^2}\text{Vol}(N) + \frac{1}{8\pi^2}\int_N (\text{Scal}_{g_N} - 6)\text{dvol}_N \\ &= \frac{1}{8\pi^2}\int_N (\text{Scal}_{g_N} - 2)\,\text{dvol}_N. \end{aligned}$$

This result was obtained by Dillen and Kühnel in [DK05, p. 191] in the case of cones in $\mathbb{R}^N$. Observe that the value of $\tau(N)$ depends on the mean scalar curvature of N.

Example 4.6. (Flat Manifolds) Let us suppose that (N, g_N) is a compact odd dimensional flat Riemannian manifold i.e. that $R \equiv 0$. It follows that

$$C^{2k}((R-D)^k) = (-1)^k C^{2k}(D^k).$$

But we know the contractions of the curvature tensor of the sphere by Equation (2.9):

$$C^{2k}(D^k) = \frac{((2k)!)^2}{2^k}\binom{n-1}{2k},$$

therefore

$$\begin{aligned}\tau(N) &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k A(n,k)\int_N C^{2k}(D^k)\mathrm{dvol}_N \\ &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k A(n,k)\frac{((2k)!)^2}{2^k}\binom{n-1}{2k}\mathrm{Vol}(N) \\ &= C\cdot\mathrm{Vol}(N).\end{aligned}$$

Example 4.7. (Space forms) As in the preceding example, if (N, g_N) is an odd dimensional space of constant sectional curvature $\lambda \in \mathbb{R}$, the invariant $\tau(N)$ is given by a function of λ times the volume of N. Recall that the curvature tensor of such a manifold is given by

$$R = \lambda D.$$

Hence we have

$$\begin{aligned}C^{2k}((R-D)^k) &= C^{2k}((\lambda D - D)^k) \\ &= (\lambda-1)^k C^{2k}(D^k) \\ &= \frac{(\lambda-1)^k((2k)!)^2}{2^k}\binom{n-1}{2k}.\end{aligned}$$

Thus $\tau(N)$ is given by

$$\begin{aligned}\tau(N) &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} A(n,k)\int_N \frac{(\lambda-1)^k((2k)!)^2}{2^k}\binom{n-1}{2k}\mathrm{dvol}_N \\ &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(\lambda-1)^k((2k)!)^2}{2^k}\binom{n-1}{2k}A(n,k)\mathrm{Vol}(N) \\ &= \phi(\lambda)\cdot\mathrm{Vol}(N),\end{aligned}$$

the function $\phi(\lambda)$ being a polynomial in the variable λ.
Then if $(N, g_N) = (\mathbb{S}^{n-1}, g_1)$, with g_1 the metric on the unit sphere with constant sectional curvature $\lambda = 1$, then the invariant $\tau(\mathbb{S}^{n-1})$ is simply given by

$$\tau(\mathbb{S}^{n-1}) = A(n,0)\mathrm{Vol}(\mathbb{S}^{n-1}) = \frac{(n-1)!}{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)} = 1.$$

This value for $\tau(\mathbb{S}^{n-1})$ is confirmed by Theorem 4.2 applied to $\mathbb{R}^n$ seen as the standard cone over $\mathbb{S}^{n-1}$

$$\tau(\mathbb{S}^{n-1}) = \chi(\mathbb{R}^n) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{Pf}(\Omega) = \chi(\mathbb{R}^n) = 1.$$

Remark 4.8. (Extension to odd dimensions) The invariant $\tau(N)$ is actually defined also when N is even dimensional. For an arbitrary $(n-1)$-dimensional compact Riemannian manifold (N, g_N) we set

$$\tau(N) = (-1)^n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} A(n,k) \int_N C^{2k}((R-D)^k) \mathrm{dvol}_N. \tag{4.4}$$

We believe that for even dimensional N, this invariant simplifies to give only the Euler-characteristic of N via the Gauss-Bonnet-Chern theorem. For a 2-dimensional N we get

$$\tau(N) = -A(3,0) \int_N \mathrm{dvol}_N - A(3,1) \int_N C^2(R-D) \mathrm{dvol}_N.$$

As before we have

$$\begin{aligned} C^2(R-D) &= \sum_{i,j=1}^{2} (R-D)(e_i, e_j)(e_i, e_j) \\ &= \sum_{i,j=1}^{2} R(e_i, e_j)(e_i, e_j) - \sum_{i,j=1}^{2} D(e_i, e_j)(e_i, e_j) \\ &= \mathrm{Scal}_{g_N} - 2 \\ &= 2K_{\mathrm{Gauss}} - 2. \end{aligned}$$

Therefore

$$\begin{aligned} \tau(N) &= -\frac{1}{4\pi}\mathrm{Vol}(N) - \frac{1}{8\pi} \int_N (2K_{\mathrm{Gauss}} - 2)\, \mathrm{dvol}_N \\ &= -\frac{1}{4\pi} \int_N K_{\mathrm{Gauss}} \mathrm{dvol}_N \\ &= -\frac{1}{2}\chi(N), \end{aligned}$$

where we have used the Gauss-Bonnet Theorem for N in the last equality. Observe that this result is consistent with Theorem 4.2 since if (M, g) is a complete non compact 3-dimensional manifold with one conical end of link (N, g_N) then the left-hand side of Equation is simply $-\chi(M)$ since the Pfaffian of a 3-dimensional vanishes. But it is known that the Euler-characteristic of an odd dimensional manifold with boundary is half the Euler-characteristic of its boundary. In our case we thus have

$$\tau(N) = -\frac{1}{2}\chi(N) = -\chi(M).$$

Relation between $\tau(N)$ and the Lipschitz-Killing curvatures of N

Knowing the close relation between the contractions of powers of the curvature tensor of a manifold and its Lipschitz-Killing curvatures, it not surprising that the invariant $\tau(N)$ can actually be written as a sum of Lipschitz-Killing curvatures of N:

Theorem 4.9. Let (N, g_N) be a compact n-dimensional Riemannian manifold. Then

$$\tau(N) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \lambda_{n,k}\mathcal{K}_{2k}(N), \tag{4.5}$$

where the $\lambda_{n,k}$ are given by

$$\lambda_{n,k} = \sum_{j=k}^{\lfloor\frac{n-1}{2}\rfloor} (-1)^{n+j-k} \frac{(n-1-2k)!j!(2j)!}{(n-1-2j)!(j-k)!2^{j-k}} A(n,j), \tag{4.6}$$

with $A(n, j)$ defined in Equation 4.3.

Proof. Since the double-forms R and D are of type $(2, 2)$ their exterior product commutes without changing i.e. $R \wedge D = D \wedge R$ (c.f. Equation (2.1)). Thus the k-th exterior power of $(R - D)$ can be expanded using Newton's binomial:

$$(R - D)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} R^j \wedge D^{k-j}.$$

It follows that the $2k$-th contraction of $(R - D)^k$ is given by

$$\begin{aligned} C^{2k}((R-D)^k) &= \sum_{i_1,\ldots,i_{2k}=1}^{n-1} (R-D)^k(e_{i_1},\ldots,e_{i_{2k}})(e_{i_1},\ldots,e_{i_{2k}}) \\ &= \sum_{i_1,\ldots,i_{2k}=1}^{n-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left(R^j \wedge D^{k-j}\right)(e_{i_1},\ldots,e_{i_{2k}})(e_{i_1},\ldots,e_{i_{2k}}) \end{aligned}$$

Using the definition of the wedge of double-forms and the fact that the $C^{2k-2j}(D^{k-j})$ are known from Proposition 2.18 we can rewrite this last expression as

$$C^{2k}((R-D)^k) = \sum_{j=0}^{k} \mu_{n,k,j} C^{2j}(R^j).$$

Hence, since we have extended the definition of $\tau(N)$ to compact manifolds of arbitrary dimension in Remark 4.8, we have

$$\begin{aligned}
\tau(N) &= (-1)^n \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} A(n,k) \int_N C^{2k}((R-D)^k)\mathrm{dvol}_N \\
&= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} (-1)^n A(n,k) \sum_{j=0}^{k} \mu_{n,k,j} j!(2j)! \int_N \kappa_{2j}(N) \\
&= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \lambda_{n,k} \mathcal{K}_{2k}(N),
\end{aligned}$$

where $\lambda_{n,k}$ is a complicated expression obtained from the constants $A(n,k)$, $\mu_{n,k,j}$ and by changing the double sum as a simple sum.
The constants $\lambda_{n,k}$ being universal, a strategy to calculate them explicitly is to compute the invariant $\tau(N)$ on a particular example whose Lipschitz-Killing curvatures are known. Consider the family of manifolds given by $(N_a, g_a) = (\mathbb{S}_a^{n-1}, a^2 g_{\mathbb{S}^{n-1}})$ consisting of spheres of constant sectional curvature $\frac{1}{a^2}$. By Proposition 2.18 we know that:

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \lambda_{n,k} \mathcal{K}_{2k}(\mathbb{S}_a^{n-1}) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \lambda_{n,k} \frac{(n-1)!\alpha_{n-1}}{2^k k!(n-1-2k)!} a^{n-1-2k}.$$

On the other hand we have by Example 4.7 (which be easily adapted to odd dimensions):

$$\begin{aligned}
\tau(\mathbb{S}_a^{n-1}) &= (-1)^n \phi\left(\frac{1}{a^2}\right) \mathrm{Vol}(\mathbb{S}_a^{n-1}) \\
&= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{(-1)^n (a^{-2}-1)^k ((2k)!)^2}{2^k} \binom{n-1}{2k} A(n,k) \alpha_{n-1} a^{n-1} \\
&= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \sum_{j=0}^{k} \frac{((2k)!)^2 \alpha_{n-1}}{2^k} \binom{n-1}{2k} A(n,k) \binom{k}{j} a^{n-1-2j} (-1)^{n+k-j}
\end{aligned}$$

then by comparing the coefficients of the powers of a we finally get after "simplifications":

$$\lambda_{n,k} = \sum_{j=k}^{\lfloor\frac{n-1}{2}\rfloor} (-1)^{n+j-k} \frac{(n-1-2k)!j!(2j)!}{(n-1-2j)!(j-k)!2^{j-k}} A(n,j)$$

$$= \sum_{j=k}^{\lfloor\frac{n-1}{2}\rfloor} \frac{(-1)^{n+j-k}(n-1-2k)!}{(j-k!)2^{j-k}\pi^{\frac{n-1}{2}}\Gamma\left(\frac{n-2j+1}{2}\right)}$$

□

Although those constants are rather complicated, they are calculable at least for small dimensions. Some values are given in Table 4.2. Observe that for odd dimension only the top Lipschitz-Killing curvature is involved in the expression for τ, which confirms the result obtained in Remark 4.8.

k\n	2	3	4	5	6	7	8
0	$\frac{1}{2\pi}$	0	$-\frac{1}{4\pi^2}$	0	$\frac{3}{8\pi^3}$	0	$-\frac{15}{16\pi^4}$
1	-	$-\frac{1}{4\pi}$	$\frac{1}{4\pi^2}$	0	$-\frac{1}{8\pi^3}$	0	$\frac{3}{16\pi^4}$
2	-	-	-	$-\frac{1}{8\pi^2}$	$\frac{1}{8\pi^3}$	0	$-\frac{1}{16\pi^4}$
3	-	-	-	-	-	$-\frac{1}{16\pi^3}$	$\frac{1}{16\pi^4}$

Figure 4.2: Some values of $\lambda_{n,k}$.

In particular, recalling that for an even dimensional compact Riemannian manifold M^{2p} the last total Lipschitz-Killing curvature is given by

$$\mathcal{K}_{2p}(M) = (2\pi)^p \chi(M)$$

we find that

(a) if $\dim(N) = 1$, then

$$\tau(N) = \frac{1}{2\pi}\mathcal{K}_0(N) = \frac{\mathrm{Vol}(N)}{2\pi},$$

(b) if $\dim(N) = 2$, then we recover the result of Remark 4.8

$$\tau(N) = -\frac{1}{4\pi}\mathcal{K}_2(N) = -\frac{1}{4\pi}(2\pi\chi(N)) = -\frac{1}{2}\chi(N),$$

(c) if $\dim(N) = 3$, then we recover the result of Example 4.5:

$$\tau(N) = -\frac{1}{4\pi^2}\mathcal{K}_0(N) + \frac{1}{4\pi^2}\mathcal{K}_2(N) = \frac{1}{8\pi^2}\int_N (\mathrm{Scal}_{g_N} - 2)\mathrm{dvol}_N$$

(d) if $\dim(N) = 4$, then

$$\tau(N) = -\frac{1}{8\pi^2}\mathcal{K}_4(N) = -\frac{1}{8\pi^2}\left((2\pi)^2\chi(N)\right) = -\frac{1}{2}\chi(N);$$

(e) if $\dim(N) = 5$, then

$$\begin{aligned}\tau(N) &= \frac{3}{8\pi^3}\mathcal{K}_0(N) - \frac{1}{8\pi^3}\mathcal{K}_2(N) + \frac{1}{8\pi^3}\mathcal{K}_4(N)\\ &= \frac{3\mathrm{Vol}(N)}{8\pi^3} - \frac{1}{16\pi^3}\int_N \mathrm{Scal}_{g_N}\mathrm{dvol}_N + \frac{1}{8\pi^3}\mathcal{K}_4(N);\end{aligned}$$

(f) if $\dim(N) = 6$, then

$$\tau(N) = -\frac{1}{16\pi^3}\mathcal{K}_6(N) = -\frac{1}{16\pi^3}((2\pi)^3\chi(N)) = -\frac{1}{2}\chi(N);$$

As it is suggested in Table 4.2 and in the last examples if the dimension n is odd, or equivalently if $\dim(N)$ is even, then $\tau(N)$ seems to be equal to $-\frac{1}{2}\chi(N)$. The next proposition shows that it is actually a general property:

Proposition 4.10. Let (N, g_N) be a compact even dimensional Riemannian manifold. Then

$$\tau(N) = -\frac{1}{2}\chi(N). \tag{4.7}$$

Proof. Denote by $n = 2p$ the dimension of N. Then

$$\begin{aligned}\lambda_{2p+1,k} &= \sum_{j=k}^{p}(-1)^{j-k+1}\frac{(2p-2k)!}{(j-k)!2^{2p+1-k}\pi^p\Gamma\left(p-j+1\right)}\\ &= \frac{(2p-2k)!}{2^{2p+1-k}\pi^p}\sum_{j=k}^{p}(-1)^{j-k+1}\frac{1}{(j-k)!(p-j)!}\\ &= -\frac{(2p-2k)!}{2^{2p+1-k}\pi^p(p-k)!}\sum_{i=0}^{p-k}(-1)^i\binom{p-k}{i},\end{aligned}$$

but the alternating sum of the binomial coefficient vanishes, therefore if $k \neq p$ we have $\lambda_{2p+1,k} = 0$ and if $k = p$ then

$$\lambda_{2p+1,k} = -\frac{1}{2^{p+1}\pi^p} = -\frac{1}{2(2\pi)^p}.$$

Hence using the fact that $\mathcal{K}_{2p}(N) = (2\pi)^p\chi(N)$ by Equation 2.12, we finally get

$$\tau(N) = \sum_{k=0}^{p}\lambda_{n,k}\mathcal{K}_{2k}(N) = -\frac{1}{2(2\pi)^p}(2\pi)^p\chi(N) = -\frac{1}{2}\chi(N).$$

□

4.3 Proof of Theorem

Let us now prove Theorem 4.2. The manifold M being of finite topological type it can be written as

$$M = K \cup (N \times (1, \infty)) = K \cup E,$$

where $K \subset M$ is compact with $N := \partial K$ connected and $E := N \times (1, \infty)$. Let us denote by g_N the restriction of the ambient metric g to N and by g_E the restriction of g to E which can actually be written explicitly as

$$g_E = t^2 g_N + dt^2,$$

since E is supposed to be conical. An exhaustion of M by compact manifolds with boundary is given by

$$M_t := K \cup (N \times (1, t]) = K \cup E_t,$$

with $E_t = N \times (1, t]$ and the boundary of M_t is given by $N_t := \partial M_t = N \times \{t\}$. Endowed with the Riemannian metric $g_t := g|_{N_t} = t^2 g_N$, the manifold (N_t, g_t) is an $(n-1)$-dimensional compact isometrically embedded submanifold of M_t.
Let now $(e_1, \ldots, e_{n-1})$ be an orthonormal oriented moving frame on an open subset $U \subset N$ and let $e_n = \frac{\partial}{\partial t}$ be the arclength vector field on $(1, \infty)$. This gives rise to an orthonormal oriented moving frame on $U \times (1, \infty)$ by setting

$$\overline{e}_i = \frac{1}{t} e_i \quad \text{and} \quad \overline{e}_n = e_n.$$

Denote by θ^i and $\overline{\theta}^A$ the dual forms to e_i and $\overline{e}_A$ and as usual the associated connection and curvature forms of N and M are denoted by ω^i_j, $\overline{\omega}^A_B$, Ω^i_j and $\overline{\Omega}^A_B$. Since $(M_t, g|_{M_t})$ is isometrically embedded in (M, g), the connection and curvature forms of M_t obviously coincide with $\overline{\omega}^A_B$ and $\overline{\Omega}^A_B$. Observe that at each point of the boundary N_t, the vector field $\overline{e}_n$ is unitary and outward-pointing. Therefore by the Gauss-Bonnet-Chern Theorem with outward-pointing unit normal vector field (Equation (3.11)) we have

$$\chi(M_t) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{M_t} \mathrm{Pf}(\Omega) = \int_{\overline{e}_n(N_t)} \overline{\Pi}. \tag{4.8}$$

By Corollary 1.30 we know that the Pfaffian $\mathrm{Pf}(\Omega)$ vanishes on E, therefore the total curvature is well-defined:

$$\int_M |\mathrm{Pf}(\Omega)| = \int_K |\mathrm{Pf}(\Omega)| < +\infty,$$

since K is compact. Moreover $\chi(M_t) = \chi(M)$ for all $t > 1$, hence letting $t \to \infty$ we can rewrite the Equation (4.8) as

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = \lim_{t \to \infty} \int_{N_t} \overline{e}_n^* \overline{\Pi}. \tag{4.9}$$

Now recall that since the dimension is even the integrand $\overline{\Pi}$ of the boundary term is given by

$$\overline{\Pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2^n \pi^{\frac{n-1}{2}} k! \Gamma\left(\frac{n-2k+1}{2}\right)} \overline{\Phi}_k, \tag{4.10}$$

where

$$\overline{\Phi}_k = \pi_2^* \left(\sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k}}^{\sigma_{2k-1}} \wedge \overline{\omega}_n^{\sigma_{2k+1}} \wedge \ldots \wedge \overline{\omega}_n^{\sigma_{n-1}} \right). \tag{4.11}$$

Observe the similarity of $\tilde{\Phi}_k$ with the expression of the Lipschitz-Killing curvatures given in Proposition 2.12.

Let us denote by ${}^t\omega_j^i$ and ${}^t\Omega_j^i$ the connection and curvature forms of the manifold (N_t, g_t). Those forms are constructed from the moving frame $(\frac{1}{t}e_1, \ldots, \frac{1}{t}e_{n-1})$. The Riemannian manifold $E = N \times (1, \infty)$ is a warped-product manifold as described in section 1.10. The discussion in this section lead to the following equations which give expressions for the connection and curvature forms of M_t in terms of the connection and curvature forms of N:

$$\begin{cases} \overline{\omega}_j^i = \omega_j^i \\ \overline{\omega}_n^i = \theta^i \\ \overline{\Omega}_j^i = \Omega_j^i - \theta^i \wedge \theta^j, \\ \overline{\Omega}_n^i = 0. \end{cases} \tag{4.12}$$

In the same discussion we also established equation (1.16) which is the tensor version of the latter:

$$\overline{R} = t^2 \left(R - D \right),$$

where $D = \frac{1}{2} g_N \owedge g_N$.

Using the second equation of (4.12) we can first express $\overline{\Phi}_k$ as

$$\overline{\Phi}_k = \pi_2^* \left(\sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k}}^{\sigma_{2k-1}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_{n-1}} \right)$$

Since we integrate over the boundary of M_t we have to restrict this form to the submanifold N_t. But on this manifold we have:

$$\begin{aligned}
\overline{e}_n^*\overline{\Phi}_k &= \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k}}^{\sigma_{2k-1}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_{n-1}} \\
&= \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}_{\sigma_1\sigma_2} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k-1}\sigma_{2k}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_{n-1}} \\
&= \frac{1}{2^k} \sum_{\sigma\in\mathfrak{S}_{n-1}} \sum_{\tau\in\mathfrak{S}_{2k}} \varepsilon_\tau \overline{R}_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}} \cdots \overline{R}_{\sigma_{2k-1}\sigma_{2k}\sigma_{\tau_{2k-1}}\sigma_{\tau_{2k}}} \overline{\theta}^1 \wedge \ldots \wedge \overline{\theta}^{2k} \wedge \theta^{2k+1} \wedge \ldots \wedge \theta^{n-1} \\
&= \frac{t^{2k}}{2^k} \sum_{\sigma\in\mathfrak{S}_{n-1}} \sum_{\tau\in\mathfrak{S}_{2k}} \varepsilon_\tau \overline{R}_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}} \cdots \overline{R}_{\sigma_{2k-1}\sigma_{2k}\sigma_{\tau_{2k-1}}\sigma_{\tau_{2k}}} \mathrm{dvol}_N,
\end{aligned}$$

since $\overline{\theta}^i = t\theta^i$. By Proposition 2.23 we know that

$$C^{2k}(\overline{R}^k) = \frac{(2k)!}{2^{2k}(n-1-2k)!} \sum_{\sigma\in\mathfrak{S}_{n-1}} \sum_{\tau\in\mathfrak{S}_{2k}} \varepsilon_\tau \overline{R}_{\sigma_1\sigma_2\sigma_{\tau_1}\sigma_{\tau_2}} \cdots \overline{R}_{\sigma_{2k-1}\sigma_{2k}\sigma_{\tau_{2k-1}}\sigma_{\tau_{2k}}}$$

this is equal to

$$\overline{e}_n^*\tilde{\Phi}_k = \frac{2^k(n-1-2k)!}{(2k)!} t^{2k} C^{2k}(\overline{R}^k)\mathrm{dvol}_N.$$

Now by Lemma 2.24 we know that the contractions of $\overline{R}^k$ are related to the contractions of $(R-D)^k$ by the equation

$$C_M^{2k}(\overline{R}^k) = \frac{1}{t^{2k}} C_N^{2k}((R-D)^k),$$

where we have denoted by C_M the contraction operator on M and by C_N the contraction operator on N to avoid any confusion. Therefore we can write the boundary term of the Gauss-Bonnet-Chern Theorem as

$$\begin{aligned}
\overline{e}_n^*\overline{\Phi}_k &= \frac{2^k(n-1-2k)!}{(2k)!} t^{2k} C_M^{2k}(\overline{R}^k)\mathrm{dvol}_N \\
&= \frac{2^k(n-1-2k)!}{(2k)!} C_N^{2k}((R-D)^k)\mathrm{dvol}_N
\end{aligned}$$

Observe that this expression does not depend on t any more. Using Equation (4.10) we obtain the following expression for the form $\overline{\Pi}$:

$$\begin{aligned}
\overline{e}_n^*\overline{\Pi} &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{1}{2^n\pi^{\frac{n-1}{2}} k!\Gamma\left(\frac{n-2k+1}{2}\right)} \frac{2^k(n-1-2k)!}{(2k)!} C_N^{2k}((R-D)^k)\mathrm{dvol}_N \\
&= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} A(n,k) C_N^{2k}((R-D)^k)\mathrm{dvol}_N,
\end{aligned}$$

with

$$A(n,k) = \frac{(n-1-2k)!}{2^{n-k}\pi^{\frac{n-1}{2}}k!(2k)!\Gamma\left(\frac{n-2k+1}{2}\right)}.$$

Hence Equation (4.9) becomes

$$\begin{aligned}\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}}\int_M \mathrm{Pf}(\Omega) &= \lim_{t\to\infty}\int_{N_t} \overline{e}_n^*\overline{\Pi} \\ &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} A(n,k)\int_N C_N^{2k}((R-D)^k)\mathrm{dvol}_N,\end{aligned}$$

which completes the proof.

4.4 Manifolds with Asymptotically Conical Ends

The geometric hypotheses of Theorem 4.2 can be relaxed. Indeed, since we are interested in the asymptotic behaviour of the boundary term

$$\int_{\overline{e}_n(N_t)} \overline{\Pi},$$

we can look at conditions that ensure this boundary term to converge towards the one we obtained in Theorem 4.2. This convergence is holds if the metric g on each end $E = N \times (1,\infty)$ of M and its derivatives converge to the conical metric.
We summarize here some of the main results about asymptotically conical manifolds and we refer to Appendix A for more details.
Let (N, g_N) be a compact $n-1$-dimensional Riemannian manifold and set $M = N \times (1,\infty)$. Denote by $\overline{g}$ the conical metric i.e. the warped-product

$$\overline{g} = t^2 g_N + dt^2.$$

For $t \in (1,\infty)$ we write $N_t = N \times \{t\}$.

Definition 4.11. A Riemannian metric g on E is *asymptotically conical at order r and with (decreasing) rate α* if there exists a function $\rho : (1,\infty) \to \mathbb{R}$ such that $\rho(t) = o(t^{-\alpha})$ as $t \to \infty$ and such that for all $0 \le k \le r$ and $h = g - \overline{g}$ we have

$$\left\|\overline{\nabla}^k h\right\|_{\overline{g}} \le \rho \tag{4.13}$$

where this notation means that at every point $(x,t) \in N \times (1,\infty)$ the following inequality holds

$$\left\|(\overline{\nabla}^k h)_{(x,t)}\right\|_{\overline{g}} \le \rho(t).$$

A Riemannian manifold (E, g) endowed with an asymptotically conical metric is said to be *asymptotically conical*.

We will be mostly interested in the case where $r = 2$, as it will provide a control both on the connection and on the curvature of g with respect to the connection and the curvature of $\overline{g}$. Let ∇ be the Levi-Civita connection associated to g and $\overline{R}$ (resp. R) be the curvature tensor associated to $\overline{g}$ (resp. to g). Let $(\overline{e}_1, \dots, \overline{e}_n)$ be an orthonormal directly oriented moving frame for $\overline{g}$ on an open set $U \times (1, \infty) \subset M$ such that $\overline{e}_1, \dots, \overline{e}_{n-1}$ are tangent to N_t and $\overline{e}_n = \frac{\partial}{\partial t}$. Then the metrics $\overline{g}$ and g can be written as

$$\overline{g} = \delta_{ij}\overline{\theta}^i \otimes \overline{\theta}^j \quad \text{and} \quad g = g_{ij}\overline{\theta}^i \otimes \overline{\theta}^j,$$

where $(\overline{\theta}^1, \dots, \overline{\theta}^n)$ is the dual coframe. In Appendix A several estimates about the rate of convergence of the components of the connection and the curvature of g are computed. The following Proposition summarizes those results and the reader may find its proof directly in the Appendix.

Proposition 4.12. The following convergences hold

(a) If $r = 0$ we have

$$|h_{ij}| \leq \rho \quad \text{and} \quad \mathrm{dvol}_g = (1 + f)\mathrm{dvol}_{\overline{g}},$$

where f is a smooth function on $U \times (1, \infty)$ such that at each point $(x, t) \in U \times (1, \infty)$ we have $|f(x, t)| \leq \rho(t)$.

(b) if $r = 1$ we have in addition to (a) that

$$|\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k| \leq C \cdot \rho;$$

(c) if $r = 2$ we have in addition of (a) and (b) that

$$|R_{jkl}^i - \overline{R}_{jkl}^i| \leq C \cdot \rho;$$

This control on the geometry at infinity of M yields the same theorem as in the conical case.

Theorem 4.13. Let (M, g) be an even n-dimensional complete orientable Riemannian of finite topological type. Assume that M has one end $E = N \times (1, \infty)$ with $N \subset M$ compact and moreover that E is asymptotically conical at order at least 2. Then the total curvature is well-defined:

$$\int_M |\mathrm{Pf}(\Omega)| < +\infty,$$

and the Gauss-Bonnet defect of M is given by

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = \tau(N), \tag{4.14}$$

Proof. By Proposition 4.12 there exist smooth functions $\zeta^i_{jk}, \eta^i_{jkl}$ such that

$$\Gamma^i_{jk} = \overline{\Gamma}^i_{jk} + \zeta^i_{jk},$$
$$R^i_{jkl} = \overline{R}^i_{jkl} + \eta^i_{jkl},$$

and $|\zeta^i_{jk}|, |\eta^i_{jkl}| \leq C \cdot \rho$. The components Γ^i_{jk} and R^i_{jkl} are computed with respect to the orthonormal moving frame obtained from $(\overline{e}_1, \ldots, \overline{e}_n)$ by applying the Gram-Schmidt process (see Appendix A for further details).
First, let us show that the total curvature is well-defined. Recall that $\Omega \equiv 0$ if n is odd and if n is even

$$\begin{aligned}\Omega &= \frac{(-1)^{n/2}}{2^n \pi^{n/2}(n/2)!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{n-1}}_{\sigma_n} \\ &= \frac{(-1)^{n/2}}{2^{2n} \pi^{n/2}(n/2)!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma R^{\sigma_1}_{\sigma_2 i_1 i_2} \cdots R^{\sigma_{n-1}}_{\sigma_n i_{n-1} i_n} \theta^{i_1} \wedge \ldots \wedge \theta^{i_n} \\ &= \frac{(-1)^{n/2}}{2^{2n} \pi^{n/2}(n/2)!} \left(\sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma \varepsilon_\tau R^{\sigma_1}_{\sigma_2 \tau_1 \tau_2} \cdots R^{\sigma_{n-1}}_{\sigma_n \tau_{n-1} \tau_n} \right) \mathrm{dvol}_g.\end{aligned}$$

Moreover, by Proposition 4.12 we know that $\mathrm{dvol}_g = (1+f)\mathrm{dvol}_{\overline{g}}$ with $|f| \leq \rho$. Therefore, replacing R^i_{ijkl} by $\overline{R}^i_{jkl} + \eta^i_{jkl}$ in the above expression for Ω, we get on $U \times (1, \infty)$:

$$\begin{aligned}\Omega &= \frac{(-1)^{n/2}}{2^{2n} \pi^{n/2}(n/2)!} \left(\sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma \varepsilon_\tau \overline{R}^{\sigma_1}_{\sigma_2 \tau_1 \tau_2} \cdots \overline{R}^{\sigma_{n-1}}_{\sigma_n \tau_{n-1} \tau_n} \right) \mathrm{dvol}_{\overline{g}} + \frac{(-1)^{n/2}}{2^{2n} \pi^{n/2}(n/2)!} G(1+f)\mathrm{dvol}_{\overline{g}} \\ &= \overline{\Omega} + \frac{(-1)^{n/2}}{2^{2n} \pi^{n/2}(n/2)!} G(1+f)\mathrm{dvol}_{\overline{g}} \\ &= \frac{(-1)^{n/2}}{2^{2n} \pi^{n/2}(n/2)!} G(1+f)\mathrm{dvol}_{\overline{g}}\end{aligned}$$

where $|G| \leq C \cdot \rho$ and where we have used the fact that $\overline{\Omega} \equiv 0$ on E by Proposition 1.30. Now, since the volume form $\mathrm{dvol}_{\overline{g}}$ splits on $U \times (1, \infty)$ as

$$\mathrm{dvol}_{\overline{g}} = t^{n-1} \mathrm{dvol}_{g_N} \wedge dt,$$

and since $\rho = o(t^{1-n})$ as $t \to \infty$, we have on one hand

$$\int_1^\infty \rho^k \cdot t^{1-n} dt < +\infty,$$

for all $k \geq 1$ and on the other hand,

$$\begin{aligned}\int_{U \times (1,\infty)} |\Omega| &= \frac{1}{2^{2n} \pi^{n/2}(n/2)!} \int_U \int_1^\infty |G||1+f||t^{n-1} \mathrm{dvol}_{g_N} \wedge dt| \\ &\leq \frac{C}{2^{2n} \pi^{n/2}(n/2)!} \int_U \int_1^\infty (\rho + \rho^2) t^{n-1} dt \mathrm{dvol}_{g_N} \\ &< +\infty.\end{aligned}$$

Which shows that $\int_M \Omega$ is well-defined. As in the proof of Theorem 4.2, let us consider the exhaustion $\{M_t\}_{t>1}$ of M given by $M_t = K \cup (N \times (1,t])$. The Gauss-Bonnet-Chern Theorem (the version of Equation (3.11) since e_n is outward-pointing) applied to the compact manifold with boundary M_t gives

$$\chi(M_t) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{M_t} \mathrm{Pf}(\Omega) = \int_{e_n(N_t)} \Pi.$$

Therefore we need to show that:

$$\lim_{t\to\infty} \int_{N_t} e_n^* \Pi = \tau(N).$$

Here the form Π is given by

$$\Pi = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^{k+1}}{2^n \pi^{\frac{n-1}{2}} k! \Gamma\left(\frac{n-2k+1}{2}\right)} \Phi_k,$$

where

$$\Phi_k = \pi_2^* \left(\sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \right),$$

with $\pi_2 : SM \to M$ the canonical projection. As in the case of Ω, we show that Φ_k can be written as $\overline{\Phi}_k$ (the same form but built from the $\overline{g}$-orthonormal moving frame $(\overline{e}_1, \ldots, \overline{e}_n)$) plus a residual term whose integral tends to 0 as $t \to \infty$.

$$\begin{aligned}
\Phi_k &= \pi_2^* \left(\sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \Omega^{\sigma_1}_{\sigma_2} \wedge \ldots \wedge \Omega^{\sigma_{2k-1}}_{\sigma_{2k}} \wedge \omega^{\sigma_{2k+1}}_n \wedge \ldots \wedge \omega^{\sigma_{n-1}}_n \right) \\
&= \pi_2^* \left(\frac{1}{2^k} \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma R^{\sigma_1}_{\sigma_2 i_1 i_2} \cdots R^{\sigma_{2k-1}}_{\sigma_{2k} i_{2k-1} i_{2k}} \Gamma^{\sigma_{2k+1}}_{i_{2k+1} n} \cdots \Gamma^{\sigma_{n-1}}_{i_{n-1} n} \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}} \right) \\
&= \pi_2^* \left(\frac{1}{2^k} \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma (\overline{R}^{\sigma_1}_{\sigma_2 i_1 i_2} + \eta^{\sigma_1}_{\sigma_2 i_1 i_2}) \cdots (\overline{R}^{\sigma_{2k-1}}_{\sigma_{2k} i_{2k-1} i_{2k}} + \eta^{\sigma_{2k-1}}_{\sigma_{2k} i_{2k-1} i_{2k}}) \right. \\
&\qquad \left. \cdot (\overline{\Gamma}^{\sigma_{2k+1}}_{i_{2k+1} n} + \zeta^{\sigma_{2k+1}}_{i_{2k+1} n}) \cdots (\overline{\Gamma}^{\sigma_{n-1}}_{i_{n-1} n} + \zeta^{\sigma_{n-1}}_{i_{n-1} n})\, \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}} \right) \\
&= \pi_2^* \left(\frac{1}{2^k} \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{R}^{\sigma_1}_{\sigma_2 i_1 i_2} \cdots \overline{R}^{\sigma_{2k-1}}_{\sigma_{2k} i_{2k-1} i_{2k}} \overline{\Gamma}^{\sigma_{2k+1}}_{i_{2k+1} n} \cdots \overline{\Gamma}^{\sigma_{n-1}}_{i_{n-1} n} \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}} \right. \\
&\qquad \left. + \tilde{F}_{i_1 \ldots i_{n-1}} \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}} \right),
\end{aligned}$$

with $|\tilde{F}_{i_1\ldots i_{n-1}}| \leq C \cdot \rho$ since all the remaining terms contain at least an η^i_{jkl} or a ζ^i_{jk}. Moreover, the coframe $(\theta^1, \ldots, \theta^n)$ is obtained from $(\overline{\theta}^1, \ldots, \overline{\theta}^n)$ by applying the Gram-Schmidt process for the metric g. Therefore each θ^i can be written as

$$\theta^i = \overline{\theta}^i + \beta^i_j \overline{\theta}^j,$$

with $|\beta^i_j| \leq \rho$. Hence

$$\begin{aligned}\theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}} &= \left(\overline{\theta}^{i_1} + \beta^{i_1}_{j_1}\overline{\theta}^{j_i}\right) \wedge \ldots \wedge \left(\overline{\theta}^{i_{n-1}} + \beta^{i_{n-1}}_{j_{n-1}}\overline{\theta}^{j_{n-1}}\right) \\ &= \overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}} + H^{i_1\ldots i_{n-1}}_{j_1\ldots j_{n-1}}\overline{\theta}^{j_1} \wedge \ldots \wedge \overline{\theta}^{j_{n-1}},\end{aligned}$$

with $\left|H^{i_1\ldots i_{n-1}}_{j_1\ldots j_{n-1}}\right| \leq \rho$. It follows that Φ_k can be written as

$$\begin{aligned}\Phi_k = \pi_2^* &\left(\frac{1}{2^k} \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{R}^{\sigma_1}_{\sigma_2 i_1 i_2} \cdots \overline{R}^{\sigma_{2k-1}}_{\sigma_{2k} i_{2k-1} i_{2k}} \overline{\Gamma}^{\sigma_{2k+1}}_{i_{2k+1} n} \cdots \overline{\Gamma}^{\sigma_{n-1}}_{i_{n-1} n} \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}} \right. \\ &\qquad\qquad \left. + \tilde{F}_{i_1\ldots i_{n-1}} \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}} \right) \\ = \pi_2^* &\left(\frac{1}{2^k} \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{R}^{\sigma_1}_{\sigma_2 i_1 i_2} \cdots \overline{R}^{\sigma_{2k-1}}_{\sigma_{2k} i_{2k-1} i_{2k}} \overline{\Gamma}^{\sigma_{2k+1}}_{i_{2k+1} n} \cdots \overline{\Gamma}^{\sigma_{n-1}}_{i_{n-1} n} \overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}} \right. \\ &+ \frac{1}{2^k} \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{R}^{\sigma_1}_{\sigma_2 i_1 i_2} \cdots \overline{R}^{\sigma_{2k-1}}_{\sigma_{2k} i_{2k-1} i_{2k}} \overline{\Gamma}^{\sigma_{2k+1}}_{i_{2k+1} n} \cdots \overline{\Gamma}^{\sigma_{n-1}}_{i_{n-1} n} H^{i_1\ldots i_{n-1}}_{j_1\ldots j_{n-1}} \overline{\theta}^{j_1} \wedge \ldots \wedge \overline{\theta}^{j_{n-1}} \\ &\left. + \tilde{F}_{i_1\ldots i_{n-1}} \overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}} + \tilde{F}_{i_1\ldots i_{n-1}} H^{i_1\ldots i_{n-1}}_{j_1\ldots j_{n-1}} \overline{\theta}^{j_1} \wedge \ldots \wedge \overline{\theta}^{j_{n-1}} \right) \\ = \overline{\Phi}_k &+ \pi_2^* \left(F_{i_1\ldots i_{n-1}} \overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}}\right),\end{aligned}$$

with $|F_{i_1\ldots i_{n-1}}| \leq \rho$.
Finally, integrating over N_t we get

$$\begin{aligned}\int_{N_t} e_n^* \Phi_k = \int_{N_t} \overline{e}_n^* \overline{\Phi}_k + \int_{N_t} \alpha^j_n \overline{e}_j^* \overline{\Phi}_k + \int_{N_t} F_{i_1\ldots i_{n-1}} \overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}} \\ + \int_{N_t} \alpha^j_n F_{i_1\ldots i_{n-1}} \overline{\theta}^i_1 \wedge \ldots \wedge \overline{\theta}^{i_{n-1}}.\end{aligned}$$

But we know that $\overline{\Phi}_k$ does not depend on t since it is defined on N and thus

$$\alpha^j_n \overline{e}_j^* \overline{\Phi}_k = \alpha^j_n \frac{1}{t} \overline{e}_j(0)^* \overline{\Phi}_k,$$

where $(\overline{e}_1(0), \ldots, \overline{e}_{n-1}(0))$ is the g_N-orthonormal moving frame on $V \subset N$ used to define the moving frame $(\overline{e}_1, \ldots, \overline{e}_n)$. Moreover, on the boundary N_t, the terms $\overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}}$ are either 0 if two indices are repeated or if $i_j = n$ for some j, or they are equal to $\varepsilon_{i_1 \ldots i_{n-1}} t^{n-1} \mathrm{dvol}_N$ if all indices are given by a permutation of the set $\{1, \ldots, n-1\}$. so that

$$\begin{aligned}
\left| \int_{N_t} e_n^* \Phi_k - \int_{N_t} \overline{e}_n^* \overline{\Phi}_k \right| \leq & \int_{N_t} \frac{|\alpha_n^j|}{t} \left| \overline{e}_j(0)^* \overline{\Phi}_k \right| + \int_{N_t} t^{n-1} |\varepsilon_{i_1 \ldots i_{n-1}}| |F_{i_1 \ldots i_{n-1}}| \mathrm{dvol}_N| \\
& + \int_{N_t} t^{n-1} |\alpha_n^j| |\varepsilon_{i_1 \ldots i_{n-1}}| |F_{i_1 \ldots i_{n-1}}| |\mathrm{dvol}_N| \\
\leq & \int_{N_t} \frac{\rho}{t} \left| \overline{e}_j^* \overline{\Phi}_k \right| + \int_{N_t} t^{n-1} \rho |\mathrm{dvol}_N| + \int_{N_t} t^{n-1} \rho^2 |\mathrm{dvol}_N|
\end{aligned}$$

Since the right-hand side goes to 0 as $t \to \infty$ we obtain

$$\lim_{t \to \infty} \left| \int_{N_t} e_n^* \Pi - \int_{N_t} \overline{e}_n^* \overline{\Pi} \right| = 0,$$

and by Theorem 4.2 we know that for all t we have

$$\int_{N_t} \overline{e}_n^* \overline{\Pi} = \tau(N),$$

which completes the proof since we know that

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = \lim_{t \to \infty} \int_{N_t} e_n^* \Pi.$$

□

4.5 Manifolds with Conical Singularities

After having studied the extremity of the cone lying "at infinity" it is time to take a closer look at the other side. Let us give the definition of a manifold with conical singularities.

Definition 4.14. A *Riemannian manifold with conical singularities* is a metric space $(\hat{M}, d)$ such that

(a) There exists a finite set $\Sigma = \{p_1, \ldots, p_r\} \subset \hat{M}$ such that its complement $M := \hat{M} \backslash \Sigma$ is a smooth Riemannian manifold, that is M is a smooth manifold and the distance d is induced by a Riemannian metric g.

(b) Each point $p_i \in \Sigma$ admits a neighbourhood $\hat{U}_i \subset \hat{M}$ which is homeomorphic to the cone over a connected and compact $(n-1)$-dimensional manifold N_i:

$$\hat{U}_i \cong C(N_i) = N_i \times [0,1) \big/ N_i \times \{0\}.$$

(c) The above homeomorphism is assumed to be a diffeomorphism on $U_i := \hat{U}_i \setminus \{p_i\}$.

The point p_i is called a *standard conical singularity* of $\hat{M}$ with *link* (N_i, g_i) if the metric g on U_i is the standard cone metric i.e.

$$g = \overline{g} := t^2 g_i + dt^2.$$

More generally, the point p_i is called a *conical singularity* of $\hat{M}$ with *link* (N_i, g_i) if the metric g on U_i can be written as

$$g = t^2 g_i + dt^2 + h_i = \overline{g} + h_i,$$

where h_i is a bilinear form satisfying for $0 \leq k \leq 2$:

$$\|\overline{\nabla}^k h_i\|_{\overline{g}} \leq \rho, \tag{4.15}$$

where $\rho : (0,1) \to \mathbb{R}$ is a smooth function such that $\rho = o(t)$ as $t \to 0$.

Remark 4.15. Observe that $\hat{M}$ is topologically a manifold if and only if each N_i is homeomorphic to a sphere $\mathbb{S}^{n-1}$ and $\hat{M}$ is a Riemannian manifold if and only if each link (N_i, g_i) is isometric to the sphere with its standard metric.

Remark 4.16. Although the condition on the convergence of h and its covariant derivatives is quite similar to the one in the definition of an asymptotically conical end, it is remarkable that the convergence in the present case is significantly weaker. Indeed there is no assumption on the rate of convergence of ρ towards 0 in this case, while for asymptotically conical ends we needed that $\rho = o(t^{1-n})$ as $t \to \infty$. This difference is essentially due to the fact that the volume form of the cone $C(N)$ splits locally as $t^{n-1}\mathrm{dvol}_N \wedge dt$ and this factor t^{n-1} diverges as $t \to \infty$ but converges to 0 as $t \to 0$.

4.6 Standard Conical Singularities

Let us first deal with the case of standard conical singularities. The theorem we are willing to show is the following:

Theorem 4.17. Let $(\hat{M}, d)$ be an even dimensional Riemannian manifold with one standard conical singularity $p \in \hat{M}$ of link (N, g_N). Then we have

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = -\tau(N).$$

Proof. The proof is essentially the same as the proof of the conical end case. The only difference is that we have to use the other version of the Gauss-Bonnet-Chern Theorem as this time we will consider an inward pointing vector field at each point

of the boundary of the exhaustion (we refer to the proof of Theorem 4.2 for the notations).
By definition of a standard conical singularity we know that there exists an open neighbourhood $U \subset M = \hat{M} \setminus \{p\}$ of p such that U with the induced metric is isometric to the standard cone on the link N:

$$U \cong N \times (0,1) \quad \text{with metric} \quad g = t^2 g_N + dt^2.$$

Then M can be written as $K \cup U$ where K is a compact submanifold with boundary and let us consider the exhaustion of M given by

$$M_t = K \cup (N \times [t,1)),$$

with boundary $N_t := \partial M_t = N \times \{t\}$. Now let $(e_1, \ldots, e_{n-1})$ be an oriented orthonormal moving frame on an open subset $V \subset N$ and set

$$\overline{e}_i = \frac{1}{t} e_i \quad \text{and} \quad \overline{e}_n = \frac{\partial}{\partial t},$$

which is an oriented orthonormal moving frame on $V \times (0,1) \subset M$. In contrast with the proof of Theorem 4.2, the vector field $\overline{e}_n$ is *inward-pointing* at each point of N_t, therefore by applying the Gauss-Bonnet-Chern Theorem 3.9 to M_t we get

$$\chi(M_t) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{M_t} \mathrm{Pf}(\Omega) = \int_{\overline{e}_n(N_t)} \tilde{\Pi},$$

where

$$\tilde{\Pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^{k+1}}{2^n \pi^{\frac{n-1}{2}} k! \Gamma\left(\frac{n-2k+1}{2}\right)} \tilde{\Phi}_k$$

with

$$\tilde{\Phi}_k = \pi_2^* \left(\sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k}}^{\sigma_{2k-1}} \wedge \overline{\omega}_n^{\sigma_{2k+1}} \wedge \ldots \wedge \overline{\omega}_n^{\sigma_{n-1}} \right),$$

where we recall that $\pi_2 : SM \to M$ is the canonical projection. Note that here $\overline{\omega}_B^A$ and $\overline{\Omega}_B^A$ denote the connection and curvature forms with respect to the moving frame $(\overline{e}_1, \ldots, \overline{e}_n)$ and they are not to be confused with the quantities that are used in subsection 3.3.5 to define $\overline{\Pi}$.
Now, the relations 4.12 take the following form with respect to our moving frame $(\overline{e}_1, \ldots, \overline{e}_n)$:

$$\begin{cases} \overline{\omega}_j^i = \omega_j^i, \\ \overline{\omega}_n^i = -\theta^i, \\ \overline{\Omega}_j^i = \Omega_j^i - \theta^i \wedge \theta^j, \\ \overline{\Omega}_n^i = 0. \end{cases} \tag{4.16}$$

Therefore

$$\tilde{\Phi}_k = (-1)^{n-1-k} \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k}}^{\sigma_{2k-1}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_{n-1}},$$

and thus

$$\begin{aligned}\tilde{\Pi} &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2^n \pi^{\frac{n-1}{2}} k! \Gamma\left(\frac{n-2k+1}{2}\right)} \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon_\sigma \overline{\Omega}_{\sigma_2}^{\sigma_1} \wedge \ldots \wedge \overline{\Omega}_{\sigma_{2k}}^{\sigma_{2k-1}} \wedge \theta^{\sigma_{2k+1}} \wedge \ldots \wedge \theta^{\sigma_{n-1}} \\ &= \overline{\Pi}.\end{aligned}$$

Then it suffices to imitate the rest of the proof of Theorem 4.2 to relate $\tilde{\Pi}$ to $\tau(N)$ and obtain

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\overline{\Omega}) = \int_{\overline{e}_n(N)} \overline{\Pi} = -\int_N (-\overline{e}_n)^* \overline{\Pi} = -\tau(N),$$

since $-\overline{e}_n$ is outward-pointing. □

4.7 Conical Singularities

As in the case of asymptotically conical ends, we construct on an open subset $V \times (0,1)$ with $V \subset N$ a $\overline{g}$-orthonormal moving frame $(\overline{e}_1, \ldots, \overline{e}_n)$ where $\overline{e}_i$ is tangent to N for all $1 \leq i \leq n-1$ and $\overline{e}_n = \frac{\partial}{\partial t}$, with t the arclength parameter on $(0,1)$. Observe that this time the last vector field of the frame $\overline{e}_n$ is inward-pointing instead of outward-pointing. Imitating the arguments of Appendix A, we apply the Gram-Schmidt process to the frame to obtain a g-orthonormal moving frame $(e_1, \ldots, e_n)$. The assumption (4.15) on the norm of $\|\overline{\nabla}^k h\|_{\overline{g}}$ implies, as in the case of asymptotically conical ends, the following convergences for $\rho = o(t)$ as $t \to 0$:

(a) For $k = 0$ then $|h_{ij}| \leq \rho$ and the volume forms are related by

$$\mathrm{dvol}_g = (1 + f)\mathrm{dvol}_{\overline{g}},$$

with $f : V \times (0,1) \to \mathbb{R}$ a smooth function such that for all $(x,t) \in V \times (0,1)$ we have $|f(x,t)| \leq \rho(t)$.

(b) For $k = 1$ then there exist smooth function ζ_{jk}^i such that

$$\Gamma_{jk}^i = \overline{\Gamma}_{jk}^i + \zeta_{jk}^i,$$

and $|\zeta_{jk}^i| \leq C\rho$.

(c) For $k = 2$ then there exist smooth functions η^i_{jkl} such that

$$R^i_{jkl} = \overline{R}^i_{jkl} + \eta^i_{jkl},$$

and $|\eta^i_{jkl}| \leq C\rho$.

We are ready to prove the last part of the Main Theorem 7. The proof is similar to the argument presented in the case of asymptotically conical ends, the difference lying in the fact that we look at the limit as t goes to 0 instead of ∞.

Theorem 4.18. Let $\hat{M}$ be an even dimensional compact Riemannian manifold with one conical singularity p and let (N, g_N) be its link. Then the total curvature is well-defined:

$$\int_M |\mathrm{Pf}(\Omega)| < +\infty,$$

and

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = -\tau(N) \tag{4.17}$$

The proof is the same as the one of Theorem 4.13 except for the fact that the function ρ does not have the same behaviour in the neighbourhood of 0.

Remark 4.19. Note that in the following proof, the quantities denoted with a bar (especially the forms $\overline{\Pi}$ and $\overline{\Phi}_k$) refer to the moving frame orthonormal with respect to the standard conical metric $\overline{g}$ and *not* to the forms constructed in Chapter 3.

Proof. By definition of a manifold with a conical singularity, $M = \hat{M} \setminus \{p\}$ is a smooth Riemannian manifold and there exists a neighbourhood $\hat{U} \subset \hat{M}$ of p such that there is a diffeomorphism between $U = \hat{U} \setminus \{p\} \subset M$ and the cone over the link (N, g_N):

$$U \cong N \times (0, 1),$$

on which the metric can be written as

$$g = t^2 g_N + dt^2 + h$$

As in the case of (asymptotically) conical ends, we consider an exhaustion of M defined by

$$M_t = (M \setminus U) \cup (N \times [t, 1)),$$

for $t \in (0, 1)$. Each M_t is a compact manifold with boundary $N_t := \partial M_t = N \times \{t\}$. Let $\overline{g}$ be as usual the standard cone metric on $N \times (0, 1)$. Let $(\overline{e}_1, \ldots, \overline{e}_n)$ be a $\overline{g}$-orthonormal moving frame on an open subset $V \times (0, 1) \subset M$ with $V \subset N$ constructed as before by taking vectors $\overline{e}_i$ that are tangent to N for all $1 \leq i \leq n-1$ and by taking $\overline{e}_n = \frac{\partial}{\partial t}$, where t is the arclength parameter on $(0, 1)$. As in Appendix

A, we apply the Gram-Schmidt process to this frame with respect to the metric g to obtain a g-orthonormal moving frame $(e_1, \ldots, e_n)$ that satisfies

$$\begin{aligned} e_i &= \overline{e}_i + \alpha_i^j \overline{e}_j \quad \text{with} \quad |\alpha_i^j| \leq \rho, \\ \theta^i &= \overline{\theta}^i + \beta_j^i \overline{\theta}^j \quad \text{with} \quad |\beta_j^i| \leq \rho. \end{aligned}$$

As in the proof of Theorem 4.13 the form Ω can be written as

$$\Omega = \frac{(-1)^{n/2}}{2^{2n}\pi^{n/2}(n/2)!} G(1+f)\mathrm{dvol}_{\overline{g}},$$

where G is a smooth function satisfying $|G| \leq C \cdot \rho$. Therefore

$$\begin{aligned} \int_{V\times(0,1)} |\Omega| &= \frac{1}{2^{2n}\pi^{n/2}(n/2)!} \int_{V\times(0,1)} |G||1+f||\mathrm{dvol}_{\overline{g}}| \\ &\leq \frac{C}{2^{2n}\pi^{n/2}(n/2)!} \int_{V\times(0,1)} (\rho+\rho^2)t^{n-1}|\mathrm{dvol}_N \wedge dt| \\ &= \frac{C}{2^{2n}\pi^{n/2}(n/2)!} \int_V \int_0^1 (\rho+\rho^2)t^{n-1} dt \mathrm{dvol}_N \\ &< +\infty. \end{aligned}$$

Recall that $|\Omega|$ is the natural density associated to Ω (see Remark 4.1). Note that the condition on $\rho = o(1)$ as $t \to 0$ could easily be weakened without changing the result but this stronger control is needed hereafter. The Gauss-Bonnet-Chern Theorem 3.7 for the manifold M_t with the chosen inward-pointing vector field e_n gives

$$\chi(M_t) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{M_t} \mathrm{Pf}(\Omega) = \int_{e_n(N_t)} \tilde{\Pi}.$$

We also know that $\chi(M_t) = \chi(M)$ and that

$$\lim_{t\to 0} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{M_t} \mathrm{Pf}(\Omega) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega).$$

Recall that in the proof of Theorem 4.13 we obtained an expression for Φ_k in terms of $\overline{\Phi}_k$ plus a residual term. Reproducing the same argument we get

$$\Phi_k = \overline{\Phi}_k + \pi_2^* \left(F_{i_1 \ldots i_{n-1}} \overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}} \right),$$

with $|F_{i_1\ldots i_{n-1}}| \leq C \cdot \rho$. Therefore, writing $e_n = \overline{e}_n + \alpha_n^j \overline{e_j}$ we conclude that

$$\begin{aligned} \int_{N_t} e_n^* \Phi_k = \int_{N_t} \overline{e}_n^* \overline{\Phi}_k + \int_{N_t} \alpha_n^j \overline{e}_j^* \overline{\Phi}_k + \int_{N_t} F_{i_1\ldots i_{n-1}} \overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}} \\ + \int_{N_t} \alpha_n^j F_{i_1\ldots i_{n-1}} \overline{\theta}_1^i \wedge \ldots \wedge \overline{\theta}^{i_{n-1}}. \end{aligned}$$

But we know that $\overline{\Phi}_k$ does not depend on t since it is defined on N and thus

$$\alpha_n^j \overline{e}_j^* \overline{\Phi}_k = \alpha_n^j \frac{1}{t} \overline{e}_j(0)^* \overline{\Phi}_k,$$

where $(\overline{e}_1(0), \ldots, \overline{e}_{n-1}(0))$ is the g_N-orthonormal moving frame on $V \subset N$ used to define the moving frame $(\overline{e}_1, \ldots, \overline{e}_n)$. Moreover, on the boundary N_t, the terms $\overline{\theta}^{i_1} \wedge \ldots \wedge \overline{\theta}^{i_{n-1}}$ are either 0 if two indices are repeated or if $i_j = n$ for some j, or they are equal to $\varepsilon_{i_1 \ldots i_{n-1}} t^{n-1} \mathrm{dvol}_N$ if all indices are given by a permutation of the set $\{1, \ldots, n-1\}$. Hence

$$\left| \int_{N_t} e_n^* \Phi_k - \int_{N_t} \overline{e}_n^* \overline{\Phi}_k \right| \leq \int_{N_t} \frac{|\alpha_n^j|}{t} |\overline{e}_j(0)^* \overline{\Phi}_k| + \int_{N_t} t^{n-1} |\varepsilon_{i_1 \ldots i_{n-1}}| |F_{i_1 \ldots i_{n-1}}| |\mathrm{dvol}_N| \\ + \int_{N_t} t^{n-1} |\alpha_n^j| |\varepsilon_{i_1 \ldots i_{n-1}}| |F_{i_1 \ldots i_{n-1}}| |\mathrm{dvol}_N|.$$

The right-hand converges to 0 as $t \to 0$ if $\lim_{t \to 0} \frac{|\alpha_n^j|}{t} = 0$, which is precisely the case since $|\alpha_n^j| \leq \rho = o(t)$ as $t \to 0$. Since we know that

$$\int_{N_t} \overline{e}_n^* \overline{\Pi} = -\tau(N),$$

we finally get that

$$\lim_{t \to 0} \int_{N_t} e_n^* \Pi = -\tau(N).$$

□

4.8 Proof of the Main Theorem

Recall that the main theorem is the following:

Main Theorem. Let $(\hat{M}, g)$ be a complete even dimensional Riemannian manifold with finitely many conical singularities $\{p_1, \ldots, p_r\}$ and finitely many asymptotically conical ends $\{E_1, \ldots, E_s\}$ (at order 2 with decreasing rate $n-1$). Then the total curvature of $\hat{M}$ is finite and we have

$$\chi(M) - \int_M \Omega = -\sum_{i=1}^{r} \tau(N_i) + \sum_{j=1}^{s} \tau(N_j),$$

where the N_k are the links of the conical singularities and of the asymptotically conical ends.

Let us resume what has been done in the previous sections. We proved in the Theorems 4.2 and 4.13 that if a complete even-dimensional Riemannian manifold

(M, g) of finite topological type has an asymptotically conical end with link N, then its Gauss-Bonnet defect is equal to $\tau(N)$. In the Theorems 4.17 and 4.18 we showed that this statement still holds if $(\hat{M}, d)$ is an even-dimensional Riemannian manifold with one conical singularities. In each of these cases we can actually assume that the manifold has a finite number of asymptotically conical ends or conical singularities and simply add the resulting τ.

Example 4.20. (Cone on the unit sphere) The simplest example having both a conical end and a conical singularity is the standard cone on a sphere $\mathbb{S}^{n-1}$, i.e.

$$\hat{M} = C(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1} \times [0, \infty) \Big/ \mathbb{S}^{n-1} \times \{0\},$$

endowed with the metric $g = t^2 g_{\mathbb{S}^{n-1}} + dt^2$. Then both the end and the singularity have the same link $N = \mathbb{S}^{n-1}$ and we know from Example 4.7 that $\tau(\mathbb{S}^{n-1}) = 1$. Moreover, since $M = \mathbb{S}^{n-1} \times (0, \infty)$ has the same homotopy type as $\mathbb{S}^{n-1}$, we have $\chi(M) = \chi(\mathbb{S}^{n-1}) = 0$ if n is even. Therefore, since the Pfaffian vanishes, the Main Theorem is verified:

$$\chi(M) - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \mathrm{Pf}(\Omega) = 0 = -\tau(N) + \tau(N).$$

Observe once again that the standard cone on the unit sphere $\mathbb{S}^{n-1}$ is isometric to $\mathbb{R}^n$ with its standard metric, so this is what the Main Theorem looks like for $\mathbb{R}^n$ seen as a standard cone having a conical singularity at the origin and a conical end.

4.9 Consequences of the Main Theorem

4.9.1 Total curvature in dimension 4

We present a consequence of the Main Theorem which is in sharp contrast with the 2-dimensional Cohn-Vossen inequality. Indeed we show that in dimension 4, there is no topological obstruction for the total curvature of a complete manifold with a conical end.

Corollary 4.21. Let (M^4, g) be a complete 4-dimensional Riemannian manifold of finite topological type with one conical end. Then for every $\lambda \in \mathbb{R}$ there exists a metric Riemannian $\tilde{g}$ which is conformal to g and such that

$$\frac{1}{4\pi^2} \int_M \mathrm{Pf}(\Omega) = \lambda.$$

In order to prove this result, we introduce the *normalized Hilbert-Einstein* functional, which is defined for a compact smooth n-dimensional Riemannian manifold (N, g) by

$$\mathcal{E}_N(g) = \frac{1}{2} \frac{1}{V(g)^{n-2/n}} \int_N S_g \mathrm{dvol}_g, \tag{4.18}$$

where $V(g) = \mathrm{Vol}(N, g)$ is the standard volume of (N, g). Note that this functional is scale invariant i.e. $\mathcal{E}_N(\lambda^2 g) = \mathcal{E}_N(g)$ for any constant $\lambda > 0$. The infimum of $\mathcal{E}_N(g)$ over the conformal class of g is known to be the *the conformal Yamabe energy of* g and has been largely studied in the context of the Yamabe problem. In our case we are interested in the supremum of $\mathcal{E}_N(g)$ within a conformal class. More precisely, we show that in dimension ≥ 3, this supremum is arbitrary large:

Lemma 4.22. For any compact Riemannian manifold (N, g) of dimension $n \geq 3$, we have

$$\sup_{\varphi} \mathcal{E}_N(\varphi^2 g) = +\infty,$$

where the supremum is taken over all smooth functions $\varphi : M \longrightarrow (0, \infty)$.

In dimension 2 the Gauss-Bonnet theorem shows that $\mathcal{E}_N(g)$ does not depend on the metric since the scalar curvature is twice the Gauss curvature.

Proof. (Lemma 4.22) Let us write a conformal deformation of g as $\tilde{g} = \varphi^2 g = u^{\frac{4}{n-2}} g$ with $u = \varphi^{\frac{n-2}{2}}$ Then the scalar curvatures S_g and $S_{\tilde{g}}$ of g and $\tilde{g}$ respectively are related by the well-known equation

$$u^{\frac{n+2}{n-2}} S_{\tilde{g}} = 4\left(\frac{n-1}{n-2}\right) \Delta_g u + S_g u,$$

where Δ_g is the Laplacian of g. The volume forms are related by $\mathrm{dvol}_{\tilde{g}} = u^{\frac{2n}{n-2}} \mathrm{dvol}_g$. Therefore, an integration over N gives

$$\int_N S_{\tilde{g}} \mathrm{dvol}_{\tilde{g}} = \int_N S_g u^2 \mathrm{dvol}_g + 4\left(\frac{n-1}{n-2}\right) \int_N |\nabla u|^2 \mathrm{dvol}_g. \tag{4.19}$$

Using Lemma 4.23 we can find a sequence $\{w_k\} \subset C^\infty(N)$ such that

$$|w_k| \leq \frac{1}{2} \quad \text{and} \quad \int_N |\nabla u|^2 \mathrm{dvol}_g \overset{k\to\infty}{\longrightarrow} \infty.$$

For the sequence $g_k = (1 + w_k)^{4/n-2} g$ of conformal deformations of g, we then have from Equation (4.19)

$$V(g_k) \leq \left(\frac{3}{2}\right)^{\frac{2n}{n-2}} V(g) \quad \text{and} \quad \int_N S_{g_k} \mathrm{dvol}_{g_k} \overset{k\to\infty}{\longrightarrow} \infty,$$

where S_{g_k} is the scalar curvature of g_k. We therefore have $\lim_{k\to\infty} \mathcal{E}_N(g_k) = +\infty$. $\square$

Lemma 4.23. On any Riemannian manifold (N, g) there exists sequence of smooth functions $w_k : N \longrightarrow \mathbb{R}$ with compact support such that $\|\nabla w_k\|_{L^2(N)} \to \infty$ while $\|w_k\|_{L^\infty(N)}$ is arbitrarily small.

Proof. Let $\mathcal{C}$ be a smooth simple closed curve in the manifold N with parametrization $\gamma : [0, 2\pi] \to \mathcal{C}$ Up to a bilipschitz transformation of the metric, one may assume that a tubular neighbourhood U of $\mathcal{C}$ in N is diffeomorphic to a torus $\mathcal{C} \times \mathbb{B}^{n-1}$ with the metric

$$g = dt^2 + h,$$

where h is the standard metric on the disk $\mathbb{B}^{n-1}$. Let us fix a constant $a > 0$ and choose a function $\eta \in C_0^\infty(\mathbb{B}^{n-1})$ with compact support such that $0 \leq \eta \leq 1 = \eta(0)$ and define the function w_k to be zero on $N \setminus U$ and

$$w_k(t, x) = a\eta(x)\sin(kt),$$

for any point $(t, x) \in U \cong \mathcal{C} \times \mathbb{B}^{n-1}$. We then have ∇w_k on $N \setminus U$ and

$$\nabla w_k = ka\cos(kt)\eta(x) + a\sin(kt)\nabla\eta,$$

in U. Therefore $\|\nabla w_k\|_{L^2(N)} = O(k)$ as $k \to \infty$ and $|w_k| \leq a$. □

The particular case of a compact 3-dimensional Riemannian manifold (N, g) allows us to write the invariant $\tau(N)$ using the normalized Hilbert-Einstein functional (4.18). Indeed on one hand we have

$$\mathcal{E}_N(g) = \frac{1}{2}\frac{1}{V(g)^{1/3}}\int_N S_g \mathrm{dvol}_g,$$

and on the other hand using Example 4.5 we know that

$$\tau(N) = \frac{1}{8\pi^2}\int_N (S_g - 2)\mathrm{dvol}_g,$$

so that we can write

$$\tau(N) = \frac{1}{4\pi^2}\left(\mathcal{E}_N(g) \cdot V(g)^{1/3} - V(g)\right). \tag{4.20}$$

Proof. (Corollary 4.21)
Let us write $M = K \cup (N \times (1, \infty))$ with $K \subset M$ compact and $N = \partial K$. On $N \times (1, \infty)$ the metric g is

$$g = t^2 g_N + dt^2,$$

with g_N a Riemannian metric on N. By the main theorem we know that

$$\frac{1}{4\pi^2}\int_M \mathrm{Pf}(\Omega) = \chi(M) - \tau(N, g_N),$$

therefore by Equation (4.20) we obtain

$$\frac{1}{4\pi^2}\int_M \mathrm{Pf}(\Omega) = \chi(M) + \frac{1}{4\pi^2}\left(V(g_N) - \mathcal{E}_N(g_N) \cdot V(g_N)^{1/3}\right).$$

On the link N we consider the following conformal change of the metric: for $\alpha > -1$ set

$$g_\alpha = (\alpha+1)^2 g_N.$$

Since the Hilbert-Einstein functional is scale invariant we have $\mathcal{E}_N(g_\alpha) = \mathcal{E}_N(g_N)$. Moreover

$$V(g_\alpha) = (\alpha+1)^3 V(g_N),$$

therefore

$$\frac{1}{4\pi^2}\int_M \mathrm{Pf}(\Omega) = \chi(M) + \frac{1}{4\pi^2}\left((\alpha+1)^3 V(g_N) - \mathcal{E}_N(g_N)(\alpha+1)V(g_N)^{1/3}\right),$$

which is a polynomial of degree 3 in the variable α. So by taking α arbitrarily large, the right-hand side of the latter equation can be made as large as desired. On the other hand, we showed in Lemma 4.22 that one can find a sequence g_k of conformal deformation of g_N such that the volume $V(g_k)$ is uniformly bounded and such that $\lim_{k\to\infty}\mathcal{E}_N(g_k) = +\infty$, which implies that

$$\lim_{k\to\infty}\left(V(g_k) - \mathcal{E}_N(g_k)V(g_k)^{1/3}\right) = -\infty.$$

□

4.9.2 Q-curvature and conformally flat 4-manifolds

It appears that in the context of conformally flat geometry, a modified version of the Pfaffian, called the Q curvature, enjoys better properties under conformal changes of the metric. This notion goes back to Thomas P. Branson [Bra85] and has been largely studied since then by people such as Sun-Yung Alice Chang, Paul Yang and many others. In this section we recover a result by Buzano and Nguyen at least in a particular case of a conformally flat 4-dimensional manifold.
Although the Q-curvature can be defined in arbitrary dimensions, we restrict ourselves to the dimension 4 to avoid unnecessary technicalities.

Definition 4.24. Let (M, g) be an 4-dimensional Riemannian manifold. Its $Q = Q_g$ *curvature* is the following scalar

$$Q_g = \frac{1}{12}\left(-\Delta_g S_g + S_g^2 - 3\|\mathrm{Ric}_g\|_g^2\right) \tag{4.21}$$

In [BN17], Buzano and Nguyen show the following theorem:

Theorem 4.25. Let $g = e^{2w} g_{\mathrm{eucl}}$ be a metric on $\mathbb{R}^4 \setminus \{0\}$ which is complete at infinity and has finite area over the origin. If g has finite total Q-curvature

$$\int_{\mathbb{R}^4} |Q| \mathrm{dvol}_g = \int_{\mathbb{R}^4} |Q| e^{4w} \mathrm{dvol}_{\mathbb{R}^4},$$

and non-negative scalar curvature at infinity and at the origin i.e.

$$\inf_{x\in\mathbb{R}^4\setminus B(0,r_2)} S_g(x) \geq 0 \quad \text{and} \quad \inf_{x\in B(0,r_1)} S_g(x) \geq 0,$$

for some $0 < r_1 \leq r_2 < \infty$, then we have

$$\chi(\mathbb{R}^4) - \frac{1}{4\pi^2}\int_{\mathbb{R}^4} Q\mathrm{dvol}_g = \nu - \mu, \tag{4.22}$$

where

$$\nu = \lim_{r\to\infty} \frac{\mathrm{Vol}_g(\partial B(0,r))^{4/3}}{4(2\pi^2)^{1/3}\mathrm{Vol}_g(B(0,r))}, \quad \mu = \lim_{r\to 0} \frac{\mathrm{Vol}_g(\partial B(0,r))^{4/3}}{4(2\pi^2)^{1/3}\mathrm{Vol}_g(B(0,r))} - 1.$$

Remark 4.26. (a) This theorem is actually a model case of their main result as they show a more general formula ([BN17, Thm 1.6]) which holds for manifolds that are not necessary conformally flat. In their case, neither the singularities nor the ends are assumed to be conical, but on the other hand they suppose the metric to be conformally flat on the ends and with non-negative scalar curvature at each singular point and at infinity on each end.

(b) Observe that both μ and ν are limits of isoperimetric ratios and do not involve the curvature tensor of the manifold, which is in sharp contrast with our invariant τ. The curvature is actually hidden in the relation between the Pfaffian and the Q-curvature.

The fact that the manifold is conformally flat yields an easy way to compare the Q-curvature and the Pfaffian of the curvature forms. To understand this relation a small digression about the decomposition of the curvature tensor is necessary.

Definition 4.27. Let (M, g) be an n-dimensional Riemannian manifold. The *Schouten tensor* of (M, g) is the following $(0, 2)$ symmetric tensor

$$A = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{S_g}{2(n-1)}g\right).$$

The Schouten tensor has the useful property that it appears in the decomposition of the curvature tensor as follows

$$R = W + A \owedge g,$$

where W is the Weyl tensor of (M, g) and $\owedge$ is the Kulkarni-Nomizu defined on p. 30. Observe that if (M, g) is conformally flat then $W \equiv 0$ and the curvature tensor writes simply as $R = A \owedge g$. Using this expression we can show that the Pfaffian can be written using elementary symmetric functions in the eigenvalues of A. This result comes from [Via00].

Lemma 4.28. Let (M^n, g) be an even dimensional conformally flat Riemannian manifold. Then
$$\mathrm{Pf}(\Omega) = (n/2)!\sigma_{n/2}(A)\mathrm{dvol}_g,$$
where $\sigma_k(A)$ denotes the k-th elementary symmetric functions in the eigenvalues of A.

Proof. Since $R = A \owedge g$ we have that the curvature forms of M are given (relatively to an orthonormal moving frame) by
$$\Omega_{ij} = \frac{1}{2}(A \owedge g)_{ijkl}\theta^k \wedge \theta^l,$$
with
$$(A \owedge g)_{ijkl} = (A \owedge g)(e_i, e_j, e_k, e_l) = A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}.$$
Therefore
$$\begin{aligned}
\Omega_{ij} &= \frac{1}{2}(A \owedge g)_{ijkl}\theta^k \wedge \theta^l \\
&= \frac{1}{2}\left(A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}\right)\theta^k \wedge \theta^l \\
&= \frac{1}{2}A_{ik}\theta^k \wedge \theta^j - \frac{1}{2}A_{jk}\theta^k \wedge \theta^i + \frac{1}{2}A_{jk}\theta^i \wedge \theta^k - \frac{1}{2}A_{ik}\theta^j \wedge \theta^k \\
&= -A_{ik}\theta^j \wedge \theta^k + A_{jk}\theta^i \wedge \theta^k.
\end{aligned}$$
Without loss of generality A is diagonal i.e. $A_{ij} = \lambda_i\delta_{ij}$ with λ_i the eigenvalues of A (no sum over the index i), and we obtain the following expression for the curvature forms
$$\Omega_{ij} = -\lambda_i\delta_{ik}\theta^j \wedge \theta^k + \lambda_j\delta_{jk}\theta^i \wedge \theta^k = (\lambda_i + \lambda_j)\theta^i \wedge \theta^j.$$
The Pfaffian can then be written using these expressions for Ω_{ij}:
$$\begin{aligned}
\mathrm{Pf}(\Omega) &= \frac{1}{2^{n/2}(n/2)!}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon_\sigma\Omega_{\sigma_1\sigma_2} \wedge \ldots \wedge \Omega_{\sigma_{n-1}\sigma_n} \\
&= \frac{1}{2^{n/2}(n/2)!}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon_\sigma(\lambda_{\sigma_1} + \lambda_{\sigma_2})\cdots(\lambda_{\sigma_{n-1}} + \lambda_{\sigma_n})\theta^{\sigma_1} \wedge \ldots \wedge \theta^{\sigma_n} \\
&= \frac{1}{2^{n/2}(n/2)!}\sum_{\sigma\in\mathfrak{S}_n}(\lambda_{\sigma_1} + \lambda_{\sigma_2})\cdots(\lambda_{\sigma_{n-1}} + \lambda_{\sigma_n})\mathrm{dvol}_g \\
&= \frac{1}{2^{n/2}(n/2)!}2^{n/2}\sum_{\sigma\in\mathfrak{S}_n}\lambda_{\sigma_1}\cdots\lambda_{\sigma_{n/2}}\mathrm{dvol}_g \\
&= \frac{1}{(n/2)!}((n/2)!)^2\sigma_2(A)\mathrm{dvol}_g \\
&= (n/2)!\sigma_2(A)\mathrm{dvol}_g.
\end{aligned}$$
□

One can show that if λ_i denote the eigenvalues of A, then

$$\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(|\mathrm{Tr}(A)|^2 - \|A\|_g^2\right) = \frac{1}{2(n-2)^2}\left(\frac{n}{4(n-1)}S^2 - \|\mathrm{Ric}_g\|^2\right).$$

In the particular case of dimension 4 this gives

$$\mathrm{Pf}(\Omega) = 2\sigma_2(A)\mathrm{dvol}_g, \quad \text{with} \quad \sigma_2(A) = \frac{1}{8}\left(\frac{1}{3}S^2 - \|\mathrm{Ric}_g\|^2\right).$$

Therefore, the Q-curvature 4.21 of a 4-dimensional conformally flat Riemannian manifold can be written as

$$Q\mathrm{dvol}_g = \mathrm{Pf}(\Omega) - \frac{1}{12}\Delta_g S_g \mathrm{dvol}_g. \tag{4.23}$$

Now that we have an explicit relation between the Pfaffian and the Q-curvature, we show that in the case of $\mathbb{R}^4$ endowed with the metric $g = f^2 \cdot \sum_{i=1}^4 dx_i^2$, where f is a smooth function on $\mathbb{R}^4$ satisfying

$$f(x) = \begin{cases} 1 & \text{if } \|x\| \leq \frac{1}{2}, \\ \|x\|^\alpha & \text{if } \|x\| \geq 1, \end{cases}$$

then our Main Theorem implies Theorem 4.25. The Riemannian manifold $(\mathbb{R}^4, g)$ is conformally flat and has one conical end with link $(\mathbb{S}^3, g_\alpha)$. Indeed, on the end $\mathbb{R}^4 \setminus B(0,1)$ the metric takes the form

$$t^{2\alpha} g_{\mathrm{eucl}} = t^{2\alpha}(t^2 g_2 + dt^2) = s^2 g_\alpha + ds^2,$$

where g_1 is the standard metric on $\mathbb{S}^3$, $g_\alpha = (\alpha+1)^2 g_1$ and $s = \frac{1}{\alpha+1} t^{\alpha+1}$. So the restriction of g to the end is conical. By Proposition 1.33 the scalar curvature of this cone is given by

$$S_g = \frac{1}{t^2}\left(S_{g_\alpha} - 6\right) = \frac{1}{t^2}\left(\frac{6}{(\alpha+1)^2} - 6\right) = \frac{6}{t^2}\left(\frac{1}{(\alpha+1)^2} - 1\right),$$

with S_{g_α} being the scalar curvature of $(\mathbb{S}^3, g_\alpha)$. It follows from this expression for the scalar curvature that the Laplacian of S_g vanishes on $\mathbb{R}^4 \setminus B(0,1)$ since $\Delta_g(t^{-2}) = 0$. Therefore the only contribution of the integral of the Laplacian is on the unit ball $B(0,1)$. Therefore by the divergence formula we have for some $r > 1$

$$\begin{aligned} \int_{B(0,1)} \Delta_g S_g \mathrm{dvol}_g &= \int_{B(0,t)} \Delta_g S_g \mathrm{dvol}_g \\ &= \int_{\partial B(0,t)} g(\nabla S_g, N)\mathrm{dvol}_g, \end{aligned}$$

with N the outward-pointing normal unit vector field. But knowing the expression for S_g on the cone it is easy to compute derivative of the scalar curvature:

$$\nabla S_g = -\frac{12}{t^3}\left(\frac{1}{(\alpha+1)^2}-1\right)\frac{\partial}{\partial t}.$$

Moreover the outward-pointing unit normal N at a point of $\partial B(0,t) = S(0,t)$ is given by $N = \frac{\partial}{\partial t}$ and the volume form of g is given by

$$\mathrm{dvol}_g = (\alpha+1)^3 t^3 \mathrm{dvol}_{g_1},$$

so that finally we have

$$\begin{aligned}\int_{B(0,1)} \Delta_g S_g \mathrm{dvol}_g &= -\int_{\mathbb{S}^3} \frac{12}{t^3}\left(\frac{1}{(\alpha+1)^2}-1\right)(\alpha+1)^3 t^3 \mathrm{dvol}_{g_1} \\ &= -24\pi^2\left((\alpha+1)-(\alpha+1)^3\right),\end{aligned}$$

where we have used the fact that the euclidean volume of $\mathbb{S}^3$ is $2\pi^2$. We have by Equation (4.23) and by the Main Theorem that

$$\begin{aligned}\chi(\mathbb{R}^4) - \frac{1}{4\pi^2}\int_{\mathbb{R}^4} Q\mathrm{dvol}_g &= \chi(\mathbb{R}^4) - \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\mathrm{Pf}(\Omega) + \frac{1}{48\pi^2}\int_{\mathbb{R}^4}\Delta_g S_g \mathrm{dvol}_g \\ &= \tau(\mathbb{S}^3, g_\alpha) + \frac{1}{48\pi^2}\int_{\mathbb{R}^4}\Delta_g S_g\mathrm{dvol}_g \\ &= \tau(\mathbb{S}^3, g_\alpha) - \frac{1}{2}\left((\alpha+1)-(\alpha+1)^3\right)\end{aligned}$$

But by Example 4.5 we know that

$$\begin{aligned}\tau(\mathbb{S}^3, g_\alpha) &= \frac{1}{8\pi^2}\int_{\mathbb{S}^3}(S_{g_\alpha}-2)\mathrm{dvol}_{g_\alpha} \\ &= \frac{1}{8\pi^2}\int_{\mathbb{S}^3}\left(\frac{6}{(\alpha+1)^2}-2\right)\mathrm{dvol}_{g_\alpha} \\ &= \frac{1}{2}\left(3(\alpha+1)-(\alpha+1)^3\right).\end{aligned}$$

Therefore the Gauss-Bonnet defect using the Q-curvature instead of the Pfaffian is given in our case by

$$\chi(\mathbb{R}^4) - \int_{\mathbb{R}^4} Q\mathrm{dvol}_g = \frac{1}{2}\left(3(\alpha+1)-(\alpha+1)^3\right) - \frac{1}{2}\left((\alpha+1)-(\alpha+1)^3\right) = \alpha+1$$

On the other hand, by Theorem 4.25 we have

$$\chi(\mathbb{R}^4) - \frac{1}{4\pi^2}\int_{\mathbb{R}^4} Q\mathrm{dvol}_g = \nu,$$

where

$$\nu = \lim_{t\to\infty} \frac{\mathrm{Vol}(\partial B_t(0), g)^{4/3}}{4(2\pi^2)^{1/3}\mathrm{Vol}(B_t(0), g)}.$$

But we have

$$\mathrm{Vol}(\partial B_t(0), g)^{4/3} = \left(t^3(\alpha+1)^3 2\pi^2\right)^{4/3} = r^4(\alpha+1)^4 2^{4/3}\pi^{8/3},$$

and

$$\mathrm{Vol}(B_t(0), g) = (\alpha+1)^3 \frac{r^4}{4} 2\pi^2,$$

hence

$$\begin{aligned}\nu &= \lim_{t\to\infty} \frac{\mathrm{Vol}(\partial B_t(0), g)^{4/3}}{4(2\pi^2)^{1/3}\mathrm{Vol}(B_t(0), g)} \\ &= (\alpha+1)\frac{2^{4/3}\pi^{8/3}}{2\pi^2 2^{1/3}\pi^{2/3}} \\ &= \alpha+1.\end{aligned}$$

4.9.3 A Remark About Singularities in Dimension 2

Remark 4.29. We finally rapidly explain why equation (6) is a special case of the Main Theorem. Following the terminology and notations in [HT92], one says that a conformal metric g on a compact Riemann surface S has a *simple singularity* of order $\beta \in \mathbb{R}$ at a point p if there is a local complex coordinate z in a neighbourhood of p such that in that neighbourhood

$$g = e^{2u}|z - z(p)|^{2\beta}|dz|^2$$

where u is a bounded function with integrable Laplacian. Using polar coordinates, we easily see that if $\beta > -1$ then p is a conical singularity with cone angle $\theta = 2\pi(1+\beta)$. Replacing the coordinate z by $w = 1/|z-p|$, one also sees that if $\beta < -1$, then a punctured neighborhood of p is a conical end with cone angle $\theta = -2\pi(1+\beta)$. Note that in both case the cone angle θ is the length of the link.

To derive Formula (6) from the Main Theorem, we now consider a compact surface S with conical singularities $\{p_1, \dots, p_r\}$ of order $\beta_i > -1$ and conical ends $\{p_{r+1}, \dots, p_m\}$ of order $\beta_j < -1$ (more precisely one obtains a conical end after removing the point p_j for $r+1 \le j < m$). We then have

$$\tau_k = \frac{1}{2\pi}\theta_k = \begin{cases} (\beta_k+1) & \text{if } k \le r \\ -(\beta_k+1) & \text{if } k > r. \end{cases}$$

With $M = S \setminus \{p_1, \dots p_r, q_1, \dots q_s\}$, we have from the main Theorem

$$\begin{aligned}\chi(S) - \frac{1}{2\pi}\int_S KdA &= (m + \chi(M)) - \frac{1}{2\pi}\int_M KdA \\ &= m - \sum_{i=1}^{r} \tau_i + \sum_{j=r+1}^{m} \tau_j \\ &= -\sum_{k=1}^{m} \beta_k.\end{aligned}$$

Appendices

Appendix A

On Asymptotically Conical Manifolds

This appendix is devoted to demonstrating the results about asymptotically conical manifolds that we used in the proof of Theorem 4.13. Let us recall the definition: let (N, g_N) be a compact $n-1$-dimensional Riemannian manifold and set $E = N \times (0, \infty)$. Denote by $\overline{g}$ the conical metric i.e. the warped-product product metric given by

$$\overline{g} = t^2 g_N + dt^2.$$

For $t \in (0, \infty)$ we write $N_t = N \times \{t\}$ the slice at height t.

Definition A.1. A Riemannian metric g on E is *asymptotically conical at order r and with (decreasing) rate α* if there exists a function $\rho : (1, \infty) \to \mathbb{R}$ such that $\rho(t) = o(t^{-\alpha})$ as $t \to \infty$ and such that for all $0 \leq k \leq r$ and $h = g - \overline{g}$ we have

$$\left\| \overline{\nabla}^k h \right\|_{\overline{g}} \leq \rho \tag{A.1}$$

where this notation means that at every point $(x, t) \in N \times (1, \infty)$ the following inequality holds

$$\left\| (\overline{\nabla}^k h)_{(x,t)} \right\|_{\overline{g}} \leq \rho(t).$$

A Riemannian manifold (E, g) endowed with an asymptotically conical metric is said to be asymptotically conical.

Our purpose requires to have a control over the curvature, so that the order is chosen to be at least $r = 2$ and this control has to be sufficiently strong, meaning that the decreasing rate must be at least $\alpha = n - 1$. In some particular cases the rate can be improved, but in general this value can not be lowered.

Remark A.2. In [CEV17], the authors require that at order r the rate of the k-th covariant derivative is exactly $\alpha = k$ or in other words they assume that

$$\left\| \overline{\nabla}^k h \right\|_{\overline{g}} = o(t^{-k}) \quad \text{as} \quad t \to \infty,$$

for all $0 \leq k \leq r$. Observe that this hypothesis implies that at order 0, there is no specification of the convergence rate of the norm of h towards 0. One can also compare with the asymptotic conditions given in the definition of an asymptotically conical end in [Con11].

The bilinear form h can be seen as a perturbation of the conical metric and we shall see how this modification affects the connection and the curvature. It appears that the condition (A.1) at order 2 ensures the convergence of the components Γ^i_{jk} of the connection ∇ and R^i_{jkl} the curvature tensor R with respect to a g-orthonormal moving frame towards the components $\overline{\Gamma}^i_{jk}$ of the connection $\overline{\nabla}$ and $\overline{R}^i_{jkl}$ of the curvature tensor $\overline{R}$ with respect to a $\overline{g}$-orthonormal frame.

A.1 Asymptotically Conical Manifolds at Order Zero

In this context, the moving frame approach requires two different moving frames, each of them being orthonormal with respect to either g or $\overline{g}$. So on $U \times (1, \infty) \subset M$ let $(\overline{e}_1, \ldots, \overline{e}_n)$ be an orthonormal frame such that the $\overline{e}_i$'s are tangent to N_t for $1 \leq i \leq n-1$ and $\overline{e}_n = \frac{\partial}{\partial t}$. Denote by $(\overline{\theta}^1, \ldots, \overline{\theta}^n)$ the dual coframe and let g_{ij} and h_{ij} be the components of g and h with respect to $\overline{g}$ i.e

$$g = g_{ij} \overline{\theta}^j \otimes \overline{\theta}^j \quad \text{and} \quad h = h_{ij} \overline{\theta}^i \otimes \overline{\theta}^j = (g_{ij} - \delta_{ij}) \overline{\theta}^i \otimes \overline{\theta}^j.$$

The condition (A.1) for $r = 0$ implies that the components of h vanish asymptotically:

$$|h_{ij}| \leq \|h\|_{\overline{g}} \leq \rho.$$

This implies that the metrics $\overline{g}$ and g are bounded with respect to each other, or in other words:

Proposition A.3. Two norms $\| \cdot \|_{\overline{g}}$ and $\| \cdot \|_g$ are equivalent norms.

Proof. Since $\sup_{x \in N_t} |h_{ij}| \to 0$ as $t \to 0$, there exists $C_1, C_2 > 0$ such that

$$|h(X, Y)| \leq C_2 |\overline{g}(X, Y)|$$

for all $X, Y \in \Gamma(M)$. Hence for any vector field $X \in \Gamma$ we have on one hand

$$\|X\|_g^2 = g(X, X) = \overline{g}(X, X) + h(X, X) \leq (1 + C_2) \|X\|_{\overline{g}}^2.$$

and on the other hand

$$\|X\|_{\overline{g}}^2 = \overline{g}(X, X) = g(X, X) - h(X, X) \leq (1 + C_1) \|X\|_g^2$$

which can be summed up as

$$\frac{1}{\sqrt{1+C_2}}\|X\|_g \leq \|X\|_{\overline{g}} \leq \sqrt{1+C_1}\|X\|_g.$$

The same argument can be carried out for higher order tensors, showing that $\|\cdot\|_g$ and $\|\cdot\|_{\overline{g}}$ are equivalent on $\mathrm{Tens}_l^k(M)$ for all $0 \leq k, l \leq n$. $\square$

In order to be able to compare the connections and the curvature tensors, it is convenient to introduce another moving frame which is orthonormal with respect to g. Applying the Gram-Schmidt process for the metric g to the frame $(\overline{e}_1, \ldots, \overline{e}_n)$ we obtain an orthonormal frame $(e_1, \ldots, e_n)$ on $U \times (1, \infty)$. This gives

$$e_i = \frac{\overline{e}_i - \sum_{j=1}^{i-1} \mathrm{proj}_{\overline{e}_j}(\overline{e}_i)}{\|\overline{e}_i - \sum_{j=1}^{i-1} \mathrm{proj}_{\overline{e}_j}(\overline{e}_i)\|_g}$$

where the projection is given for $1 \leq j \leq i-1$ by

$$\mathrm{proj}_{\overline{e}_j}(\overline{e}_i) = \frac{g(\overline{e}_j, \overline{e}_i)}{g(\overline{e}_j, \overline{e}_j)}\overline{e}_j = \frac{g_{ij}}{g_{jj}}\overline{e}_j = \frac{h_{ij}}{1+h_{jj}}\overline{e}_j.$$

Set

$$\lambda_i = \left\| \overline{e}_i - \sum_{j=1}^{i-1} \mathrm{proj}_{\overline{e}_j}(\overline{e}_i) \right\|_g = \left\| \overline{e}_i - \sum_{j=1}^{i-1} \frac{h_{ij}}{1+h_{jj}}\overline{e}_j \right\|_g$$

This new frame can be seen as the old frame plus a perturbation:

$$e_i = \overline{e}_i + \sum_{j=1}^{i} \alpha_i^j \overline{e}_i,$$

where the coefficients α_i^j for $1 \leq j \leq i \leq n$ are given by

$$\alpha_i^i = \frac{1-\lambda_i}{\lambda_i} \quad \text{and} \quad \alpha_i^j = \frac{-h_{ij}}{\lambda_i(1+h_{jj})}.$$

Since

$$\begin{aligned}
\lambda_i^2 &= 1 + h_{ii} - 2\sum_{j=1}^{i-1} \frac{h_{ij}^2}{1+h_{jj}} + \sum_{j,k=1}^{i-1} \frac{h_{ij}h_{ik}(\delta_{jk}+h_{jk})}{(1+h_{jj})(1+h_{kk})} \\
&= 1 + h_{ii} - \sum_{j=1}^{i-1} \frac{h_{ij}^2}{1+h_{jj}} + \sum_{\substack{j,k=1 \\ j\neq k}}^{i-1} \frac{h_{ij}h_{ik}h_{jk}}{(1+h_{jj})(1+h_{kk})} \\
&\leq 1 + |h_{ii}| + \sum_{j=1}^{i-1} \frac{|h_{ij}^2|}{|1+h_{jj}|} + \sum_{\substack{j,k=1 \\ j\neq k}}^{i-1} \frac{|h_{ij}||h_{ik}||h_{jk}|}{|1+h_{jj}||1+h_{kk}|} \\
&\leq 1 + \rho + \rho^2 \sum_{j=1}^{i-1} \frac{1}{|1+h_{jj}|} + \rho^3 \sum_{\substack{j,k=1 \\ j\neq k}}^{i-1} \frac{1}{|1+h_{jj}||1+h_{kk}|} \\
&\leq 1 + \rho + C_1\rho^2 + C_2\rho^3,
\end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$, it follows that the rate of convergence of λ_i toward 1 is

$$\lambda_i \leq \sqrt{1+\rho+C_1\rho^2+C_2\rho^3} = 1 + \frac{\rho}{2}.$$

This means for the coefficient α_i^i that

$$|\alpha_i^i| = \left|\frac{1}{\lambda_i} - 1\right| \leq |\lambda_i - 1| + o(|\lambda_i - 1|) \leq \rho.$$

The coefficients α_i^j for $1 \leq j < i \leq n$ have the same asymptotic beahviour:

$$\begin{aligned}
|\alpha_i^j| &= \frac{|h_{ij}|}{|\lambda_i||1+h_{jj}|} \\
&\leq \rho\left(1+\frac{\rho}{2}\right)(1+\rho) \\
&\leq \rho.
\end{aligned}$$

It is clear that the new associated coframe $(\theta^1, \dots, \theta^n)$ satisfies a similar equation:

$$\theta^i = \overline{\theta}^i + \sum_{j=1}^{i} \beta_j^i \overline{\theta}^j \quad \text{with} \quad |\beta_j^i| \leq \rho.$$

Using this, a relation between the volume forms of $\overline{g}$ and g can be worked out. Once again, we can see the quantity depending on g as a perturbation of the quantity depending on $\overline{g}$.

Lemma A.4. There exists $f \in C^\infty(U \times (1,\infty))$ such that $|f| \leq \rho$ and

$$\mathrm{dvol}_g = (1+f)\mathrm{dvol}_{\overline{g}}.$$

Proof. We have

$$\begin{aligned}
\mathrm{dvol}_g &= \theta^1 \wedge \ldots \wedge \theta^n \\
&= \left(\overline{\theta}^1 + \sum_{j=1}^{1} \beta^1_j \overline{\theta}^j\right) \wedge \left(\overline{\theta}^2 + \sum_{j=1}^{2} \beta^2_j \overline{\theta}^j\right) \wedge \ldots \wedge \left(\overline{\theta}^n + \sum_{j=1}^{n} \beta^n_j \overline{\theta}^j\right) \\
&= \overline{\theta}^1 \wedge \ldots \wedge \overline{\theta}^n + f\overline{\theta}^1 \wedge \ldots \wedge \overline{\theta}^n \\
&= (1+f)\mathrm{dvol}_{\overline{g}}
\end{aligned}$$

and f is a sum of combinatorial terms, each of them containing at least a factor β^i_j so that $|f| \le \rho$. □

A.2 Asymptotically Conical Manifolds at Order One

As it can be expected, the condition (A.1) for $r = 1$ provides a control on both the metric and its connection. Let $\overline{\nabla}$ (resp. ∇) be the Levi-Civita connection associated to $\overline{g}$ (resp. to g). It is known that the difference of two connections is tensorial, therefore we define a tensor A by setting for $X, Y \in \Gamma(M)$

$$A(X,Y) = \nabla_X Y - \overline{\nabla}_X Y.$$

The norm of the tensor A is then controlled by the derivatives of h with respect to $\overline{\nabla}$.

Proposition A.5. Let $X, Y, Z \in \Gamma(M)$, then

$$2g(A(X,Y),Z) = (\overline{\nabla}_X h)(Y,Z) + (\overline{\nabla}_Y h)(X,Z) + (\overline{\nabla}_Z h)(X,Y) \tag{A.2}$$

Proof. By applying twice the Koszul formula we have

$$\begin{aligned}
2g(A(X,Y),Z) &= 2g(\nabla_X Y, Z) - 2g(\overline{\nabla}_X Y, Z) \\
&= 2g(\nabla_X Y, Z) - 2\overline{g}(\overline{\nabla}_X Y, Z) - 2h(\overline{\nabla}_X Y, Z) \\
&= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) - g([X,Z],Y) \\
&\quad - g([Y,Z],X) - X\overline{g}(Y,Z) - Y\overline{g}(X,Z) + Z\overline{g}(X,Y) \\
&\quad - \overline{g}([X,Y],Z) + \overline{g}([X,Z],Y) + \overline{g}([Y,Z],X) - 2h(\overline{\nabla}_X Y, Z)
\end{aligned}$$

The covariant derivative of h is the $(3,0)$-tensor defined by

$$(\overline{\nabla} h)(Y,Z,X) = (\overline{\nabla}_X h)(Y,Z) = Xh(Y,Z) - h(\overline{\nabla}_X Y, Z) - h(Y, \overline{\nabla}_X Z),$$

therefore $Xh(Y,Z) = (\overline{\nabla}_X h)(Y,Z) + h(\overline{\nabla}_X Y, Z) + h(Y, \overline{\nabla}_X Z)$. Similarly we find

$$\begin{aligned}
Yh(X,Z) &= (\overline{\nabla}_Y h)(X,Z) + h(\overline{\nabla}_Y X, Z) + h(X, \overline{\nabla}_Y Z), \\
Zh(X,Y) &= (\overline{\nabla}_Z h)(X,Y) + h(\overline{\nabla}_Z X, Y) + h(X, \overline{\nabla}_Z Y).
\end{aligned}$$

Finally, we use the fact that $\overline{\nabla}$ is the Levi-Civita connection associated to $\bar{g}$ and is therefore torsion-free to write the Lie brackets as the difference $[X,Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X$. Replacing all this in the expression for $2g(A(X,Y),Z)$ we get after all the simplifications

$$2g(A(X,Y),Z) = (\overline{\nabla}_X h)(Y,Z) + (\overline{\nabla}_Y h)(X,Z) + (\overline{\nabla}_Z h)(X,Y)$$

□

This shows that

$$\|A\|_g \leq C \cdot \rho, \tag{A.3}$$

for some constant $C \in \mathbb{R}$ since by Proposition A.3 we have

$$\begin{aligned}
\|A\|_g &= \sup_{X,Y,Z \in S_g M} |g(A(X,Y),Z)| \\
&\leq \frac{1}{2} \sup_{X,Y,Z \in S_g M} \left|(\overline{\nabla}_X h)(Y,Z) + (\overline{\nabla}_Y h)(X,Z) + (\overline{\nabla}_Z h)(X,Y)\right| \\
&\leq \frac{3}{2} \|\overline{\nabla} h\|_g \\
&\leq C \|\overline{\nabla} h\|_{\overline{g}}.
\end{aligned}$$

and by assumption $\|\overline{\nabla} h\|_{\overline{g}} \leq \rho$. It can now be proved that there is a convergence of ∇ to $\overline{\nabla}$ in the sense that the components of ∇ with respect to a g-orthonormal moving frame converge to the components of $\overline{\nabla}$ with respect to a $\overline{g}$-orthonormal moving frame. Let $(\overline{e}_1, \ldots, \overline{e}_n)$ and $(e_1, \ldots, e_n)$ be the moving frames defined above and recall that

$$\Gamma_{ij}^k = g(\nabla_{e_i} e_j, e_k) \quad \text{and} \quad \overline{\Gamma}_{ij}^k = \overline{g}(\overline{\nabla}_{\overline{e}_i} \overline{e}_j, \overline{e}_k).$$

Let us first establish a technical result on the convergence of the derivatives of the coefficients α_i^i and α_i^j.

Lemma A.6. We have for all $1 \leq j \leq i$

$$\|d\alpha_i^j\|_{\overline{g}} \leq C \cdot \rho.$$

Proof. Recall that for $1 \leq j \leq i \leq n$ the coefficients α_i^j are given by

$$\alpha_i^i = \frac{1 - \lambda_i}{\lambda_i} \quad \text{and} \quad \alpha_i^j = \frac{-h_{ij}}{\lambda_i (1 + h_{jj})}, \quad \text{where} \quad \lambda_i = \left\| \overline{e}_i - \sum_{j=1}^{i-1} \frac{h_{ij}}{1 + h_{jj}} \overline{e}_j \right\|_g .$$

hence

$$\begin{aligned}
\overline{\nabla} \alpha_i^i &= d\alpha_i^i = -\frac{1}{\lambda_i^2} d\lambda_i \\
\overline{\nabla} \alpha_i^j &= d\alpha_i^j = -\frac{dh_{ij}}{\lambda_i (1 + h_{jj})} + \frac{h_{ij} d\lambda_i}{\lambda_i^2 (1 + h_{jj})} - \frac{h_{ij} dh_{jj}}{\lambda_i^2 (1 + h_{jj})^2}.
\end{aligned}$$

It is therefore necessary to compute the total derivative of λ_i. We use the fact that

$$d\lambda_i = \frac{1}{2\lambda_i} d(\lambda_i^2)$$

so that it is enough to calculate $d\lambda_i^2$:

$$\begin{aligned}
d(\lambda_i^2) = dh_{ii} - 2\sum_{j=1}^{i-1} &\frac{2h_{ij}dh_{ij}(1+h_{jj}) - h_{ij}^2 dh_{jj}}{(1+h_{jj})^2} \\
&+ \sum_{j,k=1}^{i-1} \frac{(dh_{ij}h_{ik}(\delta_{jk}+h_{jk}) + h_{ij}dh_{ik}(\delta_{jk}+h_{jk}) + h_{ij}h_{ik}dh_{jk})\,(1+h_{jj})(1+h_{kk})}{(1+h_{jj})^2(1+h_{kk})^2} \\
&- \sum_{j,k=1}^{i-1} \frac{h_{ij}h_{ik}(\delta_{jk}+h_{jk})(dh_{jj}(1+h_{kk}) + (1+h_{jj})dh_{kk})}{(1+h_{jj})^2(1+h_{kk})^2} \\
= dh_{ii} - 2\sum_{j=1}^{i-1} &\frac{h_{ij}}{1+h_{jj}} dh_{ij} + \sum_{j=1}^{i-1} \frac{h_{ij}^2}{(1+h_{jj})^2} dh_{jj} + \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ik}h_{jk}}{(1+h_{jj})(1+h_{kk})} dh_{ij} \\
&+ \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{jk}}{(1+h_{jj})(1+h_{kk})} dh_{ik} + \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{ik}}{(1+h_{jj})(1+h_{kk})} dh_{jk} \\
&- \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{ik}h_{jk}}{(1+h_{jj})^2(1+h_{kk})} dh_{jj} - \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{ik}h_{jk}}{(1+h_{jj})(1+h_{kk})^2} dh_{kk}.
\end{aligned}$$

Therefore the norm of λ_i^2 satisfies

$$\begin{aligned}
\|d(\lambda_i^2)\|_{\overline{g}} \leq \|dh_{ii}\|_{\overline{g}} &+ 2\sum_{j=1}^{i-1} \frac{|h_{ij}|}{|1+h_{jj}|} \|dh_{ij}\|_{\overline{g}} + \sum_{j=1}^{i-1} \frac{h_{ij}^2}{(1+h_{jj})^2} \|dh_{jj}\|_{\overline{g}} \\
&+ \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{|h_{ik}||h_{jk}|}{|1+h_{jj}||1+h_{kk}|} \|dh_{ij}\|_{\overline{g}} + \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{|h_{ij}||h_{jk}|}{|1+h_{jj}||1+h_{kk})|} \|dh_{ik}\|_{\overline{g}} \\
&+ \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{|h_{ij}||h_{ik}|}{|1+h_{jj}||1+h_{kk}|} \|dh_{jk}\|_{\overline{g}} + \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{|h_{ij}||h_{ik}||h_{jk}|}{(1+h_{jj})^2|1+h_{kk}|} \|dh_{jj}\|_{\overline{g}} \\
&+ \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{|h_{ij}||h_{ik}||h_{jk}|}{|1+h_{jj}|(1+h_{kk})^2} \|dh_{kk}\|_{\overline{g}} \\
&\leq \rho + C_1\rho^2 + C_2\rho^3 + C_3\rho^4,
\end{aligned}$$

since for all $1 \leq j \leq i \leq n$ the following bounds hold

$$|h_{ij}| \leq \rho, \quad \|dh_{ij}\|_{\overline{g}} \leq \rho \quad \text{and} \quad \frac{1}{|1+h_{jj}|^k} \quad \text{is bounded for all } k.$$

So the norm of the $d\alpha_i^i$ satisfies

$$\begin{aligned}\|d\alpha_i^i\|_{\overline{g}} &= \frac{1}{2\lambda_i^3}\|d\lambda_i^2\|_{\overline{g}} \\ &\leq \frac{1}{2\lambda_i^3}\left(\rho + C_1\rho^2 + C_2\rho^3 + C_3\rho^4\right) \\ &\leq C\cdot\rho,\end{aligned}$$

because $\frac{1}{\lambda_i}$ is bounded. It remains to deal with the case of $d\alpha_i^j$ for $j < i$, but from the expression for $d\alpha_i^j$ we deduce that

$$\begin{aligned}\|d\alpha_i^j\|_{\overline{g}} &\leq \frac{1}{\lambda_i|1+h_{jj}|}\|dh_{ij}\|_{\overline{g}} + \frac{|h_{ij}|}{\lambda_i^2|1+h_{jj}|}\|d\lambda_i\|_{\overline{g}} + \frac{|h_{ij}|}{\lambda_i^2(1+h_{jj})^2}\|dh_{jj}\|_{\overline{g}} \\ &\leq \frac{\rho}{\lambda_i|1+h_{jj}|} + \frac{\rho}{2\lambda_i^3|1+h_{jj}|}\|d(\lambda_i)^2\|_{\overline{g}} + \frac{\rho^2}{\lambda_i^2(1+h_{jj})^2} \\ &\leq \frac{\rho}{\lambda_i|1+h_{jj}|} + \frac{\rho^2 + C_1\rho^3 + C_2\rho^4 + C_3\rho^5}{2\lambda_i^3|1+h_{jj}|} + \frac{\rho^2}{\lambda_i^2(1+h_{jj})^2} \\ &\leq C\cdot\rho.\end{aligned}$$

□

This technical lemma is now used to show the anticipated result.

Proposition A.7. We have

$$|\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k| \leq C\cdot\rho.$$

Proof. So far we have the following relations between the objects defined (M, g) and $(M, \overline{g})$:

$$\begin{cases} g = \overline{g} + h, \\ e_i = \overline{e}_i + \alpha_i^\mu\overline{e}_\mu, \\ \nabla = \overline{\nabla} + A, \end{cases}$$

where h, α_i^j and A all vanish asymptotically in the sense describe above. Therefore

$$\begin{aligned}\Gamma_{ij}^k &= g(\nabla_{e_i}e_j, e_k) \\ &= \overline{g}(\nabla_{e_i}e_j, e_k) + h(\nabla_{e_i}e_j, e_k) \\ &= \overline{g}(\overline{\nabla}_{e_i}e_j, e_k) + \overline{g}(A(e_i, e_j), e_k) + h(\nabla_{e_i}e_j, e_k)\end{aligned} \tag{A.4}$$

First of all, by Equation (A.3) the term $\overline{g}(A(e_i, e_j), e_k)$ is bounded by

$$\overline{g}(A(e_i, e_j), e_k) \leq \|A\|_{\overline{g}} \leq C\cdot\rho$$

Now let us study more carefully the terms $\overline{g}(\overline{\nabla}_{e_i}e_j, e_k)$ and $h(\nabla_{e_i}e_j, e_k)$. First by replacing all the e_l by $\overline{e}_l + \alpha_l^\mu \overline{e}_\mu$ in $\overline{g}(\overline{\nabla}_{e_i}e_j, e_k)$ we get

$$\begin{aligned}\overline{g}(\overline{\nabla}_{e_i}e_j, e_k) = {} & \overline{g}(\overline{\nabla}_{\overline{e}_i}\overline{e}_j, \overline{e}_k) + \overline{g}(\alpha_i^\mu \overline{\nabla}_{\overline{e}_\mu}\overline{e}_j, \overline{e}_k) + \overline{g}(\overline{\nabla}_{\overline{e}_i}(\alpha_j^\nu \overline{e}_\nu), \overline{e}_k) + \overline{g}(\alpha_i^\mu \overline{\nabla}_{\overline{e}_\mu}(\alpha_j^\nu \overline{e}_\nu), \overline{e}_k) \\ & + \overline{g}(\overline{\nabla}_{\overline{e}_i}\overline{e}_j, \alpha_k^\rho \overline{e}_\rho) + \overline{g}(\alpha_i^\mu \overline{\nabla}_{\overline{e}_\mu}\overline{e}_j, \alpha_k^\rho \overline{e}_\rho) + \overline{g}(\overline{\nabla}_{\overline{e}_i}(\alpha_j^\nu \overline{e}_\nu), \alpha_k^\rho \overline{e}_\rho) \\ & + \overline{g}(\alpha_i^\mu \overline{\nabla}_{\overline{e}_\mu}(\alpha_j^\nu \overline{e}_\nu), \alpha_k^\rho \overline{e}_\rho)\end{aligned}$$

The first term on the right-hand side is precisely $\overline{\Gamma}_{ij}^k$. Moreover, all terms that do not involve a derivative of α_ν^μ will tend to zero asymptotically since the $\overline{\Gamma}_{ij}^k$ are bounded (see Section 1.10) and since $|\alpha_\nu^\mu| \leq \rho$. It remains to deal with the terms of the form $\overline{g}(\overline{\nabla}_{\overline{e}_k}(\alpha_i^j \overline{e}_j), \overline{e}_l)$, but by Lemma A.6 we know that the derivatives of α_i^j have the following asymptotics

$$\|d\alpha_i^j\|_{\overline{g}} \leq C \cdot \rho.$$

So we obtain the bounds

$$\begin{aligned}&|\overline{g}(\alpha_i^\mu \overline{\nabla}_{\overline{e}_\mu}\overline{e}_j, \overline{e}_k)| \leq |\alpha_i^\mu| \overline{\Gamma}_{\mu j}^k \leq \rho \cdot \overline{\Gamma}_{\mu j}^k. \\ &|\overline{g}(\overline{\nabla}_{\overline{e}_k}(\alpha_i^j \overline{e}_j), \overline{e}_l)| \leq |\overline{e}_k(\alpha_i^l)| + |\alpha_i^j| \overline{\Gamma}_{kj}^l \leq \|d\alpha_i^l\|_{\overline{g}} + \rho \cdot \overline{\Gamma}_{\mu j}^k \leq \tilde{C} \cdot \rho\end{aligned}$$

Repeating the process of replacing all the e_l by $\overline{e}_l + \alpha_l^\mu \overline{e}_\mu$ in $h(\nabla_{e_i}e_j, e_k)$, we again find that

$$|h(\nabla_{e_i}e_j, e_k)| = C \cdot \rho,$$

for the same reasons (and since $|h_{ij}| \leq \|h\|_{\overline{g}} \leq \rho$). Thus

$$|\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k| \leq C \cdot \rho.$$

□

A.3 Asymptotically Conical Manifolds at Order Two

By adding a control the asymptotic behaviour of the second derivative of h, it can be expected to obtain convergence results at the level of the curvature tensor R of (M, g).
As in the previous section, let us begin with a technical lemma, this time about the second derivatives of α_i^j.

Lemma A.8. For all $1 \leq j \leq i \leq n$ we have

$$\|\overline{\nabla} d\alpha_i^j\|_{\overline{g}} \leq C \cdot \rho.$$

Proof. As before we treat separately α_i^i and α_i^j for $j < i$. The computations made in the proof of Lemma A.6 lead to the following expression for the derivative of α_i^i:

$$
\begin{aligned}
d\alpha_i^i &= -\frac{1}{2\lambda_i^3} d(\lambda_i^2) \\
&= -\frac{1}{2\lambda_i^3} \Bigg(dh_{ii} - 2\sum_{j=1}^{i-1} \frac{h_{ij}}{1+h_{jj}} dh_{ij} + \sum_{j=1}^{i-1} \frac{h_{ij}^2}{(1+h_{jj})^2} dh_{jj} + \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ik}h_{jk}}{(1+h_{jj})(1+h_{kk})} dh_{ij} \\
&\qquad + \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{jk}}{(1+h_{jj})(1+h_{kk})} dh_{ik} + \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{ik}}{(1+h_{jj})(1+h_{kk})} dh_{jk} \\
&\qquad - \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{ik}h_{jk}}{(1+h_{jj})^2(1+h_{kk})} dh_{jj} - \sum_{\substack{j,k=1\\ j\neq k}}^{i-1} \frac{h_{ij}h_{ik}h_{jk}}{(1+h_{jj})(1+h_{kk})^2} dh_{kk} \Bigg).
\end{aligned}
$$

It is not difficult to see that the norm of the covariant derivative of each of the terms in brackets will be bounded by $C \cdot \rho$ if not a higher power of ρ. The term with slowest rate of convergence is

$$\|\overline{\nabla} h_{ii}\|_{\overline{g}} \leq \tilde{C} \cdot \rho.$$

Therefore

$$\|\overline{\nabla} d\alpha_i^i\|_{\overline{g}} \leq C \cdot \rho.$$

For $1 \leq j < i \leq n$ we know that

$$d\alpha_i^j = -\frac{dh_{ij}}{\lambda_i(1+h_{jj})} + \frac{h_{ij}d\lambda_i}{\lambda_i^2(1+h_{jj})} - \frac{h_{ij}dh_{jj}}{\lambda_i^2(1+h_{jj})^2},$$

hence

$$
\begin{aligned}
\|\overline{\nabla} d\alpha_i^j\|_{\overline{g}} \leq{}& \frac{1}{\lambda_i^2|1+h_{jj}|}\|d\lambda_i\|_{\overline{g}}\|dh_{ij}\|_{\overline{g}} + \frac{1}{\lambda_i(1+h_{jj})^2}\|dh_{jj}\|_{\overline{g}}\|dh_{ij}\|_{\overline{g}} + \frac{1}{\lambda_i|1+h_{jj}|}\|\overline{\nabla} dh_{ij}\|_{\overline{g}} \\
&+ \frac{1}{\lambda_i|1+h_{jj}|}\|dh_{ij}\|_{\overline{g}}\|d\lambda_i\|_{\overline{g}} + \frac{2|h_{ij}|}{\lambda_i^3|1+h_{jj}|}\|d\lambda_i\|_{\overline{g}}^2 \\
&+ \frac{|h_{ij}|}{\lambda_i^2|1+h_{jj}|}\|dh_{jj}\|_{\overline{g}}\|d\lambda_i\|_{\overline{g}} + \frac{|h_{ij}|}{\lambda_i^2|1+h_{jj}|}\|\overline{\nabla} d\lambda_i\|_{\overline{g}} \\
&+ \frac{1}{\lambda_i^2(1+h_{jj})^2}\|dh_{ij}\|_{\overline{g}}\|dh_{jj}\|_{\overline{g}} + \frac{2|h_{ij}|}{\lambda_i^3(1+h_{jj})^2}\|d\lambda_i\|_{\overline{g}}\|dh_{jj}\|_{\overline{g}} \\
&+ \frac{2|h_{ij}|}{\lambda_i^2|1+h_{jj}|^3}\|dh_{jj}\|_{\overline{g}}^2 + \frac{|h_{ij}|}{\lambda_i^2(1+h_{jj})^2}\|\overline{\nabla} dh_{jj}\|_{\overline{g}} \\
\leq{}& C \cdot \rho.
\end{aligned}
$$

Which completes the proof. □

This technical lemma allows us to show the convergence of the components of R to $\overline{R}$ in the following sense.

Proposition A.9. We have

$$|R^i_{jkl} - \overline{R}^i_{jkl}| = C \cdot \rho.$$

Proof. Recall that by Lemma 1.14 we have the following expression for the components of the curvature tensor:

$$R^i_{jkl} = e_k(\Gamma^i_{lj}) - e_l(\Gamma^i_{kj}) + \Gamma^\mu_{lj}\Gamma^i_{k\mu} - \Gamma^\mu_{kj}\Gamma^i_{l\mu} + (\Gamma^\mu_{lk} - \Gamma^\mu_{kl})\,\Gamma^i_{\mu j}. \tag{A.5}$$

Moreover by Proposition A.7 there exist smooth functions $\eta^i_{jk} \in C^\infty(U \times (1,\infty))$ such that $|\eta^i_{jk}| \leq C \cdot \rho$ and

$$\Gamma^i_{jk} = \overline{\Gamma}^i_{jk} + \eta^i_{jk}.$$

Therefore by replacing Γ^i_{jk} by $\overline{\Gamma}^i_{jk} + \eta^i_{jk}$, and e_k by its expression as a "perturbed frame" i.e. $e_k = \overline{e}_k + \alpha^\nu_k \overline{e}_k$, we get from Equation (A.5):

$$\begin{aligned}
R^i_{jkl} &= (\overline{e}_k + \alpha^\nu_k \overline{e}_\nu)(\overline{\Gamma}^i_{lj} + \eta^i_{lj}) - (\overline{e}_l + \alpha^\nu_l \overline{e}_\nu)(\overline{\Gamma}^i_{kj} + \eta^i_{kj}) + (\overline{\Gamma}^\mu_{lj} + \eta^\mu_{lj})(\overline{\Gamma}^i_{k\mu} + \eta^i_{k\mu}) \\
&\quad - (\overline{\Gamma}^\mu_{kj} + \eta^\mu_{kj})(\overline{\Gamma}^i_{l\mu} + \eta^i_{l\mu}) + (\overline{\Gamma}^\mu_{lk} + \eta^\mu_{lk} - \overline{\Gamma}^\mu_{kl} - \eta^\mu_{kl})(\overline{\Gamma}^i_{\mu j} + \eta^i_{\mu j}) \\
&= \Big(\overline{e}_k(\overline{\Gamma}^i_{lj}) - \overline{e}_l(\overline{\Gamma}^i_{kj}) + \overline{\Gamma}^\mu_{lj}\overline{\Gamma}^i_{k\mu} - \overline{\Gamma}^\mu_{kj}\overline{\Gamma}^i_{l\mu} + (\overline{\Gamma}^\mu_{lk}) - \overline{\Gamma}^\mu_{kl})\overline{\Gamma}^i_{\mu j}\Big) \\
&\quad + \overline{e}_k(\eta^i_{lj}) + \alpha^\nu_k \overline{e}_\nu(\overline{\Gamma}^i_{lj}) + \alpha^\nu_k \overline{e}_\nu(\eta^i_{lj}) - \overline{e}_l(\eta^i_{kj}) - \alpha^\nu_l \overline{e}_\nu(\overline{\Gamma}^i_{kj}) - \alpha^\nu_l \overline{e}_\nu(\eta^i_{kj}) \\
&\quad + \overline{\Gamma}^\mu_{lj}\eta^i_{k\mu} + \overline{\Gamma}^i_{k\mu}\eta^\mu_{lj} + \eta^\mu_{lj}\eta^i_{k\mu} - \overline{\Gamma}^\mu_{kj}\eta^i_{l\mu} - \eta^\mu_{kj}\overline{\Gamma}^i_{l\mu} - \eta^\mu_{kj}\eta^i_{l\mu} + \overline{\Gamma}^\mu_{lk}\eta^i_{\mu j} + \overline{\Gamma}^i_{\mu j}\eta^\mu_{lk} \\
&\quad + \eta^\mu_{lk}\eta^i_{\mu j} - \overline{\Gamma}^\mu_{kl}\eta^i_{\mu j} - \overline{\Gamma}^i_{\mu j}\eta^\mu_{kl} - \eta^\mu_{kl}\eta^i_{\mu j}.
\end{aligned}$$

Therefore the difference of the components of R and $\overline{R}$ is given by

$$\begin{aligned}
R^i_{jkl} - \overline{R}^i_{jkl} &= \overline{e}_k(\eta^i_{lj}) + \alpha^\nu_k \overline{e}_\nu(\overline{\Gamma}^i_{lj}) + \alpha^\nu_k \overline{e}_\nu(\eta^i_{lj}) - \overline{e}_l(\eta^i_{kj}) - \alpha^\nu_l \overline{e}_\nu(\overline{\Gamma}^i_{kj}) - \alpha^\nu_l \overline{e}_\nu(\eta^i_{kj}) \\
&\quad + \overline{\Gamma}^\mu_{lj}\eta^i_{k\mu} + \overline{\Gamma}^i_{k\mu}\eta^\mu_{lj} + \eta^\mu_{lj}\eta^i_{k\mu} - \overline{\Gamma}^\mu_{kj}\eta^i_{l\mu} - \eta^\mu_{kj}\overline{\Gamma}^i_{l\mu} - \eta^\mu_{kj}\eta^i_{l\mu} + \overline{\Gamma}^\mu_{lk}\eta^i_{\mu j} + \overline{\Gamma}^i_{\mu j}\eta^\mu_{lk} \\
&\quad + \eta^\mu_{lk}\eta^i_{\mu j} - \overline{\Gamma}^\mu_{kl}\eta^i_{\mu j} - \overline{\Gamma}^i_{\mu j}\eta^\mu_{kl} - \eta^\mu_{kl}\eta^i_{\mu j}.
\end{aligned}$$

The terms that do not involve a derivative of $\overline{\Gamma}^i_{jk}$ or η^i_{jk} are already known to be bounded by $C \cdot \rho$ since the $\overline{\Gamma}^i_{jk}$ are bounded in t and $|\eta^i_{jk}| \leq C \cdot \rho$. So it remains to show that we have

$$\begin{aligned}
|\overline{e}_l(\overline{\Gamma}^i_{jk})| &= |d\overline{\Gamma}^i_{jk}(\overline{e}_l)| \leq C \cdot \rho, \\
|\overline{e}_l(\eta^i_{jk})| &= |d\eta^i_{jk}(\overline{e}_l)| \leq C \cdot \rho.
\end{aligned}$$

First recall from Section 1.10 that for $1 \leq i,j,k \leq n-1$, the $\overline{\Gamma}^i_{jk}$ are the components of the connection of the manifold N and therefore do not depend on t. Moreover if $1 \leq j,k \leq n-1$

$$\overline{\Gamma}^n_{jk} = \begin{cases} 0 & \text{if } j \neq k, \\ -\frac{1}{t} & \text{if } i = j. \end{cases}$$

Hence we have the following possibilites

$$\overline{e}_l(\overline{\Gamma}^i_{jk}) = \begin{cases} \text{does not depend on } t & \text{if } 1 \leq i,j,k,l \leq n-1, \\ 0 & \text{if } 1 \leq i,j,k \leq n-1, \quad l = n \\ 0 & \text{if } i = n, \quad 1 \leq j,k,l \leq n-1, \quad j \neq k, \\ t^{-2} & \text{if } i = l = n, \quad 1 \leq j = k \leq n-1. \end{cases}$$

In any case, we have

$$|\alpha_l^\nu \overline{e}_\nu(\overline{\Gamma}^i_{jk})| \leq C \cdot \rho.$$

Finally, we we deal with the derivatives of the η^k_{ij}. Expanding Equation (A.4) to its maximum we find the following (rather long) expression for the η^k_{ij}:

$$\begin{aligned}
\eta^k_{ij} &= \Gamma^k_{ij} - \overline{\Gamma}^k_{ij} \\
&= \alpha_i^\mu \overline{\Gamma}^k_{\mu j} + \overline{e}_i(\alpha_j^k) + \alpha_j^\nu \overline{\Gamma}^k_{i\nu} + \alpha_i^\mu \overline{e}_\mu(\alpha_j^k) + \alpha_i^\mu \alpha_i^\nu \overline{\Gamma}^k_{\mu\nu} + \alpha_k^\mu \overline{\Gamma}^\mu_{ij} + \alpha_i^\mu \alpha_k^\nu \overline{\Gamma}^\nu_{\mu j} \\
&\qquad + \alpha_k^\mu \overline{e}_i(\alpha_j^\mu) + \alpha_k^\mu \alpha_j^\nu \overline{\Gamma}^\mu_{i\nu} + \alpha_i^\mu \alpha_k^\nu \overline{e}_\mu(\alpha_j^\nu) + \alpha_i^\mu \alpha_j^\nu \alpha_k^\rho \overline{\Gamma}^\rho_{\mu\nu} + A^k_{ij} + \alpha_k^\nu A^\nu_{ij} \\
&\qquad + \alpha_j^\nu A^k_{i\nu} + \alpha_j^\nu \alpha_k^\mu A^\mu_{i\nu} + \alpha_i^\mu A^k_{\mu j} + \alpha_i^\mu \alpha_k^\nu A^\nu_{\mu j} + \alpha_i^\mu \alpha_j^\nu A^k_{\mu\nu} + \alpha_i^\mu \alpha_j^\nu \alpha_k^\rho A^\rho_{\mu\nu} \\
&\qquad + \overline{\Gamma}^\mu_{ij} h_{\mu k} + \alpha_k^\mu \overline{\Gamma}^\nu_{ij} h_{\nu\mu} + \overline{e}_i(\alpha_j^\mu) h_{\mu k} + \alpha_j^\mu \overline{\Gamma}^\nu_{i\mu} h_{\nu k} + \overline{e}_i(\alpha_j^\mu) \alpha_k^\nu h_{\mu\nu} + \alpha_j^\mu \alpha_k^\nu \overline{\Gamma}^\rho_{i\mu} h_{\rho\nu} \\
&\qquad + \alpha_i^\nu \overline{\Gamma}^\mu_{\nu j} h_{\mu k} + \alpha_i^\nu \alpha_k^\mu \overline{\Gamma}^\rho_{\nu j} h_{\rho\mu} + \alpha_i^\nu \overline{e}_\nu(\alpha_j^\mu) h_{\mu k} + \alpha_i^\nu \alpha_j^\mu \overline{\Gamma}^\rho_{\nu\mu} h_{\rho k} + \alpha_i^\nu \alpha_k^\rho \overline{e}_\nu(\alpha_j^\mu) h_{\mu\rho} \\
&\qquad + \alpha_i^\nu \alpha_k^\rho \alpha_j^\mu \overline{\Gamma}^\sigma_{\nu\mu} h_{\sigma\rho},
\end{aligned}$$

where A^k_{ij} and h_{ij} are the components of A and h with respect to $(\overline{e}_1, \ldots, \overline{e}_n)$. We have to show that the derivative in the direction $\overline{e}_l$ of every term on the right-hand side of the above expression bounded by a constant times the function ρ plus possibly a map which is little o of ρ. Fortunately, the following asymptotic behaviours are already known:

- $\|dh_{ij}\|_{\overline{g}} \leq \rho$ for all $1 \leq i,j \leq n$;
- $\|d\alpha_j^i\|_{\overline{g}} \leq C \cdot \rho$ for all $1 \leq i,j \leq n$;
- $\|A\|_{\overline{g}} \leq C \cdot \rho$;
- $|\alpha_l^\nu \overline{e}_\nu(\overline{\Gamma}^i_{jk})| \leq C \cdot \rho$;

Therefore differentiating the above expression for η^k_{ij} in the direction $\overline{e}_l$, we note that it is enough to show that

$$\begin{aligned}
|\overline{e}_l(\overline{e}_k(\alpha_j^i))| &\leq C \cdot \rho, \\
|\overline{e}_l(A^k_{ij})| &\leq C \cdot \rho,
\end{aligned}$$

as $\rho \to 0$. But by Lemma A.8 we have

$$\begin{aligned}|\overline{e}_l(\overline{e}_k(\alpha_j^i))| &\leq |(\overline{\nabla} d\alpha_i^j)(\overline{e}_k, \overline{e}_l)| + |\overline{\Gamma}_{lk}^{\mu} d\alpha_i^j(\overline{e}_\mu)| \\ &\leq \|\overline{\nabla}^2 h\|_{\overline{g}} + \|\overline{\nabla} h\|_{\overline{g}} \\ &\leq C \cdot \rho.\end{aligned}$$

Finally, we deal with the derivatives of A_{ij}^k:

$$\begin{aligned}\overline{e}_l(A_{ij}^k) &= \overline{\nabla}_{\overline{e}_l}(\overline{g}(A(\overline{e}_i, \overline{e}_j), \overline{e}_k)) \\ &= \overline{\nabla}_{\overline{e}_l}\left(g(A(\overline{e}_i, \overline{e}_j), \overline{e}_k)\right) - \overline{\nabla}_{\overline{e}_l} h(A(\overline{e}_i, \overline{e}_j), \overline{e}_k) \\ &= \frac{1}{2}\overline{\nabla}_{\overline{e}_l}\left(\overline{\nabla}_{\overline{e}_i} h(\overline{e}_j, \overline{e}_k) + \overline{\nabla}_{\overline{e}_j} h(\overline{e}_i, \overline{e}_k) + \overline{\nabla}_{\overline{e}_k} h(\overline{e}_i, \overline{e}_j)\right) - \overline{\nabla}_{\overline{e}_l} h(A(\overline{e}_i, \overline{e}_j), \overline{e}_k),\end{aligned}$$

where the last equation comes from Proposition A.5. Hence

$$|\overline{e}_l(A_{ij}^k)| \leq \tilde{C}\|\overline{\nabla}^2 h\|_{\overline{g}} + \|\overline{\nabla} h\|_{\overline{g}}\|A\|_{\overline{g}} \leq C \cdot \rho,$$

which completes the proof. □

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